Lecture 18: Inverse Z transform

- Consider a causal signal $f[n]$ that may, or may not be absolutely summable.

  - Let $g[n] = (r)^n f[n]$ for some $r > 0$ such that

$$\sum_{n=0}^{\infty} |g[n]| = \sum_{n=0}^{\infty} |r^n f[n]| < \infty$$

  - Then the DTFT of $g[n]$ exists and is given by

$$G(e^{j\omega}) = \sum_{n=-\infty}^{\infty} g[n] e^{-j\omega n} = \sum_{n=0}^{\infty} g[n] e^{-j\omega n}$$

$$= \sum_{n=0}^{\infty} r^n f[n] e^{-j\omega n}$$

$$= \sum_{n=0}^{\infty} f[n] (r^{-1} e^{j\omega})^{-n}$$

FIX

$$= Z_1\{f[n]\}\Big|_{z = r^{-1} e^{j\omega}}$$ for r fixed so that $|z = re^{j\omega}| = |r| \in$ ROC.

  - The inverse DTFT is then

$$g[n] = r^n f[n] = \frac{1}{2\pi} \int_{-\pi}^{\pi} G(e^{j\omega}) e^{j\omega n} \, d\omega$$

multiply through by r^{-n}

$$r^{-n} g[n] = f[n] = \frac{1}{2\pi} \int_{-\pi}^{\pi} G(e^{j\omega}) (r^{-1} e^{j\omega})^n \, d\omega$$

  - Substitute for $G(e^{j\omega})$

FIX

$$f[n] = \frac{1}{2\pi} \int_{-\pi}^{\pi} \underbrace{\left[\sum_{n=0}^{\infty} f[n] (r^{-1} e^{j\omega})^{-n} \right]}_{F(z)\big|_{z = r^{-1} e^{j\omega}}} (r^{-1} e^{j\omega})^n \, d\omega$$

• Let $z = r^{-1} e^{j\omega}$ then since r is constant $dz = +j r^{-1} e^{+j\omega} d\omega$

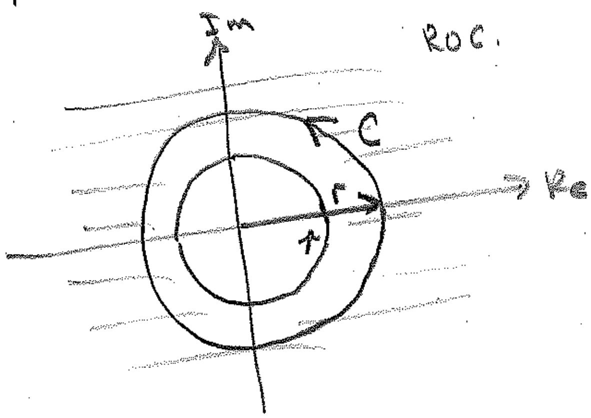
or $d\omega = \frac{1}{j} r e^{-j\omega} dz$

AND $f[n] = \frac{1}{2\pi} \oint F(z) \underbrace{(r^{-1} e^{j\omega})^n \left[\frac{1}{j} r e^{-j\omega}\right]} dz$

z^{n-1}
$(r^{-1}e^{j\omega})^n (r^{-1}e^{j\omega})^{-1}$
$(r^{-1}e^{j\omega})^n (r e^{-j\omega})$

$= \frac{1}{2\pi j} \oint_C F(z) z^{n-1} dz$ for ~~r such that~~ $z = re^{j\omega} \in$ ROC.

NOTE! NOT Bromwitch contour



A contour integral in complex plane.

Note: $re^{j\omega} \in$ ROC implies $r^n f[n]$ is absolutely summable.

– Example: Recall that $Z_1\{(\mu)^n u[n]\} = \frac{z}{z-\mu}$ for $|z| > |\mu|$

Then $x[n] = Z_1^{-1}\left\{\frac{z}{z-\mu}\right\} = \frac{1}{2\pi j} \oint_C \frac{z}{z-\mu} z^{n-1} dz$ for $C = re^{j\omega}$, $r > |\mu|$, $\omega \in [-\pi, \pi]$

$= \frac{1}{2\pi j} \oint_C \frac{z^n}{z-\mu} dz$ Singularity @ μ

$= \frac{1}{\cancel{2\pi j}} \cancel{2\pi j} [K]$

where K is the residue. $(z-\mu) \frac{z^n}{z-\mu} \Big|_{z=\mu} = \mu^n$

$x[n] = \mu^n$ for $n \geq 0$

Since causal $x[n] = (\mu)^n u[n]$

— What about Bilateral Z transform inverse.

Write a general signal as anticausal + causal.

$$x[n] = \underbrace{x[n]u[-(n+1)]}_{\text{anticausal}, x_a[n]} + \underbrace{x[n]u[n]}_{\text{causal}, x_c[n]}$$

Aside: $u[-(n+1)] = u[-n]\big|_{n \to n+1}$ advanced by 1

$u[n-1]\big|_{n \to -n} \neq u[-(n+1)]$

$$Z_2\{x[n]\} = \sum_{n=-\infty}^{-1} x_a[n] z^{-n} + \sum_{n=0}^{\infty} x_c[n] z^{-n}$$

$$= \sum_{n=1}^{\infty} x_a[-n] z^{n} + Z_1\{x_c[n]u[n]\}$$

$$= \sum_{n=0}^{\infty} x_a[-n] z^{n} - x_a[0] z^{-0} + Z_1\{x_c[n]u[n]\}$$

$$= Z_1\{x_a[-n]\}\Big|_{|z|>r \Rightarrow |z|<r,\ z \to z^{-1}} - \underbrace{x_a[0]}_{\substack{=0 \\ \text{by defn.}}} + Z_1\{x_c[n]\}$$

— Example

$$Z_2\{-r^n u[-(n+1)]\} = \sum_{n=-\infty}^{-1} -r^n z^{-n}$$

$$= \sum_{n=1}^{\infty} -r^{-n} z^{n}$$

$$= -\lim_{N \to \infty} \frac{(r^{-1}z)^{N+1} - (r^{-1}z)^1}{r^{-1}z - 1}$$

$$= -\frac{-r^{-1}z}{r^{-1}z - 1} \quad \text{if } |r^{-1}z| < 1 \text{ or } |z| < |r|$$

$$= \frac{z}{z-r} \quad |z| < |r|$$

Note: Same as $Z\{r^n u[n]\}$ but ROC is $|z| < |r|$ v/s $|z| > |r|$

Thus the only way to tell causal from anticausal is via ROC.

— An example with complex roots.

$$X(z) = \frac{z}{z^2+1} = \frac{z}{(z+j)(z-j)} \quad \text{for } |z| > 1$$

Causal ROC thus,

$$\underset{n \geq 0}{x[n]} = \frac{1}{2\pi j} \oint_C X(z)\, z^{n-1}\, dz$$

C = unit circle radius > 1.

$$= \frac{1}{2\pi j} \oint_C \frac{z^n}{(z+j)(z-j)}\, dz$$

$$= \frac{1}{2\pi j} (2\pi j) \left[K_1 + K_2 \right]$$

$$K_1 = \left. \frac{j z^n}{z-j} \right|_{z=-j} = \frac{(-j)^n}{-2j} = -\frac{1}{2j}(-j)^n$$

$$K_2 = \left. \frac{j z^n}{z+j} \right|_{z=j} = \frac{(j)^n}{2j} = \frac{1}{2j}(j)^n$$

$$x[n] = \frac{1}{2j}(j)^n - \frac{1}{2j}(-j)^n \quad n \geq 0$$

$$= \frac{1}{2j}\left(e^{j\pi/2}\right)^n - \frac{1}{2j}\left(e^{-j\pi/2}\right)^n \quad n \geq 0$$

$$= \sin\left(\frac{\pi}{2} n\right) \quad n \geq 0$$, since causal.

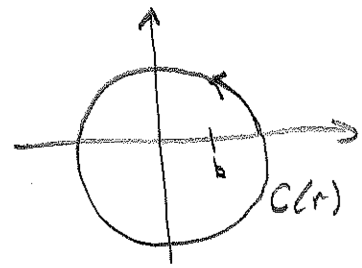
$$x[n] = j \sin\left(\frac{\pi}{2} n\right) u[n]$$

— Example: Given the transfer function

$$H(z) = \frac{z+a}{z+b} \qquad |z| > |b| \qquad a, b \in \mathbb{R}$$

Find the impulse response $h[n]$.

Solution: Since ROC corresponds to causal signal.



$C(r)$ $r > |b|$

$$\underset{n \geq 0}{h[n]} = \frac{1}{2\pi j} \oint_C H(z) z^{n-1} \, dz$$

$$= \frac{1}{2\pi j} \oint_c \frac{z+a}{z+b} z^{n-1} \, dz$$

NOTE: take care to simplify to rational function

$$= \frac{1}{2\pi j} \oint_c \frac{(z+a)z^n}{z(z+b)} \, dz$$

$$= \frac{1}{2\pi j} 2\pi j \left[k_1 + k_2 \right]$$

$$k_1 = \left. \frac{(z+a)z^n}{z+b} \right|_{z=0} = \begin{cases} \frac{a}{b} & n=0 \\ 0 & \text{else} \end{cases}$$

$$k_2 = \left. \frac{(z+a)z^n}{z} \right|_{z=-b} = \frac{(a-b)(-b)^n}{-b}$$

$$= \frac{(b-a)}{b} (-b)^n$$

Thus $h[n] = \frac{a}{b} \delta[n] + \frac{(b-a)}{b} (-b)^n u[n]$

— Example: Linear Time Invariant DT System with

$$h[n] = \left(\tfrac{1}{4}\right)^n u[n]$$

and input $x[n] = \cos\left(\frac{\pi}{4} n\right) u[n]$

What is the Z transform of the output, $Y(z)$?

• From last time.

$$H(z) = \frac{z}{z - \frac{1}{4}} \qquad X(z) = \frac{z(z - \cos(\frac{\pi}{4}))}{z^2 - 2\cos(\frac{\pi}{4})z + 1}$$

AND $$Y(z) = H(z)X(z) = \frac{z^2(z - \cos(\frac{\pi}{4}))}{(z - \frac{1}{4})(z^2 - 2\cos(\frac{\pi}{4})z + 1)} \qquad |z| > 1$$

— To find $y[n] = \mathcal{Z}^{-1}\{Y(z)\}$ we need roots of

$$z^2 - 2\cos(\tfrac{\pi}{4})z + 1 \qquad \cos(\tfrac{\pi}{4}) = \frac{1}{\sqrt{2}}$$

$$(z - \tfrac{1}{\sqrt{2}} + j\tfrac{1}{\sqrt{2}})(z - \tfrac{1}{\sqrt{2}} - j\tfrac{1}{\sqrt{2}})$$

$$y[n] = \frac{1}{2\pi j}\oint_C \frac{(z - \frac{1}{\sqrt{2}})\, z^{n+1}}{(z-\frac{1}{4})(z - \frac{1}{\sqrt{2}} + j\frac{1}{\sqrt{2}})(z - \frac{1}{\sqrt{2}} - j\frac{1}{\sqrt{2}})}\, dz \qquad C = \text{unit circle radius} > 1.$$

$$= \frac{1}{2\pi j}\, 2\pi j\,[k_1 + k_2 + k_3]$$

$$k_1 = \left.\frac{(z - \frac{1}{\sqrt{2}})z^{n+1}}{z^2 - \frac{2}{\sqrt{2}}z + 1}\right|_{z=\frac{1}{4}} = \frac{(\frac{1}{4} - \frac{1}{\sqrt{2}})(\frac{1}{4})^{n+1}}{\frac{1}{16} - \frac{2}{\sqrt{2}}\cdot\frac{1}{4} + 1} = A\left(\frac{1}{4}\right)^n \qquad A \in \mathbb{R}$$

$$k_2 = \left.\frac{(z - \frac{1}{\sqrt{2}})z^{n+1}}{(z-\frac{1}{4})(z - \frac{1}{\sqrt{2}} - j\frac{1}{\sqrt{2}})}\right|_{z=\frac{1}{\sqrt{2}}-j\frac{1}{\sqrt{2}}} = \frac{(\frac{1}{\sqrt{2}} - j\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}})(\frac{1}{\sqrt{2}} - j\frac{1}{\sqrt{2}})^{n+1}}{(\frac{1}{\sqrt{2}} - j\frac{1}{\sqrt{2}} - \frac{1}{4})(\frac{1}{\sqrt{2}} - j\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}} - j\frac{1}{\sqrt{2}})}$$

$$B \in \mathbb{C} \quad = B\left(\frac{1}{\sqrt{2}} - j\frac{1}{\sqrt{2}}\right)^n$$

$$k_3 = \left.\frac{(z - \frac{1}{\sqrt{2}})z^{n+1}}{(z-\frac{1}{4})(z - \frac{1}{\sqrt{2}} + j\frac{1}{\sqrt{2}})}\right|_{z=\frac{1}{\sqrt{2}}+j\frac{1}{\sqrt{2}}} = \frac{(\frac{1}{\sqrt{2}} + j\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}})(\frac{1}{\sqrt{2}} + j\frac{1}{\sqrt{2}})^{n+1}}{(\frac{1}{\sqrt{2}} + j\frac{1}{\sqrt{2}} - \frac{1}{4})(\frac{1}{\sqrt{2}} + j\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}} + j\frac{1}{\sqrt{2}})}$$

$$= C\left(\frac{1}{\sqrt{2}} + j\frac{1}{\sqrt{2}}\right)^n \qquad C \in \mathbb{C}$$

$$y[n] = \left[A\left(\tfrac{1}{4}\right)^n + B\left(\tfrac{1}{\sqrt{2}} - j\tfrac{1}{\sqrt{2}}\right)^n + C\left(\tfrac{1}{\sqrt{2}} + j\tfrac{1}{\sqrt{2}}\right)^n\right]u[n]$$

next time we will see a way to reduce this using properties + tables of transforms.