- Contour integration in complex plane.

• Recall the definition of a parametric curve in the complex plane.

C: For $\rho \in \mathbb{R}$ $s(\rho) = x(\rho) + jy(\rho)$ for $a \le \rho \le b$

• The integral of a complex function $f(s)$ along C is the complex number

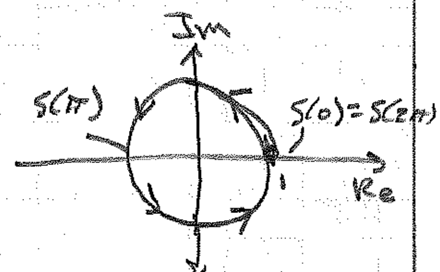
$$\int_C f(s)\,ds = \int_a^b f(s(\rho))\,s'(\rho)\,d\rho$$

where $s'(\rho) = \dfrac{ds}{d\rho}$

Example: Let $f(s) = \dfrac{1}{s}$ and C be the unit circle

$s(\rho) = \cos(\rho) + j\sin(\rho)$ $\quad 0 \le \rho \le 2\pi$

$= e^{j\rho}$



then $f(s(\rho)) = \dfrac{1}{s(\rho)} = \dfrac{1}{e^{j\rho}}$

and $s'(\rho) = je^{j\rho}$

$$\int_C \frac{1}{s}\,ds = \int_0^{2\pi} e^{-j\rho} j e^{j\rho}\,d\rho = j\int_0^{2\pi} d\rho = 2\pi j$$

Another example: $f(s) = s^2$ $\quad$ C = line from $(-1-j)$ to $(1+j)$

$s(\rho) = \rho + j\rho \quad -1 \le \rho \le 1$

$f(s(\rho)) = [s(\rho)]^2 = (\rho + j\rho)(\rho + j\rho)$

$= \rho^2 + 2j\rho^2 - \rho^2 = j2\rho^2$

$s'(\rho) = 1 + j$ AND $\displaystyle\int_C s^2\,ds = \int_{-1}^{1} j2\rho^2(1+j)\,d\rho$

$$= 2j(1+j)\int_{-1}^{1} \rho^2\,d\rho$$

example cont.

$$\int_C s^2\,ds = 2(j-1)\ \frac{1}{3}\rho^3\Big|_{-1}^{1}$$

$$= \frac{2(j-1)}{3}\left(1-(-1)\right) = \frac{4}{3}(j-1)$$

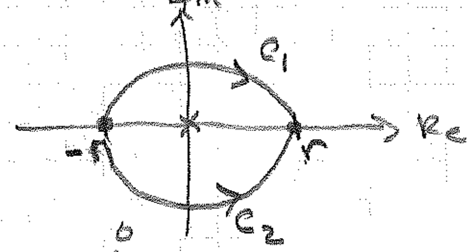
— In general the $\int$ from point s_0 to s_1 depends on path taken.

Example: let $s_0 = -r + 0j \quad s_1 = r + 0j \quad r > 0$ fixed

and C_1 = clockwise arc from $s_0 \to s_1$

C_2 = counterclockwise arc from $s_0 \to s_1$

$$f(s) = \frac{1}{s}$$



$$\int_{C_1} \frac{1}{s}\,ds = \int_{\pi}^{0} \underbrace{\frac{1}{re^{j\rho}}}_{f(s(\rho))} \cdot \underbrace{jre^{j\rho}}_{s'(\rho)}\,d\rho = j\int_{\pi}^{0} d\rho = j(0-\pi) = -j\pi$$

$$\int_{C_2} \frac{1}{s}\,ds = \int_{-\pi}^{0} j\,d\rho = j\left(0-(-\pi)\right) = j\pi$$

NOT Same.

but $\int C_1 - \int C_2 = 2\pi j$

Another Example: $f(s) = s^2 \quad C_1 \,\&\, C_2$ same as previous ex.

$$f(s(\rho)) = [s(\rho)]^2 = [re^{j\rho}]^2 = r^2 e^{j2\rho}$$

$$\int_{C_1} s^2\,ds = \int_{\pi}^{0} r^2 e^{j2\rho} \cdot jre^{j\rho}\,d\rho = jr^3 \int_{\pi}^{0} e^{j3\rho}\,d\rho$$

$$= \frac{2}{3}r^3$$

$$\int_{C_2} s^2\,ds = \int_{-\pi}^{0} r^2 e^{j2\rho} \cdot jre^{j\rho}\,d\rho = jr^3 \int_{-\pi}^{0} e^{j3\rho}\,d\rho$$

same

$$= \frac{2}{3}r^3$$

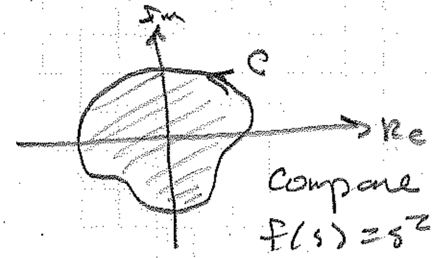
- Fundamental theorem of Complex Analysis.

  Let $f(s) = F'(s)$ over some region of $\mathbb{C}$ where f is analytic

  Given a curve C from s_0 to s_1

$$\int_C f(s)\,ds = F(s)\Big|_{s_0}^{s_1} = F(s_1) - F(s_2)$$

- For a closed, simple curve C $(s_0 = s_1)$ where f is analytic on and inside C,



  then

$$\oint_C f(s)\,ds = F(s_0) - F(s_1) = 0$$

Compare $f(s) = s^2$

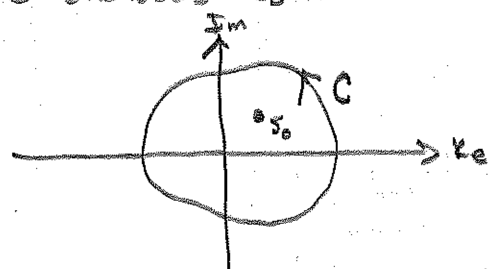
$1/s$

This is Cauchy's integral theorem. $f(s) = \frac{1}{s}$

- Now consider a curve C and function $f(s)$ analytic on and inside C, and a function

$$g(s) = \frac{f(s)}{s - s_0}$$ where $s_0 \in \mathbb{C}$ is a singularity AND C encloses s_0

  Then



$$\oint_C g(s)\,ds = \oint_C \frac{f(s)}{s - s_0}\,ds$$

$$= 2\pi j\, f(s_0)$$

<u>Residue Theorem</u>

  - $f(s_0)$ is called the <u>residue</u>, the value of the analytic $f(s_0)$ at the singularity of $g(s)$.

- For the functions we will be interested in integrating, (rational functions) this is all we need, since integration is a linear operator, and $f(s) =$ s^n, a, e^{st}

– Example $f(s) = 1$ $\quad g(s) = \frac{f(s)}{s+1} = \frac{1}{s+1}$

has singularity at $s_0 = -1$

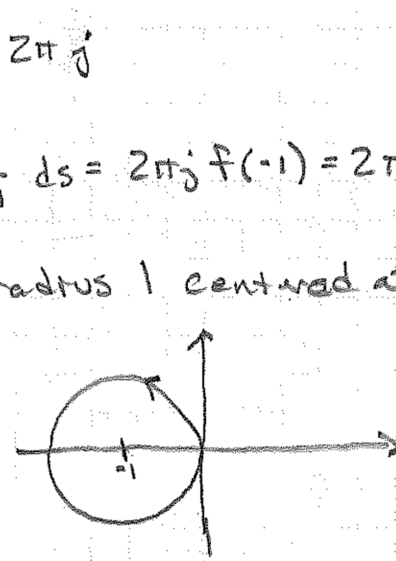
We showed earlier that $\oint_C \frac{1}{s}\, ds = 2\pi j$

The residue theorem tells us $\oint_C \frac{1}{s+1}\, ds = 2\pi j f(-1) = 2\pi j$

Lets verify that. Let C = circle of radius 1 centered at -1

$$s(\rho) = (\cos(\rho) - 1) + j\sin(\rho)$$

$$s'(\rho) = -\sin(\rho) + j\cos(\rho)$$



$$g(s(\rho)) = \frac{1}{s(\rho)+1} = \frac{1}{\cos(\rho) + j\sin(\rho)}$$

$$\int_0^{2\pi} \frac{-\sin(\rho) + j\cos(\rho)}{\cos(\rho) + j\sin(\rho)}\, d\rho = \int_0^{2\pi} \frac{j e^{j\rho}}{e^{j\rho}}\, d\rho = j\int_0^{2\pi} \rho\, d\rho = 2\pi j$$

– Examples where $f(s)$ not 1 but still analytic inside C.

These integrals are much harder to do by hand but residue theorem makes them easy.

- $f(s) = s^n \quad n \in \mathbb{Z}^+ \qquad g(s) = \frac{s^n}{s+1}$

  we will assume without proof that s^n is analytic. we did $n=1$

  Let C be the unit circle centered at $(-1, 0)$

  Then $$\oint_C \frac{f(s)}{s+1}\, ds = \oint_C \frac{s^n}{s+1}\, ds = 2\pi j f(-1) = 2\pi j (-1)^n$$

- similarly if $f(s) = e^{st}$, $t > 0$ (we showed analytic for $t=1$)

$$\oint_C \frac{f(s)}{s+1}\, ds = \oint_C \frac{e^{st}}{s+1}\, ds = 2\pi j f(-1) = 2\pi j e^{-t}$$

- What if there are more than 1 singularity?

Let $g(s) = \dfrac{f(s)}{(s-s_1)(s-s_2)\cdots(s-s_N)}$ $f(s)$ analytic

AND C is closed, simple contour containing all singularities.

Then
$$\oint_C g(s)\,ds = 2\pi j \sum_{n=1}^{N} K_n$$

where K_n is n^{th} residue $K_n = (s-s_n)\,g(s)\Big|_{s=s_n}$

Look familiar?

Example $f(s) = se^{st}$ $\quad g(s) = \dfrac{f(s)}{s^2+3s+2} = \dfrac{f(s)}{(s+1)(s+2)}$

Let C be circle of radius 3 centered at origin, which encloses $(-1,0)$ and $(-2,0)$

$$\oint_C g(s)\,ds = 2\pi j\left[K_1 + K_2\right]$$

$$K_1 = (s+1)\,g(s)\Big|_{s=-1} = \frac{f(s)}{s+2}\Big|_{s=-1} = \frac{se^{st}}{s+2}\Big|_{s=-1} = \frac{-1e^{-t}}{1} = -e^{-t}$$

$$K_2 = (s+2)\,g(s)\Big|_{s=-2} = \frac{f(s)}{s+1}\Big|_{s=-2} = \frac{se^{st}}{s+1}\Big|_{s=-2} = \frac{-2e^{-2t}}{-1} = 2e^{-2t}$$

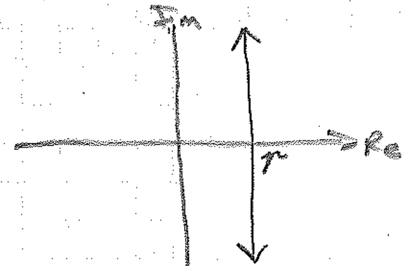
Finally $\displaystyle\oint_C \frac{se^{st}}{s^2+3s+2}\,ds = 2\pi j\left[-e^{-t} + 2e^{-2t}\right]$

- We will be interested in 2 specific integrals.
  - Inverse Laplace: $x(t) = \frac{1}{2\pi j}\int_{\gamma - j\infty}^{\gamma + j\infty} X(s)e^{st}\,ds$ for $t > 0$
    $s \in \mathbb{C}$
  - Inverse Z transform: $x[n] = \frac{1}{2\pi j}\oint X(z)z^{n-1}\,dz$ for $n > 0$
    $z \in \mathbb{C}$

  where in both cases $X(s)$ and $X(z)$ will be rational complex functions.

- To use the residue theorem for the Inverse Laplace transform we define the Bromwitch contour:

$$\int_{\gamma - j\infty}^{\gamma + j\infty} X(s)e^{st}\,ds$$

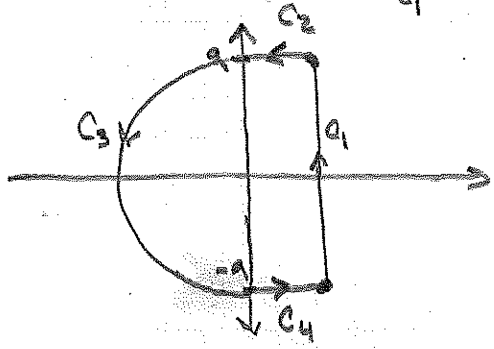


$\gamma >$ largest singularity of $X(s)$

becomes

$$\oint_C X(s)e^{st}\,ds = \lim_{a\to\infty} \int_{C_1} + \int_{C_2} + \int_{C_3} \int_{C_4}$$

where



- So as long as z^{n+1} and e^{st} are analytic then

$X(s) = \frac{P(s)e^{st}}{Q(s)}$ for polynomials $P(s)$ $Q(s)$ are analytic except at poles of $Q(s)$.

$X(z) = \frac{P(z)z^{n-1}}{Q(z)}$ "