ECE 3704 Lecture #3 (Review)

Time domain analysis of CT + DT LTI Systems

- Recall from 2714 that linearity and time invariance sifting property leads directly to convolution integral for CT system analysis

$$x(t) = \int_{-\infty}^{\infty} x(\tau)\,\delta(t-\tau)\,d\tau \longrightarrow \boxed{h(t)} \longrightarrow y(t) = \int_{-\infty}^{\infty} x(\tau)\,h(t-\tau)\,d\tau$$

$$\equiv x(t) * h(t) \equiv h(t) * x(t)$$

- This approach to analysis works for any LTI system stable or unstable, as long as the integral converges.

- We can build up a table of convolutions by evaluating the convolution integral for a set of primitive functions.

Examples:

- $\delta(t) * x(t) = \int_{-\infty}^{\infty} \delta(\tau)\,x(t-\tau)\,d\tau = x(t)$

- $u(t) * u(t) = \begin{cases} 0 & t < 0 \\ \int_0^t d\tau & t \geq 0 \end{cases} = \begin{cases} 0 & t < 0 \\ t & t \geq 0 \end{cases} = t\,u(t)$

- $e^{at} u(t) * u(t) = \begin{cases} 0 & t < 0 \\ \int_0^t e^{a\tau}\,d\tau & t \geq 0 \end{cases} = \begin{cases} 0 & t < 0 \\ \frac{1}{a}(e^{at} - 1) & t \geq 0 \end{cases} = \frac{e^{at} - 1}{a} u(t)$

  $a \neq 0$

- etc.

— These tables plus a set of properties allows us to perform convolution on a wide array of signals without integration directly.

- commutivity $\quad x_1(t) * x_2(t) = x_2(t) * x_1(t)$
- distributivity $\quad x_1(t) * [x_2(t) + x_3(t)] = x_1(t) * x_2(t) + x_1(t) * x_3(t)$
- associativity $\quad x_1(t) * [x_2(t) * x_3(t)] = [x_1(t) * x_2(t)] * x_3(t)$
- time shift $\quad x_1(t-T_1) * x_2(t-T_2) = x_1(t) * x_2(t)\Big|_{t \to t-T_1-T_2}$

  $T_1, T_2 \in \mathbb{R}$
- multiplicative scaling $\quad [a x_1(t)] * [b x_2(t)] = ab\,[x_1(t) * x_2(t)]$

  $a, b \in \mathbb{C}$

— An Example

- $x_1(t) = 5e^{4t}u(t) + 7e^{-t}u(t-1)$

  $x_2(t) = u(t)$

$$
\begin{aligned}
(x_1 * x_2)(t) &= [5e^{4t}u(t) + 7e^{-t}u(t-1)] * u(t) \\
&= [5e^{4t}u(t)] * u(t) + [7e^{-t}u(t-1)] * u(t) && \text{Distributive} \\
&= 5\,[e^{4t}u(t)] * u(t) + 7\,[e^{-t}u(t-1)] * u(t) && \text{scaling} \\
&= 5\,[e^{4t}u(t)] * u(t) + 7e^{-1}[e^{-t}u(t)] * u(t)\Big|_{t \to t-1} && \text{timeshift} \\
&= 5\,\frac{e^{4t}-1}{4}u(t) + 7e^{-1}\,\frac{e^{-t}-1}{-1}u(t)\Big|_{t \to t-1} && \text{simplify} \\
&= \frac{5}{4}(e^{4t}-1)u(t) - 7e^{-1}(e^{-(t-1)}-1)u(t-1)
\end{aligned}
$$

- Recall an important convolution is when
  $x(t) = e^{st}$ $s \in \mathbb{C}$ any $h(t)$ is arbitrary impulse response

$$y(t) = e^{st} * h(t) = h(t) * e^{st}$$

$$= \int_{-\infty}^{\infty} h(\tau)\, e^{s(t-\tau)}\, d\tau \quad \text{Defn. conv.}$$

$$= \int_{-\infty}^{\infty} h(\tau)\, e^{-s\tau} e^{st}\, d\tau = e^{st} \int_{-\infty}^{\infty} h(\tau)\, e^{-s\tau}\, d\tau$$

$$= e^{st} \underline{\underline{H(s)}}$$

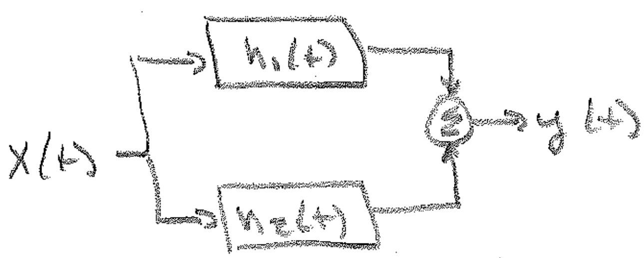
- $H(s)$ is Eigenvalue for signal e^{st}
- $H(s)$ is called Transfer function or System function AND is the Laplace transform of impulse response.

- CT systems can also be represented by block diagrams

$x(t) \rightarrow \boxed{h(t)} \rightarrow y(t)$ basic block $y(t) = h(t) * x(t)$

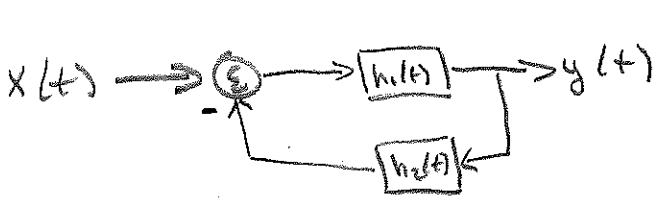
$x(t) \rightarrow \boxed{h_1(t)} \rightarrow z(t) \rightarrow \boxed{h_2(t)} \rightarrow y(t)$ series combination

$$y(t) = \underbrace{(h_1(t) * h_2(t))}_{h(t) \text{ of combination}} * x(t)$$



parallel Combination

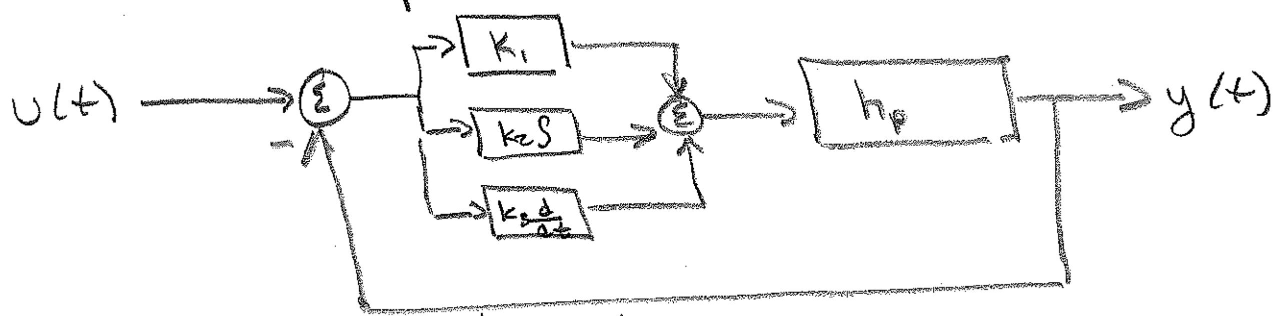
$$y(t) = \underbrace{(h_1(t) + h_2(t))}_{h(t)} * x(t)$$



Feedback Motif

- The most basic block for CT systems is $\boxed{\int} = \boxed{1/s} = \boxed{s^{-1}}$
  Integrator

— Block Diagrams are a nice way to visualize systems. For example



PID Controller for "Plant" with impulse response $h_p(t)$.

— For DT systems convolution becomes a sum

$$x[n]=\sum_{m=-\infty}^{\infty} x[m]\,\delta[n-m] \longrightarrow \boxed{h[n]} \longrightarrow y[n]=\sum_{m=-\infty}^{\infty} x[m]\,h[n-m]$$

- As for CT case, this analysis method work for any LTI system, as long as sum converges.

— We can build up a similar table of results.

- $\delta[n] * x[n] = \displaystyle\sum_{m=-\infty}^{\infty} \delta[m]\,x[n-m] = x[n]$

- $u[n] * u[n] = \begin{cases} 0 & n<0 \\ \displaystyle\sum_{m=0}^{n} (1) & n\geq 0 \end{cases} = \begin{cases} 0 & n<0 \\ n+1 & n\geq 0 \end{cases} = n\,u[n]$

- $r^n u[n] * u[n] = \begin{cases} 0 & n<0 \\ \displaystyle\sum_{m=0}^{n} r^m \end{cases} = \begin{cases} 0 & n<0 \\ \dfrac{r^{n+1}-1}{r-1} & n>0 \end{cases} = \dfrac{r^{n+1}-1}{r-1}\,u[n]$

  $r \neq 1$

- etc.

- Such tables plus the following properties allow us to quickly analyze a variety of systems.
  - communivity : $x_1[n] * x_2[n] = x_2[n] * x_1[n]$
  - distributivity: $x_1[n] * (x_2[n] + x_3[n]) = x_1[n] * x_2[n] + x_1[n] * x_3[n]$
  - associativity : $\{x_1[n] * x_2[n]\} * x_3[n] = x_1[n] * \{x_2[n] * x_3[n]\}$
  - index shift: $x_1[n-n_1] * x_2[n-n_2] = x_1[n] * x_2[n] \Big|_{n \to n-n_1-n_2}$
  - scaling : $\{a x_1[n]\} * \{b x_2[n]\} = ab\{x_1[n] * x_2[n]\}$

- An example : $x_1[n] = z^n u[n-4] \quad x_2[n] = 5u[n+1]$

$$
\begin{aligned}
x_1[n] * x_2[n] &= \{z^n u[n-4]\} * \{5u[n+1]\} \\
&= \{z^4 z^{n-4} u[n-4]\} * \{5u[n+1]\} \\
&= z^4(5)\{z^n u[n]\} * \{u[n]\} \Big|_{\substack{n \to n-4+1 \\ n-3}} \\
&= 80 \left\{ \frac{z^{n+1}-1}{z-1} u[n] \right\} \Big|_{n \to n-3} \\
&= 80 (z^{n-2} - 1) u[n-3]
\end{aligned}
$$

- Another important example is $x[n] = z^n \quad z \in \mathbb{C}$

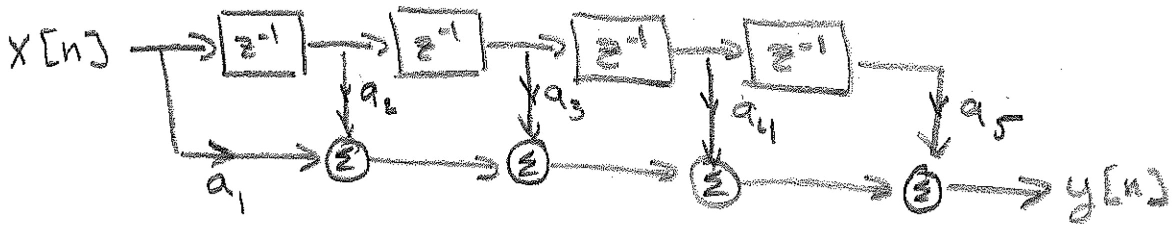
$$
\begin{aligned}
y[n] &= h[n] * x[n] \\
&= \sum_{m=-\infty}^{\infty} h[m] z^{n-m} \\
&= \sum_{m=-\infty}^{\infty} h[m] z^n z^{-m} \\
&= z^n \sum_{m=-\infty}^{\infty} h[m] z^{-m} \equiv z^n H(z)
\end{aligned}
$$

- $H(z)$ is Eigenvalue for z^n
- Called transfer function
- Z transform of $h[n]$

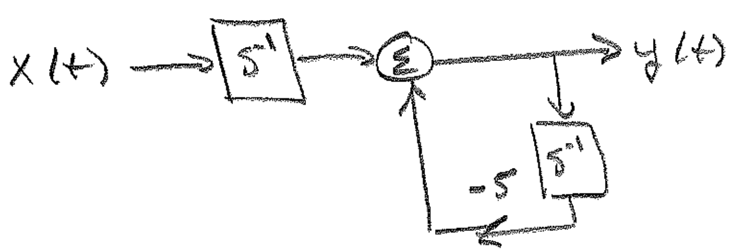
- DT systems can also be represented by block diagrams. The basic motifs and properties are same as for CT systems.
- The most basic block for DT systems is delay

$$\to \boxed{D} \to \; = \; \to \boxed{z^{-1}} \to$$

- Example: FIR Filter 4^{th} order



- Examples of time-domain analysis.
  - Given system below where $x(t) = e^{-5t} u(t)$. Find $y(t)$



- reading the Block Diagram: $y(t) = -5\int y + \int x$
- Taking derivative of both sides $y' = -5y + x$
- IN standard LCCDE form: $y' + 5y = x$
- Corresponding $h(t) = e^{-5t} u(t)$
- output: $y(t) = h(t) * x(t)$

$$= \left[e^{-5t} u(t)\right] * \left[e^{-5t} u(t)\right]$$

$$\boxed{y(t) = t e^{-5t} u(t)}$$ From Table.

- Given 4^{th} order FIR Filter BD from before find $y[n]$ if $x[n] = u[n] - u[n-5]$

— Reading BD: $y[n] = a_5\, x[n-4] + a_4\, x[n-3] + a_3 x[n-2] + a_2 x[n-1] + a_1\, x[n]$

– corresponding $h[n] = a_5 \delta[n-4] + a_4 \delta[n-3] + a_3 \delta[n-2] + a_2 \delta[n-1] + a_1 \delta[n]$

– output $y[n] = h[n] * x[n]$

$$
\begin{aligned}
&= a_5\, \delta[n-4] * u[n] - a_5 \delta[n-4] * u[n-5] \\
&+ a_4\, \delta[n-3] * u[n] - a_4\, \delta[n-3] * u[n-5] \\
&+ a_3\, \delta[n-2] * u[n] - a_3\, \delta[n-2] * u[n-5] \\
&+ a_2\, \delta[n-1] * u[n] - a_2\, \delta[n-1] * u[n-5] \\
&+ a_1\, \delta[n] * u[n] - a_1\, \delta[n] * u[n-5]
\end{aligned}
$$

$$
\begin{aligned}
y[n] = &\; a_5\, u[n-4] - a_5 u[n-9] \\
&+ a_4\, u[n-3] - a_4\, u[n-8] \\
&+ a_3\, u[n-2] - a_3\, u[n-7] \\
&+ a_2\, u[n-1] - a_2\, u[n-6] \\
&+ a_1\, u[n] - a_1\, u[n-5]
\end{aligned}
$$