ECE 3704 Lecture # 2 (Review)

Modeling and Characterization of Signals and Systems.

- Signals are modeled as functions $f: A \to B$ where A, B are sets.
- examples
  - $A = \mathbb{R}$, $B = \mathbb{R}$ CT Analog Signals
  - $A = \mathbb{Z}$, $B = \mathbb{R}$ DT real-valued signal
  - $A = \mathbb{Z}$, $B = \mathbb{C}$ DT complex-valued signal
  - $A = \mathbb{Z}$, $B = b_N 2^N + b_{N-1} 2^{N-1} + b_{N-2} 2^{N-2} \cdots b_0$ Digital Signal.
    $b_i \in \{0,1\}$
  - $A \equiv \mathbb{R}$, $B \equiv \mathbb{R}^2$ Two-channel CT Analog Signal (e.g. stereo audio)
  - $A \equiv \{0,1,2,\cdots N\} \times \{0,1,2 \ldots M\}$ Digital Image, grayscale 8 bit
    $B \equiv \{0,1,2 \ldots 255\}$

- To model signals we build up from primitive signals using transformations.
  - Example Primitives for CT signals
    - $\delta(t)$ delta function
    - $u(t)$ step function
    - e^{st} $s \in \mathbb{C}$ the complex exponential

      Recall $s = \alpha + j\beta$ $\quad e^{st} = e^{(\alpha + j\beta)t}$

      $\alpha \in \mathbb{R}$ real $\quad = e^{\alpha t} e^{j\beta t}$

      $\beta \in \mathbb{R}$ imag $\quad = e^{\alpha t}\left[\cos(\beta t) + j\sin(\beta t)\right]$

  - By transforming primitive signals we can model more complex ones.

    Ex: $\cos(\omega t) = \frac{1}{2} e^{j\omega t} + \frac{1}{2} e^{-j\omega t}$; $\quad x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t}$ $\quad a_k \in \mathbb{C}$

— we can also think about decomposing signals rather than building them up. The most usefull is the Fourier decomposition

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega)\, e^{j\omega t}\, d\omega$$

Which applies to many, but as we will see, not all signals. For example $x(t) = e^{t}u(t)$ cannot be decomposed into pure sinusoids.

— Example primitives for DT signals.

- $\delta[n]$ "delta"
- $u[n]$ "step"
- $z^n \quad z \in \mathbb{C} \qquad z = re^{j\theta} \quad r, \theta \in \mathbb{R}$

$$z^n = r^n e^{j\theta n}$$
$$= r^n\left[\cos(\theta n) + j\sin(\theta n)\right]$$

— Example models for DT signals.

- $x[n] = (r)^n u[n] \quad r \in \mathbb{R}.$
- $x[n] = (r)^n \cos(\omega n)u[n] \quad r \in \mathbb{R} \quad \omega \in \mathbb{R}.$

$$= (r)^n\left[\tfrac{1}{2}e^{j\omega n} + \tfrac{1}{2}e^{-j\omega n}\right]u[n]$$
$$= \left[\tfrac{1}{2}(r)^n e^{j\omega n} + \tfrac{1}{2}(r)^n e^{-j\omega n}\right]u[n]$$
$$= \tfrac{1}{2}z^n u[n] + \tfrac{1}{2}(z^*)^n u[n] \qquad z = re^{j\theta}$$

— The corresponding Fourier decomposition for DT signals is the discrete Fourier.

$$x[n] = \frac{1}{2\pi}\int_{-\pi/2}^{\pi/2} X(e^{j\omega})e^{j\omega n}\,d\omega$$

which again applies to some, but not all signals of interest.

— Systems either

1) produce signals
2) measure signals
3) transform signals (inputs) into other signals (outputs)

— CT systems transform CT signals to CT signals

— DT systems transform DT signals to DT signals

— Hybrid systems convert between to/from CT/DT signals

$x(t) \rightarrow \boxed{} \rightarrow x[n]$ — sampling

$x[n] \rightarrow \boxed{} \rightarrow x(t)$ — reconstruction.

— Major classification of systems.

- memoryless or dynamic
- causal, anti-causal, non-causal
- invertable, non-invertable
- BIBO stable or unstable
- Time-varying or time-invariant
- Linear or Non linear.

— the focus of 2214 and this course are Linear Time-Invariant systems (LTI) in CT and DT.

- CT systems can be represented by
  - Differential Equations,
    e.g. damped pendulum $a\ddot{y} + b\dot{y} + c\sin(y) = x(t)$
- CT LTI systems can be represented by
  - LCCDE with zero initial conditions
  - impulse response $h(t)$
  - block diagram
  - transfer Function $H(s)$ * Focus of 3704.
- Stable CT LTI systems can also be represented by their frequency response

$$H(j\omega) = \frac{Y(j\omega)}{X(j\omega)} = \mathcal{F}\{h(t)\}$$

- DT systems can in general be represented by difference equations.
  e.g. logistic equation $y[n+1] = a\,x[n](1 - x[n])$
- DT LTI systems can be represented by
  - LCCDE with zero auxillary conditions
  - impulse response $h[n]$
  - block diagram
  - transfer Function $H(z)$ * Focus of 3704
- Stable DT LTI systems can be represented by Frequency Response $H(e^{j\omega}) = \dfrac{Y(e^{j\omega})}{X(e^{j\omega})}$

- ECE 2714 leaves open two major questions
  1) how do we deal with signals that do not have Fourier transforms
  2) how do we deal with unstable systems.
- the first question is seemingly not that important as signals without Fourier transforms grow faster than practically useful.
- The second question is more relevant.
  - how do we prevent unstable systems when designing?
  - How can we stabalize unstable systems.
- The first and second questions are related since the impulse response $h(t)$, $h[n]$ of unstable systems do NOT have Fourier Transforms.

- we will address these issues by introducing two new tools
  - Laplace Transform
  - Z-transform.

– Some example Problems.

1) Given the following LCCDE determine the impulse response, $h(t)$.

$$D^3 y + 6D^2 y + 11Dy + 6y = Dx + x$$

$$Q(D) = D^3 + 6D^2 + 11D + 6 \qquad P(D) = D + 1 + \overset{b_0}{0}D^2$$

$$= (D+1)(D+2)(D+3)$$

$$y_n(t) = C_1 e^{-t} + C_2 e^{-2t} + C_3 e^{-3t} \qquad y_n(0) = 0$$

$$y_n'(t) = -C_1 e^{-t} - 2C_2 e^{-2t} - 3C_3 e^{-3t} \qquad y_n'(0) = 0$$

$$y_n''(t) = C_1 e^{-t} + 4C_2 e^{-2t} + 9C_3 e^{-3t} \qquad y_n''(0) = 1$$

Solve
$$\begin{bmatrix} 1 & 1 & 1 \\ -1 & -2 & -3 \\ 1 & 4 & 9 \end{bmatrix} \begin{bmatrix} C_1 \\ C_2 \\ C_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \Rightarrow \begin{matrix} C_1 = 1/2 \\ C_2 = -1 \\ C_3 = 1/2 \end{matrix}$$

$$h(t) = \overset{0}{\cancel{b_0}}\,\delta(t) + [P(D)\, y_n]\,u(t)$$

$$= \left[-\tfrac{1}{2}e^{-t} + 2e^{-2t} - \tfrac{3}{2}e^{-3t} + \tfrac{1}{2}e^{-t} - 2e^{2t} + \tfrac{1}{2}e^{-3t}\right]u(t)$$

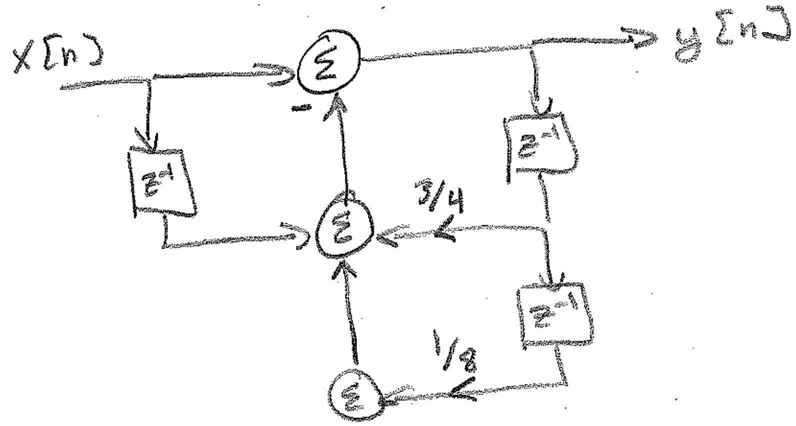
$$= \left[\underset{\uparrow}{0}\,e^{-t} + e^{-2t} - e^{-3t}\right]u(t)$$

why?

☆ We will see Laplace gives us intuition about solutions like this and in many cases simplifies the analysis.

2) Find the impulse response that corresponds to the block diagram

$\boxed{z^{-1}}$ = delay block



$$y[n] = -\frac{3}{4} y[n-1] - \frac{1}{8} y[n-2] + x[n] - x[n-1]$$

$$y[n+2] + \frac{3}{4} y[n+1] + \frac{1}{8} y[n] = x[n+2] - x[n+1]$$

$$Q(E) = E^2 + \frac{3}{4}E + \frac{1}{8} \qquad P(E) = E^2 - E + 0$$

$$(E + \tfrac{1}{2})(E + \tfrac{1}{4})$$

$$y_h[n] = C_1(-\tfrac{1}{2})^n + C_2(-\tfrac{1}{4})^n$$

$$a_2 = 0 \quad b_2 = 1/8$$

$$h[n] = \frac{a_2}{b_2}\delta[n] + y_h[n]u[n]$$

$$= C_1(-\tfrac{1}{2})^n u[n] + C_2(-\tfrac{1}{4})^n u[n]$$

$$h[n] = -\frac{3}{4} h[n-1] - \frac{1}{8} h[n-2] + \delta[n] - \delta[n-1]$$

$$h[0] = -\frac{3}{4}(0) - \frac{1}{8}(0) + (1) - (0) = 1 \quad = C_1 + C_2$$

$$h[1] = -\frac{3}{4}(1) - \frac{1}{8}(0) + (0) - 1 = \frac{-7}{4} \quad = -\frac{1}{2}C_1 - \frac{1}{4}C_2$$

$$\Downarrow$$

$$h[2] = -\frac{3}{4}(-\tfrac{7}{4}) - \frac{1}{8}(1) + 0 + 0 = \frac{21}{16} - \frac{1}{8} = \frac{19}{16} \checkmark$$

$$C_1 = 6$$

$$C_2 = -5$$

$$\boxed{h[n] = 6(-\tfrac{1}{2})^n u[n] - 5(-\tfrac{1}{4})^n u[n]}$$

Check

$$h[2] = 6(-\tfrac{1}{2})^2 - 5(-\tfrac{1}{4})^2 = \frac{6}{4} - \frac{5}{16} = \frac{19}{16} \checkmark$$