ECE 4624: Meeting 7

Time Domain Analysis of DT LTI Systems

Chris Wyatt

2025-09-18

Today we review how to perform time-domain analysis of DT LTI systems using convolution and review the concept of an Eigenfunction.

READING:

- PM 2.3 and 2.6
- ECE 2714 Supplementary Notes Chapters 9 and 13

Topics:

- Convolution
- DT time-domain analysis
- Correlation
- Eigen-Function for DT systems

Review DT LTI systems and superposition property

If a DT system is LTI then the superposition property holds. Given a system where

$$x_i[n] \mapsto y_i[n] \ \forall \ i$$

then

$$\sum_i a_i x_i[n] \mapsto \sum_i a_i y_i[n]$$

Convolution Sum

Let $h[n]$ be the DT impulse response. Then if the system is LTI the output for an input $x[n]$ is given by

$$y[n] = \sum_{m=-\infty}^{\infty} x[m]h[n-m] = x[n] * h[n]$$

This is called the convolution sum.

Graphical View of the Convolution Sum.

Example

Find the convolution of two unit step functions.

Example

Find the convolution if the signals

$$x_1[n] = \gamma^n \, u[n] \text{ for } \gamma \in \mathbb{R}$$

and

$$x_2[n] = u[n]$$

Properties of DT Convolution

There are several useful properties of convolution. We do not prove these here, but it is not terribly difficult to do so. Given signals $x_1[n]$, $x_2[n]$, and $x_3[n]$:

Commutative Property The ordering of the signals does not matter.

$$x_1[n] * x_2[n] = x_2[n] * x_1[n]$$

Distributive Propery Convolution is distributed over addition.

$$x_1[n] * (x_2[n] + x_3[n]) = (x_1[n] * x_2[n]) + (x_1[n] * x_3[n])$$

Associative Property The order of convolution does not matter.

$$x_1[n] * (x_2[n] * x_3[n]) = (x_1[n] * x_2[n]) * x_3[n]$$

Index Shift Given $x_3[n] = x_1[n] * x_2[n]$ then for index shifts $m_1, m_2 \in \mathbb{R}$

$$x_1[n - m_1] * x_2[n - m_2] = x_3[n - m_1 - m_2]$$

Multiplicative Scaling Given $x_3[n] = x_1[n] * x_2[n]$ then for constants $a, b \in \mathbb{C}$

$$(a\, x_1[n]) * (b\, x_2[n]) = a\, b\, x_3[n]$$

Example

Suppose an LTI system has an impulse response

$$h[n] = \delta[n+1] - 2\delta[n] + \delta[n-1]$$

What is the output if the input is

$$x[n] = \left(\frac{1}{2}\right)^{|n|}$$

DT Convolution of Finite-Length Signals

- For finite-length signals, DT convolution gives us an algorithm to determine their convolution.
- Suppose the signal x_1 is non-zero only over the interval $[N_1, M_1]$, and the signal x_2 is non-zero only over the interval $[N_2, M_2]$.
- The *length* of the signals are $L_1 = M_1 - N_1 + 1$ and $L_2 = M_2 - N_2 + 1$ respectively.
- The non-zero terms of the convolution sum (when the signals overlap) is then the range $[N_1 + N_2, M_1 + M_2]$ and the sum can be truncated as:

$$x_1[n] * x_2[n] = \sum_{m=N_1+N_2}^{M_1+M_2} x_1[m]x_2[n-m]$$

- It is common to shift both signals so that they both start at index 0 , zero-padding them both to have length $L = L_1 + L_2 - 1$.
- Then the convolution becomes

$$y = x_1 * x_2 = \sum_{m=0}^{L-1} x_1[m]x_2[L+n-m]$$

  where the indexing of x_2 is modulo the signal length.
- The resulting signal after convolution, y, is also of length L, and can then be shifted back to start at $N_1 + N_2$.

Example

Compute the convolution of the DT signals $\{1, -1, 1\}$ and $\{1, 1, 1, 1\}$

Code for Convolution of two finite length signals

For the example above:

```
const unsigned int L = 6; //L1=3, L2=4, so L=6
double x1[L] = {1., -1., 1., 0, 0, 0};
double x2[L] = {1., 1., 1., 1., 0, 0};
double y[L];

for(int n = 0; n < L; n++){
   double sum = 0.;
   for(int m = 0; m < L; m++){
      int idx = (L+n-m) % L;
      sum += x1[m]*x2[idx];
   }
   y[n] = sum;
}
```

Correlation

An operation very similar to convolution is *correlation*. Given two signals $x[n]$ and $y[n]$

- their cross-correlation is

$$r_{xy}[l] = \sum_{n=-\infty}^{\infty} x[n]y[n-l]$$

  where l is called the "lag".

- note that

$$r_{yx}[l] = \sum_{n=-\infty}^{\infty} y[n]x[n-l] = r_{xy}[-l]$$

- if $x = y$ we get the auto-correlation $r_{xx}[l]$
- the normalized autocorrelation and cross-correlation are given by

$$\rho_{xx}[l] = \frac{r_{xx}[l]}{r_{xx}[0]}$$

$$\rho_{xy}[l] = \frac{r_{xy}[l]}{\left(r_{xx}[l] \cdot r_{yy}[l]\right)^{\frac{1}{2}}}$$

Correlation of input and output of LTI system

Given an LTI system with impulse response $h[n]$ then

$$r_{yx}[l] = h[n] * r_{xx}[l]$$

Applications of correlation

- "radar" equation
- The Fourier transform of the auto/cross correlation is the power spectral density. This forms the basis of working with random signals.

Eigenfunction for DT-LTI Systems

Let $x[n] = z^n$ for $z \in \mathbb{C}$, then $y[n] = h[n] * x[n] = x[n] * h[n]$ and by the definition of DT convolution

$$\begin{aligned} y[n] &= \sum_{m=-\infty}^{\infty} h[m]x[n-m] \\ &= \sum_{m=-\infty}^{\infty} h[m]z^{n-m} = \sum_{m=-\infty}^{\infty} h[m]z^n z^{-m} \\ &= z^n \sum_{m=-\infty}^{\infty} h[m]z^{-m} \\ &= z^n H(z) \end{aligned}$$

where $H(z) = \sum_{m=-\infty}^{\infty} h[m]z^{-m}$ is the *Z Transform* of the impulse response, $h[n]$.

$H(z)$ is called the *transfer function* or *Eigenvalue* of the system and z^n is the *Eigenfunction* for DT LTI systems.