Lecture 25: State Space Analysis in DT

- Similar to CT systems, DT systems have a state-space representation as well.

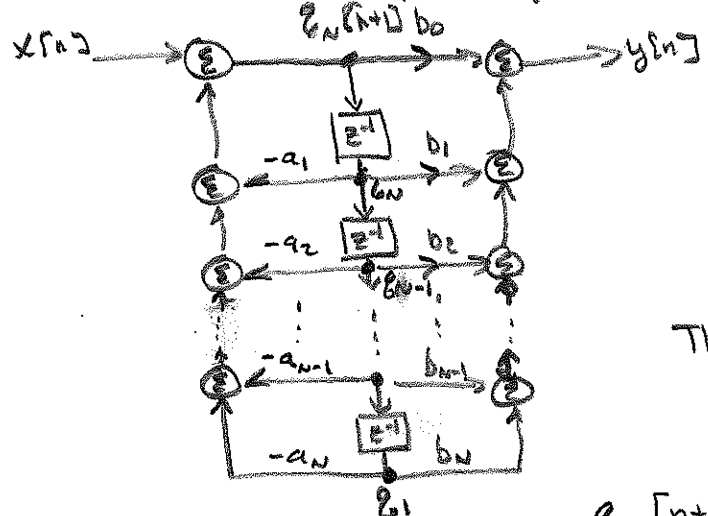
- Consider the DT transfer function

$$H(z) = \frac{\sum_{k=0}^{N} b_k z^{N-k}}{z^N + \sum_{k=1}^{N} a_k z^{N-k}} = \frac{b_0 z^N + b_1 z^{N-1} + \cdots b_N}{z^N + a_1 z^{N-1} + \cdots a_N}$$

This is equivalent to the LCCDE in advance form

$$y[n+N] + a_1 y[n+N-1] + \cdots a_N y[n] = b_0 x[n+N] + b_1 x[n+N-1] + \cdots b_N x[n]$$

In Direct Form II the corresponding BD is



Let output of each delay be a state variable

$q_i \quad i = 1, 2, \ldots N$

Then $q_i[n+1] = q_{i+1}[n]$

for $i = 1$ to $N-1$

$$q_N[n+1] = -\sum_{k=1}^{N} a_k q_{N+1-k}[n] + x[n]$$

the state equation

AND $$y[n] = \sum_{k=1}^{N} b_k q_{N+1-k}[n] + b_0 q_N[n+1]$$

$$= (b_N - b_0 a_N) q_1[n] + (b_{N-1} - b_0 a_{N-1}) q_2[n] + \cdots (b_1 - b_0 a_1) q[n] + b_0 x[n]$$

- collecting the state variables into a vector.

$$\begin{bmatrix} q_1[n+1] \\ q_2[n+2] \\ \vdots \\ q_N[n+2] \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & & & \\ & & & & & \\ & & & & & \end{bmatrix}}_{A} \begin{bmatrix} q_1[n] \\ q_2[n] \\ \vdots \\ q_N[n] \end{bmatrix} + \underbrace{\begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix}}_{B} x[n]$$

- The ouput is $y[n] = \begin{bmatrix} b_N - b_0 a_N \\ b_{N-1} - b_0 a_{N-1} \\ \vdots \\ b_1 - b_0 a_1 \end{bmatrix}^T \begin{bmatrix} q_1[n] \\ q_2[n] \\ \vdots \\ q_N[n] \end{bmatrix} + \underbrace{b_0 x[n]}_{D}$

C^T

- In most general form

$$q[n+1] = A\,q[n] + B\,x[n]$$
$$y[n] = C\,q[n] + D\,x[n]$$

where $x[n] \in \mathbb{C}^M$ are M-inputs
$y[n] \in \mathbb{C}^P$ are P-outputs
$q[n] \in \mathbb{C}^N$ are N statevariables

$A \in \mathbb{C}^{N\times N}$ $\quad C \in \mathbb{C}^{P\times N}$
$B \in \mathbb{C}^{N\times M}$ $\quad D \in \mathbb{C}^{P\times M}$

- As in CT this set of equations is NOT Unique, a different realization would correspond to a different basis for the Matrices.

- Solving DT State Space equations using Convolution.

• Given the state equation $q[n+1] = A q[n] + B x[n]$

Then $q[n] = A q[n-1] + B x[n-1]$ (1)
$q[n-1] = A q[n-2] + B x[n-2]$ (2)
$\vdots$
$q[1] = A q[0] + B x[0]$

• Subst (2) → (1) gives $q[n] = A(A q[n-2] + B x[n-2]) + B x[n-1]$
$= A^2 q[n-2] + AB x[n-2] + B x[n-1]$

• Continuing in this way we obtain

$$q[n] = A^n q[0] + A^{n-1} B x[0] + A^{n-2} B x[1] + \cdots B x[n-1]$$

$$= \underbrace{A^n q[0]}_{\text{Zero-input solution}} + \underbrace{\sum_{m=0}^{n-1} A^{n-1-m} B x[m]}_{A^{n-1} u[n-1] * B x[n]}$$

since $n-1 \geq 0$
$n \geq 1$

Zero-state solution.

The basic operation is raising matrix A to power.

– deriving A^n, recall the Eigen decomposition of A

$$A = V \Lambda V^{-1}$$

V = columns of eigenvectors
Λ = diagonal of Eigenvalues.

Then $A^n = (V\Lambda V^{-1})(V\Lambda V^{-1}) \cdots (V\Lambda V^{-1})$

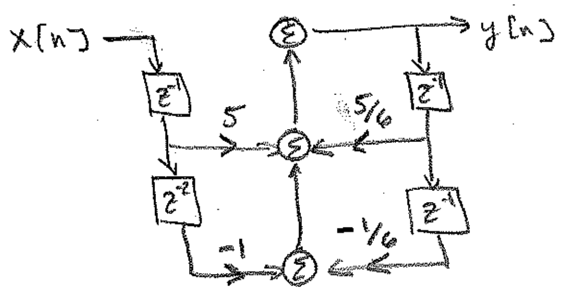
$$= V\Lambda^n V^{-1} \qquad \text{where } \Lambda^n = \begin{bmatrix} \lambda_1^n & & 0 \\ & \lambda_2^n \ddots & \\ 0 & & \lambda_n^n \end{bmatrix}$$

– Example: Consider the system

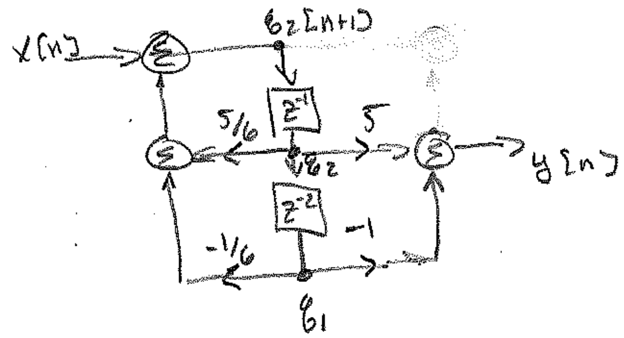
$$y[n] = \frac{5}{6} y[n-1] - \frac{1}{6} y[n-2] + 5x[n-1] - x[n-2]$$

Assume zero A.C.

- In Direct Form I



- In Direct Form II



- Reading off state equations

$$q_1[n+1] = q_2$$
$$q_2[n+1] = \frac{5}{6} q_2 - \frac{1}{6} q_1 + x[n]$$

- Thus $q[n+1] = \begin{bmatrix} 0 & 1 \\ -\frac{1}{6} & 5/6 \end{bmatrix} q[n] + \begin{bmatrix} 0 \\ 1 \end{bmatrix} x[n]$

- The output is $y[n] = 5q_2 - q_1$ or $y[n] = \begin{bmatrix} -1 & 5 \end{bmatrix} q[n]$

- To find solution we need an expression for A^n, thus we need the Eigen decomposition of A

  – To find Eigenvalues solve $|\lambda I - A| = 0$

$$\begin{vmatrix} \lambda & -1 \\ -1/6 & \lambda - 5/6 \end{vmatrix} = \lambda(\lambda - 5/6) + \frac{1}{6} = 0 \qquad \text{Thus } \lambda_1 = \frac{1}{3} \quad \lambda_2 = \frac{1}{2}$$

$$= \lambda^2 - 5/6\,\lambda + \frac{1}{6} = 0$$

$$(\lambda - \tfrac{1}{3})(\lambda - \tfrac{1}{2}) = 0$$

• To get the eigen vectors find nullspaces

$$(A-\lambda_1 I)v_1 = 0 \qquad\qquad (A-\lambda_2 I)v_2 = 0$$

$$\begin{bmatrix} -1/3 & 1 \\ -1/6 & \cancel{\tfrac{5}{6}-\tfrac{1}{3}}\ \tfrac{1}{2} \end{bmatrix} v_1 = 0 \qquad\qquad \begin{bmatrix} -1/2 & 1 \\ -1/6 & \cancel{\tfrac{5}{6}-\tfrac{1}{2}}\ \tfrac{1}{3} \end{bmatrix} v_2 = 0$$

$$v_1 = \begin{bmatrix} 3 \\ 1 \end{bmatrix} \qquad\qquad v_2 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

$$V = \begin{bmatrix} 3 & 2 \\ 1 & 1 \end{bmatrix} \qquad V^{-1} = \begin{bmatrix} 1 & -2 \\ -1 & 3 \end{bmatrix}$$

AND
$$A^n = \begin{bmatrix} 3 & 2 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} (1/3)^n & 0 \\ 0 & (1/2)^n \end{bmatrix}\begin{bmatrix} 1 & -2 \\ -1 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} 3 & 2 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} (\frac{1}{3})^n & -2(\frac{1}{3})^n \\ -(\frac{1}{2})^n & 3(\frac{1}{2})^n \end{bmatrix} = \begin{bmatrix} 3(\frac{1}{3})^n - 2(\frac{1}{2})^n & -6(\frac{1}{3})^n + 6(\frac{1}{2})^n \\ (\frac{1}{3})^n - (\frac{1}{2})^n & -2(\frac{1}{3})^n + 3(\frac{1}{2})^n \end{bmatrix}$$

• The convolution is then $\sum_{m=0}^{n-1} A^{n-1-m} B x[m]$ $\qquad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ for $n \geq 1$

— An alternate approach is to use the delay form

$$q[n] = A q[n-1] + B x[n-1]$$
$$q[n-1] = A q[n-2] + B x[n-2]$$
$$q[n-2] = A q[n-3] + B x[n-3]$$

etc,

$$q[0] = A q[-1] + B x[-1]$$

• substituting as before gives

$$q[n] = A^n q[-1] + \sum_{m=-1}^{n} A^{n-m} B x[m] = A^n q[-1] + A^n u[n] * B x[n]$$

where $A^n u[n] * B x[n] = \sum_{m=-1}^{n} A^{n-m} B x[m] = \sum_{m=0}^{n} A^{n-m} x[m]$ if x is causal since $x[-1] = 0$.

— Rather than work this out in detail lets consider the z-transform approach.

— solving DT state equations using Z-transform

$$Z\{q[n+1]\} = Z\{A\,q[n]\} + Z\{B\,x[n]\}$$

$$zQ(z) - zq[0] = A\,Q(z) + B\,X(z)$$

• solving for $Q(z)$ $\quad (zI - A)\,Q(z) = zq[0] + B\,X(z)$

$$Q(z) = (zI - A)^{-1} z q[0] + (zI - A)^{-1} B\,X(z)$$

• Then $q[n] = \underbrace{Z^{-1}\{(zI-A)^{-1} z\, q[0]\}}_{\text{zero-input}} + \underbrace{Z^{-1}\{(zI-A^{-1})^{-1} B\,X(z)\}}_{\text{zero-state}}$

NOTE $(zI - A)^{-1} z = (I - z^{-1}A)^{-1} = Z\{A^n\}$ so $A^n = Z^{-1}\{(I - z^{-1}A)^{-1}\}$

— Using same example as before, $B\,X(z) = \begin{bmatrix} 0 \\ \frac{z}{z-1} \end{bmatrix}$ $\quad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

$$(zI - A) = \begin{bmatrix} z & -1 \\ +1/6 & z - 5/6 \end{bmatrix}$$

$$(zI - A)^{-1} = \frac{1}{z(z - 5/6) + \frac{1}{6}} \begin{bmatrix} z - 5/6 & 1 \\ -1/6 & z \end{bmatrix}$$

$$(zI - A)^{-1} B\,X(z) = \frac{1}{z^2 - 5/6\,z + \frac{1}{6}} \begin{bmatrix} \frac{z}{z-1} \\ \frac{z^2}{z-1} \end{bmatrix} = \begin{bmatrix} \frac{z}{(z-1)(z^2 - 5/6\,z + 1/6)} \\ \frac{z^2}{(z-1)(z^2 - 5/6\,z + 1/6)} \end{bmatrix}$$

$$q_1[n] = Z^{-1}\left\{ \frac{z}{(z-1)(z-\frac{1}{3})(z-\frac{1}{2})} \right\}$$

$$q_2[n] = Z^{-1}\left\{ \frac{z^2}{(z-1)(z-\frac{1}{3})(z-\frac{1}{2})} \right\}$$

recall $q_1[n+1] = q_2[n]$
$q_1[n] = q_2[n-1]$
$Q_1(z) = z^{-1} Q_2(z)$

For q_1 we need PFE $\quad \frac{Q_1(z)}{z} = \frac{1}{(z-1)(z-\frac{1}{3})(z-\frac{1}{2})} = \frac{k_1}{z-1} + \frac{k_2}{z-1/3} + \frac{k_3}{z-1/2}$

— Example cont.

$$K_1 = \frac{1}{(z-\frac{1}{3})(z-\frac{1}{2})}\Bigg|_{z=1} = \frac{1}{(\frac{2}{3})(\frac{1}{2})} = 3$$

$$K_2 = \frac{1}{(z-1)(z-\frac{1}{2})}\Bigg|_{z=\frac{1}{3}} = \frac{1}{(-\frac{2}{3})(-\frac{1}{6})} = 9$$

$$K_3 = \frac{1}{(z-1)(z-\frac{1}{3})}\Bigg|_{z=\frac{1}{2}} = \frac{1}{(-\frac{1}{2})(\frac{1}{6})} = -12$$

Thus $q_1[n] = 3u[n] + 9\left(\frac{1}{3}\right)^n u[n] - 12\left(\frac{1}{2}\right)^n u[n]$

- To find q_2 do PFE of $\dfrac{Q_2(z)}{z} = \dfrac{z}{(z-1)(z-\frac{1}{3})(z-\frac{1}{2})} = \dfrac{K_1}{z-1} + \dfrac{K_2}{z-\frac{1}{3}} + \dfrac{K_3}{z-\frac{1}{2}}$

$$K_1 = \frac{z}{(z-\frac{1}{3})(z-\frac{1}{2})}\Bigg|_{z=1} = 3 \qquad K_3 = \frac{z}{(z-1)(z-\frac{1}{3})}\Bigg|_{z=\frac{1}{2}} = -6$$

$$K_2 = \frac{z}{(z-1)(z-\frac{1}{2})}\Bigg|_{z=\frac{1}{3}} = 3$$

Thus $q_2[n] = 3u[n] + 3\left(\frac{1}{3}\right)^n u[n] - 6\left(\frac{1}{2}\right)^n u[n]$

- To find output we note $y[n] = -q_1[n] + 5q_2[n]$

$$y[n] = (-3+15)u[n] + (-9+15)\left(\tfrac{1}{3}\right)^n u[n] + (12-30)\left(\tfrac{1}{2}\right)^n u[n]$$

$$= 12u[n] + 6\left(\tfrac{1}{3}\right)^n u[n] - 18\left(\tfrac{1}{2}\right)^n u[n]$$