Inverse Laplace Transform

- Consider a causal function $f(t)$ that may or may NOT be absolutely integrable.

Let $g(t) = f(t)e^{-ct}$ for some $c>0$ such that

$$\int_0^\infty |g(t)|\,dt = \int_0^\infty |f(t)e^{-ct}|\,dt < \infty$$

Then the Fourier Transform of $g(t)$ exists

$$G(\omega) = \int_{-\infty}^{\infty} g(t)\,e^{-j\omega t}\,dt = \int_0^\infty f(t)e^{-ct}e^{-j\omega t}dt$$

$$= \int_0^\infty f(t)\,e^{-(c+j\omega)t}\,dt.$$

$$= \mathcal{L}_1\{f(t)\}\Big|_{s=c+j\omega} \qquad c \text{ fixed s.t. } c+j\omega \in ROC$$

The inverse Fourier Transform is

$$g(t) = f(t)e^{-ct} = \frac{1}{2\pi}\int_{-\infty}^{\infty} G(\omega)e^{j\omega t}\,d\omega$$

multiply through by e^{ct}

$$e^{ct}g(t) = f(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} G(\omega)\,e^{ct}e^{j\omega t}\,d\omega$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} G(\omega)\,e^{(c+j\omega)t}\,d\omega$$

Substitute $G(\omega)$

$$f(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\underbrace{\left[\int_0^\infty f(\tau)\,e^{-(c+j\omega)\tau}\,d\tau\right]}_{F(s)|_{s=c+j\omega}} e^{(c+j\omega)t}\,d\omega$$

Cont. $$f(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(c+j\omega)\, e^{(c+j\omega)t}\, d\omega$$

Let $s = c + j\omega$ since c constant $ds = 0 + j\,d\omega$

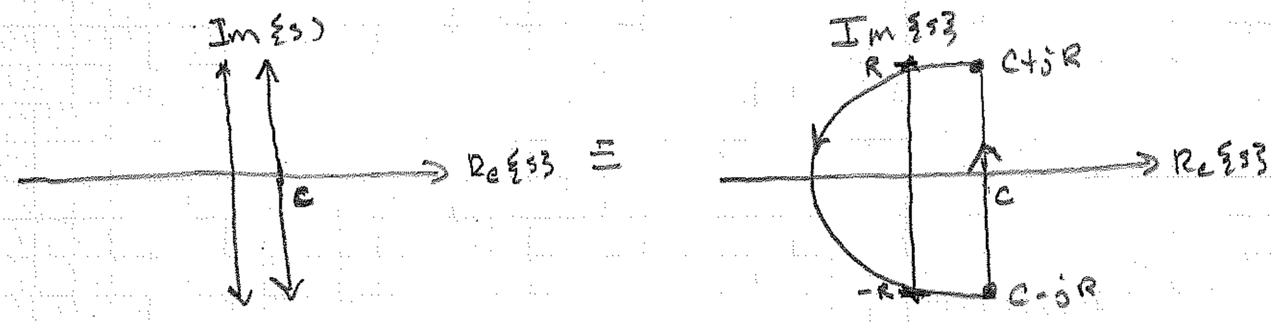
$$d\omega = \frac{1}{j}\, ds$$

AND $$f(t) = \frac{1}{2\pi j} \int_{c-j\infty}^{c+j\infty} F(s)\, e^{st}\, ds$$ A complex integral.

- Note: we can think of c as the real values such that $f(t)e^{-ct}$ has a Fourier Transform. This is the ROC.

- Does the above integral look like our contour integrals from previous lectures?

Yes, if we use the Bromwitch contour.



remember, only one point at ∞ in complex plane.

limit $R \to \infty$

$$f(t) = \frac{1}{2\pi j} \int_{c+j\infty}^{c+j\infty} F(s)\, e^{st}\, ds = \lim_{R \to \infty} \oint_{C(R)} F(s)\, e^{st}\, ds$$

where $C(R)$ is Bromwitch contour.

This allows us to use the method of residues since $c \in$ ROC implies $C(R)$ encloses all singularities of $F(s)$.

Example: Recall that $\mathcal{L}\{x(t)\} = \mathcal{L}\{e^{-at}u(t)\}$ $a \in \mathbb{R}$

was $\dfrac{1}{s+a}$ $\quad Re\{s\} > -a$

Then $$x(t) = \underbrace{\mathcal{L}^{-1}}_{\text{one-sided}}\left\{\frac{1}{s+a}\right\} = \frac{1}{2\pi j}\int_{c-j\infty}^{c+j\infty} \frac{1}{s+a}e^{st}\,ds \quad \text{for } c > -a$$

$$= \lim_{R\to\infty} \frac{1}{2\pi j}\oint_{C(R)} \frac{e^{st}}{s+a}\,ds. \quad c > -a$$

Now from last time since e^{st} is analytic and $C(R)$ encloses singularity @ $-a$, from Residue Theorem

$$\oint \frac{e^{st}}{s+a}\,ds = 2\pi j\, k_1$$

$$k_1 = (s+a)\frac{e^{st}}{s+a}\bigg|_{s=-a} = e^{st}\bigg|_{s=-a} = e^{-at} \quad t > 0$$

Thus $$x(t) = \frac{1}{\cancel{2\pi j}} \cdot \cancel{2\pi j}\, e^{-at} \quad t > 0$$
$$= e^{-at}u(t).$$

What about inverse of Bilateral Laplace?

- Given a non-causal signal $x(t)$ we can write it as

$$x(t) = \underbrace{x_1(t)u(-t)}_{\text{anticausal part}} + \underbrace{x_2(t)u(t)}_{\text{causal part.}} \quad t=0$$

$$\mathcal{L}_2\{x(t)\} = \int_{-\infty}^{0} x_1(t)e^{-st}\,dt + \int_{0}^{\infty} x_2(t)e^{-st}\,dt$$

$$= \int_{0}^{\infty} x_1(-t)e^{st}\,dt + \mathcal{L}\{x_2(t)\}$$

cont.
$$\mathcal{L}_2\{x(t)\} = \underbrace{\mathcal{L}_1\{x_1(-t)\}}_{X_1(s)}\Big|_{s=-s} + \underbrace{\mathcal{L}_1\{x_2(t)\}}_{X_2(s)} \quad \text{for } L < Re\{s\} < U$$

To use this for the inverse we apply this in reverse,

For causal signal $\quad X(s) = \underset{Re\{s\}<U}{X_1(s)} + \underset{Re\{s\}>L}{X_2(s)}$

and
$$x(t) = \mathcal{L}_1^{-1}\{X_1(-s)\}\Big|_{t \to -t} + \mathcal{L}_1^{-1}\{X_2(s)\}$$

• Example: Recall $\mathcal{L}_2\{e^{-|t|}\} = \underset{Re\{s\}<1}{\dfrac{-1}{s-1}} + \underset{Re\{s\}>-1}{\dfrac{1}{s+1}} = X(s)$

Then
$$x(t) = \mathcal{L}_2^{-1}\{X(s)\} = \mathcal{L}_1^{-1}\left\{\frac{+1}{s+1}\right\}\Big|_{t \to -t} + \mathcal{L}_1^{-1}\left\{\frac{1}{s+1}\right\}$$

From our previous result $a=1$
$$x(t) = e^{-t}u(t)\Big|_{t \to -t} + e^{-t}u(t)$$
$$= e^{t}u(-t) + e^{-t}u(t) = e^{-|t|}.$$

• Example: $X(s) = \dfrac{1}{s} \quad Re\{s\} > 0$

$$x(t) = \frac{1}{2\pi j}\int_{c-j\infty}^{c+j\infty} X(s)e^{st}\,ds \quad c > 0$$

$$= \frac{1}{2\pi j}\oint_{C=\text{unit circle}} \frac{e^{st}}{s}\,ds = \frac{1}{2\pi j}\,2\pi j\,K \qquad K = s\frac{e^{st}}{s}\Big|_{s=0} = 1$$

$$= 1 \quad t > 0$$

$x(t) = u(t)$.

Example: $X(s) = \dfrac{10}{s^2+5s+6}$ $\quad Re\{s\} > -2$

First we use PFE.

$$X(s) = \frac{10}{(s+2)(s+3)} = \frac{A}{s+2} + \frac{B}{s+3}$$

$$A = \left.\frac{10}{s+3}\right|_{s=-2} = \frac{10}{1} = 10$$

$$B = \left.\frac{10}{s+2}\right|_{s=-3} = \frac{10}{-1} = -10$$

Then $x(t) = \mathcal{L}^{-1}\left\{\dfrac{10}{s+2}\right\} + \mathcal{L}^{-1}\left\{\dfrac{-10}{s+3}\right\}$

Using Residue theorem.

$$x(t) = \frac{10}{2\pi j}\int_{c-j\infty}^{c+j\infty} \frac{e^{st}}{s+2}\, ds - \frac{10}{2\pi j}\int_{c-j\infty}^{c+j\infty} \frac{e^{st}}{s+3}\, ds$$

$c > -2 \qquad\qquad c > -2$

$$= \frac{10}{2\pi j}\oint_C \frac{e^{st}}{s+2}\, ds - \frac{10}{2\pi j}\oint_C \frac{e^{st}}{s+3}\, ds$$

C = unit circle radius > 3

$$= \frac{10}{2\pi j}\, 2\pi j\, e^{-2t} \quad \frac{-10}{2\pi j}\, 2\pi j\, e^{-3t}$$

$$= 10e^{-2t} - 10e^{-3t} \quad t > 0$$

$$x(t) = 10\left(e^{-2t} - e^{-3t}\right)u(t).$$

Example with complex singularities,

$$X(s) = \frac{s}{s^2+2s+5} \qquad Re\{s\} > -1$$

Using PFE

$$X(s) = \frac{s}{(s+1+j2)(s+1-j2)} = \frac{A}{(s+1+j2)} + \frac{B}{s+1-j2}$$

$$A = \frac{-1-j2}{-1-j2+1-j2} = \frac{-1-j2}{-j4} = \frac{1}{2} - \frac{1}{4}j$$

$$B = \frac{-1+j2}{-1+j2+1+j2} = \frac{-1+j2}{j4} = \frac{1}{2} + \frac{1}{4}j$$

Then $x(t) = \mathcal{L}_1^{-1}\left\{\frac{\frac{1}{2}-\frac{1}{4}j}{s+1+j2}\right\} + \mathcal{L}_1^{-1}\left\{\frac{\frac{1}{2}+\frac{1}{4}j}{s+1-j2}\right\}$

Using Residues,

$$x(t) = \frac{1}{2\pi j}\int_{-j\infty}^{j\infty} \frac{\frac{1}{2}-\frac{1}{4}j}{s+1+j2} e^{st}\, ds + \frac{1}{2\pi j}\int_{-j\infty}^{j\infty} \frac{\frac{1}{2}+\frac{1}{4}j}{s+1-j2} e^{st}\, ds$$

Let $\gamma = 0 > -1$

Let C be unit circle w/ radius > 5 (encloses $-1 \pm j2$)

$$x(t) = \frac{1}{2\pi j} 2\pi j\, k_1 + \frac{1}{2\pi j} 2\pi j\, k_2$$

$$k_1 = \left(\frac{1}{2} - \frac{1}{4}j\right) e^{(-1-j2)t} \qquad k_2 = \left(\frac{1}{2} + \frac{1}{4}j\right) e^{(-1+j2)t}$$

$$x(t) = k_1 + k_2 = \left(\frac{1}{2} - \frac{1}{4}j\right) e^{(-1-j2)t} + \left(\frac{1}{2} + \frac{1}{4}j\right) e^{(-1+j2)t}$$

for $t > 0$

With some effort you can write this as

$$x(t) = e^{-t}\left(\cos(2t) - \frac{1}{2}\sin(2t)\right)$$