Lecture 21: Systems analysis using Z-transform

- The transfer function gives us a 4th way to represent DT LTI systems

$$\begin{tikzcd} & \text{LCCDE} \arrow[dl, leftrightarrow] \arrow[dr, leftrightarrow] & \\ h[n] \arrow[rr, leftrightarrow] \arrow[dr, leftrightarrow] & & H(z) \\ & \text{BD} \arrow[ur, leftrightarrow] & \end{tikzcd}$$

$$H(z) = \frac{P(z)}{Q(z)} = \frac{\sum_{k=0}^{M} b_k z^k}{\sum_{k=0}^{N} a_k z^k} = \frac{\prod_{k=0}^{M}(z - z_k)}{\prod_{k=0}^{N}(z - p_k)}$$

for $r_1 < |z| < r_2$ — coefficient form — zero-pole form.

- The values of z for which $H(z)=0$ are called zero's
- The singularities of z are called poles.
- Typically plotted using o = zero x = pole with duplicates indicated by (#)
- If system is causal $M \le N$ and ROC $|z| > r$ with all poles inside circle of radius r, i.e. outermost pole is at radius r.

$$\arg\max_k |p_k| = r$$

- If the system is stable, ROC includes unit circle and all poles fall inside unit circle.
- <u>If</u> the system is stable then it has a frequency response

$$H(e^{j\omega}) = H(z)\Big|_{z=e^{j\omega}}$$ the TF evaluated on unit circle.

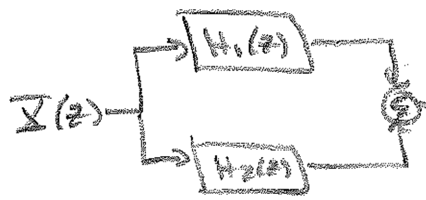
- There exists an analogue to the Routh-Herwitz criteria for DT TF's $H(z)$ called Jury criteria.

– Recall from 2714 the DT block Diagrams. These are simplified using the TF concept

$X(z) \rightarrow \boxed{H(z)} \rightarrow Y(z) = H(z)\,X(z)$

series: $X(z) \rightarrow \underbrace{\boxed{H_1(z)} \rightarrow \boxed{H_2(z)}}_{H(z) = H_1(z) \cdot H_2(z)} \rightarrow Y(z) = H_1(z)\,H_2(z)\,X(z)$

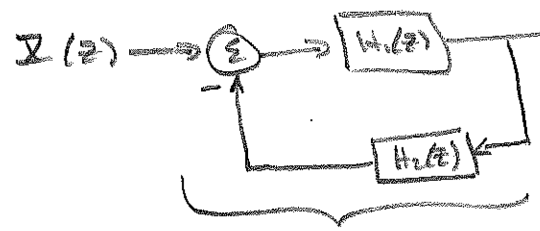
parallel:



$Y(z) = (H_1(z) + H_2(z))\,X(z)$

$H(z) = H_1(z) + H_2(z)$

feedback:



$$Y(z) = \frac{H_1(z)}{1 + H_1(z)H_2(z)}\,X(z)$$

$$H(z) = \frac{H_1(z)}{1 + H_1(z)H_2(z)}$$

– The most basic building blocks are gain/multiplier

$\xrightarrow{a} \;\equiv\; \rightarrow \boxed{H(z) = a} \rightarrow$

and unit delay $H(z) = z^{-1}$

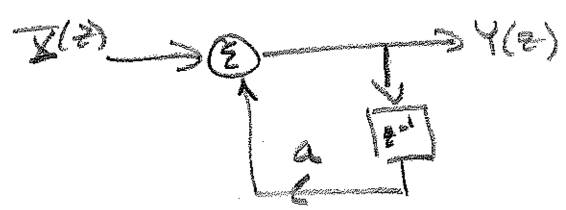
$\rightarrow \boxed{z^{-1}} \rightarrow \;\equiv\; \rightarrow \boxed{1/z} \rightarrow$

– We will use these extensively when talking about implementing digital filters.

Simple example $H(z) = \dfrac{z}{z - a} = \dfrac{1}{1 - az^{-1}} \Rightarrow Y(z) = \dfrac{1}{1 - az^{-1}}\,X(z)$

$(1 - az^{-1})\,Y(z) = X(z)$

$Y(z) = az^{-1}\,Y(z) + X(z)$



Example: Given the LCCDE

$$y[n] = -\frac{1}{8} y[n-2] + \frac{1}{4} y[n-1] + x[n] - x[n-1] \quad \text{zero A.C.}$$

Find TF $H(z)$, determine if system is stable. Then find

$y_1[n]$ if $x_1[n] = \left(\frac{3}{4}\right)^n u[n]$

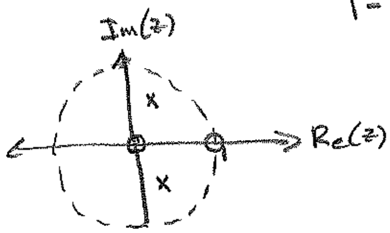
$y_2[n]$ if $x_2[n] = \cos\left(\frac{\pi}{6} n\right)$

Solution: Taking z-transform of LCCDE

$$Y(z) = -\frac{1}{8} z^{-2} Y(z) + \frac{1}{4} z^{-1} Y(z) + X(z) - z^{-1} X(z)$$

$$\left(1 - \frac{1}{4} z^{-1} + \frac{1}{8} z^{-2}\right) Y(z) = \left(1 - z^{-1}\right) X(z)$$

$$H(z) = \frac{Y(z)}{X(z)} = \frac{1 - z^{-1}}{1 - \frac{1}{4} z^{-1} + \frac{1}{8} z^{-2}} = \frac{z(z-1)}{z^2 - \frac{1}{4} z + \frac{1}{8}}$$



↳ has complex roots $\frac{1}{8} \pm j \frac{\sqrt{7}}{8} = p_{1,2}$

$|p_1| = |p_2| < 1$ ∴ stable system.

— Now if $x_1[n] = \left(\frac{3}{4}\right)^n u[n]$ $\quad X_1(z) = \dfrac{z}{z - 3/4}$

(NOTE causal)

$$Y_1(z) = H(z) X_1(z) = \frac{z(z-1)}{z^2 - \frac{1}{4} z + \frac{1}{8}} \cdot \frac{z}{z - 3/4} = \frac{z^2 (z-1)}{\left(z - \frac{3}{4}\right)\left(z^2 - \frac{1}{4} z + \frac{1}{8}\right)}$$

To find $y_1[n] = \mathcal{Z}^{-1}\{Y_1(z)\}$ using either PFE and table or by definition.

• Using PFE of $\dfrac{Y(z)}{z} = \dfrac{z(z-1)}{\left(z - \frac{3}{4}\right)\left(z^2 - \frac{1}{4} z + \frac{1}{8}\right)} = \dfrac{A}{z - \frac{3}{4}} + \dfrac{Bz + C}{z^2 - \frac{1}{4} z + \frac{1}{8}}$

$$A = \left.\frac{z(z-1)}{z^2 - \frac{1}{4} z + \frac{1}{8}}\right|_{z = \frac{3}{4}} = \frac{\frac{3}{4}\left(\frac{3}{4} - 1\right)}{\frac{9}{16} - \frac{3}{16} + \frac{2}{16}} = \frac{-\frac{3}{16}}{\frac{8}{16}} = -\frac{3}{8}$$

Example cont.

To Find C Let $z=0$ $\quad \frac{A}{-\frac{3}{4}} + \frac{C}{1/8} = \frac{0}{(\frac{1}{8})(-\frac{3}{4})} = 0$

$$-\frac{4}{3}A + 8C = 0 \Rightarrow C = -\frac{1}{16}$$

To Find B consider $Y_1(z)$ when $z^{-1} \to 0$

$$\frac{A}{1-\frac{3}{4}z^{-1}} + \frac{B + Cz^{-1}}{1-\frac{1}{4}z^{-1}+\frac{1}{8}z^{-2}} = \frac{1(1-z^{-1})}{(1-\frac{1}{4}z^{-1}+\frac{1}{8}z^{-1})(1-\frac{3}{4}z^{-1})}$$

$$A + B = 1 \Rightarrow B = 1 - A = 1 + \frac{3}{8} = \frac{11}{8}$$

Thus $Y_1(z) = \dfrac{-3/8\, z}{z - 3/4} + \dfrac{z(\frac{11}{8}z - \frac{1}{16})}{z^2 - \frac{1}{4}z + \frac{1}{8}}$

The first term is easy $Z_1^{-1}\left\{-3/8 \frac{z}{z-3/4}\right\} = -\frac{3}{8}\left(\frac{3}{4}\right)^n u[n]$

To match the second term to a table, write it as

$$\frac{z(Az+B)}{z^2 + 2az + |\gamma|^2} \quad \text{so } A = \frac{11}{8} \quad B = -\frac{1}{16}$$
$$a = -\frac{1}{8} \quad |\gamma|^2 = \frac{1}{8}$$

then Z_1^{-1} is $\quad r|\gamma|^n \cos(\omega_0 n + \theta) u[n]$

$$r = \left[\frac{A^2|\gamma|^2 + B^2 - 2AaB}{|\gamma|^2 - a^2}\right]^{1/2} \qquad \omega_0 = \cos^{-1}\left(\frac{-a}{|\gamma|}\right)$$

$$\theta = \tan^{-1}\left(\frac{Aa - B}{A(|\gamma|^2 - a^2)^{1/2}}\right)$$

Another approach is to write it as

$$Z_1\{A|\gamma|^n \cos(\omega_0 n)u[n] + B|\gamma|^n \sin(\omega_0 n)u[n]\}$$

$$= \frac{Az(z - |\gamma|\cos(\omega_0))}{z^2 - 2|\gamma|\cos(\omega_0)z + |\gamma|^2} + \frac{B|\gamma|\sin(\omega_0)z}{z^2 - 2|\gamma|\cos(\omega_0)z + |\gamma|^2}$$

— Or use definition: $y[n] = -\frac{3}{8}(\frac{3}{4})^n u[n] + A\left(\frac{1}{8} + j\frac{\sqrt{7}}{8}\right)^n u[n] + B\left(\frac{1}{8} - j\frac{\sqrt{7}}{8}\right)^n u[n]$

They are all equivalent.

– What about when input is $\cos(\frac{\pi}{6} n)$?

Since the system is stable we have $H(e^{j\omega}) = H(z)\Big|_{z=e^{j\omega}}$

and $y[n] = \left|H(e^{j\omega})\right|_{\omega=\frac{\pi}{6}} \cos\left(\frac{\pi}{6} n + \theta\right)$ $\qquad \theta = \angle H(e^{j\omega})\Big|_{\omega=\pi/6}$

$$H(e^{j\omega}) = \frac{e^{j\omega}(e^{j\omega}-1)}{(e^{j2\omega}) - \frac{1}{4}e^{j\omega} + 1/8} \qquad H(e^{j\frac{\pi}{6}}) = \frac{e^{j\frac{\pi}{6}}\left(e^{j\frac{\pi}{6}} - 1\right)}{e^{j\pi/3} - \frac{1}{4}e^{j\pi/6} + \frac{1}{8}}$$

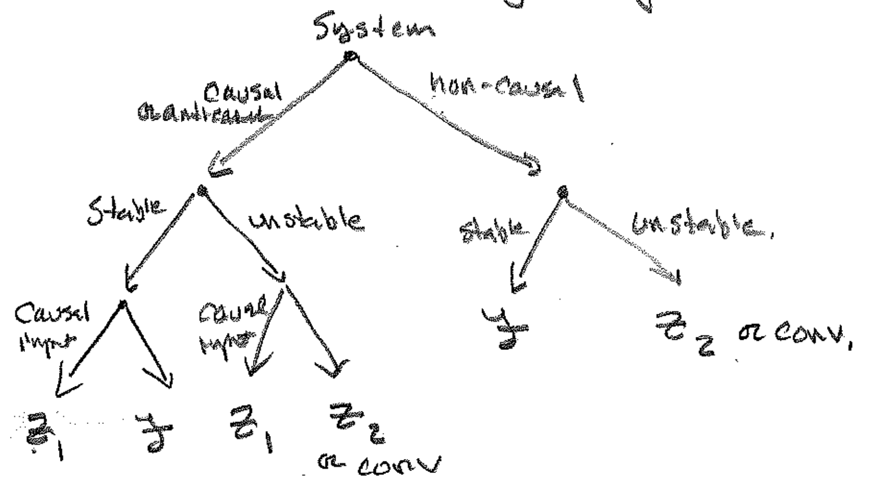
– What if system is NOT stable and $x[n] = \cos(\frac{\pi}{6} n)$?

$y[n]$ does not exist.

– What if system is not stable and $x[n] = \cos(\frac{\pi}{6} n) u[n]$

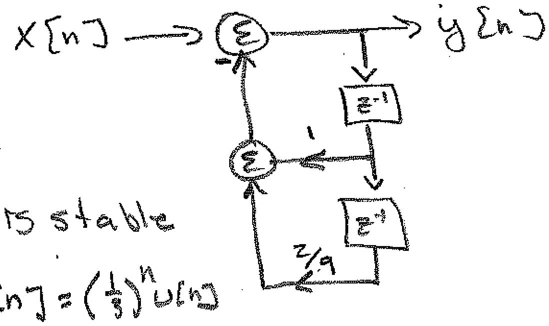
Then $y[n] = Z_1^{-1}\left\{ H(z) X(z) \right\}$

– Thus we have a similar tree of analysis techniques for DTLTI sys.



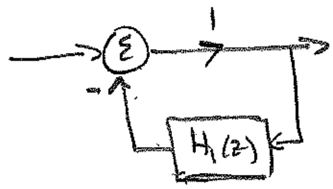
The "sweet spot" for z transform is causal systems with causal inputs, or causal, stable systems with sinusoidal input via $H(z) \to H(e^{j\omega})$.

- Given the BD.



- Find $H(z)$
- Determine if system is stable
- Find $y[n]$ if $x[n] = (\frac{1}{3})^n u[n]$

– There are multiple ways to find $H(z)$. Lets combine blocks



$$H_1(z) = z^{-1} + \frac{2}{9} z^{-2}$$

then NOTE $$H(z) = \frac{1}{1 + H_1(z)} = \frac{1}{1 + z^{-1} + \frac{2}{9} z^{-2}} = \frac{z^2}{z^2 + z + \frac{2}{9}}$$

$$(z + \tfrac{2}{3})(z + \tfrac{1}{3})$$

– Stability roots of $Q(z)$ = poles are $-\frac{2}{3}, -\frac{1}{3}$ whose magnitude < 1.

thus stable

– Given input $x[n] = (\frac{1}{3})^n u[n] \xrightarrow{z} X(z) = \frac{z}{z - 1/3}$

AND $$Y(z) = H(z)X(z) = \frac{z^2}{z^2 + z + \frac{2}{9}} \cdot \frac{z}{z - 1/3} = \frac{z^3}{(z + \frac{2}{3})(z + \frac{1}{3})(z - 1/3)}$$

To find $y[n]$ do PFE of $\frac{Y(z)}{z}$

$$\frac{A}{z + 2/3} + \frac{B}{z + 1/3} + \frac{C}{z - 1/3} = \frac{z^2}{(z + \frac{2}{3})(z + \frac{1}{3})(z - 1/3)}$$

$$A = \left.\frac{z^2}{(z + 1/3)(z - 1/3)}\right|_{z = -2/3} =$$

$$B = \left.\frac{z^2}{(z + \frac{2}{3})(z - 1/3)}\right|_{z = -1/3} =$$

$$C = \left.\frac{z^2}{(z + \frac{2}{3})(z + 1/3)}\right|_{z = 1/3} =$$

- Then $y[n] = A\left(-\frac{2}{3}\right)^n u[n] + B\left(-\frac{1}{3}\right)^n u[n] + C\left(\frac{1}{3}\right)^n u[n]$ 21 7