Lecture 24: State Space Analysis in CT

- Thus far we have only considered single input, single output (SISO) systems.

$$x(t) \to \boxed{} \to y(t) \qquad \begin{array}{l} x: \mathbb{R} \to \mathbb{R} \\ y: \mathbb{R} \to \mathbb{R} \end{array}$$

Described by: N^{th} order LCCDE, impulse response $h(t)$, transfer function $H(s)$, and if stable, frequency response $H(j\omega)$.

- In more complex systems we may be interested in
  - multiple inputs
  - multiple outputs,
  - internal signals
- Another name for signal is "state" and we actually have had multiple states in our systems. To see this consider a second order LCCDE

$$a \frac{d^2y}{dt^2}(t) + b \frac{dy}{dt}(t) + c\, y(t) = x(t)$$

This can be represented by two first order LCCDE

Let $z_1(t) = y(t)$ and $z_2(t) = \frac{dy}{dt} = \frac{dz_1}{dt}$

then $\frac{dz_2}{dt} = \frac{d^2y}{dt}$ and the LCCDE is $a\frac{dz_2}{dt} + b z_2 + c z_1 = x$

solving for $\frac{dz_2}{dt} = -\frac{b}{a} z_2(t) - \frac{c}{a} z_1(t) + \frac{1}{a} x(t)$

and noting $\frac{dz_1}{dt} = z_2$ gives the system of ODEs.

$$\dot{z}_1(t) = z_2(t)$$
$$\dot{z}_2(t) = -\frac{b}{a} z_2(t) - \frac{c}{a} z_1(t) + \frac{1}{a} x(t)$$

It is traditional to use "dot" notation in state space since there is only first derivatives.

Collecting z_1 & z_2 into a vector $z = \begin{bmatrix} z_1 \\ z_2 \end{bmatrix}$

$$\dot{z} = \begin{bmatrix} 0 & 1 \\ -\frac{c}{a} & -\frac{b}{a} \end{bmatrix} z + \begin{bmatrix} 0 \\ 1/a \end{bmatrix} x$$

the z_1, z_2 are called internal state variables,

$$= A z + B x \quad \text{State Eq}$$

$$y = \begin{bmatrix} 1 & 0 \end{bmatrix} z = C z \quad \text{Observation Eq}$$

A is a 2×2 matrix

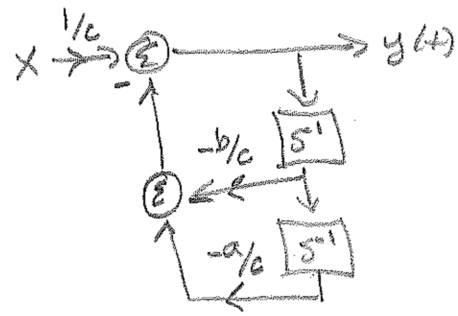
B is a 2×1 matrix

C is a 1×2 matrix, "observation"

— To see where these internal states have been hiding consider the direct form I implementation of this system

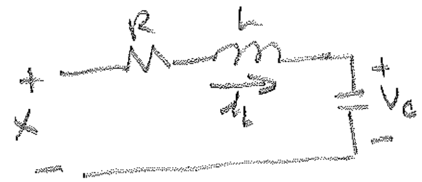
$$a\iint y + b\int y + cy = x$$

$$y = -\frac{a}{c}\iint y - \frac{b}{c}\int y + \frac{1}{c}x$$



the output of the integrators correspond to internal state variables.

— In a circuit with more than two energy storage elements



i_L & V_c are internal states

By choosing to treat one, or the other or both, I get a different output,

i.e. $y = \begin{bmatrix} i_L \\ V_c \end{bmatrix}$

— In the general case a Linear State Space representation is

$$\dot{z} = Az + Bx$$
$$y = Cz + Dx$$

$z \in \mathbb{R} \to \mathbb{R}^n$ states

$x \in \mathbb{R} \to \mathbb{R}^m$ inputs

$y \in \mathbb{R} \to \mathbb{R}^p$ outputs

$\to \mathbb{R}$ could be $\to \mathbb{C}$

with $A \in \mathbb{R}^{n\times n}$ $B \in \mathbb{R}^{n\times m}$ $C \in \mathbb{R}^{p\times n}$ $D \in \mathbb{R}^{p\times m}$

- This gives us a way to represent and analyze linear systems with multiple inputs and multiple outputs,
- They also give a fuller description to the system (controlability + observability)

— Some immediate questions arise
- how do we find state equations?
- how do we do the equivalent of convolution?
- Does Laplace still work?
- what does this gain us?

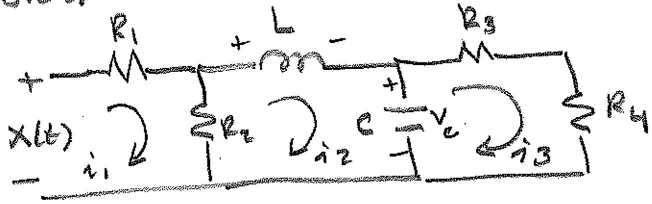
Note; that when $m=1$, $p=1$ and $n=$ order of system we have the same SISO system we have been studying.

i.e. a LCCDE.

- Lets look at three ways to obtain a state equation.

1) Electrical Circuits
   a) choose all capacitor voltages and inductor currents as state variables,
   b) Choose a set of loop currents and express the state variables in terms of these
   c) Write loop equations and eliminate all but state variables.

Example;



a) Let z_1 = inductor curret
z_2 = capacitor voltage

b) $z_1 = i_2$ and $z_2 = V_c$ (5)

c) loop equations

$$R_1 i_1 + R_2(i_1 - i_2) = X \quad (1)$$
$$R_2(i_1 - i_2) + L\dot{i}_2 + V_c = 0 \quad (2)$$
$$-z_2 + R_3 i_3 + R_4 i_3 = 0 \quad (3)$$
$$C\dot{V}_c = i_2 - i_3 \quad (4)$$

- Subst (5) → (1) gives $i_1 = \frac{R_2}{R_1+R_2} z_1 + \frac{1}{R_1+R_2} X$ (6)

- Subst (5) → (2) gives $R_2 i_1 - R_2 z_1 + L\dot{z}_1 + z_2 = 0$ (7)

- Solve (3) for i_3 gives $i_3 = \frac{1}{R_3+R_4} z_2$ (8)

- Solve (4) $\dot{z}_2 = \frac{1}{C} z_1 - \frac{1}{C} i_3$ (9)

- Subst (8) → (9) gives $\dot{z}_2 = \frac{1}{C} z_1 - \frac{1}{C(R_3+R_4)} z_2$

- (6) → (7) gives $\dot{z}_1 = \left(\frac{R_2}{L} + \frac{R_2}{L(R_1+R_2)}\right) z_1 - \frac{1}{L} z_2 + \frac{R_2}{R_1+R_2} X.$

2) From $H(s)$ a) implement as BD in terms of integrators

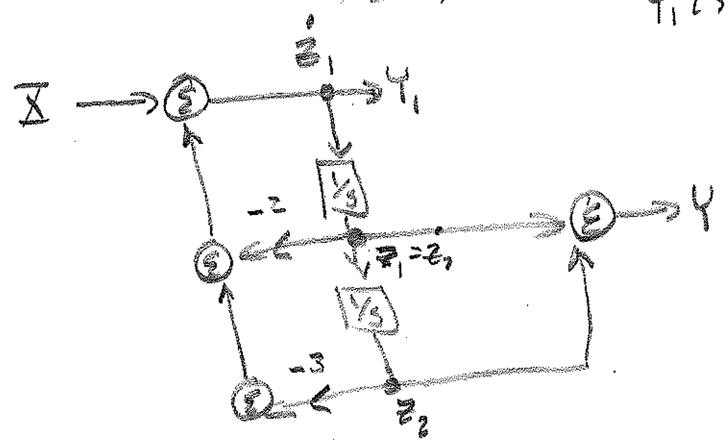
b) choose output of integrators as state variables

c) write down state space equations.

Example: $H(s) = \dfrac{s+1}{s^2+2s+3} = \dfrac{s^2}{s^2+2s+3} \cdot \left(\dfrac{1}{s}+\dfrac{1}{s^2}\right) = H_1(s)\,H_2(s)$

$H_1(s) = \dfrac{1}{1+2s^{-1}+3s^{-2}} = \dfrac{Y_1(s)}{X(s)} \Rightarrow Y_1(s) + 2s^{-1}Y(s) + 3s^{-2}Y(s) = X(s)$

$Y_1(s) = -2s^{-1}Y(s) - 3s^{-2}Y(s) + X(s)$



$H_2(s) = \left(\dfrac{1}{s} + \dfrac{1}{s^2}\right) = \dfrac{Y(s)}{Y_1(s)} \Rightarrow Y(s) = \dfrac{1}{s}Y_1(s) + \dfrac{1}{s^2}Y_1(s)$

Let $z_1 = \int y_1$ $\quad z_2 = \int z_1$ then

$\dot{z}_1 = x - 2z_1 - 3z_2$ and $\dot{z}_2 = z_1$

with $y = z_1 + z_2$

thus $\dot{z} = \begin{bmatrix} -2 & -3 \\ 1 & 0 \end{bmatrix} z + \begin{bmatrix} 1 \\ 0 \end{bmatrix} x$

$y = \begin{bmatrix} 1 & 1 \end{bmatrix} z$

3) From LCCDE

a) assign state variables to $y, \frac{dy}{dt}, \frac{d^2y}{dt^2}, \ldots$ etc, up to $\frac{d^{N-1}y}{dt^{z}}$

b) write state equations,

Example: $$\frac{d^3y}{dt^3} + \frac{d^2y}{dt^2} - 4\frac{dy}{dt} + 2y = X.$$

Let $z_1 = y$ $z_2 = \frac{dy}{dt}$, $z_3 = \frac{d^2y}{dt^2}$

then $$\dot{z}_3 + z_3 - 4z_2 + 2z_1 = x \Rightarrow \dot{z}_3 = -2z_1 + 4z_2 - z_3 + x$$
$$\dot{z}_1 = z_2 \quad , \quad \dot{z}_2 = z_3$$

AND $$\dot{z} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -2 & 4 & -1 \end{bmatrix} z + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} x$$

- Now, how do we solve state equations?
- First using Laplace.
  - the i^{th} state equation is of form
$$\dot{z}_i = a_{i1} z_1 + a_{i2} z_2 + \cdots a_{in} z_n + b_{i1} x_1 + b_{i2} x_2 + \cdots$$

Taking Laplace transform
$$z_i(t) \overset{\mathcal{L}}{\longleftrightarrow} Z_i(s)$$
$$\dot{z}_i(t) \overset{\mathcal{L}}{\longleftrightarrow} s Z_i(s) - z_i(0)$$
$$x_i(t) \overset{\mathcal{L}}{\longleftrightarrow} X_i(s)$$

then $$s Z_i(s) - z_i(0) = a_{i1} Z_1(s) + a_{i2} Z_2(s) + \cdots a_{in} Z_N(s) + b_{i1} X_1(s) + \cdots$$

collecting all equations $$s Z(s) - z(0) = A Z(s) + B X(s)$$

Rearranging $$(sI - A) Z(s) = z(0) + B X(s)$$

See Matlab toolbox ilaplace

AND $$Z(s) = (sI - A)^{-1} z(0) + (sI - A)^{-1} B X(s)$$

$$z(t) = \underbrace{\mathcal{L}^{-1}\{(sI-A)^{-1} z(0)\}}_{\text{zero-input}} + \underbrace{\mathcal{L}^{-1}\{(sI-A)^{-1} B X(s)\}}_{\text{zero-state.}}$$

— In time Domain $\dot{z} = Az + Bx$

The solution is easy to write down

$$z(t) = \underbrace{e^{At} z(0)}_{\text{zero-input response.}} + \underbrace{\int_0^t e^{A(t-\tau)} B x(\tau)\, d\tau}_{\text{convolution = zero-state response.}}$$

but requires matrix exponentials.

- For $a \in \mathbb{R}$ $\quad e^{at} = 1 + at + \frac{a^2}{2} t^2 + \frac{a^3}{3!} t^3 + \cdots = \sum_{k=0}^{\infty} \frac{a^k t^k}{k!}$

- for $A \in \mathbb{R} \times \mathbb{R}$ similarly

$$e^{At} = I + At + \frac{A^2}{2} t^2 + \cdots = \sum_{k=0}^{\infty} \frac{A^k t^k}{k!}$$

- Consider the Eigen decomposition of A, since A is always square.

$$A = V \Lambda V^{-1} \quad \text{where } V = [v_1\ v_2 \cdots v_n]$$

$$\Lambda = \begin{bmatrix} \lambda_1 & & & 0 \\ & \lambda_2 & & \\ & & \ddots & \\ 0 & & & \lambda_n \end{bmatrix}$$

and $Av_i = \lambda_i v_i \quad v_i = i^{th}$ Eigenvector

$\lambda_i = i^{th}$ Eigenvalue.

since $A^k = (V\Lambda V^{-1})(V\Lambda V^{-1})(\quad)\cdots(V\Lambda V^{-1})$

$= V\Lambda^k V^{-1}$

$$e^{At} = V e^{\Lambda t} V^{-1} \quad \text{where } e^{\Lambda t} = \begin{bmatrix} e^{\lambda_1 t} & & & 0 \\ & e^{\lambda_2 t} & & \\ & & \ddots & \\ 0 & & & e^{\lambda_n t} \end{bmatrix}$$

Note the integral of a matrix $C \in \mathbb{R}^{m \times n}$

$$\int C(t)\, dt = \left[\int C_{ij}(t)\, dt \right]_{m \times n}$$

In our case $C(\tau) = e^{A(t-\tau)} B x(\tau)$

Example: $\dot{z}_1 = z_2$

$\dot{z}_2 = -3z_2 - 2z_1 + x$ or $\dot{z} = Az + Bx$ $A = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix}$ $B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

Assuming zero-initial conditions.

- To find Eigenvalues of A solve $|A - \lambda I| = 0$

$A - \lambda I = \begin{bmatrix} -\lambda & 1 \\ -2 & -3-\lambda \end{bmatrix}$ $|A - \lambda I| = -\lambda(-3-\lambda) + 2$
$= 3\lambda + \lambda^2 + 2$ $\Rightarrow \lambda^2 + 3\lambda + 2 = 0$
$(\lambda+1)(\lambda+2) = 0$
$\lambda = -1, -2.$

- To find Eigenvectors $Av_1 = \lambda_1 v_1 \Rightarrow (A - \lambda_1 I)v_1 = 0$
$Av_2 = \lambda_2 v_2 \Rightarrow (A - \lambda_2 I)v_2 = 0$

gives systems $\begin{bmatrix} 1 & 1 \\ -2 & -2 \end{bmatrix} v_1 = 0$ $\begin{bmatrix} 2 & 1 \\ -2 & -1 \end{bmatrix} v_2 = 0$

whose nullspaces are $v_1 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$ $v_2 = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$

normalizing gives $\frac{v_1}{|v_1|} = \frac{1}{\sqrt{2}}\begin{bmatrix} -1 \\ 1 \end{bmatrix}$ $\frac{v_2}{|v_2|} = \frac{1}{\sqrt{5}}\begin{bmatrix} -1 \\ 2 \end{bmatrix}$

- thus $A = \frac{1}{\sqrt{2}\sqrt{5}} \begin{bmatrix} -1 & -1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} -1 & 0 \\ 0 & -2 \end{bmatrix} \begin{bmatrix} -1 & -1 \\ 1 & 2 \end{bmatrix}^{-1}$

$= \frac{1}{\sqrt{2}\sqrt{5}} \begin{bmatrix} -1 & -1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} -1 & 0 \\ 0 & -2 \end{bmatrix} \begin{bmatrix} -2 & -1 \\ 1 & 1 \end{bmatrix}$

— Now suppose $x(t) = e^{-3t}u(t)$ then $Bx(t) = \begin{bmatrix} 0 \\ e^{-3t}u(t) \end{bmatrix}$

$e^{A(t-\tau)} Bx(\tau) = \frac{1}{\sqrt{2}\sqrt{5}} \begin{bmatrix} -1 & -1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} e^{-(t-\tau)} & 0 \\ 0 & e^{-2(t-\tau)} \end{bmatrix} \begin{bmatrix} -2 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ e^{-3\tau}u(\tau) \end{bmatrix}$

$= \frac{1}{\sqrt{2}\sqrt{5}} \begin{bmatrix} -1 & -1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} e^{-(t-\tau)} & 0 \\ 0 & e^{-2(t-\tau)} \end{bmatrix} \begin{bmatrix} -2e^{-3\tau}u(\tau) \\ e^{-3\tau}u(\tau) \end{bmatrix}$

$= \frac{1}{\sqrt{2}\sqrt{5}} \begin{bmatrix} -1 & -1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} -2e^{-(t-\tau)}e^{-3\tau} \\ e^{-2(t-\tau)}e^{-3\tau} \end{bmatrix}$

$= \frac{1}{\sqrt{2}\sqrt{5}} \begin{bmatrix} 2e^{-(t-\tau)}e^{-3\tau} - e^{-2(t-\tau)}e^{-3\tau} \\ -2e^{-(t-\tau)}e^{-3\tau} + 2e^{-2(t-\tau)}e^{-3\tau} \end{bmatrix}$ gives $z \int$ for z_1 & z_2