Your name: ________________________________

Even and odd

Our first experience with definitions, examples, theorems, and proofs

Overview

Definitions are important to read and understand by looking at examples. Many proofs are little more than working with the definitions and rewriting things. With a bit of practice, these become very routine. This activity has you work through two definitions, a few examples, and then some proofs. Everything relies on the definitions, so keep coming back to them.

Note 1. The *integers* are positive and negative counting numbers $\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots$

Definition 2. Even. An integer n is *even* if there exists an integer k for which $n = 2k$.

Definition 3. Odd. An integer n is *odd* if there exists an integer k for which $n = 2k + 1$.

Note 4. 19 meets the definition to be odd because 19 is an integer, $19 = 2(9)+1$, and 9 is an integer.

Example 5. Check that 12 meets the definition to be even by writing $12 = 2k$ for an appropriate value of k and make sure k is an integer.

Example 6. Does -9 meet the definition to be odd? Write $-9 = 2k + 1$.

Example 7. Does 0 meet the definition to be even? Write $0 = 2k$ and check the value of k.

Example 8. Does 1.73 meet the definition to be odd? Explain.

Note 9. Suppose that m is an integer. Then $2m + 1$ is an integer and it is odd because it meets the definition to be odd. Also, $2m + 2$ is even because it is an integer and can be rewritten as $2(m + 1)$, which is of the form $2k$ where $k = m + 1$, which is an integer.

Show 10. Suppose that m is an integer. Show that $2m + 6$ is even by rewriting it until it meets the definition to be even. Connect your expressions with = signs, not implication signs $\Rightarrow$.

Show 11. Suppose that m is an integer. Show that $4m + 9$ is odd by rewriting it until it meets the definition to be odd. Connect your expressions with = signs.

Stop. Compare your answers to the questions above with the other people in your group before you move on. Resolve any differences in your answers.

Show 12. Suppose that m is even. Then $m = 2k$ for some integer k. Show that $m+8$ is even by rewriting it as $2k + 8$ and continuing until it is 2 times an integer. Connect your expressions with = signs.

Guided proof 13. You already know that the sum of two even numbers is even. Fill in the blanks to produce a proof of this fact, using Definition 2 of even.

a. Let m and n be even integers.

b. There exist integers j and k such that $m =$__________ and $n =$__________, by Definition 2.

c. Thus, $m + n =$ __________________________ $= 2(j + k)$.

d. This number meets the definition to be even because $j + k$ is an __________ and because $m + n$ is __________ times an integer.

e. We saw that if m and n are even, then $m + n$ is even. We made no further assumption about m and n. Thus, the sum of any two even numbers is even.

Guided proof 14. Fill in the blanks to show that the product of two even numbers is even.

a. Let m and n __________________________

b. There exist __________________________ such that $m =$__________ and $n =$__________, by __________.

c. Thus, $mn =$_______________________________________ $= 2($__________$)$.

d. This number satisfies the definition to be even because ________________ is an integer and mn is __________________________.

e. We saw that ____. We made no ____. Thus, ________________________________.

Prove 15. Show that the sum of two odd numbers is even. Follow the examples above and remember that good form is critically important in proofs.

a.

b.

c.

d.

e. (Yes, you need to write this every time! It's how we make generalizations.)

Prove 16. Show that the product of two odd numbers is odd. Follow the examples above and remember that good form is critically important in proofs.

a.

b.

c.

d.

e.

Group work 17. Let m and n be even integers. Does this imply that $m = 2k$ and $n = 2k$ for some integer k? Discuss with members of your group.

Find specific values of m and n for which $m = 2k$ and $n = 2k$ for some integer k does happen.

Find specific values of m and n for which $m = 2k$ and $n = 2k$ for some integer k does not happen.

How does proof 13 deal with this?

Prove 18. Show that the square of an even number is even. That is, if m is even, then m^2 is even. Follow the models above and use good form.

Prove 19. Show that the square of an odd number is odd. That is, if m is odd, then m^2 is odd.

Question 20. What does it mean that an integer is a multiple of 4? Give your own definition analogous to the definitions of even and odd.

Prove 21. Let m be even. Show that $m^2 + 2m + 4$ is a multiple of 4. Use good form.

Question 22. Let m be even. Can $m^2 + 2m + 4$ be a multiple of 8? Explain.

Challenges. Here are some statements that are harder to prove because they require a bit more than simply restating the definitions. See if you can make a good argument for them.

Question 23. Pretend for a minute that there is an integer m which is both even and odd. Work with the definitions to see that this really must be fantasyland.

Prove 24. If m is an integer, then m is even or m is odd. That is, it has to be one of the two, there is no third possibility.
Here is one suggestion. 0 is even. If n is even, then $n+1$ is odd. If n is odd, then $n+1$ is even. This should cover all positive integers. Also, if n is even, then $-n$ is even, which tells us about negative integers.

Prove 25. If m is an integer and m^2 is odd, then m is odd. **Hint:** There are two cases to check, the case in which m is even and the case in which m is odd. You may wish to refer back to 18, 19, and 24.

Your name: ______________________________

Sum and dot product of 3–dimensional vectors

Overview

In Calculus III and Linear Algebra, we define vectors and work with them. They have a geometric interpretation, but here we will simply give an algebraic definition of 3–dimensional vectors and some operations on them and work with their algebraic properties. This activity illustrates proofs in which all that is needed is the definition and a "rewrite" proof, where you can work forward and backward to show a series of equalities. Notice how we often use the same definition twice in one proof, once to "unpack" and the second time to "re–pack."

Definition 26. 3–dimensional vector. A three–dimensional vector is an ordered triple $\langle a_1, a_2, a_3 \rangle$, where a_1, a_2, and a_3 are real numbers. The numbers a_1, a_2, and a_3 are called *components* of the vector.

Notation 27. A 3–dimensional vector $\langle a_1, a_2, a_3 \rangle$ is often denoted by a single letter with an arrow over the top, like this $\vec{a}$. When it is written like $\langle a_1, a_2, a_3 \rangle$ it is said to be in *open form.* The commas and brackets are part of the definition and are important.

Definition 28. Equality of 3–dimensional vectors. 3–dimensional vectors $\langle a_1, a_2, a_3 \rangle$ and $\langle b_1, b_2, b_3 \rangle$ are equal if $a_1 = b_1, a_2 = b_2$, and $a_3 = b_3$. Note: The order of the numbers is important.

Definition 29. Sum of 3–dimensional vectors. The sum of 3–dimensional vectors $\langle a_1, a_2, a_3 \rangle$ and $\langle b_1, b_2, b_3 \rangle$ is the 3–dimensional vector $\langle a_1 + b_1, a_2 + b_2, a_3 + b_3 \rangle$. We write $\vec{a} \oplus \vec{b}$ for the sum of $\vec{a}$ and $\vec{b}$, using a new symbol so we don't confuse addition of vectors with addition of real numbers.

Example 30. Is $\langle 3, 9, 12 \rangle$ a 3–dimensional vector? Explain.

Is it equal to $\langle 12, 3, 9 \rangle$? Explain.

Example 31. Is $\langle \sqrt{3}, \sqrt[3]{9}, \sqrt{-12} \rangle$ a 3–dimensional vector? Explain.

Example 32. Is $\langle 8, 13.35321, \pi, -7 \rangle$ a 3–dimensional vector? Explain.

Example 33. Is $\langle 3 + 9 + 12 \rangle$ a 3–dimensional vector? Explain.

Example 34. Is $\langle \begin{bmatrix} 6 & 0 \\ 2 & 5 \end{bmatrix}, -4, 7 \rangle$ a 3–dimensional vector? Explain.

Example 35. Let x be a real number. Is $\langle \frac{14}{3}, 2 - 7x, \sqrt{16} \rangle$ a 3–dimensional vector? Explain.

Stop. Compare your answers to the questions above with the members of your group. Make sure you agree on everything.

Example 36. Calculate the sum of $\vec{c} = \langle 12, -5, 3\rangle$ and $\vec{d} = \langle 6, 4, -11\rangle$. Start by writing $\vec{c} \oplus \vec{d} = \ldots$ and write the vectors in open form next.

Calculate $\vec{d} \oplus \vec{c}$ in the same way.

Show 37. You are going to show that addition of 3–dimensional vectors is commutative. Fill in the blanks. This is a "rewrite" proof. You can work forward from the top, backward from the bottom, or a bit of both. Let $\vec{a}$ and $\vec{b}$ be 3–dimensional vectors. Then,

$$\begin{aligned}
\vec{a} \oplus \vec{b} &= \langle \quad , \quad , \quad \rangle \oplus \langle \quad , \quad , \quad \rangle \\
&= \langle \qquad , \qquad , \qquad \rangle \\
&= \langle \qquad , \qquad , \qquad \rangle \\
&= \langle \quad , \quad , \quad \rangle \oplus \langle \quad , \quad , \quad \rangle \\
&= \vec{b} \oplus \vec{a}
\end{aligned}$$

We have seen that $\vec{a} \oplus \vec{b} = \vec{b} \oplus \vec{a}$. We made no further assumption about $\vec{a}$ and $\vec{b}$. Thus, for all 3–dimensional vectors $\vec{a}$ and $\vec{b}$, we know that $\vec{a} \oplus \vec{b} = \vec{b} \oplus \vec{a}$. Thus, addition of 3–dimensional vectors is commutative.

Show 38. Go back to each line of the proof above and give exactly one reason for the equality on that line at the very right side of the line. The first one is "Write in open form." Two of them are Definition 29. In the middle you will use the fact that addition of real numbers is commutative. Thus, at the heart of it, commutativity of vector addition comes from commutativity of addition of real numbers.

Show 39. Show that addition of 3–dimensional vectors is associative. Start with arbitrary 3–dimensional vectors $\vec{a}, \vec{b}$, and $\vec{c}$. Write $(\vec{a} \oplus \vec{b}) \oplus \vec{c}$ and rewrite it until it becomes $\vec{a} \oplus (\vec{b} \oplus \vec{c})$. Take small steps and write exactly one reason for each equality. Since you know what equality you need to show, you can work forward from the top, backward from the bottom, or both.

Let $\vec{a}, \vec{b}$, and $\vec{c}$ be ____.

$$\begin{aligned}
(\vec{a} \oplus \vec{b}) \oplus \vec{c} &= \\
&= \\
&= \langle (\quad + \quad) + \quad , (\quad + \quad) + \quad , (\quad + \quad) + \quad \rangle \\
&= \langle \quad + (\quad + \quad), \quad + (\quad + \quad), \quad + (\quad + \quad) \rangle \\
&= \\
&= \\
&= \vec{a} \oplus (\vec{b} \oplus \vec{c})
\end{aligned}$$

We have seen that ...

Stop. Compare your argument to the rest of the members of your group. Make sure that you agree on absolutely every step and every justification.

Definition 40. Scalar product for 3–dimensional vectors. Let c be a real number and let $\vec{a} = \langle a_1, a_2, a_3 \rangle$ be a 3–dimensional vector. The *scalar product* of c and $\vec{a}$ is a 3–dimensional vector defined as:

$$c\vec{a} = \langle ca_1, ca_2, ca_3 \rangle.$$

Example 41. Let $c = 3$ and $\vec{a} = \langle 7, -4, \sqrt{2} \rangle$. Calculate $c\vec{a}$, starting by writing $c\vec{a} = 3\langle 7, -4, \sqrt{2} \rangle = \ldots$.

Example 42. Calculate $\pi\langle 9, 4, 1 \rangle =$

Example 43. Calculate $(2 + \sqrt{3})\langle 5, b, c \rangle =$

Show 44. You are going to show that the scalar product is distributive over vector addition. First use the word "Let" to settle on one real number c and two 3–dimensional vectors, $\vec{a}$ and $\vec{b}$. Then start with the expression $c(\vec{a} \oplus \vec{b})$ and rewrite it three times. Then, move to the last expression and work backwards, until you meet in the middle. Provide one reason for each equality, on the right, on the same line as the equality. At the end, follow the model to conclude that you have shown distributivity in general.

Let ...

$$c(\vec{a} \oplus \vec{b}) =$$

$$= c\vec{a} \oplus c\vec{b}$$

We have seen that ...

Stop. Check over what everyone in your group has done, and make sure that you completely agree.

Show 45. Show that the scalar product is distributive over real number addition. Start with "Let." Write $(c+d)\vec{a}$ and rewrite it until it equals $c\vec{a} \oplus d\vec{a}$. Work forward from the top and backward from the bottom. Provide one reason for each equality. At the end, follow the model to conclude that this shows distributivity in general. Explain why some addition signs are $+$ and others are $\oplus$.

Definition 46. Zero vector. The vector $\langle 0,0,0\rangle$ is a special 3–dimensional vector, called the *zero vector*. We denote it by $\vec{0}$.

Definition 47. Additive inverse. Let $\vec{a}$ be a 3–dimensional vector, with open form $\langle a_1, a_2, a_3\rangle$. Define a new vector by $-\vec{a} = \langle -a_1, -a_2, -a_3\rangle$. It is called the *additive inverse* of $\vec{a}$.

Show 48. Let $\vec{a}$ be a 3–dimensional vector. Use a rewrite proof to show that $\vec{a} \oplus \vec{0} = \vec{a}$. This is called the *additive identity* property. It's not very exciting. Make a general conclusion.

Show 49. Let $\vec{a}$ be a 3–dimensional vector, and let $-\vec{a}$ be its additive inverse. Use good form to show that $\vec{a} \oplus (-\vec{a}) = \vec{0}$. This is called the *additive inverse* property. This is also not very exciting. Make a general conclusion.

Definition 50. Dot product of 3–dimensional vectors. The dot product of 3–dimensional vectors $\langle a_1, a_2, a_3\rangle$ and $\langle b_1, b_2, b_3\rangle$ is the real number $a_1b_1 + a_2b_2 + a_3b_3$.

Notation 51. The dot product of 3–dimensional vectors $\vec{a}$ and $\vec{b}$ is denoted $\vec{a} \bullet \vec{b}$.

Example 52. Calculate the dot product of $\vec{a} = \langle 12, -5, 3\rangle$ and $\vec{b} = \langle 6, 4, -11\rangle$ Do this by writing

$$\begin{aligned} \vec{a} \bullet \vec{b} &= \langle a_1, a_2, a_3\rangle \bullet \langle b_1, b_2, b_3\rangle \\ &= a_1b_1 + a_2b_2 + a_3b_3 \end{aligned}$$

and then substituting in the numbers. This makes the calculation just a matter of rewriting, so it is a good way to do calculations like this.

Show 53. Show that the dot product is commutative, just as multiplication of real numbers is commutative. Start with "Let". Write one expression at the top of the space below, and write your goal expression at the bottom, and then work forward and backward until you have a rewrite proof. Follow the models from previous examples, and be sure to make a general conclusion.

Example 54. Calculate $\vec{a} \bullet \vec{0}$. Is this a general result? If so, make your calculation into a general result.

Show 55. Show that the dot product is distributive over vector addition. That is, show that $(\vec{a} \oplus \vec{b}) \bullet \vec{c} = \vec{a} \bullet \vec{c} + \vec{b} \bullet \vec{c}$. Start with "Let". Write the first expression at the top, the last expression at the bottom, and then work forward and backward. Also explain why one addition symbol is $\oplus$ and the other is $+$.

Show 56. Let $\vec{a}$ and $\vec{b}$ be 3–dimensional vectors and let c be a real number. Show in general that $c(\vec{a} \bullet \vec{b}) = (c\vec{a}) \bullet \vec{b} = \vec{a} \bullet (c\vec{b})$. Since there are two equalities to show, think about how you will organize the equalities.

Your name: ______________________________

The Division Algorithm

Dividing integers with remainders will form the basis for several things we want to prove.

Overview

We would like to distribute n objects evenly among k people and find out how many are left over. We will investigate a procedure for doing this, which is called division, even though there will be no fractions in this activity. Procedures that are guaranteed to work are called *algorithms* after the 9th century Persion mathematician al-Khwarizmi, who worked on procedures for arithmetic. The division algorithm itself dates to Euclid's *Elements* from around 300 BC.

Example 57. You are the dealer in a card game that has 37 cards. (It's not a standard deck of cards.) There are 5 people playing, and everyone needs to end up with the same number of cards. Dealing one card to each player leaves 32 cards in your hands. Write down the numbers 37, 32, and continue until you cannot deal out any more cards evenly:

Let r denote the number of cards left over at the end, and let q denote the number of times you subtracted 5, which is also the number of cards that each person got. Notice that $0 \leq 5q \leq 37$. You see that $37 - 5q = r$, which you can rewrite as $37 = 5q + r$. Fill in q and r and write out these two equations.
$37 - 5q = r$ becomes:

$37 = 5q + r$ becomes:

Example 58. Now you're playing a card game that you have not played before, and you haven't taken the time to count how many cards are in the deck. You are the dealer again, and there are 5 people who need cards. Let n denote the number of cards in the deck. Imagine that you repeat the procedure from the previous example until you can no longer deal out cards evenly. Again, let r denote the number of cards you have left over at the end and let q denote the number of cards that each person got.

What inequalities do we know about the possible values of r?

What inequalities do we know about the possible values of q?

Write the relationship between n, 5, q, and r analogous to $37 - 5q = r$ and the expression analogous to $37 = 5q + r$.

The last expression accounts for where all of the n cards have gone; some are dealt out, some are left in your hands. Write out a sentence that explains this.

Example 59. Continuing to divide by 5, complete this sentence: Given an integer $n > 0$, there exist integers q and r where (list properties of q here) ________________________ and (list properties of r here) ________________________, such that (write the relationship between n, q, and r analogous to $37 = 5q + r$) ________________________.

Example 60. Once again, we have n cards, but now there are k people playing, where $k > 0$ is an integer. You are dealing. Describe in words how you will deal out the cards and when you will stop.

Describe in words how many cards you will have left over.

Using q to denote the number of cards each person gets and r to denote the number of cards left over, write out the relationship between $n, k, q,$ and r, and write inequalities concerning q and r.

Stop. Compare your work to the others in your group and reconcile any differences.

Question 61. What happens when $k = 1$? What is r? What is q?

Question 62. What happens when $k = 0$? What inequality must r satisfy? Can you satisfy $n = qk + r$? For what value(s) of q? Describe in words, and take your time to get it right, because this explains why we don't divide by 0.

Note 63. As above, suppose that n and k are integers that are greater than 0. Suppose you find integers q and r for which $n = qk + r$ and $0 \leq r < k$. Suppose your friend tries to do the same thing and finds integers q_2 and r_2 for which $n = q_2k + r_2$ and $0 \leq r_2 < k$. Must it be the case that $q = q_2$ and $r = r_2$? That is, are the values of q and r *unique*? If you think about dealing n cards to k people, it's pretty clear that you and your friend will get the same values of q and r, but how can we see this without thinking about card dealing? It will take a few steps.

Show 64. Suppose that r and r_2 are integers for which $0 \leq r < k$ and $0 \leq r_2 < k$. Show that $-k < r - r_2 < k$. Starting and ending expressions are shown; your goal is to provide crystal clear intermediate steps with no extra steps. Note: You can add inequalities that run the same direction; for example, if $a < b$ and $c \leq d$, then $a + c < b + d$.

Suppose ① $0 \leq r < k$ and ② $0 \leq r_2 < k$.

Thus $-k < r - r_2 < k$.

Show 65. Suppose that n and k are integers with $k > 0$, that q, r, q_2, and r_2 are integers such that $n = qk + r$ and $n = q_2k + r_2$, and that $0 \leq r < k$ and $0 \leq r_2 < k$. Use 64 to show that $q = q_2$ and $r = r_2$. Starting and ending points are suggested. This is not simply a rewrite proof; it requires a spark of genius to complete it, so work on scratch paper, write down everything you know including 64,work together, and be patient. Write a final argument here.

To start, note that because $n = qk + r$ and $n = q_2k + r_2$, we have $qk + r = q_2k + r_2$.

Thus $q = q_2$ and $r = r_2$.

Theorem 66. The Division Algorithm. Let n and k be integers greater than 0. Then,

1. **Existence.** There exist integers q and r for which $n = qk + r$ and for which $0 \leq r < k$.
2. **Uniqueness.** The numbers q and r are unique: there is only one way to choose q and r so that $n = qk + r$ and $0 \leq r < k$.

The number q is called the *quotient* and r is called the *remainder*. Note that there are two parts to the theorem. You have proven this theorem above in two problems. The existence part was proven in ___________. The uniqueness part was proven in ___________. Check that your group agrees.

Example 67. Rewrite Theorem 66 for the case where $k = 2$. Be specific about the possible values of r.

Let n be an integer greater than 0. Then,

1. **Existence:**
2. **Uniqueness:**

Prove 68. Let n be an integer greater than 0. Recall the definitions of even and odd. Use the existence part of the Division Algorithm with $k = 2$ to show that n must satisfy at least one of these definitions.

Prove 69. Let n be an integer greater than 0. Use the uniqueness part of the Division Algorithm with $k = 2$ to conclude that if n is even, then it cannot be odd. Also, if n is odd, it cannot be even. Thus, each integer is even or odd, never both.

Note 70. Now we will divide negative numbers by positive numbers, with remainder.

Example 71. Start with -37, add 5 to get -32, and add 5 repeatedly, writing down the numbers you come to, until you reach a number from 0 to 4.

Count the number of 5's that you added to write $-37 = 5q + r$; you fill in q and r. Note the sign of the quotient q.

This represents division of a negative number by 5, with remainder. How does it differ from division of a positive number with remainder?

In what ways is it the same as division of a positive number with remainder?

Prove 72. Let n be an integer less than 0. Let k be an integer greater than 0. Add k to n repeatedly until you reach a number from 0 to $k - 1$.

How can you be sure that you will ever get all the way to non-negative numbers?

How can you be sure that you won't jump over the numbers $0, 1, \ldots, k - 1$ and keep adding k forever?

Use these facts to argue that you can write $n = qk + r$ for some integer q and a small value of r.

What inequalities do you know for q and r?

Prove 73. Let n be an integer less than 0 and let k be an integer greater than 0. Scrutinize your proof of 65. We did not assume $n \geq 0$. Will your proof work for $n < 0$? Explain.

Theorem 74. Rewrite Theorem 66 to reflect what you know now about both positive and negative values of n. Be specific about the assumptions on n and k.

Show 75. Let n be an integer and suppose that $n = 3m + 1$ for some integer m. Identify k, q, and r. Use the Division Algorithm to argue that n cannot be written as $n = 3k$ where k is an integer. Thus, n is not a multiple of 3. Take your time and work together to write a really clear argument.

Which part(s) of the Division Algorithm did you use, existence, uniqueness, or both? ____________________

Show 76. Let n be an integer and suppose that n^2 is a multiple of 3. For example, n^2 could be 36. List two more possible values of n^2:
We would like to conclude that n is a multiple of 3. It's hard to do this directly, but it can be done indirectly. Use the Division Algorithm to write $n = 3q + r$. Start with "There exist integers ...." and be specific about the possible values of r.

Your previous step gives three cases to consider. For each case, write the expression for n, then use algebra to compute n^2.

Case 0: $r = 0$, $n = 3q$, so $n^2 =$

Case 1: $r = 1$, $n =$ _______, so $n^2 =$

Case 2:

Knowing that n^2 is in fact a multiple of 3, use your cases above to rule out one or more of the cases, and so rule out one or more of the possible values of r. Can you conclude that n is a multiple of 3? This is harder than you might think to explain clearly; work hard on it.

Show 77. Suppose n is an integer. Can n^2 be of the form $3m + 2$ where m is an integer? Examine the cases in the previous question carefully.

Example 78. In the triple of consecutive integers 5, 6, 7, exactly one number is a multiple of 3. Circle it. Same thing for 13, 14, 15. Circle the multiple of 3. Write down 5 more triples of consecutive integers. Do you always get a multiple of 3? Can you get more than one multiple of 3?

Show 79. Let n be an integer and consider the numbers n, $n+1$, and $n+2$. Show that exactly one of these is a multiple of 3. Use three cases, $n = 3k$, $n = 3k+1$, and $n = 3k+2$, and follow the guide below.

Case 1: $n = 3k$. Then $n+1 =$ ____________________ and $n+2 =$ ____________________.
Exactly one of these is a multiple of three (circle it). Use the Division Algorithm to argue that the other two are not multiples of 3.

Case 2: $n = 3k+1$. Then $n+1 =$ ____________________ and $n+2 =$ ____________________.
Exactly one of these is a multiple of three (circle it). Use the Division Algorithm to argue that the other two are not multiples of 3.

Case 3: $n = 3k+2$. Then $n+1 =$ ____________________ and $n+2 =$ ____________________.

Example 80. Think about pairs of consutive even numbers, like 8 and 10, or 14 and 16. One number is a multiple of 4 (circle it), the other is not. Check five more pairs.

Show 81. Let n be even and consider the numbers n and $n+2$. Use two cases to show that exactly one of these is a multiple of 4. What two cases? You will need a new idea.

Example 82. Let $n = 1$ and compute $n^3 - n$. Let $n = 3$ and compute $n^3 - n$. Let $n = 5$ and compute $n^3 - n$.

Show 83. Let n be an odd integer. Show that $n^3 - n$ is a multiple of 24. Here you will need a few sparks of genius. Use scratch paper to brainstorm different approaches that you could try, then try the one that looks the most promising. Fortunately there are several ways to do this proof.

Your name: ________________________________

Exploring inequalities

Overview

In this activity you will explore properties of inequalities, but without proving the inequalities. The point here is to use examples and counterexamples to sharpen your intuition about inequalities and their properties. Be adventurous when you look for counterexamples. If you find a counterexample, put a box around it. If the conclusion about inequalities seems to be correct, put a big check mark next to it.

Question 84. Is the statement $7 \leq 7$ true? Explain.

Question 85. Is the statement $7 < 7$ true? Explain.

Question 86. Is the statement $6 \leq 7$ true? Explain.

Question 87. Suppose $a < 7$. Can you conclude that $a \leq 7$? This is counterintuitive for many people. Writing $a \leq 7$ does not mean, "I certify that a really could equal 7." Instead, it means more like "I certify that the value of a can be at most 7" or that "$a > 7$ is false." Again, is it also true that $a \leq 7$?

Question 88. Suppose $a \leq 7$. Can you be certain that $a < 7$? A good technique is to write down five numbers satisfying $a \leq 7$ and see if they also satisfy $a < 7$. Try to find a counterexample. If you find one, put a box around it, otherwise put a check mark.

Question 89. Suppose $a < b$ and $b \leq c$. Is it guaranteed that $a < c$? Work with examples if it helps, and look for a counterexample.

Question 90. Suppose $a < b$ and $b \leq c$. Is it guaranteed that $a \leq c$? Work with examples if it helps, and look for a counterexample.

Question 91. Suppose $a \leq b$ and $b \leq c$. Is it guaranteed that $a < c$? Work with examples if it helps, and look for a counterexample.

Question 92. Suppose $a > 12$. Consider the inequality $-a > -12$. Write down five numbers satisfying $a > 12$ and check whether or not they satisfy $-a > -12$. Look for a counterexample. If you find a counterexample, put a box around it. If the result is OK, put a check mark.

Question 93. Suppose $a > 12$. Consider the inequality $-a < -12$. Write down five numbers satisfying $a > 12$ and check whether or not they satisfy $-a < -12$. If you find a counterexample, put a box around it.

Question 94. Suppose $c < 5$. Use examples to check whether $c^2 < 25$. If you find a counterexample, put a box around it and consider whether an additional condition on c would guarantee $c^2 < 25$.

Question 95. Suppose $c < 7$ and $d \leq 8$. Use examples to check whether $c + d < 15$. If you find a counterexample, put a box around it and consider whether an additional condition on c and d would guarantee $c + d < 15$.

Question 96. Suppose $c < 3$ and $d \leq 4$. Use examples to check whether $cd < 12$. If you find a counterexample, put a box around it and consider whether an additional condition on c and d would guarantee $cd < 12$.

Question 97. Suppose $a \leq b$ and $c \geq 0$. Use examples to check whether $ac \leq bc$, as above.

Question 98. Suppose $a \leq b$ and $c \leq d$. Use examples to check whether $a + c < b + d$, as above. If you find a counterexample, put a box around it and consider whether an additional condition would guarantee $a + c < b + d$.

Question 99. Suppose $a < b$ and $c \leq d$. Use examples to check whether $ac < bd$. If you find a counterexample, put a box around it and consider whether an additional condition would guarantee $ac < bd$.

Question 100. Suppose $a \leq b$. Use examples to check whether $a^2 < b^2$. If you find a counterexample, put a box around it and consider whether an additional condition would guarantee $a^2 < b^2$.

Show 101. If $1 < p$, cite a general result from above to conclude that $5 < 5p$.

Show 102. Suppose p is an integer with $-5 < 5p < 5$. Without dividing by 5, check whether or not it is possible that $p = 1$, $p > 1$, $p = -1$, $p < -1$. Conclude that $p = 0$. If you use properties of inequalities, cite the number from above that you are using.

Question 103. Find an integer $n > 0$ for which $\frac{1}{n} < 0.1$.

Question 104. Find the smallest integer $n > 0$ for which $\frac{1}{n} < 0.03$.

Question 105. Find the smallest integer $n > 0$ for which $\frac{1}{n} < 0.0002$.

Question 106. Let $\varepsilon > 0$. Describe a procedure for finding the smallest integer $n > 0$ for which $\frac{1}{n} < \varepsilon$.

Question 107. Suppose $a < b$. Use examples to check whether $\frac{1}{a} < \frac{1}{b}$. If you find a counterexample, put a box around it and consider whether an additional condition would guarantee $\frac{1}{a} < \frac{1}{b}$.

Question 108. Suppose $a \leq b$. Use examples to check whether $\frac{1}{a} \geq \frac{1}{b}$. If you find a counterexample, put a box around it and consider whether an additional condition would guarantee $\frac{1}{a} \geq \frac{1}{b}$.

Your name: ______________________________

Contrapositive, process of elimination, contradiction

Overview

A central part of mathematics is identifying logical statements and showing which statements imply other statements. This activity introduces logical statements and four ways to prove implications: direct proof, contrapositive, the process of elimination, and proof by contradiction. You may have seen this same material presented using truth tables, but this particular activity specifically avoids truth tables.

Definition 109. Logical statement. A *logical statement* is a sentence that is either true or false. Sometimes logical statements have an unknows such as n, but for each value of n, the statement is either true or false. We often label logical statements with capital letters.

Example 110. For each of the logical statements below, write its truth value T or F. If the sentence is not a logical statement, explain why not.

a. P: 18 is even

b. Q: 19 is even

c. R: 19 is a large number

d. S: 13 is prime

e. T: $2^5 - 1$ is prime

f. U: $\sqrt{2}$ is rational

Example 111. For each of the logical statements below, give five values of the integer n for which the statement is true, if possible, and five values of the integer n for which the statement is false, if possible.

a. $2^n - 1$ is prime	T: $n =$	F: $n =$
b. n is a perfect square	T: $n =$	F: $n =$
c. n^2 is a prime number	T: $n =$	F: $n =$
d. $n^2 + 3n + 1$ is odd	T: $n =$	F: $n =$

Definition 112. Conjunction, logical and. The *conjunction* of two logical statements P and Q is a new logical statement denoted $P \wedge Q$ which is true when both P and Q are true, and false otherwise. It is usually read as "and". The symbol $\wedge$ is only used between logical statements. We don't write "Suppose $x > 3 \wedge \leq 9$" but rather "Suppose $x > 3 \wedge x \leq 9$."

Definition 113. Disjunction, logical or. The *disjunction* of two logical statements P and Q is a new logical statement denoted $P \vee Q$ which is true when P is true, when Q is true, or when both are true, but false when both are false. The symbol $\vee$ is only used between logical statements. Instead of writing "Suppose $n = 3 \vee 5$," you could write "Suppose $n = 3 \vee n = 5$."

Exercise 114. Give the truth value T or F of each of the new statements below, using the statements from the previous page.

a. $P \vee Q$

b. $Q \wedge S$

c. $P \wedge Q \wedge T$

d. $P \vee (Q \wedge U)$

Definition 115. Negation. The *negation* of a logical statement P is a new statement denoted $\neg P$ which is true when P is false and false when P is true. $\neg P$ is read as "not P".

Exercise 116. Give the truth value T or F of each of the new statements below.

a. $\neg Q$

b. $P \wedge \neg Q$

c. $Q \vee \neg S$

Definition 117. Implication. For logical statements P and Q, we say that P implies Q and write $P \to Q$ if P being true guarantees that Q is true.

Exercise 118. For each line below, identify the statement corresponding to P and the statement corresponding to Q in the implication $P \to Q$. In every case, n is an integer.

a. If n is even, this implies that n^2 is even.

b. If n^2 is odd, then n is odd.

c. If n is odd, then $n^3 - n$ is a multiple of 24.

Definition 119. Direct proof. A *direct proof* of an implication is where we start with the statement P and use the information in it together with a series of valid logical steps to show that Q is true. This establishes that $P \to Q$. We have seen a number of direct proofs, including proofs by rewriting.

Prove 120. Let n be an integer. Consider $S : n$ is even and $T : n^2 + 6n + 7$ is odd. Write a direct proof that $S \to T$. Start with "Let n ..."

Example 121. Let n be an integer. Consider $A : n^2$ is even and $B : n$ is even. Try to write a direct proof that $A \to B$. If you don't see a way to do it, you can stop trying.

Definition 122. Contrapositive. Here is another way to think about showing $P \to Q$. You need to be sure that it never happens that P is true but Q is false. You can do this by showing that whenever Q is false, P is also false. In other words, show that $\neg Q \to \neg P$. This is called *proof by contrapositive*. It may be easier to find a direct proof that $\neg Q \to \neg P$ than it is to show $P \to Q$.

Exercise 123. Let n be an integer. Consider $A : n^2$ is even and $B : n$ is even. State $\neg B$ and $\neg A$, then show that $\neg B \to \neg A$. State what you have shown in terms of n^2 and n.

Exercise 124. Let n be an integer. Consider $P : n^2 + 8n + 9$ is even and $Q : n$ is odd. State $\neg Q$ and $\neg P$ and show $\neg Q \to \neg P$.

Exercise 125. Let n be an integer. Consider $S : n^2$ is a multiple of 3 and $T : n$ is a multiple of 3. State $\neg T$ and $\neg S$. By the Division Algorithm, there are two ways that $\neg T$ can happen. Write down each one, and prove that in either case, they imply $\neg S$.

Definition 126. Rational. A real number is said to be *rational* if it can be written as $\frac{a}{b}$ where a and b are integers and $b \neq 0$; this is the quotient of two integers.

Definition 127. Irrational. A real number is said to be *irrational* if it cannot be written as the quotient of two integers.

Exercise 128. Let $r \neq 0$. Show that if r is irrational, then $\frac{1}{r}$ is irrational.

Exercise 129. Suppose that a is an irrational number. Show that $5a$ is irrational.

Exercise 130. Suppose that a is an irrational number. Let b be a rational number not equal to 0. Show that ba is irrational.

Note 131. Showing that a statement is false. Sometimes we want to show that a statement P is false. Here is a method to do that. Pretend for a minute that P is true, and use rules of algebra, previously-proven results, theorems, etc. to make a series of logical implications $P \to Q$, $Q \to R$, $R \to S$, $S \to T$ until you arrive at a statement T that you know to be false. Then you have shown that $P \to T$. But since T is false, then you can be certain that P is false. It can be difficult to identify a specific statement that is false; work hard to accomplish this, because it can really help to clarify the proof.

Note 132. When proving that statement P is false, we start by writing "Pretend for a minute that P is true." We do not really believe that P is true, but it helps to pretend that it's true as you make a chain of implications starting from P.

Prove 133. Let n be an integer. Consider the statement P : n is both even and odd. Complete the following proof that P is false.
Pretend for a minute that P is true. Then $n = 2k$ and $n = 2j + 1$ for ...

But k and j are integers, and so $k - j$ is an integer, so $k - j = \frac{1}{2}$ is false. Thus, P must be false.

Prove 134. Consider the statements $P : n$ is a positive integer and $Q : n^2$ is 6. Prove that $P \wedge Q$ is false by first pretending that it is true. One idea is that n would have to be larger than 2 but smaller than 3.

Note 135. In the previous problem, we saw that $P \wedge Q$ is false, but we cannot say which of the two statements is false. If n is an integer, then $Q : n^2 = 6$ is false. If $n^2 = 6$, then n is not an integer, so P is false. The next example shows a slightly different approach where you can make a solid conclusion.

Example 136. Let n be an integer and suppose that n is even. Prove that the statement Q: n is odd is false.
Pretend for a minute that Q is true. The argument in 133 leads us to a false statement. Thus, Q must be false.
What is different here is that before we encounter the statement Q, we have supposed that n is an integer and n is even, both of which can be true. In that context, we can see that Q is false.

Guided proof 137. Let R be the statement that $\sqrt{2}$ is rational. You will show that R is false, by making a series of deductions that lead to a conclusion that is known to be false.
Pretend for a minute that R is true, that is, that $\sqrt{2}$ is rational.
Then there exist integers p and q for which $\sqrt{2} = \frac{p}{q}$, and we can arrange it so that p and q are not both even. (If they were both even numbers, we would factor out 2 from each until they are not both even.)
Using algebra, $2q^2 = p^2$. Thus, p^2 is ________________. Thus, p is ____________ by result ____________ and so can be written as $p =$ ____________ for some ____________.
Using algebra, $q^2 =$ ______________________, and so q^2 is ____________. Thus q is ____________.
But we know that this is false, because ________________________________. Thus, the statement R is false. Note that because $\sqrt{2}$ is not rational, it must be irrational.

Prove 138. Suppose there are 20 kids playing musical chairs, with 19 chairs. When the music stops, at least one kid will not have a chair to sit on. Show that the statement M : "all kids have a chair to sit on by themselves" is false.
Pretend for a moment that M is true. Let k denote the number of kids and let c denote the number of chairs. (What is the relationship between k and c?)

Definition 139. Composite. An integer $n > 1$ is *composite* if it can be written as $n = ab$ where a and b are integers with $1 < a, b < n$.

Definition 140. Prime. An integer $n > 1$ is *prime* if it is not composiite, that is, its only non-negative integer factors are 1 and itself.

Example 141. List the prime numbers less than 20.

Example 142. Write 24 as a product of prime factors.

Prove 143. Let S be the statement that there are finitely many prime numbers. Show that S is false by filling in the blanks.
Pretend for a minute that S is true, so there are only finitely many ____________________________. Let k be how many prime numbers there are, and call the prime numbers $p_1, p_2, p_3, \ldots, p_k$. Consider the number $n = p_1p_2p_3 \cdots p_k + 1$. Then n is larger than all prime numbers and so n is not ______________, so it must be _____________________. By considering factors of n, at least one factor must be a ______________ number. But by the ____________________________ part of the ____________________________, n is not a multiple of p_1, n is not a multiple of p_2, etc. Thus, n is not a multiple of a prime number. We have arrived at the false statement that n is composite and yet has no prime factors. Thus, statement S must be false, and now we know that there are infinitely many prime numbers.

Definition 144. Process of elimination. Consider logical statements P, Q, and R and suppose we know that $P \vee Q \vee R$ is true. Suppose now that we show that Q is false and R is false. We can conclude that P is true. Hopefully that is obviously true. If not, one can use truth tables to make it extra clear, which is one place where truth tables really help.

Prove 145. Let n be an integer. Suppose that n^2 is a multiple of 5. Use the Division Algorithm to produce statements P, Q, R, S, and T of the form $n = 5k + r$ for different values of r, so that $P \vee Q \vee R \vee S \vee T$ is true. Then show that Q, R, S, and T are false, and conclude that P is true, so that n is a multiple of 5. Do a really good job on these cases, because you'll use them a few more times in the next questions.

Example 146. List the first 11 perfect squares, $0, 1, 4, 9, \ldots$.

Prove 147. Use the cases from 145 to argue that perfect squares of integers can only end in the decimal digits 0, 1, 4, 5, 6, 9, and never in 2, 3, 7, 8.

Prove 148. The result in 145 can also be shown with a contrapositive proof.
Let n be an integer. Consider A: n^2 is a multiple of 5 and B: n is a multiple of 5. Clearly state $\neg B$ and $\neg A$. Rewrite $\neg B$ in terms of the cases from 145, and then argue that each case iimplies $\neg A$. You may use the cases from 145 without rewriting them.

Definition 149. Proof by contradiction. One way to prove that a statement P is true is to pretend for a minute that $\neg P$ is true and argue to a false statement, conclude that $\neg P$ is false, and thus establish that P is true. It may be easiest to think of this as a proof by the process of elimination: we know that $P \vee \neg P$ is true, and we are eliminating $\neg P$.

Note 150. When using proof by contradiction that P is true, we will write "Pretend for a minute that $\neg P$ is true" and find a chain of implications resulting in a statement that is false. Sometimes it is difficult to put your finger on what specific statement is false, but you realize that two or more statements are true, but cannot be true at the same time. That is the nature of a contradiction. If you can put your finger on a specific statement that is false, that is better.

Guided proof 151. Let L be the statement "There is a largest integer." Prove that L is false.
Pretend for a minute that L is true. Write n for the largest integer. Consider $n + 1$.

Thus, L is false.

Prove 152. Show that $\sqrt{5}$ is irrational, following the proof that $\sqrt{2}$ is irrational. Start by pretending for a minute that $\sqrt{5}$ is rational and argue to a false statement or a contradiction.

Your name: ________________________________

Special words in mathematics

A short guide to how to use certain words

Overview

Using the right words in the right situation shows that you understand the logical structure of what you are writing. It also makes it clear to the reader what you mean.

Definition 153. Let. The word "Let" has two main uses in mathematics, both of them in proofs.

a. The word "Let" is used to introduce a new variable or other object and give it a specific value. This is often used in proofs where you need to show the existence of some object, but is also used in many other contexts.

- Let $f(x) = \sin(x) + \cos(x)$.
- Let $x = \frac{-b+\sqrt{b^2-4ac}}{2a}$.
- Let $n = \frac{1}{a} + 1$, rounded up to the next integer.

The word "set" can also be used here in place of "let."

b. The word "Let" can also be used to introduce a new variable having a particular property:

- Let n be even.
- Let $a > 0$.
- Let $r \in \mathbb{Q}$.
- Let $0 < x < 1$.

These statements cause the variable to take on a specific value. We don't know the specific value, only that it has the property we make it have. This is very useful when writing a generic proof that is supposed to work for all values of the variable having the property.

A very important point is that whenever you use the word "Let", you change the value of the variable. So for example, if you start a proof by saying "Let $a > 0$," then a becomes a specific real number. Based on this a, you may construct other variables like b which depend on a. For example, $b = 1/a$. Later in the same proof, if you say, "Let $a > 1$," then this changes the value of a, and any variable that depends on a will lose its connection. Instead, you may want to think of $a > 1$ as a case to consider and use the word "suppose."

Definition 154. Suppose. The word "Suppose" has two main uses in mathematics.

a. The word "Suppose" can be used to introduce cases in a proof, for example, to restrict consideration of an already–introduced variable to a smaller range of variables. Using "suppose" this way does not introduce a new variable or change the value of the variable.

- Let $a > 0$. Case 1. Suppose $a \geq 1$. ... Case 2. Suppose $0 < a < 1$. ...
- Let $x \in [3, 7]$. Case 1. Suppose that $x < 4$. ... Case 2. Suppose that $x \geq 4$.

b. The word "Suppose" is also used to introduce a logical statement at the beginning of a theorem or proof.

- Suppose that the function f is continuous on the interval $[a, b]$.
- Suppose that n is an odd integer.

Definition 155. Assume. The word "Assume" is most often used to introduce a proof by contradiction. Because it is helpful to know that a proof by contradiction is coming, it is helpful to use familiar wording. You can say things like:

a. Assume that $\sqrt{2}$ is rational.

b. Assume for the sake of contradiction that $\sqrt{2}$ is rational.

c. Pretend for a minute that $\sqrt{2}$ is rational. (Recommended in this class, but unconventional outside this class.)

Remark 156. The word "Any" is ambiguous and it is best to avoid using it. Sometimes it means "for all" and sometimes it means "there exists" and sometimes you just can't tell. Consider this example:

a. Let a be a real number. Suppose that $n > a$ for any non-negative integer n. This would be true **for all** non-negative integers n if $a = -5$. But it would be true for **some** non-negative integer n if $a = 10$. The meaning is ambiguous.

Example 157. A badly told story. Amanda was a sophomore in college. One day after class, she went to study in the park. She walked past a family at a picnic table and headed toward a shady tree. Barney said, "This next test is going to be really hard!" Amanda told Barney to relax.

Who is Barney? We haven't been introduced. How does he know Amanda? Were they walking together? Were they meeting to study?

Writing a proof is a bit like tellling a story. It's important to introduce the variables you use. Don't let a variable barge in without introduction like Barney did. Make sure to relate a new variable to existing variables.

Example 158. Show that for all real numbers $a > 0$, there is an integer n with $\frac{1}{n^2} < a$.

Let $a > 0$.

Case 1. Suppose that $a \geq 1$. ____________________ Then $\frac{1}{n^2} = \frac{1}{4} < 1 \leq a$, and so $\frac{1}{n^2} < a$.

Case 2. Suppose that $a < 1$. Let n be the next integer larger than $\frac{1}{\sqrt{a}}$. Then $n > \frac{1}{\sqrt{a}}$. Squaring both sides, $n^2 > \frac{1}{a}$. Taking reciprocals, $\frac{1}{n^2} < a$, as desired.

In each case, we have shown the existence of an integer n with the desired property. Thus, for all $a > 0$, there is an integer n with $\frac{1}{n^2} < a$.

Note that in Case 1, the variable n has not been introduced. Fill in the blank to do that properly.

Your name: ______________________________

Integer-valued functions

Overview

This activity introduces some functions from the real numbers to the integers, and asks you to establish some of their properties.

Exercise 159. Think of a function f with the following properties: First, $f : \mathbb{R} \to \mathbb{Z}$, meaning that the input to f is a real number, and the output from f will always be an integer. Second, f takes on the following values:

$$\begin{array}{rclcrcr} f(0.5) & = & 1 & \qquad & f(-3.2) & = & -3 \\ f(0.9) & = & 1 & & f(-10) & = & -10 \\ f(1) & = & 1 & & f(-9.5) & = & -9 \\ f(1.1) & = & 2 & & f(18.2) & = & 19 \end{array}$$

Humans have an amazing ability to generalize from examples like this. Describe what f does to a generic input number x.

We will need the letter f for other functions. Decide among yourselves on new notation for f. You can use special symbols, as with the notation $|x|$ or $n!$, or you can use multiple letters, as with $\sin(x)$ or $\ln(x)$.

Exercise 160. Think of a function g with the following properties: First, $g : \mathbb{R} \to \mathbb{Z}$. Second, g takes on the following values:

$$\begin{array}{rclcrcr} g(0.5) & = & 0 & \qquad & g(-3.2) & = & -4 \\ g(0.9) & = & 0 & & g(-10) & = & -10 \\ g(1) & = & 1 & & g(-9.5) & = & -10 \\ g(1.1) & = & 1 & & g(18.2) & = & 18 \end{array}$$

Describe what g does to a generic input number x.

Decide among yourselves on new notation for g.

Can you write g in terms of f? If so, how? If not, why not?

Exercise 161. Think of a function h with the following properties: First, $h : \mathbb{R} \to \mathbb{Z}$. Second, h takes on the following values:

$$\begin{array}{rclcrcr} h(0.5) & = & 1 & \qquad & h(-3.2) & = & -3 \\ h(0.9) & = & 1 & & h(-10) & = & -9 \\ h(1) & = & 2 & & h(-9.5) & = & -9 \\ h(1.1) & = & 2 & & h(18.2) & = & 19 \end{array}$$

Describe what h does to a generic input number x.

Can you write h in terms of f or g or both? If so, how? If not, why not?

Exercise 162. Using the notation introduced on the previous page, several inequalities are listed below. They might be true for all x, or they might fail for some values of x. If an inequality is true for all x, say so. If it fails for some x values, gives a specific example of x, called a *counterexample*, and calculate the quantities in the inequality to explain the counterexample.

a. $x \leq f(x)$

b. $x < f(x)$

c. $f(x) - 1 < x \leq f(x)$

d. $g(x) \leq x$

e. $g(x) < x$

f. $g(x) \leq x < g(x) + 1$

g. $0 \leq x - g(x) < 1$

h. $x < h(x)$

Your name: ______________________________

Construction of an object with a property

Overview

Many proofs in mathematics require us to show that an object having a certain property exists. In many cases, you can tell exactly how to make, or construct, the desired object.

Prove 163. Let $x > 0$ be a real number.

a. Construct a real number a with $0 < a < x$. That is, write a formula to compute a in terms of x, then check that $0 < a < x$.

b. When $x = 0.0006$, what number does your formula produce for a?

c. Construct real numbers a and b with $0 < a < b < x$.

d. When $x = 0.0006$, what numbers does your formula produce for a and b?

e. Find two different ways to construct a real number c satisfying $x < c$.

Prove 164. Given a real number x, describe a procedure that results in an integer n with $n > x$.

What result does your procedure give for $x = 13.1$? $x = 18$? $x = -22.2$?

Prove 165. **a.** Find the smallest integer n with $n^2 > 17$.

b. Find the smallest integer n with $n^2 > 177$.

c. Describe a procedure for finding the smallest integer n with $n^2 > 1777$.

d. Given an integer $k > 1$, describe a short procedure to find the smallest integer n with $n^2 > k$.

e. Describe a procedure for finding the largest integer n with $n^2 < 1777$.

Prove 166. Given a real number a, construct an integer n with $a \leq n < a + 1$. You may wish to consider two cases: Case 1, suppose a is an integer. Case 2, suppose a is not an integer.

What value does your procedure give when $x = 13.1$? $x = 18$? $x = -22.2$?

Prove 167. Let a and b be real numbers with $a < b$. Show that there exists a real number c with $a < c < b$. Describe how to construct c and then work with inequalities to show that it works.

Prove 168. Let $a > 0$. Show that there exists an integer $n > 0$ with $\frac{1}{n} < a$. Describe how to construct n and then show that it works.

Prove 169. Suppose that $x = 4.32\overline{764}$, meaning that the decimal expansion has 764 repeating forever. Show that x is a rational number. Start by explaining what needs to exist and what properties it needs. **Hint:** Look at 100000x - 100x.

Prove 170. Suppose that x is a real number with a repeating decimal expansion (of d repeating digits starting after the pth decimal place). Show that x is a rational number.

Prove 171. Let a and b be real numbers with $b - a > 1$. Show that there exists an integer n with $a < n < b$. Describe how to construct n and then show that it works.

Prove 172. Let a and b be real numbers with $a < b$. Show that there exists a rational number r with $a < r < b$. Describe how to construct r and then show that it works.

Your name: ______________________________

Introduction to set theory

This activity introduces sets, ways to write them, and the relations between them.

Overview

Many important ideas in mathematics are expressed using sets, and many proofs come down to dealing with sets in the right way. This activity was co-authored by Johanson Berlie.

Definition 173. Set. A set is a well-defined collection of distinct objects. These objects are called **elements** or **members** of the set.

Example 174. Consider all students registered for at least one credit hour at this university this semester. The objects are students, and there is a clear criterion for deciding which students we have in mind, so the collection is well defined. It's OK that we don't have a list of the students, we can still talk about the set.

Example 175. Consider all the days last spring when it was somewhat gloomy. Here, the objects are days that are somewhat gloomy. Since 'somewhat gloomy' is not defined precisely, this collection is not a set.

Exercise 176. Which of the following are sets? Consider whether the set is well defined and explain your thinking. Mark sets with a check mark, non-sets with an X. If the set is small enough, list out its elements.

a. Your Facebook friends right now

b. Your high school friends on graduation day

c. All the stars in the Milky Way galaxy

d. All the small stars in the Milky Way galaxy

e. The ten most important characters in the Harry Potter books

f. The days in a year with exactly 20mm of rainfall

g. The days in 2016 in Bowling Green, Ohio with less than 20mm of rainfall

h. English letters that are vowels

i. Planets in our solar system

j. Construct your own example or non-example of a set and explain why it is or is not a set.

Definition 177. Elements of a set. If A is a set and an object x is an element of A, we write $x \in A$. The symbol $\in$ looks a bit like a letter E and stands for "is an element of." Thus $x \in A$ should be read as "x is an element of A." If x is not an element of A, we write $x \notin A$. Sets are usually denoted by capital letters and their elements by lower case letters.

Definition 178. Tabular form. We represent a set in **tabular form** by listing out its elements, separated by commas and enclosed in braces $\{\ \}$. The order in which we list the elements does not matter, only which objects are elements of the set. If the set has too many elements to list, establish a pattern and use $\ldots$.

Example 179. If the set C consists of the primary colors, we could write $C = \{\text{red, blue, yellow}\}$ or $C = \{\text{yellow, red, blue}\}$, because the order in which we list the elements doesn't matter.

Remark 180. For convenience or from lack of attention, sometimes an element is repeated when listing in tabular form; this does not change the actual elements of a set. Thus, the tabular forms $\{a, b\}$ and $\{a, b, b, a\}$ both refer to the same set.

Example 181. The set of Fibonacci numbers can be written $F = \{1, 1, 2, 3, 5, 8, 13, \ldots\}$ or as $F = \{1, 2, 3, 5, 8, 13, \ldots\}$. The first way is how people usually write the Fibonacci numbers, the second way recognizes that since a set is a collection of distinct elements, listing 1 twice doesn't change the set. Be careful with the idea of establishing a pattern: If someone doesn't know what the Fibonacci numbers are, they might not be able to tell you the next element of the set.

Exercise 182. List out the next five elements of the set of Fibonacci numbers.

Definition 183. Set-builder form. We represent a set in **set-builder form** by stating the properties which its elements must satisfy.

Example 184. If the set C consists of the primary colors, then we can write $C = \{x : x \text{ is a primary color}\}$. In this notation we wrote x as a temporary name for an element of the set, and wrote the condition that x needs to satisfy after the colon character :. Sometimes people use a vertical line | instead of a colon.

Exercise 185.

a. Express the set $A = \{x : x \text{ is a "home row" character on a keyboard}\}$ in tabular form.

b. Express the set $B = \{A, L, G, E, B, R\}$ in set-builder form.

c. Express the set $P = \{2, 3, 5, 7, 11, \ldots, 97\}$ in set-builder form.

d. Express the set $T = \{x : x \text{ is a power of } 2\ \}$ in tabular form.

e. Suppose that $R = \{x | x \text{ is a zero of } f(x) = 5x^3 - 2x^2 + 7x - 1\}$. Suppose that $u \in R$. Write the equation that we know that u satisfies.

f. Suppose that $M = \{m : m \text{ is a multiple of } 7\ \}$. Let $k \in M$. Without using the word "multiple," what do we know about k?

Definition 186. Empty set. A set which contains no elements is called the **null set** or **empty set**. We denote it by the symbol $\varnothing$.

Example 187. The set of real-valued solutions of the equation $x^2 + 1 = 0$ is empty, since there is no real number that solves the equation.

Definition 188. Singleton set. A set which contains only a single element is called a **singleton set**.

Example 189. The set of mountains on earth with height over 29,000 feet is a singleton set, since Mt. Everest is the only element of the set.

Exercise 190. For the following questions, identify the sets in the context of the definitions above.

a. The set of all mountains in the state of Ohio above 2000 feet in height. This might require an internet search.

b. If Jill has classes on Mondays, Wednesdays and Fridays and has work on Wednesdays and Saturdays, describe the set of days on which Jill has both work and classes?

c. Suppose we draw two lines in the plane and consider the intersection of the two lines, that is, the set of all points that are on both lines. Can this set be empty? If so, draw a picture. Can this set be a singleton? If so, draw a picture. Can this set be anything else? If so, draw a picture and describe it.

Definition 191. Subset of a set. If every element in a set A is also a member of a set B, then A is called a **subset** of B and we write $A \subseteq B$.

Remark 192. If A is a set, then $A \subseteq A$, because every element of A is also a member of A.

Remark 193. The notation $\subseteq$ is much like the inequality symbol $\leq$ for real numbers. We know that $3 \leq 5$ and writing $x \leq 5$ means that x could be any number up to and including 5. It is always true that $x \leq x$ when x is a real number. When you see the statement $A \subseteq B$, think that A is a subset of B, and possibly equal to B.

Remark 194. The null set is a subset of every set: Suppose A is a set. Then it is true that every element of $\varnothing$ is a member of A. People sometimes say this is "vacuously true," because $\varnothing$ doesn't have any elements to bother with. We write $\varnothing \subseteq A$.

Definition 195. Not a subset. When A is not a subset of B, we write $A \not\subseteq B$. This happens when there is an element of A that is not an element of B.

Example 196. If $A = \{$ green, yellow, red, black, dog, cat, mouse$\}$ and $B = \{$dog, cat$\}$ then $B \subseteq A$. However, $A \not\subseteq B$ because, for example, mouse $\in A$ but mouse $\notin B$.

Exercise 197. **a.** If A is the set of all cars manufactured by a Japanese car company and B is the set of all Toyota sedans, then what is the relationship between A and B?

b. Let M be the set of people you have communicated with on social media in the last week, and let C be the set of people you are taking a class with now. Is $M \subseteq C$? If not, name one person in M but not in C. Is $C \subseteq M$? If not, name one person in C but not in M.

c. Let V be the set of people who voted in the last US presidential election, and let C be the set of US citizens. Is $V \subseteq C$? Under what condition would we have $V \not\subseteq C$?

Definition 198. Proper subset. If every element in a set A is also a member of a set B, and yet B contains at least one element that is not in A, then A is called a **proper subset** of B and we write $A \subset B$. Sometimes people write $A \subsetneq B$. Generally speaking, when you want to show that $A \subset B$, you need to check that $A \subseteq B$ and that A and B are not equal.

Remark 199. The notation $\subset$ is much like the strict inequality symbol $<$ for real numbers. We know that $3 < 5$ is true and writing $x < 5$ means that x could be any number up to but not including 5. It is never true that $x < x$ when x is a real number. When you see the statement $A \subset B$, think that A is a subset of B but not equal to B. That means that B has an element that A does not have.

Exercise 200. If A = $\{$ green, yellow, red, black, dog, cat, mouse$\}$ and B = $\{$dog, cat$\}$ then $B \subset A$. Put your finger on why this is true.

Exercise 201. If A is a set, explain why $A \subset A$ is not true. We could write $A \not\subset A$.

Exercise 202. **a.** Suppose that L is the set of US citizens who voted legally in the last presidential election and C is the set of US citizens at that time. Explain why $L \subseteq C$ is true. Explain why $L \subset C$ is true.

b. Consider a class at the University. Let R be the set of students who are registered for the class and let F be the set of people who take the final exam.

What needs to happen at the final exam to make $R = F$?

What needs to happen to make $F \subset R$?

What needs to happen to make $R \subset F$?

Which of the three possibilities do you think is most likely to happen?

Least likely? Why?

Your name: ______________________________

Set subsets and equality

This activity works on showing that one set is a subset of another, and showing equality between two sets.

Exercise 203. Let T be the set of all multiples of 3, and let S be the set of all multiples of 6.
List at least 5 elements of T:
List at least 5 elements of S:
Determine which of the following set inclusions is true. If not true, list at least one element which serves as a counterexample.

- **a.** $T \subseteq S$
- **b.** $S \subseteq T$

If one of the following set inclusions is true, list five elements that account for the strict set inclusion.

- **a.** $T \subset S$
- **b.** $S \subset T$

Exercise 204. Let C denote the set of composite numbers and let E denote the set of even numbers. Determine which of the following set inclusions is true. If not true, list five elements which serve as counterexamples.

- **a.** $C \subseteq E$
- **b.** $E \subseteq C$

Exercise 205. Let $\mathbb{R}$ denote the real numbers, $\mathbb{Z}$ denote the integers, and $\mathbb{Q}$ denote the rational numbers. Express the most informative set inclusions between these sets. You do not need to prove them.

Problem 206. Let $\mathbb{Q}$ denote the rational numbers. Let $A = \{x \in \mathbb{R} : x \text{ solves } x^2 = a \text{ where } a \text{ is an integer and } a \geq 0\}$. List out at least 5 elements of A: ____________________
Show that $A \not\subseteq \mathbb{Q}$. Make your logic crystal clear.

Problem 207. Continuing the previous problem, show that $\mathbb{Q} \not\subseteq A$. Make your logic crystal clear.

Guided proof 208. When showing that $A \subseteq B$, you need to show that every element of A is also an element of B. Here is how to do it. Let $x \in A$. Use this fact and the definitions of A and B to show that $x \in B$. Since you made no further assumptions about x, this shows that every element of A is also an element of B. Note: Your proof must start with "Let $x \in A$." and must get to "Thus $x \in B$." before generalizing.

Guided proof 209. When showing that $A \subset B$, you need to show that $A \subseteq B$ and you need to show that there is an element of B that is not in A. If you can construct such an element, do that, because that shoud make the proof clearer to the reader.

Guided proof 210. Let $\mathbb{Z}$ denote the integers and $\mathbb{Q}$ denote the rational numbers. Recall that $\mathbb{Q} = \{x : x = \frac{p}{q}$ for some integers p and q with $q \neq 0\}$. Show that $\mathbb{Z} \subset \mathbb{Q}$ following the model.

a. Let $m \in \mathbb{Z}$. m meets the definition to be in $\mathbb{Q}$ because we can write it as ______________________. Thus $m \in \mathbb{Q}$. We made no further assumption about $m \in \mathbb{Z}$. Thus for all $m \in Z$, we know that $m \in \mathbb{Q}$. Thus $\mathbb{Z} \subseteq \mathbb{Q}$.

b. To see that $\mathbb{Z} \subset \mathbb{Q}$, note that ____________ is in $\mathbb{Q}$ but is not in $\mathbb{Z}$.

Prove 211. Let $\mathbb{R}$ denote the real numbers and $\mathbb{C}$ denote the complex numbers. Recall that $\mathbb{C} = \{a + ib : a, b \in \mathbb{R}\}$ where $i = \sqrt{-1}$, which is not a real number. Show that $\mathbb{R} \subset \mathbb{C}$ following the model in the previous problem.

a.

b.

Prove 212. Let $2\mathbb{Z} = \{m \in \mathbb{Z} :$ there exists $j \in \mathbb{Z}$ such that $m = 2j\}$ and $6\mathbb{Z} = \{m \in \mathbb{Z} :$ there exists $j \in \mathbb{Z}$ such that $m = 6j\}$. Show that $6\mathbb{Z} \subset 2\mathbb{Z}$, following the model in 210.

Definition 213. Intervals of real numbers.. Let a and b be real numbers with $a \leq b$. We define the following four types of intervals:

a. $(a, b) = \{x \in \mathbb{R} : a < x < b\}$. We say that both endpoints are open.

b. $[a, b) = \{x \in \mathbb{R} : a \leq x < b\}$. The compound inequality $a \leq x \leq b$ means $a \leq x$ and $x \leq b$.

c. $(a, b] = \{x \in \mathbb{R} : a < x \leq b\}$

d. $[a, b] = \{x \in \mathbb{R} : a \leq x \leq b\}$. We say that both endpoints are closed.

Exercise 214. Sketch the following intervals on separate number lines. To indicate open endpoints, draw an open circle like $\circ$. To indicate closed endpoints, draw a closed circle like $\bullet$.

a. $(2, 9)$

b. $[2, 7)$

c. $(3, 5]$

d. $[0, 1]$

Note 215. Inequalities between real numbers have the *transitivity* property: If $a \leq b$ and $b \leq c$, then we can conclude that $a \leq c$. Similar inequalities are true with $\geq, <$, and $>$.

Exercise 216. Using standard interval notation, show that $(4, 9] \subset [2, 9]$. Begin with "Let $x \in (4, 9]$." then rewrite this as a compound inequality, then rewrite as two separate inequalities. Note when you use transitivity. Make sure to show that $(4, 9]$ is a proper subset of $[2, 9]$.

Definition 217. Set equality. For sets A and B, we say that A equals B when the elements of A are exactly the same as the elements of B. We write $A = B$ when A and B are equal sets.

Example 218. The set consisting of all colors of a rainbow and the set consisting of colors of white light observed through a prism are equal sets.

Guided proof 219. One way to show that $A = B$ is to *show inclusion both ways.* That means to show that $A \subseteq B$ and $B \subseteq A$. Here is how you do it.

1. To show $A \subseteq B$: Let $x \in A$. Use this fact and the definitions of A and B to show that $x \in B$. Having made no further assumption about $x \in A$, you can conclude that $A \subseteq B$.

2. To show $B \subseteq A$: ____________________. Use this fact and the definitions of A and B to show that _____________. Having made no further assumption about _____________, you can conclude that _____________.

Exercise 220. Let $A = \{x : x \text{ solves } ax = b \text{ where } a \text{ and } b \text{ are integers and } a \neq 0\}$. Let $\mathbb{Q} = \{x : x \text{ is a rational number}\}$. Show that $A = \mathbb{Q}$ by showing set inclusion both ways.

a. Let $x \in A$. Then there exist a and b such that _____________ and $a \neq 0$. Dividing through by a, $x =$ _____________ where a and b are integers and a is not zero. Thus, $x \in \mathbb{Q}$. Since x was arbitrary, $A \subseteq \mathbb{Q}$.

b. Let $x \in \mathbb{Q}$.

Exercise 221. Let $A = \{x : (x-3)^2 - 1 < 3\}$ and let $B = (1, 5)$. Show that $A = B$ by showing set inclusion both ways.

Exercise 222. Let $A = \{x : -x^2 + 5x + 14 \geq 0\}$ and let $B = [-2, 7]$. Show that $A = B$ by showing set inclusion both ways.

Problem 223. Let $2\mathbb{Z} = \{m \in \mathbb{Z} : \text{there exists } j \in \mathbb{Z} \text{ such that } m = 2j\}$. Let $3\mathbb{Z} = \{m \in \mathbb{Z} : \text{there exists } j \in \mathbb{Z} \text{ such that } m = 3j\}$, and similarly with other sets like $5\mathbb{Z}$ and $15\mathbb{Z}$. Show that $2\mathbb{Z} \cap 3\mathbb{Z} = 6\mathbb{Z}$ by showing set inclusion both ways.

Problem 224. Write out all elements in $6\mathbb{Z} \cap 8\mathbb{Z} \cap \{1, 2, 3, \ldots, 100\}$.

Exercise 225. Let $\mathbb{R}^3$ denote the set of all 3-dimensional vectors and let $S = \{v : v = t_1\langle 1, 0, 0\rangle + t_2\langle 1, 1, 0\rangle + t_3\langle 1, 0, 1\rangle$ where t_1, t_2, t_3 are real numbers $\}$.

a. Show that $S \subseteq \mathbb{R}^3$.

b. Show that $\mathbb{R}^3 \subseteq S$. Now, *you* determine what values of t_1, t_2, and t_3 will work to make v have the form of an element of S.

Your name: ______________________________

Quantifiers and nested quantifiers

Overview

Many statements that we want to prove are supposed to be true **for all** objects having a certain property. Other times, we want to show that **there exists** an object having a property. 'For all' and 'there exists' are called quantifiers. Sometimes the two occur together, as in a statement that 'for every object A, there is an object B having a certain relationship to A.' Then we say that the quantifiers are *nested*. In this activity you will work with quantifiers and nested quantifiers.

Definition 226. For all. The phrase *for all* means that what follows is supposed to be true for all values of the indicated variable, and will probably require a generic proof to cover all possibilities. Alternative words are "for every" or "for each." Sometimes people write "for any" but please avoid that because it can be ambiguous. The notation $\forall$ is often used to represent "for all". The format is "for all (introduce a variable and optionally put restrictions on the variable), we have (property satisfied by the variable)." In this course, use the words "we have" except when "for all" is followed by "there exists."

Definition 227. There exists. The phrase *there exists* claims that an object with a certain property can be shown to exist. Often, you prove existence by constructing the object that is needed, but occasionally the proof works differently. The notation $\exists$ is often used to represent "there exists." The format is "there exists (introduce a variable and optionally put restrictions on the variable) such that (property satisfied by the variable)." In this course, write out the words "such that."

Exercise 228. Rewrite English sentences symbolically, and rewrite symbolic statements as English sentences. The first ones are done for you. Pay close attention to the format for writing the statements.

a. Every prime number is greater than 1. Solution: $\forall$ prime n, we have $n > 1$.

b. There is a real number x for which $x^2 = 2$. Solution: $\exists\, x \in \mathbb{R}$ such that $x^2 = 2$. The set $\mathbb{R}$ is called the *universe* for the variable x.

c. $\forall\, x \in A$, we have $x^2 = x$.

d. The equation $x = \sin(x)$ has an integer-valued solution.

Exercise 229. Write the following statements symbolically:

a. For every a, there is a b for which $b^2 = a$

b. For every b, there is an a for which $b^2 = a$

c. For every a and every b, we have $b^2 = a$

d. There exists an a and there exists a b such that $b^2 = a$

Exercise 230. For each of the statements in the previous problem, decide whether it is true or false if the universe for both a and b is the set of non–negative integers. If false, give specific numbers as counterexamples using "Let ..."

a.

b.

c.

d.

Remark 231. Now you will write proofs that involve nested quantifiers. For each, first write the statement with $\forall$ and $\exists$. Note the order in which the quantifiers occur, because that will have a big impact on the structure of the proof. To prove a "for all" statement, use "Let ..." to introduce a generic instance of the variable that satisfies whatever restriction is imposed. If your proof works for a generic value of the variable, then it will work for all variables satisfying the restriction. To prove a "there exists" statement, construct the required variable in terms of other variables, also using "Let ..."

Prove 232. Show that for all odd integers m and n, there exists an integer p such that $m + n = 2p$.
In symbols: $\forall$ integers m and n, __________________________

Start your proof by writing "Let m and n be odd integers." A few steps later, you will be able to construct the integer p. At the end, generalize by noting that you made no further assumptions about m and n, so your result is true for all odd integers m and n.

Prove 233. Show that for every rational number $r \neq 0$, there exists a rational number s such that $rs = 1$. Remember to generalize at the end. Is there more than one possible value for s?
In symbols:

Prove 234. Show that for every real number y there is a value of x for which $y = 2x - 5$.
In symbols:

Prove 235. (Calculus required.) Show that for every continuous function $f : \mathbb{R} \to \mathbb{R}$, there exists a function F with $F' = f$ and $F(0) = 0$. Is there more than one possible choice for F?
In symbols:

Prove 236. (Linear algebra required.) Suppose that the 3 by 3 matrix A is invertible. (The matrix A is given; you don't need to say "Let A be a matrix" or construct A.) Show that for all 3-dimensional vectors b, the equation $Ax = b$ has a solution x, which is also a 3-dimensional vector. Is there more than one possible value for x?
In symbols:

Exercise 237.

a. Show that there exists an integer n such that for all integers p with $p \leq 0$, we have $2^n > p$. First define n, then show that it works for all p by starting with "Let $p \leq 0$ be an integer."

b. Show that for every integer $p > 0$, there is an integer n with $2^n > p$. It's OK if 2^n is much larger than p; you don't need the smallest possible n, just one that works.

c. Using the integer n that you constructed in (a) or (b), show that for all $m > n$, we have $2^m > p$.

d. Write three quantifiers to express what you have shown about p, n, and m in (a), (b), and (c).

Exercise 238.

a. Suppose that $n^2 > \frac{1}{a}$. Cite a property of inequalities to show that $n^2 + 10 > \frac{1}{a}$.

b. Show that for every real number $a > 0$, there is an integer n with $\frac{1}{n^2+10} < a$.

c. Using the integer n that you constructed in (b), show that for all $m > n$, we have $\frac{1}{m^2+10} < a$.

d. Write three quantifiers to express what you have shown about a, n, and m in (b) and (c).

Exercise 239.

a. Let $a > 0$. Solve the inequality $0 < \frac{1}{x^2-100} < a$ for x.

b. Show that for every real number $a > 0$, there is an integer n with $0 < \frac{1}{n^2-100} < a$.

c. Using the integer n that you constructed, show that for all $m > n$, we have $\frac{1}{m^2-100} < a$.

d. Write three quantifiers to express what you have shown about a, n, and m in (b) and (c).

Exercise 240. Show that there exists a number a such that for all $x \in \mathbb{R}$, $5\sin(x) + 7\cos(3x) < a$. Note that in this problem, you first construct a and then you show a "for all" statement.
In symbols:

Exercise 241. Show that for the integers, there exists a number a for which, for all integers b, $ab = b$.
In symbols:

Remark 242. The negation of a logical statement P is denoted by $\neg P$. It is true when P is false and false when P is true. When the logical statement begins with a quantifier, we can think through the result of negation. In what follows, $Q(x)$ is a logical statement whose truth value depends on the value of x. For example, $Q(x)$ could be the statement "$x^2 = x$."

1. Consider $\neg\forall\ x \in A$, we have $Q(x)$. This means that it is not true that all x values "work," so there must be an x value that does not work, that is, $\exists\ x \in A$ such that $\neg Q(x)$.

2. Consider $\neg \exists\, x \in A$ such that $Q(x)$. Try as we might, we cannot find a value of x that "works," so it must be that all values of x fail to work. That is, $\forall\, x \in A$, we have $\neg Q(x)$.

Note that in both cases, negating the quantifier can be done quite mechanically: negation turns $\forall$ into $\exists$ and it turns $\exists$ into $\forall$. Also note that in both cases we keep the restriction $x \in A$ and instead negate the property $Q(x)$ that x is supposed to have. When negating nested quantifiers, after flipping the first quantifier, the negation applies to the next quantifier, which then flips, and so on.

Exercise 243. Negate the following statements.

a. $\exists$ integer k such that $k^2 < k$

b. $\exists\, x \geq 0$ such that $\cos(x) > e^x$

c. $\forall\, y \in \mathbb{R}$, $\exists\, x \in \mathbb{R}$ such that $x^2 = y$

d. $\forall\, a > 0$, $\exists$ integer n such that $\forall\, m > n$, $\sin(m) < a$

e. $\forall\, a > 0$, $\exists\, d > 0$ such that $\forall\, x \in (a - d, a + d)$, we have $|\cos(x) - \cos(a)| < a$

Exercise 244. Write the following statements symbolically. Introduce new notation as you need it. The first one is done for you.

a. Every state has a city named Springfield. Use variables s and c.

Solution: $\forall$ state s, $\exists$ city c in s such that NameOf(c) = Springfield.

b. Every bridge has a weight limit. Use variables b and w.

c. There is an integer that is larger than every other integer. Use variables m and n.

d. Every broken clock is right twice a day. Use variables c, t_1, t_2.

e. Every married couple can find a tax deduction. Use variables m and d.

f. Between every two locations in the US, there is a shortest driving route. Use variables L_1, L_2, r, and R.
Solution: $\forall$ locations L_1 and L_2 in the US, $\exists$ route r such that r starts at L_1 and ends at L_2 and $\forall$ route R such that R starts at L_1 and ends at L_2, $\text{length}(r) \leq \text{length}(R)$.
Why is the second $\forall$ needed?

Your name: ______________________________

Union and intersection of sets

We introduce the union and intersection of sets and learn how to prove statements about them.

Note: The notation $x \in A$ is usually read "x is an element of A." The symbol $\in$ looks like the letter E because it stands for "element". As far as I can tell, the E is not there to mean "x exists in A". Being an element is the point, not existing.

Definition 245. Union of sets. The *union* of sets A and B is a new set, consisting of all elements which belong to A or to B or to both. We denote this new set by $A \cup B$. The logical statement "$x \in A \cup B$" is true when "$x \in A$ or $x \in B$" is true.

Definition 246. Intersection of sets. The *intersection* of sets A and B is a new set, consisting of all elements which belong to both A and B. We denote this new set by $A \cap B$. The logical statement "$x \in A \cap B$" is true when "$x \in A$ and $x \in B$" is true.

Example 247. Let C be the set of Computer Science majors and M be the set of Mathematics majors. Then $C \cap M$ is the set of students double majoring in Computer Science and Mathematics (a powerful combination!) while $C \cup M$ is the set of students majoring in one, the other, or both majors.

Example 248. Find $[1,5] \cup (3,7)$. Draw a diagram on a number line to illustrate.

Example 249. Find $[1,5] \cap (3,7)$. Draw a diagram on a number line to illustrate.

Problem 250. Let $3\mathbb{Z} = \{m \in \mathbb{Z} : \text{there exists } j \in \mathbb{Z} \text{ such that } m = 3j\}$. Let $5\mathbb{Z} = \{m \in \mathbb{Z} : \text{there exists } j \in \mathbb{Z} \text{ such that } m = 5j\}$, and similarly with other sets like $7\mathbb{Z}$ and $12\mathbb{Z}$.

a. Find $3\mathbb{Z} \cap 5\mathbb{Z}$ and write it in the most convenient form you can.

b. Find $3\mathbb{Z} \cup 5\mathbb{Z}$ and write it in the most convenient form you can.

Remark 251. Remember that to show $S \subseteq T$, you need to show that for all x in S, we have x in T. In symbols, $\forall x \in S, x \in T$. To do a proof like that, you need to start with "Let $x \in S$." and you need to end with something like "Thus $x \in T$. Since $x \in S$ was arbitrary, $S \subseteq T$."

Prove 252. Suppose that $A \cup B \subseteq C$.

a. Show that $A \subseteq C$ following the model.

Let $x \in A$. Then ____________ by definition of union. Thus ____________ since $A \cup B \in C$.

Since ____________ was arbitrary, ____________.

b. Show that $B \subseteq C$. Use good form.

Remark 253. For sets A, B, and C, to show that $A \cap B \subseteq C$, you need to start with "Let $x \in A \cap B$." This tells you that $x \in A$ and $x \in B$. Using those two pieces of information, you need to show that $x \in C$. You will usually need to use both pieces of information.

Guided proof 254. Show that $[1, 5] \cap (3, 7) \subseteq (3, 5]$.
Let $x \in [1, 5] \cap (3, 7)$. Then $x \in$ _____________ and $x \in$ _____________ by definition of ______________.
Thus, $1 \leq x \leq 5$ and ____________________.
Thus, ______ $<$ ______ $\leq$ ______
Thus, $x \in (3, 5]$. Since $x \in [1, 5] \cap (3, 7)$ was arbitrary, we conclude that $[1, 5] \cap (3, 7) \subseteq (3, 5]$.

Prove 255. Suppose that $A \subseteq C$ and $B \subseteq C$. Show that $A \cap B \subseteq C$ following the model above.
Let _____________

Thus _____________. Since _____________ was arbitrary, _______________________________________.

Prove 256. Show that $2\mathbb{Z} \cap 7\mathbb{Z} \subseteq 14\mathbb{Z}$. Use good form. You do not need to prove any results about multiples or divisibility, simply use what you know is true from the definition of these sets.
Let ____________________

Thus _____________. Since ____________________ was arbitrary, _________________________________.

Remark 257. To show that $A \cup B \subseteq C$, you start with "Let $x \in A \cup B$." This tells you that $x \in A$ or $x \in B$. This is not all that much to work with, since maybe only one of these is true, and you don't know which one. No matter which one is true, you need to show that $x \in C$. You should do this by considering two cases. Case 1, when $x \in A$. Case 2, when $x \in B$. In both cases, show that $x \in C$ and then no matter which case applies, you get the result you need. In the end, you show that $A \subseteq C$ and $B \subseteq C$.

Guided proof 258. Suppose that $A \subseteq C$ and $B \subseteq C$. Show that $A \cup B \subseteq C$.
Let $x \in A \cup B$. Then $x \in A$ or $x \in B$ or both, by _________________________________.
Case 1. Suppose that $x \in A$. Then $x \in C$ because _____________.
Case 2. Suppose that $x \in B$. Then _____________ because _____________.
In both cases, _________. Since _____________ was arbitrary, we conclude that _____________

Prove 259. Show that $[1, 5] \cup (3, 7) \subseteq [1, 7)$. Use good form. You will use transitivity for inequalities.

Prove 260. Suppose that $B \subseteq A$. Show that $A \cup B \subseteq A$. Use good form.

Prove 261. Suppose that $A \cup B \subseteq A$. Show that $B \subseteq A$. Use good form.

Remark 262. To show that $A \subseteq B \cap C$, start with "Let $x \in A$ and show that $x \in B$ and $x \in C$, so you can conclude that $x \in B \cap C$. Make sure to write that you have shown that $x \in B \cap C$ before you make a general conclusion.

Prove 263. Show that $14\mathbb{Z} \subseteq 2\mathbb{Z} \cap 7\mathbb{Z}$. Use transitivity and make a general conclusion at the end.
Let _____________

Thus _____________. Since _______ was arbitrary, ____.

Prove 264. Show that $(3, 5] \subseteq [1, 5] \cap (3, 7)$. Use good form and make a general conclusion at the end.

Remark 265. To show that $A \subseteq B \cup C$, start with "Let $x \in A$ and show that $x \in B$ or that $x \in C$. Often, some x values are in B and others are in C, and so you will want to introduce two cases that split the values of x into two groups. Unfortunately, the cases cannot be "Case 1: Suppose $x \in B$" and "Case 2: Suppose $x \in C$" because you do not yet know that those case cover all possibiliities. Instead, you may need to use cases like "Case 1: Suppose $x < 0$" and "Case 2: Suppose $x \geq 0$." The details will be different for each problem. Each case must end with "Thus $x \in B \cup C$."

Prove 266. Show that $[1, 7) \subseteq [1, 5] \cup (3, 7)$. Use good form and make a general conclusion at the end.

Prove 267. Show that $(3,8) \cup [6,9] = (3,9]$. There are two steps. Use inequalities and transitivity, not statements like $(3,8) \subseteq (3,9]$.

a. Step 1. Show that $(3,8) \cup [6,9] \subseteq (3,9]$.

b. Step 2. Show that $(3,9] \subseteq (3,8) \cup [6,9]$.

Prove 268. Show that $(3,8) \cap [6,9] = [6,8)$. There are two steps.

a. Step 1.

b. Step 2.

Your name: ______________________________

Mathematical Induction

Proving that a claim is true for all $n = 1, 2, 3, \ldots$ by building on previous results.

Overview

One important task in mathematics is to find regular patterns and prove that they hold. The main method we use for this is mathematical induction. This activity was co-authored by Ying-Ju Chen.

Theorem 274. Mathematical induction
For each integer $n = 1, 2, 3, \ldots$, let $P(n)$ denote a true/false logical statement involving n.

(i) (The basis step) Prove that $P(1)$ is true.

(ii) (The inductive step) For each $n = 1, 2, 3, \ldots$, suppose that $P(n)$ is true, and use $P(n)$ to prove that $P(n+1)$ is true.

From the above two steps, we can conclude that $P(n)$ is true for all $n = 1, 2, 3, \ldots$.

Note 275. Proving the inductive step is usually done as a "rewrite" proof, where you start with the left hand side of what you want to show and rewrite until you come to the desired right hand side. Often, some quantity in the statement $P(n+1)$ can be written in terms of a similar quantity in the statement $P(n)$ plus a new part. You will always use the fact that $P(n)$ is true.

Guided proof 276. Use mathematical induction to show that 3^n is odd for all $n = 1, 2, 3, \ldots$.

a. State $P(n)$: $P(n)$ is that ________________________

b. Basis step: $P(1)$ is that __________________. This is true because __________________.

c. State $P(n+1)$: $P(n+1)$ is that ________________________

d. Inductive step: Let $n \geq 1$. suppose that $P(n)$ is true. Show that $P(n+1)$ is true. You may use facts you have already proven about odd numbers.

Because $P(n)$ is true, __________________. Now $3^{n+1} =$ ____________, which is odd because ____. Thus $P(n+1)$ is true. Since $n \geq 1$ was arbitrary, by mathematical induction, $P(n)$ is true for all $n \geq 1$.

Question 277. With induction proofs, $P(n)$ is always a true/false logical statement. A student working on an induction problem wrote $P(n+1) = P(n) + \frac{1}{n^2}$. How can you tell that this must be wrong?

Exercise 278. Fill in the table using your powers of pattern recognition.

n	1	2	3	4	5	6	7	$\cdots$	n	$n+1$
nth odd integer	1	3	5	7				$\cdots$		
sum of first n odd integers	1	4						$\cdots$		

Column n of the table contains a conjecture about the sum of the first n odd integers. In the next problem, you will use mathematical induction to prove it.

Guided proof 279. Prove the conjecture in 278 using mathematical induction.

a. State $P(n)$: $P(n)$ is that "the sum of the first n odd integers equals ______________"

b. Basis step: $P(1)$ is that the sum of the 1st odd integer is 1^2. This is true because the sum is 1 and because $1^2 = 1$.

c. Write out $P(n+1)$: $P(n+1)$ is that "the sum of the first ____"

d. Inductive step: Let $n \geq 1$. suppose that $P(n)$ is true, and use that to show that $P(n+1)$ is true.

the sum of the first $n+1$ odd integers

= the sum of the first n odd integers plus ______________

= _____________________ + _____________________since $P(n)$ is true

= _____________________by algebra

Thus $P(n+1)$ is true. Since ______________ was arbitrary, by ____________________________ we conclude that the sum of the first n odd integers equals ______________ for all $n = 1, 2, 3, \ldots$.

Notation 280. Summation notation for $a_1+a_2+a_3+\cdots+a_n$ is $\displaystyle\sum_{k=1}^{n} a_k$. For example, $\sum_{k=1}^{n} k = 1+2+\cdots+n$.

Example 281. Use summation notation and the formula for the nth odd integer in 278 to rewrite $P(n)$ in 279: $P(n)$ is that _____________________ = _______

Exercise 282. Fill in the blanks to practice splitting off the last term of a sum.

a. $\displaystyle\sum_{k=1}^{n+1} k^2 = \left(\sum_{k=1}^{n} k^2\right) + $ ______________

b. $\displaystyle\sum_{k=1}^{n+1} \frac{1}{k^3} = \left(\sum_{k=1}^{n} \frac{1}{k^3}\right) + $ ______________

Show 283. Show that $\sum_{k=1}^{n} 4k - 3 = n(2n-1)$ for all $n = 1, 2, 3, \ldots$.

a. State $P(n)$: $P(n)$ is that

b. Basis step: $P(1)$ is that This is true because: _____________________

c. State $P(n+1)$: $P(n+1)$ is that

Suggestion: Use algebra to simplify the right hand side.

d. Inductive step: Let $n \geq 1$. suppose that $P(n)$ is true, and use that to show that $P(n+1)$ is true.

$\displaystyle\sum_{k=1}^{n+1} 4k - 3 = \sum_{k=1}^{n}$ _____________________ + _____________________

= _____________________ + _____________________ since $P(n)$ is true

=

=

Thus, _____________________. Since ...

Stop. Compare your proofs with the other people in your group before you move on.

Show 284. Use mathematical induction to show that $\sum_{k=1}^{n} 5^k = \frac{5}{4}(5^n - 1)$ for all integers $n = 1, 2, 3, \ldots$.

a. State $P(n)$:

b. Basis step:

c. State $P(n+1)$:

Suggestion: Multiply out the right hand side.

d. Inductive step: Let $n \geq 1$. suppose that $P(n)$ is true, and use that to show that $P(n+1)$ is true.

Note 285. The basis step need not use $n = 1$, for example, it can use $n = -3, n = 0$, or $n = 100$.

Show 286. Use mathematical induction to show that $2n + 1 < 2^n$ for all integers n with $n \geq 4$.

a. State $P(n)$:

b. Basis step: $P(4)$ is that: Check: ____________________

c. State $P(n+1)$:

d. Inductive step: Let $n \geq 4$. suppose that $P(n)$ is true, and use that to show that $P(n+1)$ is true.

$$\begin{aligned} 2(n+1)+1 = \underline{\hspace{4cm}} &= \underline{\hspace{4cm}} \\ &< \underline{\hspace{4cm}} \qquad \text{since } P(n) \text{ is true} \\ &< 2^n + 2^n \qquad \text{since } \underline{\hspace{3cm}} \\ &= 2^{n+1}. \end{aligned}$$

Thus, ____________________. Since ...

Show 287. Use mathematical induction to show that $5^n > 2^n + 3^n$ for all integers n with $n \geq 2$. Use the same format as above.

a.

b.

c.

d.

Show 288. Use induction to prove Bernoulli's inequality: For all $x \in \mathbb{R}$, if $1+x > 0$, then $(1+x)^n \geq 1+nx$ for all $n = 0, 1, 2, \ldots$. Use the same format as above.
Let x be such that $1 + x > 0$.

Where did you use the assumption that $1 + x > 0$?

Show 289. Use induction to prove that $\frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \frac{1}{3\cdot 4} + \cdots + \frac{1}{n\cdot(n+1)} = \frac{n}{n+1}$ for all positive integers n. Start by writing $P(n)$ using summation notation.

Note 290. It is possible to show that the statement in 289 is true without using mathematical induction, but using an algebraic technique. How?

Show 291. For each $n \in \mathbb{Z}^+$, let $P(n)$ denote the statement "$n^2 + 5n + 1$ is an even integer."

a. State $P(n + 1)$

b. Suppose that $P(n)$ is true, and use that to prove that $P(n + 1)$ is true.

c. For which n is $P(n)$ actually true?

d. What is moral of this exercise?

Show 292. Use induction to prove that $n^3 - n$ is a multiple of 6 for all integers $n = 0, 1, 2, \ldots$. Do the induction step as a "rewrite" proof.

Show 293. Use induction to prove that $11^n - 4^n$ is a multiple of 7 for all $n = 0, 1, 2, \ldots$. Do the induction step as a "rewrite" proof. **Hint:** Use the equation $11^n - 4^n = 7k$ once.

Example 294. Write the following numbers as multiples of 5 plus a remainder from 0 to 4.

$37 = 5 \cdot$ ______ + ______
$38 = 5 \cdot$ ______ + ______
$39 = 5 \cdot$ ______ + ______
$40 = 5 \cdot$ ______ + ______
$41 = 5 \cdot$ ______ + ______

Prove 295. Let $k > 0$ be an integer. For each integer n, let $P(n)$ be the statement: "There exist integers q and r with $0 \leq r < k$ such that $n = kq + r$." Use mathematical induction to show that $P(n)$ is true for all integers n.

a. Show that $P(0)$ is true.

b. Let n be an integer. Suppose that $P(n)$ is true, so that $n = kq + r$ for some integers q and r, with $0 \leq r < k$. Show that there exist integers q' and r' with $0 \leq r' < k$ so that $n + 1 = kq' + r'$, and thus conclude that $P(n+1)$ is true. Note that you will define q' and r' in terms of q and r. It is helpful to do this with two cases, depending on the value of r:

Case 1. Suppose $0 \leq r < k - 1$.

Case 2. Suppose $r = k - 1$.

c. Suppose that $P(n)$ is true and show that $P(n-1)$ is true. It is helpful to do this with two cases.

Use steps b and c and the idea of mathematical induction to conclude the proof that $P(n)$ is true for all n.

Show 296. Use mathematical induction to prove that $\sum_{k=1}^{n} k = \frac{1}{2}n(n+1)$ for all integers $n = 1, 2, 3, \ldots$.

Show 297. Use mathematical induction to prove that $\sum_{k=1}^{n} k^2 = \frac{1}{6}n(n+1)(2n+1)$ for all integers $n = 1, 2, 3, \ldots$.

Show 298. Let r be a real number not equal to 1. Use induction to prove that $\sum_{k=0}^{n} r^k = \frac{1-r^{n+1}}{1-r}$ for all integers $n = 0, 1, 2, \ldots$. Note where you use the assumption on r.

Show 299. Use mathematical induction to show that $n! > 3^n$ for all $n = 7, 8, 9, \ldots$.

Show 300. For $n = 2, 3, \ldots$, let $P(n)$ be the statement "n is prime or n can be written as the product of prime factors less than n. Supposing that $P(2), P(3), \ldots, P(n)$ are true, show that $P(n+1)$ is true, and thus conclude that $P(n)$ is true for all $n = 2, 3, \ldots$. This is called *strong induction.*

Show 301. Prove that $1^2 - 2^2 + 3^2 - 4^2 + 5^2 + \cdots - (2n)^2 + (2n+1)^2 = (n+1)(2n+1)$ for all $n = 0, 1, 2, \ldots$. Start by writing $P(n)$ using summation notation, starting from $k = 0$.

Your name: ______________________________

Infinite unions and intersections

Overview

We are working with infinitely many sets of real numbers. These exercises will give you practice with sets and teach you things about the real numbers as well.

Definition 302. Union. Let $A_1, A_2, \ldots$ be sets. The *union* of $A_1, A_2, \ldots$, which is denoted $\bigcup_{n=1}^{\infty} A_n$, is all elements which are in A_n for some $n = 1, 2, 3, \ldots$.

Definition 303. Intersection. Let $A_1, A_2, \ldots$ be sets. The *intersection* of $A_1, A_2, \ldots$, which is denoted $\bigcap_{n=1}^{\infty} A_n$, is all elements which are in A_n for all $n = 1, 2, 3, \ldots$.

Problem 304. Use quantifiers to express what it means that $x \in \bigcup_{n=1}^{\infty} A_n$.
Solution: $\exists n$ such that $x \in A_n$. In words, there is at least one n for which x is in A_n; that is what it takes to be in the union.

Problem 305. Use quantifiers to express what it means that $x \in \bigcap_{n=1}^{\infty} A_n$.

Exercise 306. To show $A \subseteq \bigcap_{n=1}^{\infty} B_n$: Let $x \in A$, and show that $x \in B_n$ for all $n = 1, 2, 3, \ldots$. What kind of proof do you need to write in order to show that $x \in B_n$ for all $n = 1, 2, 3, \ldots$? What is the first line of that part of the proof?

Exercise 307. To show $A \subseteq \bigcup_{n=1}^{\infty} B_n$: Let $x \in A$, and show that $x \in B_n$ for some $n = 1, 2, 3, \ldots$. In other words, show that $\exists n$ such that $x \in B_n$. What kind of proof do you need to write in order to show that $x \in B_n$ for some $n = 1, 2, 3, \ldots$? What is going to have to happen first in the proof?

Remark 308. To show $\bigcap_{n=1}^{\infty} A_n \subseteq B$: Let $x \in \bigcap_{n=1}^{\infty} A_n$. Then you know that $x \in A_n$ for all n, which is a lot of information about x. Use this information to show that $x \in B$. Exactly how that will work depends on the problem.

Remark 309. To show $\bigcup_{n=1}^{\infty} A_n \subseteq B$: Let $x \in \bigcap_{n=1}^{\infty} A_n$. Then you know that $x \in A_n$ for some n, but you don't know which n, so that is not very informative. There are infinitely many cases, one for each possible value of n. The best you can do is to start with, "Suppose $x \in A_n$" and work forward from there to show that $x \in B$. In the end, this is the same as showing that $A_n \subseteq B$ for all n.

Problem 310. Let $A = \displaystyle\bigcup_{n=1}^{\infty} [n, n+1)$.

a. Write A in open form, listing out the first five sets in the union:

$A =$ ________ $\cup$ ________ $\cup$ ________ $\cup$ ________ $\cup$ ________ $\cup \cdots$

b. Figure out what interval A is equal to and call the new interval B. $B =$ ____________.

c. Show that $A \subseteq B$.

Let $x \in A$. Then $x \in [n, n+1)$ for ____________________.

Thus, $x \in B$. Since $x \in A$ was arbitrary, $A \subseteq B$.

d. Show that $B \subseteq A$.

Let $x \in B$. You need to show that there exists an n for which $x \in [n, n+1)$. You need to construct the value of n, starting with x.

Thus, $x \in [n, n+1)$, and so $x \in A$. Since $x \in B$ was arbitrary, $B \subseteq A$.

Problem 311. Let $A = \bigcup_{n=1}^{\infty} [-n, n]$.

a. Write A in open form, listing out the first five sets in the union.

b. Figure out what interval A is equal to and call the new interval B. $B =$ ____________________.

c. Show that $A \subseteq B$. Use good form.

d. Show that $B \subseteq A$. Use good form.

Problem 312. Let $A = \bigcap_{n=1}^{\infty} (-n, n)$.

a. Write A in open form, listing out the first five sets in the intersection.

b. Figure out what interval A is equal to and call the interval B. $B =$ ______________.

c. Show that $A \subseteq B$.

d. Show that $B \subseteq A$.

Let $x \in B$. You need to show that $x \in (-n, n)$ for all n. How do you do that?

Problem 313. Let $A = \displaystyle\bigcup_{n=0}^{\infty} [n, n^2]$.

a. List out the first five sets in this union, as you did above. Draw them on a number line if it helps.

b. Make a conjecture about how you can write A in a simpler way and call the new set B. It should be the union of three simpler things.

c. Show that $A \subseteq B$. Let $x \in A$. Then $x \in [n, n^2]$ for some $n = 0, 1, 2, \ldots$.. You want to show that $x \in B$. You can do this with three cases, depending on whether $n = 0$, $n = 1$, or $n > 1$. Each case needs to end with $x \in B$.

d. Show that $B \subseteq A$. Let $x \in B$. There are three cases, and in each one, you will need to construct n so that $x \in [n, n^2]$. Each case needs to end with $x \in A$.

Problem 314. Let $A = \displaystyle\bigcup_{r \in \mathbb{Q}} (r - \tfrac{1}{10}, r + \tfrac{1}{10})$. Here, $\mathbb{Q}$ is the set of all rational numbers. Think of a simpler way to describe the set A, then prove your conjecture by showing set containment both ways.

Problem 315. Let n be an integer greater than 0.

a. Show that $[5 + \frac{1}{n}, 6] \subseteq (5, 6]$

b. Show that $(5, 6] \subseteq [5, 6]$.

Problem 316. Let $A = \bigcup_{n=1}^{\infty} [\frac{1}{n}, 1]$. List out the first five sets in this union, as you did above. Draw a picture of them above a number line. Make a conjecture about what interval A is equal to, call the new interval B, then show that $A = B$ by showing containment both ways. You will need to use this property of real numbers: if $x > 0$, then there exists a positive integer n with $0 < \frac{1}{n} < x$.

Problem 317. Suppose that $x \leq 5 + \frac{1}{n}$ for all $n = 1, 2, 3, \ldots$. Show that $x \leq 5$. **Hint:** Consider different types of proof including direct, contrapositive, contradiction, etc.

Problem 318. Let $A = \bigcap_{n=1}^{\infty} [0, 1 + \frac{1}{n}]$. List out the first five sets in this intersection, as you did above. Draw a picture of them above a number line. Make a conjecture about what interval A is equal to, call the new set B, then prove that $A = B$ by showing containment both ways.

Problem 319. Let $a < b$. Show that $\bigcup_{n=1}^{\infty}[a, b - \frac{1}{n}] = [a, b)$. (If $b - \frac{1}{n} < a$, the interval is empty.) Draw pictures, then show set inclusion both ways.

Problem 320. Let $a < b$. Show that $\bigcap_{n=1}^{\infty}(a - \frac{1}{n}, b + \frac{1}{n}) = [a, b]$. Draw pictures, then show set inclusion both ways.

Problem 321. Let $A = \bigcup_{k \in \mathbb{Z}}(k, k+1)$.

1. Draw out some of the intervals on a number line.

2. Make a conjecture about what set A is. You do not need to prove the conjecture.

Problem 322. For $n = 2, 3, 4, \ldots$, let $A_n = \{2n, 3n, 4n, \ldots\}$.

a. Write out the first five of the A_n.

b. Let $B = \bigcup_{n=2}^{\infty} A_n$. Describe the set B in simpler terms, perhaps by writing out the smallest 10 elements of B, then describe B in a sentence.

c. What is $\mathbb{N}\backslash B$? Remember that $\mathbb{N} = \{0, 1, 2, 3, \ldots\}$.

Problem 323. Use quantifiers to write down what it means that $x \in \bigcup_{n=1}^{\infty} A_n$. Use quantifiers to express what it means that $x \notin \bigcup_{n=1}^{\infty} A_n$ by negative quantifiers and rewriting until the expression is as simple as possible.

Problem 324. Use quantifiers to write down what it means that $x \in \bigcap_{n=1}^{\infty} A_n$. Use quantifiers to express what it means that $x \notin \bigcap_{n=1}^{\infty} A_n$ by negative quantifiers and rewriting until the expression is as simple as possible.

Definition 325. Set complement. If A is a set, then A^c is all elements under consideration that are not in A. For example, if $A = [0, 8)$, then $A^c = (-\infty, 0) \cup [8, \infty)$.

Problem 326. de Morgan's law. Show that $(\bigcup_{n=1}^{\infty} A_n)^c = \bigcap_{n=1}^{\infty} A_n^c$ by writing logical expressions for x being in the set on the left side and for the right side. Start by writing a logical expression that means the same thing as $x \in (\bigcup_{n=1}^{\infty} A_n)^c$ and work with it until it is a logical expression for $x \in \bigcap_{n=1}^{\infty} A_n^c$. When you write the proof this way, you do not need to show containment both ways to show that the two sets are equal.

$x \in (\bigcup_{n=1}^{\infty} A_n)^c$ means $\neg(\exists$integer n such that $x \in A_n)$, which means ...

Your name: ____________________________

Deriving properties of inequalities

We can define the $<$ relation for real numbers and establish its properties.

Overview

In this activity, we back up to the point after the real numbers have been constructed, but before subtraction and inequalities have been defined. We define the $<$ relation and prove a number of useful properties that it satisfies. Since the $>$ relation is so similar, we will not define it or show its properties.

Remark 327. Most of us first learned numbers by counting, using 1, 2, 3, . . . , which we will call *positive integers*. Later, we learned about addition of positive integers and multiplication of positive integers. Both operations give back positive integers; we say that the set of positive integers is *closed* under addition and multiplication. Later, we learned about zero, negative numbers, rational numbers, and real numbers. It is not always made clear, but the negative integers can be constructed from the positive integers, the rationals from the integers, and the reals from the rationals. In this activity, we assume that the real numbers have been constructed and have been shown to have their usual algebraic properties, and work from there to prove some basic (and very familiar) facts.

Note 328. Let $\mathbb{R}$ denote the set of real numbers, and denote addition and multiplication of real numbers in the usual ways. **Addition** has these properties: commutativity ($a + b = b + a$), associativity ($a + (b + c) = (a + b) + c$), additive identity (there exists a unique real number called 0 for which $a + 0 = a$ for all $a \in \mathbb{R}$), and additive inverse (for each number a in $\mathbb{R}$, there exists a unique real number $-a$ for which $a + (-a) = 0$). **Multiplication** has these properties: commutativity ($ab = ba$), associativity ($a(bc) = (ab)c$), multiplicative identity (there exists a unique real number called 1, with $1 \neq 0$, such that $a \cdot 1 = a$ for all a in $\mathbb{R}$), multiplicative inverse (for each a in $\mathbb{R}$ with $a \neq 0$, there exists a unique number called a^{-1} for which $a \cdot a^{-1} = 1$. **Addition and multiplication** are related by the distributive property: $((a + b)c = ac + bc)$.

Note 329. In this activity, subtraction is not defined, so be careful not to use it!

Show 330. Let a be a real number. Justify each line in the following proof to show that $0 \cdot a = 0$.

$$\begin{aligned}
a + (-a) &= 0 \\
1 \cdot a + (-a) &= 0 \\
(0 + 1) \cdot a + (-a) &= 0 \\
(0 \cdot a + 1 \cdot a) + (-a) &= 0 \\
(0 \cdot a + a) + (-a) &= 0 \\
0 \cdot a + (a + (-a)) &= 0 \\
0 \cdot a + 0 &= 0 \\
0 \cdot a &= 0
\end{aligned}$$

Show 331. People sometimes ask if the additive inverse $(-a)$ is the same as the product $(-1)\cdot a$, where (-1) is the additive inverse of 1. It's true, and here is how you show it; fill in steps and write the justifications at the right side of each line.

$$\begin{aligned}
a + (-1) \cdot a &= 1 \cdot a + (-1) \cdot a \\
&= (1 + (-1)) \cdot a \\
&= \\
&= 0,
\end{aligned}$$

This shows that $(-1) \cdot a$ is the additive inverse of a, because that number is unique.

Show 332. You might think that it is obvious that $(-1)(-1) = 1$, where (-1) is the additive inverse of 1, but this takes a few steps. Fill in steps and write justifications.

$$\begin{aligned} (-1) + (-1)(-1) &= (-1)(1) + (-1)(-1) \\ &= (-1)(1 + (-1)) \\ &= \\ &= 0 \end{aligned}$$

Why does this show that $(-1)(-1)$ is the additive inverse of -1?

Show 333. The additive inverse of a sum works out nicely. Let a and b be real numbers and think about the additive inverse of $a + b$. Write justifications to the right of each statement.

$$\begin{aligned} -(a + b) &= (-1)(a + b) \\ &= (-1)(a) + (-1)(b) \\ &= (-a) + (-b) \end{aligned}$$

Show 334. Let $a \in \mathbb{R}$. The statement $-(-a) = a$ is just a statement about additive inverses. Prove that it is true.

Definition 335. Positive real numbers. By construction, the real numbers have a subset $\mathbb{R}^+$, called the *positive real numbers,* for which:

a. If $a, b \in \mathbb{R}^+$, then $a + b \in \mathbb{R}^+$. ($\mathbb{R}^+$ is closed under addition.)

b. If $a, b \in \mathbb{R}^+$, then $a \cdot b \in \mathbb{R}^+$. ($\mathbb{R}^+$ is closed under multiplication.)

c. For every real number a, either $a \in \mathbb{R}^+$ or $(-a) \in \mathbb{R}^+$ or $a = 0$. Exactly one of the three happens.

Note that the positive real numbers are exactly analogous to the positive integers that you learned first. We can't use interval notation to write what $\mathbb{R}^+$ is, because intervals are defined in terms of inequalities, and we have not defined inequalities yet!

Exercise 336. Under each property in 335, write a sentence that states it in plain English

Show 337. Let $a \in \mathbb{R}$ and suppose that $a \neq 0$. Show that $a \cdot a \in \mathbb{R}^+$, justifying each step, citing previous definitions or results by number. **Hint:** Use a proof by cases, using the two remaining cases in 335c. For future reference, this gives us a new way to show that a number is in $\mathbb{R}^+$.

a. Suppose that $a \in$ _______.

b. Suppose that $(-a) \in$ ______.

Show 338. Show that $1 \in \mathbb{R}^+$. Justify each step.

Show 339. Show that $(-1) \notin \mathbb{R}^+$. **Hint:** Pretend for a minute that $(-1) \in \mathbb{R}^+$ and use 335a.

Definition 340. Less than. Let a and b be real numbers. We write that $a < b$ if $b + (-a) \in \mathbb{R}^+$.

Note 341. All of the following problems rely on Definition 340, so you will use it over and over. Note that $>$ has not been defined yet, so be careful not to use it.

Show 342. Show that $-1 < 0$.

Show 343. Show that $1 < 1$ is not true. Thus, the $<$ relation is not reflexive. Justify each step, citing previous results by number.

Show 344. Show that $0 < 1$.

Show that $1 < 0$ is not true. Thus, the $<$ relation is not symmetric.

Show 345. Show that the $<$ relation on $\mathbb{R}$ is transitive. Follow good form by first letting a, b, c be real numbers and supposing that $a < b$ and $b < c$, then showing $a < c$. Justify each step by number. In this proof, you are likely to use the fact that $(-b) + b = 0$, which is the additive inverse property.

Show 346. Let $a, b \in \mathbb{R}$ and suppose that $a < b$. Show that $-b < -a$. Justify each step.

Show 347. Let $a, b, c \in \mathbb{R}$. Suppose that $a < b$. Show that $a + c < b + c$. Justify each step.

Show 348. Let $a, b, c, d \in \mathbb{R}$. Suppose that $a < b$ and $c < d$. Show that $a + c < b + d$. Justify each step.

Show 349. Let a, b, c be real numbers. Suppose that $a < b$ and $0 < c$. Show that $ac < bc$.

Show 350. Let a, b, c be real numbers. Suppose that $a < b$ and $c < 0$. Show that $bc < ac$.

Show 351. Let $a, b \in \mathbb{R}$ and suppose that $0 < a$ and $b < 0$. Use a previous result to show that $ab < 0$.

Show 352. Let $a \in \mathbb{R}$ and suppose that $0 < a$. Show that $0 < a^{-1}$. Here a^{-1} is the multiplicative inverse of a. **Hint:** This one take a bit more effort than the previous ones. Note that division has not been defined yet, so just use addition and multiplication.

Show 353. Let $a, b \in \mathbb{R}$ and suppose that $0 < a$ and $a < b$. Show that $b^{-1} < a^{-1}$.

Your name: ______________________________

Construction of the real numbers

Construction of the real numbers using Dedekind cuts

Overview

Many things can be defined and written about that don't actually exist; unicorns, little green men from Mars, and others may come to mind. To this point in your mathematical career, you have worked with real numbers and used many of their properties, but how do we know that they really exist? The mathematical answer is that we *construct* them from simpler numbers and show that they have the right properties.

Note 354. Natural numbers. At some point in your life you learned the counting numbers 1, 2, 3, ... Then you learned to add and multiply, and these operations have the familiar algebraic properties like commutativity, associativity, distributivity. Their biggest claim to fame: No matter how high you count, there is always a next number. Note, however, that we are not *defining* the natural numbers.

Definition 355. Zero. Subtraction problems like $9 - 5$ have answers that are whole numbers, but subtraction problems like $3 - 3$ and $12 - 12$ call for a new number. You can define 0 as $3 - 3$ or as $12 - 12$; there are many ways to write this new number.

Definition 356. Negative numbers. Subtraction problems like $5 - 9$ and $8 - 12$ need another set of new numbers to be defined. You can define -4 to be $5 - 9$ but also $8 - 12$. Anytime you want to work with -4, you can substitute in $5 - 9$ instead. Or $8 - 12$.

Exercise 357. With negative numbers, we say that $a - b = c - d$ if $a + d = c + b$. Check that this is the case for $5 - 9$ and $8 - 12$. It's important that to check, and you only need to work with natural numbers.

Definition 358. Integers. The natural numbers, zero, and the negative numbers make up the integers. Each integer can be written as $a - b$ where a and b are natural numbers. The integers are closed under addition and multiplication, and these operations have the usual algebraic properties like commutativity and associativity, additive inverses, additive identity, and multiplicative identity.

Definition 359. Rational numbers. Division problems like $15 \div 5$ have answers that are integers, but problems like $5 \div 15$ need yet more new numbers to be defined. For some reason people decided to write the new numbers as $\frac{5}{15}$ but we could have chosen some other notation like $(5, 15)$ or $5\#15$. At any rate, these new "rational" numbers are made up of two integers, the second of which needs to be non-zero. The rules for rational numbers are worth noting in some detail:

- **a.** Rational numbers $\frac{a}{b}$ and $\frac{c}{d}$ are *equal* if $ad = bc$, which is a matter of integer multiplication.
- **b.** We say that $\frac{a}{b} < \frac{c}{d}$ if $ad < bc$, which again comes down to integers
- **c.** The sum of $\frac{a}{b}$ and $\frac{c}{d}$ is the rational number $\frac{ad+bc}{bd}$. Note that you only need to use integer arithmetic to find the sum of two rational numbers.
- **d.** The product of $\frac{a}{b}$ and $\frac{c}{d}$ is the rational number $\frac{ac}{bd}$.

It's important to note that everything about these new rational numbers is defined in terms of integers. There are multiple ways to define each rational number; $\frac{3}{12} = \frac{1}{4}$ for example. The rational numbers can be shown to have the usual algebraic properties, always because the integers have the property. Bonus: non-zero rational numbers have multiplicative inverses. The set of all rational numbers is denoted by $\mathbb{Q}$.

Exercise 360. Use Definition 359 for each part.

a. Check that $\frac{1}{3} = \frac{5}{15}$ using the definition.

b. Check that $\frac{1}{4} < \frac{2}{7}$ using the definition.

c. Add $\frac{1}{3} + \frac{1}{4}$ using the definition.

d. Show that $\frac{a}{b} + \frac{c}{d} = \frac{c}{d} + \frac{a}{b}$ using a rewrite proof. Proofs of other algebraic properties are similar.

Definition 361. Complex numbers. Once the real numbers have been defined, we can define the complex numbers by letting $i = \sqrt{-1}$ and then thinking about numbers of the form $a + ib$ where a and b are real numbers.

Remark 362. To construct the integers, the rational numbers, and the complex numbers, you put together two numbers of a simpler sort. As it happens, constructing the real numbers is harder. The basic idea is this: to refer to a real number like π, think of the rational numbers 3, 3.1, 3.14, 3.141, 3.1415, and all other rational numbers less than π. Then π is the "top" of this set of rational numbers. This is how we can use rational numbers to get our hands on real numbers like π that are not rational. In fact, we will literally *define* real numbers to be sets of rational numbers like this.

Remark 363. Here are a few examples of the sets we'll be using. After each set, describe it in words.

a. $A = \{q \in \mathbb{Q} : q < 0\}$

b. $B = \{q \in \mathbb{Q} : q \leq 7\}$

c. $C = \{q \in \mathbb{Q} : q^3 < 5\}$

d. $D = \{q \in \mathbb{Q} : q < 0 \text{ or } q^2 < 2\}$

It's OK to use words like "cube root of 5" in your description, even though that is not a rational number, and so has not been constructed yet.

Definition 364. Closed below. A set A of rational numbers is said to be *cl4osed below* if for all $q \in A$, for all $p \in \mathbb{Q}$ with $p < q$, we have $p \in A$. In words, if A contains the rational number q, then it contains every rational number less than q as well.

Exercise 365. Is the set $\{\frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \ldots\}$ closed below? Why or why not?

Remark 366. Note that "closed below" is unusual in that it has two "for all" quantifiers. To prove that a set A is closed below, follow this format: "Let $q \in A$. Let $p \in \mathbb{Q}$ such that $p < q$. Show that $p \in A$.

Exercise 367. Show that each of the following sets is closed below, following Remark 366.

a. $A = \{q \in \mathbb{Q} : q < 0\}$. **Hint:** Use transitivity.

b. $B = \{q \in \mathbb{Q} : q \leq 7\}$

c. $C = \{q \in \mathbb{Q} : q^3 < 5\}$. Do not write $\sqrt[3]{5}$, just use rational numbers and integers.

d. $D = \{q \in \mathbb{Q} : q < 0 \text{ or } q^2 < 2\}$. **Hint:** Consider two cases, $p < 0$ and $p \geq 0$. Do not write $\sqrt{2}$.

Remark 368. Sets C and D illustrate how we can use sets of rational numbers to point to a number that we know is irrational, in this case $\sqrt[3]{5}$ and $\sqrt{2}$. The irrational number we have in mind is at the "top" of the set, just like 0 is at the "top" of the set A.

Definition 369. Dedekind cut. A set $A \subseteq \mathbb{Q}$ is called a *Dedekind cut* if all of the following happen:

i. A is closed below

ii. A has no greatest element, meaning that for all $p \in A$, there is a number $q \in A$ with $p < q$. Keep in mind that q has to be a rational number.

iii. There exists an integer m with $m \in A$

iv. There exists an integer n with $n \notin A$

Exercise 370. Check each of the sets below to see if it is a Dedekind cut. You have already shown that they are closed below. If the set is a Dedekind cut, show that. If not, explain why not.

a. $A = \{q \in \mathbb{Q} : q < 0\}$. Let $m =$ ______________. Let $n =$ ______________. To show (ii), let $p \in A$. Let $q =$ ______________.

b. $B = \{q \in \mathbb{Q} : q \leq 7\}$

c. $C = \{q \in \mathbb{Q} : q^3 < 5\}$. Let $m =$ ______________. Let $n =$ ______________. To show (ii), given $p \in A$, I suggest you consider two cases. When $p < 1$, let $q = 1$. When $p \geq 1$, let $q = p + c$ where $c = \frac{5-p^3}{100}$ when $p \geq 1$. Calculate $(p+c)^3$ and use the fac that $c < 1$, so $c^2 < c$ and $c^3 < c$. Also keep in mind that $p < 2$.

d. $D = \{q \in \mathbb{Q} : q < 0 \text{ or } q^2 < 2\}$.

Exercise 371. Let A and B be Dedekind cuts. Define a new set C by $C = \{z : \text{there exist } a \in A \text{ and } b \in B \text{ such that } z = a + b\}$. Show that C is a Dedekind cut by checking all four requirements in 369.

1. C is closed below. Let $d \in C$, and let $c \in \mathbb{Q}$ with $c < d$. We need to show that $c \in C$, so we need to write it as the sum of an element of A and an element of B. Because __________, there exist $a \in A$ and $b \in B$ such that $d = a + b$. Let $p = a - (d - c)/2$ and $q = b - (d - c)/2$. It's clear that p and q are ______________ numbers and that $p < a$ and that ______________. Since $p < a$ and A is closed below, we know that $p \in A$. Since $q < b$ and _____________________, we know that ______________. Finally, $p + q =$ ____ $= c$, and so ______________.
2. C has no greatest element. Let $p \in C$. Show that there is a number $q \in C$ with $p < q$.
3. There exists an integer m with $m \in C$. **Hint:** Use $m_A \in A$ and $m_B \in B$ from 369(c).
4. There exists an integer n with $n \notin C$. **Hint:** Use $n_A \in A$ and $n_B \in B$. Then $a < n_A$ for all $a \in A$ and $b < n_B$ for all $b \in B$. Then $a + b < n_A + n_B$ for all $a \in A$ and all $b \in B$. Now ... why does that mean that $n_A + n_B$ is not in C?

Definition 372. Real numbers. We will refer to each Dedekind cut as a "real number." The set of real numbers will be written as $\mathbb{R}$.

Remark 373. Yes, you read that right. A real number is being defined as nothing more, and nothing less, than a Dedekind cut, which is a set of rational numbers. This is simply one way to use rational numbers to describe and work with real numbers. It was easier to define 0 as $3 - 3$ or to define fractions as things you get from pairs of integers. You'll get used to it.

Definition 374. Equality of real numbers. If A and B are real numbers, we say that A and B are equal if $A \subseteq B$ and $B \subseteq A$. We write $A = B$.

Definition 375. Less than for real numbers. If A and B are real numbers we say that A is less than B if $A \subset B$. We write $A < B$.

Definition 376. The real number zero. Let $\mathbf{0} = \{q \in \mathbb{Q} : q < 0\}$.

Definition 377. The real number one. Let $\mathbf{1} = \{q \in \mathbb{Q} : q < 1\}$.

Definition 378. Addition of real numbers. Let A and B be real numbers. The sum of A and B is the real number $C = \{z : \text{there exist } a \in A \text{ and } b \in B \text{ such that } z = a + b\}$. This set is a Dedekind cut as explained in 371. We write $A \oplus B$ for the sum.

Prove 379. Show that $\mathbf{0} \oplus \mathbf{1} = \mathbf{1}$ by showing set inclusion both ways.

a. Show $\mathbf{0} \oplus \mathbf{1} \subseteq \mathbf{1}$. Let $c \in \mathbf{0} \oplus \mathbf{1}$. Then $c = a + b$ for some $a \in \mathbf{0}$ and $b \in \mathbf{1}$. Thus $a < 0$ and $b < 1$. Thus, ______________ and so $c \in \mathbf{1}$.

b. Show $\mathbf{1} \subseteq \mathbf{0} \oplus \mathbf{1}$. Let $c \in \mathbf{1}$. Then ______________. Write $c = \frac{c-1}{2} + \frac{c+1}{2}$ and check that this means that $c \in \mathbf{0} \oplus \mathbf{1}$.

Prove 380. Commutativity. Let A and B be real numbers. Show that $A \oplus B = B \oplus A$. Instead of showing set inclusion both ways, do this as a rewrite proof:

$$\begin{aligned} A \oplus B &= \{a + b : a \in A, b \in B\} \\ &= \{b + a : \\ &= \end{aligned}$$

Prove 381. Associativity. Let A, B, and C be real numbers. Show that $A \oplus (B \oplus C) = (A \oplus B) \oplus C$. Do this as a rewrite proof, abbreviating the definition of the sum.

$$\begin{aligned} A \oplus (B \oplus C) &= A \oplus \{b + c : b \in B, c \in C\} \\ &= \{a + (b + c) : \\ &= \end{aligned}$$

Prove 382. Additive identity. Let A be a real number. Show that $A \oplus \mathbf{0} = A$.

a. Let $p \in A \oplus \mathbf{0}$. Then $p = a + b$ where $a \in$ ______________ and b ______________. **Hint:** You will use the fact that A is closed below.

b. Let $p \in A$. We need to write p as the sum of an element a of A and a rational number b less than zero. That means that a will be greater than p. Fortunately, such a number exists because A has no greatest element.

Let $a \in A$ such that $p < a$. Let $b = p - a$. Then ...

Definition 383. Additive inverse.. Let A be a real number. Define a new set $-A$ by

$$-A = \{b - c : b, c \in Q, b < 0, c \notin A\}$$

Think of b as being very close to 0, so the elements of $(-A)$ are like the negatives of the rational numbers bigger than the elements of A. It may be hard to understand this set intuitively, and so it may be easier just to work with the definition and not try to get the intuition straight.

Guided proof 384. Show that $-A$ is a Dedekind cut.

i. A is closed below. Let $q \in -A$. Let $p \in \mathbb{Q}$ with $p < q$. Since $q \in -A$, we can write $q = b - c$ where $b < 0$ and $c \notin A$. Check that $p = b + (p - q) - c$ by substituting in for q. Thus, we have written $p = b' - c'$ where $b' < 0$ and $c' \notin A$, by setting $b' =$______________ and $c' =$______________. This tells us that $p \in -A$, so $-A$ is closed below.

ii. A has no greatest element. Let $p \in -A$. Then we can write $p = b - c$ where $b < 0$ and $c \notin A$. Let $q = (b/2) - c$. Check that $q \in -A$ and that $p < q$.

iii. There exists an integer m with $m \in -A$. Check that $-1 - n_A$ where $n_A \notin A$ from 369(d) will work.

iv. Challenge: There exists an integer n with $n \notin -A$. Probably $-m_A + 1$ from 369(c) will work.

Prove 385. Additive inverse property. Let A be a real number. Show that $A \oplus (-A) = \mathbf{0}$.

a. Let $z \in A \oplus (-A)$. Then $z = p + q$ where $p \in A$ and $q = b - c$ where $b < 0$ and $c \notin A$. That is, $z = p + b - c$. Since $c \notin A$, we know that $c > p$, because __________________________. Then $z < 0$ because __________________________. Thus $z \in \mathbf{0}$, and so $A \oplus (-A) \subseteq \mathbf{0}$.

b. Let $z \in \mathbf{0}$. Then z is a rational number and $z < 0$. We need to write $z = p + q$ where $p \in A$ and $q \in (-A)$, and q needs to be written as $b - c$ where $b < 0$ and $c \notin A$. If z is close to 0, then p and c will need to be close to the "top" of the set A. The next result will show that there exists $p \in A$ and $c \notin A$ with $p - c = z/2$. Also let $b = z/2$. Then $z = z/2 + z/2 = p - c + b$ as required. So $z \in A \oplus (-A)$ and thus $\mathbf{0} \subseteq A \oplus (-A)$.

Guided proof 386. Let A be a real number. Let $c > 0$ be a rational number. Then there exists $a \in A$ and $b \notin A$ such that $b - a = c$.
By the definition of a Dedekind cut, there are integers $m \in A$ and $n \notin A$. Consider the numbers $m + kc$ for $k = 0, 1, 2, \ldots$. Draw a picture of these on a number line below. When $k = 0$, $m + kc \in A$. When $k > (n-m)/c$, $m + kc > n$ and so $m + kc \notin A$. Thus, for some value of k, $m + kc \in A$ but $m + (k+1)c \notin A$. Let $a = m + kc$ and $b = m + (k+1)c$.

Challenge 387. $\mathbf{0}$ is unique. That is, if there is another real number Z for which $A \oplus Z = A$ for all real numbers A, then $Z = \mathbf{0}$.

Challenge 388. Given a real number A, the additive inverse $-A$ is unique.

Definition 389. Multiplication of positive real numbers. Let A and B be real numbers with $\mathbf{0} < A$ and $\mathbf{0} < B$. The product of A and B is the real number $\{z : z \in \mathbb{Q}$ and $z \leq 0$ or there exists $a \in A$ with $a > 0$ and $b \in B$ with $b > 0$ such that $z = ab\}$. We write $A \otimes B$ to denote the product.

Remark 390. It is easiest to define multiplication of positive real numbers. The definition is very much like the definition of the sum, only a bit more complicated because of the need to include the negative rational numbers.

Challenge 391. Show that $A \otimes B$ is a Dedekind cut.

Prove 392. Multiplicative identity for positive real numbers. Let A be a real number with $0 < A$. Then $A \otimes \mathbf{1} = A$.

Prove 393. Distributivity for positive real numbers. Let A, B, and C be real numbers with $0 < A$, $0 < B$, and $0 < C$. Show that $A \otimes (B \oplus C) = (A \otimes B) \oplus (A \otimes C)$.

Definition 394. Multiplicative inverse of positive real numbers. Let A be a real number with $0 < A$. Define A^{-1} to be $\{z : z \in \mathbb{Q} \text{ and } z \leq 0 \text{ or } 1/z \in A\}$.

Prove 395. Show that A^{-1} is a Dedekind cut.

Prove 396. Show that $A \otimes A^{-1} = \mathbf{1}$

Prove 397. Let $D = \{q \in \mathbb{Q} : q < 0 \text{ or } q^2 < 2\}$ and let $E = \{q \in \mathbb{Q} : q < 2\}$. Show that $D \otimes D = E$. This confirms that D corresponds to the square root of 2.

Prove 398. Let A and B be real numbers. Show that $A \otimes B = B \otimes A$.

Prove 399. Let A, B, and C be real numbers. Show that $A \otimes (B \otimes C) = (A \otimes B) \otimes C$.

Prove 400. Prove that the multiplicative identity $\mathbf{1}$ is unique

Prove 401. Prove that the multiplicative inverse A^{-1} is unique

Definition 402. Multiplication of non-positive real numbers.. Let A and B be real numbers.

- **a.** If $0 < A$ and $B < 0$, then $A \times B = A \times (-B)$
- **b.** If $A < 0$ and $0 < B$, then $A \times B = (-A) \times B$
- **c.** If $A < 0$ and $B < 0$, then $A \times B = (-A) \times (-B)$

Prove 403. Show that the properties of multiplication extend to multiplication of non-positive numbers.

Definition 404. Least upper bound. Let S be a collection of real numbers. A number B is the least upper bound of S if $A \leq B$ for all $A \in S$, and if for all other upper bounds C of S, we have $B \leq C$.

Prove 405. Given a set S of real numbers, the number $B = \cup_{A \in S} A$ is the least upper bound of S. Thus, every collection of real numbers has a least upper bound.

Your name: ______________________________

Set union, intersection, Venn diagram, complement, difference

Overview

An introduction to set unions and intersections, Venn diagrams to represent sets, and set complements and differences. Co-authored by Johanson Berlie.

Definition 420. Disjoint sets. If two sets A and B have no elements in common, we say that they are **disjoint** sets.

Example 421. The set consisting of all World War II veterans and the set of all millennials are disjoint sets.

Definition 422. Comparable sets. Two sets A and B are **comparable** if $A \subseteq B$ or $B \subseteq A$. If this is not the case, then they are said to be **not comparable**.

Example 423. The sets A = {General Motors, Toyota, Ford, Renault} and B = {Tesla, Fisker} are not comparable. If we define a set C = {Ford, Toyota}, then A and C are comparable, since $C \subseteq A$.

Exercise 424. For the following questions, compare the sets in the context of the definitions above. Are they equal? Are they disjoint? Are they comparable? Is one a subset of the other? In every case, explain why.

a. The set A of US citizens and the set B of people in the US right now.

b. Think of the set A of US citizens and the set J of citizens of Japan. Do they have any elements in common, or are they disjoint? This might require an internet search.

c. The set $A = \{\divideontimes, \boxtimes, \blacktriangleleft, \boxplus\}$ and the set $B = \{\boxplus, \blacktriangleleft, \divideontimes, \boxtimes\}$

d. The set of humans that have been to more than one planet and the set of humans that have been to Pluto.

e. The set G consisting of the members of the Green Bay Packers on the first play of a game and the set M consisting of the members of the Manchester United soccer team in play during a game.

f. The set J consisting of all employees of the Department of Justice and the set F consisting of all federal employees.

g. Suppose you know that $A \subseteq B$ and also that $B \subseteq A$. What more can you say about A and B now? Why?

h. Is it possible for the universal set to be disjoint from any of its subsets? Explain why or why not.

Definition 425. Union of sets. The **union** of sets A and B is a new set, consisting of all elements which belong to A or to B or to both. We denote this by $A \cup B$.

Definition 426. . The **intersection** of sets A and B is a new set, consisting of all elements which belong to both A and B. We denote this by $A \cap B$.

Example 427. Let C be the set of Computer Science majors and M be the set of Mathematics majors. Then $C \cap M$ is the set of students double majoring in Computer Science and Mathematics (a powerful combination!) while $C \cup M$ is the set of students majoring in one, the other, or both majors.

Exercise 428. Answer the following, expressing the answers as sets where possible.

a. If $P = \{red, blue, yellow\}$, $S = \{purple, green, orange\}$, $M = \{red, green, blue\}$, find:

b. $P \cup S$

c. $P \cap M$

d. $P \cup S \cup M$

e. $P \cap S \cap M$

f. What is $A \cap A$? What about $A \cup A$?

g. Is it always true that $A \cup B = B \cup A$? Explain.

h. If $A \subseteq B$ then what is $A \cup B$?

i. Is $A \cap B$ a subset of A? Is $A \cap B$ a subset of B? Explain.

j. What is $\varnothing \cap A$? Does it depend on the set A? Explain why or why not.

k. What is $\varnothing \cup A$? Does it depend on the set A? Explain why or why not.

l. If $A \cup B = \varnothing$, can we say anything about A or B? Is there something special about them? Explain.

m. If A and B are disjoint, what can we say about $A \cap B$?

n. If $A \subseteq B$ and B and C are disjoint, then what about A and C?

o. If U denotes the universal set, is it true that $U \cap A = A \cap \varnothing$? Explain.

Definition 429. Universal set. If all of the sets under discussion are subsets of a fixed set, this set is called the **universal set** or **universe of discourse** and denoted by **U**. Sometimes there is more than one possibility for U.

Example 430. If we were studying the citizenship of people around the world, the universal set would consist of all the people on earth. Citizens of the US would be one interesting set, citizens of Canada would be another. Do these two sets have any elements in common? If so, how could that happen?

Example 431. If we were studying binary stars, the universal set would be all the stars in the universe.

Exercise 432.

a. When thinking of prime numbers, what is a good choice for the universal set?

b. When solving linear equations like $5x + 7 = 3$, what a good choice for the universal set?

c. When solving quadratic equations like $x^2 + 5x + 8 = 0$, what is the universal set?

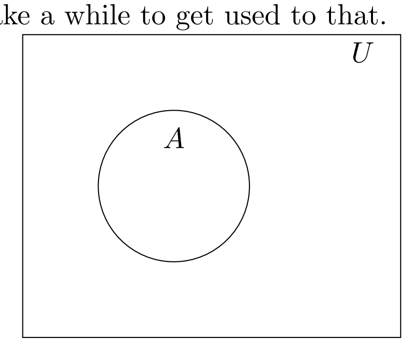
d. Is it possible for the universal set to be empty? Explain why or why not.

e. If two sets A and B are subsets of a given universal set, U, is it possible that $A = U$ or $B = U$? Explain why or why not.

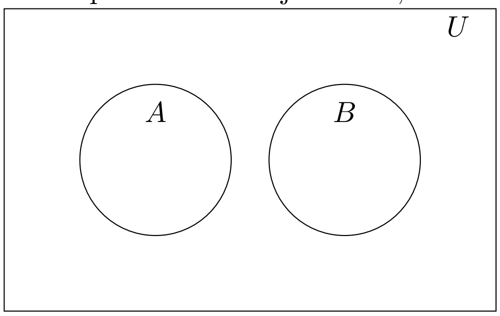
Definition 433. Venn diagram. A Venn diagram (also called **primary diagram**, **set diagram** or **logic diagram**) is a picture that shows all possible logical relations between a finite collection of sets. These diagrams depict elements as points in the plane, and sets as regions inside closed curves.

Remark 434. This definition might seem complicated, but for our purposes we will think of Venn diagrams as circles that represent sets. The following examples will help make this clear.

Example 435. To represent a single set using a Venn diagram, we draw a single circle (representing the set) inside a rectangle (representing the universal set). We usually write the label for a given set inside the curve that represents it. Note that the set A could be empty but we still draw it with a circle; it may take a while to get used to that.



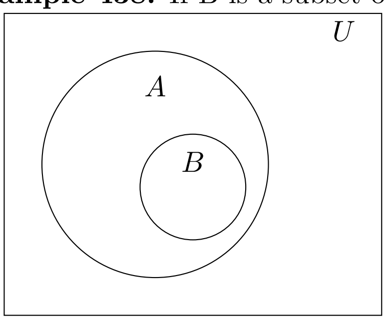
Example 436. The representation in the above example can be extended to any finite number of sets. If want to represent two disjoint sets, A and B, we can represent them as below.



The circles do not overlap, to indicate that the sets are disjoint. Note that A or B or both could actually be empty sets, a bit like looking down into empty paper bags.

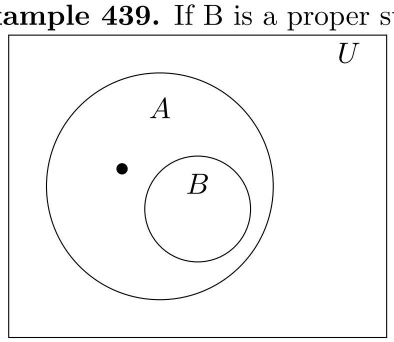
Remark 437. The power of Venn diagrams is that they make it easy to understand set relations and operations. Let's look at a few examples.

Example 438. If B is a subset of A, that is, if $B \subseteq A$ we can represent this as below.



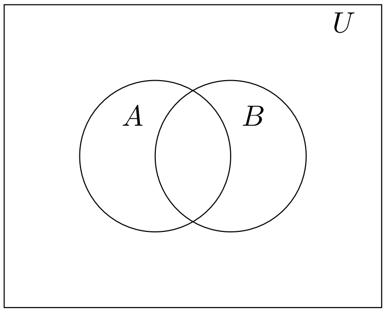
Note that it is possible that $B = A$; the blank areas in the diagram may or may not contain points; they may be empty.

Example 439. If B is a proper subset of A, we can represent this with a dot outside B, but within A.

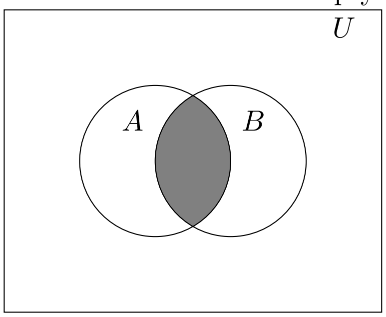


Now it's clear that every element of B is also an element of A, but that there is an element of A that is not an element of B.

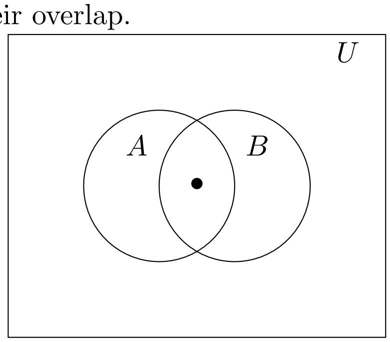
Example 440. The generic picture of two sets A and B shows them overlapping, but not completely:



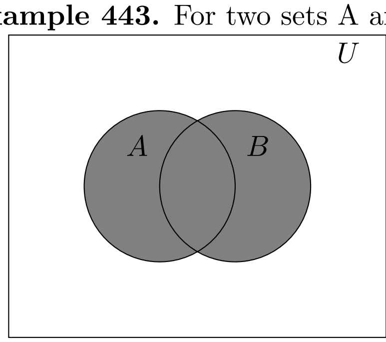
Example 441. To indicate the intersection of A and B, we shade $A \cap B$ as below, even if it is possible that the intersection is an empty set:



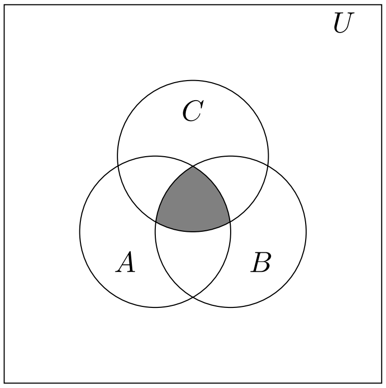
Example 442. If we know that A and B, are not disjoint, we indicate this by a dot inside the region of their overlap.



Example 443. For two sets A and B, we represent $A \cup B$ as below:



Example 444. For three sets A, B and C, we draw them to allow for all possible intersections. We represent $A \cap B \cap C$ as below, even though it may actually be an empty set:



Exercise 445. For the following questions, represent your answers as Venn diagrams, where possible. If a Venn diagram is not possible, explain why.

a. Make a Venn diagram for sets A and B, showing that A and B are disjoint, and shade $A \cup B$.

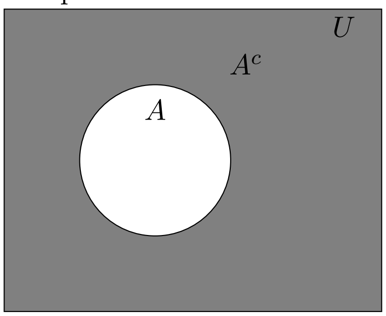
b. Make a Venn diagram for sets A, B, and C, showing that B and C are disjoint. Shade the set $(A \cap B) \cup C$.

c. Make a Venn diagram for sets A, B, and C, showing that $A \cap B \cap C$ is empty, and shading $(A \cap B) \cup (A \cap C) \cup (B \cap C)$.

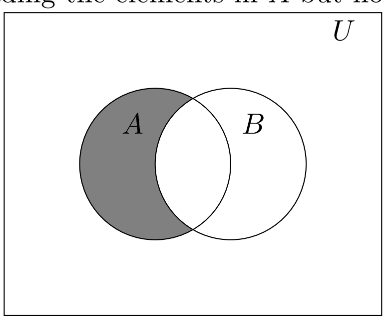
d. Make a Venn diagram for sets A, B, and C, showing that A and B are not disjoint and that $C \subset B$ but C is disjoint from A.

e. Make a Venn diagram for sets A, B, and C, showing that $A \cap B \subset C$, but $C \subset A \cup B$.

Definition 446. Set complement. The **complement** of a set A is the set of elements that do not belong to A. Therefore, it is every element of the universal set U that is not an element of A. We denote the complement of A as A^c or A'. Visually, we represent this as:



Definition 447. Set difference. The **difference** of sets A and B is the set of elements which belong to A but not to B. We denote this as $A - B$ or $A \setminus B$. Visually, we represent this by drawing A and B and shading the elements in A but not in B.



Exercise 448. For the following questions, start by drawing a Venn diagram for sets A and B, then shade the desired set or sets. For more than one set, use different shading for each one.

a. $(A \cap B)^c$, given that A and B are not disjoint.

b. $(A^c \cup B^c)$, given that A and B are not disjoint.

c. A^c and $A^c \setminus B$.

d. Supposing that $B \subset A$, shade in $A \setminus B$.

Your name: ______________________________

Operations on sets

This activity works with set identities and relates them to logic.

Overview

Sets are absolutely fundamental to mathematics. This chapter focuses on building up set identities, relationships between sets that are always true.

Problem 449. Let A and B be sets. Show that $(A \cup B)^c = A^c \cap B^c$ by showing set inclusion both ways. The first part is done for you. This is one of de Morgan's laws. Draw a really nice Venn diagram to illustrate.

- Let $x \in (A \cup B)^c$. Then $x \notin A \cup B$. So $x \notin A$ and $x \notin B$. That means that $x \in A^c$ and $x \in B^c$, and so $x \in A^c \cap B^c$. Since x was arbitrary, $(A \cup B)^c \subseteq A^c \cap B^c$.
- Let $x \in A^c \cap B^c$.

Problem 450. Let A and B be sets. Show that $(A \cap B)^c = A^c \cup B^c$ by showing set inclusion both ways. This is the other one of de Morgan's laws. Draw a really nice Venn diagram to illustrate.

Problem 451. Let A and B be sets. Let P be the logical statement $x \in A$, and let Q be the logical statement $x \in B$. Use P and Q and logic symbols ($\wedge$ for *and*, $\vee$ for *or*, $\neg$ for *not*) to translate statements about sets into logic statements:

1. $x \in A \cup B$ is ____________________

2. $x \in A^c$ is ____________________

3. $x \in B^c$ is ____________________

4. $x \in A^c \cap B^c$ is ____________________

5. $x \in (A \cup B)^c$ is ____________________

In the white space above and to the right, make a truth table for P, Q, and each of the other logical statements in the previous problem to establish that $x \in (A \cup B)^c$ is logically equivalent to $x \in A^c \cap B^c$. Compare the truth values in the columns corresponding to $x \in (A \cup B)^c$ to the Venn diagram you made above. Explain how they agree.

Problem 452. Let A and B be sets. Follow the previous exercise to use a truth table to show that $x \in (A \cap B)^c$ is logically equivalent to $x \in A^c \cup B^c$. Compare the truth table to the Venn diagram again.

Problem 453. Let D, E, and F be sets. Use one of de Morgan's laws that you showed above to establish that $(D \cup E \cup F)^c = D^c \cap E^c \cap F^c$. This proof works by rewriting, not by showing inclusion both ways. **Hint:** Let $A = D \cup E$ and $B = F$.

Problem 454. Let D, E, and F be sets. Use one of de Morgan's laws to show that $(D \cap E \cap F)^c = D^c \cup E^c \cup F^c$.

Problem 455. Let A, B, and C be sets. Use logical statements P, Q, and R and a truth table to show that $x \in A \cup (B \cap C)$ is logically equivalent to $x \in (A \cup B) \cap (A \cup C)$. Be sure to define P, Q, and R at the beginning.

Problem 456. Let A, B, and C be sets. Show that $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$ by showing inclusion both ways. When you encounter a union, use a proof by cases. For example, if you know that $x \in A \cup B$, one case is that $x \in A$, the other is that x is not in A, but $x \in B$. Organize your writing carefully to make the steps of this argument really clear.

Definition 457. Set difference. Let A and B be sets. The *set difference* $A \backslash B$ is the set $A \cap B^c$, which is all points that are in A but not in B. Draw a Venn diagram to illustrate this definition.

Definition 458. Symmetric difference. Let A and B be sets. The *symmetric difference* of A and B is the set $A \triangle B = (A \backslash B) \cup (B \backslash A)$. Draw a Venn diagram to illustrate this definition.

Problem 459. Consider again the logical statements from 451. Write a logical statement that is equivalent to $x \in A \triangle B$. Make a truth table with 4 rows, labeled 1, 2, 3, 4, and three columns, one for $x \in A$, one for $x \in B$, and the third for $x \in A \triangle B$. Draw a Venn diagram and label the regions in it 1, 2, 3, 4 so that they correspond to the truth table.

Problem 460. Let A and B be sets. Show that $A \triangle B = B \triangle A$ by showing set inclusion both ways. Draw a nice Venn diagram to illustrate.

Problem 461. Let A, B, and C be sets. Show that $(A \triangle B) \triangle C = A \triangle (B \triangle C)$ in three ways.

1. Draw separate Venn diagrams for the two sets.

2. Show set inclusion both ways.

3. Convert inclusion in $A \triangle B$, $B \triangle C$, and other sets to logical statements and use a truth table to show the equality.