Lecture 20: Solving LCCDE using Z transform

- Recall from 2714 that a linear, constant coefficient difference equation is a relationship between shifted versions of input and output.

  - Advance Form $\displaystyle\sum_{k=0}^{N} a_k y[n+k] = \sum_{k=0}^{M} b_k x[n+k]$

    e.g. $2y[n+1] - y[n] = x[n+1]$ $\quad N = M = 1$

  - Delay Form $\displaystyle\sum_{k=0}^{N} a_k y[n-P+k] = \sum_{k=0}^{M} b_k x[n-P+k]$ $\quad P = \max(N, M)$

  - Recursive form $\displaystyle y[n] = -\sum_{k=1}^{N} \frac{a_k}{a_0} y[n-P+k] + \sum_{k=0}^{M} \frac{b_k}{a_0} x[n-P+k]$

- Given $x[n]$ and $N+M$ auxillary/Initial conditions we wish to solve the Difference Equation, i.e. find expression for $y[n]$

  The solution can be divided into two-parts.

  - Zero-input due to A.C. only $y_{zi}[n]$
    also called natural response

  - Zero-state due to input only $y_{zs}[n]$ also called forced response.

  Thus $y[n] = y_{zi}[n] + y_{zs}[n]$

  When the A.C. are zero $y[n] = y_{zs}[n]$ and the system is LTI.

- The time/index shift property gives us a straight forward approach to solving LCCDE's.

- Recall the index shift property of z transform is

$$x[n] \overset{z}{\longleftrightarrow} X(z)$$

• Delay (right shift)

$$x[n-k]u[n-k] \overset{z}{\longleftrightarrow} z^{-k}X(z)$$

$$x[n-k]u[n] \longleftrightarrow z^{-k}X(z) + \underbrace{z^{-k}\sum_{n=1}^{k} x[-n]z^{n}}_{\text{z transform of } k-1 \text{ values before } n=0}$$

$k>0$

• Advance (left shift)

$$x[n+k]u[n] \longleftrightarrow z^{k}X(z) - \underbrace{z^{k}\sum_{n=0}^{k-1} x[n]z^{-n}}_{\text{z transform of } k-1 \text{ values from } n=0 \; k-1}$$

$k>0$

For Delay form we need $k-1$ values before $n=0$ as A.C.
" Advance form " " " " after $n=-1$ as A.C.

which form we use depends on A.C. given (before $n=0$ or after $n=0$)
typically delay or recursive form.

- Example: Given $y[n+1] + ay[n] = x[n+1]$ and $y[-1] = B$

Then rewrite in dely form

$$y[n] + ay[n-1] = x[n]$$

Take z transform $\quad Y(z) + a\left[z^{-1}Y(z) + y[-1]\right] = X(z)$

Solve for $Y(z) \Rightarrow \quad Y(z)(1+az^{-1}) = -ay[-1] + X(z)$

$$Y(z) = \frac{-ay[-1]}{1+az^{-1}} + \frac{X(z)}{1+az^{-1}}$$

$$= \underbrace{\frac{-ay[-1]z}{z+a}}_{Y_{zi}(z)} + \underbrace{\frac{z}{z+a}X(z)}_{Y_{zs}(z)}$$

$$Y_{zs}(z) = H(z)Y(z)$$

$H(z)$ ↑ transfer Function

– Example cont. Suppose $x[n] = (\frac{1}{2})^n u[n]$ and $B = 1$

then $$Y(z) = \frac{-az}{z+a} + \frac{z}{z+a} \cdot \frac{z}{z-\frac{1}{2}}$$

$$y_{zi}[n] = Z^{-1}\left\{\frac{-az}{z+a}\right\} = -a(-a)^n u[n]$$

$$y_{zs}[n] = Z^{-1}\left\{\frac{z^2}{(z+a)(z-\frac{1}{2})}\right\}$$

Two cases: $a \neq -\frac{1}{2}$ Then $$\frac{Y_{zs}(z)}{z} = \frac{A}{z+a} + \frac{B}{z-\frac{1}{2}}$$

$$A = \left.\frac{z}{z-\frac{1}{2}}\right|_{z=-a} = \frac{-a}{-a-\frac{1}{2}} = \frac{2a}{2a+1}$$

$$B = \left.\frac{z}{z+a}\right|_{z=\frac{1}{2}} = \frac{1/2}{1/2+a} = \frac{1}{2a+1}$$

$$y_{zs}[n] = Z^{-1}\left\{\frac{Az}{z+a} + \frac{Bz}{z-\frac{1}{2}}\right\} = \frac{2a}{2a+1}(-a)^n u[n] + \frac{1}{2a+1}\left(\frac{1}{2}\right)^n u[n]$$

If $a = \frac{1}{2}$ then $$Y_{zs}(z) = \frac{z^2}{(z-1/2)^2}$$

$$\frac{Y_{zs}(z)}{z} = \frac{z}{(z-1/2)^2} = \frac{A}{(z-1/2)^2} + \frac{B}{(z-1/2)}$$

$$A = z\Big|_{z=1/2} = 1/2 \qquad B = \frac{d}{dz}[z]\Big|_{z=1/2} = 1$$

$$Y_{zs} = \frac{\frac{1}{2}z}{(z-\frac{1}{2})^2} + \frac{z}{z-\frac{1}{2}}$$

and
$$y_{zs}[n] = n\left(\frac{1}{2}\right)^n u[n] + \left(\frac{1}{2}\right)^n u[n]$$
$$= (n+1)\left(\frac{1}{2}\right)^n u[n]$$

- When A.C = 0 then $y_{zi}[n] = 0$ and $y[n] = y_{zs}[n]$

$$Y(z) = Y_{zs}(z) = H(z)\,X(z)$$

This allows us to easily determine the transfer function of a LCCDE description. and saves LCCDE $\rightarrow h[n] \rightarrow H(z)$

$$\text{LCCDE} \xrightarrow{\text{direct.}} H(z)$$

- Example: a second order system is given by

$$y[n] = \left(\tfrac{1}{4}\right) y[n-2] - \left(\tfrac{1}{8}\right) y[n-1] + x[n] - x[n-1]$$

with zero auxiliary conditions. What is the transfer function of the system?

- Taking Z transform

$$Y(z) = \left(\tfrac{1}{4}\right) z^{-2} Y(z) - \left(\tfrac{1}{8}\right) z^{-1} Y(z) + X(z) - z^{-1} X(z)$$

$$\left(1 + \tfrac{1}{8} z^{-1} - \tfrac{1}{4} z^{-2}\right) Y(z) = \left(1 - z^{-1}\right) X(z)$$

$$H(z) = \frac{Y(z)}{X(z)} = \frac{1 - z^{-1}}{1 + \frac{1}{8} z^{-1} - \frac{1}{4} z^{-2}} = \frac{z(z-1)}{z^2 + \frac{1}{8} z - \frac{1}{4}}.$$

- Where do the A.C. come from? Suppose the previous example represents a filter. To compute $y[n]$ we need the previous two values of output $y[n-1]$ and $y[n-2]$ in addition to current and previous input $x[n]$, $x[n-1]$

  - When we initialize the filter we have to choose the values in the output buffer and input buffer. These are the A.C.

  - Thus it makes sense for us to just set them to zero. (zero initialize the arrays).

- Stability of LCCDE: To determine the stability we check poles of $H(z)$.

$$H(z) = \frac{\prod_{k=0}^{M} (z - \beta_k)}{\prod_{k=0}^{N} (z - \alpha_k)}$$

β_k are <u>zeros</u> of TF

α_k are <u>poles</u> of TF
the singularity of $H(z)$

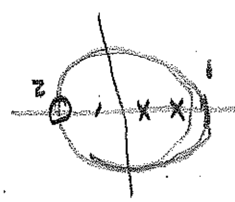
- A system is stable if $\forall k \; |\alpha_k| < 1$ (inside unit circle)

- A system is marginally stable if <u>non-repeated</u> poles on unit circle and remaining poles inside unit circle.

- A system is unstable if there are repeated poles on unit circle.

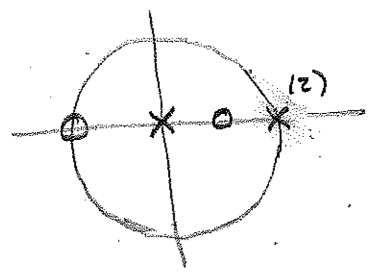
- Examples,

$$H(z) = \frac{k(z+1)^2}{(z-\frac{1}{2})(z-\frac{3}{4})}$$



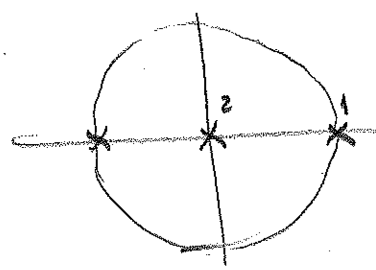
stable.

$$H(z) = \frac{k(z+1)(z-\frac{1}{2})}{z(z^2+2z+1)}$$
$$z(z-1)^2$$



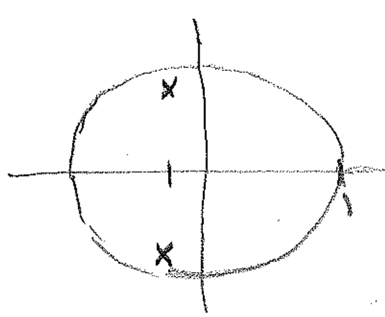
unstable

$$H(z) = \frac{z^2+\frac{1}{2}z-4}{z^2(z-1)(z+1)}$$



marginally stable

$$H(z) = \frac{1}{z^2+\frac{1}{2}z+1}$$
$$(z+\tfrac{1}{4}+j\tfrac{\sqrt{15}}{4})(z+\tfrac{1}{4}-j\tfrac{\sqrt{15}}{4})$$



Stable.

- As in CT we have to take care when pole/zero cancellation occurs, but now it is due to quantization effects in filter coefficients.

- Relationship of Z-transform to DTFT.

  Recall for a causal system $H(z)$ $|z| > r$

  If the system is in addition stable then all poles inside unit circle and $r < 1$. Thus the ROC includes the unit circle $z = e^{j\omega}$ $\omega \in \mathbb{R}$, and

$$H(e^{j\omega}) = H(z)\Big|_{z = e^{j\omega}} \quad \text{for stable, causal systems.}$$

- As in CT, DT anlysis typically proceeds as

  System description ⟶ $H(z)$ —stable?⟶ $H(\omega)$ ⟶ Fourier Analysis.
  $h[n]$, LCCDE
  B, D
  $H(z)$ —unstable⟶ Z-transform (easiest when both input & system causal)

- Inverse Systems. As with Laplace the z-transform gives us a principled approach to finding the inverse of a system.

$$x[n] \longrightarrow \boxed{H_1(z)} \longrightarrow \boxed{H_2(z)} \longrightarrow y[n] = x[n]$$

implies $H_2(z) = \dfrac{1}{H_1(z)} = H_1^{-1}(z)$ if $H_1(z) = \dfrac{P(z)}{Q(z)}$, $H_2(z) = \dfrac{Q(z)}{P(z)}$

NOTE zeros of H_1 become poles of H_2
thus the inverse is stable only if zeros of H_1 inside unit circle.