What you might say on the first day of class

This class is designed to help you develop key skills in math that will help you succeed in higher level, more abstract courses. My goal is that by working hard in this course, you will become unstoppable in your future math courses.

There are two main components to the course: learning how to read a math book on your own, and practicing the basic steps in mathematics, where we start with examples and non–examples, make a definition, consider more examples, make a conjecture, see if we can find a counterexample to the conjecture, make another conjecture, and if we can find a proof, call the result a theorem.

The simplest way to describe this course is that it is the "proof course." A better way to describe it is that it focuses on the core ideas of mathematics, over and over again, in different settings. The core idea of mathematics is to make definitions of mathematical objects, consider examples and non–examples of those definitions to make sure we understand them well, make conjectures about what we think might be true, and then try to prove those things. It is all about understanding mathematical ideas and how they fit together. Proofs are a part of that, but they are only a part. This course will not have the same feel as algebra and calculus courses, which are heavier on calculations and don't have quite as many different ideas or different types of examples. This course is much more about skills than content.

Most people learn by doing. Me talking at the board is not the same as you learning. Most of your time in the course will be spent working on activities as a group while I go from group to group, seeing how you are doing, answering questions, and making suggestions. You will sometimes hand the activities in at the end of class so that I can read over them and make comments, then give them back to you at the beginning of the next class. There is no need to rush through the activities. Take your time, think about what you're doing. There are often multiple correct ways to do a problem. It's not question of "what I want" as the teacher, but what works. Work with your group to make sure you all understand everything along the way. You do not need to finish the activity; I always try to add extra material at the end so that no group runs out of things to do.

Outside of class, you will be reading the textbook, taking notes on what you read, and solving some exercises. You will bring your notes to class and they will be read over and returned by the start of the next class to make sure that you are doing the reading and thinking. Your first assignment is to read Chapter 1 of the textbook and turn in your notebook on Wednesday. Each chapter will also have a short reading quiz. I will not lecture over the material in the book; that is one of the keys to you learning how to read a book on your own.

There will be a final exam, but rather than mid-term exams, there will be quizzes over the material in activities, once you have had a chance to get good at it.

Your name: ______________________________

Even and odd

Our first example of definitions, examples, theorems, and proofs

Overview

Definitions are important to read and understand by looking at examples. Many proofs are little more than working with the definitions and rewriting things. With a bit of practice, these become very routine. This activity has you work through two definitions, a few examples, and then some proofs. Everything relies on the definitions, so keep coming back to them. We will use a similar model many times during the semester.

Definition 1. Even. An integer n is *even* if there exists an integer k for which $n = 2k$.

Definition 2. Odd. An integer n is *odd* if there exists an integer k for which $n = 2k + 1$.

Note 3. 19 meets the definition to be odd because 19 is an integer and $19 = 2(9)+1$.

Example 4. Check that 12 meets the definition to be even by writing $12 = 2k$ for an appropriate value of k and make sure k is an integer.

Example 5. Does -9 meet the definition to be odd? Write $-9 = 2k + 1$.

Example 6. Does 0 meet the definition to be even? Write $0 = 2k$ for an integer k.

Example 7. Does 1.73 meet the definition to be odd? Explain.

Note 8. Suppose that m is an integer. Then $2m + 1$ is an integer and it is odd because it meets the definition to be odd. Also, $2m + 2$ is even because it is an integer and can be rewritten as $2(m + 1)$, which is of the form $2k$ where $k = m + 1$, which is an integer.

Show 9. Suppose that m is an integer. Show that $2m+6$ is even by rewriting it until it meets the definition to be even. Connect your statements with $=$ signs.

Show 10. Suppose that m is an integer. Show that $4m + 9$ is odd by rewriting it until it meets the definition to be odd. Connect your statements with $=$ signs.

Stop.

Compare your answers to the questions above with the other people in your group before you move on. Resolve any differences in your answers.

Show 11. Suppose that m is even. Then $m = 2k$ for some integer k. Show that $m+8$ is even by rewriting it as $2k + 8$ and continuing until it is 2 times an integer. Connect your statements with = signs.

Note 12. You already know that the sum of two even numbers is even. The next item guides you through a proof of this fact, using the definitions of even and odd above.

Guided proof 13. Suppose that m and n are even. Fill in the blanks to show that $m + n$ is even.

1. There exist integers j and k such that $m =$__________ and $n =$__________.

2. Thus, $m + n =$ ___________________________ $= 2(j + k)$.

3. This number meets the definition to be even because it is an __________ and because $(j + k)$ is an __________.

4. We saw that if m and n are even, then $m + n$ is even. We made no further assumption about m and n. Thus, the sum of any two even numbers is even.

Guided proof 14. Suppose that m is even and n is odd. Fill in the blanks to show that mn is even. Use the previous exercise as a model.

1. There exist ___________________________ such that $m =$__________ and $n =$__________.

2. Thus, $mn =$____ $= 2($__________$)$.

3. This number satisfies the definition to be even because ________________ and ________________.

4. We saw that ____. We made no ____. Thus, ___________________________________.

Prove 15. Let m and n be odd. Follow the examples above as a model to show that mn is odd by rewriting mn until it meets the definition to be odd. Good form is critically important in proofs.

1.

2.

3.

4.

Prove 16. Let m and n be odd. Follow the examples above as a model to show that $m + n$ is even by rewriting it. Good form is critically important in proofs.

1.

2.

3.

4.

Group work 17. Step 1 in each of your proofs uses the definition of even or odd to write m and n in a more useful, more informative way. How does this help your proof along? Discuss with members of your group.

Group work 18. Step 3 in each of your proofs also uses the definition. How does this use of the definition differ from what happens in Step 1? Discuss with members of your group.

Prove 19. Let m be odd. Follow the models above to prove that m^2 is odd. Use good form.

Prove 20. Let m be even. Follow the models above to prove that m^2 is even. Use good form.

Question 21. What does it mean that an integer is a multiple of 4? Give your own definition analogous to the definitions of even and odd.

Prove 22. Let m be even. Show that $m^2 + 2m + 4$ is a multiple of 4.

Question 23. Let m be even. Can $m^2 + 2m + 4$ be a multiple of 8? Explain.

Challenges. Here are some statements that are harder to prove, because they require a bit more than simply restating the definitions. See if you can make a good argument for them.

Question 24. Pretend for a minute that there is an integer m which is both even and odd. Work with the definitions to see that this really must be fantasyland.

Prove 25. If m is an integer, then m is even or m is odd; it has to be one of the two, there is no third possibility.
Here is one suggestion. 0 is even. If n is even, then $n+1$ is odd. If n is odd, then $n+1$ is even. This should cover all positive integers. Also, if n is even, then $-n$ is even, which tells us about negative integers.

Prove 26. If m is an integer and m^2 is odd, then m is odd. **Hint:**: There are two cases to check, the case in which m is even and the case in which m is odd.

Your name: ______________________________

Background questions – Math 3280

Please do your best with these questions and turn this sheet in on Wednesday.

1. How comfortable are you already with the "definition, example, theorem, proof" progression in mathematics classes?

2. What kinds of experiences have you had in the past with proofs?

3. Have you ever had success reading a mathematics textbook and really learning from it? If so, please tell what book, what course, and what made it work. If not, please tell me what you think prevented you from being able to read the book.

4. How do you have Canvas configured, to send you emails or texts or what? How does that work for you?

5. If you take the elevator to the fourth floor, do you turn right or left to get to my office?

6. What are the two flyers on my door about?

7. Do you have a hard copy of the textbook that you can read? Have you been able to get a PDF file of the first chapter from the library?

8. Do you have any interest in going to graduate school? Please explain.

9. Do you have any questions or concerns about the coursework?

10. Do you have any questions or concerns about the grading?

11. What is the most likely reason that you will miss class? I'm just curious.

12. What mathematics courses have you already taken in college?

13. What mathematics courses are you taking this semester? It's OK to just list the numbers, like Math 3410.

14. Including this one, how many semesters until you graduate?

15. Please let me know anything you think I should know about you. I'll read it all. Sometimes people like to tell about their hobbies, movies they like, other academic interests, clubs they're in, where they're from, etc.

Your name: ______________________________

Sum and dot product of 3–dimensional vectors

We define a new mathematical object and do some work with it.

Overview

In Calculus III and Linear Algebra, one defines vectors and works with them. They have a geometric interpretation, but here we will simply give an algebraic definition and work with their algebraic properties. This activity illustrates proofs in which all that is needed is the definition and some rewriting. Notice how we often use the same definition twice in one proof, once to "unpack" and the second time to "re–pack."

Definition 27. 3–dimensional vector. A three–dimensional vector is an ordered triple $\langle a_1, a_2, a_3 \rangle$, where a_1, a_2, and a_3 are real numbers.

Notation 28. A 3–dimensional vector $\langle a_1, a_2, a_3 \rangle$ is often denoted by a single letter with an arrow over the top, like this $\vec{a}$. When it is written like $\langle a_1, a_2, a_3 \rangle$ it is said to be in *open form.*

Definition 29. Equality of 3–dimensional vectors. 3–dimensional vectors $\langle a_1, a_2, a_3 \rangle$ and $\langle b_1, b_2, b_3 \rangle$ are equal if $a_1 = b_1, a_2 = b_2$, and $a_3 = b_3$. The order of the numbers is important.

Definition 30. Sum of 3–dimensional vectors. The sum of 3–dimensional vectors $\langle a_1, a_2, a_3 \rangle$ and $\langle b_1, b_2, b_3 \rangle$ is the 3–dimensional vector $\langle a_1 + b_1, a_2 + b_2, a_3 + b_3 \rangle$. We write $\vec{a} \oplus \vec{b}$, using a new symbol so we don't confuse addition of vectors with addition of real numbers.

Example 31. Is $\langle 3, 9, 12 \rangle$ a 3–dimensional vector? Explain. Is it equal to $\langle 12, 3, 9 \rangle$? Explain.

Example 32. Is $\langle \sqrt{3}, \sqrt[3]{9}, \sqrt[4]{-12} \rangle$ a 3–dimensional vector? Explain.

Example 33. Is $\langle 8, 13.35321, \pi, -7 \rangle$ a 3–dimensional vector? Explain.

Example 34. Is $\langle 1, \pm 7, \heartsuit \rangle$ a 3–dimensional vector? Explain.

Example 35. Is $\langle 3 + 9 + 12 \rangle$ a 3–dimensional vector? Explain.

Example 36. Is $\langle \begin{bmatrix} 6 & 0 \\ 2 & 5 \end{bmatrix}, -4, 7 \rangle$ a 3–dimensional vector?

Example 37. Let x be a real number. Is $\langle \frac{14}{3}, 2 - 7x, \sqrt{16} \rangle$ a 3–dimensional vector? Explain.

Stop. Compare your answers to the questions above with the members of your group. Make sure you agree on everything.

Example 38. Calculate the sum of $\vec{a} = \langle 12, -5, 3\rangle$ and $\vec{b} = \langle 6, 4, -11\rangle$. Start by writing $\vec{a} \oplus \vec{b} = \ldots$ and write the vectors in open form next.

Show 39. We are going to show that addition of 3–dimensional vectors is commutative. We will do this by rewriting. Follow the model.
Let $\vec{a}$ and $\vec{b}$ be 3–dimensional vectors. Then,

$$\begin{aligned}
\vec{a} \oplus \vec{b} &= \langle \quad , \quad , \quad \rangle \oplus \langle \quad , \quad , \quad \rangle \\
&= \langle \qquad , \qquad , \qquad \rangle \\
&= \langle \qquad , \qquad , \qquad \rangle \\
&= \langle \quad , \quad , \quad \rangle \oplus \langle \quad , \quad , \quad \rangle \\
&= \vec{b} \oplus \vec{a}
\end{aligned}$$

We have seen that $\vec{a} \oplus \vec{b} = \vec{b} \oplus \vec{a}$. We made no further assumption about $\vec{a}$ and $\vec{b}$. Thus, for all 3–dimensional vectors $\vec{a}$ and $\vec{b}$, we know that $\vec{a} \oplus \vec{b} = \vec{b} \oplus \vec{a}$. Thus, addition of 3–dimensional vectors is commutative.

Show 40. Go back to each line of the proof above and give a reason for the equality on that line at the very right side of the line. The first one is “Write in open form.” Somewhere in the middle you will use the fact that addition of real numbers is commutative. Thus, at the heart of it, commutativity of vector addition comes from commutativity of addition of real numbers.

Show 41. You are going to show that addition of 3–dimensional vectors is associative. Let $\vec{a}, \vec{b}$, and $\vec{c}$ be 3–dimensional vectors. Start with $(\vec{a} \oplus \vec{b}) \oplus \vec{c}$ and rewrite it until it becomes $\vec{a} \oplus (\vec{b} \oplus \vec{c})$ following the model of the previous proof, starting with the word “Let”. Also write explanations for each step. Conclude, following the model, that you have shown that vector addition is associative. This last part is very important.

Stop. Compare your argument to the rest of the members of your group. Make sure that you agree on absolutely every step and every justification.

Definition 42. Scalar product for 3–dimensional vectors. Let c be a real number and let $\vec{a} = \langle a_1, a_2, a_3 \rangle$ be a 3–dimensional vector. The *scalar product* of c and $\vec{a}$ is defined as:

$$c\vec{a} = \langle ca_1, ca_2, ca_3 \rangle.$$

Example 43. Let $c = 3$ and $\vec{a} = \langle 7, -4, \sqrt{2} \rangle$. Calculate $c\vec{a}$, starting by writing $c\vec{a} = \ldots$.

Example 44. Calculate $\pi\langle 9, 4, 1 \rangle$.

Example 45. Calculate $(2 + \sqrt{3})\langle 5, b, c \rangle$.

Show 46. You are going to show that the scalar product is distributive over vector addition. First use the word "Let" to settle on one real number c and two 3–dimensional vectors, $\vec{a}$ and $\vec{b}$. Then write $c(\vec{a} \oplus \vec{b})$ and rewrite it until it equals $c\vec{a} \oplus c\vec{b}$. Provide a reason for each step, as in the proofs above. At the end, follow the model to conclude that you have shown distributivity in general.

Show 47. You are going to show that the scalar product is distributive over real number addition. Start with "Let." Write $(c + d)\vec{a}$ and rewrite it until it equals $c\vec{a} \oplus d\vec{a}$. Provide justifications for each step. At the end, follow the model to conclude that this shows distributivity in general.

Stop. Check over what everyone in your group has done, and make sure that you are in complete agreement.

Definition 48. Zero vector. The vector $\langle 0,0,0\rangle$ is a special 3–dimensional vector, called the *zero vector*. We denote it by $\vec{0}$.

Definition 49. Additive inverse. Let $\vec{a}$ be a 3–dimensional vector, with open form $\langle a_1, a_2, a_3\rangle$. Define a new vector by $-\vec{a} = \langle -a_1, -a_2, -a_3\rangle$. It is called the *additive inverse* of $\vec{a}$.

Show 50. Let $\vec{a}$ be a 3–dimensional vector. Show that $\vec{a} \oplus \vec{0} = \vec{a}$. It's not very exciting. Make a general conclusion.

Show 51. Let $\vec{a}$ be a 3–dimensional vector, and let $-\vec{a}$ be its additive inverse. Show that $\vec{a} \oplus (-\vec{a}) = \vec{0}$. This is also not very exciting. Make a general conclusion.

Definition 52. Dot product of 3–dimensional vectors. The dot product of 3–dimensional vectors $\langle a_1, a_2, a_3\rangle$ and $\langle b_1, b_2, b_3\rangle$ is the real number $a_1b_1 + a_2b_2 + a_3b_3$.

Notation 53. The dot product of 3–dimensional vectors $\vec{a}$ and $\vec{b}$ is denoted $\vec{a} \bullet \vec{b}$.

Example 54. Calculate the dot product of $\vec{a} = \langle 12, -5, 3\rangle$ and $\vec{b} = \langle 6, 4, -11\rangle$ Do this by writing

$$\begin{aligned} \vec{a} \bullet \vec{b} &= \langle a_1, a_2, a_3\rangle \bullet \langle b_1, b_2, b_3\rangle \\ &= a_1b_1 + a_2b_2 + a_3b_3 \end{aligned}$$

and then substituting in the numbers. This makes the calculation just a matter of rewriting.

Show 55. You will show that the dot product is commutative, just as multiplication of real numbers is commutative. This time, you write the first line, "Let $\vec{a}$ and $\vec{b}$ be ...." Follow the models from previous examples, and be sure to make a general conclusion.

Show 56. You will show that the dot product is distributive over vector addition. That is, you want to show that $(\vec{a} \oplus \vec{b}) \bullet \vec{c} = \vec{a} \bullet \vec{c} + \vec{b} \bullet \vec{c}$. Start with "Let ...". Please explain why one addition symbol is $\oplus$ and the other is $+$.

Example 57. Calculate $\vec{a} \bullet \vec{0}$. Is this a general result?

Show 58. Let $\vec{a}$ and $\vec{b}$ be 3–dimensional vectors and let c be a real number. Show in general that $c(\vec{a} \bullet \vec{b}) = (c\vec{a}) \bullet \vec{b} = \vec{a} \bullet (c\vec{b})$. Since there are two equalities to show, you might want to think about how you will go about it.

Your name: ________________________________

Quantifier assessment

This is not part of your grade in the course, but it will be checked for correctness.

A. Write the following statements symbolically:

1. For every a, there is a b for which $b^2 = a$

2. For every b, there is an a for which $b^2 = a$

3. For every a and every b, it is the case that $b^2 = a$

4. There exists an a and there exists a b such that $b^2 = a$

B. Which of the statements in the previous problem are true if the universe for both a and b is the set of non–negative integers? If not true, explain why not.

1.

2.

3.

4.

C. Negate the statements from problem A.

1.

2.

3.

4.

D. Write the following statements symbolically:

1. Every rose has a thorn.

2. Every married couple with a child gets a tax deduction.

Your name: ______________________________

The Division Algorithm

Dividing integers with remainders will form the basis for several things we want to prove.

Overview

We would like to distribute n objects evenly among k people and find out how many are left over. We will investigate a procedure for doing this, which is called division, even though there will be no fractions in this activity. Procedures that are guaranteed to work are called *algorithms* after the 9th century Persion mathematician al-Khwarizmi, who worked on procedures for arithmetic. The division algorithm itself dates to Euclid's *Elements* from around 300 BC.

Example 59. You are the dealer in a card game that has 37 cards. (It's not a standard deck of cards.) There are 5 people playing, and everyone needs to end up with the same number of cards. Dealing one card to each player leaves 32 cards in your hands. Write down the numbers 37, 32, and continue until you cannot deal out any more cards evenly. Let r denote the number of cards left at the end, and let q denote the number of times you subtracted 5, which is also the number of cards that each person got. You see that $37 - 5q = r$, which you can rewrite as $37 = 5q + r$. Fill in q and r and write out these two equations.

Example 60. Now you're playing a card game that you have not played before, and you haven't taken the time to count how many cards are in the deck. You are the dealer again, and there are 5 people who need cards. Let n denote the number of cards in the deck. Imagine that you repeat the procedure from the previous example until you can no longer deal out cards evenly. Again, let r denote the number of cards you have left at the end and q denote the number of cards that each person got. What do we know for sure about the possible values of r? What do we know for sure about the possible values of q? Write the relationship between n, 5, q, and r analogous to $37 - 5q = r$ and the expression analogous to $37 = 5q + r$. The last expression accounts for where all of the n cards have gone; some are dealt out, some are left in your hands. Write out a sentence that explains this.

Example 61. Continuing to divide by 5, complete this sentence: Given an integer $n \geq 0$, there exist integers q and r where (list properties of q here) __________________________ and (list properties of r here) __________________________, such that (write the relationship between n, q, and r analogous to $37 = 5q + r$) __________________________.

Example 62. Once again, we have n cards, but now there are k people playing, where $k > 0$ is an integer. Your friend is dealing. Write instructions telling your friend how to deal the cards out, when to stop dealing, and say what you can about how many cards she will have left over and how many cards each person will get. Using q to denote the number of cards each person gets and r to denote the number of cards left over, write out the relationship between n, k, q, and r, and write inequalities concerning q and r.

Stop. Compare your work to the others in your group. Work to get the best possible phrasing of 62.

Question 63. What happens when $k = 0$? Can you satisfy $n = qk + r$? What goes wrong?

Note 64. As above, suppose that n and k are integers that are greater than 0. Suppose you find integers q and r for which $n = qk + r$ and $0 \leq r < k$. Suppose your friend tries to do the same thing and finds integers q_2 and r_2 for which $n = q_2k + r_2$ and $0 \leq r_2 < k$. Must it be the case that $q = q_2$ and $r = r_2$? That is, are the values of q and r *unique*? If you think about dealing n cards to k people, it's pretty clear that you and your friend will get the same values of q and r, but how could we see this without thinking about card dealing? It will take a few steps.

Show 65. Suppose that r and b are integers for which $0 \leq r < k$ and $0 \leq b < k$. Carefully combine these inequalities to show that $-k < r - b < k$. Your goal is to provide a crystal clear argument with no extra steps. You can add inequalities that run the same direction; for example, if $a < b$ and $c < d$, then $a + c < b + d$.

Show 66. Suppose that n and k are integers, that $n \geq 0$ and $k > 0$, and that q, r, q_2, and r_2 are integers such that $n = qk + r$ and $n = q_2k + r_2$ and that $0 \leq r < k$ and $0 \leq r_2 < k$. Set $qk + r$ equal to $q_2k + r_2$ and use 65 argue that $q = q_2$ and $r = r_2$. It will probably not be obvious how to get started; this proof requires a spark of genius. Use scratch paper, write down everything you know, including 65, and then write a final argument here.

Theorem 67. The Division Algorithm. Let n and k be integers greater than 0. There are two parts to the theorem.

1. **Existence.** There exist integers q and r for which $n = qk + r$ and for which $0 \leq r < k$.
2. **Uniqueness.** The numbers q and r are unique; $n = qk + r$ is the only way to write n as a multiple of k plus a remainder from $0, 1, 2, \ldots, k - 1$.

The number q is called the *quotient* and r is called the *remainder*. You have proven this theorem above in two problems. The existence part was proven in __________. The uniqueness part was proven in __________.

Example 68. Rewrite Theorem 67 for the case where $k = 2$. Be specific about the possible values of r.

Prove 69. Let n be an integer greater than 0. Recall the definitions of even and odd. Use the existence part of the division algorithm with $k = 2$ to show that n must satisfy at least one of these definitions.

Prove 70. Let n be an integer greater than 0. Use the uniqueness part of the division algorithm with $k = 2$ to conclude that if n is even, then it cannot be odd. Also, if n is odd, it cannot be even. Thus, each integer is even or odd, not both.

Note 71. Now we will divide negative numbers by positive numbers, with remainder.

Example 72. Start with -37, add 5 to get -32, and add 5 repeatedly, writing down the numbers you come to, until you reach a number between 0 and 4. Count the number of 5's that you added to write $-37+5k = r$, where you fill in k and r. Rewrite this as $-37 = 5q+r$ and note the sign of q. This represents division of a negative number with remainder. How does it differ from division of a positive number with remainder?

Prove 73. Let n be an integer less than 0. Let k be an integer greater than 0. Add k to n repeatedly until you reach a number between 0 and $k-1$. Can you be sure that you will ever get all the way to non-negative numbers? Can you be sure that you don't jump over the numbers $0, 1, \ldots, k-1$ and keep adding k forever? Use the result to argue that you can write $n + pk = r$ for some integer p, and rearrange it to read $n = qk + r$. What do you know about the values of q and r?

Prove 74. Let n be an integer less than 0 and let k be an integer greater than 0. Scrutinize your proof of 66. The problem assumed that $n \geq 0$, but where did the proof use $n \geq 0$? If it did not, then the proof was more general than we needed at the time. What can we conclude about uniqueness when writing $n = qk + r$ for negative values of n?

Show 75. Suppose that n is an integer and $n = 3m + 1$ where m is an integer. Use the division algorithm to argue clearly that n cannot be a multiple of 3.

Show 76. Let n be an integer and suppose that n^2 is a multiple of 3. We would like to conclude that n is a multiple of 3. Let's use the contrapositive: We'll show that if n is not a multiple of 3, then n^2 is not a multiple of 3. Use the division algorithm to explain that there are three ways to write n as $3q + r$, and two of these make n not be a multiple of 3. In each of these two cases, compute n^2 and check that n^2 is not a multiple of 3, again using the division algorithm.

Show 77. Suppose n is an integer. Can n^2 be of the form $3m + 2$ where m is an integer? Examine the cases in the previous question carefully.

Show 78. Let n be an integer and consider the numbers n, $n+1$, and $n+2$. Show that exactly one of these is a multiple of 3. Use three cases, $n = 3k$, $n = 3k+1$, and $n = 3k+2$, and follow the guide below.

Case 1: $n = 3k$. Then $n + 1 =$ ____________________ and $n + 2 =$ ____________________.
Exactly one of these is a multiple of three (circle it). Use the division algorithm to argue that the other two are not multiples of 3.

Case 2: $n = 3k + 1$. Then $n + 1 =$ ____________________ and $n + 2 =$ ____________________.
Exactly one of these is a multiple of three (circle it). Use the division algorithm to argue that the other two are not multiples of 3.

Case 3: $n = 3k + 2$. Then $n + 1 =$ ____________________ and $n + 2 =$ ____________________.

Show 79. Let n be even and consider the numbers n and $n+2$. Use two cases to show that exactly one of these is a multiple of 4.

Show 80. Let n be an odd integer. Show that $n^3 - n$ is a multiple of 24. Here again, you will need a spark of genius. Use scratch paper to brainstorm different approaches that you could try, then try the one that looks the most promising.

Your name: ______________________________

The Pigeonhole Principle

Surprising results can come from simple counting.

Overview

The Pigeonhole Principle says that if we have more pigeons than pigeonholes to put them in, then at least one pigeonhole must contain more than one pigeon. This idea was stated by Johann Peter Gustav Lejeune Dirichlet in 1834 and so is sometimes called Dirichlet's Box Principle.

Problem 81. If N pigeons are placed in n pigeonholes and $N > n$, then one of the pigeonholes must contain two or more pigeons. **Hint:** Prove by contradiction. That is, pretend for a minute that none of the pigeonholes has more than one pigeon. That leads to a contradiction.

Note 82. In each problem below, identify the "pigeonholes" and the rule you use to put each "pigeon" into a "pigeonhole." The first one is done for you. Then write a nice solution of the problem.

Problem 83. A bag contains M&M's in six different colors: Brown, Yellow, Green, Red, Orange and Blue. How many M&M's do you need to take out of the bag in order to have at least two of the same color? How many do you need to take out of the bag if you want to have three of the same color?
Pigeonholes: The colors Brown, Yellow, Green, Red, Orange, and Blue.
Rule: Put each M&M in a pigeonhole according to its color.

Problem 84. Prove that no matter how we choose 51 distinct natural numbers from $\{1, 2, 3, \ldots, 100\}$, at least two of them must be consecutive.
Pigeonholes: The sets $\{1, 2\}, \{3, 4\}, \ldots, \{99, 100\}$.
Rule:

Problem 85. Prove that given ten integers we can choose two of them such that their difference is divisible by nine.
Pigeonholes:
Rule: Calculate the remainder when dividing by 9.

Problem 86. Prove that if six distinct numbers are selected from the set $\{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$ then some set of two of them add up to eleven.
Pigeonholes:
Rule:

Problem 87. A classroom floor is painted white and black. Is it always possible to find two points of the same color, exactly one foot apart? **Hint:** Think about an equilateral triangle.
Pigeonholes:
Rule:

Problem 88. If you have even more pigeons, you sometimes need more than two pigeons in each pigeonhole. Suppose we need to place N items into n boxes and $N > n$. Then at least one box must contain at least $\lceil \frac{N}{n} \rceil$ items, which is $\frac{N}{n}$ rounded up to the nearest integer. **Hint:** Let N_i be the number of items in box i, where $1 \leq i \leq n$. Then $N = N_1 + N_2 + \ldots + N_n$. Assume for the sake of contradiction that $N_i \leq \lceil \frac{N}{n} \rceil - 1$, for all $i = 1, 2, \ldots, n$. Prove the fact that $\lceil \frac{N}{n} \rceil < \frac{N}{n} + 1$ and use it to reach a contradiction. **Suggestion:** Write $N = qn + r$ using the Division Algorithm.

Problem 89. The human head contains fewer than 150,000 hairs. Show that there are at least 50 people who all have the same number of hairs on their heads in New York City. You can assume that New York City has a population greater than 8,000,000 people.
Pigeonholes:
Rule:

Anonymous!

Class survey

I would like your feedback to improve the course and learn from it. Many thanks in advance!

1. You have been asked to read the textbook, and we have checked your notes to make sure this is happening. Would you recommend that other faculty do the same in their courses? **yes** **no**
Please explain.

Would you recommend that other students read their textbooks the same way in other courses? **yes** **no**
Please explain.

2. To what extent do you minimize distractions when you are reading the textbook? Do you turn off your cell phone? Do you study alone? I'm curious how hard you try to minimize distractions which will break your chain of thought, and whether it helps you understand the book more.

3. In what way(s) have you changed how you work with the textbook in other courses that you are taking? Do you read them more? Differently?

4. In class, we work through activities without much "lecture." How does this work for you?

5. Is there anything we should change about the class?

6. What is going well in the class, so that we should not change it?

7. Is this class making good progress on making you unstoppable in your other math courses?

8. What else could we do to make you unstoppable in your other math courses?

9. Do you look forward to coming to class? Why or why not?

Your name: ______________________________

Square roots of prime numbers are irrational

This is a classic example of proof by contradiction.

Overview

Most students are familiar with the fact that $\sqrt{2}$ is irrational, but few can prove it. Having read a proof of this fact in your textbook or online, the starting point is to re-create the proof from memory, then to move on to showing that $\sqrt{3}$ is irrational. The proof is similar and yet different.

Definition 90. Irrational. A real number is said to be *irrational* if it cannot be written as the quotient of two integers.

Note 91. If you are new to proof by contradiction, you might prefer to start the next proof by writing "Let's pretend for a minute that $\sqrt{2}$ can be written as $\frac{p}{q}$ where p and q are integers." This makes it extra clear that you don't particularly believe that $\sqrt{2}$ is rational, you are just exploring what would happen if that were true. When you arrive at a contradiction, you realize it's time to stop pretending; $\sqrt{2}$ must be irrational.

Prove 92. Prove that $\sqrt{2}$ is irrational by contradiction. The proof begins with "Assume for the sake of contradiction that $\sqrt{2}$ can be written as $\frac{p}{q}$ where p and q are integers." Argue to a contradiction.

Show 93. A key step in the proof is that if n is an integer and n^2 is even, then n is even. You may have already shown this, using a proof by contradiction (which begins, "Assume for the sake of contradiction that n is odd.") or a proof by contrapositive (which begins, "Let us show the contrapositive, that if n is odd, then n^2 is odd.") or a proof by cases (which begins, "There are two possibilities for n, that n is even or that n is odd.") Whichever one you have already seen, choose a different one and write the proof here.

Show 94. Mimic the proof that $\sqrt{2}$ is irrational to show that $\sqrt{3}$ is irrational.

Show 95. A key step in the proof that $\sqrt{3}$ is irrational is the fact for an integer n that: if n^2 is a multiple of 3, then n is a multiple of 3. Prove this by contradiction, starting with "Assume for the sake of contradiction that n is not a multiple of 3".

Show 96. Now write a proof by contrapositive that if n^2 is a multiple of 3, then n is a multiple of 3. Clearly state what the contrapositive is, then prove it.

Show 97. Now write a proof by cases.

Show 98. Suppose that n is an integer and that n^2 is a multiple of 5. Show that n is a multiple of 5. Make the logic of your proof crystal clear.

Show 99. Show that $\sqrt{5}$ is irrational. Now that you are getting good at proofs like this, try to write a picture perfect proof.

Show 100. Suppose that n is an integer and that n^2 is a multiple of p, where p is a prime number. Make a good start on showing that n is a multiple of p.

Show 101. Suppose that p is a prime number. Show that $\sqrt{p}$ is irrational, assuming that the previous result is true.

Show 102. Suppose that n is odd and suppose that $n^3 - n$ is a multiple of 24. Show that $(n+2)^3 - (n+2)$ is also a multiple of 24.

Show 103. Check that when $n = 1$, $n^3 - n$ is a multiple of 24. Use this together with the previous problem to conclude that $n^3 - n$ is a multiple of 24 for additional values of n. What values of n can your argument cover?

Show 104. Write an argument that will cover all other odd values of n

Your name: ______________________________

Examples of sets and relations between them

This activity introduces sets, ways to write them, and the relations between them.

Overview

Many familiar ideas can be expressed using sets. We begin with examples of sets and the relations between them.

Problem 105. Let $A = \{x : x \text{ solves } ax = b \text{ where } a \text{ and } b \text{ are integers and } a \neq 0\}$. Let $\mathbb{Q} = \{x : x \text{ is a rational number }\}$. Show that $A = \mathbb{Q}$ by showing set inclusion in both directions. The first part is done for you; read that carefully.

- Let $x \in A$. Then there exist a and b such that $ax = b$ and $a \neq 0$. Dividing through by a, $x = \frac{b}{a}$ where a and b are integers and a is not zero. Thus, $x \in \mathbb{Q}$. Since x was arbitrary, $A \subseteq \mathbb{Q}$.
- Let $x \in \mathbb{Q}$.

Problem 106. Let $A = \{f : f \text{ is a continuous function from } \mathbb{R} \text{ to } \mathbb{R}\}$. Let $B = \{f : f \text{ is a differentiable function from } \mathbb{R} \text{ to } \mathbb{R}\}$. Determine whether $A \subset B$, $B \subset A$, $A \subseteq B$, or $B \subseteq A$ and then write a clean argument that it is so. Remember that to show $\subset$, you need an example of an element that is in one set but not in the other.

Problem 107. Let $A = \{x \in \mathbb{R} : x \text{ solves } x^2 = a \text{ where } a \text{ is an integer and } a \geq 0\}$. Show that $A \not\subset \mathbb{Q}$. Make your logic crystal clear.

Problem 108. Continuing the previous problem, show that $\mathbb{Q} \not\subset A$. Make your logic crystal clear.

Problem 109. Let $E = \{m \in \mathbb{Z} : \text{there exists } j \in \mathbb{Z} \text{ such that } m = 2j\}$. Let $O = \{m \in \mathbb{Z} : \text{there exists } k \in \mathbb{Z} \text{ such that } m = 2k + 1\}$. Show that $E \cap O = \emptyset$ by letting $m \in E \cap O$ and showing that this leads to a contradiction.

Problem 110. Continuing the previous problem, show that $E \cup O = \mathbb{Z}$ by showing set inclusion both ways.

Problem 111. Let $2\mathbb{Z} = \{m \in \mathbb{Z} : \text{there exists } j \in \mathbb{Z} \text{ such that } m = 2j\}$. Let $3\mathbb{Z} = \{m \in \mathbb{Z} : \text{there exists } j \in \mathbb{Z} \text{ such that } m = 3j\}$, and similarly with other sets like $5\mathbb{Z}$ and $15Z$. Show that $2\mathbb{Z} \cap 3\mathbb{Z} = 6\mathbb{Z}$ by showing set inclusion both ways.

Problem 112. Write out all elements in $6\mathbb{Z} \cap 8\mathbb{Z} \cap \{1, 2, 3, \ldots, 100\}$.

Note 113. Inequalities between real numbers have the *transitivity* property: If $a \leq b$ and $b \leq c$, then we can conclude that $a \leq c$. Similar inequalities are true with $\geq, <$, and $>$.

Problem 114. Suppose that $x > 4$. Argue that $x \geq 2$.

Problem 115. Using standard interval notation, show that $(4, 9] \subset [2, 9]$. Begin with "Let $x \in (4, 9]$." then rewrite this as a compound inequality, then rewrite as two separate inequalities. Use transitivity along the way. Make sure to

Problem 116. Using standard interval notation, show that $[2,6) \cap [3,8) = [3,6)$ by showing set inclusion both ways. As above, write compound inequalities, then individual inequalities, then compound inequalities again. Use a number line to illustrate.

Problem 117. Show that $[2,6) \cup [3,8) = [2,8)$ by showing set inclusion both ways.

Your name: ______________________________

Operations on sets

This activity works with set identities and relates them to logic.

Overview

Sets are absolutely fundamental to mathematics. This chapter focuses on building up set identities, relationships between sets that are always true.

Problem 118. Let A and B be sets. Show that $(A \cup B)^c = A^c \cap B^c$ by showing set inclusion both ways. The first part is done for you. This is one of de Morgan's laws. Draw a really nice Venn diagram to illustrate.

- Let $x \in (A \cup B)^c$. Then $x \notin A \cup B$. So $x \notin A$ and $x \notin B$. That means that $x \in A^c$ and $x \in B^c$, and so $x \in A^c \cap B^c$. Since x was arbitrary, $(A \cup B)^c \subseteq A^c \cap B^c$.
- Let $x \in A^c \cap B^c$.

Problem 119. Let A and B be sets. Show that $(A \cap B)^c = A^c \cup B^c$ by showing set inclusion both ways. This is the other one of de Morgan's laws. Draw a really nice Venn diagram to illustrate.

Problem 120. Let A and B be sets. Let P be the logical statement $x \in A$, and let Q be the logical statement $x \in B$. Use P and Q and logic symbols ($\wedge$ for *and*, $\vee$ for *or*, $\neg$ for *not*) to translate statements about sets into logic statements:

1. $x \in A \cup B$ is ____________________

2. $x \in A^c$ is ____________________

3. $x \in B^c$ is ____________________

4. $x \in A^c \cap B^c$ is ____________________

5. $x \in (A \cup B)^c$ is ____________________

In the white space above and to the right, make a truth table for P, Q, and each of the other logical statements in the previous problem to establish that $x \in (A \cup B)^c$ is logically equivalent to $x \in A^c \cap B^c$. Compare the truth values in the columns corresponding to $x \in (A \cup B)^c$ to the Venn diagram you made above. Explain how they agree.

Problem 121. Let A and B be sets. Follow the previous exercise to use a truth table to show that $x \in (A \cap B)^c$ is logically equivalent to $x \in A^c \cup B^c$. Compare the truth table to the Venn diagram again.

Problem 122. Let D, E, and F be sets. Use one of de Morgan's laws that you showed above to establish that $(D \cup E \cup F)^c = D^c \cap E^c \cap F^c$. This proof works by rewriting, not by showing inclusion both ways. **Hint:** Let $A = D \cup E$ and $B = F$.

Problem 123. Let D, E, and F be sets. Use one of de Morgan's laws to show that $(D \cap E \cap F)^c = D^c \cup E^c \cup F^c$.

Problem 124. Let A, B, and C be sets. Use logical statements P, Q, and R and a truth table to show that $x \in A \cup (B \cap C)$ is logically equivalent to $x \in (A \cup B) \cap (A \cup C)$. Be sure to define P, Q, and R at the beginning.

Problem 125. Let A, B, and C be sets. Show that $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$ by showing inclusion both ways. When you encounter a union, use a proof by cases. For example, if you know that $x \in A \cup B$, one case is that $x \in A$, the other is that x is not in A, but $x \in B$. Organize your writing carefully to make the steps of this argument really clear.

Definition 126. Set difference. Let A and B be sets. The *set difference* $A \backslash B$ is the set $A \cap B^c$, which is all points that are in A but not in B. Draw a Venn diagram to illustrate this definition.

Definition 127. Symmetric difference. Let A and B be sets. The *symmetric difference* of A and B is the set $A \triangle B = (A \backslash B) \cup (B \backslash A)$. Draw a Venn diagram to illustrate this definition.

Problem 128. Consider again the logical statements from 120. Write a logical statement that is equivalent to $x \in A \triangle B$. Make a truth table with 4 rows, labeled 1, 2, 3, 4, and three columns, one for $x \in A$, one for $x \in B$, and the third for $x \in A \triangle B$. Draw a Venn diagram and label the regions in it 1, 2, 3, 4 so that they correspond to the truth table.

Problem 129. Let A and B be sets. Show that $A \triangle B = B \triangle A$ by showing set inclusion both ways. Draw a nice Venn diagram to illustrate.

Problem 130. Let A, B, and C be sets. Show that $(A \triangle B) \triangle C = A \triangle (B \triangle C)$ in three ways.

1. Draw separate Venn diagrams for the two sets.

2. Show set inclusion both ways.

3. Convert inclusion in $A \triangle B$, $B \triangle C$, and other sets to logical statements and use a truth table to show the equality.

Your name: ____________________________

Infinite unions, intersections, and a few other things

This activity is a prequel to working with infinite unions and intersections.

Overview

Infinite unions and intersections take a bit of getting used to. Fortunately, we can understand them with quantifiers.

Definition 131. Union. Let $A_1, A_2, \ldots$ be sets, with universe X. The *union* of $A_1, A_2, \ldots$, which is denoted $\bigcup_{n=1}^{\infty} A_n$, is all elements of X which are in A_n for some $n = 1, 2, 3, \ldots$.

Definition 132. Intersection. Let $A_1, A_2, \ldots$ be sets, with universe X. The *intersection* of $A_1, A_2, \ldots$, which is denoted $\bigcap_{n=1}^{\infty} A_n$, is all elements of X which are in A_n for all $n = 1, 2, 3, \ldots$.

Problem 133. Use quantifiers to express what it means that $x \in \bigcup_{n=1}^{\infty} A_n$.
Solution: $\exists n, x \in A_n$. In words, there is at least one n for which x is in A_n; that is what it takes to be in the union.

Problem 134. Work with quantifiers to express what it means that $x \notin \bigcup_{n=1}^{\infty} A_n$. Negate the previous expression and use rules of quantifiers to rewrite it, one small step at a time, until it is as simple as possible.

Problem 135. Use quantifiers to express what it means that $x \in \bigcap_{n=1}^{\infty} A_n$.

Problem 136. Work with quantifiers to express what it means that $x \notin \bigcap_{n=1}^{\infty} A_n$. Negate the previous expression, then rewrite again using complements.

Stop. Go back to each of the four preceding problems and write a sentence explaining the logic of the last expression that you wrote down and how it relates to the expression you started with.

Problem 137. de Morgan's law. Show that $\left(\bigcup_{i\in I} A_i\right)^c = \bigcap_{i\in I} A_i^c$ by writing logical expressions for x being in the set on the left side and for the right side. Note that here the union is over sets A_i where the index i comes from an index set I, but the logic is the same as in the previous problems. Start by writing a logical expression that means the same thing as $x \in \left(\bigcup_{i\in I} A_i\right)^c$ and work with it until it is a logical expression for $x \in \bigcap_{i\in I} A_i^c$. When you write the proof this way, you do not need to show containment both ways to show that the two sets are equal.

$x \in \left(\bigcup_{i\in I} A_i\right)^c$ means $\neg(\exists i \in I, x \in A_i)$, which means ...

Problem 138. Show that $(3, \infty) \subset [3, \infty)$; these are both intervals on the real number line. Remember that when you show $\subset$ there are two things to show: containment, and that there is an element of one set that is not an element of the other. Solve this problem by letting $x \in (3, \infty)$ and writing that information as the logical statement "$x > 3$ is true".

Problem 139. Show that $[2, 5) \cap (3, 7) \subseteq (3, 5)$ using inequalities. Start by letting $x \in [2, 5) \cap (3, 7)$.

Problem 140. Let $x > 0$. Show that there exists an integer n such that $0 < \frac{1}{n} < x$. **Hint:** Look at $\frac{1}{x}$ and round up. **Another hint:** Suppose that $x = 0.31$. What value of n works?

Problem 141. Let n be an integer greater than 0. Show that $[\frac{1}{n}, 1] \subseteq (0, 1] \subseteq [0, 1]$ by working with inequalities. Then show that $[\frac{1}{n}, 1] \subset (0, 1] \subset [0, 1]$ by looking at individual points.

Your name: ______________________________

Infinite operations on sets

Unions and intersections of infinitely many sets

Overview

We are working with sets of real numbers. These exercises will give you practice with sets and teach you things about the real numbers as well.

Problem 142. Let $A = \bigcup_{n=1}^{\infty} [n, n+1)$. As indicated below, write A in open form, listing out the first 5 sets in the union. Figure out what interval A is equal to, call the interval B, then show that $A = B$ by showing containment both ways, as indicated below.

$A =$ _________ $\cup$ _________ $\cup$ _________ $\cup$ _________ $\cup$ _________ $\cup \cdots$

$B =$ ____________

1) Let $x \in A$. Then __________________ for some $n = 1, 2, 3, \ldots$.

Thus $x \in B$. Since $x \in A$ was arbitrary, $A \subseteq B$. (Always end a proof of inclusion this way!)

2) Let $x \in B$. **Note:** You need to show that there exists an n for which $x \in [n, n+1)$. Tell us how to get the right value of n, starting with x.

Problem 143. Let $A = \bigcup_{n=1}^{\infty} [-n, n]$. As indicated below, write A in open form, listing out the first 5 sets in the union. Figure out what interval A is equal to, call the interval B, then show that $A = B$ by showing containment both ways, as indicated below.

$A =$ _________ $\cup$ _________ $\cup$ _________ $\cup$ _________ $\cup$ _________ $\cup \cdots$

$B =$ ____________

1) Let $x \in A$. Then __________________ for some $n = 1, 2, 3, \ldots$.

2) Let $x \in B$. **Note:** You need to show that there exists an n for which $x \in [-n, n]$. Tell us how to get the right value of n, starting with x.

Problem 144. Let $A = \bigcap_{n=1}^{\infty} (-n, n)$. As indicated below, write A in open form, listing out the first 5 sets in the intersection. Figure out what interval A is equal to, call the interval B, then show that $A = B$ by showing containment both ways, as indicated below.

$A =$ __________ $\cap$ __________ $\cap$ __________ $\cap$ __________ $\cap$ __________ $\cap \cdots$

$B =$ ______________

1) Let $x \in A$. Then ____________________ for all $n = 1, 2, 3, \ldots$.

2) Let $x \in B$.

Problem 145. For $n = 2, 3, 4, \ldots$, let $A_n = \{2n, 3n, 4n, \ldots\}$.

1. Write out the first five of the A_n.

2. Let $B = \bigcup_{n=2}^{\infty} A_n$. Describe the set B in simpler terms, perhaps by writing out the smallest 10 elements of B, then describe B in a sentence.

3. What is $\mathbb{N} \backslash B$? Remember that $\mathbb{N} = \{0, 1, 2, 3, \ldots\}$.

Problem 146. Let $A = \bigcup_{n=0}^{\infty} [n, n^2]$. List out the first five sets in this union, as you did above. Draw them on a number line if it helps. Make a conjecture about how you can write A in a simpler way, call the new set B, then prove that $A = B$ by showing containment in both directions. In each direction, you will need to use three cases.

Problem 147. Let $A = \bigcup_{n=1}^{\infty} [\frac{1}{n}, 1]$. List out the first five sets in this union, as you did above. Draw a picture of them above a number line. Make a conjecture about what interval A is equal to, call the new set B, then show that $A = B$ by showing containment both ways. You will need to use this property of real numbers: if $x > 0$, then there exists a positive integer n with $0 < \frac{1}{n} < x$.

Problem 148. Let $A = \bigcup_{r \in \mathbb{Q}} (r - \frac{1}{10}, r + \frac{1}{10})$. Here, $\mathbb{Q}$ is the set of all rational numbers. Think of a simpler way to describe the set A, then prove your conjecture by showing set containment both ways.

Problem 149. Let $A = \bigcap_{n=1}^{\infty} [0, 1 + \frac{1}{n}]$. List out the first five sets in this intersection, as you did above. Draw a picture of them above a number line. Make a conjecture about what interval A is equal to, call the new set B, then prove that $A = B$ by showing containment both ways.

Hint: You may want to show that $A \subseteq B$ by showing the logically equivalent statement that $B^c \subseteq A^c$ (thinking of the universal set as $[0, \infty)$ so you can avoid negative numbers). This is the same as the contrapositive: suppose that $x \notin B$, then show that $x \notin A$. You may find it useful to keep in mind that if $x > 0$, then there exists an integer n for which $0 < \frac{1}{n} < x$.

Problem 150. Let $A = \bigcup_{k \in \mathbb{Z}} (k, k+1)$.

1. Draw out some of the intervals on a number line.

2. Make a conjecture about what set A is.

Problem 151. Let $a < b$. Show that $\bigcup_{n=1}^{\infty} [a, b - \frac{1}{n}] = [a, b)$. Draw pictures, then show set inclusion both ways. If $b - \frac{1}{n} < a$, the interval is empty.

Problem 152. Let $a < b$. Show that $\bigcap_{n=1}^{\infty} (a - \frac{1}{n}, b + \frac{1}{n}) = [a, b]$. Draw pictures, then show set inclusion both ways.

Your name: ________________________________

Homework on set theory

20 points

Let A, B, and C be sets. Make your logic crystal clear. You can use the back of this sheet of paper.

1. Suppose that $A \subseteq C$ and $B \subseteq C$. Show that $A \cup B \subseteq C$.

2. Suppose that $A \cup B \subseteq C$. Show that $A \subseteq C$ and $B \subseteq C$.

3. Suppose that $A \cup B \subseteq A$. Show that $B \subseteq A$.

4. Suppose that $B \subseteq A$. Show that $A \cup B \subseteq A$.

Your name: ______________________________

Mathematical Induction

Proving that a claim is true for all n

Overview

One important task in mathematics is to find and distinguish regular patterns or sequences. The main method we use to prove results involving positive integers or about sequences is mathematical induction.

Theorem 153. Mathematical induction

For each integer n, let $P(n)$ denote an assertion involving n.

(i) (The basis step) Prove that $P(1)$ is true.

(ii) (The inductive step) For each $n = 1, 2, 3, \ldots$, assume that $P(n)$ is true, and use $P(n)$ to prove that $P(n+1)$ is true.

From the above two steps, we can conclude that $P(n)$ is true for all $n = 1, 2, 3, \ldots$.

Note 154. Usually, the assertion $P(n+1)$ can be written in terms of the assertion $P(n)$ plus a new part. Use what you know from $P(n)$ being true to make it easier to prove what you need to show.

Guided proof 155. Show that 3^n is odd for all $n = 1, 2, 3, \ldots$.

1. State $P(n)$: $P(n)$ is that ________________________

2. Basis step: $P(1)$ is that __________________. This is true because __________________.

3. State $P(n+1)$: $P(n+1)$ is that ________________________

4. Inductive step: Let $n \geq 1$. Assume that $P(n)$ is true. Show that $P(n+1)$ is true. You may use facts you have already proven about odd numbers.

Example 156. List the first 7 positive odd integers in the table below, and write out the sum of the first n positive odd numbers in the third row. For example, when $n = 3$, the sum is $1 + 3 + 5$.

n	1	2	3	4	5	6	7	$\cdots$	n	$n+1$
odd numbers	1	3	5	7				$\cdots$		
sum	1	4						$\cdots$		

Write out a formula for the nth odd integer in terms of n. Also write a formula for the $n+1$st odd integer in terms of n. Make conjecture about the sum of the first n odd integers, which will look like this: $1 + 3 + 5 + \cdots + (2n-1) =$ ____________.

Guided proof 157. Prove the conjecture in 156 using mathematical induction.

1. State $P(n)$: $P(n)$ is that $1 + 3 + 5 + \cdots + (2n-1) =$ ____________

2. Basis step: $P(1)$ is that __________________. This is true because __________________.

3. Write out $P(n+1)$: $P(n+1)$ is that ____________________________________

4. Inductive step: Let $n \geq 1$. Assume that $P(n)$ is true, and use that to show that $P(n+1)$ is true.

By ________________________________ we conclude that $1+3+5+\cdots+(2n-1) =$ ___________ for all $n = 1, 2, 3, \ldots$.

Notation 158. The standard notation for the sum of $a_1, a_2, \ldots, a_n$ is $a_1 + a_2 + a_3 + \cdots + a_n = \sum_{k=1}^{n} a_k$ and the notation for the product of $a_1, a_2, \ldots, a_n$ is $a_1 \cdot a_2 \cdot a_3 \cdots \cdot a_n = \prod_{k=1}^{n} a_k$.

Example 159. Rewrite the result in 157 using the standard notation for the sum.

Show 160. Show that $\sum_{k=1}^{n}(4k-3) = n(2n-1)$ for all positive integers n.

1. State $P(n)$: $P(n)$ is that:

2. Basis step: $P(1)$ is that:

3. State $P(n+1)$: $P(n+1)$ is that:

4. Inductive step: Let $n \geq 1$. Assume that $P(n)$ is true, and use that to show that $P(n+1)$ is true.

Show 161. Use mathematical induction to show that $\sum_{k=1}^{n} 5^k = \frac{5}{4}(5^n - 1)$ for all positive integers n.

1. State $P(n)$: $P(n)$ is that:

2. Basis step: $P(1)$ is that:

3. State $P(n+1)$: $P(n+1)$ is that:

4. Inductive step: Let $n \geq 1$. Assume that $P(n)$ is true, and use that to show that $P(n+1)$ is true.

Stop. Compare your proofs with the other people in your group before you move on.

Note 162. The basis step need not use $n = 1$, for example, it can use $n = -3, n = 0$, or $n = 100$.

Show 163. Use mathematical induction to show that $2n + 1 < 2^n$ for all integers n with $n \geq 4$.

1. State $P(n)$: $P(n)$ is that:

2. Basis step: $P(4)$ is that:

3. State $P(n+1)$: $P(n+1)$ is that:

4. Inductive step: Let $n \geq 4$. Assume that $P(n)$ is true, and use that to show that $P(n+1)$ is true.

Show 164. Use mathematical induction to show that $5^n > 2^n + 3^n$ for all integers n with $n \geq 2$. Use the same format as above.

Show 165. Use induction to prove Bernoulli's inequality: For $x \in \mathbb{R}$, if $1 + x > 0$, then $(1+x)^n \geq 1 + nx$ for all $n = 0, 1, 2, \ldots$. Use the same format as above.

Show 166. Use induction to prove that $\frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \frac{1}{3\cdot 4} + \cdots + \frac{1}{n\cdot(n+1)} = \frac{n}{n+1}$ for all positive integers n.

Note 167. In Problem 166, we should be able to show that the statement is true without using mathematical induction. How?

Show 168. For each $n \in \mathbb{Z}^+$, let $P(n)$ denote the assertion "$n^2 + 5n + 1$ is an even integer."

1. Prove that $P(n+1)$ is true whenever $P(n)$ is true.

2. For which n is $P(n)$ actually true?

3. What is moral of this exercise?

Show 169. Use induction to prove that $n^3 - n$ is a multiple of 6 for all integers $n = 0, 1, 2, \ldots$.

Show 170. Use induction to prove that $11^n - 4^n$ is a multiple of 7 for all $n = 0, 1, 2, \ldots$.

Show 171. Prove that $1^2 - 2^2 + 3^2 - 4^2 + 5^2 + \cdots - (2n)^2 + (2n+1)^2 = (n+1)(2n+1)$ for all $n = 0, 1, 2, \ldots$.
Hint: It would be helpful to write down $P(0)$ and $P(1)$ first.

Note 172. We also can use mathematical induction to show some propositions about sets.

Show 173. Prove that if $A_1, A_2, \ldots, A_n$ and $B_1, B_2, \ldots, B_n$ are sets such that $A_j \subseteq B_j$ for $j = 1, 2, \ldots, n$, then $\bigcap_{j=1}^{n} A_j \subseteq \bigcap_{j=1}^{n} B_j$. **Hint:** In the initial step, show that if $A_1 \subseteq B_1$ and $A_2 \subseteq B_2$, then $A_1 \cap A_2 \subseteq B_1 \cap B_2$.

Show 174. Prove that if $A_1, A_2, \ldots, A_n$ and B are sets, then $(A_1 \cup A_2 \cup \cdots \cup A_n) \cap B = (A_1 \cap B) \cup (A_2 \cap B) \cdots \cup (A_n \cap B)$ for any given positive integer n. **Hint:** In the initial step, we have to show the distributive property of intersection over union of sets: $(A_1 \cup A_2) \cap B = (A_1 \cap B) \cup (A_2 \cap B)$. We have already shown that earlier in the semester. You will also use this property in the inductive step.

Note 175. In 173 and 174, instead of showing $P(1)$ is true, we show $P(2)$ is true in the initial step. But why? Explain!

Your name: ______________________________

Set theory practice

More practice working with sets

Overview

Write really detailed proofs with crystal clear logic. In particular, when showing that $A \subseteq B$, start with "Let $x \in A$," show that x is in B, and then say, "Since $x \in A$ was arbitrary, $A \subseteq B$."

Show 176. Let A, B, and C be sets. Suppose that $A \subseteq B$ and $B \subseteq C$. Show that $A \subseteq C$. **Note:** This *shows* transitivity but it does not *use* transitivity.

Show 177. Let A, B, and C be sets. Suppose that $A \subset B$ and $B \subset C$. Show that $A \subset C$. **Note:** Here we have strict set inclusion, so you will need to show that A is not equal to C.

Show 178. Let A, B, and C be sets. Show that $C \subseteq A$ and $C \subseteq B$ if and only if $C \subseteq A \cap B$. **Note:** "If and only if" means there are two things to show:

1. Suppose that $C \subseteq A$ and $C \subseteq B$. Show that $C \subseteq A \cap B$.

2. Suppose that $C \subseteq A \cap B$. Show that $C \subseteq A$ and $C \subseteq B$.

Show 179. Let A and B be sets. Show that $A \cap B = B$ if and only if $B \subseteq A$.

Show 180. Let $A_1, A_2, A_3, \ldots$ and $B_1, B_2, B_3, \ldots$ be sets. Suppose that $A_n \subseteq B_n$ for all $n = 1, 2, 3, \ldots$. Show that $\bigcap_{n=1}^{\infty} A_n \subseteq \bigcap_{n=1}^{\infty} B_n$

Show 181. Let $A_1, A_2, A_3, \ldots$ and $B_1, B_2, B_3, \ldots$ be sets. Suppose that $A_n \subseteq B_n$ for all $n = 1, 2, 3, \ldots$. Show that $\bigcup_{n=1}^{\infty} A_n \subseteq \bigcup_{n=1}^{\infty} B_n$

Show 182. Let A and $B_1, B_2, B_3, \ldots$ be sets. Suppose that $A \subseteq \bigcap_{n=1}^{\infty} B_n$. Show that $A \subseteq B_n$ for all $n = 1, 2, 3, \ldots$. Start the proof with "Let $n \geq 1$ be an integer" and be sure to end the proof by generalizing over n. The second step in the proof is "Let $x \in A$."

Your name: ______________________________

The power set and the Cartesian product

Useful constructions with sets.

Overview

The power set is our first example of thinking hard about collections of sets. The Cartesian product is used often when you want ordered pairs or ordered triples of numbers or other objects.

Problem 183. Write out the members of the following power sets. It may be helpful to do #3, #4, then #2, #1, and finally #5.

1. $S = \emptyset$. $\mathcal{P}(S) =$

2. $S = \{1\}$. $\mathcal{P}(S) =$

3. $S = \{1, 2\}$. $\mathcal{P}(S) =$

4. $S = \{1, 2, 3\}$. $\mathcal{P}(S) =$

5. $S = \{1, 2, 3, 4\}$. $\mathcal{P}(S) =$

6. $S = \{1, 2, 3, 4, 5\}$. $\mathcal{P}(S) =$

Question 184. If S has n elements, how many members will $\mathcal{P}(S)$ have? Explain as well as you can.

Problem 185. Write the appropriate symbol between the entities, or mark the statement as true or false. Give an explanation for anything that is not obvious enough.

1. $1 \qquad \mathcal{P}(\{1, 2, 3\})$
2. $[3, 10] \qquad \mathbb{Z}$
3. $[3, 10] \qquad \mathbb{R}$
4. $\mathbb{Q} \qquad \mathbb{R}$
5. $\mathbb{Q} \qquad \mathcal{P}(\mathbb{R})$
6. $[3, 10] \qquad \mathcal{P}(\mathbb{R})$
7. $\mathbb{N} \qquad \mathbb{R}$
8. $\emptyset \qquad \mathbb{R}$
9. $\emptyset \qquad \mathcal{P}(\mathbb{R})$
10. $\{\emptyset\} \subseteq A$?
11. $\emptyset \subset \mathcal{P}(A)$?

Question 186. Suppose that S is a set. Then $\mathcal{P}(S)$ is also a set, but if we let $A \in \mathcal{P}(S)$, then A is also a set. Explain how this can be. What is the relationship between A and S?

Problem 187. Let I be a set, and for each i in I, let B_i be a set. Show that $\mathcal{P}\left(\bigcap_{i\in I} B_i\right) = \bigcap_{i\in I} \mathcal{P}(B_i)$. Let A be an element of the set on the left-hand side. Notice that A is a set. Argue that it is an element of the set on the right-hand side. Let A be an element of the set on the right-hand side ...

Problem 188.

1. Sketch the Cartesian product $A = [1, 3] \times [2, 5]$.

2. Sketch the Cartesian product $B = [2, 4] \times [1, 3]$.

3. Sketch the intersection $A \cap B$.

4. It seems that $A \cap B$ is also a Cartesian product. Identify the sets whose product is $A \cap B$.

5. What is $([1, 3] \cap [2, 4]) \times ([2, 5] \cap [1, 3])$?

Your name: ______________________________

Relations

Often treated as the little brother to functions, relations have unsuspected depth.

Overview

You are already familiar with a number of relations, including $<, \leq, =, \geq$, and $>$ for real numbers, plus $\subset$, $\subseteq$, and $=$ for sets. Many other relations can be defined. The most useful ones are called equivalence relations; they are analogous to equality for numbers and for sets. They partition the space into equivalence classes, which are very useful in a number of ways.

Definition 189. Relation. A *relation* on a set X is a subset S of $X \times X$.

Notation 190. Suppose that S is a relation on a set X. That is, suppose that S is a subset of $X \times X$, which means that S is a set of points of the form (x, y), where $x \in X$ and $y \in X$. Rather than write $(x, y) \in S$, we usually write $x \sim y$. How to read this out loud? There is no perfect solution. I would suggest that you read it as "x tilde y" because $\sim$ is the tilde that appears above the n in some Spanish words.

Problem 191. You are going to write out the subsets of $\{1, 2, 3, 4\}$ and then draw arrows between them to indicate the proper subset relation. You might want to lay the sets out in a nice order to make the arrows easy to draw and to read. What is the set X on which this relation is defined?

Definition 192. Reflexive. A relation $\sim$ is *reflexive* if $x \sim x$ for all x in X.

Definition 193. Symmetric. A relation $\sim$ is *symmetric* if $x \sim y$ implies $y \sim x$.

Definition 194. Transitive. A relation $\sim$ is *transitive* if $x \sim y$ and $y \sim z$ implies $x \sim z$.

Note 195. Sometimes people have a hard time remembering the words reflexivity, symmetry, and transitivity. Notice that they are in alphabetical order, and that they involve 1, 2, or 3 objects at a time, respectively.

Definition 196. Equivalence relation. A relation $\sim$ is called an *equivalence relation* if it is reflexive, symmetric, and transitive. Note that equality is an equivalence relation on the set of real numbers.

Definition 197. Equivalence class. Suppose that $\sim$ is an equivalence relation. Fix x in X. The set of all elements y for which $x \sim y$ is called the *equivalence class containing* x.

Problem 198. Let $X = \mathbb{Z}^+$ and say that $x \sim y$ if y is divisible by x. People often write $x|y$ for this relation and say that x divides y.

1. Check whether this relation is reflexive. If so, prove that it is, starting with "Let $x \in \mathbb{Z}^+$." If not, give a counterexample.

2. Check whether this relation is symmetric. If so, prove that it is, starting with "Let $x, y \in \mathbb{Z}^+$ and suppose that $x \sim y$." If not, give a counterexample.

3. Check whether this relation is transitive. If so, prove that it is, starting with "Let $x, y, z \in \mathbb{Z}^+$ and suppose that $x \sim y$ and $y \sim z$." If not, give a counterexample.

4. Thinking of the relation as a set of ordered pairs, write out ten different ordered pairs satisfying the relation, and graph them on the xy plane.

Problem 199. Consider all cities in the US that have population over 30,000. For each of the following relations, determine whether they are reflexive, symmetric, and/or transitive. Provide a counterexample for any property that fails to hold. If all three hold, the relation is an equivalence relation. In that case, identify the equivalence classes and tell how many such classes there are.

1. Say that $x \sim y$ if the names of cities x and y start with the same letter.

2. Say that $x \sim y$ if x and y are in the same state.

3. Say that $x \sim y$ if cities x and y are within 50 miles of each other.

Problem 200. Let X be the set of all English words. Say that $x \sim y$ if the letters in x and y appear on the same number keys on a cell phone, in the same order. For example, BAR $\sim$ CAP.

1. Check whether this relation is reflexive.

2. Check whether this relation is symmetric.

3. Check whether this relation is transitive.

4. If all three properties hold, describe the equivalence classes, and tell what the equivalence class of BAR is.

Problem 201. Let $X = \mathbb{Z}$. Say that $x \sim y$ if x and y has the same remainder as y when they are divided by 3. Then, for example, $13 \sim 19$ and $9 \sim 27$.

1. Show that this relation is reflexive, symmetric, and transitive. Do this in general, starting with "Let."

2. Describe all elements of the equivalence class containing 0.

3. Describe the other equivalence classes. How many are there?

Problem 202. Consider the set X of all non–zero 3–dimensional vectors. For $\vec{a}$ and $\vec{b}$ in X, say that $\vec{a} \sim \vec{b}$ if there exists a constant c for which $\vec{a} = c\vec{b}$.

1. Show that this relation is reflexive, starting with "Let." Tell what c is.

2. Show that this relation is symmetric. You will need two values of c.

3. Show that this relation is transitive. Here there will be three values of c.

4. This is an equivalence relation. Describe the equivalence classes. The collection of all equivalence classes is called *projective space*.

5. Could you use the angle between lines to define a distance between equivalence classes? What would the maximum distance be?

Problem 203. Consider the set of all English words. Say that $x \sim y$ if one can be obtained from the other by changing exactly one letter. For example, BAT $\sim$ CAT but BAT $\nsim$ CAR. Check whether this relation is reflexive, symmetric, and/or transitive. Provide counterexamples if necessary.

Problem 204. Consider the set of all functions on the real line. That is, consider the set of all $f : \mathbb{R} \to \mathbb{R}$. Say that $f \sim g$ if f and g are equal except at a finite number of points. For example, if $f(x) = x^2$ and $g(x) = \begin{cases} x^2, & x \neq 0 \\ 5, & x = 0 \end{cases}$, then $f \sim g$. Show that this is an equivalence relation. How can you describe the equivalence classes?

Note 205. Problems 10.1 and 10.3 from Daepp and Gorkin[1] are particularly good at this stage in the course.

[1]Reading, Writing, and Proving: A Closer Look at Mathematics, 2011, by Ulrich Daepp and Pamela Gorkin

Homework problems, week 12

Due on (put date here).

Write up solutions of each of the problems below. They are designed to be straightforward problems. The goal is to come as close to perfection in your solutions as you can.

- Do not take shortcuts.
- If you need to show that something is true for all n, or for all x, y, start the proof with "Let ..."
- If you need cases, explain what the cases are and why they cover all the possibilities.
- If you are doing a proof by contradiction, start that part by saying "Assume ..."
- If you are doing a proof by contrapositive, tell what P and Q are, and that you will be showing that $\neg Q$ implies $\neg P$.
- Take small steps in each proof, and explain each step.
- Follow good form.
- If your proof started with "Let ..." it will probably end by saying "We made no further assumption ..."

Here are the problems to do. You can write them in your notebook or on separate paper.

1. Show that if n is an integer and $7n$ is odd, then n is odd. **Hint:** Be clear what facts you are using about even and odd numbers.

2. Without consulting your book or your notes, prove that $\sqrt{2}$ is irrational. I mean it. Do this from memory. You should be able to write a very nice proof, with no missing steps.

3. Let x and y be real numbers, and suppose that the product xy is irrational. Show that either x or y (or both) must be irrational. **Hint:** You can do this. Be patient, think about it.

4. Let $A = \{2k+1 : k \in \mathbb{Z}\}$ and let $B = \{2m-11 : m \in \mathbb{Z}\}$. Show that $A = B$ by showing containment both ways. **Hint:** Use good form!

5. Let $A = \{(x, y) \in \mathbb{R}^2 : y = 5x/7 - 2/7\}$ and $B = \{(x, y) \in \mathbb{R}^2 : 5x - 7y = 2\}$. Show that $A = B$ by showing containment both ways.

6. Let $A = \{m \in \mathbb{Z} : m = 15k \text{ for some } k \in \mathbb{Z}\}$, let $B = \{m \in \mathbb{Z} : m = 35j \text{ for some } j \in \mathbb{Z}\}$, and let $C = \{m \in \mathbb{Z} : m = 105n \text{ for some } n \in \mathbb{Z}\}$. Show that $A \cap B = C$ by showing containment both ways. One direction is easier than the other. Label one of them "the easy direction" and the other "the hard direction". **Hint:** Yes, we worked on a problem just like this in class. Don't go back and find it, work through this one on your own. **Another hint:** In the hard direction, you should come to something like $3k = 7j$ where j and k are integers. You will need to conclude that j is a multiple of 3. If you are up for the challenge, show this using the division algorithm. Don't use any ideas about prime factorization.

Your name: ______________________________

Inequalities

We can define the $<$ relation for real numbers and establish its properties.

Overview

Most students at your level take the real numbers as things that simply exist and have a number of properties such as commutativity. In fact, the real numbers can be *constructed* from the rational numbers, and the rational numbers from the integers, and the integers from the positive integers. In this activity, we back up to the point that the real numbers have been constructed, but before inequalities have been defined. We define the $<$ relation and prove a number of useful properties that it satisfies. Since the $>$ relation is so similar, we will not define it or show its properties.

Note 206. Let $\mathbb{R}$ denote the set of real numbers, and denote addition and multiplication of real numbers in the usual ways. **Addition** has these properties: commutativity $(a+b=b+a)$, associativity $(a+(b+c)=(a+b)+c)$, additive identity (there exists a unique real number called 0 for which $a+0=a$ for all $a \in \mathbb{R}$), and additive inverse (for each number a in $\mathbb{R}$, there exists a unique real number $-a$ for which $a+(-a)=0$). **Multiplication** has these properties: commutativity $(ab=ba)$, associativity $(a(bc)=(ab)c)$, multiplicative identity (there exists a unique real number called 1, with $1 \neq 0$, such that $a \cdot 1 = a$ for all a in $\mathbb{R}$), multiplicative inverse (for each a in $\mathbb{R}$ with $a \neq 0$, there exists a unique number called a^{-1} for which $a \cdot a^{-1} = 1$. **Addition and multiplication** are related by the distributive property: $((a+b)c = ac + bc)$.

Definition 207. Subtraction. Let a and b be real numbers. The difference of a and b, denoted $a-b$, is the real number $a+(-b)$, where $(-b)$ denotes the additive inverse of b.

Note 208. At the end of this activity, you will see how to establish the following useful properties regarding additive inverses and subtraction:

a. $a \cdot 0 = 0$ for all real numbers a.

b. The additive inverse $(-a)$ is equal to $(-1) \cdot a$, where (-1) is the additive inverse of 1

c. $(-1)(-1) = 1$.

d. The additive inverse of $a+b$ is $(-a)+(-b)$. Using subtraction notation, $-(a+b) = -a-b$.

e. $-(-a) = a$.

You can use subtraction as usual in this activity, but if you would like to avoid subtraction notation and just use additive inverses, that is worth attempting.

Definition 209. Positive real numbers. By construction, the real numbers have a subset $\mathbb{R}^+$, called the *positive real numbers,* for which:

a. If $a, b \in \mathbb{R}^+$, then $a+b \in \mathbb{R}^+$. ($\mathbb{R}^+$ is closed under addition.)

b. If $a, b \in \mathbb{R}^+$, then $a \cdot b \in \mathbb{R}^+$. ($\mathbb{R}^+$ is closed under multiplication.)

c. For every real number a, either $a \in \mathbb{R}^+$ or $(-a) \in \mathbb{R}^+$ or $a = 0$. Exactly one of the three happens.

Under each property above, write a sentence that states it in plain English. Think of $\mathbb{R}^+$ as being the positive, non–zero numbers. We can't use interval notation to write what $\mathbb{R}^+$ is, because intervals are defined in terms of inequalities, and we have not defined inequalities yet!

Show 210. Let $a \in \mathbb{R}$ and suppose that $a \neq 0$. Show that $a \cdot a \in \mathbb{R}^+$. When you use properties from 208 or 209, cite them by number. **Hint:** There are two cases left in 209c.

Show 211. Show that $1 \in \mathbb{R}^+$. Be careful to cite any previous properties that you use.

Show 212. Show that $(-1) \notin \mathbb{R}^+$. **Hint:** Assume that $(-1) \in \mathbb{R}^+$ and use 209a.

Definition 213. Less than. Let a and b be real numbers. We write that $a < b$ if $b - a \in \mathbb{R}^+$.

Note 214. All of the following problems rely on Definition 213, so you will use it over and over. Note that $>$ has not been defined yet, so be careful not to use it.

Show 215. Show that $-1 < 0$. **Hint:** use 208d.

Show 216. Show that $1 < 1$ is not true. Thus, the $<$ relation is not reflexive. When you use properties from 208 or 209, cite them by number.

Show 217. Show that $0 < 1$ but that $1 < 0$ is not true. Thus, the $<$ relation is not symmetric.

Show 218. Show that the $<$ relation on $\mathbb{R}$ is transitive. Follow good form by first letting a, b, c be real numbers and supposing that $a < b$ and $b < c$. When you use properties from 208 or 209, cite them by number. You may enjoy ticking off the properties of the real numbers that you use. For example, in this proof, you are likely to use the fact that $(-b) + b = 0$, which is the additive inverse property.

Show 219. Let $a, b \in \mathbb{R}$ and suppose that $a < b$. Show that $-b < -a$. When you use properties from 208 or 209, cite them by number.

Show 220. Let $a, b, c \in \mathbb{R}$. Suppose that $a < b$. Show that $a + c < b + c$. When you use properties from 208 or 209, cite them by number.

Show 221. Let $a, b, c, d \in \mathbb{R}$. Suppose that $a < b$ and $c < d$. Show that $a + c < b + d$. When you use properties from 208 or 209, cite them by number.

Show 222. Let a, b, c be real numbers. Suppose that $a < b$ and $0 < c$. Show that $ac < bc$.

Show 223. Let a, b, c be real numbers. Suppose that $a < b$ and $c < 0$. Show that $bc < ac$.

Show 224. Let $a, b \in \mathbb{R}$ and suppose that $0 < a$ and $b < 0$. Use a previous result to show that $ab < 0$.

Show 225. Let $a \in \mathbb{R}$ and suppose that $0 < a$. Show that $0 < a^{-1}$. Here a^{-1} is the multiplicative inverse of a. **Hint:** This one take a bit more effort than the previous ones. Note that division has not been defined yet, so just use addition, subtraction, and multiplication.

Show 226. Let $a, b \in \mathbb{R}$ and suppose that $0 < a$ and $a < b$. Show that $b^{-1} < a^{-1}$.

Note 227. Below, you are asked to prove basic properties of additive inverses and subtraction.

Show 228. Let a be a real number. Show that $a \cdot 0 = 0$.

Show 229. People sometimes ask if the additive inverse $(-a)$ is the same as $(-1) \cdot a$, where (-1) is the additive inverse of 1. It's true, and here is how you show it; you should fill in steps and write the justifications at the right side of each line.

$$\begin{aligned} a + (-1) \cdot a &= 1 \cdot a + (-1) \cdot a \\ &= (1 + (-1)) \cdot a \\ &= \\ &= 0, \end{aligned}$$

This shows that $(-1) \cdot a$ is the additive inverse of a.

Show 230. You might think that it is obvious that $(-1)(-1) = 1$, where (-1) is the additive inverse of 1, but this takes a few steps beyond the properties of the real numbers in 206. Write justifications and complete the following steps to show it.

$$\begin{aligned} (-1) + (-1)(-1) &= (-1)(1) + (-1)(-1) \\ &= (-1)(1 + (-1)) \\ &= \\ &= 0, \end{aligned}$$

which shows that $(-1)(-1)$ is the additive inverse of -1, which is 1.

Show 231. The additive inverse of a sum works out nicely. Let a and b be real numbers and think about the additive inverse of $a+b$. Write justifications to the right of each statement.

$$\begin{aligned} -(a+b) &= (-1)(a+b) \\ &= (-1)(a)+(-1)(b) \\ &= (-a)+(-b) \end{aligned}$$

Show 232. Let $a \in \mathbb{R}$. The statement $-(-a)=a$ is just a statement about additive inverses. Prove that it is true.

Your name: ______________________________

Quiz on inequalities

5 points

Definition Positive real numbers. By construction, the real numbers have a subset $\mathbb{R}^+$, called the *positive real numbers,* for which:

a. If $a, b \in \mathbb{R}^+$, then $a + b \in \mathbb{R}^+$. ($\mathbb{R}^+$ is closed under addition.)

b. If $a, b \in \mathbb{R}^+$, then $a \cdot b \in \mathbb{R}^+$. ($\mathbb{R}^+$ is closed under multiplication.)

c. For every real number a, either $a \in \mathbb{R}^+$ or $(-a) \in \mathbb{R}^+$ or $a = 0$. Exactly one of the three happens.

Definition Less than. Let a and b be real numbers. We write that $a < b$ if $b - a \in \mathbb{R}^+$.

Show. Let a, b, c be real numbers. Suppose that $a < b$ and $0 < c$. Show that $ac < bc$. Take very small steps and be careful to cite justifications for every single step.

Your name: ________________________________

Quiz on induction

15 points

For each problem below, clearly state $P(1), P(k)$, and $P(k+1)$ as logical statements with double quotes around them. When proving that $P(k)$ being true implies that $P(k+1)$ is true, do not write down $P(k+1)$ as if it were true, but rather start with one side and work with it until it turns into the other side.

Show. Use induction to show that for $n > 0$, 8 divides $5^n + 2(3^{n-1}) + 1$. **Hint:** As in other proofs of divisibility, add and subtract to be able to use $P(n)$ to simplify $P(n+1)$.

Show. On the back of this piece of paper, use induction to show that for all $n \geq 1$, we have that $1(1!) + 2(2!) + \cdots + n(n!) = (n+1)! - 1$.

Your name: ______________________________

Absolute value and related functions

A careful development of the properties of the absolute value function.

Overview

The absolute value function is easy to understand for numbers like 9 and -13, but it's harder to show its properties because our intuition works so hard to see all variables as having positive values. In this activity, we will not use the standard notation for the absolute value function and will have to keep our intuition at bay. We will instead rely completely on the definition. **When you use a property of inequalities, cite it by number.**

Definition 233. Absolute value. The function $f : \mathbb{R} \to \mathbb{R}$ defined by

$$f(x) = \begin{cases} x, & \text{if } x \geq 0 \\ -x, & \text{if } x < 0 \end{cases}$$

is called the *absolute value* function.

Notation 234. In this activity, do not use the standard notation for absolute value, not even once. Every time you work with the absolute value function, use and cite the definition.

Show 235. Show that $f(ab) = f(a)f(b)$ for all real numbers a and b. Follow the model.

Let a and b be real numbers. There are four cases.

1. Suppose that $a \geq 0$ and $b \geq 0$. Then $ab \geq 0$ so $f(ab) = ab$ and $f(a) = a$ and $f(b) = b$, so $f(ab) = ab = f(a)f(b)$.

2. Suppose that $a \geq 0$ and $b < 0$.

3. Suppose that $a < 0$ and $b \geq 0$.

4. Suppose that $a < 0$ and $b < 0$.

In each case, we see that ___________________________. We made no further assumptions about _____________, thus ____.

Show 236. Following the model above, show that $f(f(a)) = f(a)$ for all real numbers a.

Show 237. Show that $f(-a) = f(a)$ for all real numbers a.

Show 238. Show that $f(a - b) = f(b - a)$ for all real numbers a and b.

Show 239. Show that $f(a) \geq 0$ for all real numbers a.

Show 240. Follow the model in 235 to show that $f(a + b) \leq f(a) + f(b)$ for all real numbers a and b. When a and b have different signs, consider two cases, $a + b \geq 0$ and $a + b < 0$. You will probably want to show that if $b < 0$, then $b < -b$. Make a good, solid argument using transitivity of $<$.

Show 241. Show that for real numbers a and b, $f(a) \leq b$ if and only if $-b \leq a \leq b$. Remember that an "if and only if" proof has two directions. In both directions, you will have to consider two cases, $a \geq 0$ and $a < 0$. Note that the statement $-b \leq a \leq b$ is equivalent to ($-b \leq a$ and $a \leq b$).

Show 242. Show that for all real numbers a and b, $f(a) \geq b$ if and only if ($a \geq b$ or $a \leq -b$).

Show 243. Show that for all real numbers a and b, $f(a) \leq f(a-b) + f(b)$. **Hint:** Look at $f((a-b)+b)$.

Show 244. Show that for all real numbers a and b, $f(a) - f(b) \leq f(a-b)$ and also $f(b) - f(a) \leq f(b-a)$.

Show 245. Show that for all real numbers a and b, $f(a-b) \geq f(f(a) - f(b))$.

Definition 246. Minimum function. The function $h : [0, \infty) \to \mathbb{R}$ defined by

$$h(x) = \begin{cases} x, & \text{if } x \leq 1 \\ 1, & \text{if } x > 1 \end{cases}$$

can be called the minimum function.

Show 247. Show that for all real numbers a, $h(a) = 0$ if and only if $a = 0$.

Show 248. Show that if $a \leq b$, then $h(a) \leq h(b)$.

Show 249. Show that if $h(a) < h(b)$, then $a < b$.

Show 250. Show that for all real numbers a and b, $h(a + b) \leq h(a) + h(b)$. **Hint:** Use a proof by cases. But what are the cases?

Your name: ______________________________

Functions

One-to-one, onto and bijective functions

Overview

You are comfortable working with functions already. There are various ways of describing functions. Here you will learn the formal definition of a function.

Definition 251. Function. Let X and Y be sets. A function f from X to Y is a relation from X to Y that satisfies:

1. for each $x \in X$ there is a $y \in Y$ such that $(x, y) \in f$, and
2. if $(x, y) \in f$ and $(x, z) \in f$, then $y = z$.

The set X is called the domain of f and the set Y is called the codomain of f.

Notation 252. We write $f : X \to Y$ to describe a function f from X to Y and we write $f(x) = y$ instead of $(x, y) \in f$.

Definition 253. Injective Functions. Let X and Y be sets and let $f : X \to Y$ be a function. The function f is said to be injective or one-to-one if whenever $x_1, x_2 \in X$ are such that $x_1 \neq x_2$ then $f(x_1) \neq f(x_2)$.

Definition 254. Surjective Functions. Let X and Y be sets and let $f : X \to Y$ be a function. The function f is said to be surjective or onto if for each $y \in Y$ there exists an $x \in X$ such that $f(x) = y$.

Definition 255. Bijective Functions. Let X and Y be sets and let $f : X \to Y$ be a function. The function f is said to be bijective if it is both injective and surjective.

Example 256. Let $X = \{Monday, \diamondsuit, \sqrt{\pi}, purple\}$ and $Y = \{\alpha, \heartsuit, fun\}$ be sets and define the relation f from X to Y by $f = \{(Monday, fun), (\diamondsuit, \alpha), (\sqrt{\pi}, fun), (purple, fun)\}$. Draw a diagram to illustrate the relation. Is the relation f a function? Prove your answer by using the definition.

Problem 257. Let $X = \{Cleveland, Chicago, LosAngeles, Miami\}$ be a set of American cities and let $Y = \{Cavaliers, Heat, Lakers, Bulls, Clippers\}$ be a set of NBA teams.

a) Provide an example of a relation that is a function from X to Y and draw a diagram to illustrate your example.

b) Provide an example of a relation from X to Y that is not a function and draw a diagram to illustrate your example.

c) Provide an example of an injective function from X to Y. Draw a diagram to illustrate your example.

d) Show that there are no surjective functions from X to Y.

Note 258. The next problem states an equivalent definition for injectivity. This definition is very useful when proving that a function is injective.

Problem 259. Let X and Y be two sets and let $f : X \to Y$ be a function. Then f is injective if and only if for all $x_1, x_2 \in X$ such that $f(x_1) = f(x_2)$ we have $x_1 = x_2$.

Problem 260. Let $f : \mathbb{N} \to \mathbb{N}$ be a function defined by $f(n) = 2n + 1$. Show that the function f is injective but not surjective.

Hint: Use the previous problem to prove injectivity. In order to prove that f is not surjective you need to find an $m \in \mathbb{N}$ which cannot be written as $2n + 1$.

Problem 261. Let $f : \mathbb{N} \to \mathbb{Z}$ be a function defined by

$$f(n) = \begin{cases} \frac{n}{2}, & \text{if } n \text{ is even} \\ \frac{-(n+1)}{2}, & \text{if } n \text{ is odd} \end{cases}$$

Show that f is a bijective function.

Hint: To show that f is injective let $n_1, n_2 \in \mathbb{N}$ be such that $f(n_1) = f(n_2)$ and look at all the possible cases according to the parity of n_1 and n_2.

To prove the surjectivity let $m \in \mathbb{Z}$. Consider the two possible cases; one when $m \geq 0$ and the other one when $m < 0$. Then in each case find an $n \in \mathbb{N}$ such that $f(n) = m$.

Review questions for the final quiz

Taken from a variety of sources without attribution.

Overview

The final quiz will consist of approximately 6 questions and will be worth approximately 60 points. I will try to make sure that it can be done by a prepared student in 2 hours. The best way to prepare is to work out problems on the review sheet and on the handouts that we have had in class.

Key things to review are all in-class activities about set theory and induction, plus the homework on set theory.

Induction problems

1. Show that $1 + 2 + 2^2 + 2^3 + \cdots + 2^n = 2^{n+1} - 1$ for all $n \geq 0$.

2. Show that $\sum_{j=0}^{n} r^j = \frac{r^{n+1}-1}{r-1}$ for all $n \geq 0$ and all real numbers $r \neq 1$. (There is an easier formula when $r = 1$!)

3. Show that $2^n < n!$ for all $n \geq 4$.

4. Show that $3^n < n!$ for all $n \geq 7$.

5. Show that $n! < n^n$ for all $n > 1$.

6. Show that $1 \cdot 2 + 2 \cdot 3 + 3 \cdot 4 + \cdots + n(n+1) = \frac{n(n+1)(n+2)}{3}$ for all $n \geq 1$.

7. Show that $1 + \frac{1}{4} + \frac{1}{9} + \cdots + \frac{1}{n^2} < 2 - \frac{1}{n}$ for all $n \geq 2$.

8. Show that $n^5 - n$ is a multiple of 5 for all n.

9. Show that $n^2 - 1$ is a multiple of 8 for all odd n. **Hint:** You could try showing $P(1)$ and then show that $P(n)$ implies $P(n+2)$.

10. Draw n lines in the plane such that no two lines are parallel and no three lines go through a common point. Show that this divides the plane into $\frac{n^2+n+2}{2}$ regions. **Hint:** How many regions does the $n+1$st line add?

11. Show that $1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{2^n} \geq 1 + \frac{n}{2}$. This is too hard for the final quiz, but you may enjoy working on it. Notice that increasing n by 1 will double the number of terms, unlike most of the other problems you have worked on. This result shows that the harmonic series diverges.

12. Show that 5^n is odd for all $n \geq 0$.

13. Use the product rule to show that, for every integer $n \geq 1$, the derivative of x^n is nx^{n-1}.

Set theory and other problems

1. Show that $[2,5)\cap(3,7)=(3,5)$ by showing inclusion both ways. Start by letting $x\in[2,5)\cap(3,7)$, so that $x\in[2,5)$ and $x\in(3,7)$, then write it as $2\le x<5$ and $3<x<7$. There are four inequalities here. Soon enough you can conclude that $x\in(3,5)$. Then show containment the other way as well.

2. Let $x>0$. Show that there exists an integer n such that $0<\frac{1}{n}<x$.

3. Show that $\bigcup_{n=1}^{\infty}[\frac{1}{n},1]=(0,1]$ by showing inclusion in both directions.

4. Show that $A\cup B=A$ if and only if $B\subseteq A$.

5. Show that $A\subseteq C$ and $B\subseteq C$ if and only if $A\cup B\subseteq C$.

6. Show that $A\cap(B\cup C)=(A\cap B)\cup(A\cap C)$ by showing containment in both directions and using cases. Then show it again using logical statements such as $P=$ "$x\in A$" and a truth table. Try to do this without looking back at problems we did in class.

7. For each $n=1,2,3,\ldots$, let A_n be a set. Show de Morgan's law: $\left(\bigcup_{n=1}^{\infty}A_n\right)^c=\bigcap_{n=1}^{\infty}A_n^c$ by showing set containment in both directions.

8. Show that if n is odd, then n^2+2n-7 is a multiple of 4.

9. Let a,b, and c be integers. Suppose that b is a multiple of a or c is a multiple of a. Show that bc is a multiple of a.

10. Let a and b be integers. Suppose that a is a multiple of b and that b is a multiple of a. Show that $a=\pm b$.

11. Suppose that n is an integer and n^2 is a multiple of 5. Show that n is a multiple of 5. Try to do this without looking back at your notes!

12. Suppose that n is an odd integer. Show that n^3-25n is a multiple of 24. This is similar to something we did in class. See if you can do it that way. Can you do it by induction instead? Work with $P(n)$ and $P(n+2)$. Which way is easier?

13. Show that an integer n cannot be both even and odd. What kind of proof did you use?

14. Suppose that m and n are integers and that $3m=7n$. Show that n is a multiple of 3. Do this by writing n as $3q+r$ for different possible values of r.

15. Give a complete proof that $\sqrt{3}$ is irrational.

Your name: ______________________________

Final quiz

60 points

Please use one side of a sheet of paper for each problem. If you are totally stuck, you can ask for a hint, but it may cost you a little something. Good luck!

1. Show that $1^2 + 2^2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6}$ for all $n = 1, 2, 3, \ldots$. In other words, show that $\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}$. Use good form. **Hint:** Factor out $(n+1)$, don't multiply it out.

2. Answer this question on the back of the paper.

Use mathematical induction to show that $7^n - 1$ is divisible by 6 for all $n = 0, 1, 2, \ldots$. Use good form.

3. Suppose that $A \subset B$. Show that $A \cap B \subset B$. You can draw a Venn diagram, but write a regular proof considering an arbitrary value of x. **Hint:** Show inclusion and also show that the sets are not equal.

Suppose that $A \cap B \subset B$. Does this imply that $A \subset B$? If so, show it. If not, draw a Venn diagram that shows what else could happen.

4. Answer this question on the back of the paper.

Show that $\bigcup_{n=1}^{\infty}(\frac{1}{n}, 2) = (0, 2)$.

5. Show that $[3,7] \cup (5,9) = [3,9)$ by showing set inclusion both ways.

6. Answer this question on the back of the paper.

Suppose that m and n are integers and that $3m = 7n$. Show that n is a multiple of 3. Do this by writing n as $3q + r$ for different possible values of r and figuring out which values of r make sense.

Reading assignment #1

Due on the second day of class. 10 points

The idea is to read Chapter 1 of the textbook by Daepp and Gorkin[2]. The assignment is to read it in a particular way. It may take 3 hours to get it done, but you will learn something in those three hours, and you will start to develop a very important skill.

Get a copy of Chapter 1, "The How, When, and Why of Mathematics." Get out your notebook or some paper. Go somewhere quiet, where you won't be interrupted for a while. Turn off your phone so you aren't disturbed. Don't listen to music that will distract you, and make sure there is no TV or youtube on where you can see it or hear it.

Put the notebook or paper right in front of you. Put the textbook itself a bit farther away. Make note of the time that you start reading in your notebook, maybe in the left margin. Read the first paragraph of the chapter, then write one or more sentences in your notes which capture the main idea(s) of the paragraph.

Read the second paragraph, about Geogre Pólya's list of guidelines. Look up the list in the Appendix. Consider writing them in your notebook, or abbreviated versions of them.

Continue to write a sentence summarizing each paragraph. I believe that if you are not writing, you are probably not thinking as hard as you need to. Read slowly. If you run into a word you don't know, google it or look it up in a dictionary. If you really don't know it, write the definition in your notebook. It is OK to spend 15 minutes on each page of the book. Really. It is not a goal of the course to learn how to read faster. The goal is to learn how to get more out of the time you spend reading. If you stop to take a break, note the time that you stopped and the time you start again.

Read Exercise 1.1 and the text that walks you through Pólya's guidelines. Use your notebook to try to solve the puzzle yourself. I've printed the alphabet twice for you. That should save you a little time.

A second example starts on page 3 of the textbook. As you read it, draw diagrams in your notebook. Yes, there are diagrams printed in the textbook, but you will think harder about the diagram and understand more if you draw your own.

Example 1.2 asks a question. Read the question and see if you can answer it on your own, without reading further in the book.

On page 7, you will see that solutions of the exercises are provided. Resist the urge to turn your brain off and just read the solutions. That is not what they are there for!

Read through each of the problems that begin on page 8 in the book. Figure out what each problem is asking for and write that in your notes. If you can solve the problem, do that. If not, that's OK.

Problems 1.1 to 1.8 look nice.

You might start Problem 1.9 by trying some possible values for n.

Problem 1.10 doesn't interest me. Does it interest you?

Can you draw the region described in Problem 1.11?

Problem 1.12 is good. Would it help to make a graph?

Problem 1.13 seems silly. Do you like it anyway?

Read the Tips on Doing Homework. At the end, tally up how much time you have spent on reading this chapter. Write this number in your notebook and remember the number when you come to class.

[2] Reading, Writing, and Proving: A Closer Look at Mathematics, 2011, by Ulrich Daepp and Pamela Gorkin

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Reading assignment, Chapter 2

Due on Wednesday, September 2.

Read Chapter 2 of the book by Daepp and Gorkin. As with Chapter 1,

1. Read somewhere quiet, minimizing distractions from phones and friends

2. Note the times that you start and stop reading, and add up the minutes

3. Read with a pencil in your hand and your notebook open in front of you

4. Write a sentence to summarize each paragraph, re-draw diagrams, work out examples and exercises on your own

5. Look up words you don't know, and write down ones you really don't know

6. Read slowly. You are not reading a comic book or a newspaper. It is not a goal of this class for you to learn how to read faster. The goal is to learn how to get more out of the time you spend reading, and to learn to concentrate for longer periods of time.

7. At the end, tally up how much time you have spent on reading this chapter. Write this number in your notebook and remember the number when you come to class.

You will read about "statements." Focus on the ones about mathematical things, and don't worry too much about interpreting the ones that are non-mathematical.

Note that on page 14, there is a statement about the color of the cover of the book. Books from Springer used to be plain yellow, but the authors must not have realized that someone would put a big blue bar on the cover of this edition of the book. Just imagine that the book cover is all yellow.

Fill out every truth table that is suggested in the chapter. Truth tables are an excellent way to get great clarity about complicated combinations of statements. The idea is to consider every possible combination of True and False for the basic statements. For example, if there are two statements, P and Q, there will be four rows in the table, running through the four possible combinations of True and False for P and Q. On page 21, there is a truth table for three statements, P, Q, and R. It has eight rows.

The most important use of truth tables is to tell when two complicated combinations of logical expressions are, in fact, the same.

For me, the hardest thing about truth tables is making columns for implications like $P \to Q$. Here is the best way I know to think about them. Each row of the truth table for P and Q covers one combination of truth values for P and Q. Some of these combinations are consistent with the implication that P implies Q. For example, when P is True and Q is True, this is consistent with $P \to Q$, so we put T in the $P \to Q$ column. The row in which P is True and Q is False, however, is inconsistent with the implication $P \to Q$, so we put F in that row. The cases in which P is False are a bit different, but they are also consistent with $P \to Q$, since $P \to Q$ only has anything to say about P and Q when P is True. So we put T in those rows too.

Problems 1 to 7 are good, so please do those. Rather than working on problems 9-21, I would much prefer that you spend your time making the truth tables I describe below.

1. Make a truth table for $\neg(P \vee Q)$ and $\neg P \wedge \neg Q$.

2. Make a big truth table for P, Q, R, $P \wedge (Q \vee R)$, $P \vee (Q \wedge R)$, $(P \wedge Q) \vee (P \wedge R)$, and $(P \vee Q) \wedge (P \vee R)$. Which of these are equal? How can you rememeber that?

Reading assignment, Chapter 3

Read Chapter 3 of the textbook by Daepp and Gorkin. As with Chapters 1 and 2,

1. Read somewhere quiet, minimizing distractions from phones and friends

2. Note the time that you start and stop reading, and add up the minutes

3. Read with a pencil in your hand and your notebook open in front of you

4. Write a sentence to summarize each paragraph, re–draw diagrams, work out examples and exercises on your own

5. Look up words you don't know, and write down ones you really don't know

6. Read slowly. You are not reading a comic book or a newspaper. It is not a goal of this class for you to learn how to read faster. The goal is to learn how to get more out of the time you spend reading, and to learn to concentrate for longer periods of time.

7. At the end, tally up how much time you have spent on reading this chapter. Write this number in your notebook and remember the number when you come to class.

Theorem 3.1 lists three properties of logical statements. Please make truth tables for each of them to check that they are tautologies. Also add de Morgan's laws from Theorem 2.9. Then you'll have the whole set. Having de Morgan's laws handy should make Exercise 3.2 easier.

How can you remember the distributive property?

The contrapositive is really important. See if you can explain it just by thinking about $P \to Q$ and $\neg Q \to \neg P$, without using truth tables.

Theorem 3.3 is proven using the contrapositive. This is a very useful method of proof. Please note that it differs from proof by contradiction.

Read about the converse, and make sure never to confuse an implication with its converse.

Problems 2, 3, 4, 9, 14, 16, 18, 5, 6, 8, 19, 15 are good to work on, in that order. Work through at least half of these problems.

Chapters 1 to 5 are mostly there to help develop proof techniques. After Chapter 5, we will spend more of our time on definitions, examples, theorems, and proofs. Use your time now to develop basic logic and proof techniques that will help you for the rest of the semester and beyond!

Reading assignment, Chapter 4

Due on Friday, September 18.

Read and understand Chapter 4 of the textbook by Daepp and Gorkin. As with previous chapters,

1. Read somewhere quiet, minimizing distractions from phones and friends

2. Note the time that you start and stop reading, and add up the minutes

3. Read with a pencil in your hand and your notebook open in front of you

4. Write a sentence to summarize each paragraph, re–draw diagrams, work out examples and exercises on your own

5. Look up words you don't know, and write down ones you really don't know

6. Read slowly. You are not reading a comic book or a newspaper. It is not a goal of this class for you to learn how to read faster. The goal is to learn how to get more out of the time you spend reading, and to learn to concentrate for longer periods of time.

7. At the end, tally up how much time you have spent on reading this chapter. Write this number in your notebook and remember the number when you come to class.

This is a very important chapter, one with real substance. Hopefully you will feel that way when you read it, and will enjoy it more as a result. It is a little bit about set theory, but mostly about quantifiers.

This chapter has a large number of very dense expressions involving quantifiers, implications, and logical operators. Slow way down when you run into one of them. Pick them apart in your mind and then write them down so they are crystal clear. Every symbol is important. It's a bit like when you're reading someone your credit card number or you're giving your phone number to someone you really want to call you. Every symbol is important.

Exercises 4.1, 4.2, 4.3, and 4.6 are all useful to do. 4.2(a) is harder than they make out, because you not only want to write that a solution x exists, but that if y is also a solution, then $x = y$. The discussion that begins at the bottom of page 36 is very important, negating statements with quantifiers.

There are 20 problems. The more of them you do, the better, of course, but you may not be able to work through all of them. **Please at least do problems # 1–7 and 20.** Read # 11. Does this joke work on your friends?

Pay attention to the phrase "only if" which appears in Problem 12. It is often used in a way that can be confusing. Compare these two statements for example, in which R means Race and P means prize:

1. I will race if there is a prize offered. $P \to R$. This is the most common way that people use the word "if." The prize will make me race.

2. I will race only if there is a prize offered. $R \to P$. People say this sort of thing pretty often too, but it's a bit less clear unless you think about it carefully. Part of the problem is the time order in which things happen, because the racing comes *after* the prize is offered. "If you see me racing, you can be sure that there was a prize offered. (But offering a prize is no guarantee that I will race.)"

Reading assignment, Chapter 5

Due Friday, September 25.

Read and understand Chapter 5 of the textbook by Daepp and Gorkin. As with previous chapters,

1. Read somewhere quiet, minimizing distractions from phones and friends
2. Note the time that you start and stop reading, and add up the minutes
3. Read with a pencil in your hand and your notebook open in front of you
4. Write a sentence to summarize each paragraph, re–draw diagrams, work out examples and exercises on your own
5. Look up words you don't know, and write down ones you really don't know
6. Read slowly. You are not reading a comic book or a newspaper. It is not a goal of this class for you to learn how to read faster. The goal is to learn how to get more out of the time you spend reading, and to learn to concentrate for longer periods of time.
7. At the end, tally up how much time you have spent on reading this chapter. Write this number in your notebook and remember the number when you come to class.

This chapter walks you through a number of types of proofs and gives examples of each. **Rewrite these proofs in your notes, in your own words as much as possible, so that you make them yours.** By the end of reading the chapter, you should **know** the proof that the square root of 2 is irrational and you should know the other proofs as well.

It might help, in your notes, to make a list of proof techniques from the chapter and from previous chapters. What chapter talked about proof by contrapositive? Is that in Chapter 5? What about truth tables? You can prove things with those. What kinds of things?

Read and understand Problem 1. It is important.

Read the other problems, find the ones that are easy, and do them. This may seem like a strange assignment, but I really mean it. Think about each problem (if you can get through all of them), and make sure that if a problem is easy, that you recognize that and write out the solution. Don't worry if a problem looks hard but turns out to be easy. That happens all the time. But hopefully you will spot a number of them that really are easy, and do them. We will go over these problems in class the following week.

Reading assignment, Chapter 6

Due on Wednesday, October 7

Read and understand Chapter 6 of the textbook by Daepp and Gorkin. As with previous chapters,

1. Read somewhere quiet, minimizing distractions from phones and friends

2. Note the time that you start and stop reading, and add up the minutes

3. Read with a pencil in your hand and your notebook open in front of you

4. Write a sentence to summarize each paragraph, re–draw diagrams, work out examples and exercises on your own

5. Look up words you don't know, and write down ones you really don't know

6. Read slowly. You are not reading a comic book or a newspaper. It is not a goal of this class for you to learn how to read faster. The goal is to learn how to get more out of the time you spend reading, and to learn to concentrate for longer periods of time.

7. At the end, tally up how much time you have spent on reading this chapter. Write this number in your notebook and remember the number when you come to class.

Many proofs in other classes involve showing that two sets are equal, or that one set is a subset of another, or that one set is not a subset of another. Work really hard on this chapter and it will pay you dividends for a long, long time.

Make sure to add value when you take notes. Write new thoughts, new questions, new comments.

This chapter introduces sets, subsets, equality of sets, and how to tell what the members of a set are. As you read, take time to write out at leart 5 members of each set that is introduced. Note that A being a subset of B is the same as the logical implication $x \in A$ implies $x \in B$. There is a tight connection between statements in set theory and logical statements. Here is another: Set A being equal to set B is the same as the logical implications $x \in A$ if and only if $x \in B$.

There are many examples in this chapter. Work through them by rewriting them and adding useful steps in your notes.

On page 64, intersections, unions, and complements of sets are introduced. As you read about them, explain in your notes how these relate logical statements such as $x \in A$ and $x \in B$ to $x \in A \cap B$.

You may enjoy reading about the paradoxes on page 67. Give them a try. Even if they are not your cup of tea, try to see what the issue is.

Problems $1 - 9$ are essential. Do them.

Problem 10 is a good thought problem. Think about it and write your answer.

Starting with Problem 11, there are things for you to prove. I would be happy to see you do many of these by yourself.

Reading assignment, Chapter 7

Due Monday, November 2.

Read and understand Chapter 7 of the textbook by Daepp and Gorkin called "Operations on sets."

This is a short chapter, all about working with sets. You can approach these problems in a number of ways. Often it helps to draw a nice Venn diagram and get the right intuitive idea for what is being claimed, but don't stop there. You can also just focus on letting $x \in A$ or whatever and working with that, without thinking about Venn diagrams.

Most of the chapter is devoted to one example, showing that, if A, B, and C are sets, then $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$. The book suggests working forward from one side, and backward from the other, just as people sometimes build a bridge by starting at each bank of a river and meeting in the middle.

It also suggests breaking into cases at some point. For example, if $x \in A \cup (B \cap C)$, you can consider the case $x \in A$, which is great because then it's pretty clear that $x \in (A \cup B) \cap (A \cup C)$. But you also need to consider the case $x \notin A$, so that $x \in B \cap C$. But that's helpful, because then $x \in B$ and $x \in C$, and pretty soon it is clear that $x \in (A \cup B) \cap (A \cup C)$.

Do problem 7.1, parts a, c, d, e, f. Take your time and use really good form so that the proof is crystal clear. Notice that part (c) (statement 18 in the theorem) is an "if and only if" statement, so it has two parts. It's going to look something like this:

1. Suppose that $A \subseteq B$. We want to show that $(X \backslash B) \subseteq (X \backslash A)$. Let $x \in X \backslash B$. Then $x \notin B$. (More steps here.) Thus, $x \in X \backslash A$, and so $(X \backslash B) \subseteq (X \backslash A)$.

2. Suppose that $(X \backslash B) \subseteq (X \backslash A)$. We want to show that $A \subseteq B$. Let $x \in A$. (More steps here.) Thus, $x \in B$.

Do problem 7.4.

Do problem 7.6.

I guess that these problems are a bit dull, but it really is helpful to be good at proving things about sets.

As with the previous chapters,

1. Read somewhere quiet, minimizing distractions from phones and friends

2. Note the time that you start and stop reading, and add up the minutes

3. Read with a pencil in your hand and your notebook open in front of you

4. Write a sentence to summarize each paragraph, re-draw diagrams, work out examples and exercises on your own

5. Look up words you don't know, and write down ones you really don't know

6. Read slowly. You are not reading a comic book or a newspaper. It is not a goal of this class for you to learn how to read faster. The goal is to learn how to get more out of the time you spend reading, and to learn to concentrate for longer periods of time.

7. At the end, tally up how much time you have spent on reading this chapter. Write this number in your notebook and remember the number when you come to class.

Reading assignment, Chapter 8

Make a good effort by Monday, November 9; due on Friday, November 13.

Read and understand Chapter 8 of the textbook by Daepp and Gorkin, called "More on operations on sets."

This chapter is a challenge. You will really need to use all the reading skills you have been practicing when you read this chapter. The ideas are harder, and some are really hard, but not impossible. Just slow yourself down and write things out in lots of detail.

Example 8.2(a) would be a great one to write out concrete fractions with different values of p and q to understand the sets A_q and then the union of these sets. For Example 8.2(b), do the same to understand what the sets B_i are, and then what their intersection is. No shortcuts! Write out elements for each set.

Exercise 8.3 is also good.

In the middle of page 82 the phrase "collection of subsets of X" appears. This is a very new, very difficult concept; do not underestimate how tricky it can be, but patiently think about it and keep coming back to it. For example, $\mathcal{A}$ might be all intervals of the form $[k, k+1]$ and you might want to take the union of all such intervals, or the intersection.

Exercise 8.4 is excellent. Draw pictures until everything is crystal clear. Exercise 8.5 is also excellent.

Rewrite the proofs of Examples 8.6 and 8.7 to make them your own. Really.

Exercises 8.9 and 8.10 are also excellent. Do them on your own, then compare to the solutions in the book.

Do problems 1, 2, and 3.

Here is a challenge problem. Let $a < b$. Show that $\bigcup_{n=1}^{\infty}[a, b - \frac{1}{n}] = [a, b)$. Draw pictures, then show set inclusion both ways.

Here is another challenge problem. Let $a < b$. Show that $\bigcap_{n=1}^{\infty}[a, b + \frac{1}{n}) = [a, b]$. Draw pictures, then show set inclusion both ways.

As with the previous chapters,

1. Read somewhere quiet, minimizing distractions from phones and friends
2. Note the time that you start and stop reading, and add up the minutes
3. Read with a pencil in your hand and your notebook open in front of you
4. Write a sentence to summarize each paragraph, re-draw diagrams, work out examples and exercises on your own
5. Look up words you don't know, and write down ones you really don't know
6. Read slowly. You are not reading a comic book or a newspaper. It is not a goal of this class for you to learn how to read faster. The goal is to learn how to get more out of the time you spend reading, and to learn to concentrate for longer periods of time.
7. At the end, tally up how much time you have spent on reading this chapter. Write this number in your notebook and remember the number when you come to class.

Reading assignment, Chapter 9

Due in the tenth week of class.

Read and understand Chapter 9 of the textbook by Daepp and Gorkin, called "The Power Set and the Cartesian Product." This is the last chapter on plain set theory. It should stretch your mind in a few new directions. Prepare to move slowly and think carefully.

When A is a set, the power set of A is the collection of all subsets of A. Read Example 9.1 and do Exercise 9.3 and then **do Problem 9.1.** Work through Exercise 9.2 and then **do Problem 9.2.** Problem 9.2 is hard, but excellent for you. Take it very slowly. Work through Exercise 9.4 and then **do Problem 9.5. Do Problem 9.8.**

Do Problem 9.11. For 9.11, you have already seen the power set of a set containing 2 elements and 3 elements. **Hint:** When you are making a subset of a set A, for each element of A, you have to decide whether it goes in or out of the subset. There are two choices (in or out) each time. If the hint doesn't help you, write out the power set of $\{1, 2, 3, 4\}$, then read the hint again. Hopefully you don't have to write out the power set of $\{1, 2, 3, 4, 5\}$!

You are already very familiar with one Cartesian product: making ordered pairs (x, y) of real numbers is the Cartesian product $\mathbb{R} \times \mathbb{R}$, which you know better as the xy plane. Every problem involving Cartesian products of sets containing real numbers can be depicted as points in the xy plane. Make a graph in every case. This will help your intuition. When there are only finitely many points, like with $\{0, 1\} \times \{2, 3\}$, also list out all of the (x, y) pairs.

Answer these questions: Who is the Cartesian product named after? Why, exactly?

Work through Exercise 9.5 a, b, e.

For Theorem 9.7, draw A and C as intervals on the x axis and draw B and D as intervals on the y axis, then draw out the sets in the statement of the theorem on two separate sets of axes. Make sure you are crystal clear about what these sets are, and you will be close to mastering Cartesian products.

Do Problem 9.12. It connects Cartesian products to things you learned in geometry.

Do Problem 9.17a. Notice that this is an "if and only if" proof, and it has three set equalities to show. Suppose that $A \times B = C \times D$ and show that $A = C$ and $B = D$ by showing containment each way. Here is one part of the argument: Let $x \in A$. Also let $y \in B$. Then $(x, y) \in A \times B = C \times D$, and so $x \in C$. Thus $A \subseteq C$. After that part is done, suppose that $A = C$ and $B = D$ and argue that $A \times B = C \times D$.

Think about Problem 9.19.

As with the previous chapters,

1. Read somewhere quiet, minimizing distractions from phones and friends
2. Note the time that you start and stop reading, and add up the minutes
3. Read with a pencil in your hand and your notebook open in front of you
4. Write a sentence to summarize each paragraph, re-draw diagrams, work out examples and exercises on your own
5. Look up words you don't know, and write down ones you really don't know
6. Read slowly.
7. At the end, tally up how much time you have spent on reading this chapter. Write this number in your notebook and remember the number when you come to class.

Reading assignment, Chapter 10

Due in the eleventh week of class.

Read and understand Chapter 10 of the textbook by Daepp and Gorkin, called "Relations."

The main definition for Chapter 10 appears at the end of Chapter 9, on page 93. Here is the deal. A *relation* S from a set X to a set Y is a subset of $X \times Y$. If $Y = X$, we say the relation is a relation on X. At the beginning of Chapter 10, we see that we are going to be only working with relations on a set X.

Suppose that S is a relation on a set X. That is, suppose that S is a subset of $X \times X$, which means that S is a set of points of the form (x, y), where $x \in X$ and $y \in X$. Rather than write $(x, y) \in S$, we usually write $x \sim y$. How to read this out loud? There is no perfect solution. I would suggest that you read it as "x tilde y" (because $\sim$ is the tilde that appears above the n in some Spanish words).

Suppose that $X = \mathbb{R}$ and let $S = \{(x, y) : x \leq y\}$. Then $x \sim y$ means that $(x, y) \in S$, which means that $x \leq y$. In this way, we see that $\leq$ is a relation on $\mathbb{R}$. **Write out the set S corresponding to the relations $<$, $\leq$, $=$, $\geq$, and $>$. Then also sketch these as regions in the xy plane.**

Note that relations are between two elements. Thus, "divisible by 4" is not a relation. However, if $X = \mathbb{Z}^+$, you could say that $x \sim y$ if y is divisible by x, and then you would have a relation. People often write $x|y$ for this relation and say that x divides y. Call this relation S. **Write out at least ten of the ordered pairs in S, using at least five different values of x.**

Read Exercises 10.1 and 10.2.

Read the definitions of reflexive, symmetric, and transitive. A relation that satisfies all three is called an equivalence relation. This is where most of the action is with relations. **Do Problem 10.2. You should start every part of the problem by writing down examples.** For example, for (a), the example $3 < 3$ will tell you whether the relation is reflexive, $3 < 5$ and $5 < 3$ will tell you about symmetry, and $3 < 5, 5 < 7$, and $3 < 7$ will get you started on transitivity.

Read Example 10.3, then **do Problem 10.3.** Use examples to check reflexivity, symmetry, and transitivity.

Equivalence relations are very important, as are equivalence classes. An equivalence relation is like the equality relation (=), but applied to other contexts. Here is an example that is useful. Think of the integers, $\mathbb{Z}$. Say that $x \sim y$ if x and y have the same remainder when you divide by 2. Then $6 \sim 22$ and $31 \sim 7$. This relation is reflexive, because $x \sim x$. It is symmetric because if $x \sim y$ then $y \sim x$. And it is transitive because if $x \sim y$ and $y \sim z$, then $x \sim z$. Now we can say that 6 is equivalent to 22, and 31 is equivalent to 7, according to this definition of equivalence. The equivalence class that contains 6 and 22 is all even numbers, and the equivalence class containing 31 and 7 is all odd numbers. Let this sink into your mind, and you will start to see that it makes for a useful way to organize things, when an equivalence relation is available.

Do Problem 10.1 Start by writing out examples for the pairs (x, y) and (w, z). Think about lines and circles in the plane.

1. Read somewhere quiet, minimizing distractions from phones and friends
2. Note the time that you start and stop reading, and add up the minutes
3. Read with a pencil in your hand and your notebook open in front of you
4. Write a sentence to summarize each paragraph, re-draw diagrams, work out examples and exercises on your own
5. Look up words you don't know, and write down ones you really don't know
6. Read slowly.
7. Tally up how much time you have spent on reading this chapter.

Reading assignment, Chapter 18

Reading due on November 30, notes and exercises due on December 2.

Read and understand Chapter 18 of the textbook by Daepp and Gorkin, called "Mathematical Induction," up to the statement, but not the proof, of Theorem 18.6.

Mathematical induction and recursion play an important role especially in discrete mathematics. Prepare to move slowly and think carefully. To understand the proof of Theorem 18.1, you will need **Well-ordering principle of the natural numbers**: Every nonempty subset of the natural numbers contains a minimum.

Read Theorem 18.1 and then **do Problem 18.1** and **Problem 18.3**. Follow the steps in Theorem 18.1, defining the assertion $P(n)$ for the problem first. You will need the condition "$P(n)$ is true" to show the induction step. Work through Exercises 18.3 to 18.5 and then **do Problem 18.9** without going back to Exercise 18.5. You can do it!

Recursion is a very useful tool to define functions, sequences and sets. Before you move to Theorem 18.6, read the definition of n factorial for $n \in \mathbb{N}$. Write out $3!, 4!$ and $5!$. As an exercise, simplify $\frac{6!}{2!4!}$. More generally, simplify $\frac{n!}{m!(n-m)!}$ where n and m are two positive integers with $n \geq m$. These fractions are called *binomial coefficients* and are useful in probability.

Here is another example of using recursion: Let $n \in \mathbb{Z}^+$. Consider the function $S(n) = S(n-1) + n$ with $S(0) = 0$. Write out $S(1), S(2), S(3)$ and $S(4)$. Can you figure out what this function does for us? Together with Problem 18.1, you should be able to see the connection between induction and recursion.

Theorem 18.6 shows the existence and uniqueness of a recursive function $g : N \to X$ given a function $f : X \to X$ and $a \in X$, where X is a nonempty set. The function g satisfies

(i) The base step: $g(0) = a$, and

(ii) The recursive step: $g(n+1) = f(g(n))$ for all $n \in \mathbb{N}$.

The proof of Theorem 18.6 is too long for us to read this semester. You can come back it later.
As with the previous chapters,

1. Read somewhere quiet, minimizing distractions from phones and friends

2. Note the time that you start and stop reading, and add up the minutes

3. Read with a pencil in your hand and your notebook open in front of you

4. Write a sentence to summarize each paragraph, re-draw diagrams, work out examples and exercises on your own

5. Look up words you don't know, and write down ones you really don't know

6. Read slowly.

7. At the end, tally up how much time you have spent on reading this chapter. Write this number in your notebook and remember the number when you come to class.

Your name: ______________________________

Quiz on even and odd and three–dimensional vectors

20 points

Work hard to write really nice proofs.

Definition 1. Even. An integer n is *even* if there exists an integer k for which $n = 2k$.

Definition 2. Odd. An integer n is *odd* if there exists an integer k for which $n = 2k + 1$.

Definition 3. Three–dimensional vector. A three–dimensional vector is an ordered triple $\langle a_1, a_2, a_3 \rangle$, where a_1, a_2, and a_3 are real numbers.

Definition 4. Sum of 3–dimensional vectors. The sum of 3–dimensional vectors $\langle a_1, a_2, a_3 \rangle$ and $\langle b_1, b_2, b_3 \rangle$ is the 3–dimensional vector $\langle a_1 + b_1, a_2 + b_2, a_3 + b_3 \rangle$. We write $\vec{a} \oplus \vec{b}$, using a new symbol so we don't confuse addition of vectors with addition of real numbers.

Definition 5. Scalar product for 3–dimensional vectors. Let c be a real number and let $\vec{a} = \langle a_1, a_2, a_3 \rangle$ be a 3–dimensional vector. The *scalar product* of c and $\vec{a}$ is defined as:

$$c\vec{a} = \langle ca_1, ca_2, ca_3 \rangle.$$

Show. Show that the product of two odd numbers is odd.

Show. On the back of this sheet, show that the scalar product is distributive over vector addition. That is, show that $c(\vec{a} \oplus \vec{b}) = c\vec{a} \oplus c\vec{b}$.

Your name: ____________________________

Quiz on sum and dot product of 3–dimensional vectors

10 points

Definition 3–dimensional vector. A three–dimensional vector is an ordered triple $\langle a_1, a_2, a_3\rangle$, where a_1, a_2, and a_3 are real numbers.

Definition Equality of 3–dimensional vectors. 3–dimensional vectors $\langle a_1, a_2, a_3\rangle$ and $\langle b_1, b_2, b_3\rangle$ are equal if $a_1 = b_1, a_2 = b_2$, and $a_3 = b_3$. The order of the numbers is important.

Definition Sum of 3–dimensional vectors. The sum of 3–dimensional vectors $\langle a_1, a_2, a_3\rangle$ and $\langle b_1, b_2, b_3\rangle$ is the 3–dimensional vector $\langle a_1 + b_1, a_2 + b_2, a_3 + b_3\rangle$. We write $\vec{a} \oplus \vec{b}$, using a new symbol so we don't confuse addition of vectors with addition of real numbers.

Show. Show that addition of 3–dimensional vectors is commutative. Start with "Let," take one step at a time, write the justification for the step, and make a general conclusion.

Definition Dot product of 3–dimensional vectors. The dot product of 3–dimensional vectors $\langle a_1, a_2, a_3\rangle$ and $\langle b_1, b_2, b_3\rangle$ is the real number $a_1b_1 + a_2b_2 + a_3b_3$.

Show. On the other side of this sheet of paper, show that the dot product is distributive over vector addition. That is, show that $(\vec{a} \oplus \vec{b}) \bullet \vec{c} = \vec{a} \bullet \vec{c} + \vec{b} \bullet \vec{c}$. Start with "Let . . . ," take one step at a time, write the justification for the step, and make a general conclusion. Please also explain why one addition symbol is $\oplus$ and the other is $+$.

Possible questions for the quiz over the Division Algorithm

This will be a 10–point quiz. I will choose two of the following four problems for the quiz. I strongly suggest that you write out solutions for each of these before Thursday, and that you try to do them without consulting your notes. Rediscover the arguments, and you will own them.

Then, some hours later, write them again on a fresh sheet of paper. This is the best way to learn them.

I will be happy to look at your practice solutions before class on Thursday. If you have questions, you may ask them by email.

1. Let $n > 0$ and $k > 0$ be integers. Argue that there exist integers q and r such that $n = qk + r$ and $0 \leq r < k$. You can phrase the argument in terms of dealing out n cards to k people, or in terms of starting with n and subtracting k repeatedly.

2. Let $n > 0$ and $k > 0$ be integers. Suppose that there exist integers q and r for which $n = qk + r$ and $0 \leq r < k$, and at the same time that there exist integers a and b for which $n = ak + b$ and $0 \leq b < k$. Show that $q = a$ and $r = b$. This shows that there is at most one way to write $n = qk + r$ with $0 \leq r < k$.

3. Let k be an integer. Show that exactly one of the integers $k, k+1, k+2, k+3$ is a multiple of 4. **Note:** I may ask instead for you to show that exactly one of the numbers $k, k+1, k+2$ is a multiple of 3, or that exactly one of the numbers $k, k+1, k+2, k+3, k+4$ is a multiple of 5. The argument is basically the same in every case.

4. Let n be an odd integer. Show that $n^3 - n$ is a multiple of 24.

Your name: ______________________________

Quiz on things related to the Division Algorithm

10 points

1. Let $n > 0$ and $k > 0$ be integers. Suppose that there exist integers q and r for which $n = qk + r$ and $0 \leq r < k$, and at the same time that there exist integers a and b for which $n = ak + b$ and $0 \leq b < k$. Show that $q = a$ and $r = b$. This shows that there is at most one way to write $n = qk + r$ with $0 \leq r < k$.

2. Let k be an integer. Show that exactly one of the integers $k, k + 1, k + 2, k + 3$ is a multiple of 4. You can use the back of this sheet of paper.

Your name: ______________________________

Quiz on even and odd

15 points

Work hard to write really nice proofs.

1. Show that the product of two odd numbers is odd.

2. Show that for all integers n, the quantity $n^2 + 10n + 21$ is either odd or is a multiple of 4.

3. The numbers $0, 1, 4, 9, 16, 25, \ldots$ are called perfect squares. The differences between consecutive perfect squares are $1, 3, 5, 7, 9, \ldots$. Show that the difference between consecutive perfect squares is always an odd number.

Your name: ______________________________

Quiz on irrationality of square roots of odd primes

20 points

Show. Show that if n is an integer and n^2 is a multiple of 47, then n is a multiple of 47. You choose the type of proof you want to do. Whatever you do, there are too many cases to check one by one, so organize your thoughts efficiently.

Show. On the other side of this sheet of paper, show that $\sqrt{47}$ is irrational. I recommend a proof by contradiction.

Your name: ________________________________

Quiz on infinite set operations

5 points

1. Let $B = \bigcup_{n=0}^{\infty}[n, n^2]$. List out the first five or more sets in this union. Draw them on a number line if it helps.

Let $C = \{0\} \cup \{1\} \cup [2, \infty)$. Show that $B = C$ by showing containment in both directions. You will need to use three cases in each direction to deal with $0, 1$, and the rest.

Your name: ______________________________

Quiz on a new operation with 3–dimensional vectors

20 points

Definition Sum of 3–dimensional vectors. The sum of 3–dimensional vectors $\langle a_1, a_2, a_3\rangle$ and $\langle b_1, b_2, b_3\rangle$ is the 3–dimensional vector $\langle a_1 + b_1, a_2 + b_2, a_3 + b_3\rangle$.

Definition The *twist product* of 3–dimensional vectors $\langle a_1, a_2, a_3\rangle$ and $\langle b_1, b_2, b_3\rangle$ is the 3–dimensional vector $\langle a_1b_3, a_2b_2, a_3b_1\rangle$. It is denoted $\vec{a} * \vec{b}$.

Example. For example, $\langle 1, 3, 6\rangle * \langle 2, 7, 10\rangle = \langle 1 \cdot 10, 3 \cdot 7, 6 \cdot 2\rangle = \langle 10, 21, 12\rangle$.

Show. Show that the twist product is distributive over vector addition. That is, show that $(\vec{a} \oplus \vec{b}) * \vec{c} = \vec{a} * \vec{c} \oplus \vec{b} * \vec{c}$. Start with "Let ...," take one step at a time, write the justification for the step, and make a general conclusion.

Show. Prove or disprove: "The twist product is commutative."

Show. On the other side of this piece of paper, show that for all 3–dimensional vectors $\vec{a}$ and $\vec{b}$ and real numbers c, $\vec{a} * (c\vec{b}) = (c\vec{a}) * \vec{b} = c(\vec{a} * \vec{b})$. Use parentheses *every* time three things are multiplied together.

Your name: ______________________________

Quiz on a new operation with 3–dimensional vectors

20 points

Definition Sum of 3–dimensional vectors. The sum of 3–dimensional vectors $\langle a_1, a_2, a_3\rangle$ and $\langle b_1, b_2, b_3\rangle$ is the 3–dimensional vector $\langle a_1 + b_1, a_2 + b_2, a_3 + b_3\rangle$.

Definition Duplicate product. The *duplicate product* of 3–dimensional vectors $\langle a_1, a_2, a_3\rangle$ and $\langle b_1, b_2, b_3\rangle$ is the 3–dimensional vector $\langle a_1b_1, a_2b_3, a_3b_3\rangle$. (That is not a typo, b_3 is used twice. That is why it is called the duplicate product.) It is denoted $\vec{a} * \vec{b}$.

Example. For example, $\langle 1, 3, 6\rangle * \langle 5, 2, 4\rangle = \langle 1 \cdot 5, 3 \cdot 4, 6 \cdot 4\rangle = \langle 5, 12, 24\rangle$.

Show. Show that the duplicate product is distributive over vector addition. That is, show that $(\vec{a} \oplus \vec{b}) * \vec{c} = \vec{a} * \vec{c} \oplus \vec{b} * \vec{c}$. Start with "Let ...," take one step at a time, write the justification for the step, and make a general conclusion.

Show. Prove or disprove: "The duplicate product is commutative." (You use a proof to prove, a counterexample to disprove.)

Show. On the other side of this piece of paper, show that for all 3–dimensional vectors $\vec{a}$ and $\vec{b}$ and real numbers c, $\vec{a} * (c\vec{b}) = (c\vec{a}) * \vec{b} = c(\vec{a} * \vec{b})$. Use parentheses *every* time three things are multiplied together.

Your name: ______________________________

Quiz on a new operation with 3–dimensional vectors

20 points

Definition Sum of 3–dimensional vectors. The sum of 3–dimensional vectors $\langle a_1, a_2, a_3\rangle$ and $\langle b_1, b_2, b_3\rangle$ is the 3–dimensional vector $\langle a_1 + b_1, a_2 + b_2, a_3 + b_3\rangle$.

Definition The *twist product* of 3–dimensional vectors $\langle a_1, a_2, a_3\rangle$ and $\langle b_1, b_2, b_3\rangle$ is the 3–dimensional vector $\langle a_1 b_3, a_2 b_2, a_3 b_1\rangle$. It is denoted $\vec{a} * \vec{b}$.

Example. For example, $\langle 1, 3, 6\rangle * \langle 2, 7, 10\rangle = \langle 1 \cdot 10, 3 \cdot 7, 6 \cdot 2\rangle = \langle 10, 21, 12\rangle$.

Show. Show that the twist product is distributive over vector addition. That is, show that $(\vec{a} \oplus \vec{b}) * \vec{c} = \vec{a} * \vec{c} \oplus \vec{b} * \vec{c}$. Start with "Let ...," take one step at a time, write the justification for the step, and make a general conclusion.

Show. Prove or disprove: "The twist product is commutative."

Show. On the other side of this piece of paper, show that for all 3–dimensional vectors $\vec{a}$ and $\vec{b}$ and real numbers c, $\vec{a} * (c\vec{b}) = (c\vec{a}) * \vec{b} = c(\vec{a} * \vec{b})$. Use parentheses *every* time three things are multiplied together.

Your name: ______________________________

Quiz on some problems from Chapter 5 of Daepp and Gorkin

15 points

1. Let x and y be real numbers. Use the triangle inequality to show that $||x| - |y|| \leq |x - y|$.

2. Prove or refute the following conjecture: There are no positive integers x and y such that $x^2 - y^2 = 10$. You can use the back of the sheet if you like.