Lecture 19: Properties of Z transform

Today we focus on some important properties of Z transform. These are usefull because

- properties + table of transforms is a convienient way to perform forward and inverse transforms without iteration
- the techniques used are often employed when solving problems.
- the properties and proofs provide some intuition about transform.

- Linearity Property: Given $x_1[n] \xrightarrow{Z} X_1(z)$ with ROC R_1

$x_2[n] \xrightarrow{Z} X_2(z)$ with ROC R_2

then for constants a, b

$$a x_1[n] + b x_2[n] \overset{Z}{\longleftrightarrow} a X_1(z) + b X_2(z)$$

$$\text{ROC} = R_1 \cap R_2$$

Proof: From definition $Z\{a x_1[n] + b x_2[n]\} =$

$$\sum_{n=-\infty}^{\infty} (a x_1[n] + b x_2[n]) z^{-n} = a \sum_{n=-\infty}^{\infty} x_1[n] z^{-n} + b \sum_{n=-\infty}^{\infty} x_2[n] z^{-n}$$

$$= a X_1(z) + b X_2(z)$$

Example: What is Z transform of

$$x[n] = 3\left(\tfrac{1}{2}\right)^n u[n] + 2\left(\tfrac{1}{4}\right)^n u[n] + u[n]$$

Recall that $Z\{r^n u[n]\} = \dfrac{z}{z-r}$ for $|z| > r$

Using linearity property $X(z) = 3\dfrac{z}{z-\frac{1}{2}} + 2\dfrac{z}{z-\frac{1}{4}} + \dfrac{z}{z-1}$

Example: What is inverse Z transform of $X(z) = \dfrac{5z^2 - 11z}{z^2 - 5z + 6}$ $|z| > 3$

We note ROC corresponds to causal signal and can use PFE, but note not strictly proper rational function.

Thus we would have to expand as

$$X(z) = A + \frac{B}{z-2} + \frac{C}{z-3}$$ but second terms are not in form $\dfrac{z}{z-r}$.

When using PFE with Z transform it is usefull to do PFE of $\dfrac{X(z)}{z}$ then do inverse.

Example cont. $\frac{X(z)}{z} = \frac{5z-11}{(z-2)(z-3)} = \frac{A}{z-2} + \frac{B}{z-3}$

$A = \left.\frac{5z-11}{z-3}\right|_{z=2} = \frac{10-11}{2-3} = 1$

$B = \left.\frac{5z-11}{z-2}\right|_{z=3} = \frac{15-11}{1} = 4$

Then $X(z) = \frac{Az}{z-2} + \frac{Bz}{z-3}$

$= \frac{z}{z-2} + 4\frac{z}{z-3}$

and $x[n] = (2)^n u[n] + 4(3)^n u[n]$

Note the original approach is not incorrect. $X(z) = A + \frac{B}{z-2} + \frac{C}{z-3}$

$$\begin{array}{r} 5 \\ z^2-5z+6 \overline{)\,5z^2-11z} \\ -\underline{5z^2-25z+30} \\ 14z-30 \end{array}$$

so $A = 5$

and $\frac{B}{z-2} + \frac{C}{z-3} = \frac{14z-30}{z^2-5z+6}$

$B = \left.\frac{14z-30}{z-3}\right|_{z=2} = \frac{28-30}{-1} = 2$

$C = \left.\frac{14z-30}{z-2}\right|_{z=3} = \frac{42-30}{1} = 12$

using

$Z\{\gamma^{n-1} u[n-1]\} = \frac{\gamma z}{z-\gamma}$

AND $x[n] = 5\delta[n] + 2(2)^{n-1}u[n-1] + 12(3)^{n-1}u[n-1]$

at $n=0$ $x[0] = 5+0+0 = 5 = 2+3$

at $n=1$ $x[1] = 0 + 2(2)^0 + 12(3)^0 = 2 + 12$

etc.

But the second approach is a bit more work.

- time shift: let $x[n] \xrightarrow{z} X(z)$ with ROC R.

then $x[n-k] \overset{z}{\longleftrightarrow} z^{-k}X(z)$

- if advance $(k<0)$ then if $x[n] \neq 0$ for $n = 0, 1 \ldots k-1$
  then ROC $= R - \{\infty\}$ (remove ∞)

- if delay $(k>0)$ then if $x[n] \neq 0$ for $k-1, \ldots -1$ ($x[n]$ not causal)
  then ROC $= R - \{0\}$ (remove 0)

Note: we will use this property a lot next time when we discuss solving LCCDE, so I will save those examples for then.

– Time reversal $\quad x[n] \overset{z}{\longleftrightarrow} X(z) \quad$ ROC $R_x = r_1 < |z| < r_2$

$$x[-n] \overset{z}{\longleftrightarrow} X(z^{-1}) \quad \text{ROC} = \frac{1}{r_2} < |z| < \frac{1}{r_1}$$

Proof: $X(z) = \sum_{n=-\infty}^{\infty} x[n] z^{-n}$ $\quad$ Let $v[n] = x[-n]$ then

$$V(z) = \sum_{n=-\infty}^{\infty} v[n] z^{-n} = \sum_{n=-\infty}^{\infty} x[-n] z^{-n}$$

$$= \sum_{m=+\infty}^{-\infty} x[m] z^{m} = \sum_{m=-\infty}^{\infty} x[m] (z^{-1})^{-m}$$

$$= X(z^{-1})$$

Example: What is $\mathcal{Z}\{r^{-n} u[-n]\}$

Recall $\mathcal{Z}\{r^n u[n]\} = \frac{z}{z-r} \quad |z| > |r|$

Then $\mathcal{Z}\{r^{-n} u[-n]\} = \frac{z^{-1}}{z^{-1} - r} = \frac{1}{1 - rz} \quad |z| < \frac{1}{|r|}$

– Multiplication by exponential = scaling in z domain.

$$x[n] \overset{z}{\longleftrightarrow} X(z) \quad \text{ROC} \quad r_1 < |z| < r_2$$

$$r^n x[n] \overset{z}{\longleftrightarrow} X\left(\frac{z}{r}\right) \quad \text{ROC} \quad |r| r_1 < |z| < |r| r_2$$

Example: We previously derived $\mathcal{Z}\{\cos(\omega_0 n) u[n]\} = \frac{z(z - \cos(\omega_0))}{z^2 - 2\cos(\omega_0) z + 1}$

$|z| > 1$

Then $\mathcal{Z}\{r^n \cos(\omega_0 n) u[n]\} = \left. \frac{z(z - \cos(\omega_0))}{z^2 - 2\cos(\omega_0) z + 1} \right|_{z \to \frac{z}{r}}$

$$= \frac{\frac{z}{r}\left(\frac{z}{r} - \cos(\omega_0)\right)}{\left(\frac{z}{r}\right)^2 - 2\cos(\omega_0)\left(\frac{z}{r}\right) + 1} \quad \text{multiply by } \frac{r^2}{r^2}$$

$$= \frac{z(z - r\cos(\omega_0))}{z^2 - 2r\cos(\omega_0) z + r^2} \quad |z| > |r|$$

— Differentiation in z-domain

$$x[n] \overset{z}{\longleftrightarrow} X(z) \qquad \text{ROC } R$$

$$n\,x[n] \overset{z}{\longleftrightarrow} -z\frac{dX}{dz}(z) \qquad \text{ROC } R \text{ (same)}$$

Example: $\mathcal{Z}\{n\,u[n]\} = -z\frac{dX}{dz}$ where $X(z) = \frac{z}{z-1}$ $|z|>1$

$$\frac{dX}{dz} = \frac{(z-1)(1) - z(1)}{(z-1)^2} = \frac{-1}{(z-1)^2}$$

Thus $\mathcal{Z}\{n\,u[n]\} = -z \cdot \frac{-1}{(z-1)^2} = \frac{z}{(z-1)^2}$ $|z|>1$

— ★ Convolution property: Let $x_1[n] \overset{z}{\longleftrightarrow} X_1(z)$ ROC R_1

$x_2[n] \overset{z}{\longleftrightarrow} X_2(z)$ ROC R_2

then $x_1[n] * x_2[n] \overset{z}{\longleftrightarrow} X_1(z)X_2(z)$ ROC $= R_1 \cap R_2$ unless pole-zero cancel.

Example: Consider a DT LTI system with impulse response

$h[n] = \left(\frac{1}{2}\right)^n u[n]$ and input $x[n] = \cos\left(\frac{\pi}{4}n\right)u[n]$

Determine output. $y[n] = h[n] * x[n]$

Then $Y(z) = H(z)\cdot X(z)$ $\qquad y[n] = \mathcal{Z}^{-1}\{Y(z)\}$

• From Table $H(z) = \frac{z}{z-\frac{1}{2}}$ $\qquad X(z) = \frac{z\left(z-\cos\left(\frac{\pi}{4}\right)\right)}{z^2 - 2\cos\left(\frac{\pi}{4}\right)z + 1}$

Then $Y(z) = \frac{z}{z-1/2} \cdot \frac{z\left(z-\cos\left(\frac{\pi}{4}\right)\right)}{z^2-2\cos\left(\frac{\pi}{4}\right)z+1}$

we need to find $y[n] = \mathcal{Z}^{-1}\{Y(z)\}$

write $\frac{Y(z)}{z} = \frac{A}{z-1/2} + \frac{Bz+C}{z^2-2\cos\left(\frac{\pi}{4}\right)z+1} = \frac{z\left(z-\cos\left(\frac{\pi}{4}\right)\right)}{\left(z-\frac{1}{2}\right)\left(z^2-2\cos\frac{\pi}{4}z+1\right)} = U(z)$

Example cont.

$$A = \left.\frac{z\left(z - \cos\frac{\pi}{4}\right)}{z^2 - 2\cos\frac{\pi}{4} z + 1}\right|_{z=\frac{1}{2}} = \frac{\frac{1}{2}\left(\frac{1}{2} - \cos\frac{\pi}{4}\right)}{\left(\frac{1}{4}\right) - 2\cos\frac{\pi}{4}\left(\frac{1}{2}\right) + 1}$$

To find C let $z = 0$ then $\frac{A}{-\frac{1}{2}} + \frac{C}{1} = 0 \Rightarrow C = 2A$

To find B take $\lim_{z\to\infty} z\,U(z)$

$$\frac{Az}{z-\frac{1}{2}} + \frac{Bz^2 + Cz}{z^2 - 2\cos\frac{\pi}{4} z + 1} = \frac{z^2\left(z - \cos\frac{\pi}{4}\right)}{z^2 - 2\cos\left(\frac{\pi}{4}\right) z + 1}$$

$$\frac{A}{1 - \frac{1}{2z}} + \frac{B + \frac{C}{z}}{1 - 2\cos\frac{\pi}{4}\cdot\frac{1}{z} + \frac{1}{z^2}} = \frac{z - \cos\left(\frac{\pi}{4}\right)}{1 - 2\cos\frac{\pi}{4}\frac{1}{z} + \frac{1}{z^2}}$$

$$\lim_{z\to\infty} \quad A + B = -\cos\left(\frac{\pi}{4}\right) \Rightarrow B = -\left(A + \cos\frac{\pi}{4}\right)$$

Then $Y(z) = \dfrac{Az}{z - \frac{1}{2}} + \dfrac{z(Bz + C)}{z^2 - 2\cos\left(\frac{\pi}{4}\right)z + 1}$

$$y[n] = A\left(\tfrac{1}{2}\right)^n u[n] + D\cos\left(\tfrac{\pi}{4}n\right)u[n] + E\sin\left(\tfrac{\pi}{4}n\right)u[n]$$

$$\mathcal{Z}\left\{D\cdot\cos\left(\tfrac{\pi}{4}n\right)\right\} = D\,\frac{z\left(z - \cos\left(\frac{\pi}{4}\right)\right)}{z^2 - 2\cos\left(\frac{\pi}{4}\right)z + 1}$$

$$\mathcal{Z}\left\{E\sin\left(\tfrac{\pi}{4}n\right)\right\} = E\cdot\frac{\sin\left(\frac{\pi}{4}\right)z}{z^2 - 2\cos\left(\frac{\pi}{4}\right) + 1}$$

$$Dz\left(z - \cos\left(\tfrac{\pi}{4}\right)\right) + E\sin\left(\tfrac{\pi}{4}\right)z = z(Bz + C)$$

$$Dz^2 - Dz\cos\left(\tfrac{\pi}{4}\right) + E\sin\left(\tfrac{\pi}{4}\right)z = Bz^2 + Cz$$

$$Dz^2 + \left(E\sin\left(\tfrac{\pi}{4}\right) - D\cos\left(\tfrac{\pi}{4}\right)\right)z = Bz^2 + Cz$$

$D = B \qquad E\sin\left(\frac{\pi}{4}\right) - D\cos\left(\frac{\pi}{4}\right) = C$

$$E = \frac{C + B\cos\left(\frac{\pi}{4}\right)}{\sin\left(\frac{\pi}{4}\right)}$$

– Initial Value property: If $x[n]$ is causal

then $x[0] = \lim_{z \to \infty} X(z)$

This can be used to check the value of inverse Z-transform and gives rationale for ROC of causal signal to include $|z| \Rightarrow \infty$

– Summation property $x[n] \xleftrightarrow{z} X(z)$

then $\sum_{m=-\infty}^{n} x[m] \xleftrightarrow{z} \frac{z}{z-1} X(z)$

– Final Value property $x[n] \xleftrightarrow{z} X(z)$

$\lim_{n \to \infty} x[n] = \lim_{z \to 1} (z-1) X(z)$ when poles of $(z-1)X(z)$ inside unit circle.

Example: Recall from 2714 that an Finite Impulse Response (FIR) filter has impulse response

$$h[n] = \sum_{k=0}^{M} b_k \delta[n-k]$$

Using the table $Z\{\delta[n]\} = 1$ and time-shift property

$$H(z) = \sum_{k=0}^{M} b_k z^{-k} = b_0 + b_1 z^{-1} + b_2 z^{-2} + \cdots b_M z^{-M}$$

$$= \frac{b_0 z^{M} + b_1 z^{M-1} + \cdots b_M}{z^M}$$

This TF has M zeros an M poles at origin. Thus is always stable.