Lecture 22. Digital Filter Design (Tutorial)

- Note todays lecture is a preview of material you would learn more deeply in 4624: DSP AND Filter Design.'

- Recall from 2714 that realizable filters must be causal if implemented in real-time.

Example: The ideal LowPASS filter has a frequency response

$$H(\omega) = \begin{cases} 1 & 2\pi k - \omega_c < \omega < 2\pi k + \omega_c \\ 0 & \text{else} \end{cases} \qquad k = 0, 1, 2, \ldots$$

The corresponding impulse response is $h[n] = \frac{1}{2\pi} \int_{-\omega_c}^{\omega_c} e^{j\omega n} \, d\omega = \frac{1}{\pi n} \sin(\omega_c n)$

which is not causal and thus not realizable.

- As in CT realizable filters must meet the Paley-Wiener conditions
  - a) have $H(\omega) = 0$ only at finite number of frequencies
  - b) cannot have $H(\omega) = 0$ over finite interval
  - c) transition between bands cannot be zero width
  - d) cannot choose $|H(\omega)|$ and $\angle H(\omega)$ independently.

- Thus we restrict our designs to those described by stable LCCDE

$$y[n] = \sum_{k=1}^{N} -a_k y[n-k] + \sum_{k=0}^{M} b_k x[n-k]$$

$$H(z) = \frac{\sum_{k=0}^{M} b_k z^{-k}}{1 + \sum_{k=1}^{N} a_k z^{-k}} \xRightarrow{\text{if stable}} H(\omega) = \frac{\sum_{k=0}^{M} b_k e^{-j\omega k}}{1 + \sum_{k=1}^{N} a_k e^{-j\omega k}}.$$

- the two major class of DT Filters are.

- FIR $a_k = 0$
  - can have linear phase
  - always stable
  - efficient implementation
  - startup time is finite
  - generally require higher order to get same performance.

$\Longleftrightarrow$

- IIR
  - cannot have linear phase
  - can have lower order than equivalent FIR.
  - lower complexity means can accomodate faster sample times.

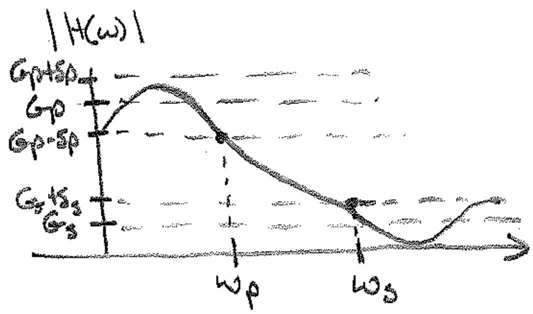
— FIR filters are designed by computational approaches. We specify a desired FR $H_d(e^{j\omega})$ and compute b_k's that approximate that desired response.

$$H(e^{j\omega}) = \sum_{k=0}^{M} b_k e^{-j\omega k} \quad \text{is a polynomial in } z^{-1} = e^{-j\omega}$$

thus we find polynomial coefficients such that $H(e^{j\omega}) \approx H_d(e^{j\omega})$

- MAJOR Strategies
  - Windowing
  - Frequency Sampling
  - Chebyshev Approximation

— Parameters of a Filter Design.



ω_p = pass band edge

ω_s = stop band edge.

G_p = pass band gain
δ_p = pass band ripple
G_s = stop band gain
δ_s = stop band ripple
} often specified in dB.

— FIR design by windowing.

1) Given desired $H_d(e^{j\omega}) \xrightarrow{\mathcal{F}^{-1}} h_d[n]$, choose M

2) truncate and window $h[n] = h_d[n] \cdot w[n]$

$$H(e^{j\omega}) = H_d(e^{j\omega}) * W(e^{j\omega})$$

where $w[n]$ is a window function $w[n] = 0$ for $n \notin 0, 1, \cdots M-1$

- rectangular window $w[n] = u[n] - u[n-M-1]$ leads to ringing.
- can use Blackman, Hamming, or Hanning windows to reduce ringing.

3) $b_k = h[k]$

This approach is simple but does not give us control directly over $\omega_p, \omega_s, \delta_p, \delta_s$, they depend on M & $w[n]$ chosen.

- FIR Design by Frequency Sampling.
  1) Sample the desired $H_d(e^{j\omega})$ at equally spaced frequencies

$$\omega_k = \frac{2\pi}{M}(k+\alpha) \qquad \alpha = 0 \text{ or } 1/2 \qquad k = 0, 1, \cdots \tfrac{M-1}{2} \;\; M \text{ odd}$$
$$k = 0, 1, \cdots \tfrac{M}{2} - 1 \;\; M \text{ even}$$

  2) Fit Polynomial $\sum_{k=0}^{M} b_k z^{-k}$ to those samples.

  • drawback is no control over ripple, fit can wildly vary in between samples

- FIR Design by Chebyshev Approximation. Use parks-McClellan algorithm to find coefficients. Details are outside course scope but you can use firpm command in Matlab.

  • The advantage is this gives an equiripple filter.

$\boxed{\text{DEMO of firpm}}$

- IIR filters can be designed by converting Analog filters.

Recall a causal, stable CT system has a transfer function

$$H(s) = \frac{P(s)}{Q(s)} = \frac{\sum_{k=0}^{M} \beta_k s^k}{\sum_{k=0}^{N} \alpha_k s^k} \quad \text{for } Re(s) > r \quad r < 0.$$

Equivalently the impulse response $h(t) = \mathcal{L}_1^{-1}\{H(s)\}$

AND LCCDE is of form $\sum_{k=0}^{N} \alpha_k D^{(k)} y(t) = \sum_{k=0}^{M} \beta_k D^{(k)} x(t)$

  • Given a stable CT system implementing a filter we can convert it to a DT filter using $H(s)$, $h(t)$, or the LCCDE

- Remember IIR filters cannot have linear phase since they would be unstable.

  Linear phase requires FIR.

- IIR requires we just accept non-linear phase.

— IIR Design by sampling the LCCDE.

- Consider the backward difference approximation to derivative

$$\frac{dy}{dt} \approx \frac{y(t) - y(t-nT)}{T}$$

- Sampling this at $t = nT$ gives $\quad \frac{dy}{dt}(nT) \approx \frac{y(nT) - y(nT-T)}{T}$

$$= \frac{y[n] - y[n-1]}{T}$$

- Comparing the transfer functions between CT & DT

$$\mathcal{L}\left\{\frac{dy}{dt}\right\} = s\,Y(s) \qquad Z\left\{\frac{y[n]-y[n-1]}{T}\right\} = \frac{1-z^{-1}}{T}\,Y(z)$$

$$H(s) = s \qquad s = \frac{1-z^{-1}}{T} \qquad H(z) = \frac{1-z^{-1}}{T}$$

- Thus given Analog Filter $H(s)$ $\quad H(z) = H(s)\Big|_{s = \frac{1-z^{-1}}{T}}$. This maps LHP of s-plane to circle in z plane centered at $\frac{1}{2}$ radius $\frac{1}{2}$ since $z = \frac{1}{1-sT}$, thus stable.

— IIR Design by sampling impulse response.

- Let $h(t)$ be the impulse response of the analog filter.

- Ideally sample $h(t)$ to get $h[n] = h(nT) \doteq h(t) * \sum_{k=-\infty}^{\infty} \delta(t-nT)$

  where aliasing occurs if $\frac{2\pi}{T} < 2\omega_{max}$ (Nyquist).

- Comparing Laplace and Z-transforms.

$$H(s) = \int_0^{\infty} h(t)\, e^{-st}\, dt \qquad H(z) = \sum_{n=0}^{\infty} h[n]\, z^{-n}$$

$$\approx T \sum_{n=0}^{\infty} h(nT)\, e^{-snT} \qquad \text{implies } z = e^{sT}$$

- Thus the system maps LH of s-plane to inside unit circle and RH of s-plane to outside unit circle, AND poles of $H(z)$, $z_k = e^{p_k T}$ for poles p_k of $H(s)$

— IIR Design using Bilinear Transform

- Both of previous approaches have drawbacks, such that the mapping from s-plane to z-plane is not conformal.

- By using a trapezoidal approximation to the integral, one arrives at the mapping $s = \frac{2}{T}\left(\frac{1-z^{-1}}{1+z^{-1}}\right) = \frac{2}{T}\left(\frac{z-1}{z+1}\right)$ the bilinear map.

This maps all analog frequencies into discrete freq range $-\pi, \pi$ by compressing them.

- To design an IIR filter using Bilinear map simply let

$$H(z) = H(s)\Big|_{s=\frac{2}{T}\left(\frac{z-1}{z+1}\right)}$$

— Example: 3rd order lowpass Butterworth Analog filter has

$$H(s) = \frac{\omega_c^3}{(s-p_1)(s-p_2)(s-p_3)} \qquad p_1 = \omega_c e^{j2\pi/3} \quad p_2 = \omega_c \quad p_3 = \omega_c e^{-j2\pi/3}$$

Using Bilinear transform with sample spacing T gives

$$H(z) = \frac{\omega_c^3}{\left(\frac{2}{T}\frac{z-1}{z+1} - p_1\right)\left(\frac{2}{T}\frac{z-1}{z+1} - p_2\right)\left(\frac{2}{T}\frac{z-1}{z+1} - p_3\right)}$$

e.g. $\omega_c = 100$ $\quad p_1 = -50 + j\,86.6$ $\quad p_2 = -100$ $\quad p_3 = -50 - j\,86.6$.

$T = \frac{1}{1000}$

$$H(z) = \frac{\omega_c^3 (z+1)^3}{(z-\alpha_1)(z-\alpha_2)(z-\alpha_3)}$$

$\alpha_1 = 0.8238 + j\,0.2318$

$\alpha_2 = 0.8238 + j\,0.2318$

$\alpha_3 = 0.7265$

$\boxed{\text{Matlab DEMO}}$

butter

cheb1ord

cheby1