

# ECE 4624: Meeting 2

## Continuous-Time (CT) and Discrete-Time (DT) Signals and Systems

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Today we define CT and DT Signals and how to characterize them. Then we review the basic notion of a system as a transformation between signals.

Readings:

- ▶ PM 2.1 and 2.2
- ▶ Chapter 2 ECE 2714 Supplementary Notes
- ▶ Chapter 3 ECE 2714 Supplementary Notes

Topics:

- |                                |                                |
|--------------------------------|--------------------------------|
| ▶ Signals as Functions         | ▶ Basic Systems                |
| ▶ Primitive Models             | ▶ Input Signals                |
| ▶ Impulse Function             | ▶ Output or Response Signals   |
| ▶ Step Function                | ▶ Impulse Response             |
| ▶ Complex Exponential          | ▶ Step Response                |
| ▶ Basic Signal Transformations | ▶ Complex Exponential Response |
| ▶ Characterization of Signals  |                                |

# Signals as Functions

In order to reason about signals mathematically we need a representation or *model*. Signals are modeled as functions, mappings between sets

$$f : A \rightarrow B$$

where  $A$  is a set called the *domain* and  $B$  is a set called the *range*.

The most basic classification of signals depends on the sets that makeup the domain and co-domain. We will be interested in two versions of the domain, the reals denoted  $\mathbb{R}$  and the integers denoted  $\mathbb{Z}$ . We will be interested in two versions of the co-domain, the reals  $\mathbb{R}$  and the set of complex numbers  $\mathbb{C}$ .

# Analog Signals

If the function  $f : \mathbb{R} \rightarrow \mathbb{R}$ , we call this an analog or real, continuous-time signal, e.g. a voltage at time  $t \in \mathbb{R}$ ,  $v(t)$ . We will write these as  $x(t)$ ,  $y(t)$ , etc. The units of  $t$  are seconds.

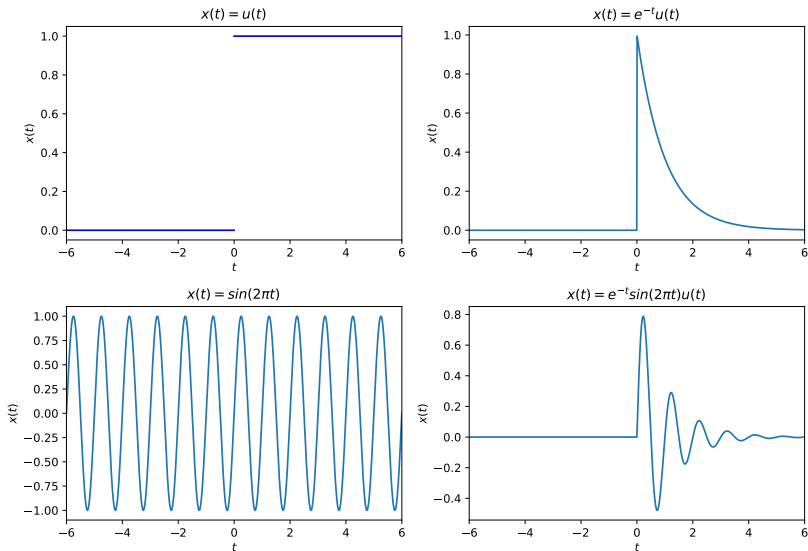


Figure 1: Example plots of analog signals.

## Real, Discrete-time Signal

If the function  $f: \mathbb{Z} \rightarrow \mathbb{R}$ , we call this a real, discrete-time signal, e.g. the temperature every day at noon. We will write these as  $x[n]$ ,  $y[n]$ , etc. Note  $n$  is dimensionless.

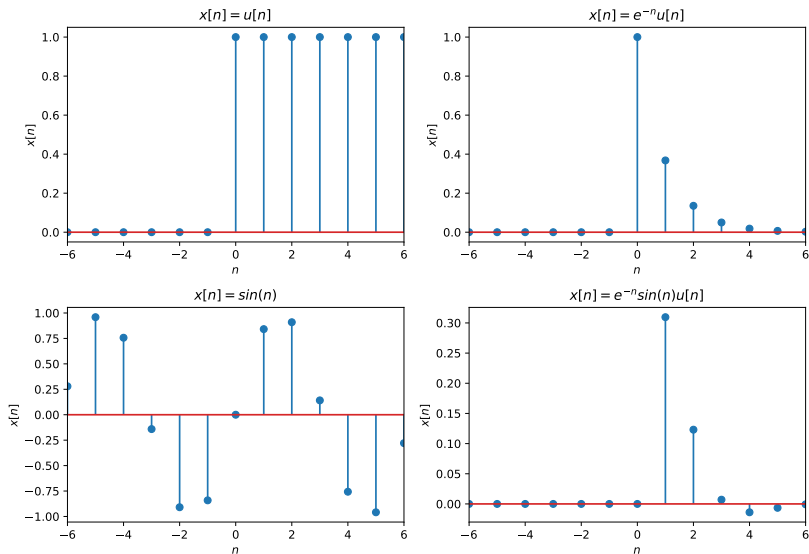


Figure 2: Example plots of real-valued, discrete-time signals.

## Some other possibilities

►  $f : \mathbb{R} \rightarrow \mathbb{Z}$ , discrete-valued, continuous-time signals

►  $f : \mathbb{Z} \rightarrow \mathbb{Z}$ , discrete-valued, discrete-time signals

The co-domain can also be complex.

►  $f : \mathbb{R} \rightarrow \mathbb{C}$ , complex-valued, continuous-time signals, e.g.

$$x(t) = e^{j\omega t} = \cos(\omega t) + j \sin(\omega t)$$

►  $f : \mathbb{Z} \rightarrow \mathbb{C}$ , complex-valued, discrete-time signals, e.g.

$$x[n] = e^{j\omega n} = \cos(\omega n) + j \sin(\omega n)$$

Since the domains  $\mathbb{R}$  and  $\mathbb{Z}$  are usually interpreted as time, we call these *time-domain* signals. In the time-domain, when the co-domain is  $\mathbb{R}$  we call these real signals. All physical signals are real. However pairs of complex signals are important mathematical models in linear systems theory.

# Digital Signals

We are ultimately interested in signals

$$f : \mathbb{Z} \rightarrow \mathbb{Q}$$

where the range  $\mathbb{Q} \subset \mathbb{Z}^+$  is the set of unsigned  $N$ -bit values

$$\sum_{i=-A}^B b_i 2^{-i}$$

- ▶  $b_i \in \{0, 1\}$  is the  $i$ th bit
- ▶  $b_{-A}$  is the most-significant bit (MSB)
- ▶  $b_B$  is the least-significant bit (LSB)
- ▶ the location of the decimal point is implied
- ▶ signed values have various representations

However, these functions are hard to work with mathematically, so we generally work with DT signals and look at practical consequences separately.

# Primitive Models

We mathematically model signals by combining elementary/primitive functions, for example:

- ▶ polynomials:  $x(t) = t$ ,  $x(t) = t^2$ , etc.
- ▶ transcendental functions:  $x(t) = e^t$ ,  $x(t) = \sin(t)$ ,  $x(t) = \cos(t)$ , etc.
- ▶ piecewise functions, e.g.

$$x(t) = \begin{cases} f_1(t) & t < 0 \\ f_2(t) & t \geq 0 \end{cases}$$



# Unit Step Function

To model turning signals on we often use a piecewise function, the unit-step

In continuous-time:

$$u(t) = \begin{cases} 0 & t < 0 \\ 1 & t \geq 0 \end{cases}$$

Note: some texts define the step function at  $t = 0$  to be 0 or  $\frac{1}{2}$ .

In discrete-time:

$$u[n] = \begin{cases} 0 & n < 0 \\ 1 & n \geq 0 \end{cases}$$

Note: there is no ambiguity about the value at  $n = 0$ .

## Unit Impulse Function

An important signal in linear system theory is the impulse function:

It is easily defined in discrete-time:

$$\delta[n] = \begin{cases} 0 & n \neq 0 \\ 1 & n = 0 \end{cases}$$

However in continuous-time it is defined using generalized functions. Two definitions:

$$\delta(t) = \lim_{\epsilon \rightarrow 0} \begin{cases} \frac{1}{2\epsilon} & |t| < \epsilon \\ 0 & \text{else} \end{cases}$$

$$\delta(t) = \lim_{\epsilon \rightarrow 0} \frac{1}{\sqrt{2\pi\epsilon}} e^{-\frac{t^2}{2\epsilon^2}}$$

Note the area under each definition is always one.

## CT Impulse Function cont.

In practice we can often use the following heuristic definition and some properties, without worrying about the distribution functions.

$$\delta(t) = \begin{cases} 0 & t \neq 0 \\ \infty & t = 0 \end{cases}$$

- ▶ The area under the unit impulse is unity since by definition

$$\int_{-\infty}^{\infty} \delta(t) dt = 1$$

- ▶ Sampling property:  $x(t)\delta(t - t_0) = x(t_0)\delta(t - t_0)$
- ▶ Sifting Property:

$$\int_a^b x(t)\delta(t - t_0) dt = x(t_0)$$

for any  $a < t_0 < b$ .

## Relationships between the unit step and impulse in CT

We previously defined the unit step function. The impulse can be defined in terms of the step:

$$\delta(t) = \frac{du}{dt}$$

and vice-versa

$$u(t) = \int_{-\infty}^t \delta(\tau) d\tau$$

using the notion of distributions, e.g.

$$u(t) = \int_{-\infty}^t \delta(\tau) d\tau = \lim_{\epsilon \rightarrow 0} \int_{-\infty}^t \frac{1}{\sqrt{2\pi\epsilon}} e^{-\frac{\tau^2}{2\epsilon^2}} d\tau = \lim_{\epsilon \rightarrow 0} \frac{1}{2} \left( 1 + \operatorname{erf} \left( \frac{t}{\sqrt{2}\epsilon} \right) \right)$$

## Variations on the step and impulse in CT

We can apply additional transformations to the impulse and step functions to get other useful signals, e.g.

► ramp

$$r(t) = \int_{-\infty}^t u(\tau) d\tau = tu(t)$$

► causal pulse of width  $\epsilon$

$$p(t) = u(t) - u(t - \epsilon)$$

► non-causal pulse of width  $2\epsilon$

$$p(t) = u(t + \epsilon) - u(t - \epsilon)$$

## Relationships between the unit step and impulse in DT

Some useful properties of the DT impulse function are:

► Energy is 1:  $\sum_{n=-\infty}^{\infty} \delta[n] = 1$

► Sampling:  $x[n]\delta[n - n_0] = x[n_0]\delta[n - n_0]$

► Sifting:  $\sum_{n=-\infty}^{\infty} x[n]\delta[n - n_0] = x[n_0]$

The impulse can be defined in terms of the step and vice-versa:

$$\delta[n] = u[n] - u[n - 1]$$

$$u[n] = \sum_{m=-\infty}^n \delta[m]$$

or

$$u[n] = \sum_{k=0}^{\infty} \delta[n - k]$$

## Complex Exponential in CT

One of the most important signals in systems theory is the complex exponential:

$$x(t) = C e^{at}$$

where the parameters  $C, a \in \mathbb{C}$  in general.

► When  $C$  and  $a$  are both real ( $\Im(C) = \Im(a) = 0$ ), we have the familiar exponential.

## Complex Exponential in CT cont.

To get the pure sinusoidal case, let  $C \in \mathbb{R}$  and  $a$  be purely imaginary:  $a = j\omega_0$ :

$$x(t) = Ce^{j\omega_0 t}$$

where  $\omega_0$  is the frequency (in radians/sec). This is called the complex sinusoid.

By Euler's identity:

$$e^{j\omega_0 t} = \cos(\omega_0 t) + j \sin(\omega_0 t)$$

and

$$\Re(x(t)) = \cos(\omega_0 t) = \frac{1}{2} (e^{j\omega_0 t} + e^{-j\omega_0 t})$$

$$\Im(x(t)) = \sin(\omega_0 t) = \frac{1}{2j} (e^{j\omega_0 t} - e^{-j\omega_0 t})$$

are both real sinusoids.



## Complex Exponential in CT cont.

When the parameter  $C$  is complex we get a phase shift. Again let  $a = j\omega_0$ . When  $C$  is complex we can write it as  $C = Ae^{j\phi}$  where  $A = |C|$  and  $\phi = \angle C$ . Then

$$x(t) = Ae^{j\phi}e^{j\omega_0 t} = Ae^{j(\omega_0 t + \phi)}$$

and

$$\Re(x(t)) = A \cos(\omega_0 t + \phi)$$

$$\Im(x(t)) = A \sin(\omega_0 t + \phi)$$

Since  $\sin$  is a special case of  $\cos$ , i.e.  $\cos(\theta) = \sin(\theta + \frac{\pi}{2})$ , the general real sinusoid is

$$A \cos(\omega_0 t + \phi)$$

- ▶  $A$  is called the amplitude
- ▶  $\omega_0$  is again the frequency in radians/sec.
- ▶  $\phi$  is called the phase shift and is related to a time shift  $T_s$  by

$$\phi = \omega_0 T_s$$

## Complex Exponential in DT

The DT Complex Exponential is defined in a similar fashion to the CT version, but with some important differences.

The general DT complex exponential is given by the expression:

$$x[n] = Ce^{\beta n}$$

where in general  $C \in \mathbb{C}$  and  $\beta \in \mathbb{C}$ .

It is usually more convenient to write this as

$$x[n] = C\alpha^n$$

where  $\alpha = e^{j\theta}$  is a complex number  $\alpha = \cos(\theta) + j\sin(\theta)$ .

# Basic Transformations (focusing on DT)

We can also apply transformations to signals to increase their modeling flexibility.

- ▶ magnitude scaling

$$x_2[n] = ax_1[n]$$

for  $a \in \mathbb{R}$ .

- ▶ time differences

$$x_2[n] = x_1[n] - x_1[n-1]$$

- ▶ running sums

$$x_2[n] = \sum_{m=-\infty}^n x_1[m]$$

- ▶ sums

$$y[n] = \sum_i x_i[n]$$

## Basic DT Transformations cont.

- ▶ multiplication (modulation)

$$y[n] = x_1[n]x_2[n]$$

- ▶ time index shift

$$x_2[n] = x_1[n + m]$$

- ▶ if  $m < 0$  it is called a *delay*

- ▶ if  $m > 0$  it is called an *advance*

- ▶ time reversal

$$x_2[n] = x_1[-n]$$

## Basic DT Transformations cont.

### ► decimation

$$y[n] = x[mn]$$

for  $m \in \mathbb{Z}^+$ .

- e.g. for  $m = 2$  only keep every other sample
- e.g. for  $m = 3$  only keep every third sample
- etc.

### ► interpolation

$$y[n] = \begin{cases} x\left[\frac{n}{m}\right] & n = 0, \pm m, \pm 2m \dots \\ 0 & \text{else} \end{cases}$$

When  $m = 2$  this inserts a zero sample between every sample of the signal.

# Characterization of CT Signals

There are a few basic ways of characterizing signals.

- ▶ A CT signal is causal if  $x(t) = 0 \ \forall t < 0$ .
- ▶ A CT signal is anti-causal or acausal if  $x(t) = 0 \ \forall t \geq 0$ . A signal can be written as the sum of a causal and anti-causal signal.
- ▶ A CT signal is periodic if  $x(t) = x(t+T) \ \forall t$  for a fixed parameter  $T \in \mathbb{R}$  called the *period*. The simplest periodic signals are those based on the sinusoidal functions.
- ▶ A CT signal is even if  $x(t) = x(-t) \ \forall t$ .
- ▶ A CT signal is odd if  $x(t) = -x(-t) \ \forall t$ .

Any CT signal can be written in terms of an even and odd component

$$x(t) = x_e(t) + x_o(t)$$

where  $x_e(t) = \frac{1}{2} \{x(t) + x(-t)\}$  and  $x_o(t) = \frac{1}{2} \{x(t) - x(-t)\}$ .

## Characterization of CT Signals cont.

- ▶ The *energy* of a CT signal  $x(t)$  is defined as a measure of the function

$$E_x = \lim_{T \rightarrow \infty} \int_{-T}^T |x(t)|^2 dt .$$

- ▶ The *power* of a CT signal is the energy averaged over an interval as that interval tends to infinity.

$$P_x = \lim_{T \rightarrow \infty} \frac{1}{2T} \int_{-T}^T |x(t)|^2 dt .$$

Signals can be characterized based on their energy or power:

- ▶ Signals with finite, non-zero energy and zero power are called *energy signals*.
- ▶ Signals with finite, non-zero power (and by implication infinite energy) are called *power signals*.

# Characterization of DT Signals

Similarly there are a few basic ways of characterizing DT signals.

- ▶ A DT signal is *causal* if  $x[n] = 0 \ \forall n < 0$ .
- ▶ A DT signal is *anti-causal* or acausal if  $x[n] = 0 \ \forall n \geq 0$ .
- ▶ A DT signal can be written as the sum of a causal and anti-causal signal.
- ▶ A DT signal is periodic if  $x[n] = x[n + N] \ \forall n$  for a fixed period  $N \in \mathbb{Z}$ .
- ▶ A DT signal is even if  $x[n] = x[-n] \ \forall n$ .
- ▶ A DT signal is odd if  $x[n] = -x[-n] \ \forall n$ .

Any DT signal can be written in terms of an even and odd component

$$x[n] = x_e[n] + x_o[n]$$

where  $x_e[n] = \frac{1}{2} \{x[n] + x[-n]\}$  and  $x_o[n] = \frac{1}{2} \{x[n] - x[-n]\}$



## Characterization of DT Signals cont.

Analogous to CT signals, the energy of a DT signal is

$$E_x = \lim_{N \rightarrow \infty} \sum_{-N}^N |x[n]|^2 .$$

The power of a DT signal is the energy averaged over an interval as that interval tends to infinity.

$$P_x = \lim_{N \rightarrow \infty} \frac{1}{2N+1} \sum_{-N}^N |x[n]|^2 .$$

DT Signals with finite, non-zero energy and zero power are called *energy signals*.

DT Signals with finite, non-zero power (and by implication infinite energy) are called *power signals*.

# Systems

A system is an interconnected set of components or sub-systems. Mathematically a system is a transformation,  $T$ , between one or more signals, a rule that maps functions to functions.

► single input - single output (SISO) system.

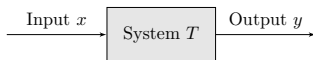


Figure 3: SISO Block Diagram

We will focus on single input - single output systems.

## Other System Types

- ▶ single input - multiple output (SIMO) system



Figure 4: SIMO Block Diagram

- ▶ general case, multiple input - multiple output (MIMO)

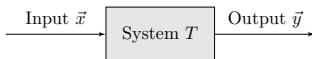


Figure 5: MIMO Block Diagram

MIMO systems can be handled using state-based descriptions.

## Systems cont.

- If both input and output are CT signals, it is a CT system.

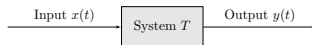


Figure 6: Generic Block Diagram of CT System

- If both input and output are DT signals, it is a DT system.



Figure 7: Generic Block Diagram of a DT System

## Systems cont.

- If input and output are not both CT or DT signals, it is a hybrid CT-DT system.

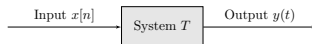


Figure 8: Generic Block Diagram of a Hybrid DT/CT System



Figure 9: Generic Block Diagram of a Hybrid CT/DT System

We will focus on Hybrid and DT systems.

# System Input and Output

- ▶ The input to a SISO system is a signal.
- ▶ The output from a SISO system, or its *response*, is also a signal
- ▶ There are three canonical responses considered:
  - ▶ Impulse Response
  - ▶ Step Response
  - ▶ Complex Exponential or sinusoidal response

For LTI systems the last two can be derived from the first.

- ▶ The impulse response for CT systems is denoted  $h(t)$ .



Figure 10: Impulse Response of CT System

- ▶ The impulse response for DT systems is denoted  $h[n]$ .

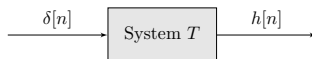


Figure 11: Impulse Response of DT System

# CT system representations

We can mathematically represent, or model, systems multiple ways.

- ▶ purely mathematically - in time domain we generically use
  - ▶ for CT systems: differential equations e.g.

$$y'' + ay' + by = x$$

- ▶ for DT systems: difference equations e.g.

$$y[n] = ay[n-1] + by[n-2] + x[n]$$

- ▶ graphically, using a mixture of math and block diagrams

Mathematical models:

- ▶ provide abstraction, removing (often) irrelevant detail.
- ▶ can be more or less detailed, an *internal* v.s. *external* (block box) description
- ▶ are not unique with respect to instantiation (implementation)
- ▶ are limited to the regime they were designed for

# System properties and classification

Choosing the right kind of system model is important. Here are some important properties that allow us to broadly classify systems.

- ▶ Memory
- ▶ Invertability
- ▶ Causality
- ▶ Stability
- ▶ Time-invariance
- ▶ Linearity

Our focus is on linear, time-invariant (LTI) systems. Such systems can be represented completely by:

- ▶ an LCCDE,
- ▶ an impulse response, or
- ▶ a transfer function.

When stable they can also be represented by their frequency response.



## Reminders and Next Actions

- ▶ Read PM 4.2
- ▶ Refer to as needed Chapters 14 and 16 of ECE 2714 Supplementary Notes
- ▶ Reminder: PS #1 is due Sept. 10