# ECE 4624: Meeting 6

## DT Systems as Linear, Constant-Coefficient Difference Equations and Impulse Response

Chris Wyatt

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Today we look at DT systems as difference equations and define the correspondence between linear, constant-coefficient difference equations (LCCDE) and linear, time-invariant (LTI) DT systems. We then see how to determine the impulse response of DT LTI systems. Today's lecture is a review of prerequisite material from ECE 2714.

READING:

- PM 4.1.4 and Chapter 6
- 2714 Supplementary Notes 5 and 7

Topics:

- Properties and Classification of DT systems
- Difference Equations
- Impulse Response of LCCDE

# DT system representations

We can mathematically represent, or model, DT systems multiple ways.

- purely mathematically - in time domain we will use
  - linear, constant coefficient difference equations, e.g.

$$y[n] = ay[n-1] + by[n-2] + x[n]$$

  - DT impulse response $h[n]$
- purely mathematically - in frequency domain we will use
  - frequency response
  - transfer function (complex frequency, covered in ECE 3704)
- graphically, using a mixture of math and block diagrams

# System properties and classification

Choosing the right kind of system model is important. Here are some important properties that allow us to broadly classify systems.

- Memory
- Invertability
- Causality
- Stability
- Time-invariance
- Linearity

# Memory

- The output of a DT system with memory depends on previous or future inputs and is said to be *dynamic*.
- Otherwise the system is memoryless or *instantaneous*, and the output $y[n]$ at index $n$ depends only on $x[n]$.

# Invertability

- A system is invertible if there exists a system that when placed in series with the original recovers the input.

$$x[n] \mapsto Ty[n] \mapsto T^{-1}x[n]$$

where $T^{-1}$ is the inverse system of $T$.

## Causality

- A DT system is causal if the output at index $n$ depends on the input for index values at or before $n$:

$$y[n] \text{ depends on } x[m] \text{ for } m \leq n$$

- While all physical CT systems are causal, practical DT systems may not be since we can use memory to "shift time".
- For CT systems we cannot store the infinite number of values between two time points $t_1$ and $t_2$, but we can store the $n_2 - n_1$ values of a DT system between between two indices $n_1$ and $n_2$ (assuming infinite precision).

## Stability

- A DT system is (BIBO) stable if applying a bounded-input (BI)

  $$|x[n]| < \infty \;\forall\; n$$

  results in a bounded-output (BO) $x[n] \mapsto y[n]$ and

  $$|y[n]| < \infty \;\forall\; n$$

- Note, bounded in practice is limited by the physical situation, e.g. the number of bits used to store values.

# Time-invariance

- A DT system is time(index)-invariant if, given

$$x[n] \mapsto y[n]$$

then an index-shift of the input leads to the same index-shift in the output

$$x[n-m] \mapsto y[n-m]$$

# Linearity

- A DT system is linear if the output due to a sum of scaled individual inputs is the same as the scaled sum of the individual outputs with respect to those inputs.

If

$$x_1[n] \mapsto y_1[n] \text{ and } x_2[n] \mapsto y_2[n]$$

then

$$ax_1[n] + bx_2[n] \mapsto ay_1[n] + by_2[n]$$

for constants $a$ and $b$.

- Note this property extends to sums of arbitrary signals, e.g. if

  $$x_i[n] \mapsto y_i[n] \ \forall \ i \in [1 \cdots N]$$

  then given $N$ constants $a_i$, if the system is linear

  $$\sum_{i=1}^{N} a_i x_i[n] \mapsto \sum_{i=1}^{N} a_i y_i[n]$$

- This is a very important property, called *superposition*, and it simplifies the analysis of systems greatly.

# Difference Equations

- A *difference equation* is a relation among combinations of two DT functions and shifted versions of them.
- Similar to differential equations where the solution is a CT function, the solution to a difference equation is a DT function.

# LCCDE

A *linear*, *constant-coefficient*, difference equation (LCCDE) comes in three equivalent forms.

- Delay form.

$$\sum_{k=0}^{N} a_k y[n-k] = \sum_{k=0}^{M} b_k x[n-k]$$

  or

$$a_0 y[n] + a_1 y[n-1] + \cdots a_N y[n-N] = b_0 x[n] + \cdots b_M x[n-M]$$

- Advance form. Let $n \to n+N$, then the delay form becomes

$$\sum_{k=0}^{N} a_k y[n+N-k] = \sum_{k=0}^{M} b_k x[n+N-k]$$

  or

$$a_0 y[n+N] + a_1 y[n+N-1] + \cdots a_N y[n] = b_0 x[n+N] + \cdots b_M x[n+N-M]$$

- Recursive form.

$$y[n] = -\frac{a_1}{a_0} y[n-1] - \cdots - \frac{a_N}{a_0} y[n-N] + \frac{b_0}{a_0} x[n] + \cdots \frac{b_M}{a_0} x[n-M]$$

The *order* of the system is given by max of $N$ and $M$.

# Advance and Delay Operators

- Define the advance operator $E^m$ as shifting a DT function by positive $m$, i.e. $E^m x[n] = x[n+m]$
- Define the delay operator $D^m$ as shifting a DT function by negative $m$, i.e. $D^m x[n] = x[n-m]$.

# Iterative solution of LCCDEs

To perform an iterative solution we need the difference equation in delay form

$$y[n] = -\left( \frac{a_1}{a_0} y[n-1] + \cdots \frac{a_N}{a_0} y[n-N] \right) + \frac{b_0}{a_0} x[n] + \cdots \frac{b_M}{a_0} x[n-M]$$

Given an input and initial conditions for $y[n_0 - 1] \cdots y[n_0 - N]$ we can simulate $y[n]$ for $n \geq n_0$.

# Solution of the homogeneous LCCDE

- The iterative solution does not give us (directly) an analytical expression for the output at arbitrary $n$. We have to start at the initial conditions and compute our way up to $n$.
- We now consider an analytical solution when the input is zero, the solution to the *homogeneous* difference equation

$$Q(E)\,y = a_0 y[n+N] + a_1 y[n+N-1] + \cdots a_N y[n] = 0 \;.$$

given $N$ sequential auxiliary conditions on $y$.

- Similar to differential equations, the homogeneous solution depends on the roots of the characteristic equation $Q(E) = 0$ whose roots are either real or occur in complex conjugate pairs.
- Let $\lambda_i$ be the $i$-th root of $Q(E) = 0$, then the solution is of the form

$$y[n] = \sum_{i=1}^{N} C_i \lambda_i^n$$

where the parameters $C_i$ are determined from the auxiliary conditions.
- For a real system (when the coefficients of the difference equation are real) and when the roots are complex $\lambda_{1,2} = |\lambda| e^{\pm j\beta}$, it is cleaner to assume a form for those terms as

$$y[n] = C|\lambda|^n \cos(\beta n + \theta)$$

for constants $C$ and $\theta$.

## Example:

# Impulse response from LCCDE

To find the solution to $Q(E)y = P(E)x$ when $x[n] = \delta[n]$ assuming $y[n] = 0$ for $n < 0$, giving the *impulse response* $y[n] = h[n]$.

**Step 1:** Let $y_h$ be the homogeneous solution to $Q(E)y_h = 0$ for $n > N$.

**Step 2:** Assume a form for $h[n]$ given by

$$h[n] = \frac{b_N}{a_N}\delta[n] + y_h[n]u[n]$$

**Step 3:** Using the iterative procedure above find the $N$ auxiliary conditions we need by,

- first, rewrite the equation in delay form and solve for $y[n]$,
- then let $x[n] = \delta[n]$ and manually compute $h[0]$ assuming $h[n] = 0$ for $n < 0$,
- repeating the previous step for $h[1]$, continuing up to $h[N-1]$.

**Step 4:** Using the auxillary conditions in step 3, solve for the constants in the solution $h[n]$ from step 2.

Note we can check our solution for as many $n$ values as desired using the recursive form.

## Examples:

# LCCDE and impulse response for IIR filters

# LCCDE and impulse response for FIR filters

## Examples: