ECE 4624: Meeting 6

DT Systems as Linear, Constant-Coefficient Difference Equations and Impulse Response

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2025-09-08

Today we look at DT systems as difference equations and define the correspondence between linear, constant-coefficient difference equations (LCCDE) and linear, time-invariant (LTI) DT systems. We then see how to determine the impulse response of DT LTI systems. Today's lecture is a review of prerequisite material from ECE 2714.

READING:

- PM 4.1.4 and Chapter 6
- 2714 Supplementary Notes 5 and 7

Topics:

- Properties and Classification of DT systems
- Difference Equations
- Impulse Response of LCCDE

DT system representations

We can mathematically represent, or model, DT systems multiple ways.

- purely mathematically - in time domain we will use
  - linear, constant coefficient difference equations, e.g.

$$y[n] = ay[n-1] + by[n-2] + x[n]$$

  - DT impulse response $h[n]$
- purely mathematically - in frequency domain we will use
  - frequency response
  - transfer function (complex frequency, covered in ECE 3704)
- graphically, using a mixture of math and block diagrams

System properties and classification

Choosing the right kind of system model is important. Here are some important properties that allow us to broadly classify systems.

- Memory
- Invertability
- Causality
- Stability
- Time-invariance
- Linearity

Memory

- The output of a DT system with memory depends on previous or future inputs and is said to be *dynamic*.
- Otherwise the system is memoryless or *instantaneous*, and the output $y[n]$ at index n depends only on $x[n]$.

Invertability

- A system is invertible if there exists a system that when placed in series with the original recovers the input.

$$x[n] \mapsto Ty[n] \mapsto T^{-1}x[n]$$

where T^{-1} is the inverse system of T.

Causality

- A DT system is causal if the output at index n depends on the input for index values at or before n:

$$y[n] \text{ depends on } x[m] \text{ for } m \leq n$$

- While all physical CT systems are causal, practical DT systems may not be since we can use memory to "shift time".
- For CT systems we cannot store the infinite number of values between two time points t_1 and t_2, but we can store the $n_2 - n_1$ values of a DT system between between two indices n_1 and n_2 (assuming infinite precision).

Stability

- A DT system is (BIBO) stable if applying a bounded-input (BI)

$$|x[n]| < \infty \ \forall \ n$$

  results in a bounded-output (BO) $x[n] \mapsto y[n]$ and

$$|y[n]| < \infty \ \forall \ n$$

- Note, bounded in practice is limited by the physical situation, e.g. the number of bits used to store values.

Time-invariance

- A DT system is time(index)-invariant if, given

$$x[n] \mapsto y[n]$$

then an index-shift of the input leads to the same index-shift in the output

$$x[n-m] \mapsto y[n-m]$$

Linearity

- A DT system is linear if the output due to a sum of scaled individual inputs is the same as the scaled sum of the individual outputs with respect to those inputs.

If

$$x_1[n] \mapsto y_1[n] \text{ and } x_2[n] \mapsto y_2[n]$$

then

$$ax_1[n] + bx_2[n] \mapsto ay_1[n] + by_2[n]$$

for constants a and b.

- Note this property extends to sums of arbitrary signals, e.g. if

$$x_i[n] \mapsto y_i[n] \ \forall \ i \in [1 \cdots N]$$

then given N constants a_i, if the system is linear

$$\sum_{i=1}^{N} a_i x_i[n] \mapsto \sum_{i=1}^{N} a_i y_i[n]$$

- This is a very important property, called *superposition*, and it simplifies the analysis of systems greatly.

Difference Equations

- A *difference equation* is a relation among combinations of two DT functions and shifted versions of them.
- Similar to differential equations where the solution is a CT function, the solution to a difference equation is a DT function.

LCCDE

A *linear*, *constant-coefficient*, difference equation (LCCDE) comes in three equivalent forms.

- Delay form.

$$\sum_{k=0}^{N} a_k y[n-k] = \sum_{k=0}^{M} b_k x[n-k]$$

  or

$$a_0 y[n] + a_1 y[n-1] + \cdots a_N y[n-N] = b_0 x[n] + \cdots b_M x[n-M]$$

- Advance form. Let $n \to n+N$, then the delay form becomes

$$\sum_{k=0}^{N} a_k y[n+N-k] = \sum_{k=0}^{M} b_k x[n+N-k]$$

  or

$$a_0 y[n+N] + a_1 y[n+N-1] + \cdots a_N y[n] = b_0 x[n+N] + \cdots b_M x[n+N-M]$$

- Recursive form.

$$y[n] = -\frac{a_1}{a_0} y[n-1] - \cdots - \frac{a_N}{a_0} y[n-N] + \frac{b_0}{a_0} x[n] + \cdots \frac{b_M}{a_0} x[n-M]$$

The *order* of the system is given by max of N and M.

Advance and Delay Operators

- Define the advance operator E^m as shifting a DT function by positive m, i.e. $E^m x[n] = x[n+m]$
- Define the delay operator D^m as shifting a DT function by negative m, i.e. $D^m x[n] = x[n-m]$.

Iterative solution of LCCDEs

To perform an iterative solution we need the difference equation in delay form

$$y[n] = -\left(\frac{a_1}{a_0}y[n-1] + \cdots \frac{a_N}{a_0}y[n-N]\right) + \frac{b_0}{a_0}x[n] + \cdots \frac{b_M}{a_0}x[n-M]$$

Given an input and initial conditions for $y[n_0-1] \cdots y[n_0-N]$ we can simulate $y[n]$ for $n \geq n_0$.

Solution of the homogeneous LCCDE

- The iterative solution does not give us (directly) an analytical expression for the output at arbitrary n. We have to start at the initial conditions and compute our way up to n.
- We now consider an analytical solution when the input is zero, the solution to the *homogeneous* difference equation

$$Q(E)\,y = a_0 y[n+N] + a_1 y[n+N-1] + \cdots a_N y[n] = 0\ .$$

given N sequential auxiliary conditions on y.

- Similar to differential equations, the homogeneous solution depends on the roots of the characteristic equation $Q(E) = 0$ whose roots are either real or occur in complex conjugate pairs.
- Let λ_i be the i-th root of $Q(E) = 0$, then the solution is of the form

$$y[n] = \sum_{i=1}^{N} C_i \lambda_i^n$$

  where the parameters C_i are determined from the auxiliary conditions.
- For a real system (when the coefficients of the difference equation are real) and when the roots are complex $\lambda_{1,2} = |\lambda| e^{\pm j\beta}$, it is cleaner to assume a form for those terms as

$$y[n] = C|\lambda|^n \cos(\beta n + \theta)$$

  for constants C and θ.

Example:

Impulse response from LCCDE

To find the solution to $Q(E)y = P(E)x$ when $x[n] = \delta[n]$ assuming $y[n] = 0$ for $n < 0$, giving the *impulse response* $y[n] = h[n]$.

Step 1: Let y_h be the homogeneous solution to $Q(E)y_h = 0$ for $n > N$.

Step 2: Assume a form for $h[n]$ given by

$$h[n] = \frac{b_N}{a_N}\delta[n] + y_h[n]u[n]$$

Step 3: Using the iterative procedure above find the N auxiliary conditions we need by,

- first, rewrite the equation in delay form and solve for $y[n]$,
- then let $x[n] = \delta[n]$ and manually compute $h[0]$ assuming $h[n] = 0$ for $n < 0$,
- repeating the previous step for $h[1]$, continuing up to $h[N-1]$.

Step 4: Using the auxillary conditions in step 3, solve for the constants in the solution $h[n]$ from step 2.

Note we can check our solution for as many n values as desired using the recursive form.

Examples:

LCCDE and impulse response for IIR filters

LCCDE and impulse response for FIR filters

Examples: