ECE 3704 Lecture 11 Properties of Laplace Transform

- Last week we learned how to compute the forward Laplace transform and Inverse Laplace transform for causal signals using the integrals

$$X(s) = \int_0^\infty x(t) e^{-st}\, dt \quad \text{AND} \quad x(t) = \frac{1}{2\pi j} \int_{\sigma - j\infty}^{\sigma + j\infty} X(s) e^{st}\, ds$$

- Today we will go over several usefull properties of these transforms, that when combined with a table of transforms will allow us to analyze a wide variety of stable and unstable systems.

- Linearity Property. For $X_1(s) = \mathcal{L}\{x_1(t)\}$ and $X_2(s) = \mathcal{L}\{x_2(t)\}$

  Then $\mathcal{L}\{a\, x_1(t) + b x_2(t)\} = a X_1(s) + b X_2(s)$

  for $a, b \in \mathbb{C}$

  - proof: $\mathcal{L}\{a x_1(t) + b x_2(t)\} = \int_0^\infty (a x_1(t) + b x_2(t)) e^{-st}\, dt$

  $$= a \int_0^\infty x_1(t) e^{-st}\, dt + b \int_{-\infty}^{\infty} x_2(t) e^{-st}\, dt$$

  $$= a X_1(s) + b X_2(s)$$

  - Example: $x(t) = 4e^{-t} u(t) - 7 e^{-5t} u(t)$ $\quad X(s) = ?$

  $$\mathcal{L}\{x(t)\} = \mathcal{L}\{4e^{-t} u(t) - 7e^{-5t} u(t)\}$$

  $$= 4 \mathcal{L}\{e^{-t} u(t)\} - 7 \mathcal{L}\{e^{-5t} u(t)\}$$

  $$= \frac{4}{s+1} + \frac{-7}{s+5}$$

  - This also works in reverse. If $X(s) = \frac{10}{s+2} + \frac{3}{s+10} = \frac{13s + 106}{s^2 + 12s + 20}$

  $$x(t) = \mathcal{L}^{-1}\left\{\frac{10}{s+2} + \frac{3}{s+10}\right\} = 10 \mathcal{L}^{-1}\left\{\frac{1}{s+2}\right\} + 3 \mathcal{L}^{-1}\left\{\frac{1}{s+10}\right\}$$

  $$= 10 e^{-2t} u(t) + 3 e^{-10t} u(t).$$

– time shift property. For causal $x(t)$ with $X(s) = \mathcal{L}\{x(t)\}$

Then $\mathcal{L}\{x(t-t_0)\} = X(s)\, e^{-st_0}$
$t_0 \geq 0$

Note: $t_0 \geq 0$ otherwise could make signal non-causal.

Proof: $$\mathcal{L}\{x(t-t_0)\} = \int_0^\infty x(t-t_0)\, e^{-st}\, dt$$

Let $\tau = t - t_0 \Rightarrow t = \tau + t_0$
$d\tau = dt$

$$= \int_{t_0}^\infty x(\tau)\, e^{-s(\tau+t_0)}\, d\tau$$

$$= \int_0^\infty x(\tau)\, e^{-s\tau} e^{-st_0}\, d\tau \equiv e^{-st_0} X(s)$$

Example: $x(t) = u(t) - u(t-10)$, a causal pulse of 10s.

$$\mathcal{L}\{x(t)\} = \mathcal{L}\{u(t) - u(t-10)\}$$
$$= \mathcal{L}\{u(t)\} - \mathcal{L}\{u(t-10)\} \quad \text{by linearity property}$$
$$= \frac{1}{s} - \frac{1}{s} e^{-10s} \quad \text{by time shift property.}$$

Note: When doing inverse transforms, collect all terms with common shift and do PFE seperately in each.

Example: $$X(s) = \frac{e^{-4s}(s+4) + e^{-7s}(s^2+3s+2)}{(s+1)(s^2+3s+2)}$$

$$= \frac{e^{-4s}}{s^2+3s+2} + \frac{e^{-7s}}{s+4}$$

$$= e^{-4s}\left[\frac{k_1}{s+1} + \frac{k_2}{s+2}\right] + e^{-7s}\left[\frac{1}{s+4}\right]$$

$$= e^{-4s}\left[\frac{1}{s+1} + \frac{-1}{s+2}\right] + e^{-7s}\left[\frac{1}{s+4}\right]$$

$$x(t) = \left[e^{-t}u(t) - e^{-2t}u(t)\right]_{t \to t-4} + \left[e^{-4t}u(t)\right]\Big|_{t \to t-7}$$

$$= e^{-(t-4)}u(t-4) - e^{-2(t-4)}u(t-4) + e^{-4(t-7)}u(t-7)$$

— Frequency shift. Given causal signal $f(t)$ Let $x(t) = f(t)e^{s_0 t}$ for $s_0 \in \mathbb{C}$

Then $\mathcal{L}\{x(t)\} = F(s - s_0) = F(s)\Big|_{s \to s - s_0}$

Proof: Bonus Problem this week

Example: $x(t) = \cos(\omega_0 t)\, u(t)$ $\quad \omega_0 \in \mathbb{R}$. $\quad X(s) = ?$

$$x(t) = \frac{1}{2}e^{j\omega_0 t}u(t) + \frac{1}{2}e^{-j\omega_0 t}u(t) \quad \text{by Eulers}$$

$$\mathcal{L}\{x(t)\} = \frac{1}{2}\mathcal{L}\{e^{j\omega_0 t}u(t)\} + \frac{1}{2}\mathcal{L}\{e^{-j\omega_0 t}u(t)\} \quad \text{by linearity}$$

$$= \frac{1}{2}\mathcal{L}\{u(t)\}\Big|_{s \to s - j\omega_0} + \frac{1}{2}\mathcal{L}\{u(t)\}\Big|_{s \to s + j\omega_0}$$

$$= \frac{1}{2}\frac{1}{s - j\omega_0} + \frac{1}{2}\frac{1}{s + j\omega_0}$$

$$= \frac{1}{2}\left[\frac{s + j\omega_0 + s - j\omega_0}{(s - j\omega_0)(s + j\omega_0)}\right]$$

$$= \frac{1}{\cancel{2}}\left[\frac{\cancel{2}s}{s^2 + \omega_0^2}\right] = \frac{s}{s^2 + \omega_0^2}$$

— time differentiation. Let $X(s) = \mathcal{L}\{x(t)\}$ then

$\mathcal{L}\{\frac{dx}{dt}\}$ for $t \geq 0$ is $sX(s) - x(0^-)$

↑value at $t = 0^-$, for causal $x(t) = 0$, we will use this latter for LCCDE.

Repeated differentiation gives general form

$$\mathcal{L}\left\{\frac{d^n x}{dt^n}\right\} = s^n X(s) - \sum_{k=1}^{n} s^{n-k}\frac{d^{k-1}}{dt^{k-1}}x(t)\Big|_{t=0^-}$$

* We will discuss this property in detail with examples when we cover LCCDE next time.

— frequency differentiation. $x(t) \overset{\mathcal{L}}{\longleftrightarrow} X(s)$

$$-t\,x(t) \overset{\mathcal{L}}{\longleftrightarrow} \frac{dX(s)}{ds} \quad \text{complex differentiation}$$

Example: $\mathcal{L}\{t u(t)\}$ the ramp signal

$$\mathcal{L}\{t u(t)\} = -\mathcal{L}\{-t u(t)\} = -\frac{dX(s)}{ds} \quad \text{where } X(s) = \mathcal{L}\{u(t)\} = \frac{1}{s}$$

$$= -\frac{-1}{s^2} = \frac{1}{s^2}$$

— Integration property. $x(t) \overset{\mathcal{L}}{\longleftrightarrow} X(s)$

$$\int_{0^-}^{t} x(\tau)\,d\tau \overset{\mathcal{L}}{\longleftrightarrow} \frac{X(s)}{s}$$

Example: Recall the integrator block $x(t) \rightarrow \boxed{\int} \rightarrow y(t) = \int_{0^-}^{t} x(\tau)\,d\tau$

then take Laplace of input and output.

$$X(s) \rightarrow \boxed{\int} \rightarrow Y(s) = \frac{1}{s} X(s) \quad \text{which is why } \boxed{\int} = \boxed{1/s}$$

— time scaling property $x(t) \overset{\mathcal{L}}{\longleftrightarrow} X(s)$

For $a > 0$ $\quad \mathcal{L}\{x(at)\} \overset{\mathcal{L}}{\longleftrightarrow} \frac{1}{a} X\left(\frac{s}{a}\right)$

Recall this corresponds to speeding up signal.

— time convolution $*$: $x_1(t) \overset{\mathcal{L}}{\longleftrightarrow} X_1(s)$ AND $x_2(t) \overset{\mathcal{L}}{\longleftrightarrow} X_2(s)$

then $\quad x_1(t) * x_2(t) \overset{\mathcal{L}}{\longleftrightarrow} X_1(s)\, X_2(s)$

So for causal LTI system with impulse response $h(t)$.

The output in Laplace domain is

$$Y(s) = H(s) X(s) \quad \text{where } H(s) = \mathcal{L}\{h(t)\} \text{ and } X(s) = \mathcal{L}\{x(t)\}$$

To get $y(t) = \mathcal{L}^{-1}\{Y(s)\}$

$$\begin{tikzcd} x(t) \arrow[r, "h(t)"] \arrow[d, leftrightarrow] \arrow[dr, phantom, "\updownarrow"] & y(t) = x(t) * h(t) \arrow[d, leftrightarrow] \\ X(s) \arrow[r, "H(s)"'] & Y(s) = H(s) X(s) \end{tikzcd}$$

Recall $H(s)$ is transfer function

This implies $H(s) = \frac{Y(s)}{X(s)}$

Example: Suppose we have a 1^{st} order LTI system with $h(t) = 3e^{-2t}u(t)$ with input $x(t) = \cos(10t)u(t)$. Find $Y(s)$ and $y(t)$ using Laplace.

Solution:

- We note $H(s) = \mathcal{L}\{3e^{-2t}u(t)\} = \dfrac{3}{s+2}$ (linearity + table)
- From table (or proved earlier) $X(s) = \dfrac{s}{s^2+100}$
- Thus $Y(s) = H(s)X(s) = \dfrac{3}{s+2} \cdot \dfrac{s}{s^2+100}$ by convolution property
- To find $y(t) = \mathcal{L}^{-1}\{Y(s)\}$ we write

$$Y(s) = \frac{A}{s+2} + \frac{Bs+C}{s^2+100} = \frac{3s}{(s+2)(s^2+100)}$$

$$A = \left.\frac{3s}{s^2+100}\right|_{s=-2} = \frac{-6}{104} = -\frac{3}{52}$$

$$Y(0) = \frac{A}{2} + \frac{C}{100} = 0 \Rightarrow C = -50A = \frac{150}{52} = \frac{75}{26}$$

$$sY(s) = \frac{As}{s+2} + \frac{Bs^2+Cs}{s^2+100} = \frac{3s^2}{(s+2)(s^2+100)}$$

$$= \frac{A}{1+\frac{2}{s}} + \frac{B+\frac{C}{s}}{1+\frac{100}{s^2}} = \frac{3/s}{(1+2/s)(1+\frac{100}{s^2})}$$

Let $s \to \infty$ $\quad A+B=0 \Rightarrow B = -A = \frac{3}{52}$

So $$Y(s) = \frac{-3/52}{s+2} + \frac{\frac{3}{52}s + \frac{75}{26}}{s^2+100}$$

Example cont. Expand second term

$$Y(s) = -\frac{3}{52} \cdot \frac{1}{s+2} + \frac{3}{52} \cdot \frac{s}{s^2+100} + \frac{75}{26} \frac{1}{s^2+100}$$

Note $\mathcal{L}\{\sin(\omega_0 t)u(t)\} = \frac{\omega_0}{s^2+\omega_0^2}$, write third term as

$$\frac{1}{s^2+100} = \frac{1}{10} \cdot \frac{10}{s^2+100}$$

Then $Y(s) = -\frac{3}{52} \cdot \frac{1}{s+2} + \frac{3}{52} \frac{s}{s^2+100} + \frac{75}{26} \cdot \frac{1}{10} \frac{10}{s^2+100}$

AND using linearity AND Table.

$$y(t) = -\frac{3}{52} e^{-2t} u(t) + \frac{3}{52} \cos(10t)u(t) + \frac{15}{52} \sin(10t)u(t).$$

— modulation property $x_1(t) \overset{\mathcal{L}}{\longleftrightarrow} X_1(s)$ AND $x_2(t) \overset{\mathcal{L}}{\longleftrightarrow} X_2(s)$

$$x_1(t) \cdot x_2(t) \overset{\mathcal{L}}{\longleftrightarrow} \frac{1}{2\pi j} \left[X_1(s) * X_2(s) \right]$$

where $X_1(s) * X_2(s) = \int X_1(z) X_2(s-z)\, dz$ let $s = x+jy$, $z = a+jb$

$$= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} X_1(a+jb) X_2((x-a)+j(y-b))\, da\, db$$

This is much easier to use in Fourier Domain, discussed next time.

— Initial Value property. $x(0^+) = \lim_{s \to \infty} s X(s)$

if limit exists.

— Final Value property $\lim_{t \to \infty} x(t) = \lim_{s \to 0} s X(s)$

if all poles/singularities of $X(s)$ in LHP.