ECE 3704 Lecture # 4 (Review)

CT and DT Fourier Analysis

- Recall that periodic signals with period T that meet the Direchlet conditions can be decomposed as

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t} \quad \text{where } \omega_0 = \frac{2\pi}{T}, \quad a_k \in \mathbb{C}$$

CT Fourier Series

AND Fourier coefficients are given by

$$a_k = \frac{1}{T}\int_{t_0}^{t_0+T} x(t)\, e^{-jk\omega_0 t}\, dt$$

AND measure the similarity / projection of $x(t)$ onto $e^{-jk\omega_0 t}$ (between).

- A plot of $|a_k|$ and $\angle a_k$ as function of k is called spectrum

- Example: Recall that sampling of a CT signal can be thought of as multiplication by the impulse train with spacing $T_0 \in \mathbb{R}$.

$$p(t) = \sum_{m=-\infty}^{\infty} \delta(t - mT_0)$$

The F.S. coefficients are:

$$a_k = \frac{1}{T_0}\int_{-T_0/2}^{T_0/2} x(t)\, e^{jk\omega_0 t} \qquad \omega_0 = \frac{2\pi}{T_0}$$

$$= \frac{1}{T_0}\int_{-T_0/2}^{T_0/2} \delta(t)\, e^{jk\frac{2\pi}{T_0}t} = \frac{1}{T_0} e^{jk\frac{2\pi}{T_0}(0)} = \frac{1}{T_0}$$

- For non-periodic CT signals the decomposition is over an uncountably infinite set of frequencies.

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega)\, e^{j\omega t}\, d\omega \qquad \omega \in \mathbb{R}. \qquad X(j\omega) \in \mathbb{C}$$

where the Fourier Transform is

$$X(j\omega) = \int_{-\infty}^{\infty} x(t)\, e^{-j\omega t}\, dt$$

} CT Fourier Transform pair.

- A plot of $|X(j\omega)|$ and $\angle X(j\omega)$ as a function of ω is called Spectrum. Note difference to $|a_k|$ and $\angle a_k$.

- The most useful property of the CTFT is the convolution property. Let $\mathcal{F}\{h(t)\} = H(j\omega)$
$\mathcal{F}\{x(t)\} = X(j\omega)$

Then $Y(j\omega) = H(j\omega) \cdot X(j\omega)$
and $y(t) = \mathcal{F}^{-1}\{H(j\omega)\}$ so long as $H(j\omega)$ exists.

- $H(j\omega)$ is called the frequency response of the system and exists if system is BIBO stable.

$$\int_{-\infty}^{\infty} |h(t)|\, dt < \infty \implies H(j\omega) \text{ exists.}$$

($\mathcal{F}$ transform integral converges)

- This gives us an alternate route to analysis of stable systems.

$$\begin{array}{ccc}
x(t) & \xrightarrow{\quad h(t) \quad} & y(t) = h(t) * x(t) \\
\updownarrow \mathcal{F} & \updownarrow \mathcal{F} & \updownarrow \mathcal{F} \\
X(j\omega) & \xrightarrow[\quad H(j\omega) \quad]{} & Y(j\omega) = H(j\omega) \cdot X(j\omega)
\end{array}$$

— Example: Consider a LTI CT system given by $y''(t) + 12y'(t) + 20y(t) = 5x(t)$

Find $y(t)$ if $x(t) = e^{-t}u(t)$ using Fourier Analysis.

1. determine stability. Roots of $Q(D) = D^2 + 12D + 20$
$$= (D+10)(D+2)$$
are in LHP so <u>stable</u>.

That means we can proceed using Fourier Analysis.

2. Using derivative property of $\mathcal{F}\{\frac{dx}{dt}\} = (j\omega)X(j\omega)$ we can take $\mathcal{F}$ transform of both sides.

$$(j\omega)^2 Y(j\omega) + 12(j\omega)Y(j\omega) + 20Y(j\omega) = 5 \cdot X(j\omega)$$

Some algebra gives $Y(j\omega) = \underbrace{\frac{5}{(j\omega)^2 + 12(j\omega) + 20}}_{H(j\omega)} \cdot X(j\omega)$

3. Using $\mathcal{F}\int$ or table.
$$X(j\omega) = \frac{1}{j\omega + 1}$$

Thus $Y(j\omega) = \dfrac{5}{[(j\omega)^2 + 12(j\omega) + 20](j\omega+1)}$

4. To find $y(t) = \mathcal{F}^{-1}\{Y(j\omega)\}$ we can use PFE

$$Y(j\omega) = \frac{A}{(j\omega)+10} + \frac{B}{(j\omega)+2} + \frac{C}{(j\omega)+1}$$

where some algebra gives $A = \frac{5}{72}$ $B = -\frac{5}{8}$ $C = \frac{5}{9}$

- Using table of transfoms & properties

$$y(t) = \frac{5}{72}e^{-10t}u(t) - \frac{5}{8}e^{-2t}u(t) + \frac{5}{9}e^{-t}u(t).$$

— Now for <u>DT</u> periodic signals with period N.
we can decompose them as

$$x[n] = \sum_{k=N_0}^{N_0+N-1} a_k e^{jk\omega_0 n} \qquad \omega_0 = \frac{2\pi}{N} \qquad N_0 \in \mathbb{Z}$$

where DTFS coefficients are $a_k = \frac{1}{N} \sum_{n=N_0}^{N_0+N-1} x[n] e^{-jk\omega_0 n}$

— By extension for finite length $x[n]$, the DTFS of the periodically extended $x[n]$ is equivalent to the Discrete Fourier Transform DFT/FFT, $X[k] = N a_k$

— the plot of $|a_k|$ and $\angle a_k$ over one period is called the Spectrum.

— Example: Compute the DFT by hand of the finite length signal $\{1, 2, 1\}$.

1. Periodic extension $N = 3$

$$\{\ldots, 1, 2, 1, \underset{\substack{\uparrow \\ n=0}}{1}, 2, 1, 1, 2, 1, \ldots\}$$

2. $a_k = \frac{1}{N} \sum_{n=N_0}^{N_0+N-1} x[n] e^{-j\frac{2\pi}{N}kn}$ Let $N_0 = 0$, $N = 3$

$$= \frac{1}{3} \sum_{n=0}^{2} x[n] e^{-j\frac{2\pi}{3}kn}$$

$$= \frac{1}{3}\left[x[0] e^{-j\frac{2\pi}{3}(0)k} + x[1] e^{-j\frac{2\pi}{3}(1)k} + x[2] e^{-j\frac{2\pi}{3}(2)k}\right]$$

$$= \frac{1}{3}\left[1 + 2e^{-j\frac{2\pi}{3}k} + e^{-j\frac{4\pi}{3}k}\right]$$

3. $X[k] = N a_k$ compare to Matlab

$x = [1, 2, 1]$ $X = \text{fft}(x) = [4, -\frac{1}{2} - j0.866, -\frac{1}{2} + j0.866]$ ✓

— For non-periodic DT signals the Decomposition is over an uncountably infinite set of frequencies.

$$x[n] = \frac{1}{2\pi}\int_{\omega_0}^{\omega_0+2\pi} X(e^{j\omega})\, e^{j\omega n}\, d\omega \qquad \omega \in \mathbb{R} \quad X(e^{j\omega}) \in \mathbb{C}$$

where the Discrete Time Fourier Transform is

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n]\, e^{-j\omega n}$$

is always periodic in 2π.

— Again a plot of $|X(e^{j\omega})|$ an $\angle X(e^{j\omega})$ is spectrum.

— As in CT the most usefull property of DTFT is the convolution property. Let $\mathcal{F}\{h[n]\} = H(e^{j\omega})$ and $\mathcal{F}\{x[n]\} = X(e^{j\omega})$ then $Y(e^{j\omega}) = H(e^{j\omega}) \cdot X(e^{j\omega})$ and $y[n] = \mathcal{F}^{-1}\{Y(e^{j\omega})\}$ as long as $H(e^{j\omega})$ exists.

— $H(e^{j\omega})$ is called the frequency response of the DT system and exists if the system is BIBO stable

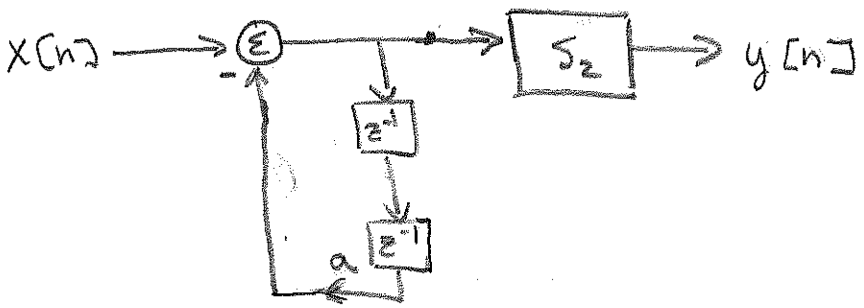
$$\sum_{n=-\infty}^{\infty} |h[n]| < \infty \implies H(e^{j\omega}) \text{ exists. (Sum converges)}$$

— This gives us an alternate Analysis route for stable systems

$$\begin{tikzcd}
x[n] \arrow[rr, "{h[n]}"] \arrow[d, leftrightarrow, "\mathcal{F}"] & {} \arrow[d, leftrightarrow, "\mathcal{F}"] & y[n] = h[n] * x[n] \arrow[d, leftrightarrow, "\mathcal{F}"] \\
X(e^{j\omega}) \arrow[rr, "H(e^{j\omega})"'] & {} & Y(e^{j\omega}) = H(e^{j\omega})\, X(e^{j\omega})
\end{tikzcd}$$

- Example: Consider the following system where #4 (6)

$a = -\frac{1}{4}$ and S_2 is variable.



Draw a block Diagram for S_2 s.t. $y[n] = x[n]$
i.e. S_2 is the inverse of the system it is in series with.

1. Let $v[n]$ be the output of the first system. Then

$$v[n] = +\frac{1}{4} v[n-2] + x[n]$$

$$v[n+2] - \frac{1}{4} v[n] = x[n+2]$$

2. To see if the first system is stable check roots of $Q(E)$

$Q(E) = E^2 - \frac{1}{4} = (E + \frac{1}{2})(E - \frac{1}{2})$. Both roots $|\frac{1}{2}| < 1$ $\therefore$ stable and we can use Fourier analysis.

3. Using delay property of DTFT $\mathcal{F}\{x[n-m]\} = e^{-j\omega m} X(e^{j\omega})$

$$e^{j2\omega} V(e^{j\omega}) - \frac{1}{4} V(e^{j\omega}) = e^{j2\omega} X(e^{j\omega})$$

AND $$\frac{V(e^{j\omega})}{X(e^{j\omega})} = \frac{e^{j2\omega}}{e^{j2\omega} - \frac{1}{4}}$$

4. We look for $H_2(e^{j\omega})$ s.t. $Y(e^{j\omega}) = H_2(e^{j\omega}) V(e^{j\omega}) = X(e^{j\omega})$

$$= H_2(e^{j\omega}) \cdot \frac{e^{j2\omega}}{e^{j2\omega} - \frac{1}{4}} \cdot X(e^{j\omega})$$

Thus $$H_2(e^{j\omega}) = \frac{e^{j2\omega} - \frac{1}{4}}{e^{j2\omega}}$$

or $y[n+2] = v[n+2] - \frac{1}{4} v[n]$

$y[n] = v[n] - \frac{1}{4} v[n-2]$