ECE 4624: Meeting 2

Continuous-Time (CT) and Discrete-Time (DT) Signals and Systems

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Today we define CT and DT Signals and how to characterize them. Then we review the basic notion of a system as a transformation between signals.

Readings:

- PM 2.1 and 2.2
- Chapter 2 ECE 2714 Supplementary Notes
- Chapter 3 ECE 2714 Supplementary Notes

Topics:

- Signals as Functions
- Primitive Models
- Impulse Function
- Step Function
- Complex Exponential
- Basic Signal Transformations
- Characterization of Signals
- Basic Systems
- Input Signals
- Output or Response Signals
- Impulse Response
- Step Response
- Complex Exponential Response

Signals as Functions

In order to reason about signals mathematically we need a representation or *model*. Signals are modeled as functions, mappings between sets

$$f : A \rightarrow B$$

where A is a set called the *domain* and B is a set called the *range*.

The most basic classification of signals depends on the sets that makeup the domain and co-domain. We will be interested in two versions of the domain, the reals denoted $\mathbb{R}$ and the integers denoted $\mathbb{Z}$. We will be interested in two versions of the co-domain, the reals $\mathbb{R}$ and the set of complex numbers $\mathbb{C}$.

Analog Signals

If the function $f : \mathbb{R} \to \mathbb{R}$, we call this an analog or real, continuous-time signal, e.g. a voltage at time $t \in \mathbb{R}$, $v(t)$. We will write these as $x(t)$, $y(t)$, etc. The units of t are seconds.

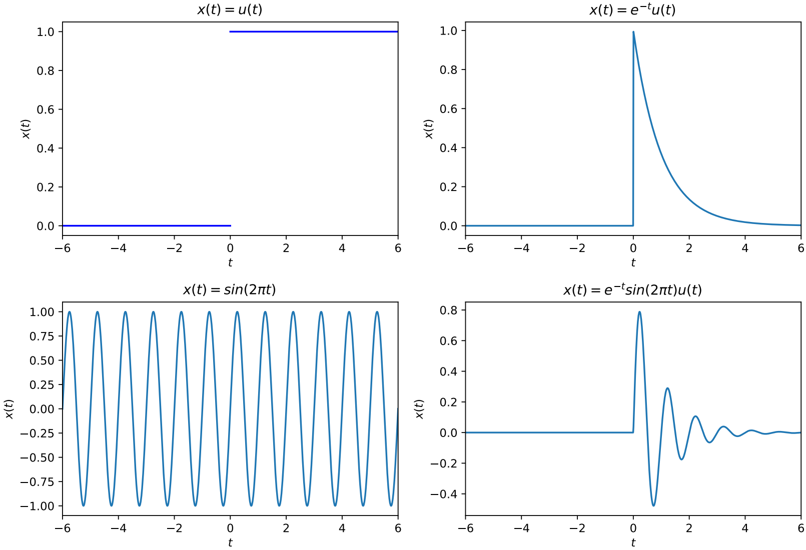

Figure 1: Example plots of analog signals

Real, Discrete-time Signal

If the function $f : \mathbb{Z} \to \mathbb{R}$, we call this a real, discrete-time signal, e.g. the temperature every day at noon. We will write these as $x[n]$, $y[n]$, etc. Note n is dimensionless.

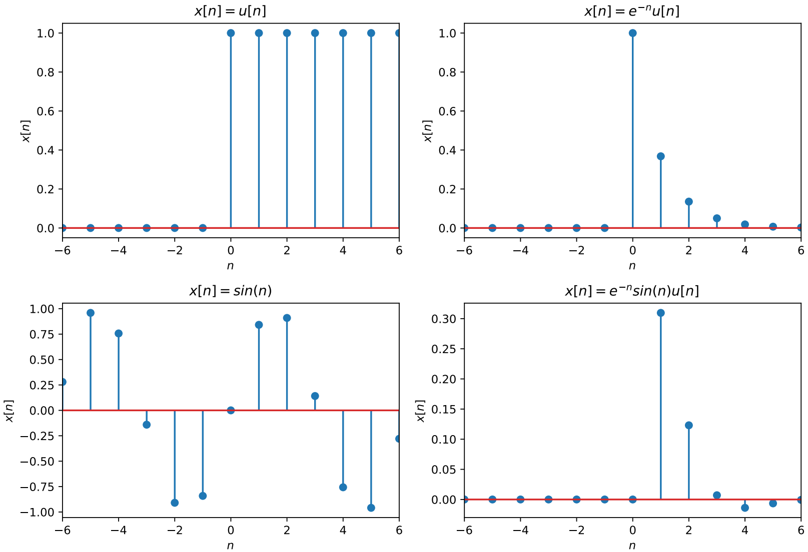

Figure 2: Example plots of real-valued, discrete-time signals.

Some other possibilities

- $f : \mathbb{R} \to \mathbb{Z}$, discrete-valued, continuous-time signals
- $f : \mathbb{Z} \to \mathbb{Z}$, discrete-valued, discrete-time signals

The co-domain can also be complex.

- $f : \mathbb{R} \to \mathbb{C}$, complex-valued, continuous-time signals, e.g.

$$x(t) = e^{j\omega t} = \cos(\omega t) + j\sin(\omega t)$$

- $f : \mathbb{Z} \to \mathbb{C}$, complex-valued, discrete-time signals, e.g.

$$x[n] = e^{j\omega n} = \cos(\omega n) + j\sin(\omega n)$$

Since the domains $\mathbb{R}$ and $\mathbb{Z}$ are usually interpreted as time, we call these *time-domain* signals. In the time-domain, when the co-domain is $\mathbb{R}$ we call these real signals. All physical signals are real. However pairs of complex signals are important mathematical models in linear systems theory.

Digital Signals

We are ultimately interested in signals

$$f : \mathbb{Z} \to \mathbb{Q}$$

where the range $\mathbb{Q} \subset \mathbb{Z}^+$ is the set of unsigned N-bit values

$$\sum_{i=-A}^{B} b_i 2^{-i}$$

- $b_i \in \{0, 1\}$ it the ith bit
- b_{-A} is the most-signifigant bit (MSB)
- b_B is the least-signifigant bit (LSB)
- the location of the decimal point is implied
- signed values have various representations

However, these functions are hard to work with mathematically, so we generally work with DT signals and look at practical consequencces separately.

Primitive Models

We mathematically model signals by combining elementary/primitive functions, for example:

- polynomials: $x(t) = t$, $x(t) = t^2$, etc.
- transendental functions: $x(t) = e^t$, $x(t) = \sin(t)$, $x(t) = \cos(t)$, etc.
- piecewise functions, e.g.

$$x(t) = \begin{cases} f_1(t) & t < 0 \\ f_2(t) & t \geq 0 \end{cases}$$

Unit Step Function

To model turning signals on we often use a piecewise function, the unit-step

In continuous-time:

$$u(t) = \begin{cases} 0 & t < 0 \\ 1 & t \geq 0 \end{cases}$$

Note: some texts define the step function at $t = 0$ to be 0 or $\frac{1}{2}$.

In discrete-time:

$$u[n] = \begin{cases} 0 & n < 0 \\ 1 & n \geq 0 \end{cases}$$

Note: there is no ambiguity about the value at $n = 0$.

Unit Impulse Function

An important signal in linear system theory is the impulse function:

It is easily defined in discrete-time:

$$\delta[n] = \begin{cases} 0 & n \neq 0 \\ 1 & n = 0 \end{cases}$$

However in continuous-time it is defined using generalized functions. Two definitions:

$$\delta(t) = \lim_{\epsilon \to 0} \begin{cases} \frac{1}{2\epsilon} & |t| < \epsilon \\ 0 & \text{else} \end{cases}$$

$$\delta(t) = \lim_{\epsilon \to 0} \frac{1}{\sqrt{2\pi}\epsilon} e^{-\frac{t^2}{2\epsilon^2}}$$

Note the area under each definition is always one.

CT Impulse Function cont.

In practice we can often use the following heuristic definition and some properties, without worrying about the distribution functions.

$$\delta(t) = \begin{cases} 0 & t \neq 0 \\ \infty & t = 0 \end{cases}$$

- The area under the unit impulse is unity since by definition

$$\int_{-\infty}^{\infty} \delta(t)\, dt = 1$$

- Sampling property: $x(t)\delta(t - t_0) = x(t_0)\delta(t - t_0)$
- Sifting Property:

$$\int_{a}^{b} x(t)\delta(t - t_0)\, dt = x(t_0)$$

  for any $a < t_0 < b$.

Relationships between the unit step and impulse in CT

We previously defined the unit step function. The impulse can be defined in terms of the step:

$$\delta(t) = \frac{du}{dt}$$

and vice-versa

$$u(t) = \int_{-\infty}^{t} \delta(\tau)\, d\tau$$

using the notion of distributions, e.g.

$$u(t) = \int_{-\infty}^{t} \delta(\tau)\, d\tau = \lim_{\epsilon \to 0} \int_{-\infty}^{t} \frac{1}{\sqrt{2\pi}\epsilon} e^{-\frac{\tau^2}{2\epsilon^2}}\, d\tau = \lim_{\epsilon \to 0} \frac{1}{2}\left(1 + \mathsf{erf}\left(\frac{t}{\sqrt{2}\epsilon}\right)\right)$$

Variations on the step and impulse in CT

We can apply additional transformations to the impulse and step functions to get other useful signals, e.g.

- ramp

$$r(t) = \int_{-\infty}^{t} u(\tau)\, d\tau = tu(t)$$

- causal pulse of width ϵ

$$p(t) = u(t) - u(t - \epsilon)$$

- non-causal pulse of width 2ϵ

$$p(t) = u(t + \epsilon) - u(t - \epsilon)$$

Relationships between the unit step and impulse in DT

Some useful properties of the DT impulse function are:

- Energy is 1: $\sum_{n=-\infty}^{\infty} \delta[n] = 1$
- Sampling: $x[n]\delta[n-n_0] = x[n_0]\delta[n-n_0]$
- Sifting: $\sum_{n=-\infty}^{\infty} x[n]\delta[n-n_0] = x[n_0]$

The impulse can be defined in terms of the step and vice-versa:

$$\delta[n] = u[n] - u[n-1]$$

$$u[n] = \sum_{m=-\infty}^{n} \delta[m]$$

or

$$u[n] = \sum_{k=0}^{\infty} \delta[n-k]$$

Complex Exponential in CT

One of the most important signals in systems theory is the complex exponential:

$$x(t) = C\,e^{at}$$

where the parameters $C, a \in \mathbb{C}$ in general.

- When C and a are both real ($\Im(C) = \Im(a) = 0$), we have the familiar exponential.

Complex Exponential in CT cont.

To get the pure sinusoidal case, let $C \in \mathbb{R}$ and a be purely imaginary: $a = j\omega_0$:

$$x(t) = Ce^{j\omega_0 t}$$

where ω_0 is the frequency (in radians/sec). This is called the complex sinusoid.

By Euler's identity:

$$e^{j\omega_0 t} = \cos(\omega_0 t) + j\sin(\omega_0 t)$$

and

$$\Re(x(t)) = \cos(\omega_0 t) = \frac{1}{2}\left(e^{j\omega_0 t} + e^{-j\omega_0 t}\right)$$

$$\Im(x(t)) = \sin(\omega_0 t) = \frac{1}{2j}\left(e^{j\omega_0 t} - e^{-j\omega_0 t}\right)$$

are both real sinusoids.

Complex Exponential in CT cont.

When the parameter C is complex we get a phase shift. Again let $a = j\omega_0$. When C is complex we can write it as $C = Ae^{j\phi}$ where $A = |C|$ and $\phi = \angle C$. Then

$$x(t) = Ae^{j\phi}e^{j\omega_0 t} = Ae^{j(\omega_0 t+\phi)}$$

and

$$\Re(x(t)) = A\cos(\omega_0 t + \phi)$$

$$\Im(x(t)) = A\sin(\omega_0 t + \phi)$$

Since $\sin$ is a special case of $\cos$, i.e. $\cos(\theta) = \sin(\theta + \frac{\pi}{2})$, the general real sinusoid is

$$A\cos(\omega_0 t + \phi)$$

- A is called the amplitude
- ω_0 is again the frequency in radians/sec.
- ϕ is called the phase shift and is related to a time shift T_s by

$$\phi = \omega_0 T_s$$

Complex Exponential in DT

The DT Complex Exponential is defined in a similar fashion the the CT version, but with some important differences.

The general DT complex exponential is given by the expression:

$$x[n] = Ce^{\beta n}$$

where in general $C \in \mathbb{C}$ and $\beta \in \mathbb{C}$.

It is usually more convenient to write this as

$$x[n] = C\alpha^n$$

where $\alpha = e^{j\theta}$ is a complex number $\alpha = \cos(\theta) + j\sin(\theta)$.

Basic Transformations (focusing on DT)

We can also apply transformations to signals to increase their modeling flexibility.

- magnitude scaling

$$x_2[n] = ax_1[n]$$

  for $a \in \mathbb{R}$.

- time differences

$$x_2[n] = x_1[n] - x_1[n-1]$$

- running sums

$$x_2[n] = \sum_{m=-\infty}^{n} x_1[m]$$

- sums

$$y[n] = \sum_{i} x_i[n]$$

Basic DT Transformations cont.

- multiplication (modulation)

$$y[n] = x_1[n]x_2[n]$$

- time index shift

$$x_2[n] = x_1[n+m]$$

  - if $m < 0$ it is called a *delay*
  - if $m > 0$ it is called an *advance*

- time reversal

$$x_2[n] = x_1[-n]$$

Basic DT Transformations cont.

- decimation
$$y[n] = x[mn]$$
for $m \in \mathbb{Z}^+$.
  - e.g. for $m = 2$ only keep every other sample
  - e.g. for $m = 3$ only keep every third sample
  - etc.
- interpolation
$$y[n] = \begin{cases} x\left[\frac{n}{m}\right] & n = 0\ ,\ \pm m, ,\ \pm 2m \cdots \\ 0 & \text{else} \end{cases}$$
When $m = 2$ this inserts a zero sample between every sample of the signal.

Characterization of CT Signals

There are a few basic ways of characterizing signals.

- A CT signal is causal if $x(t) = 0 \ \forall t < 0$.
- A CT signal is anti-causal or acausal if $x(t) = 0 \ \forall t \geq 0$. A signal can be written as the sum of a causal and anti-causal signal.
- A CT signal is periodic if $x(t) = x(t+T) \ \forall t$ for a fixed parameter $T \in \mathbb{R}$ called the *period*. The simplest periodic signals are those based on the sinusoidal functions.
- A CT signal is even if $x(t) = x(-t) \ \forall t$.
- A CT signal is odd if $x(t) = -x(-t) \ \forall t$.

  Any CT signal can be written in terms of an even and odd component

  $$x(t) = x_e(t) + x_o(t)$$

  where $x_e(t) = \frac{1}{2}\{x(t) + x(-t)\}$ and $x_o(t) = \frac{1}{2}\{x(t) - x(-t)\}$.

Characterization of CT Signals cont.

- The *energy* of a CT signal $x(t)$ is defined as a measure of the function

$$E_x = \lim_{T \to \infty} \int\limits_{-T}^{T} |x(t)|^2 dt \; .$$

- The *power* of a CT signal is the energy averaged over an interval as that interval tends to infinity.

$$P_x = \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} |x(t)|^2 dt \; .$$

Signals can be characterized based on their energy or power:

- Signals with finite, non-zero energy and zero power are called *energy signals*.
- Signals with finite, non-zero power (and by implication infinite energy) are called *power signals*.

Characterization of DT Signals

Similarly there are a few basic ways of characterizing DT signals.

- A DT signal is *causal* if $x[n] = 0 \ \forall n < 0$.
- A DT signal is *anti-causal* or acausal if $x[n] = 0 \ \forall n \geq 0$.
- A DT signal can be written as the sum of a causal and anti-causal signal.
- A DT signal is periodic if $x[n] = x[n + N] \ \forall n$ for a fixed period $N \in \mathbb{Z}$.
- A DT signal is even if $x[n] = x[-n] \ \forall n$.
- A DT signal is odd if $x[n] = -x[-n] \ \forall n$.

  Any DT signal can be written in terms of an even and odd component

$$x[n] = x_e[n] + x_o[n]$$

  where $x_e[n] = \frac{1}{2}\left\{x[n] + x[-n]\right\}$ and $x_o[n] = \frac{1}{2}\left\{x[n] - x[-n]\right\}$

Characterization of DT Signals cont.

Analogous to CT signals, the energy of a DT signal is

$$E_x = \lim_{N \to \infty} \sum_{-N}^{N} |x[n]|^2 \ .$$

The power of a DT signal is the energy averaged over an interval as that interval tends to infinity.

$$P_x = \lim_{N \to \infty} \frac{1}{2N+1} \sum_{-N}^{N} |x[n]|^2 \ .$$

DT Signals with finite, non-zero energy and zero power are called *energy signals*.

DT Signals with finite, non-zero power (and by implication infinite energy) are called *power signals*.

Systems

A system is an interconncted set of components or sub-systems. Mathematically a system is a transformation, T, between one or more signals, a rule that maps functions to functions.

- single input - single output (SISO) system.

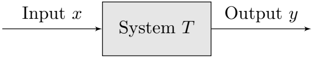

Figure 3: SISO Block Diagram

We will focus on single input - single output systems.

Other System Types

- single input - multiple output (SIMO) system

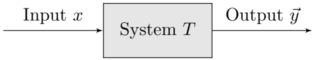

Figure 4: SIMO Block Diagram

- general case, multiple input - multiple output (MIMO)

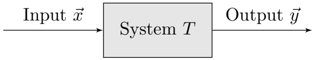

Figure 5: MIMO Block Diagram

MIMO systems can be handled using state-based descriptions.

Systems cont.

- If both input and output are CT signals, it is a CT system.

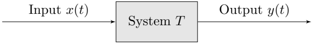

Figure 6: Generic Block Diagram of CT System

- If both input and output are DT signals, it is a DT system.

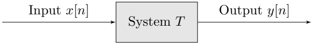

Figure 7: Generic Block Diagram of a DT System

Systems cont.

- If input and output are not both CT or DT signals, it is a hybrid CT-DT system.

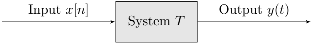

Figure 8: Generic Block Diagram of a Hybrid DT/CT System

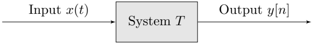

Figure 9: Generic Block Diagram of a Hybrid CT/DT System

We will focus on Hybrid and DT systems.

System Input and Output

- The input to a SISO system is a signal.
- The output from a SISO system, or its *response*, is also a signal
- There are three canonical responses considered:
  - Impulse Response
  - Step Response
  - Complex Exponential or sinusoidal response

  For LTI systems the last two can be derived from the first.
- The impulse response for CT systems is denoted $h(t)$.

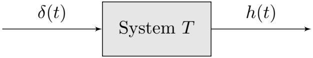

Figure 10: Impulse Response of CT System

- The impulse response for DT systems is denoted $h[n]$.

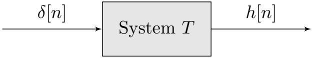

Figure 11: Impulse Response of DT System

CT system representations

We can mathematically represent, or model, systems multiple ways.

- purely mathematically - in time domain we generically use
  - for CT systems: differential equations e.g.
    $$y'' + ay' + by = x$$
  - for DT systems: difference equations e.g.
    $$y[n] = ay[n-1] + by[n-2] + x[n]$$
- graphically, using a mixture of math and block diagrams

Mathematical models:

- provide abstraction, removing (often) irrelevant detail.
- can be more or less detailed, an *internal* v.s. *external* (block box) description
- are not unique with respect to instantiation (implementation)
- are limited to the regime they were designed for

System properties and classification

Choosing the right kind of system model is important. Here are some important properties that allow us to broadly classify systems.

- Memory
- Invertability
- Causality
- Stability
- Time-invariance
- Linearity

Our focus is on linear, time-invariant (LTI) systems. Such systems can be represented completely by:

- an LCCDE,
- an impulse response, or
- a transfer function.

When stable they can also be represented by thier frequency response.

Reminders and Next Actions

- Read PM 4.2
- Refer to as needed Chapters 14 and 16 of ECE 2714 Supplementary Notes
- Reminder: PS #1 is due Sept. 10