Forward Laplace Transform

- Recall from 2714 that the Eigen function for LTI CT systems is e^{st} for $s \in \mathbb{C}$.

$$x(t) = e^{st} \longrightarrow \boxed{h(t)} \longrightarrow y(t) = x(t) * h(t) = H(s)\, e^{st}$$

  - $H(s)$ is the Eigenvalue assosiated with the Eigenfunction e^{st} at some fixed value of s.

  - $H(s)$ is the Bilateral Laplace transform of the impulse response, also called two-sided transform.

$$H(s) = \int_{-\infty}^{\infty} h(t) e^{-st}\, dt \quad \text{for } s \in R \subseteq \mathbb{C}$$

  the subset of the complex plane R is the set of values for s where the integral converges and the transform exists. We call this the region-of-convergence or ROC.

- We can take the Laplace transform of Any signal NOT just the impulse response.

- Notation $\mathcal{L}_2\{x(t)\} = X(s)$ or $x(t) \xrightarrow{\mathcal{L}_2} X(s)$

  the Forward Laplace transform.

- Lets look at some illustrative examples

Example #1: $x(t) = e^{-at} u(t)$ for $a \in \mathbb{R}$ Find $X(s) = \mathcal{L}\{x(t)\}$

$$X(s) = \int_{-\infty}^{\infty} x(t) e^{-st} dt = \int_0^{\infty} e^{-at} e^{-st} dt = \int_0^{\infty} e^{-(s+a)t} dt$$

$$= \frac{-1}{s+a} e^{-(s+a)t} \Big|_0^{\infty}$$

$$= \frac{-1}{s+a} \left[\lim_{T \to \infty} e^{-(s+a)T} - e^{-(s+a)(0)} \right]$$

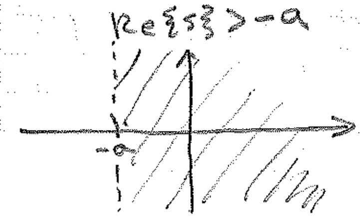
$$= \frac{-1}{s+a} \Big[\underbrace{0}_{} - 1 \Big] = \frac{1}{s+a}$$

If $Re\{s+a\} > 0$

or $Re\{s\} + Re\{a\} > 0$

$Re\{s\} > -a$

$Re\{s\} > -a$



Note this result $\mathcal{L}_2\{e^{-at}u(t)\} = \frac{1}{s+a}$ $\quad Re\{s\} > -a$

applies to any finite real a, thus

$$\mathcal{L}\{u(t)\} = \mathcal{L}_2\{e^{-at}u(t)\}\Big|_{a=0} = \frac{1}{s} \quad Re\{s\} > 0$$

– Lets compare the result in Example 1 to the CTFT definition

$$\mathcal{F}\{x(t)\} = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}\,dt = \int_{-\infty}^{\infty} x(t)e^{-st}\,dt\Bigg|_{s=j\omega}$$

The definition of the CTFT is equivalent to Laplace definition with s restricted to purely imaginary values.

What are the implications of this?

$$\mathcal{F}\{e^{-at}u(t)\} = \int_{0}^{\infty} e^{-at}e^{-j\omega t}\,dt = \int_{0}^{\infty} e^{-(j\omega+a)t}\,dt$$

$$= \frac{-1}{j\omega+a}\Big[\underbrace{\lim_{T\to\infty} e^{-(j\omega+a)T}}\; - \; e^{-(j\omega+a)0}\Big]$$

This only converges for $a > 0$.

Having a real part to s allows us to force the integral to converge (within limits), with ROC being the values that do so.

– Recall the CTFT $\mathcal{F}\{x(t)\}$ exists if $\int_{-\infty}^{\infty}|x(t)|\,dt < \infty$

When will Laplace transform exist?

IF $\int_{-\infty}^{\infty}\left|x(t)e^{ct}\right|\,dt < \infty$ for some Real c.

if $x(t)$ grows as $t \to L$, e^{ct} can counter the growth.

Consider however $x(t) = t^t$, it grows too fast.

Example 2: Non-causal pulse $x(t) = u(t+1) - u(t-1)$

$$\underline{X}(s) = \int_{-\infty}^{\infty} x(t) e^{-st}\, dt = \int_{-1}^{1} e^{-st}\, dt = -\frac{1}{s} e^{-st} \Big|_{-1}^{1}$$

$$= -\frac{1}{s} e^{-s} + \frac{1}{s} e^{s} = \frac{1}{s}\left(e^{s} - e^{-s}\right) \quad \text{for } \underline{\text{all}}\ s \in \mathbb{C}$$

– Note: The Laplace transform of a finite length signal where

$$\int_{-L}^{U} |x(t)|\, dt < \infty \qquad \begin{array}{l} L = \text{lower bound} \\ U = \text{upper bound,} \end{array}$$

will exist with ROC = entire $\mathbb{C}$ plane.

Example 3: $x(t) = e^{-t} u(t) - e^{-2t} u(t)$

$$\underline{X}(s) = \int_{0}^{\infty} e^{-t} e^{-st}\, dt - \int_{0}^{\infty} e^{-2t} e^{-st}\, dt$$

$$= \frac{1}{s+1} - \frac{1}{s+2}$$

$R_1 \equiv Re\{s\} > -1 \qquad R_2 \equiv Re\{s\} > -2$

$$= \frac{s+2-s-1}{(s+1)(s+2)} \qquad \text{with ROC} = R_1 \cup R_2 = Re\{s\} > -1$$

$$= \frac{1}{(s+1)(s+2)} = \frac{1}{s^2+3s+2} \qquad Re\{s\} > -1$$

– Note: If $x(t)$ is causal then

$$* \rightarrow \mathcal{L}_1\{x(t)\} = \int_{0}^{\infty} x(t) e^{-st}\, dt \quad \text{and ROC} = Re\{s\} > \text{constant.}$$

This is the <u>unilateral</u> or <u>one-sided</u> Laplace transform.

$\mathcal{L}_2$ v/s $\mathcal{L}_1$

When dealing with only causal signals (and thus systems) explicitly treating the ROC is NOT required.

Example 4: $x(t) = e^{-|t|}$ Note non-causal, infinite extent.

$$X(s) = \int_{-\infty}^{\infty} x(t) e^{-st}\, dt$$

$$= \int_{-\infty}^{0} e^{t} e^{-st}\, dt + \int_{0}^{\infty} e^{-t} e^{-st}\, dt \qquad x(t) = \begin{cases} e^{t} & t<0 \\ e^{-t} & t>0 \\ 1 & t=0 \end{cases}$$

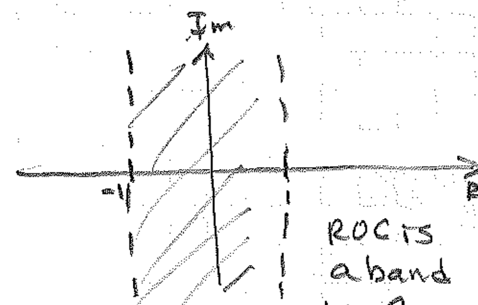
$$= \int_{-\infty}^{0} e^{-(s-1)t}\, dt + \int_{0}^{\infty} e^{-(s+1)t}\, dt$$

$$= \frac{-1}{s-1}\left[e^{-(s-1)(0)} - \lim_{T\to-\infty} e^{-(s-1)T} \right]$$

$$+ \frac{-1}{s+1}\left[\lim_{T\to\infty} e^{-(s+1)t} - e^{-(s+1)(0)} \right]$$

$$= \frac{-1}{s-1}\left[1 - \underbrace{\lim_{T\to-\infty} e^{-(s-1)T}}_{\substack{Re\{s-1\}<0 \\ Re\{s\}-1<0 \\ Re\{s\}<1}}\right] + \frac{-1}{s+1}\left[\underbrace{\lim_{T\to\infty} e^{-(s+1)T}}_{\substack{Re\{s+1\}>0 \\ Re\{s\}+1>0 \\ Re\{s\}>-1}} - 1\right]$$

$$= \underbrace{\frac{-1}{s-1}}_{\substack{\text{anti-causal} \\ \text{component} \\ Re\{s\}<1}} + \underbrace{\frac{1}{s+1}}_{\substack{\text{causal} \\ \text{component} \\ Re\{s\}>-1}}$$



ROC is a band in $\mathbb{C}$.

Note if we combine the expressions

$$X(s) = \frac{-1}{s-1} + \frac{1}{s+1} = \frac{-s-1+s-1}{(s-1)(s+1)} = \frac{-2}{s^2-1}$$

we loose the distinction between anti-causal and causal, which is why ROC needs to be kept in those cases. $-1 < Re\{s\} < 1$

Lets do more examples,

Example 5: $x(t) = \delta(t)$

$$\mathbb{X}(s) = \int_{-\infty}^{\infty} x(t) e^{-st}\, dt = \int_{-\infty}^{\infty} \delta(t) e^{-st}\, dt = e^{-s(0)} = 1$$

ROC $= \mathbb{C}$

Example 6: $x(t) = e^{j\omega_0 t}$ $\quad \omega_0 \in \mathbb{R}^+$

$$\mathbb{X}(s) = \int_{-\infty}^{\infty} e^{j\omega_0 t} e^{-st}\, dt = \int_{-\infty}^{\infty} e^{-(s-j\omega_0)t}\, dt$$

$$= \frac{-1}{s-j\omega_0} e^{-(s-j\omega_0)t} \Big/_{-\infty}^{\infty}$$

$$= \frac{-1}{s-j\omega_0}\Big[\underbrace{\lim_{T\to-\infty} e^{-(s-j\omega_0)T}}_{\substack{Re\{s-j\omega_0\}<0 \\ Re\{s\}<0}} - \underbrace{\lim_{T\to\infty} e^{-(s-j\omega_0)T}}_{\substack{Re\{s-j\omega_0\}>0 \\ Re\{s\}>0}}\Big]$$

ROC does NOT exist,

Let $s=0$, we get $\mathcal{F}\{e^{j\omega_0 t}\} = \frac{1}{2\pi}\delta(\omega-\omega_0)$

Example 7: Compare to $x(t) = e^{j\omega_0 t} u(t)$

$$\mathbb{X}(s) = \int_{0}^{\infty} e^{j\omega_0 t} e^{-st}\, dt = \int_{0}^{\infty} e^{-(s-j\omega_0)t}\, dt$$

$$= \frac{-1}{s-j\omega_0} e^{-(s-j\omega_0)t} \Big/_{0}^{\infty}$$

$$= \frac{-1}{s-j\omega_0}\Big[\underbrace{\lim_{T\to\infty} e^{-(s-j\omega_0)T}}_{Re\{s\}>0} - e^{-(s-j\omega_0)(0)}\Big]$$

$$= \frac{-1}{s-j\omega_0}\big[-1\big] = \frac{1}{s-j\omega_0} \quad Re\{s\}>0.$$

Example 8: $x(t) = \cos(\omega_0 t)\,u(t) = \frac{1}{2}e^{j\omega_0 t}u(t) + \frac{1}{2}e^{-j\omega_0 t}u(t)$

$$X(s) = \int_0^\infty x(t)e^{-st}\,dt = \frac{1}{2}\int_0^\infty e^{j\omega_0 t}e^{-st}\,dt + \frac{1}{2}\int_0^\infty e^{-j\omega_0 t}e^{-st}\,dt$$

Similar to previous example.

$$X(s) = \frac{1}{2}\int_0^\infty e^{-(s-j\omega_0)t}\,dt + \frac{1}{2}\int_0^\infty e^{-(s+j\omega_0)t}\,dt$$

$$= \frac{1}{2}\,\frac{-1}{s-j\omega_0}\,e^{-(s-j\omega_0)t}\Big|_0^\infty + \frac{1}{2}\,\frac{-1}{s+j\omega_0}\,e^{-(s+j\omega_0)t}\Big|_0^\infty$$

$$= \frac{1}{2}\left[\underset{Re\{s\}>0}{\frac{1}{s-j\omega_0}} + \underset{Re\{s\}>0}{\frac{1}{s+j\omega_0}}\right]$$

$$= \frac{1}{2}\,\frac{s+j\omega_0+s-j\omega_0}{(s-j\omega_0)(s+j\omega_0)} = \frac{s}{s^2+\omega_0^2} \quad Re\{s\}>0$$

- Note the examples thus far cover many of the signals we are interested in.

$x(t)$	$X(s)$	
$\delta(t)$	1	
$u(t)$	$1/s$	$Re\{s\}>0$
$e^{-at}u(t)$ $a\in\mathbb{R}$	$\frac{1}{s+a}$	$Re\{s\}>-a$
$\cos(\omega_0 t)u(t)$	$\frac{s}{s^2+\omega_0^2}$	$Re\{s\}>0$

$\sin(\omega_0 t)u(t)$	?
$e^{-at}\cos(\omega_0 t)u(t)$	?

The PS for this week has you derive these.