ECE 4624: Meeting 8

DT Fourier Series and Fourier Transform

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Today we will review the Fourier representation of DT signals and Fourier analysis of stable DT systems.

READING:

- PM 4.3 - 4.5
- ECE 2714 Supplementary Notes Chapters 15 and 17

Topics:

- DT Fourier Series / Discrete Fourier Transform
- DT Fourier Transform
- Frequency Domain Analysis of DT systems

A DT signal $x[n]$ is periodic if $x[n] = x[n + kN]$ for integer multiple k and fundamental period $N \in \mathbb{Z}$.

- the fundamental frequency is $\omega_0 = \frac{2\pi}{N}$ rad/sec

When $x[n]$ is periodic the complex base of the Eigenfunction becomes $z_k = e^{jk\omega_0}$, and the decomposition is a finite sum.

This gives the input-output relationship for a stable DT LTI system as

$$x[n] = \sum_{k=N_0}^{N_0+N-1} a_k e^{jk\omega_0 n} \longrightarrow y[n] = \sum_{k=N_0}^{N_0+N-1} H\left(e^{jk\omega_0}\right) a_k e^{jk\omega_0 n}$$

where $H\left(e^{jk\omega_0}\right)$ are the Eigenvalues or DT frequency response.

Fourier Series Synthesis Equation

$$x[n] = \sum_{k=N_0}^{N_0+N-1} a_k e^{jk\omega_0 n}$$

- the a_k are the Fourier series coefficients

Fourier Series Analysis Equation

$$a_k = \frac{1}{N} \sum_{n=N_0}^{N_0+N-1} x[n] e^{-j\frac{2\pi}{N}kn}$$

Finding the Fourier series coefficients using linear algebra

$$
\begin{aligned}
x[N_0] &= \sum_{k=N_0}^{N_0+N-1} a_k e^{jk\omega_0 N_0} \\
x[N_0+1] &= \sum_{k=N_0}^{N_0+N-1} a_k e^{jk\omega_0 (N_0+1)} \\
\vdots \quad &= \quad \vdots \\
x[N_0+N-1] &= \sum_{k=N_0}^{N_0+N-1} a_k e^{jk\omega_0 (N_0+N-1)}
\end{aligned}
$$

Relationship between Fourier series coefficients and Discrete Fourier Transform

Given a finite-length sequence of real or complex numbers $x[n]$, indexed from 0 to $N-1$, the *Discrete Fourier Transform* or DFT is given by

$$X[k] = \sum_{n=0}^{N-1} x[n] e^{-j\frac{2\pi}{N}kn}$$

for $k = 0, 1, 2, \cdots, N-1$. When N is a power of 2, an efficient algorithm to compute this result exists and is called the *Fast Fourier Transform* or FFT.

Note the similarity to the DT Fourier Series when $N_0 = 0$

$$a_k = \frac{1}{N} \sum_{n=N_0}^{N_0+N-1} x[n] e^{-j\frac{2\pi}{N}kn} = \frac{1}{N} \sum_{n=0}^{N-1} x[n] e^{-j\frac{2\pi}{N}kn} = \frac{1}{N} X[k]$$

Spectrum

- a plot of $|a_k|$ versus k is called the amplitude spectrum
- a plot of $\angle a_k$ versus k is called the phase spectrum
- together they are just the spectrum
- a plot of $|a_k|^2$ versus k is called the power spectrum

Properties of the DT Fourier Series

Given two signals $x[n]$ and $y[n]$ periodic in N with $\omega_0 = \frac{2\pi}{N}$, having DT Fourier coefficients a_k and b_k respectively.

- Linearity. The coefficients of the signal

  $$z[n] = Ax[n] + By[n] \text{ for constants } A, B$$

  are $Aa_k + Bb_k$
- Index Shifting. The coefficients of

  $$z[n] = x[n - n_0] \text{ are } e^{-jk\omega_0 n_0} a_k$$

  that is, it adds a phase shift.
- Frequency Shift. The coefficients of

  $$z[n] = x[n]e^{jm\omega_0 n} \text{ are } a_{k-m}$$

- Index Reversal. The coefficients of

  $$z[n] = x[-n] \text{ are } a_{-k}$$

Properties of the DT Fourier Series

- Multiplication. The coefficients of

$$z[n] = x[n] \cdot y[n] \text{ are } \sum_{m=N_0}^{N_0+N-1} a_m \cdot b_{k-m}$$

  the discrete convolution of the individual signals' coefficients.

- Convolution. The coefficients of

$$z[n] = x[n] * y[n] \text{ are } N a_k b_k$$

- Conjugate Symmetry. The coefficients of

$$z[n] = x^*[n] = \Re x[n] - j\Im x[n] \text{ are } a^*_{-k}$$

  A consequence of this property is that real, even signals have real, even a_k; and real, odd signals have purely imaginary, odd a_k. Thus if $x[n]$ is real $|a_k|$ is an even periodic function of k and $\angle a_k$ is an odd periodic function of k.

Properties of the DT Fourier Series

- Parseval's Relation. The power of the signal with Fourier series coefficients is

$$\frac{1}{N}\sum_{n=N_0}^{N_0+N-1} |x[n]|^2 \; dt = \sum_{k=N_0}^{N_0+N-1} |a_k|^2$$

DT Fourier Transform

- Recall the complex exponential z^n for $z \in \mathbb{C}$ is the Eigenfunction of DT LTI systems. If we can decompose an input into a sum of such signals, we can easily determine the output using the superposition principle. We consider such a decomposition when the input is now aperiodic, called the DT Fourier Transform (DTFT).
- In contrast to the DT Fourier series, in this case the complex exponent of the Eigenfunction becomes $z = e^{j\omega}$ a continuous variable, and the decomposition is an uncountably infinite sum (integral). This gives the input-output relationship for a stable DT LTI system as

$$x[n] = \frac{1}{2\pi} \int_{2\pi} X\left(e^{j\omega}\right) e^{j\omega n} \, d\omega \;\longrightarrow\; y[n] = \frac{1}{2\pi} \int_{2\pi} H\left(e^{j\omega}\right) X\left(e^{j\omega}\right) e^{j\omega n} \, d\omega$$

where $H\left(e^{j\omega}\right)$ are the Eigenvalues, again called the *frequency response*.