ECE 3704 Lecture 12 Solving LCCDE using Laplace

- causal LTI systems and LCCDE
- response of systems with initial conditions
- system stability and Transfer function
- Relationship of Laplace to CT Fourier Transform
- Inverse Systems.

- Recall from ECE 2714 the CT LCCDE

$$\sum_{k=0}^{N} a_k D^k y(t) = \sum_{k=0}^{M} b_k D^k x(t) \qquad M \leq N \text{ and } D^k = \frac{d^k}{dt^k}$$

$$a_0 y(t) + a_1 Dy(t) + \cdots a_N D^N y(t) = b_0 x(t) + b_1 Dx(t) + \cdots$$

corresponds to an impulse response $h(t)$.

AND That $\mathcal{L}\{h(t)\} = H(s)$ is transfer function.

- Using the derivative property of Laplace allows us to easily find $H(s)$ without first finding $h(t)$.

- Consider first case of zero-initial conditions and causal input $x(t)$, so that $y(0^-) = x(0^-) = Dy(0^-) = Dx(0^-) = \cdots = 0$.

Then the derivative property is. $\mathcal{L}\{D^n x(t)\} = s^n X(s)$.

This, plus linearity, and some algebra allows us to find the TF $H(s)$.

Example: $Dy(t) + ay(t) = bx(t) \qquad a, b \in \mathbb{C}$

$$\mathcal{L}\{Dy(t) + ay(t)\} = \mathcal{L}\{bx(t)\}$$

$$sY(s) + aY(s) = bX(s)$$

$$Y(s)(s+a) = bX(s)$$

$$\frac{Y(s)}{X(s)} = \frac{b}{s+a} \equiv H(s) \text{ from Convolution Property.}$$

– Another example: $D^2y(t) + a\,Dy(t) + b\,y(t) = c\,x(t) + d\,Dx(t)$

$a, b, c, d \in \mathbb{C}$

Taking $\mathcal{L}$ of both sides,

$$(s^2 + as + b)Y(s) = (ds + c)X(s)$$

$$H(s) = \frac{Y(s)}{X(s)} = \frac{ds + c}{s^2 + as + b}$$

– Given an input $x(t)$ or $X(s)$ (again causal) we can determine output in Laplace domain (s-domain) $Y(s) = H(s)X(s)$ where $X(s) = \mathcal{L}\{x(t)\}$ and using inverse Laplace to get $y(t)$

Example: $D^2y(t) - 16y(t) = x(t)$ where $x(t) = e^{-4t}u(t)$

$$\Downarrow$$

Find $y(t)$ $(t \geq 0)$ $\qquad X(s) = \frac{1}{s+4}$

$$(s^2 - 16)Y(s) = X(s) = \frac{1}{s+4}$$

$$Y(s) = \frac{1}{(s+4)(s^2-16)} = \frac{1}{(s+4)^2(s-4)}$$

$$y(t) = \mathcal{L}^{-1}\{Y(s)\} = \mathcal{L}^{-1}\left\{\frac{k_1}{(s+4)^2} + \frac{k_2}{s+4} + \frac{k_3}{s-4}\right\}$$

$$k_1 = \left.\frac{1}{s-4}\right|_{s=-4} = -\frac{1}{8}$$

$$k_2 = \left.\frac{d}{ds}\frac{1}{s-4}\right|_{s=-4} = \left.\frac{-1}{(s-4)^2}\right|_{s=-4} = \frac{-1}{64}$$

$$k_3 = \left.\frac{1}{(s+4)^2}\right|_{s=4} = \frac{1}{64}$$

Thus

$$y(t) = -\frac{1}{8}te^{-4t}u(t) - \frac{1}{64}e^{-4t}u(t) + \frac{1}{64}e^{4t}u(t)$$

– What if the input $x(t)$ is NOT causal?

Let $x(t) = \underbrace{x(t)u(-t)}_{x_a(t)} + \underbrace{x(t)u(t)}_{x_c(t)}$

Taking Laplace (bilateral) $X(s) = X_a(s) + X_c(s)$
$\qquad Re\{s\} < r_a \qquad Re\{s\} > r_c$

Then $Y(s) = H(s)X(s) = H(s)X_a(s) + H(s)X_c(s)$

$\underbrace{r_h < Re\{s\} < r_a}_{\text{has two components}} \qquad Re\{s\} > \max\{r_h, r_c\}$

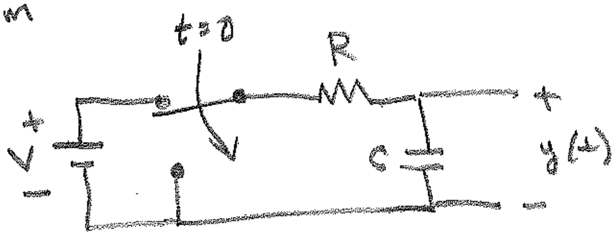
$f(t)u(-t) + g(t)u(t) + y_c(t)$

For this reason it is combursome to use Laplace for non-causal inputs even if system itself is causal.

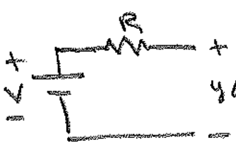
– What if the initial conditions of LCCDE are not zero?

Then NOT a LTI system. However we can still analyze the system

Example

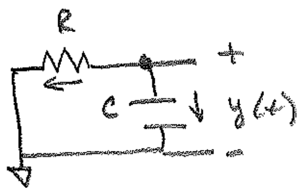


For $t<0$



$y(t) = V$

For $t \geq 0$



$-\frac{y(t)}{R} = C\frac{dy}{dt} \Rightarrow \frac{dy}{dt} = -\frac{1}{RC}y(t)$

so we have LCCDE $Dy + \frac{1}{RC}y = 0$

Taking Laplace $sY(s) - y(0^-) + \frac{1}{RC}Y(s) = 0$

$Y(s)(s + \frac{1}{RC}) = y(0^-) = V$

$Y(s) = \frac{V}{s + \frac{1}{RC}}$ and $y(t) = Ve^{-\frac{1}{RC}t} \quad t > 0$

This is called zero-input response. $\equiv x(t) = Vu(-t)$

- The above procedure is very similar to how we analyzed LCCDE in Fourier domain in 2714 with 3 exceptions.
  $s \to j\omega$, we don't have to check for stability, and $y(0^-)$ possibly non-zero.

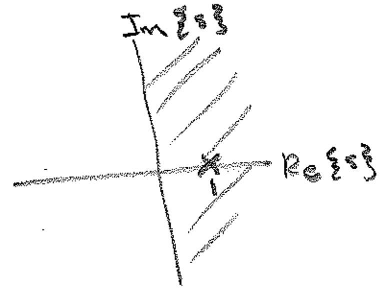
- Stability.
  - BIBO stability. Given LTI system with TF $H(s)$ the system is stable if the <u>real</u> part of <u>all</u> poles of $H(s)$ are in left-hand side of the complex plane.

• Example: $D^2 y(t) - y(t) = x(t)$ $\quad y(0^-) = x(0^-) = 0$ (zero IC)

Is the system BIBO stable?

Taking transform $s^2 Y(s) - Y(s) = X(s)$

$$H(s) = \frac{Y(s)}{X(s)} = \frac{1}{s-1}$$



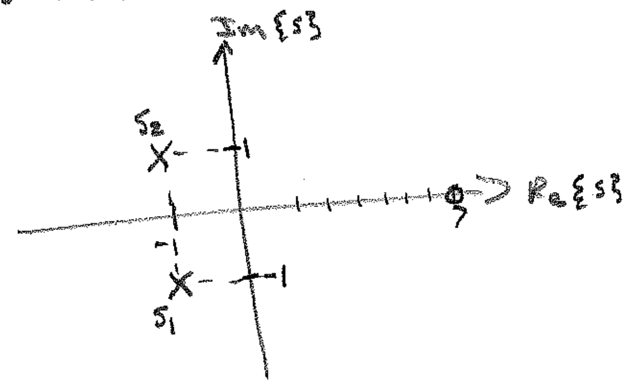
∴ NOT stable.

• Example: $D^2 y(t) + 2Dy(t) + y(t) = -7x(t) + Dx(t)$ $\quad$ zero I.C.

Taking Transform

$$s^2 Y(s) + 2s Y(s) + 1 = -7X(s) + sX(s)$$

$$H(s) = \frac{Y(s)}{X(s)} = \frac{s+7}{s^2+2s+1} = \frac{s-7}{(s+1+j)(s+1-j)}$$



$Re\{s_1\} = -1 < 0$

$Re\{s_2\} = -1 < 0$

both poles in LHP

∴ BIBO stable

Note: location of pole does not matter (in most but not all cases).

- A more nuanced look at stability.

Remark 1. In practice some care must be taken when canceling common factors in $P(s)$ + $Q(s)$ when $H(s) = \frac{P(s)}{Q(s)}$ (Pole-zero cancellation)

Why? Suppose $H(s) = \frac{(s+b_0)(s+b_1)(s+b_2)\cdots(s+b_M)}{(s+a_0)(s+a_1)(s+a_2)\cdots(s+a_N)}$

AND $b_0 = a_0$ then

$$H(s) = \frac{(s+b_1)(s+b_2)\cdots(s+b_M)}{(s+a_1)(s+a_2)\cdots(s+a_N)}$$

If $Re\{b_0\} = Re\{a_0\} > 0$ there is no issue.

But if $Re\{b_0\} = Re\{a_0\} < 0$ and $a_0 \approx b_0$

e.g. $a_0 = b_0 \pm \epsilon$ for $\epsilon \approx 0$

Then $\frac{(s+b_0)}{(s+a_0)} \neq 1 = \frac{s+b_0}{(s+b_0+\epsilon)}$ which is unstable.

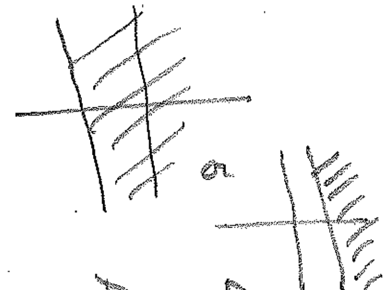
Remark 2. What about poles on imaginary axis, when $Re\{s\} = 0$

- If NO poles in RHP and one or more nonrepeated poles on imaginary Axis, system is marginally stable.

- If any poles in RHP or all poles in RHP but there is a repeated root on imaginary Axis the system is unstable.

– relationship between TF and Frequency Response.

- Note $X(s) = \int_{-\infty}^{\infty} x(t) e^{-st}\, dt$ for $Re\{s\} > \gamma$.



If $\gamma < 0$ then $s = j\omega$ is IN ROC and

$$X(s)\Big|_{s=j\omega} = \int_{-\infty}^{\infty} x(t) e^{-j\omega t}\, dt \equiv X(j\omega)$$

the CT Fourier Transform.

- When we are considering the TF $H(s)$ and a <u>stable</u> system ROC includes $s = j\omega$ and

$$H(s)\Big|_{s=j\omega} = H(j\omega) = \text{Frequency Response of system,}$$

This is often how systems analysis proceeds with non-causal inputs

System description $\rightarrow$ TF $\rightarrow$ FR $\rightarrow$ Fourier Analysis.

next time we will look at Block Diagrams & Circuits.

– Inverse systems. Recall from 2714 that an inverse system S_2 of another system S_1 is one where when placed in series the input = output

$$x(t) \rightarrow \boxed{S_1} \longrightarrow \boxed{S_2} \rightarrow x(t)$$

In terms of impulse responses this implies that

$$h(t) = h_1(t) * h_2(t) = \delta(t)$$

Inverse systems are hard to construct in this form, however taking the Laplace transform we see

$$H(s) = H_1(s) \cdot H_2(s) = 1$$

This implies that $H_2(s) = \dfrac{1}{H_1(s)}$