- Recall the basics of complex numbers
  - j is the imaginary unit, the solution to $s^2+1=0$ $\quad j=\sqrt{-1}$
  - a complex number is the combination of two real numbers x, y with the imaginary unit using addition and multiplication
    $$s = x + jy$$
  - We write $s \in \mathbb{C}$
  - $x = \mathrm{Re}\{s\}$, real part of s
  - $y = \mathrm{Im}\{s\}$, imaginary part of s
  - if $x=0$ s is purely imaginary
    if $y=0$ s is real

    NOTE $s=0+j0$ is only complex number both real and pure.
  - two complex numbers $s_1 = x_1 + jy_1$
    $s_2 = x_2 + jy_2$

    : are equal iff $x_1 = x_2$ and $y_1 = y_2$

    : $s_1 + s_2 = (x_1+x_2) + j(y_1+y_2)$

    : $s_1 \cdot s_2 = (x_1+jy_1)(x_2+jy_2)$
    $= (x_1x_2 - y_1y_2) + j(x_1y_2 + x_2y_1)$

    : $\dfrac{s_1}{s_2} = \dfrac{x_1+jy_1}{x_2+jy_2} \cdot \dfrac{x_2-jy_2}{x_2-jy_2} = \dfrac{(x_1x_2+y_1y_2)+j(x_2y_1-x_1y_2)}{x_2^2+y_2^2}$
    $s_2 \neq 0$
  - s^* is the conjugate of s $\quad s = x+jy \quad s^* = x-jy$
  - $|s|$ is the absolute value or modulus

    $s = x+jy \quad |s| = \sqrt{x^2+y^2} = \sqrt{s\,s^*} \geq 0$
  - some properties that are useful.

    $|s_1 \cdot s_2 \cdots s_n| = |s_1| \cdot |s_2| \cdots |s_n|$

    $\left|\dfrac{s_1}{s_2}\right| = \dfrac{|s_1|}{|s_2|}$ $\qquad$ NOTE: $|s_1+s_2| \neq |s_1| + |s_2|$

    $\mathrm{Re}\{s\} = \dfrac{s+s^*}{2} \qquad \mathrm{Im}\{s\} = \dfrac{s-s^*}{2j}$

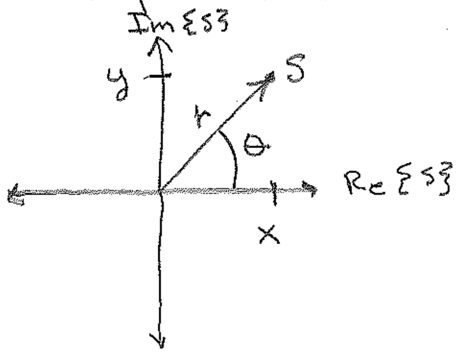
    $(s_1+s_2)^* = s_1^* + s_2^* \qquad (s_1 \cdot s_2)^* = s_1^* s_2^*$

    $\left(\dfrac{s_1}{s_2}\right)^*$ $s_2 \neq 0$ is $\dfrac{s_1^*}{s_2^*}$

- Inequalities:

$$-|s| \le \mathrm{Re}\{s\} \le |s|$$
$$-|s| \le \mathrm{Im}\{s\} \le |s|$$
$$|s_1 + s_2| \le |s_1| + |s_2|$$
$$|s_1 + s_2 + \cdots s_n| \le |s_1| + |s_2| + \cdots |s_n|$$

– Geometric Representation and Complex Plane



$r = |s|$ = norm, length, modulus magnitude

θ = argument or phase
$= \tan^{-1}\left(\frac{y}{x}\right)$

$x = r\cos(\theta)$
$y = r\sin(\theta)$

Cartesian form: $s = x + jy = r\cos(\theta) + jr\sin(\theta)$
$= r[\cos(\theta) + j\sin(\theta)]$

Polar Form: $s = re^{j\theta} = r\underbrace{[\cos(\theta) + j\sin(\theta)]}_{\text{Eulers Identity / Formula}}$

- In Polar Form:

$$s_1 \cdot s_2 = r_1 e^{j\theta_1} \cdot r_2 e^{j\theta_2} = r_1 r_2 e^{j(\theta_1+\theta_2)}$$
$$= r_1 r_2 [\cos(\theta_1+\theta_2) + j\sin(\theta_1+\theta_2)]$$
$$|re^{j\theta}| = |r|\,|e^{j\theta}| = r \cdot 1 = r.$$
$$\cos(\theta) = \frac{1}{2}e^{j\theta} + \frac{1}{2}e^{-j\theta}$$
$$\sin(\theta) = \frac{1}{2j}e^{j\theta} - \frac{1}{2j}e^{-j\theta}$$

Note: s is a vector is space spaned by Re{s} and Im(s)

Any vector rotated by 2π is indistinguishable from any other: $re^{j\theta} = re^{j(\theta + 2\pi n)}$ for $n \in \mathbb{Z}$

When $n=0$ θ is called <u>principle</u> argument. $\equiv \theta_p$.

- Complex Valued Functions $f: \mathbb{R} \to \mathbb{C}$

$$f(t) = x(t) + j\,y(t)$$

$$\frac{df}{dt} = \frac{dx}{dt} + j\frac{dy}{dt}$$

$$\int f(t)\,dt = \int x(t)\,dt + j\int y(t)\,dt + C$$

Examples: $e^{j\omega t} = \cos(\omega t) + j\sin(\omega t)$, $H(\omega) = \frac{1}{j\omega+1}$

Remark: You (should) have a good deal of experience with these functions from ECE 2714

- Complex Functions $f: \mathbb{C} \to \mathbb{C}$

Complex valued functions of complex arguments.

$$s = x + jy \qquad \underset{\substack{\uparrow \\ \text{argument}}}{f(s)} = \underbrace{\overset{\mathrm{Re}\{f\}}{u(x,y)} + j\,\overset{\mathrm{Im}\{f\}}{v(x,y)}}_{\text{value.}}$$

- Examples

Note: Sometimes Cartesian better sometimes polar

- $f(s) = s = x + jy$
- $f(s) = s^2 = (re^{j\theta})^2 = r^2 e^{j2\theta}$
- $f(s) = s^* = x - jy$
- ★ $f(s) = s^n \quad n \in \mathbb{Z} = (re^{j\theta})^n = r^n e^{j\theta n}$ — Eigenfunction for DT sys.
- ★ $f(s) = e^{st} \quad t \in \mathbb{R} = e^{(x+jy)t} = e^{xt}e^{jyt} = e^{xt}[\cos(yt) + j\sin(yt)]$ — Eigenfunction for CT sys.
- ★ $f(s) = \sum_{k=0}^{N} a_k s^k$ polynomials
- $f(s) = \sum_{k=0}^{N} (s - s_0)^k$ Power series expansion around s_0
- ★ $f(s) = \dfrac{\sum_{k=0}^{M} b_k s^k}{\sum_{k=0}^{N} a_k s^k}$ Rational Functions

— The previous examples were all single-valued functions, lets look at two interesting multi-valued functions.

- You know that for $x \in \mathbb{R}$ the solution to $x^2 = y$ is $\pm\sqrt{y}$, the function $\sqrt{}$ is multi-valued.

  Similarly for complex numbers (I'll skip the derivation)

  $$\sqrt{s} = \sqrt{x+jy} = \pm\left[\sqrt{\frac{x+\sqrt{x^2+y^2}}{2}} + j\frac{y}{|y|}\sqrt{\frac{-x+\sqrt{x^2+y^2}}{2}}\right]$$

  if $y \neq 0$

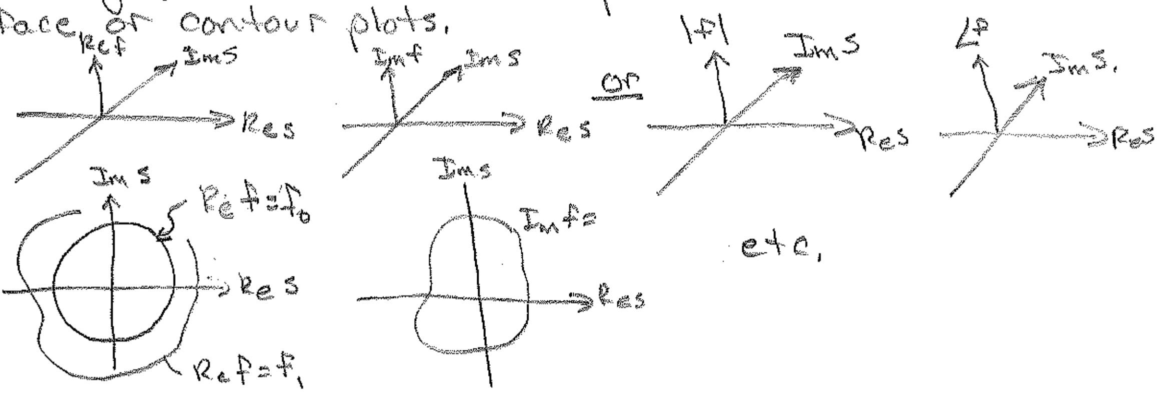
  if $y = 0$ then $\pm\sqrt{x}$ $x \geq 0$ or $\pm j\sqrt{-x}$ $x < 0$

  Thus only purely real s have purely real or purely complex square roots.

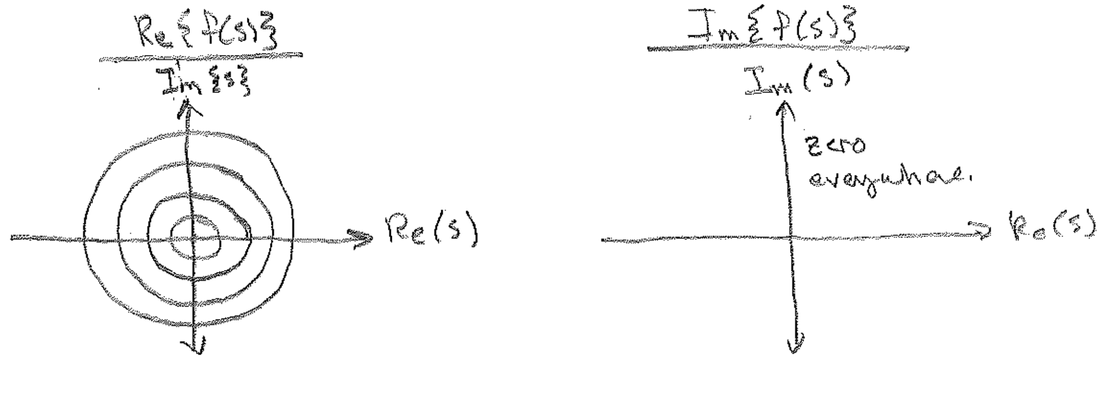
- Another example is the inverse of e^s, $\ln(s)$

  $$\ln(s) = \ln(|s|) + j(\theta_p + 2\pi n) \quad n \in \mathbb{Z} \quad \theta_p = \text{principle arg.}$$

— Visualizing complex functions. Since the argument is a pair of numbers and value is pair we have to use surface or contour plots.



- Example $f(s) = |s|^2 = x^2 + y^2$



— Limits of Complex functions

Recall from Calc I the definition of a limit for functions $f: \mathbb{R} \to \mathbb{R}$.

$$\lim_{x \to x_0} f(x) = y_0 \text{ if for some } \varepsilon > 0$$

there exists a $\delta > 0$ such that

$$0 < |x - x_0| < \delta \text{ implies } |f(x) - y_0| < \varepsilon$$

Similarly for complex functions $g: \mathbb{C} \to \mathbb{C}$

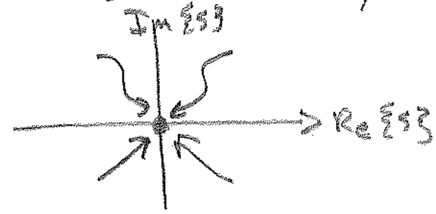
$$\lim_{s \to s_0} g(s) = w \text{ if for some } \varepsilon > 0$$

there exists a $\delta > 0$ such that

$$0 < |s - s_0| < \delta \text{ implies } |g(s) - w| < \varepsilon$$

complex modulus

except we can approach s_0 from any direction



We will use limits to define the derivative next time.

— Point at infinity. In calculus it is often useful to extend the real line to include $-\infty$ and $+\infty$ $\mathbb{R}^*$

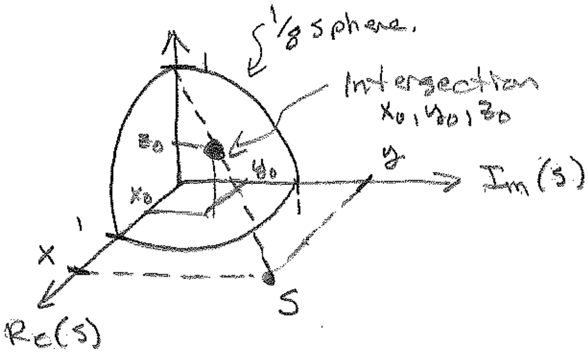
However because $\mathbb{C}$ has no ordering (2D vector space)

there is only one point at infinity.

$\mathbb{C}^* = \mathbb{C} \cup \{\infty\}$ where for $s \in \mathbb{C}$

- $s + \infty = \infty$ AND $\frac{\infty}{\infty}$, $0 \cdot \infty$, $\infty - \infty$ are <u>undefined</u>.
- $s \cdot \infty = \infty$
- $\infty + \infty = \infty$
- $\infty \cdot \infty = \infty$
- $s/\infty = 0$

- To visualize the extended Complex Plane $\mathbb{C}^*$ we can use the Riemann Sphere.



Line from s to (0,0,1) intersects sphere at some (x_0, y_0, z_0)

where
$$s = \frac{x_0 + j y_0}{1 - z_0}$$

(stereographic Projection)

In this view $s = \infty$ corresponds to point (0,0,1).

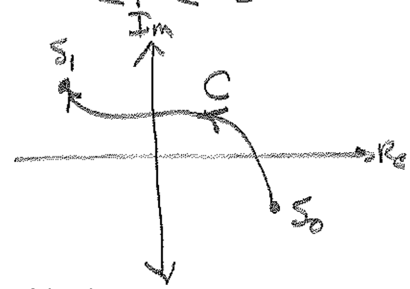
- Another concept we will use is that of curves in complex plane.

  • Let ρ be a parameterization of a curve in $\mathbb{C}$

the curve $C = S(\rho) = x(\rho) + j y(\rho)$ $a \le \rho \le b$

$S_0 = S(a)$ = initial point $x(a), y(a)$

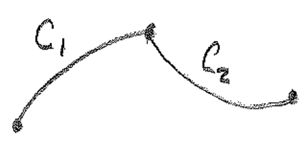
$S_1 = S(b)$ = terminal point $x(b), y(b)$



• if $S_0 = S_1$ the curve is closed

• the curve is smooth if $\frac{ds}{d\rho}$ is continuous and never 0 for $a \le \rho \le b$

• the curve is piece-wise smooth if it is smooth except at a finite number of points which join.



$C = C_1 \cup C_2$

• the curve is simple if $S(\rho_1) \neq S(\rho_2)$ for $\rho_1 \neq \rho_2$ except possibly $\rho_1 = a$, $\rho_2 = b$ (if closed)

This means there are no self intersections.

We will use curves to define the contour integration of complex functions.