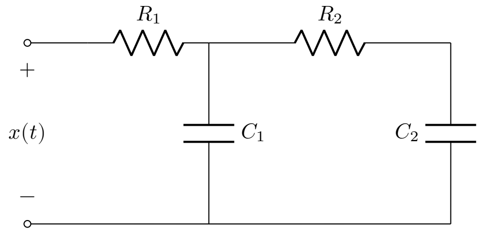
Solutions

1. Given the following circuit,



derive the state-space description in the form

$$\dot{q} = Aq + Bx$$

where q is the state vector.

$$A = \begin{bmatrix} -\left(\frac{1}{R_1C_1} + \frac{1}{R_2C_1}\right) & \frac{1}{R_2C_1} \\ \frac{1}{R_2C_2} & -\frac{1}{R_2C_2} \end{bmatrix} \qquad B = \begin{bmatrix} \frac{1}{R_1C_1} \\ 0 \end{bmatrix}$$

Solution: Let q_1 and q_2 be the states corresponding to the capacitor voltages C_1 and C_2 respectively. Consider the two clockwise loop currents i_1 and i_2. Then using KVL

$$x = R_1 i_1 + q_1 \tag{1}$$

and

$$q_1 = R_2 i_2 + q_2 \tag{2}$$

The currents are

$$C_1 \dot{q}_1 = i_1 - i_2 \tag{3}$$

$$C_2 \dot{q}_2 = i_2 \tag{4}$$

from (**1**)

$$i_1 = -\frac{1}{R_1} q_1 + \frac{1}{R_1} x \tag{5}$$

from (**2**)

$$i_2 = \frac{1}{R_2} q_1 - \frac{1}{R_2} q_1 \tag{6}$$

Substitution of (**5**) and (**6**) into (**3**) and (**4**) gives

$$\begin{aligned} \dot{q}_1 &= -\left(\frac{1}{R_1C_2} + \frac{1}{R_2C_2}\right) q_1 + \frac{1}{R_2C_1} q_2 + \frac{1}{R_1C_1} x \\ \dot{q}_2 &= \frac{1}{R_2C_2} q_1 - \frac{1}{R_2C_2} q_2 \end{aligned}$$

Thus

$$A = \begin{bmatrix} -\left(\frac{1}{R_1C_2} + \frac{1}{R_2C_2}\right) & \frac{1}{R_2C_1} \\ \frac{1}{R_2C_2} & -\frac{1}{R_2C_2} \end{bmatrix}$$

and

$$B = \begin{bmatrix} \frac{1}{R_1C_1} \\ 0 \end{bmatrix}$$

2. Given the transfer function

$$H(s) = \frac{100}{(s+5)(s^2+2s+7)} ,$$

derive the state-space description in the form

$$\dot{q} = Aq + Bx$$

$$y = Cq + Dx$$

where q is the state vector, x is the input, and y the output.

$$\boxed{A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -35 & -17 & -7 \end{bmatrix} \quad B = \begin{bmatrix} 0 \\ 0 \\ 100 \end{bmatrix} \quad C = [1\ 0\ 0] \quad D = [0]}$$

Solution:

$$H(s) = \frac{Y(s)}{X(s)} = \frac{100}{(s+5)(s^2+2s+7)}$$

or

$$s^3Y(s) + 7s^2Y(s) + 17sY(s) + 35Y(s) = 100X(s)$$

This is equivalent to the ODE

$$D^3y(t) + 7D^2y(t) + 17Dy(t) + 35y(t) = 100x(t)$$

Let $q_1 = y$, $q_2 = Dy$, and $q_3 = D^2y$, then

$$D^3y = \dot{q}_3 = -7q_3 - 17q_2 - 35q_1 + 100x$$

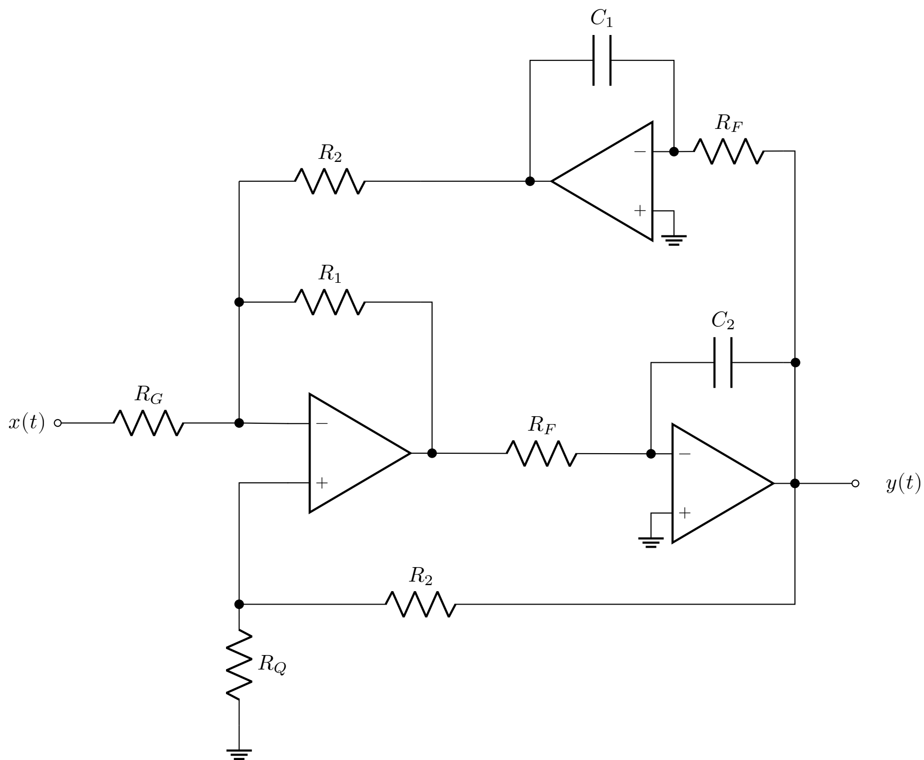
$$\dot{q}_2 = q_3$$

$$\dot{q}_1 = q_2$$

Thus

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -35 & -17 & -7 \end{bmatrix} \quad B = \begin{bmatrix} 0 \\ 0 \\ 100 \end{bmatrix} \quad C = [1\ 0\ 0] \quad D = [0]$$

3. Consider the circuit from Lab 1



Derive the state-space description in the form

$$\dot{q} = Aq + Bx$$

$$y = Cq + Dx$$

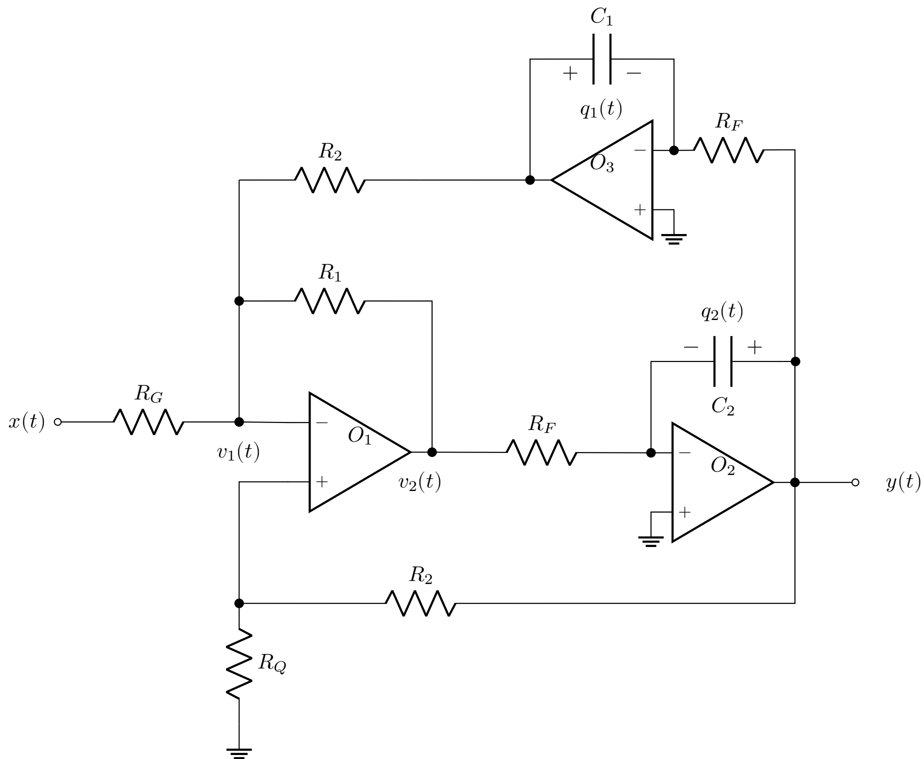
where q is the state vector, x is the input, and y the output. Then convert that description to the transfer function $H(s)$.

$$A = \begin{bmatrix} 0 & -\frac{1}{R_F C_1} \\ \frac{R_1}{R_2 C_2 R_F} & -\frac{R_1 R_2 + R_2 R_G + R_1 R_G}{R_2 R_G} \frac{R_Q}{(R_2 + R_Q) C_2 R_F} \end{bmatrix} \quad B = \begin{bmatrix} 0 \\ \frac{R_1}{C_2 R_G R_F} \end{bmatrix}$$

$$C = [0 \quad 1] \qquad D = [0 \quad 0]$$

$$H(s) = \frac{\frac{R_1}{C_2 R_F R_G} s}{s^2 + \frac{R_1 R_2 + R_2 R_G + R_1 R_G}{R_2 R_G} \frac{R_Q}{(R_2 + R_Q) C_2 R_F} s + \frac{R_1}{R_2 C_1 C_2 R_F^2}}$$

Solution: First we choose the state variables as the capacitor voltages with the orientation indicated below.



Next we write down the circuit equations. Consider the voltages at the nodes indicated above. Doing a KCL at node v_1

$$\frac{x - v_1}{R_G} + \frac{v_2 - v_1}{R_1} + \frac{q_1 - v_1}{R_2} = 0 \tag{7}$$

A KCL at the inverting input to O_2 which by the virtual ground is at a potential of zero:

$$\frac{v_2}{R_F} + C_2 \dot{q}_2 = 0 \tag{8}$$

Similartly, a KCL at the inverting input to O_3 which by the virtual ground is at a potential of zero:

$$C_1 \dot{q}_1 + \frac{q_2}{R_F} = 0 \tag{9}$$

The voltage divider at the bottom gives:

$$\frac{q_2 - v_1}{R_2} = \frac{v_1}{R_Q} \tag{10}$$

We then eliminate all the variables other than the state variables and the input (i.e. v_1, v_2, and v_3). For example, from (9) we get the first state equation

$$\dot{q}_1 = -\frac{1}{R_F C_1} q_2$$

From (8) we get

$$\dot{q}_2 = -\frac{1}{R_F C_2} v_2 \tag{11}$$

From (10) we get

$$v_1 = \frac{R_Q}{R_2 + R_Q} q_2 \tag{12}$$

Then substitute (12) into (7) and solve for v_2

$$v_2 = -\frac{R_1}{R_2} q_1 + \frac{R_1 R_2 + R_2 R_G + R_1 R_G}{R_2 R_G} \frac{R_Q}{R_2 + R_Q} q_2 - R_1 x \tag{13}$$

Then substitute (13) into (11) to get the second state equation

$$\dot{q}_2 = \frac{R_1}{R_2 C_2 R_F} q_1 - \frac{R_1 R_2 + R_2 R_G + R_1 R_G}{R_2 R_G} \frac{R_Q}{(R_2 + R_Q)\, C_2 R_F} q_2 + \frac{R_1}{C_2 R_G R_F} x$$

Thus the final form of the state equations is

$$\dot{q} = Aq + Bx$$

where

$$A = \begin{bmatrix} 0 & a \\ b & c \end{bmatrix} = \begin{bmatrix} 0 & -\frac{1}{R_F C_1} \\ \frac{R_1}{R_2 C_2 R_F} & -\frac{R_1 R_2 + R_2 R_G + R_1 R_G}{R_2 R_G} \frac{R_Q}{(R_2 + R_Q)\, C_2 R_F} \end{bmatrix} \quad B = \begin{bmatrix} 0 \\ d \end{bmatrix} = \begin{bmatrix} 0 \\ \frac{R_1}{C_2 R_G R_F} \end{bmatrix}$$

To find the transfer function we note that $y(t) = q_2(t)$. Thus the observation equation is simply

$$y = Cq + Dx = [0 \quad 1]\, q + [0 \quad 0]\, x$$

To find the transfer function we note

$$H(s) = \frac{Y(s)}{X(s)} = \frac{Q_2(S)}{X(s)}$$

Taking the Laplace transform of the state equations and solving for Q gives

$$Q(s) = (sI - A)^{-1}\, B\, x$$

where

$$(sI - A)^{-1} = \frac{1}{s^2 - cs - a\, b} \begin{bmatrix} s - c & -a \\ -b & s \end{bmatrix}$$

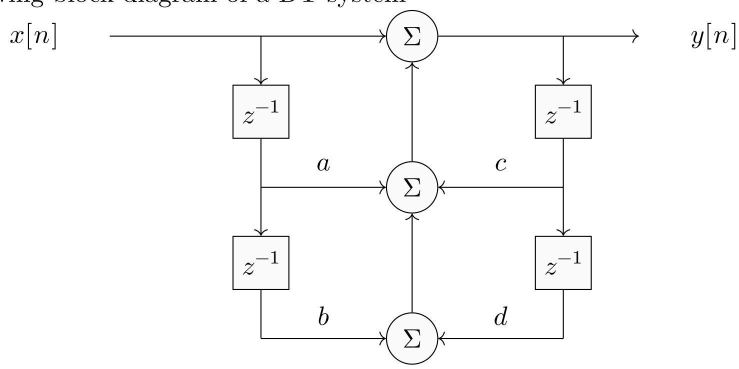
Thus

$$H(s) = \frac{ds}{s^2 - c\, s - a\, b}$$

Substituting for the constants

$$H(s) = \frac{\frac{R_1}{C_2 R_G R_F} s}{s^2 + \frac{R_1 R_2 + R_2 R_G + R_1 R_G}{R_2 R_G} \frac{R_Q}{(R_2 + R_Q)\, C_2 R_F}\, s + \frac{R_1}{R_2 C_1 C_2 R_F^2}}$$

4. Given the following block diagram of a DT system

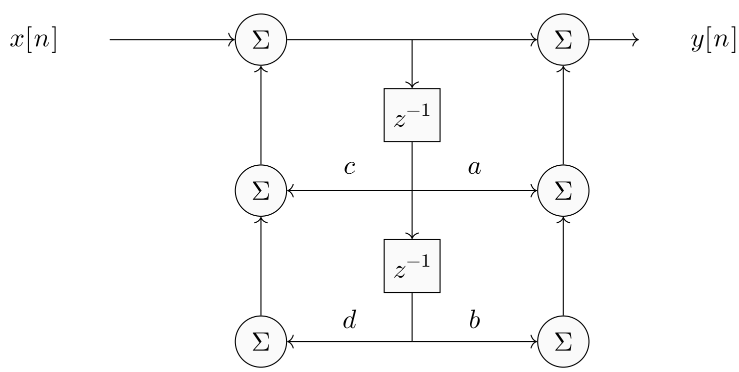


derive the state space description in the form

$$q[n+1] = Aq[n] + Bx[n]$$
$$y[n] = Cq[n] + Dx[n]$$

$$\boxed{A = \begin{bmatrix} 0 & 1 \\ d & c \end{bmatrix} \quad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \quad C = \begin{bmatrix} (b+d) & (a+c) \end{bmatrix} \quad D = [1]}$$

Solution: The given block diagram is in DFI. Reversing the order of the series combination gives the direct form II.



Let q_2 = output of second delay and q_1 = output of first delay. Then

$$\begin{aligned} q_2[n+1] &= cq_2[n] + dq_1[n] + x \\ q_1[n+1] &= q_2 \end{aligned}$$

and

$$\begin{aligned} y &= q_2[n+1] + aq_2[n] + bq_1[n] \\ &= (a+c)\,q_2 + (b+d)\,q_1 + x \end{aligned}$$

Thus

$$A = \begin{bmatrix} 0 & 1 \\ d & c \end{bmatrix} \quad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \quad C = \begin{bmatrix} (b+d) & (a+c) \end{bmatrix} \quad D = [1]$$

5. For the circuit in problem 1 where $R_1 = 100\,\mathrm{k}\Omega$, $R_2 = 50\,\mathrm{k}\Omega$, and $C_1 = C_2 = 1\,\mu\mathrm{F}$, use the derived state-space desciption to simulate the system when the input is the expression

$$x(t) = \frac{te^{-t}}{2 + \cos(10t)} u(t)$$

using the Runge-Kutta integrator (in C/C++ as demonstrated in class, or using the ode45 command in Matlab) for 8 seconds, starting at $t = 0$. Plot the input and all state trajectories, overlayed on the same plot, or as separate subplots. Submit your code as a zip file.

Solution: Substituting the values we get

$$\begin{aligned}\dot{q} &= \begin{bmatrix} -30 & 20 \\ 20 & -20 \end{bmatrix} q + \begin{bmatrix} 10 \\ 0 \end{bmatrix} x \\ &= f(t, q)\end{aligned}$$

where

$$f(t, q) = \begin{bmatrix} -30q_1 + 20q_2 + \frac{10te^{-t}}{2+\cos(10t)} \\ 20q_1 - 20q_2 \end{bmatrix}$$

When simulated the resulting state space trajectories are