Lecture 17: Forward Z transform

— Recall from 2714 the Eigen function for DT systems is the signal z^n, $z \in \mathbb{C}$

$$x[n] = z^n \longrightarrow \boxed{h[n]} \longrightarrow y[n] = h[n] * x[n]$$

$$= \sum_{m=-\infty}^{\infty} h[m]\, x[n-m]$$

$$= \sum_{m=-\infty}^{\infty} h[m]\, z^{n-m}$$

$$= z^n \sum_{m=-\infty}^{\infty} h[m]\, z^{-m} \equiv z^n H(z)$$

- $H(z)$ is the Bilateral Z-transform of $h[n]$
  - $H(z)$ is the Eigenvalue associated with z^n, called Transfer Function.
  - The value of z for which the sum converges is the Region-of-Convergence (ROC)

— Notation. We can take the z transform for any DT signal. We write

$$x[n] \xrightarrow{\mathcal{Z}} X(z) \quad \text{or} \quad X(z) = \mathcal{Z}\{x[n]\}$$

— Example: $x[n] = (\gamma)^n u[n] \qquad \gamma \in \mathbb{R}.$

$$X(z) = \sum_{n=-\infty}^{\infty} x[n] z^{-n}$$

$$= \sum_{n=0}^{\infty} (\gamma)^n z^{-n} \quad \text{since } x[n] \text{ is causal}$$

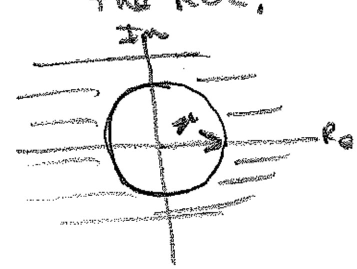
$$= \sum_{n=0}^{\infty} (\gamma z^{-1})^n = \lim_{N \to \infty} \frac{(\gamma z^{-1})^{N+1} - (\gamma z^{-1})^0}{(\gamma z^{-1}) - 1} \quad \text{by power series}$$

If $|\gamma z^{-1}| < 1$ $\quad \lim_{N \to \infty} (\gamma z^{-1})^{N+1} \longrightarrow 0 \quad \Longrightarrow \quad |\gamma z^{-1}| < 1$

$$\frac{|\gamma|}{|z|} < 1 \Rightarrow |z| > |\gamma|$$

AND $X(z) = \dfrac{-1}{(\gamma z^{-1}) - 1} = \dfrac{1}{1 - \gamma z^{-1}} = \dfrac{z}{z - \gamma}$ for $|z| > |\gamma|$

the ROC.



- Note: It will be convient to write Z transforms in terms of z or z^{-1}. They are equivalent.

— Note in previous example if $\gamma = 1$ then $x[n] = u[n]$ and $Z\{u[n]\} = \frac{z}{z-1}$

— Compare the previous example to the DTFT

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n] e^{-j\omega n} = \sum_{n=0}^{\infty} (\gamma)^n e^{-j\omega n} = \sum_{n=0}^{\infty} (\gamma e^{-j\omega})^n$$

$$= \lim_{N \to \infty} \frac{(\gamma e^{-j\omega})^{N+1} - (\gamma e^{-j\omega})^0}{\gamma e^{-j\omega} - 1}$$

Again if $|\gamma e^{-j\omega}| < 1$ then $\lim_{N \to \infty} (\gamma e^{-j\omega})^{N+1} \to 0$.

$\frac{|\gamma|}{|e^{j\omega}|} < 1 \Rightarrow |\gamma| < 1$ thus DTFT only exists for $|\gamma| < 1$.

$$X(e^{j\omega}) = \frac{-1}{\gamma e^{-j\omega} - 1} = \frac{1}{1 - \gamma e^{-j\omega}} = \frac{e^{j\omega}}{e^{j\omega} - \gamma} \quad \text{for } |\gamma| < 1.$$

- This provides some intuition as two why the Z transform exists for a broader class of signals.

Let $z = re^{j\omega}$ then $z^{-n} = r^{-n} e^{-j\omega n}$ and

$$\sum_{n=0}^{\infty} (\gamma)^n z^{-n} = \sum_{n=0}^{\infty} (\gamma)^n r^{-n} e^{-j\omega n} = \sum_{n=0}^{\infty} \left(\frac{\gamma}{r}\right)^n e^{-j\omega n}$$

$$= \text{DTFT of } \left(\frac{\gamma}{r}\right)^n$$

the radius of complex z allows us to force the DTFT to converge. The values this holds for is the ROC.

— When the signal is causal the sum is always truncated to start at 0 and we have the <u>unilateral</u> (one-sided) Z transform

$$Z_2\{x[n]\} = \sum_{n=-\infty}^{\infty} x[n] z^{-n} \qquad Z_1\{x[n]\} = \sum_{n=0}^{\infty} x[n] z^{-n}$$

bilateral / two-sided $\qquad$ Unilateral / one-sided

— Example $x[n] = (r)^{|n|}$

$$Z_2\{x[n]\} = \sum_{n=-\infty}^{\infty} x[n] z^{-n} = \underbrace{\sum_{n=-\infty}^{-1} (r^{-1})^n z^{-n}}_{\text{anticausal component}} + \underbrace{\sum_{n=0}^{\infty} r^n z^{-n}}_{\text{causal component.}}$$

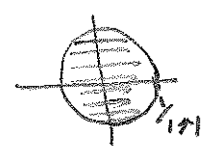
- Focusing on anticausal

$$\sum_{n=-\infty}^{-1} (r^{-1})^n z^{-n} = \sum_{n=+1}^{\infty} (r^{-1})^{-n} z^{n} = \sum_{n=1}^{\infty} (rz)^n$$

$$= \lim_{N \to \infty} \frac{(rz)^{N+1} - (rz)^1}{(rz) - 1}$$

If $|rz| < 1$ then $\lim_{N\to\infty} (rz)^{N+1} \to 0$ this implies $|r||z| < 1$
$|z| < \frac{1}{|r|}$

$$= \frac{-rz}{rz - 1} = \frac{rz}{1 - rz} \quad \text{for } |z| < \frac{1}{|r|}$$

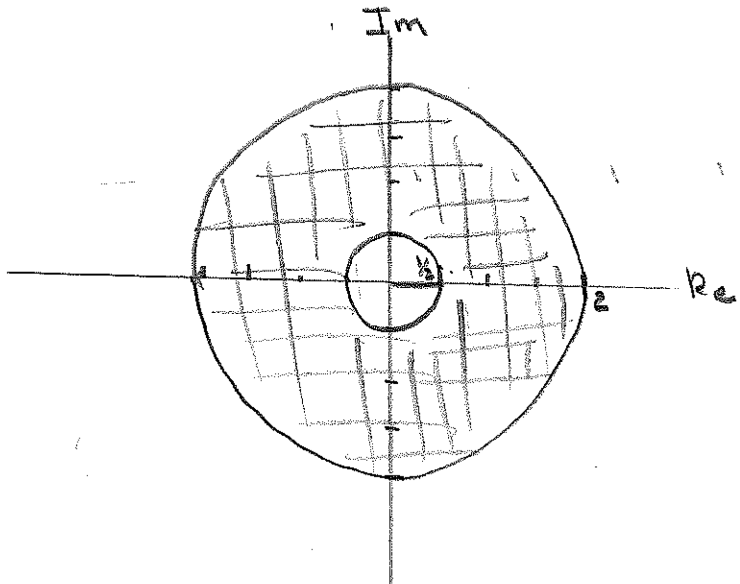


- Causal component is same as before AND

$$X(z) = \frac{rz}{1 - rz} + \frac{z}{z - r} \quad \text{for } |r| < |z| < \frac{1}{|r|}$$

Intersection of ROC's forms a ring

Consider $r = \frac{1}{2}$ then $\frac{1}{r} = 2$ and ROC is



NOTE: If all signals are causal, we can neglect the ROC, similar to Laplace.

If not then have to keep anticausal and causal separate and carry ROC through.

– Example: $x[n] = 5\left(\frac{1}{4}\right)^n u[n] + 6(2)^n u[n]$

- Since signal is causal we use the unilateral transform

$$\underline{X}(z) = \sum_{n=0}^{\infty} x(n) z^{-n} = 5\sum_{n=0}^{\infty}\left(\frac{1}{4}\right)^n z^{-n} + 6\sum_{n=0}^{\infty}(2)^n z^{-n}$$

$$= \frac{5z}{z-\frac{1}{4}} + \frac{6z}{z-2}$$

$|z| > \frac{1}{4} \cap |z| > 2$ gives $|z| > 2$ as ROC.

– Example $x[n] = \delta[n]$

$$\mathcal{Z}\{x[n]\} = \sum_{n=0}^{\infty} \delta[n] z^{-n} = z^{-0} = 1.$$

ROC is entire $\mathbb{C}$ plane.

– Example: $x[n] = u[n] - u[n-M] \qquad M \in \mathbb{Z}^+$

$$\underline{X}(z) = \sum_{n=0}^{M-1} z^{-n} = \frac{(z^{-1})^M - (z^{-1})^0}{(z^{-1}) - 1} = \frac{z^{-M} - 1}{z^{-1} - 1} = \frac{1 - z^{-M}}{1 - z^{-1}}$$

$$= \frac{z^M - 1}{z^M - z^{M-1}} = \frac{z^M - 1}{z^{M-1}(z-1)}$$

ROC is entire $\mathbb{C}$ plane.

– Example: $x[n] = e^{j\omega_0 n} u[n]$

$$\underline{X}(z) = \sum_{n=0}^{\infty} (e^{j\omega_0})^n z^{-n} = \sum_{n=0}^{\infty} (e^{j\omega_0} z^{-1})^n$$

$$= \lim_{N\to\infty} \frac{(e^{j\omega_0} z^{-1})^{N+1} - (e^{j\omega_0} z^{-1})^0}{e^{j\omega_0} z^{-1} - 1}$$

$$= \frac{-1}{e^{j\omega_0} z^{-1} - 1} \quad \text{if } |e^{j\omega_0} z^{-1}| < 1$$

$$\frac{|e^{j\omega_0}|}{|z|} < 1 \Rightarrow |z| > 1$$

$$= \frac{z}{z - e^{j\omega_0}} \qquad |z| > 1$$

— This result allows us to derive $\mathcal{Z}\{\cos(\omega_0 n)u[n]\}$

$$\mathcal{Z}\{\cos(\omega_0 n)u[n]\} = \mathcal{Z}\left\{\frac{1}{2}e^{j\omega_0 n}u[n] + \frac{1}{2}e^{-j\omega_0 n}u[n]\right\}$$

$$= \frac{1}{2}\sum_{n=0}^{\infty} e^{j\omega_0 n} z^{-n} + \frac{1}{2}\sum_{n=0}^{\infty} e^{-j\omega_0 n} z^{-n}$$

$$= \frac{1}{2}\,\frac{z}{z-e^{j\omega_0}} + \frac{1}{2}\,\frac{z}{z-e^{-j\omega_0}}$$

$$= \frac{1}{2}\left[\frac{z(z-e^{-j\omega_0}) + z(z-e^{j\omega_0})}{(z-e^{j\omega_0})(z-e^{-j\omega_0})}\right]$$

$$= \frac{1}{2}\left[\frac{z^2 - ze^{-j\omega_0} + z^2 - ze^{j\omega_0}}{z^2 - ze^{j\omega_0} - ze^{-j\omega_0} + 1}\right]$$

$$= \frac{z(z-\cos(\omega_0))}{z^2 - 2\cos(\omega_0) z + 1} \qquad |z| > 1$$

* the PS for this week asks you to derive $\mathcal{Z}\{\sin(\omega_0)u[n]\}$

— Example: Determine the $\mathcal{Z}$ transform of $x[n] = \left(\frac{1}{2}\right)^n u[n] * u[n]$ where $*$ denotes convolution.

Approach #1: perform convolution and take $\mathcal{Z}$ transform.

Approach #2: lets first derive the convolution property of $\mathcal{Z}$ transform.

— Convolution property of unilateral $\mathcal{Z}$ transform

Given two causal signals $x_1[n]$, $x_2[n]$ their convolution $x_1[n] * x_2[n]$ is a causal signal given by

$$\sum_{m=0}^{\infty} x_1[m]\,x_2[n-m]. \qquad \text{for } n \geq 0$$

The z transform is $= \sum_{n=0}^{\infty} (x_1[n] * x_2[n]) z^{-n}$

$$= \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} x_1[m]\, x_2[n-m]\, z^{-n}$$

$$= \sum_{m=0}^{\infty} x_1[m] \sum_{n=0}^{\infty} x_2[n-m]\, z^{-n}$$

Let $k = n-m$, $n = k+m$

$$= \sum_{m=0}^{\infty} x_1[m]\, z^{-m} \sum_{k=0}^{\infty} x_2[k]\, z^{-k}$$

$$= X_1(z) \cdot X_2(z).$$

— Returning to our example Let $x_1[n] = \left(\frac{1}{2}\right)^n u[n]$ $x_2[n] = u[n]$

Then $\mathcal{Z}\{x_1[n] * x_2[n]\} = X_1(z) \cdot X_2(z)$

$$= \frac{z}{z - \frac{1}{2}} \cdot \frac{z}{z-1} \quad \text{for } |z| > 1$$

$$= \frac{z^2}{(z-\frac{1}{2})(z-1)} = \frac{z^2}{z^2 - \frac{3}{2}z + \frac{1}{2}}$$

Note: if $x_1[n] = h[n]$ the impulse response of a causal LTI system and $x_2[n] = u[n]$ is the input then.

$\mathcal{Z}\{h[n]\} = H(z) = \dfrac{z}{z - \frac{1}{2}}$ is the transfer function

$\mathcal{Z}\{u[n]\} = X_2(z) = \dfrac{z}{z-1}$ is the input

if output $y[n] = h[n] * x[n]$ then

$Y(z) = H(z)\, X_2(z) = \dfrac{z^2}{z^2 - \frac{3}{2}z + \frac{1}{2}}$ is the z transform of the step response.