— Dfn: A function $f(s)$ is <u>analytic</u> in some domain $R \subseteq \mathbb{C}$ if the function is

- single valued
- has a finite complex derivative $f'(s)$ $\forall\ s \in R$ $\quad \frac{df}{ds}$

— Let $f(s) = u(x,y) + j\,v(x,y)$ $\quad s = x + jy \quad x, y \in \mathbb{R}$

f has a complex derivative if and only if

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \text{and} \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

these are called Cauchy-Riemann conditions.

If so then $\quad f'(s) = \frac{\partial u}{\partial x} + j\frac{\partial v}{\partial x} = \frac{\partial v}{\partial y} - j\frac{\partial u}{\partial y}$

— If f is analytic over all $\mathbb{C}$ it is called an <u>entire</u> function.

— If f is not analytic at a point s_0 but is analytic in the neighborhood of s_0, s_0 is called a <u>singularity</u> or <u>singular point</u>.

— Examples

- $f(s) = e^s \quad$ Let $s = x + jy \quad f(s) = e^{x+jy} = e^x e^{jy}$

Using Eulers $\quad f(s) = \underbrace{e^x\cos(y)}_{u(x,y)} + j\underbrace{e^x\sin(y)}_{v(x,y)}$

Check C-R conditions

$$\frac{\partial u}{\partial x} = e^x\cos(y) \qquad \frac{\partial u}{\partial y} = -e^x\sin(y)$$

$$\frac{\partial v}{\partial x} = e^x\sin(y) \qquad \frac{\partial v}{\partial y} = e^x\cos(y)$$

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \qquad\qquad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

$$e^x\cos(y) = e^x\cos(y)\ \checkmark \qquad -e^x\sin(y) = -e^x\sin(y)\ \checkmark$$

Thus $f'(s)$ when $f(s) = e^s$ is

$$f'(s) = e^x \cos(y) + j e^x \sin(y)$$
$$= e^x e^{jy} = e^{x+jy} = e^s \quad \text{just as if } s \in \mathbb{R}.$$

- Another example $f(s) = \frac{1}{s}$ $|s| > 0$ i.e. $s \neq 0$

$$f(s) = \frac{1}{x+jy} = \frac{1}{x+jy}\,\frac{x-jy}{x-jy} = \frac{x-jy}{x^2+y^2}$$
$$= \underbrace{\frac{x}{x^2+y^2}}_{u(x,y)} + j\,\underbrace{\frac{-y}{x^2+y^2}}_{v(x,y)}$$

$$\frac{\partial u}{\partial x} = \frac{-x^2+y^2}{(x^2+y^2)^2} \qquad \frac{\partial u}{\partial y} = \frac{-2xy}{(x^2+y^2)^2}$$

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$$\frac{\partial v}{\partial x} = \frac{2xy}{(x^2+y^2)^2} \qquad \frac{\partial v}{\partial y} = \frac{-x^2+y^2}{(x^2+y^2)^2}$$

$$f'(s) = \frac{-x^2+y^2}{(x^2+y^2)^2} + j\,\frac{2xy}{(x^2+y^2)^2}$$

With some work you can show this is $f'(s) = \frac{-1}{s^2}$

- A counter example $f(s) = s^* = \underbrace{x}_{u(x,y)} + j\,\underbrace{(-y)}_{v(x,y)}$

$$\frac{\partial u}{\partial x} = 1 \qquad \frac{\partial u}{\partial y} = 0$$

$$\frac{\partial v}{\partial x} = 0 \qquad \frac{\partial v}{\partial y} = -1$$

$\frac{\partial u}{\partial x} \neq \frac{\partial v}{\partial y}$ <u>NOT</u> analytic.

- Example: show $s_0 = -1$ is a singularity of

$$f(s) = \frac{1}{s+1} = \frac{1}{(x+jy)+1} = \frac{1}{(1+x)+jy}$$

$$= \frac{1}{(1+x)+jy}\,\frac{(1+x)-jy}{(1+x)-jy} = \frac{1+x}{(1+x)^2+y^2} + j\,\frac{-y}{(1+x)^2+y^2}$$

– Example cont. $u(x,y) = \dfrac{1+x}{(1+x)^2+y^2}$ $\quad v(x,y) = \dfrac{-y}{(1+x)^2+y^2}$

$$\frac{\partial u}{\partial x} = \frac{[(1+x)^2+y^2][1] - (1+x)2(1+x)}{[(1+x)^2+y^2]^2} = \frac{-(1+x)^2+y^2}{[(1+x)^2+y^2]^2}$$

$$\frac{\partial u}{\partial y} = \frac{[(1+x^2)+y^2][0] - (1+x)2y}{[(1+x)^2+y^2]^2} = \frac{-2(1+x)y}{[(1+x)^2+y^2]^2}$$

$$\frac{\partial v}{\partial x} = \frac{[(1+x^2)+y^2](0) + y(1+x)2}{[(1+x)^2+y^2]^2} = \frac{2(1+x)y}{[(1+x)^2+y^2]^2}$$

$$\frac{\partial v}{\partial y} = \frac{[(1+x^2)+y^2](-1) + y(2y)}{[(1+x)^2+y^2]^2} = \frac{-(1+x)^2+y^2}{[(1+x)^2+y^2]^2}$$

Check $\dfrac{\partial u}{\partial x} = \dfrac{\partial v}{\partial y}$ except at $x=-1, y=0$

$$\frac{-(1+x)^2+y^2}{[(1+x)^2+y^2]^2} \Longrightarrow \frac{0}{0} \text{ undefined.}$$

– The previous example is a rational function, a ratio of polynomials in s. These will be very important. We will need to be very adept at doing manipulations of such functions.

General Case: $f(s) = \dfrac{\sum_{k=0}^{M} b_k s^k}{\sum_{k=0}^{N} a_k s^k} = \dfrac{P(s)}{Q(s)}$

$M < N$ strictly <u>proper</u>

$M \leq N$ <u>proper</u>

$M > N$ <u>improper</u>.

• example $M=2$, $N=3$

$$f(s) = \frac{b_0 + b_1 s + b_2 s^2}{a_0 + a_1 s + a_2 s^2 + a_3 s^3}$$

- the roots of the denominator are singularities of f.
- We will often be interested in writing $P(s)/Q(s)$ as a <u>partial fraction expansion</u> of the roots,
- Recall a ratio of polynomials that is improper can be written as sum of a polynomial and a proper rational function

Example: $f(s) = \dfrac{2s^3 + 9s^2 + 11s + 2}{s^2 + 4s + 3}$

$$\begin{array}{r|l} & 2s + 1 \\ \hline s^2 + 4s + 3 & 2s^3 + 9s^2 + 11s + 2 \\ & -\left(2s^3 + 8s^2 + 6s\right) \\ \hline & 0s^3 + s^2 + 5s \\ & -\left(s^2 + 4s + 3\right) \\ \hline & s - 1 \end{array}$$

Thus $f(s) = 2s + 1 + \dfrac{s - 1}{s^2 + 4s + 3}$ = polynomial + proper rational function.

- Once a rational function is proper we can expand it it terms of roots of $Q(s)$

Example: $f(s) = \dfrac{1}{s^2 + 3s + 2} = \dfrac{1}{(s+1)(s+2)}$

$$= \frac{A}{s+1} + \frac{B}{s+2}$$

To find A, B we can "clear" the fractions

$$\frac{A}{s+1} + \frac{B}{s+2} = \frac{1}{(s+1)(s+2)}$$

$$\frac{A(s+2) + B(s+1)}{(s+1)(s+2)} = \frac{1}{(s+1)(s+2)}$$

$$(A+B)s + 2A + B = 1 + 0s.$$

$$\begin{array}{ll} A + B = 0 & 2A + B = 1 \\ A = -B & -2B + B = 1 \\ A = 1 & B = -1 \end{array}$$

$$f(s) = \frac{1}{s+1} + \frac{-1}{s+2}.$$

- clearing fractions can be tendious with higher order systems, A shortcut is to use the Heaviside "cover-up" method, or finding residues.

• CASE #1. Non-repeated roots / singularities.

$$f(s) = \frac{P(s)}{(s-\alpha_1)(s-\alpha_2)\cdots(s-\alpha_N)}$$ α_i roots of $Q(s)$

then $$f(s) = \frac{K_1}{s-\alpha_1} + \frac{K_2}{s-\alpha_2} + \cdots \frac{K_N}{s-\alpha_N}$$

To find K_i's we multiply through by that term and evaluate at $s=$ root.

$$K_i = (s-\alpha_i) f(s) \Big|_{s=\alpha_i}$$

Example: $$f(s) = \frac{s^2+2s+7}{(s+1)(s-3)(s+5)}$$ $M=2$, $N=3$ proper.

$$= \frac{K_1}{s+1} + \frac{K_2}{s-3} + \frac{K_3}{s+5}$$

$$K_1 = \frac{s^2+2s+7}{(s-3)(s+5)}\Big|_{s=-1} = \frac{1-2+7}{(-4)(4)} = \frac{6}{-16} = -\frac{3}{8}$$

$$K_2 = \frac{s^2+2s+7}{(s+1)(s+5)}\Big|_{s=3} = \frac{9+6+7}{4\cdot 8} = \frac{22}{32} = \frac{11}{16}$$

$$K_3 = \frac{s^2+2s+7}{(s+1)(s-3)}\Big|_{s=-5} = \frac{25-10+7}{(-4)(-8)} = \frac{22}{32} = \frac{11}{16}$$

This works even if roots are complex.

Example $$f(s) = \frac{1}{(s+1+j)(s+1-j)} = \frac{K_1}{s+1+j} + \frac{K_2}{s+1-j}$$

$$K_1 = \frac{1}{s+1-j}\Big|_{s=-1-j} = \frac{1}{-1-j+1-j} = \frac{-1}{2j}$$

$$K_2 = \frac{1}{s+1+j}\Big|_{s=-1+j} = \frac{1}{-1+j+1+j} = \frac{1}{2j}$$

- We will often want to avoid working with complex roots. A better approach will be to combine them into a quadratic term.

Example: $\dfrac{s+2}{(s+1)(s+1+j)(s+1-j)} = \dfrac{s+2}{(s+1)(s^2+2s+2)}$ $\quad s^3+3s^2+4s+2$

$$= \frac{A}{s+1} + \frac{Bs+C}{s^2+2s+2}.$$

$$A = \left.\frac{s+2}{s^2+2s+2}\right|_{s=-1} = \frac{-1+2}{1-2+2} = 1$$

to find B + C we can clear fractions or use another shortcut

To find C let $s=0$ $\quad \dfrac{A}{1} + \dfrac{C}{2} = \dfrac{2}{(1)(2)} = 1 \Rightarrow C = 0$

To Find B multiply by s and let $s \to \infty$

$$\frac{As}{s+1} + \frac{Bs^2+Cs}{s^2+2s+2} = \frac{A}{1+1/s} + \frac{B+\frac{C}{s}}{1+\frac{2}{s}+\frac{2}{s^2}} = \frac{\frac{1}{s}+\frac{2}{s^2}}{1+\frac{3}{s}+\frac{4}{s^2}+\frac{2}{s^3}}$$

as $s \to \infty$ we get $\dfrac{A}{1} + B = \dfrac{0}{1} \Rightarrow B = -1$

or just clear fractions and substitute A & C.

$$As^2+2As+2A + Bs^2+Cs+Bs+C = s+2$$
$$(A+B)s^2 + (2A+B+C)s + 2A+C = s+2$$
$$2A+B+C = 1$$
$$B = 1-2A-C = -1$$

Finally $f(s) = \dfrac{1}{s+1} + \dfrac{-s}{s^2+2s+2}$

- One other complication is when we have repeated roots.

$$f(s) = \frac{P(s)}{(s-\lambda)^r(s-\alpha_1)(s-\alpha_2)\cdots(s-\alpha_N)} \qquad r>1$$

$$= \frac{a_0}{(s-\lambda)^r} + \frac{a_1}{(s-\lambda)^{r-1}} + \cdots \frac{a_{r-1}}{(s-\lambda)} + \frac{k_1}{s-\alpha_1} + \frac{k_2}{s-\alpha_2} + \cdots \frac{k_N}{s-\alpha_N}$$

$$a_k = \frac{1}{k!}\frac{d^k}{ds^k}\left[(s-\lambda)^r f(s)\right]\Big|_{s=\lambda}$$

Example: $$\frac{s+3}{(s+1)^2(s+2)} = \frac{K_1}{s+2} + \frac{a_0}{(s+1)^2} + \frac{a_1}{s+1}$$

$$K_1 = \left.\frac{s+3}{(s+1)^2}\right|_{s=-2} = \frac{1}{1} = 1$$

$$a_0 = \frac{1}{0!}\frac{d^0}{ds}\left[\frac{s+3}{s+2}\right]\Bigg|_{s=-1} = \frac{1}{1}\,\frac{2}{1} = 2$$

$$a_1 = \frac{1}{1!}\frac{d}{ds}\left[\frac{s+3}{s+2}\right]\Bigg|_{s=-1}$$

$$= \left.\frac{(s+2)(1)-(s+3)(1)}{(s+2)^2}\right|_{s=-1} = \frac{1-2}{1} = -1$$

$$\therefore\ f(s) = \frac{1}{s+2} + \frac{2}{(s+1)^2} + \frac{-1}{s+1}$$

- As you might expect there is a computational tool do this for you.

Mathematica: Apart function

Matlab: partfrac (symbolic toolbox)
and
Maxima.

You can use this to check your work and help on homework, but you will need to be able to do this on exams.