Lecture 16 Analog Filter Design

— Recall from 2714 the 4 basic frequency selective filters: LowPASS, HighPASS, BANDPASS, and BANDSTOP.

— Recall also that the ideal versions of these are non-realizable because they are non-causal (violate Paley-Wiener Conditions)

Ex. Ideal LowpASS Filter has $H(\omega) = \begin{cases} 1 & |\omega| < \omega_c \\ 0 & \text{else.} \end{cases}$

Taking the inverse Fourier transform

$$h(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} H(\omega)e^{j\omega t}\,d\omega = \frac{1}{2\pi}\int_{-\omega_c}^{\omega_c} e^{j\omega t}\,d\omega = \frac{1}{2\pi}\left[\frac{1}{jt}e^{j\omega t}\Big/_{-\omega_c}^{\omega_c}\right]$$

$$= \frac{1}{2\pi jt}\left[e^{j\omega_c t} - e^{-j\omega_c t}\right] = \frac{1}{\pi t}\sin(\omega_c t)$$

This impulse response $\neq 0$ for $t<0$, thus is non-causal and physically unrealizable.

— Paley-Wiener conditions state a causal filter

a) can have $H(\omega)=0$ at only a finite number of frequencies

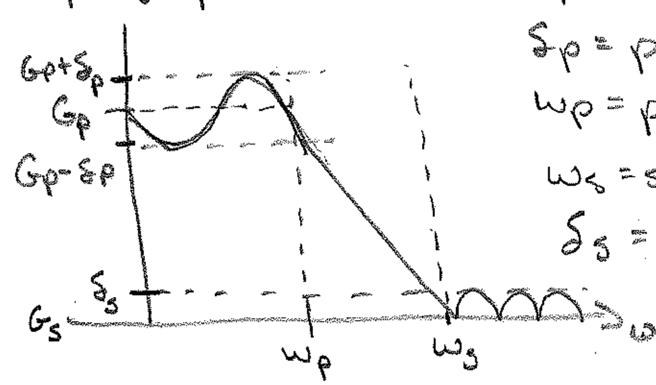
b) cannot have $H(\omega)=0$ over any finite range of frequencies

c) transition from passband to stopband cannot be zero

d) cannot specify $|H(\omega)|$ and $\angle H(\omega)$ independently.

— In practice filters are specified by one or more passbands, transition bands, and stop bands.

• Lowpass $|H(j\omega)|$



G_p = passband Gain (usually 1)

δ_p = passband ripple

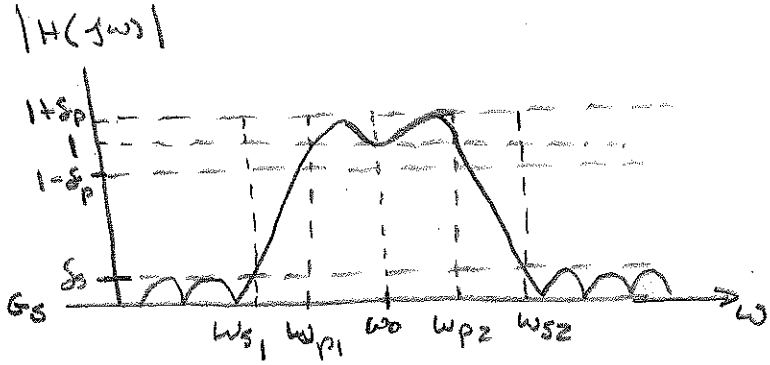
ω_p = passband frequency

ω_s = stop band frequency

δ_s = stop band ripple.

$\omega_s - \omega_p$ = transition BAND.

- BAND PASS



— We can focus on design of just low pass filters since we can transform a low pass $H_{Lp}(\omega)$ into the others

$$H_{Hp}(\omega) = 1 - H_{Lp}(\omega) \qquad H_{Bp}(\omega) = \frac{1}{2}H_{Lp}(\omega-\omega_0) + \frac{1}{2}H_{Lp}(\omega+\omega_0)$$

$$H_{Bs}(\omega) = 1 - H_{Bp}(\omega)$$

— generally G_p, δ_s, δ_p are specified in dB, frequencies in rad/s or Hz.

— Design problem is to find a transfer function $H(s)$ that is stable and has $H(\omega) \approx H_{desired}(\omega)$. The most popular approach is to use particular parametric forms for the frequency response

— Example: Butterworth lowpass filters have

$$|H(j\omega)|^2 = \frac{1}{1+\left(\frac{\omega}{\omega_c}\right)^{2N}} \qquad \text{for order } N \text{ system}$$

To find the corresponding $H(s)$ we note

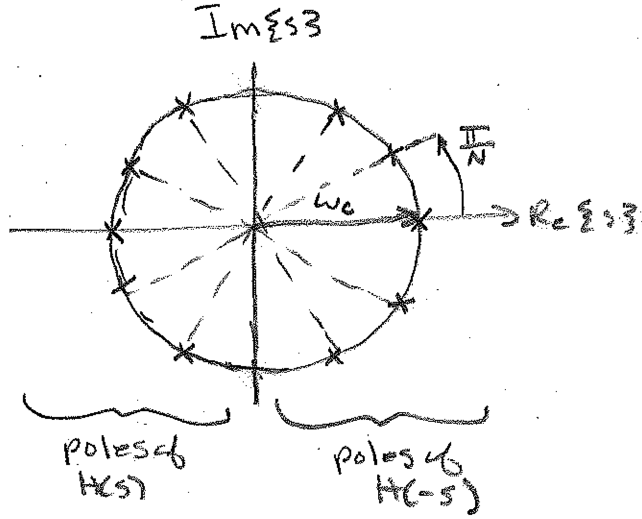
$$|H(\omega)|^2 = H(\omega)H^*(\omega)$$

Since $h(t)$ is real valued $H^*(\omega) = H(-\omega)$ thus

$$|H(\omega)|^2 = H(\omega)H(-\omega) = H(s)H(-s)\Big|_{s=j\omega}$$

Then $$H(s)H(-s) = \frac{1}{1+\left(\frac{-s^2}{\omega_c^2}\right)^N}$$

— If $H(s)$ has a pole $p_1 = \alpha_1 + j\omega_1$ then
$H(-s)$ has a pole $\bar{p}_1 = -\alpha_1 - j\omega_1$, the mirror image.



• When N is odd $\left(\frac{-s^2}{\omega_c^2}\right)^N = \frac{-s^{2N}}{\omega_c^{2N}}$ and $H(s)H(-s) = \frac{-\omega_c^{2N}}{s^{2N} - \omega_c^{2N}}$

AND poles are roots of $s^{2N} - \omega_c^{2N} = 0$

or $s^{2N} = \omega_c^{2N} = \omega_c^{2N} \underbrace{e^{j2\pi k}}_{1}$ $\Longrightarrow$ $p_k = \omega_c e^{j\pi \frac{k}{N}}$ $k \in 0, 1, .. 2N-1$

• By similar argument when N is even $p_k = \omega_c e^{j\pi \frac{2k+1}{2N}}$ $k = 0, 1, .. 2N-1$

AND $H(s) = \frac{\omega_c^N}{\prod_k (s - p_k)}$ for $\frac{\pi}{2} < \frac{k\pi}{N} < \frac{3\pi}{2}$

— To find N we use the filter specification $\omega_p, G_p, \delta_p, \omega_s, \delta_s$

working in dB Let $A = G_p - \delta_p$ $B = G_s$

Substituting and solving for N (derivation skipped)

$$N \geq \frac{\log_{10}\left[(10^{-B/10} - 1)/(10^{-A/10} - 1)\right]^{1/2}}{\log_{10}\left(\frac{\omega_s}{\omega_p}\right)}$$

where N is next largest integer.

— Example Suppose $G_p = 1 = 0\,dB$ $\quad \delta_p = 3\,dB$ $\quad \omega_p = 2\pi \cdot 5000$

$G_s = -40\,dB$ $\quad \omega_s = 2\pi\, 20\,000$

$$\left[\frac{\left(10^{\cdot 40/10} - 1\right)}{\left(10^{3/10} - 1\right)}\right]^{1/2} = 100.2327 \qquad N \geq \frac{\log_{10}(100.2327)}{\log_{10}\left(\frac{20{,}000}{5000}\right)} = 3.323$$

Thus $N = 4$

— Butterworth filters are maximally flat, and are well approximated by piecewise linear $H(s)$.

— See Matlab command butter.

— Another Example Type I Chebyshev Filters

$$|H(\omega)|^2 = \frac{1}{1 + \varepsilon^2 C_N^2\left(\frac{\omega}{\omega_p}\right)} \qquad \varepsilon > 0$$

where $C_N(v)$ is the Chebyshev polynomial of order N

$$C_N(v) = \cos\left(N \cos^{-1}(v)\right)$$

To find the coefficients we use a recursive procedure

$$C_{N+1}(v) = 2v\, C_N(v) - C_{N-1}(v)$$

derivation skipped.

where the base case of the recursion is

$$C_0(v) = 1 \qquad C_1(v) = v$$

Thus

$$C_2(v) = 2v\,C_1(v) - C_0(v) = 2v(v) - 1 = 2v^2 - 1$$

$$\begin{aligned} C_3(v) = 2v\,C_2(v) - C_1(v) &= 2v(2v^2 - 1) - v \\ &= 4v^3 - 2v - v \\ &= 4v^3 - 3v \end{aligned}$$

etc.

The design procedure is then Given ε, N, ω_p

1. Find $C_N(v)$
2. substitute $v = \frac{\omega}{\omega_p}$
3. Find poles of $H(s)$ equivalent to $|H(\omega)| = \dfrac{1}{1 + \varepsilon^2 C_N^2\left(\frac{\omega}{\omega_p}\right)}$

Step 3: Similar to Butterworth

$$|H(\omega)|^2 = H(s)H(-s)\Big|_{s=j\omega} = \frac{1}{1 + \varepsilon^2 C_N^2\left(\frac{s}{j\omega_p}\right)}$$

the poles of $H(s)H(-s)$ are solution to $1 + \varepsilon^2 C_N^2\left(\frac{p_k}{j\omega_p}\right) = 0$ for $k = 0, 1, \cdots 2N-1$. Skipping the derivation

$$p_k = j\omega_p\left[\cos(\alpha_k)\cosh(\beta_k) - j\sin(\alpha_k)\sinh(\beta_k)\right]$$

where $\alpha_k = \dfrac{(2k+1)\pi}{2N}$ $\qquad \beta_k = \dfrac{\sinh^{-1}\left(\frac{1}{\varepsilon}\right)}{N}$

$k = 0, 1, \cdots 2N-1$

— It remains to find N, ϵ that meet a desired specification
Again skipping derivation Let $A = G_p - \delta_p$ $\quad B = G_s$

$$\varepsilon = \left[10^{-A/10} - 1\right]^{1/2}$$

$$N \geq \frac{\cosh^{-1}(F)}{\cosh^{-1}(\omega_0)} \quad \text{where} \quad F = \left[\frac{10^{-B/10} - 1}{10^{-A/10} - 1}\right]^{1/2}, \quad \omega_0 = \frac{\omega_s}{\omega_p}$$

See Matlab commands cheb1ord and cheby1.

– Filter Transformations. Let $G(s)$ be the TF resulting from a lowpass Design. Define

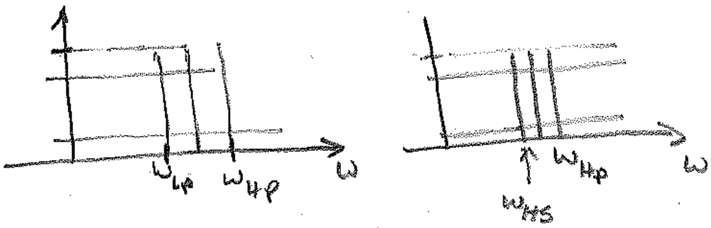
$$H(\lambda) = G(s)\Big|_{s=f(\lambda)}$$

as the transformed TF

$s \in \mathbb{C}$, $\lambda \in \mathbb{C}$

$\therefore f: \mathbb{C} \to \mathbb{C}$

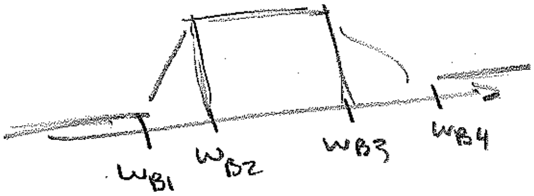
- lowpass to highpass. Define ω_{Lp} as lowpass passband edge
  ω_{Ls} as lowpass stopband edge
  ω_{Hp} as highpass passband edge
  ω_{pp} as " " stopband edge



TODO! Clear up Notation.

$$f(\lambda) = \frac{\omega_{Lp}\,\omega_{Hp}}{\lambda}$$

- lowpass to BANDPASS. Define ω_{B1} as StopBAND edge left
  ω_{B2} as PASSBAND edge left
  ω_{B3} as passBAND edge right
  ω_{B4} as stop band edge right.



$$f(\lambda) = \frac{\lambda^2 + \omega_0^2}{B\lambda}$$

$$\omega_0 = \left[\omega_{B2} \cdot \omega_{B3}\right]^{1/2}$$

$$B = \frac{\omega_{B3} - \omega_{B2}}{\omega_{Lp}}$$