Lecture 13: Circuit and Block Diagrams using Laplace

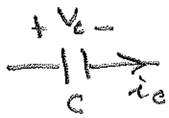
— Recall from 2714 that we can analyze electrical circuits that form LTI systems by.

1. Deriving a governing equation (LCCDE)
2. Finding the impulse response $h(t)$
3. Using convolution to determine the output $y(t)$ given an input. $x(t)$

Or alternatively using Fourier analysis

2. Determine stability and derive frequency response $H(j\omega)$
3. Use convolution theorem to find output $Y(j\omega)$ and use inverse Fourier transform to find $y(t)$

— Now that we have Laplace as a tool we can simplify this process further. The idea is to go back to circuits and take Laplace transform of each elements voltage-current model

• Capacitor 

$$i_c(t) = C\frac{dV_c}{dt}(t)$$

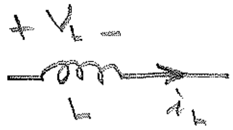
Impedance: $\frac{1}{Cs}$

$$I_c(s) = C\left[sV_c(s) - V_c(0^-)\right]$$

$$= CsV_c(s) - CV_c(0^-)$$

OR rearranging

$$V_c(s) = \frac{1}{Cs}I_c(s) + \frac{V_c(0^-)}{s}$$

• Inductor 

$$V_L(t) = L\frac{di_L}{dt}(t)$$

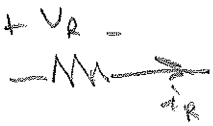
Impedance: Ls

$$V_L(s) = L\left(sI_L(s) - i_L(0^-)\right)$$

$$= LsI_L(s) - Li_L(0^-)$$

OR

$$I_L(s) = \frac{1}{Ls}V_L(s) + \frac{1}{s}i_L(0^-)$$

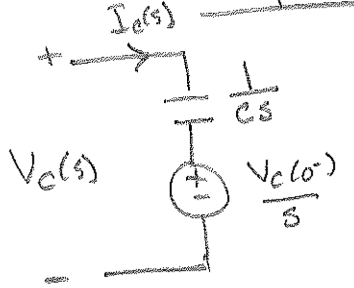
• Resistor 

$$V_R(t) = R\,i_R(t)$$

$$V_R(s) = R\,I_R(s)$$

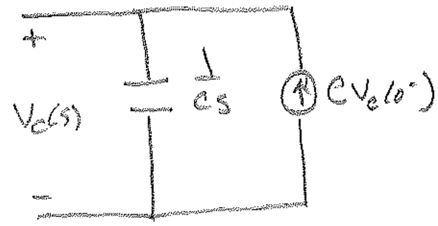
– This gives us new models in Laplace domain

Capacitor



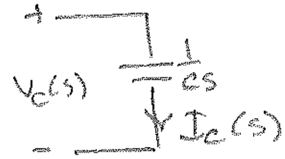
For KVL

OR



For KCL

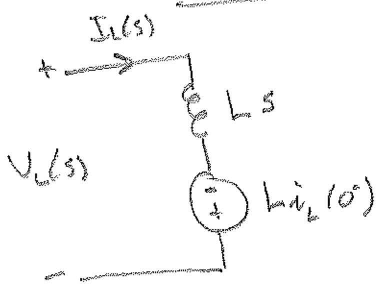
When no IC on capacitor,



$V_c(s) = \frac{1}{Cs} I_c(s)$

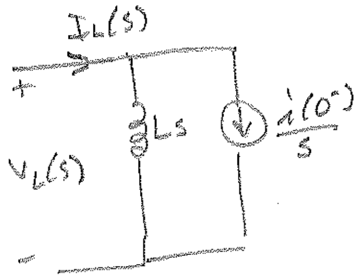
↑ impedance

Inductor



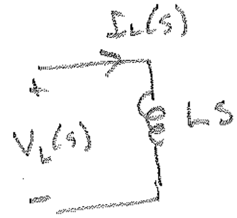
KVL

OR



KCL

When no I.C. on Inductor

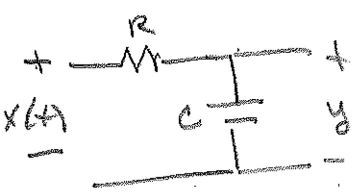


$V_L(s) = Ls\, I_L(s)$

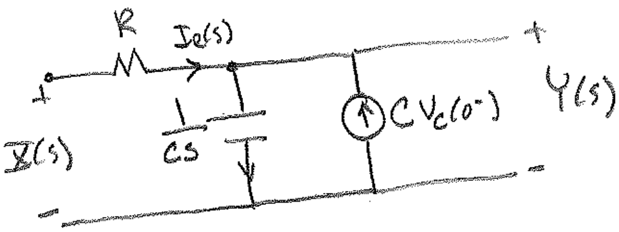
↑ impedance

– Using KVL + KCL with these models gives us a convienient way to go from circuit to transfer function directly.

- Example: RC with I.C.



$\Longrightarrow$



where $V_c(0^-) = V_0$

Using KCL: $$\frac{X(s) - Y(s)}{R} + C v_c(0^-) = \frac{Y(s)}{1/Cs} = Cs\, Y(s)$$

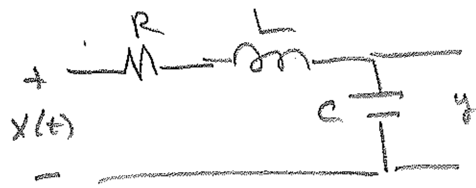
Rearranging and substituting for $V_c(0^-)$ we get

$$(1 + RCs) Y(s) = X(s) + RCV_0$$

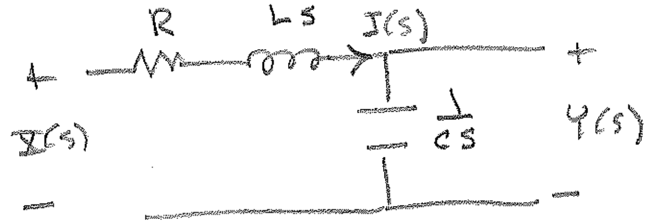
AND $$Y(s) = \frac{1}{1+RCs} X(s) + \frac{RCV_0}{1+RCs}$$

If $V_0 = 0$ $$Y(s) = \frac{1}{1+RCs} X(s) \Rightarrow H(s) = \frac{Y(s)}{X(s)} = \frac{1}{1+RCs}$$

— Example: RLC with no I.C.



$\Rightarrow$



Using KVL $\quad X(s) = RI(s) + LsI(s) + \frac{1}{Cs} I(s)$

$$Cs X(s) = RCs I(s) + CLs^2 I(s) + I(s)$$

$$Cs X(s) = (RCs + CLs^2 + 1)\, I(s)$$

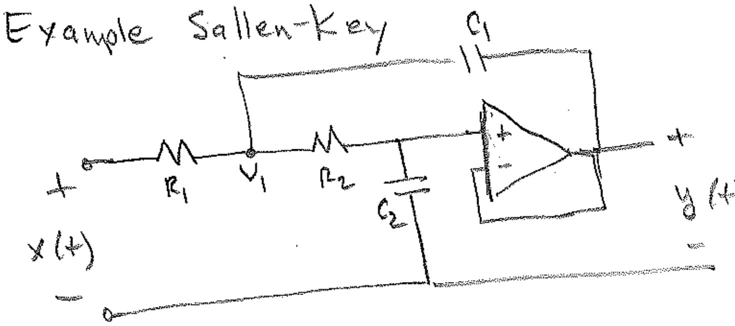
$$I(s) = \frac{Cs}{CLs^2 + RCs + 1} X(s)$$

Output is $\quad Y(s) = \frac{1}{Cs} I(s) = \frac{1}{CLs^2 + RCs + 1} X(s)$

and transfer Function $\quad H(s) = \frac{Y(s)}{X(s)} = \frac{1}{CLs^2 + RCs + 1}$

— This works with active (op-amp) circuits as well.

Example Sallen-Key



Doing KCL at node 1

$$\frac{X(s) - V_1(s)}{R_1} + \frac{Y(s) - V_1(s)}{\frac{1}{C_1 s}} = \frac{V_1(s) - Y(s)}{R_2}$$

Doing KCL at op-amp +

$$\frac{V_1(s) - Y(s)}{R_2} = \frac{Y(s)}{\frac{1}{C_2 s}}$$

The rest is just Algebra.

— Most upper level circuit courses will use these techniques.

e.g. AC-circuits.

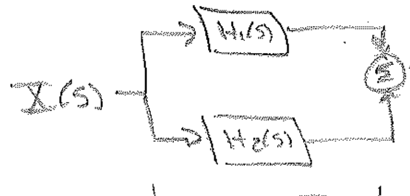
— Recall from 2714 that block diagrams can be used to model systems and implement / realize systems,

$x(t) \longrightarrow \boxed{h(t)} \longrightarrow y(t) = h(t) * x(t) \quad \Rightarrow \quad X(s) \longrightarrow \boxed{H(s)} \longrightarrow Y(s) = H(s)Y(s)$

- Series connection

$$X(s) \longrightarrow \underbrace{\boxed{H_1(s)} \longrightarrow \boxed{H_2(s)}}_{H(s) = H_1(s) H_2(s)} \longrightarrow Y(s) = H_1(s) H_2(s) X(s)$$

- parallel connection

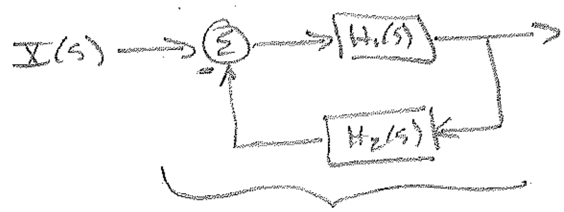


$Y(s) = (H_1(s) + H_2(s)) X(s)$

$H(s) = H_1(s) + H_2(s)$

- feedback connection

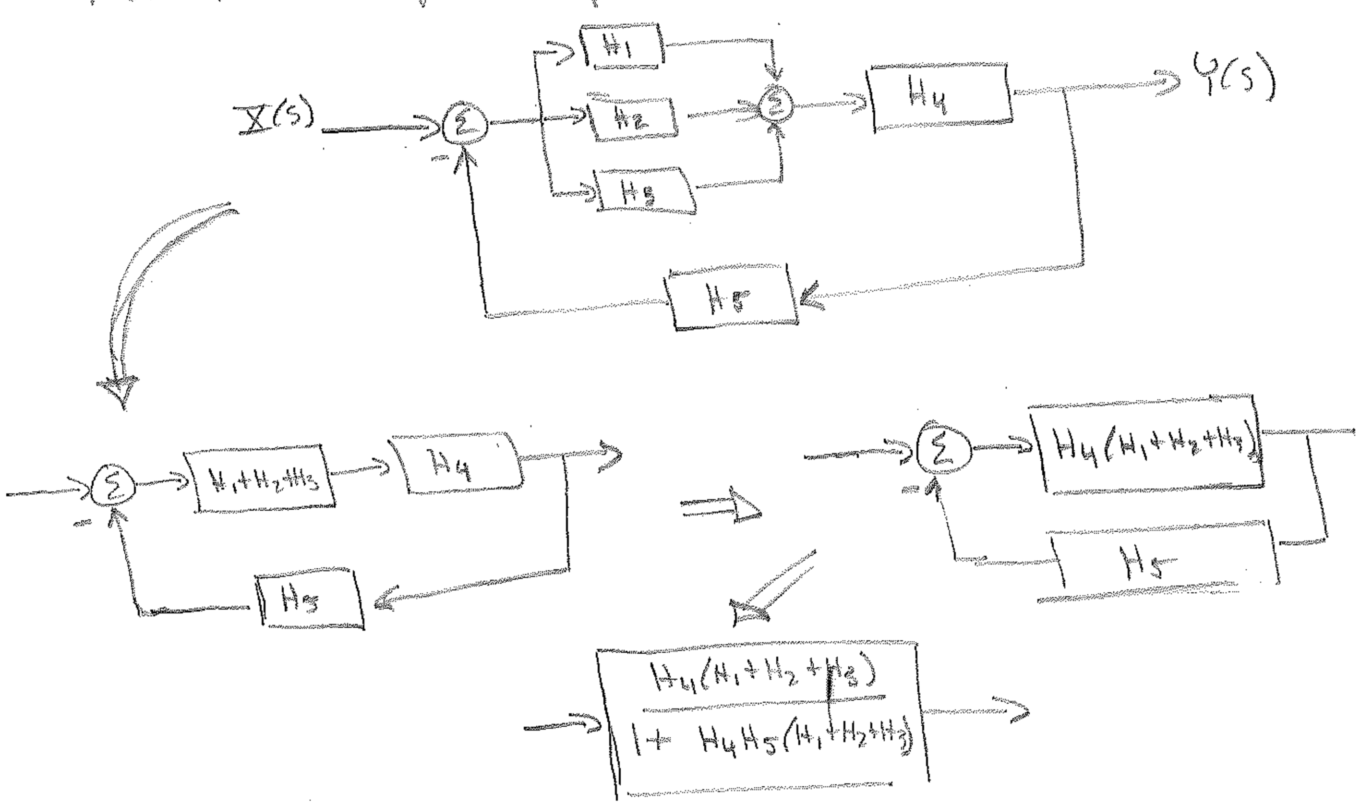
Note: much easier than in 2714 time domain.



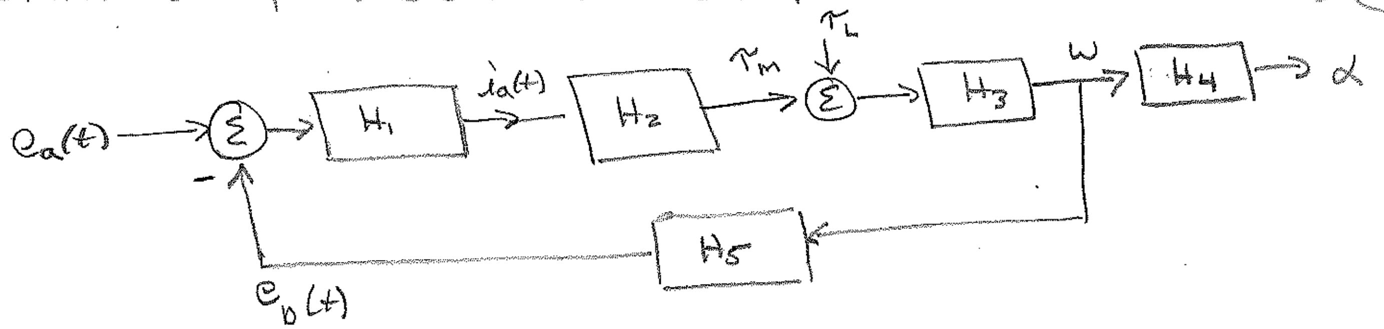
$$Y(s) = \frac{H_1(s)}{1 + H_1(s) H_2(s)} X(s)$$

$$H(s) = \frac{H_1(s)}{1 + H_1(s) H_2(s)}$$

— We can use Block diagrams to derive an overall transfer function from models of subcomponents. Example PID controller.



— Detailed Example: DC Motor model



$e_a(t)$ = armature voltage applied

$i_a(t)$ = armature current

$\tau_m(t)$ = motor tourque

$\tau_L(t)$ = load tourque

$\omega(t)$ = rotational velocity

$\alpha(t)$ = rotational position

$e_b(t)$ = back voltage (EMF)

H_1 = model of armature

H_2 = model of current to torque

H_3 = model of motor inertia and friction of bearings

H_4 = model of velocity to position

H_5 = back EMF model

$$H_1(s) = \frac{1}{R_a + L_a s}$$

R_a = armature resistance
L_a = armature inductance

6 parameters

$$H_2(s) = K_m$$

K_m = tourque gain

$$H_3(s) = \frac{1}{B_m + J_m s}$$

B_m = rotor viscous friction
J_m = rotor inertia.

$$H_4(s) = \frac{1}{s}$$

angle is integral of rotational velocity

$$H_5(s) = K_b$$

back EMF gain K_b.

The overall transfer function from input voltage to output position (disregarding τ_L) is then

$$H(s) = H_4(s) \cdot \left[\frac{H_1(s) H_2(s) H_3(s)}{1 + H_1(s) H_2(s) H_3(s) H_5(s)} \right]$$

- We can also use block diagrams to implement / realize systems. Using building blocks, the gain and integrator

$$x(t) \rightarrow \boxed{\int} \rightarrow y(t) = \int_{-\infty}^{t} x(\tau)\, d\tau \quad \Rightarrow \quad X(s) \rightarrow \boxed{1/s} \rightarrow Y(s) = \frac{1}{s} X(s)$$

- We can convert a given transfer function (say from Filter Design) to a block diagram, one of several ways.

- Example: $H(s) = \dfrac{s+a}{s^2+bs+c}$ $\quad a, b, c \in \mathbb{R}$

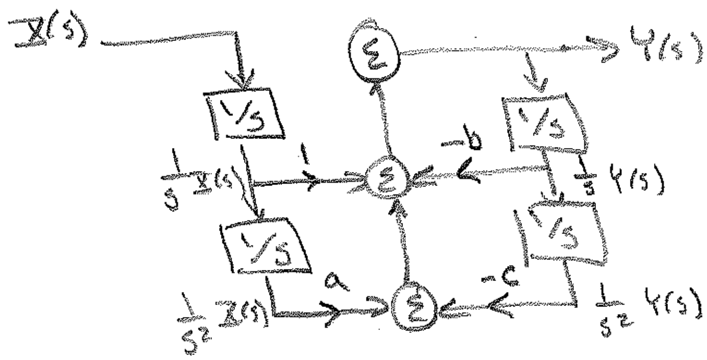
$Y(s)(s^2+bs+c) = (s+a)X(s)$ — Defn TF

$s^2 Y(s) + bsY(s) + cY(s) = sX(s) + aX(s)$ — Distribution

$Y(s) + \frac{b}{s} Y(s) + \frac{c}{s^2} Y(s) = \frac{1}{s} X(s) + \frac{a}{s^2} X(s)$ — Divide through by s^2

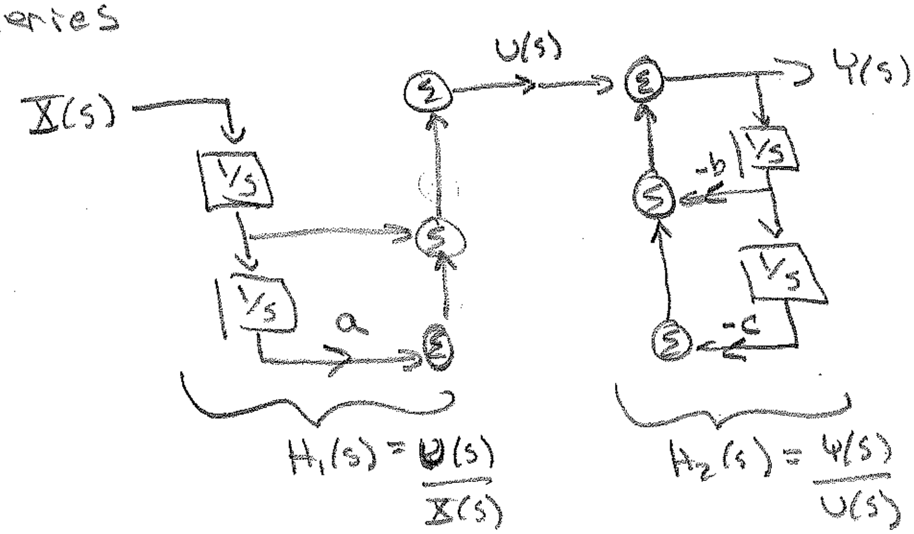
$Y(s) = -\frac{b}{s} Y(s) - \frac{c}{s^2} Y(s) + \frac{1}{s} X(s) + \frac{a}{s^2} X(s)$ — Solve for $Y(s)$

Direct translation to Block Diagram



This is called the Direct Form I implementation / realization.

- Note in the previous example we can consider this two TF's in series

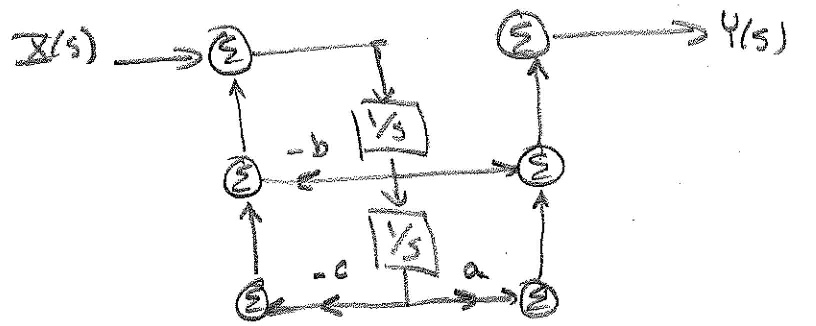


"Split" summations up.

$H(s) = H_1(s) H_2(s)$

$= H_2(s) H_1(s)$

— reassembling in opposite order and combing integrators

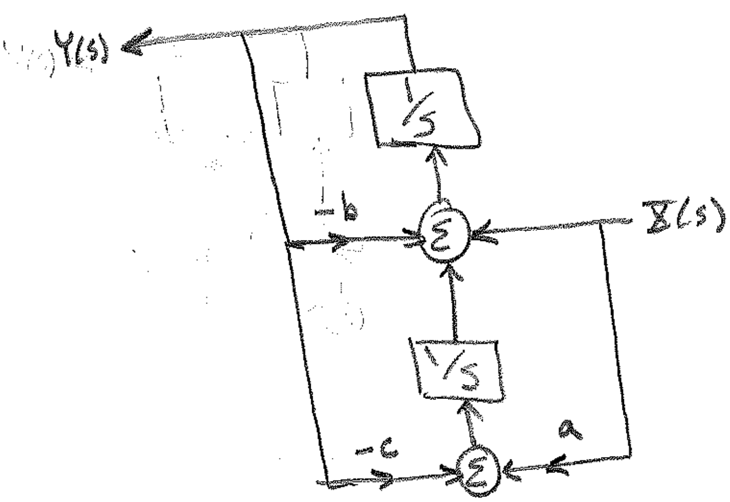


Recall each integrator and summation is an op-amp, so this saves 2-op-amps.

This is called Direct Form II.

— A result from graph theory tells us we can reverse the direction of signal flow and interchange input output to get the "transposed" Direct Form II

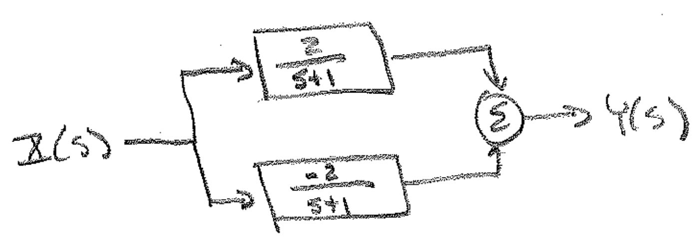
For our example:



— There are additional forms as well depending on how we factor the transfer function. For example to get a parallel implementation factor using PFE.

Example: $H(s) = \dfrac{2}{s^2+3s+2} = \dfrac{2}{(s+1)(s+2)} = \dfrac{A}{s+1} + \dfrac{B}{s+2}$

$$= \frac{2}{s+1} + \frac{-2}{s+2}$$



— The reason for so many forms is a reduction in components and reduced sensitivity to component variation.