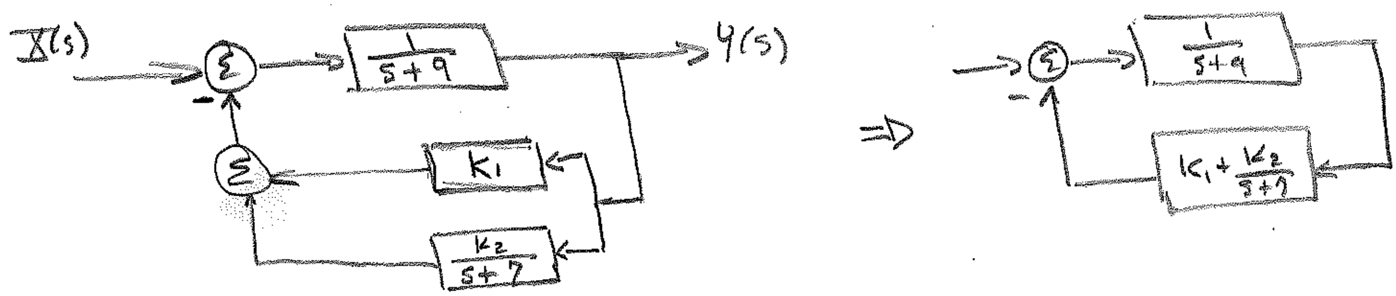
Lecture 14: Transfer Functions, Frequency Response + Bode Plots

- Recall if a system is stable then the right-most pole is in the LHP and ROC includes the imaginary Axis.

$$H(s)\Big|_{s=j\omega} = H(j\omega) \quad \text{the frequency response.}$$

- Thus we have another way to find the frequency response.
  1) Find TF , 2) determine if stable 3) evaluate $H(s)$

Example: Given the following block diagram, determine the TF $H(s)$ and the FR $H(j\omega)$ if it exists.



[Step 1] $H(s) = \dfrac{\frac{1}{s+9}}{1+\frac{1}{s+9}\left(k_1+\frac{k_2}{s+7}\right)} = \dfrac{\frac{1}{s+9}}{1+\frac{k_1(s+7)+k_2}{(s+9)(s+7)}} = \dfrac{\frac{(s+9)(s+7)}{s+9}}{(s+9)(s+7)+k_1(s+7)+k_2}$

$s^2+16s+49+k_1s+7k_1+k_2$

$$= \frac{s+7}{s^2+(16+k_1)s+49+7k_1+k_2}$$

[Step 2] $H(s)$ is stable if roots of $s^2+(16+k_1)s+49+7k_1+k_2$ are in LHP. While not too difficult here, for higher order systems, we need a better approach.

roots are $-16-k_1 \pm \sqrt{60+4k_1+k_1^2-4k_2}$ (closed loop poles)

If real part < 0 then

[Step 3] $H(j\omega) = H(s)\Big|_{s=j\omega} = \dfrac{7+j\omega}{(16+k_1)j\omega+49+7k_1+k_2-\omega^2}$

– So what is the better method? Called Routh-Hurwitz criteria

Let $H(s) = \frac{P(s)}{Q(s)}$ write $Q(s)$ in form $Q(s) = a_0 s^n + a_1 s^{n-1} + \dots a_{n-1} s + a_n$

$a_i \in \mathbb{R}$ and $a_n \neq 0$ (otherwise pole at origin)

1) If any a_i are zero, or negative with any other positive then unstable.

2) Form the Routh array

s^n	a_0	a_2	a_4	a_6	$\dots$
s^{n-1}	a_1	a_3	a_5	a_7	$\dots$
s^{n-2}	b_1	b_2	b_3	b_4	$\dots$
s^{n-3}	c_1	c_2	c_3	c_4	$\dots$
$\vdots$					
s^2	e_1	e_2			
s^1	f_1				
s^0	g_1				

where

$$b_1 = \frac{a_1 a_2 - a_0 a_3}{a_1}$$

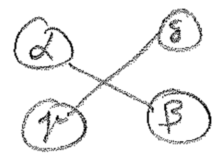
$$b_2 = \frac{a_1 a_4 - a_0 a_5}{a_1}$$

etc.

$$c_1 = \frac{b_1 a_3 - a_1 b_2}{b_1}$$

$$c_2 = \frac{b_1 a_5 - a_1 b_3}{b_1}$$

etc.



$\frac{\gamma\delta - \alpha\beta}{\gamma}$

3) all signs of 1st column $a_0, a_1, b_1, c_1 \dots$ must be positive for system to be stable

– Applying these rules to our previous example

$16 + k_1 > 0$ and $49 + 7k_1 + k_2 > 0$

$k_1 > -16$ $\qquad$ $k_2 > -49 - 7k_1$

or in terms of Routh array

s^2	1	$49 + 7k_1 + k_2$
s	$16 + k_1$	0
s^0	b_1	

$$b_1 = \frac{(16+k_1)(49+7k_1+k_2) - 0(1)}{16+k_1} = 49 + 7k_1 + k_2 > 0$$

— Another example. Given $H(s) = \dfrac{5s^2 + 7s + 1}{s^4 + 2s^3 + 3s^2 + 4s + 5}$

Find $H(j\omega)$ for system if it exists, or state why does not.

$Q(s) = s^4 + 2s^3 + 3s^2 + 4s + 5$

all coeff are positive ✓

s^4	1	3	5
s^3	2	4	0
s^2	b_1	b_2	
s^1	c_1	0	
s^0	d_1		

$$b_1 = \frac{(2)(3) - (1)(4)}{2} = 1$$

$$b_2 = \frac{(2)(5) - (1)(0)}{2} = 5$$

$$c_1 = \frac{4b_1 - 2b_2}{b_1} = \frac{4 - 10}{1} = -6$$ detect sign change. ∴ unstable

$$d_1 = \frac{c_1 b_2 - b_1(0)}{c_1} = \frac{-6(5)}{-6} = 5$$ another sign change ⟹ 2 poles in RHP.

∴ system is unstable and $H(j\omega)$ does NOT exist.

— One more $H(s) = \dfrac{-7s + 4}{s^3 + 3s^2 + 2s + 5}$

$Q(s) = s^3 + 3s^2 + 2s + 5$ all positive coeff ✓

s^3	1	2
s^2	3	5
s	b_1	0
s^0	c_1	

$$b_1 = \frac{(2)(3) - (1)(5)}{3} = \frac{1}{3}$$

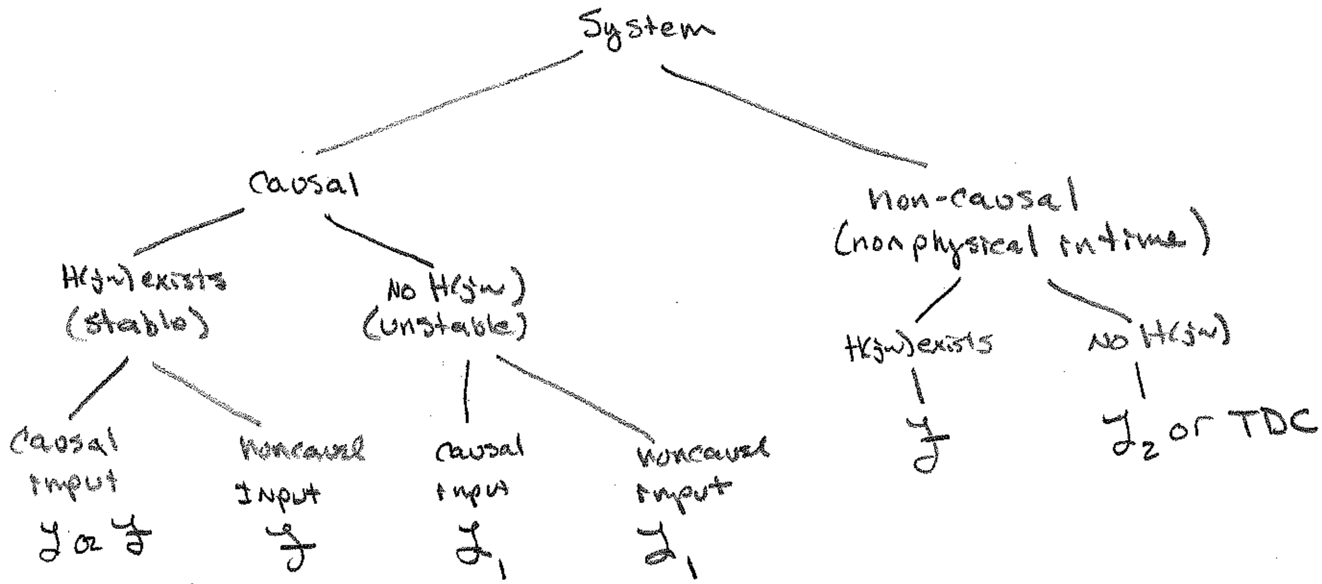
$$c_1 = \frac{5b_1 - 3(0)}{b_1} = 5$$

No sign change means system is stable.

$$H(j\omega) = \frac{-7(j\omega) + 4}{(j\omega)^3 + 3(j\omega)^2 + 2(j\omega) + 5} = \frac{4 - j7\omega}{5 - 3\omega^2 + j(2\omega - \omega^3)}$$

— A common point of confusion is around when to use which method: time-domain convolution (TDC), Fourier ($\mathcal{F}$), Laplace ($\mathcal{L}_1, \mathcal{L}_2$)

Here is my "Decision tree"



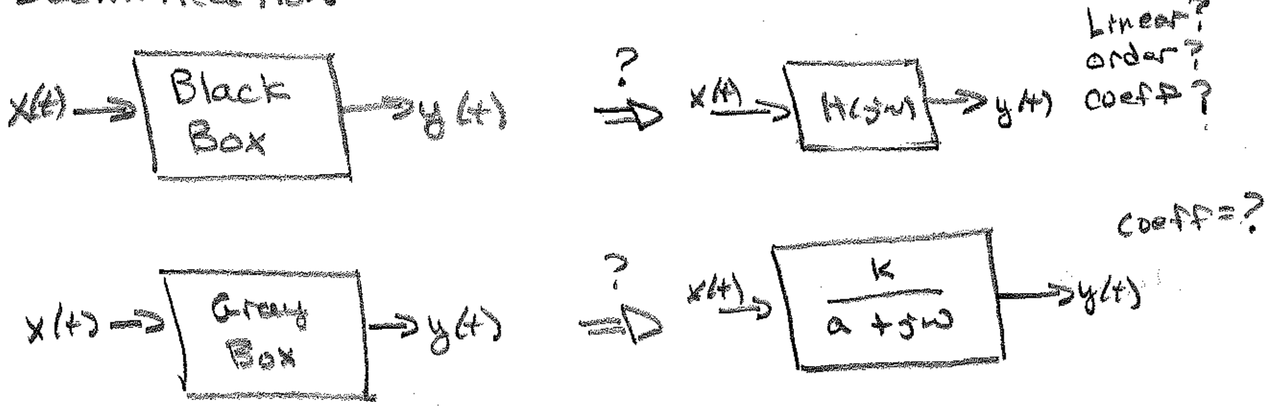
— The most common way to visualize a stable system is to use Bode plot

- semi log ω or f plot of $|H(j\omega)|$ in dB ($20 \log_{10} |H(j\omega)|$)
- semi log ω or f plot of $\angle H(j\omega)$ in radians or degrees.

This is easy to do on a computer given an expression for $H(j\omega)$

But what if we don't have an expression? How do we proceed?

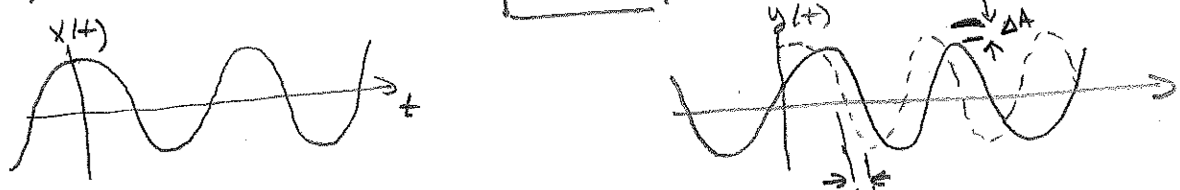
— System Identification



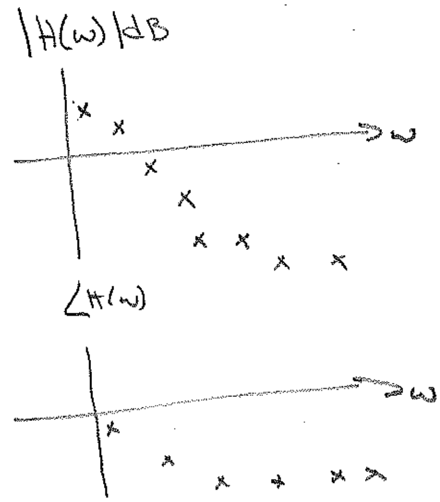
- May have model for some components / blocks but NOT others.

— A full coverage of Linear system ID is outside course scope, however we can easily measure points on $H(j\omega)$ using sinusoidal inputs and a frequency sweep.

$$x(t) = \cos(\omega_0 t) \longrightarrow \boxed{} \longrightarrow |H(\omega_0)| \cos(\omega_0 t + \angle H(\omega_0))$$



$$\underbrace{|H(\omega_0)|}_{\Delta A} \cos\Big(\omega_0 \big(t + \underbrace{\tfrac{\angle H(\omega_0)}{\omega_0}}_{\Delta t}\big)\Big)$$



— Knowledge of constucting Bode plots manually can give information about order of system, type (LP, BP, etc), and candidate transfer function.

— Consider a generic transfer function of a stable system

$$H(s) = \frac{(s+b_1)(s+b_2)\cdots(s+b_M)}{(s+a_1)(s+a_2)\cdots(s+a_N)} \qquad Re\{s\} > -\gamma \quad \gamma > 0 \quad \text{AND } N > M$$

$-b_k$ zeros, $-a_k$ poles.

Rewrite by factoring as

$$H(s) = K \frac{(1+\frac{s}{b_1})(1+\frac{s}{b_2})\cdots(1+\frac{s}{b_M})}{(1+\frac{s}{a_1})(1+\frac{s}{a_2})\cdots(1+\frac{s}{a_N})} \qquad K = \frac{b_1 b_2 b_3 \cdots b_M}{a_1 a_2 \cdots a_N} \in \mathbb{C}$$

Since stable, Let $s = j\omega$

$$H(\omega) = K \frac{(1+\frac{j\omega}{b_1})(1+\frac{j\omega}{b_2})\cdots(1+\frac{j\omega}{b_M})}{(1+\frac{j\omega}{a_1})(1+\frac{j\omega}{a_2})\cdots(1+\frac{j\omega}{a_N})}$$

Now Note the following properties of log, $\log(a^b) = b\log(a)$, $\log(ab) = \log(a) + \log(b)$, $\log(\frac{a}{b}) = \log(a) - \log(b)$

— taking Magnitude and Phase

$$|H(\omega)| = |K| \frac{\left|1+\frac{j\omega}{b_1}\right| \cdot \left|1+\frac{j\omega}{b_2}\right| \cdots \left|1+\frac{j\omega}{b_m}\right|}{\left|1+\frac{j\omega}{a_1}\right| \cdot \left|1+j\frac{\omega}{a_2}\right| \cdots \left|1+j\frac{\omega}{a_N}\right|}$$

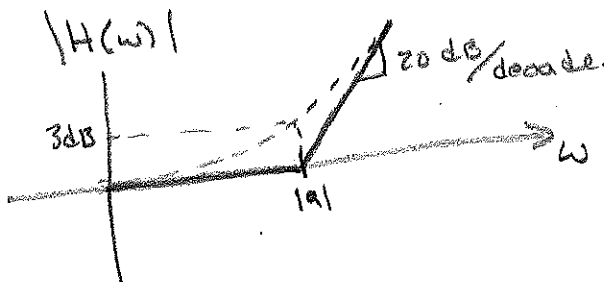
$$\angle H(\omega) = \angle K + \angle 1+j\tfrac{\omega}{b_1} + \angle 1+j\tfrac{\omega}{b_2} + \cdots \angle 1+j\tfrac{\omega}{b_m}$$
$$- \angle 1+j\tfrac{\omega}{a_1} - \angle 1+j\tfrac{\omega}{a_2} - \cdots \angle 1+j\tfrac{\omega}{a_N}$$

— Then $20\log_{10}|H(\omega)| = 20\log_{10}|K| + 20\log_{10}\left|1+j\frac{\omega}{b_1}\right| + \cdots 20\log_{10}\left|1+j\frac{\omega}{b_m}\right|$
$- 20\log_{10}\left|1+j\frac{\omega}{a_1}\right| - \cdots 20\log_{10}\left|1+j\frac{\omega}{a_N}\right|$

— For each plot we get a sum of plots for each term.

— Lets look at $\left\|1+j\frac{\omega}{a}\right|$ for some $a \in \mathbb{R}$ and convert to dB

- when $\omega << |a|$ $\quad 20\log_{10}\left|1+j\frac{\omega}{a}\right| \approx 20\log_{10}|1| = 0$ dB
- when $\omega >> |a|$ $\quad 20\log_{10}\left|1+j\frac{\omega}{a}\right| \approx 20\log_{10}|\omega| \approx 20\,\frac{\text{dB}}{\text{decade}}$



— Since plots add, if we have a mixture of real zeros & poles the plots add or subtract accordingly with slopes adding or subtracting.

Thus slope can be used to estimate system order.

– Cases not addressed above

- zero at origin $\quad 20\log_{10}|j\omega| = 20\log_{10}(\omega)$

$$\angle j\omega = \frac{\pi}{2} \text{ rad}$$

- pole at origin (marginally stable) $\quad 20\log_{10}\left|\frac{1}{j\omega}\right| = -20\log_{10}(\omega)$

$$\angle \frac{1}{j\omega} = \frac{-\pi}{2} \text{ rad}$$

- second order with complex poles.

$$H(s) = \frac{|a|^2}{(s+a)(s+a^*)} = \frac{\omega_0^2}{s^2 + 2\zeta\omega_0 s + \omega_0^2} \qquad \omega_0^2 = |a|^2$$

$$2\zeta\omega_0 = Re\{a\}$$

or $\zeta = -\frac{Re\{a\}}{|a|} \in [0, 1]$

ω_0 is natural frequency

ζ is damping ratio.

$\zeta > 1$ overdamped

$\zeta = 1$ critically damped

$\zeta < 1$ under damped.

$$20\log_{10}|H(\omega)| = -10\log_{10}\left\{\left[1-\left(\frac{\omega}{\omega_0}\right)^2\right]^2 + \left[2\zeta\left(\frac{\omega}{\omega_0}\right)\right]^2\right\}$$

$$\angle H(\omega) = -\tan^{-1}\left[\frac{2\zeta\frac{\omega}{\omega_0}}{1-\left(\frac{\omega}{\omega_0}\right)^2}\right]$$

A related parameter is the Q-factor $Q = \frac{1}{2\zeta}$