Craig Zirbel's script for the first day of class

Welcome to Math 3280, Mathematical Foundations and Techniques, better known as the proof class. I teach this class whenever I can because I love teaching it. When I tell other faculty that I'm teaching the proof course, they smile and say good for you and wish that they could be teaching it. When I tell undergraduates and people from outside the department, they often recoil in horror, as if proofs are some Medieval torture or, more likely, as if proofs are something hard that you are expected to know how to do from some innate knowledge, but have never been taught.

This class is designed to put you at ease and teach you what to do. It is designed to help you develop key skills in math that will help you succeed in higher level, more abstract courses. My goal is that by working hard in this course, you will become unstoppable in your future math courses. Most of the time in undergraduate math courses, the goal is for you to learn a bunch of new concepts, new definitions, to do exercises that are a few steps away from the definitions, and to understand some harder proofs that are important in the field. Being able to read and understand definitions is very important there, and knowing the mechanics of writing proofs is sometimes all you really need to know. This class works a lot on the mechanics of writing proofs.

There are two main components to the course: learning how to read a math book on your own, and practicing the basic steps in mathematics, where we start with examples and non–examples, make a definition, consider more examples, make a conjecture, see if we can find a counterexample to the conjecture, make another conjecture, and if we can find a proof, call the result a theorem. It is all about understanding mathematical ideas and how they fit together. Proofs are a big part of that. This course will not have the same feel as algebra and calculus courses, which are heavier on calculations and don't have quite as many different ideas or different types of examples. This course is much more about skills than content.

Most people learn by doing. Me talking at the board is not the same as you learning. Most of your time in the course will be spent working on activities as a group while I go from group to group, seeing how you are doing, answering questions, and making suggestions. You will sometimes hand the activities in at the end of class so that I can read over them and make comments, then give them back to you at the beginning of the next class. There is no need to rush through the activities. Take your time, think about what you're doing. There are often multiple correct ways to do a problem. It's not question of "what I want" as the teacher, but what works. Work with your group to make sure you all understand everything along the way. You do not need to finish the activity; I always try to add extra material at the end so that no group runs out of things to do.

Outside of class, you will be reading the textbook, taking notes on what you read, and solving some exercises. You will bring your notes to class and they will be read over and returned by the start of the next class to make sure that you are doing the reading and thinking. Your first assignment is to read Chapter 1 of the textbook and turn in your notebook next Tuesday. The assignment is posted on Canvas already. I will not lecture over the material in the book; that is one of the keys to you learning how to read a book on your own.

There will be a final exam, but rather than mid-term exams, there will be half a dozen quizzes over the material in activities, once you have had a chance to get good at it.

Syllabus for Math 3280 in Fall 2017

Course description. There are two main goals for the course:

1. Improving your ability to work with definitions, examples, counterexamples, claims, and proofs.

2. Improving your ability to read a mathematics textbook on your own.

My hope is that your new abilities in these two areas will make you unstoppable in your mathematics classes. We will spend most of class time on #1. I will design activities for us to do together in class for this purpose. Most of your time outside of class will be spent on #2. Being able to read mathematics on your own is a fantastic skill. Be sure to set aside quiet time to read the textbook.

Professor and contact information. Craig L. Zirbel. My office is room 438 in the Math building. Email is the best way to reach me, zirbel@bgsu.edu. Put "3280" in the subject line. Sending me messages on Canvas does not work well, so please avoid that. If you want to reach me quickly, try my office phone number, 419-372-7466 and leave a message.

Schedule. The class meets from 1:00 to 2:15 on Tuesdays and Thursdays in room 228 in the Mathematics building. There will be no class meeting on Tuesday, October 10 or Thursday, November 23. The last day of class will be Thursday, December 7. The final exam is scheduled on Thursday, December 14 at 1:15 PM.

Office hours. You are welcome to ask me questions in my office, which is room 438 in the mathematics building. If you are having trouble finishing activities in class, I'll ask you to finish them in my office hours. I will ask you for your available times on Thursday, then I will schedule office hours. You can also make an appointment with me. The best way to arrange a time to meet is to send an email listing a few times that would work for you. I will reply with one that works for me as well.

Textbook. The textbook for the course is *How to Read and Do Proofs*, sixth edition, by Daniel Solow, 2014. The textbook is very good and you can learn a lot from it.

Graduate assistant. Johanson Berlie will be assisting in the classroom and will help to check the notebooks. Johanson is a master's student in mathematics. Ask him about graduate school.

Coursework. Here are the main things that you will be doing:

1. Written work on in–class activities.

2. Reading, taking notes, and doing exercises from each chapter in the textbook.

3. Half a dozen quizzes, very much like the work you'll already be doing in class

4. A final exam, very similar to what we have been doing all semester long

Grading. Many things you do during the semester will have a point value attached to them. The number of points will indicate their relative importance to your grade. In–class work and reading homework will count for a larger share than in most courses, while quizzes and exams will count for a lower share. All quizzes will be announced in class at least one week ahead of time. Grades will be posted on Canvas.

Attendance. Attendance and class participation are vitally important and will contribute directly to your grade. Class time is the best time to make attempts and get immediate feedback. If you cannot attend a class, notify me as soon as possible by email or phone, before class if possible. Don't even imagine that you can miss a class without letting me know. I don't particularly need to know **why**, but I do need to know.

Academic Honesty. You will work together with members of a group on in-class activities. You must work on quizzes and exams on your own. For the reading assignments, you may work together, but you must note who you worked with, and in any case, you must write your own thoughts in your notebook.

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Topics covered.

- **a.** Proofs involving even and odd, the division algorithm, multiples of 3, 5, and irrationality of $\sqrt{2}$ and others
- **b.** Proofs involving vector operations such as sum, scalar product, dot product
- **c.** Proofs involving inequalities, definition of $<$, deriving properties of $<$
- **d.** Proofs by contrapositive, process of elimination, contradiction
- **e.** Proofs requiring a construction, for example, $\forall\ a > 0, \exists$ integer $n > 0$ such that $\frac{1}{n} < a$.
- **f.** Proofs involving set equality and subset relations, union, intersection
- **g.** Proofs using mathematical induction
- **h.** Proofs about infinite unions and intersections, for example, $\bigcup_{n=1}^{\infty}[\frac{1}{n}, 1] = (0, 1]$.

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Your name: ______________________________

Background and syllabus questions – Math 3280

Do your best with these questions and turn this sheet in on Thursday. Some of them reference information from the syllabus. This assignment is worth 10 points

1. (2 points) What mathematics courses have you already taken in college? It's OK to just list the numbers, like Math 3410.

2. (2 points) Please list all the courses you are taking this semester, aside from this one.

3. (2 points) Including this semester, how many semesters have you been at BGSU?

4. How comfortable are you with the "definition, example, theorem, proof" progression in mathematics classes?

5. What kinds of experiences have you had in the past with proofs?

6. Have you ever had success reading a mathematics textbook and really learning from it? If so, please tell what course and what made it work. If not, please tell me what you think prevented you from being able to read the book.

7. (2 points) If you take the elevator to the fourth floor, do you turn right or left to get to my office?

8. (2 points) What is the most interesting thing to you on the door of my office?

9. Do you have a hard copy of the textbook that you can read? Electronic copy?

10. Do you have any interest in going to graduate school? Please explain.

11. Do you have any questions or concerns about the coursework?

12. Do you have any questions or concerns about the grading?

13. What is the most likely reason that you will miss class? I'm just curious.

14. Please let me know anything you think I should know about you. I'll read it all. Sometimes people like to tell about their hobbies, movies they like, other academic interests, clubs they're in, where they're from, etc.

Your name: ______________________________

Even and odd

Our first experience with definitions, examples, theorems, and proofs

Overview

Definitions are important to read and understand by looking at examples. Many proofs are little more than working with the definitions and rewriting things. With a bit of practice, these become very routine. This activity has you work through two definitions, a few examples, and then some proofs. Everything relies on the definitions, so keep coming back to them.

Note 1. The *integers* are positive and negative counting numbers $\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots$

Definition 2. Even. An integer n is *even* if there exists an integer k for which $n = 2k$.

Definition 3. Odd. An integer n is *odd* if there exists an integer k for which $n = 2k + 1$.

Note 4. 19 meets the definition to be odd because 19 is an integer, $19 = 2(9)+1$, and 9 is an integer.

Example 5. Check that 14 meets the definition to be even by writing $14 = 2k$ for an appropriate value of k and make sure k is an integer.

Example 6. Does -11 meet the definition to be odd? Write $-11 = 2k + 1$.

Example 7. Does 0 meet the definition to be even? Write $0 = 2k$ and check the value of k.

Example 8. Does 1.73 meet the definition to be odd? Explain.

Note 9. Suppose that m is an integer. Then $2m + 1$ is an integer and it is odd because it meets the definition to be odd. Also, $2m + 2$ is even because it is an integer and can be rewritten as $2(m + 1)$, which is of the form $2k$ where $k = m + 1$, which is an integer.

Show 10. Suppose that m is an integer. Show that $2m - 6$ is even by rewriting it until it meets the definition to be even. Connect your expressions with $=$ signs, not implication signs $\Rightarrow$.

Show 11. Suppose that m is an integer. Show that $4m + 13$ is odd by rewriting it until it meets the definition to be odd. Connect your expressions with $=$ signs.

Stop. Compare your answers to the questions above with the other people in your group before you move on. Resolve any differences in your answers.

Show 12. Suppose that m is even. Then $m = 2k$ for some integer k. Show that $m+8$ is even by rewriting it as $2k + 8$ and continuing until it is 2 times an integer. Connect your expressions with = signs.

Guided proof 13. You already know that the sum of two even numbers is even. Fill in the blanks to produce a proof of this fact, using Definition 2 of even.

a. Let m and n be even integers.

b. There exist integers j and k such that $m =$__________ and $n =$__________, by Definition 2.

c. Thus, $m + n =$ __________________________ $= 2(j + k)$.

d. This number meets the definition to be even because $j + k$ is an __________ and because $m + n$ is __________ times an integer.

e. We saw that if m and n are even, then $m + n$ is even. We made no further assumption about m and n. Thus, the sum of any two even numbers is even.

Guided proof 14. Fill in the blanks to show that the product of two even numbers is even.

a. Let m and n __________________________

b. There exist __________________________ such that $m =$__________ and $n =$__________, by __________.

c. Thus, $mn =$____ $= 2($__________$)$.

d. This number satisfies the definition to be even because ________________ is an integer and mn is __________________________.

e. We saw that ____. We made no ____. Thus, _________________________________.

Prove 15. Show that the sum of two odd numbers is even. Follow the examples above and remember that good form is critically important in proofs.

a.

b.

c.

d.

e. (Yes, you need to write this every time! It's how we make generalizations.)

Prove 16. Show that the product of two odd numbers is odd. Follow the examples above and remember that good form is critically important in proofs.

a.

b.

c.

d.

e.

Group work 17. Let m and n be even integers. Following Definition 2, some students will write $m = 2k$ and $n = 2k$ where k is an integer. Does this accurately reflect what we know about m and n? Discuss with members of your group.

Find specific values of even m and n for which $m = 2k$ and $n = 2k$ for some integer k does happen.

Find specific values of even numbers m and n for which $m = 2k$ and $n = 2k$ for some integer k does not happen.

How does proof 13 deal with the collision of notation, where both m and n need to be written as two times an integer?

Prove 18. Show that the square of an even number is even. That is, if m is even, then m^2 is even. Write a fresh proof following the models above.

Prove 19. Show that the square of an odd number is odd. That is, if m is odd, then m^2 is odd. Write a fresh proof following the models above.

Question 20. What does it mean that an integer is a multiple of 4? Give your own definition analogous to the definitions of even and odd.

Prove 21. Let m be even. Show that $m^2 + 2m + 4$ is a multiple of 4. Use good form.

Question 22. Let m be even. Can $m^2 + 2m + 4$ be a multiple of 8? Explain.

Challenges. Here are some statements that are harder to prove because they require a bit more than simply restating the definitions. See if you can make a good argument for them.

Question 23. Pretend for a minute that there is an integer m which is both even and odd. Work with the definitions to see that this really must be fantasyland.

Prove 24. Let m be an integer. We already know that m can't be *both* even and odd, but could it be neither? Show that there is no third possibility.
Here is one suggestion. 0 is even. If n is even, then $n+1$ is odd. If n is odd, then $n+1$ is even. This should cover all positive integers. Also, if n is even, then $-n$ is even, which tells us about negative integers.

Prove 25. If m is an integer and m^2 is odd, then m is odd. **Hint:** There are two cases to check, the case in which m is even and the case in which m is odd. You may wish to refer back to 18, 19, and 24.

Your name: ______________________________

Sum and dot product of 3–dimensional vectors

Overview

In Calculus III and Linear Algebra, we define vectors and work with them. They have a geometric interpretation, but here we will simply give an algebraic definition of 3–dimensional vectors and some operations on them and work with their algebraic properties. This activity illustrates proofs in which all that is needed is the definition and a "rewrite" proof, where you can work forward and backward to show a series of equalities. Notice how we often use the same definition twice in one proof, once to "unpack" and the second time to "re–pack."

Definition 26. 3–dimensional vector. A three–dimensional vector is an ordered triple $\langle a_1, a_2, a_3\rangle$, where a_1, a_2, and a_3 are real numbers. The numbers a_1, a_2, and a_3 are called *components* of the vector.

Notation 27. A 3–dimensional vector $\langle a_1, a_2, a_3\rangle$ is often denoted by a single letter with an arrow over the top, like this $\vec{a}$. When it is written like $\langle a_1, a_2, a_3\rangle$ it is said to be in *open form.* The commas and brackets are part of the definition and are important.

Definition 28. Equality of 3–dimensional vectors. 3–dimensional vectors $\langle a_1, a_2, a_3\rangle$ and $\langle b_1, b_2, b_3\rangle$ are equal if $a_1 = b_1, a_2 = b_2$, and $a_3 = b_3$. Note: The order of the numbers is important.

Definition 29. Sum of 3–dimensional vectors. The sum of 3–dimensional vectors $\langle a_1, a_2, a_3\rangle$ and $\langle b_1, b_2, b_3\rangle$ is the 3–dimensional vector $\langle a_1 + b_1, a_2 + b_2, a_3 + b_3\rangle$. We write $\vec{a} \oplus \vec{b}$ for the sum of $\vec{a}$ and $\vec{b}$, using a new symbol so we don't confuse addition of vectors with addition of real numbers.

Example 30. Is $\langle 3, 9, 12\rangle$ a 3–dimensional vector? Explain.

Is it equal to $\langle 12, 3, 9\rangle$? Explain.

Example 31. Is $\langle \sqrt{3}, \sqrt[3]{9}, \sqrt{-12}\rangle$ a 3–dimensional vector? Explain.

Example 32. Is $\langle 8, 13.35321, \pi, -7\rangle$ a 3–dimensional vector? Explain.

Example 33. Is $\langle 3 + 9 + 12\rangle$ a 3–dimensional vector? Explain.

Example 34. Is $\langle \begin{bmatrix} 6 & 0 \\ 2 & 5 \end{bmatrix}, -4, 7\rangle$ a 3–dimensional vector? Explain.

Example 35. Let x be a real number. Is $\langle \frac{14}{3}, 2 - 7x, \sqrt{16}\rangle$ a 3–dimensional vector? Explain.

Stop. Compare your answers to the questions above with the members of your group. Make sure you agree on everything.

Example 36. Calculate the sum of $\vec{c} = \langle 12, -5, 3\rangle$ and $\vec{d} = \langle 6, 4, -11\rangle$. Start by writing $\vec{c} \oplus \vec{d} = \ldots$ and write the vectors in open form next.

Calculate $\vec{d} \oplus \vec{c}$ in the same way in a separate calculation.

Show 37. You are going to show that addition of 3–dimensional vectors is commutative. Fill in the blanks. This is a "rewrite" proof. You can work forward from the top, backward from the bottom, or a bit of both. Let $\vec{a}$ and $\vec{b}$ be 3–dimensional vectors. Then,

$$\begin{aligned}
\vec{a} \oplus \vec{b} &= \langle \qquad , \qquad , \qquad \rangle \oplus \langle \qquad , \qquad , \qquad \rangle \\
&= \langle \qquad\qquad , \qquad\qquad , \qquad\qquad \rangle \\
&= \langle \qquad\qquad , \qquad\qquad , \qquad\qquad \rangle \\
&= \langle \qquad , \qquad , \qquad \rangle \oplus \langle \qquad , \qquad , \qquad \rangle \\
&= \vec{b} \oplus \vec{a}
\end{aligned}$$

We have seen that $\vec{a} \oplus \vec{b} = \vec{b} \oplus \vec{a}$. We made no further assumption about $\vec{a}$ and $\vec{b}$. Thus, for all 3–dimensional vectors $\vec{a}$ and $\vec{b}$, we know that $\vec{a} \oplus \vec{b} = \vec{b} \oplus \vec{a}$. Thus, addition of 3–dimensional vectors is commutative.

Show 38. Go back to each line of the proof above and give exactly one reason for the equality on that line at the very right side of the line. The first one is "Write in open form." Two of them are Definition 29. In the middle you will use the fact that addition of real numbers is commutative. Thus, at the heart of it, commutativity of vector addition comes from commutativity of addition of real numbers.

Show 39. Show that addition of 3–dimensional vectors is associative. Start with arbitrary 3–dimensional vectors $\vec{a}, \vec{b}$, and $\vec{c}$. Write $(\vec{a} \oplus \vec{b}) \oplus \vec{c}$ and rewrite it until it becomes $\vec{a} \oplus (\vec{b} \oplus \vec{c})$. Take small steps and write exactly one reason for each equality. Since you know what equality you need to show, you can work forward from the top, backward from the bottom, or both.
Let $\vec{a}, \vec{b}$, and $\vec{c}$ be ____.

$$\begin{aligned}
(\vec{a} \oplus \vec{b}) \oplus \vec{c} &= \\
&= \\
&= \langle (\qquad + \qquad) + \qquad , (\qquad + \qquad) + \qquad , (\qquad + \qquad) + \qquad \rangle \\
&= \langle \qquad + (\qquad + \qquad), \qquad + (\qquad + \qquad), \qquad + (\qquad + \qquad) \rangle \\
&= \\
&= \\
&= \vec{a} \oplus (\vec{b} \oplus \vec{c})
\end{aligned}$$

We have seen that ...

Stop. Compare your argument to the rest of the members of your group. Make sure that you agree on absolutely every step and every justification.

Definition 40. Scalar product for 3–dimensional vectors. Let c be a real number and let $\vec{a} = \langle a_1, a_2, a_3 \rangle$ be a 3–dimensional vector. The *scalar product* of c and $\vec{a}$ is a 3–dimensional vector defined as:

$$c\vec{a} = \langle ca_1, ca_2, ca_3 \rangle.$$

Example 41. Let $c = 3$ and $\vec{a} = \langle 7, -4, \sqrt{2} \rangle$. Calculate $c\vec{a}$, starting by writing $c\vec{a} = 3\langle 7, -4, \sqrt{2} \rangle = \ldots$.

Example 42. Calculate $\pi\langle 9, 4, 1 \rangle =$

Example 43. Calculate $(2 + \sqrt{3})\langle 5, b, c \rangle =$

Show 44. You are going to show that the scalar product is distributive over vector addition. First use the word "Let" to settle on one real number c and two 3–dimensional vectors, $\vec{a}$ and $\vec{b}$. Then start with the expression $c(\vec{a} \oplus \vec{b})$ and rewrite it three times. Then, move to the last expression and work backwards, until you meet in the middle. Provide one reason for each equality, on the right, on the same line as the equality. At the end, follow the model to conclude that you have shown distributivity in general.

Let ...

$$c(\vec{a} \oplus \vec{b}) =$$

$$= c\vec{a} \oplus c\vec{b}$$

We have seen that ...

Stop. Check over what everyone in your group has done, and make sure that you completely agree.

Show 45. Show that the scalar product is distributive over real number addition. Start with "Let." Write $(c+d)\vec{a}$ and rewrite it until it equals $c\vec{a} \oplus d\vec{a}$. Work forward from the top and backward from the bottom. Provide one reason for each equality. At the end, follow the model to conclude that this shows distributivity in general. Explain why some addition signs are $+$ and others are $\oplus$.

Definition 46. Zero vector. The vector $\langle 0, 0, 0\rangle$ is a special 3–dimensional vector, called the *zero vector*. We denote it by $\vec{0}$.

Definition 47. Additive inverse. Let $\vec{a}$ be a 3–dimensional vector, with open form $\langle a_1, a_2, a_3\rangle$. Define a new vector by $-\vec{a} = \langle -a_1, -a_2, -a_3\rangle$. It is called the *additive inverse* of $\vec{a}$.

Show 48. Let $\vec{a}$ be a 3–dimensional vector. Use a rewrite proof to show that $\vec{a} \oplus \vec{0} = \vec{a}$. This is called the *additive identity* property. It's not very exciting. Make a general conclusion.

Show 49. Let $\vec{a}$ be a 3–dimensional vector, and let $-\vec{a}$ be its additive inverse. Use good form to show that $\vec{a} \oplus (-\vec{a}) = \vec{0}$. This is called the *additive inverse* property. This is also not very exciting. Make a general conclusion.

Definition 50. Dot product of 3–dimensional vectors. The dot product of 3–dimensional vectors $\langle a_1, a_2, a_3\rangle$ and $\langle b_1, b_2, b_3\rangle$ is the real number $a_1b_1 + a_2b_2 + a_3b_3$.

Notation 51. The dot product of 3–dimensional vectors $\vec{a}$ and $\vec{b}$ is denoted $\vec{a} \bullet \vec{b}$.

Example 52. Calculate the dot product of $\vec{a} = \langle 12, -5, 3\rangle$ and $\vec{b} = \langle 6, 4, -11\rangle$ Do this by writing

$$\begin{aligned} \vec{a} \bullet \vec{b} &= \langle a_1, a_2, a_3\rangle \bullet \langle b_1, b_2, b_3\rangle \\ &= a_1b_1 + a_2b_2 + a_3b_3 \end{aligned}$$

and then substituting in the numbers. This makes the calculation just a matter of rewriting, so it is a good way to do calculations like this.

Show 53. Show that the dot product is commutative, just as multiplication of real numbers is commutative. Start with "Let". Write one expression at the top of the space below, and write your goal expression at the bottom, and then work forward and backward until you have a rewrite proof. Follow the models from previous examples, and be sure to make a general conclusion.

Example 54. Calculate $\vec{a} \bullet \vec{0}$. Is this a general result? If so, make your calculation into a general result.

Show 55. Show that the dot product is distributive over vector addition. That is, show that $(\vec{a} \oplus \vec{b}) \bullet \vec{c} = \vec{a} \bullet \vec{c} + \vec{b} \bullet \vec{c}$. Start with "Let". Write the first expression at the top, the last expression at the bottom, and then work forward and backward. Also explain why one addition symbol is $\oplus$ and the other is $+$.

Show 56. Let $\vec{a}$ and $\vec{b}$ be 3–dimensional vectors and let c be a real number. Show in general that $c(\vec{a} \bullet \vec{b}) = (c\vec{a}) \bullet \vec{b} = \vec{a} \bullet (c\vec{b})$. Use parentheses *every* time three things are multplied together, to be clear about order of operations. Since there are two equalities to show, think about how you will organize the equalities.

Your name: ______________________________

The Division Algorithm

Dividing integers with remainders will form the basis for several things we want to prove.

Overview

We would like to distribute n objects evenly among k people and find out how many are left over. We will investigate a procedure for doing this, which is called division, even though there will be no fractions in this activity and we will avoid using division symbols like / and $\div$. The Division Algorithm dates to Euclid's *Elements* from around 300 BC. Procedures that are guaranteed to work are called *algorithms* after the 9th century Persian mathematician al-Khwarizmi, who worked on procedures for arithmetic.

Example 57. You are the dealer in a card game that has 37 cards. (It's not a standard deck of cards.) There are 5 people playing, and everyone needs to end up with the same number of cards. Dealing one card to each player leaves 32 cards in your hands. Write down the numbers 37, 32, and continue until you cannot deal out any more cards evenly:

Let r denote the number of cards left over at the end, and let q denote the number of times you subtracted 5, which is also the number of cards that each person got. Notice that $0 \leq 5q \leq 37$. You see that $37 - 5q = r$, which you can rewrite as $37 = 5q + r$. Fill in q and r and write out these two equations.
$37 - 5q = r$ becomes:

$37 = 5q + r$ becomes:

Example 58. Now you're playing a card game that you have not played before, and you haven't taken the time to count how many cards are in the deck. You are the dealer again, and there are 5 people who need cards. Let n denote the number of cards in the deck. Imagine that you repeat the procedure from the previous example until you can no longer deal out cards evenly. Again, let r denote the number of cards you have left over at the end and let q denote the number of cards that each person got.

What inequalities do we know about the possible values of r?

What inequalities do we know about the possible values of q?

Write the relationship between n, 5, q, and r analogous to $37 - 5q = r$ and the expression analogous to $37 = 5q + r$.

The last expression accounts for where all of the n cards have gone; some are dealt out, some are left in your hands. Write out a sentence that explains this.

Example 59. Continuing to divide by 5, complete this sentence: Given an integer $n > 0$, there exist integers q and r where (list properties of q here) ________________________ and (list properties of r here) ________________________, such that (write the relationship between n, q, and r analogous to $37 = 5q + r$) ________________________.

Example 60. Once again, we have n cards, but now there are k people playing, where $k > 0$ is an integer. You are dealing. Describe in words how you will deal out the cards and when you will stop.

Describe in words how many cards you will have left over.

Using q to denote the number of cards each person gets and r to denote the number of cards left over, write out the relationship between n, k, q, and r, and write inequalities concerning q and r.

Stop. Compare your work to the others in your group and reconcile any differences.

Question 61. What happens when $k = 1$? What is r? What is q?

Question 62. What happens with the cards when $k = 0$? What inequalities must r satisfy? Can you satisfy $n = qk + r$? For what value(s) of q? Describe in words, and take your time to get it right, because this explains why we don't divide by 0.

Note 63. As above, suppose that n and k are integers that are greater than 0. Suppose you find integers q and r for which $n = qk + r$ and $0 \le r < k$. Suppose your friend sets out to do the same thing and finds integers q_2 and r_2 for which $n = q_2k + r_2$ and $0 \le r_2 < k$. Must it be the case that $q = q_2$ and $r = r_2$? That is, are the values of q and r *unique*? If you think about dealing n cards to k people, it's pretty clear that you and your friend will get the same values of q and r, but how can we see this without thinking about card dealing? It will take a few steps.

Show 64. Suppose that r and r_2 are integers for which $0 \le r < k$ and $0 \le r_2 < k$. Show that $-k < r - r_2 < k$. Starting and ending expressions are shown below; your goal is to provide crystal clear intermediate steps with no extra steps. Note: You can add inequalities that run the same direction; for example, if $a < b$ and $c \le d$, then $a + c < b + d$.

Suppose ① $0 \le r < k$ and ② $0 \le r_2 < k$.

Thus $-k < r - r_2 < k$.

Show 65. Suppose that n and k are integers with $k > 0$, that q, r, q_2, and r_2 are integers such that $n = qk + r$ and $n = q_2k + r_2$, and that $0 \leq r < k$ and $0 \leq r_2 < k$. Use 64 to show that $q = q_2$ and $r = r_2$. Starting and ending points are suggested below . This is not simply a rewrite proof; it requires a spark of genius to complete it, so work on scratch paper, write down everything you know including 64, work together, and be patient. Write a final argument here.

To start, note that because $n = qk + r$ and $n = q_2k + r_2$, we have $qk + r = q_2k + r_2$.

Thus $q = q_2$ and $r = r_2$.

Theorem 66. The Division Algorithm. Let n and k be integers greater than 0. Then,

1. **Existence.** There exist integers q and r for which $n = qk + r$ and for which $0 \leq r < k$.
2. **Uniqueness.** The numbers q and r are unique: there is only one way to choose q and r so that $n = qk + r$ and $0 \leq r < k$.

The number q is called the *quotient* and r is called the *remainder*. Note that there are two parts to the theorem. You have proven this theorem above in two problems. The existence part was proven in __________. The uniqueness part was proven in __________. Check that your group agrees.

Example 67. Rewrite Theorem 66 for the case where $k = 2$. Use equalities to tell the possible values of r.

Let n be an integer greater than 0. Then,

1. **Existence:** There exist integers q and r ...
2. **Uniqueness:** The numbers q and r are ...

Prove 68. Let n be an integer greater than 0. Recall the definitions of even and odd. Use the existence part of the Division Algorithm with $k = 2$ to show that n must satisfy at least one of these definitions.

Prove 69. Let n be an integer greater than 0. Use the uniqueness part of the Division Algorithm with $k = 2$ to conclude that if n is even, then it cannot be odd. Also, if n is odd, it cannot be even. Thus, each integer is even or odd, never both.

Note 70. Now we will divide negative numbers by positive numbers, with remainder.

Example 71. Start with -37, add 5 to get -32, and add 5 repeatedly, writing down the numbers you come to, until you reach a number from 0 to 4.

Count the number of 5's that you added to write $-37 = 5q + r$; you fill in q and r. Note the sign of the quotient q.

This represents division of a negative number by 5, with remainder. How does it differ from division of a positive number with remainder?

In what ways is it the same as division of a positive number with remainder?

Prove 72. Let n be an integer less than 0. Let k be an integer greater than 0. Add k to n repeatedly until you reach a non-negative number, then stop.

How can you be sure that you will ever get all the way to a non-negative number?

How can you be sure that the first non-negative number you reach will be 0, 1, $\ldots$, $k - 1$ and not a larger number like k or $k + 1$?

What inequalities do you know for r?

Example 73. Suppose $n = 0$. Show that you can write $n = kq + r$ for integers q and r where $0 \leq r < k$.

Prove 74. Let n be an integer less than or equal to 0 and let k be an integer greater than 0. In 65, we did not assume $n \geq 0$. Scrutinize your proof of 65. Will your proof work for $n \leq 0$? Explain.

Theorem 75. Completely rewrite Theorem 66 to cover positive, negative, and zero values of n. Be specific about the assumptions on n and k.

Show 76. Let n be an integer and suppose that $n = 3m + 1$ for some integer m. Clearly $r = 1$, but should we think of k in the Division Algorithm as being 3 or as being m? $k =$ _____________
Use the Division Algorithm to argue that n cannot be written as $n = 3j$ where j is an integer. Do not re-prove this part of the Division Algorithm, just cite it carefully to make your point. Thus, if $n = 3m + 1$, then n is not a multiple of 3.

Which part(s) of the Division Algorithm did you use, existence, uniqueness, or both? ____________________

Show 77. Let n be an integer and suppose that n^2 is a multiple of 3. For example, n^2 could be 36. List two more values of n^2 that are multiples of 3:
When n^2 is a multiple of 3, we would like to conclude that n is a multiple of 3. It's hard to do this directly, but it can be done indirectly. Use the Division Algorithm to write $n = 3q + r$. Start with "There exist integers ...." and be specific about the possible values of r.

Your previous step gives three cases to consider. For each case, write the expression for n, then use algebra to compute n^2 and rewrite it as a multiple of 3 plus a remainder of 0, 1, or 2.

Case 0: $r = 0$, $n = 3q$, so $n^2 =$

Case 1: $r = 1$, $n =$ _______, so $n^2 =$

Case 2:

Knowing that n^2 is in fact a multiple of 3, use your cases above to rule out one or more of the cases, and so rule out one or more of the possible values of r. Can you conclude that n is a multiple of 3? This is harder than you might think to explain clearly; work hard on it.

Show 78. Suppose n is an integer. Can n^2 be of the form $3m + 2$ where m is an integer? Use the cases you found in the previous question.

Example 79. In the triple of consecutive integers 5, 6, 7, exactly one number is a multiple of 3. Circle it. Same thing for 13, 14, 15. Circle the multiple of 3. Write down 5 more triples of consecutive integers. Do you always get a multiple of 3? _____________ Can you get more than one multiple of 3? _____________

Show 80. Let n be an integer and consider the numbers n, $n+1$, and $n+2$. Show that exactly one of these is a multiple of 3. Use three cases, $n = 3k$, $n = 3k+1$, and $n = 3k+2$, and follow the guide below.

Case 1: $n = 3k$. Then $n+1 =$ ____________________ and $n+2 =$ ____________________.
Exactly one of these is a multiple of three (circle it). Use the Division Algorithm to argue that the other two are not multiples of 3. Work hard, be clear.

Case 2: $n = 3k+1$. Then $n+1 =$ ____________________ and $n+2 =$ ____________________.
Exactly one of these is a multiple of three (circle it). Use the Division Algorithm to argue that the other two are not multiples of 3. Work hard, be clear.

Case 3: $n = 3k+2$. Then $n+1 =$ ____________________ and $n+2 =$ ____________________.

Example 81. Think about pairs of consutive even numbers, like 8 and 10, or 14 and 16. One number is a multiple of 4 (circle it), the other is not. Check five more pairs in the same way.

Show 82. Let n be even and consider the numbers n and $n+2$. Use two cases to show that exactly one of these is a multiple of 4. What two cases? You will need a new idea.

Example 83. Let $n = 1$ and compute $n^3 - n$. Let $n = 3$ and compute $n^3 - n$. Let $n = 5$ and compute $n^3 - n$.

Show 84. Let n be an odd integer. Show that $n^3 - n$ is a multiple of 24. Here you will need a few sparks of genius. Use scratch paper to brainstorm different approaches that you could try, then try the one that looks the most promising. Fortunately there are several ways to do this proof.

Your name: ______________________________

Exploring inequalities

Overview

In this activity you will explore properties of inequalities without proving the inequalities. You will use examples and counterexamples to sharpen your intuition about inequalities and their properties. Be adventurous when you look for counterexamples; don't forget negative numbers. If you find a counterexample, put a box around it. If the conclusion seems to be correct, put a big check mark next to it.

Question 85. Is the statement $7 \leq 7$ true? Explain.

Question 86. Is the statement $7 < 7$ true? Explain.

Question 87. Is the statement $6 \leq 7$ true? Explain.

Question 88. Suppose $a < 7$. Can you conclude that $a \leq 7$? This is counterintuitive for many people. Writing $a \leq 7$ does not mean, "I certify that a really could equal 7." Instead, it means "$a < 7$ or $a = 7$". Alternatively, "$a > 7$ is false." Again, is it also true that $a \leq 7$?

Question 89. Suppose $a \leq 7$. Can you conclude that $a < 7$? A good technique is to write down five numbers satisfying $a \leq 7$ and see if they also satisfy $a < 7$. Try to find a counterexample. If you find one, put a box around it, otherwise put a check mark.

Question 90. Suppose $a < b$ and $b \leq c$. Is it guaranteed that $a \leq c$? Work with examples if it helps, and look for a counterexample. One concrete counterexample would be enough.

Question 91. Suppose $a < b$ and $b \leq c$. Is it guaranteed that $a < c$? Work with examples if it helps, and look for a counterexample. One concrete counterexample would be enough.

Question 92. Suppose $a \leq b$ and $b \leq c$. Is it guaranteed that $a < c$? Work with examples if it helps, and look for a counterexample. One concrete counterexample would be enough.

Question 93. Suppose $a > 12$. Consider the inequality $-a > -12$. Write down five numbers satisfying $a > 12$ and check whether or not they satisfy $-a > -12$. Look for a counterexample. If you find a counterexample, put a box around it. If the result is OK, put a check mark.

Question 94. Suppose $a > 12$. Consider the inequality $-a < -12$. Write down five numbers satisfying $a > 12$ and check whether or not they satisfy $-a < -12$. If you find a counterexample, put a box around it.

Question 95. Suppose $c < 5$. Use examples to check whether $c^2 < 25$. If you find a counterexample, put a box around it and look for an additional condition on c which would guarantee $c^2 < 25$.

Question 96. Suppose $c < 7$ and $d \leq 8$. Use examples to check whether $c + d < 15$. If you find a counterexample, put a box around it and look for an additional condition on c and d which would guarantee $c + d < 15$.

Question 97. Suppose $c < 3$ and $d \leq 4$. Use examples to check whether $cd < 12$. If you find a counterexample, put a box around it and look for an additional condition on c and d which would guarantee $cd < 12$.

Question 98. Suppose $a < b$ and $c > 0$. Use examples to check whether $ac < bc$, as above.

Question 99. Suppose $a \leq b$ and $c \leq d$. Use examples to check whether $a + c < b + d$, as above. If you find a counterexample, box it and look for an additional condition to guarantee $a + c < b + d$.

Question 100. Suppose $a < b$ and $c \leq d$. Use examples to check whether $ac < bd$. If you find a counterexample, put a box around it and look for an additional condition which would guarantee $ac < bd$.

Question 101. Suppose $a \leq b$. Use examples to check whether $a^2 < b^2$. If you find a counterexample, put a box around it and look for an additional condition which would guarantee $a^2 < b^2$.

Question 102. Suppose $a < b$. Use examples to check whether $\frac{1}{a} < \frac{1}{b}$. If you find a counterexample, put a box around it and look for an additional condition which would guarantee $\frac{1}{a} < \frac{1}{b}$.

Question 103. Suppose $a \leq b$. Use examples to check whether $\frac{1}{a} \geq \frac{1}{b}$. If you find a counterexample, put a box around it and look for an additional condition which would guarantee $\frac{1}{a} \geq \frac{1}{b}$.

Show 104. If $1 < p$, cite a general result from above to conclude that $5 < 5p$.

Show 105. Suppose p is an integer with $-5 < 5p < 5$. Without dividing by 5, check whether or not it is possible that $p = 1$, $p > 1$, $p = -1$, $p < -1$. Conclude that $p = 0$. If you use properties of inequalities, cite the number from above that you are using.

Question 106. Find the smallest integer $n > 0$ for which $\frac{1}{n} < 0.12$.

Question 107. Find the smallest integer $n > 0$ for which $\frac{1}{n} < 0.037$.

Question 108. Find the smallest integer $n > 0$ for which $\frac{1}{n} < 0.00026$.

Question 109. Let $a > 0$. Describe a procedure for finding the smallest integer $n > 0$ for which $\frac{1}{n} < a$.

Your name: ______________________________

Contrapositive, process of elimination, contradiction

Overview

A central part of mathematics is identifying logical statements and showing which statements imply other statements. This activity introduces logical statements and four ways to prove implications: direct proof, contrapositive, the process of elimination, and proof by contradiction. You may have seen this same material presented using truth tables, but this particular activity specifically avoids truth tables.

Definition 110. Logical statement. A *logical statement* is a sentence that is either true or false. Sometimes logical statements have an unknows such as n, but for each value of n, the statement is either true or false. We often label logical statements with capital letters.

Example 111. For each of the logical statements below, write its truth value T or F. If the sentence is not a logical statement, explain why not.

a. P: 18 is even

b. Q: 19 is even

c. R: 19 is a large number

d. S: 13 is prime

e. T: $2^5 - 1$ is prime

f. U: $\sqrt{2}$ is rational

Example 112. For each of the logical statements below, give five values of the integer n for which the statement is true, if possible, and five values of the integer n for which the statement is false, if possible.

a. $2^n - 1$ is prime	T: $n =$	F: $n =$
b. n is a perfect square	T: $n =$	F: $n =$
c. n^2 is a prime number	T: $n =$	F: $n =$
d. $n^2 + 3n + 1$ is odd	T: $n =$	F: $n =$

Definition 113. Conjunction, logical and. The *conjunction* of two logical statements P and Q is a new logical statement denoted $P \wedge Q$ which is true when both P and Q are true, and false otherwise. It is usually read as "and". The symbol $\wedge$ is only used between logical statements. We don't write "Suppose $x > 3 \wedge \leq 9$" but rather "Suppose $x > 3 \wedge x \leq 9$."

Definition 114. Disjunction, logical or. The *disjunction* of two logical statements P and Q is a new logical statement denoted $P \vee Q$ which is true when P is true, when Q is true, or when both are true, but false when both are false. The symbol $\vee$ is only used between logical statements. Instead of writing "Suppose $n = 3 \vee 5$," you could write "Suppose $n = 3 \vee n = 5$."

Exercise 115. Give the truth value T or F of each of the statements, using definitions in Example 111.

a. $P \vee Q$

b. $Q \wedge S$

c. $P \wedge Q \wedge T$

d. $P \vee (Q \wedge U)$

Definition 116. Negation. The *negation* of a logical statement P is a new statement denoted $\neg P$ which is true when P is false and false when P is true. $\neg P$ is read as "not P".

Exercise 117. Give the truth value T or F of each of the statements, using definitions in Example 111.

a. $\neg Q$

b. $P \wedge \neg Q$

c. $Q \vee \neg S$

Definition 118. Implication. For logical statements P and Q, we say that P implies Q and write $P \to Q$ if P being true guarantees that Q is true.

Exercise 119. For each line below, identify the statement corresponding to P and the statement corresponding to Q in the implication $P \to Q$. In every case, n is an integer.

a. If n is even, this implies that n^2 is even.

b. If n^2 is odd, then n is odd.

c. If n is odd, then $n^3 - n$ is a multiple of 24.

Definition 120. Direct proof. A *direct proof* of an implication is where we start with the statement P and use the information in it together with a series of valid logical steps to show that Q is true. This establishes that $P \to Q$. We have seen a number of direct proofs, including proofs by rewriting.

Prove 121. Let n be an integer. Consider $S : n$ is even and $T : n^2 + 6n + 7$ is odd. Write a direct proof that $S \to T$. Start with "Let n be an integer. Suppose that n is even."

Example 122. Let n be an integer. Consider $A : n^2$ is even and $B : n$ is even. Try to write a direct proof that $A \to B$. If you don't see a way to do it, you can stop trying.

Definition 123. Contrapositive. Here is another way to think about showing $P \to Q$. You need to be sure that it never happens that P is true but Q is false. You can do this by showing that whenever Q is false, P is also false. In other words, show that $\neg Q \to \neg P$. This is called *proof by contrapositive*. It may be easier to find a direct proof that $\neg Q \to \neg P$ than it is to show $P \to Q$.

Exercise 124. Let n be an integer. Consider $A : n^2$ is even and $B : n$ is even.
State $\neg B$:
State $\neg A$:
Show that $\neg B \to \neg A$, starting with: "Let n be an integer. Suppose that n is odd."

State what you have shown: if n^2 is even then ____________________________

Exercise 125. Let n be an integer. Consider $P : n^2 + 8n + 9$ is even and $Q : n$ is odd.
State $\neg Q$:
State $\neg P$:
Show $\neg Q \to \neg P$:

State $P \to Q$:

Exercise 126. Let n be an integer. Consider $S : n^2$ is a multiple of 3 and $T : n$ is a multiple of 3.
State $\neg T$:
State $\neg S$:
By the Division Algorithm, there are two ways that $\neg T$ can happen. Call these Case 1 and Case 2. Prove that in either case, they imply $\neg S$.

Thus, if n^2 is a multiple of 3, we know that n is a multiple of 3.

Definition 127. Rational. A real number is said to be *rational* if it can be written as $\frac{a}{b}$ where a and b are integers and $b \neq 0$; this is the quotient of two integers.

Definition 128. Irrational. A real number is said to be *irrational* if it cannot be written as the quotient of two integers.

Exercise 129. Let $r \neq 0$. Show that if r is irrational, then $\frac{1}{r}$ is irrational. Write a proof by contrapositive.

Exercise 130. Suppose that a is an irrational number. Show that $5a$ is irrational.

Exercise 131. Let b be a rational number not equal to 0. Suppose that a is an irrational number. Show that ba is irrational. There are three logical statements here. The one about b is just context, one is P and one is Q in the implication $P \to Q$. Identify P and Q, then show that $P \to Q$.

Note 132. Showing that a statement is false. Sometimes we want to show that a statement P is false. Here is a method to do that. Pretend for a minute that P is true, and use rules of algebra, previously-proven results, theorems, etc. to make a series of logical implications $P \to Q$, $Q \to R$, $R \to S$, $S \to T$ until you arrive at a statement T that you know to be false. Then you have shown that $P \to T$. But since T is false, then you can be certain that P is false. It can be difficult to identify a specific statement that is false; work hard to accomplish this, because it can really help to clarify the proof.

Note 133. When proving that statement P is false, we start by writing "Pretend for a minute that P is true." We do not really believe that P is true, but it helps to pretend that it's true as you make a chain of implications starting from P.

Prove 134. Let n be an integer. Consider the statement P : n is both even and odd. Complete the following proof that P is false.
Pretend for a minute that P is true. Then $n = 2k$ and $n = 2j + 1$ for ...

But k and j are integers, and so $k - j$ is an integer, so $k - j = \frac{1}{2}$ is false. Thus, P must be false.

Prove 135. Consider the statements P : n is a positive integer and Q : n^2 is 6. Prove that $P \wedge Q$ is false. Your calculator will tell you that $\sqrt{6} = 2.4494...$, but make an argument that does not rely on numerical approximations of square roots.
Pretend for a minute that n is a positive integer and $n^2 = 6$. Then $2^2 < n^2 < 3^2$.

Thus $2 < n < 3$. But this is false because . . .

Note 136. In the previous problem, we saw that $P \wedge Q$ is false, but we cannot say which of the two statements is false. If n is an integer, then $Q : n^2 = 6$ is false. If $n^2 = 6$, then n is not an integer, so P is false. The next example shows a slightly different approach where you can make a solid conclusion.

Example 137. Let n be an integer and suppose that n is even. Prove that the statement Q: n is odd is false.
Pretend for a minute that Q is true. The argument in 134 leads us to a false statement. Thus, Q must be false.
What is different here is that before we encounter the statement Q, we have supposed that n is an integer and n is even, both of which can be true. In that context, we can see that Q is false.

Guided proof 138. Let R be the statement that $\sqrt{2}$ is rational. You will show that R is false, by making a series of deductions that lead to a conclusion that is known to be false.
Pretend for a minute that R is true, that is, that $\sqrt{2}$ is rational.
Then there exist integers p and q for which $\sqrt{2} = \frac{p}{q}$, and we can arrange it so that p and q have no common factors. (If they had common factors, we would cancel common factors until they had no common factors.) Using algebra, $2q^2 = p^2$. Thus, p^2 is ____________________. Thus, p is ______________ by Exercise ______________ earlier in this activity, and so can be written as $p =$ ______________ for some ______________.

Using algebra, $q^2 = \frac{1}{2}p^2 =$ __________________________, and so q^2 is _____________. Thus q is _____________.

But we know that this is false, because _______________________________________. Thus, the statement R is false. Note that because $\sqrt{2}$ is not rational, it must be irrational.

Prove 139. Suppose there are 20 kids playing musical chairs, with 19 chairs. When the music stops, at least one kid will not have a chair to sit on. Show that the statement M : "all kids have a chair to sit on by themselves" is false.
Pretend for a moment that M is true. Let k denote the number of kids and let c denote the number of chairs. (What is the relationship between k and c?)

Definition 140. Composite. An integer $n > 1$ is *composite* if it can be written as $n = ab$ where a and b are integers with $1 < a, b < n$.

Definition 141. Prime. An integer $n > 1$ is *prime* if it is not composite, that is, its only non-negative integer factors are 1 and itself.

Exercise 142. List the prime numbers less than 20.

Exercise 143. Write 24 as a product of prime factors.

Prove 144. Let S be the statement that there are finitely many prime numbers. Show that S is false by filling in the blanks.
Pretend for a minute that S is true, so there are only finitely many __________________________. Let k be how many prime numbers there are, and call the prime numbers $p_1, p_2, p_3, \ldots, p_k$. Consider the number $n = p_1p_2p_3 \cdots p_k + 1$. Then n is larger than all prime numbers and so n is not _____________, so it must be ____________________. By considering factors of n, at least one factor must be a _____________ number. But by the __________________________ part of the __________________________, n is not a multiple of p_1, n is not a multiple of p_2, etc. Thus, n is not a multiple of a prime number. We have arrived at the false statement that n is composite and yet has no prime factors. Thus, statement S must be false, and now we know that there are infinitely many prime numbers.
Definition 145. Process of elimination. Consider logical statements P, Q, and R and suppose we know that $P \vee Q \vee R$ is true. Suppose we show that Q is false and R is false. We can conclude that P is true. Hopefully that is obviously true. If not, one can use truth tables to make it extra clear, which is one place where truth tables really help.

Prove 146. Let n be an integer. Suppose that n^2 is a multiple of 5. Use the Division Algorithm to produce statements P, Q, R, S, and T of the form $n = 5k + r$ for different values of r, so that $P \vee Q \vee R \vee S \vee T$ is true. Then show that Q, R, S, and T are false, and conclude that P is true, so that n is a multiple of 5. Do a really good job on these cases, because you'll use them a few more times in the next questions.
By the Division Algorithm, we can write n as _________________________________
Let $P : n = 5k$, let $Q : n = 5k + 1$, ...
Assume Q is true, then $n^2 =$ ...

Example 147. List the first 11 perfect squares, $0, 1, 4, 9, \ldots$.

Prove 148. Use the cases from 146 to argue that perfect squares of integers can only end in the decimal digits 0, 1, 4, 5, 6, 9, and never in 2, 3, 7, 8. If that doesn't make sense, let n be an integer and write $n = 10q + r$ and check n^2 in 10 cases.

Prove 149. The result in 146 can also be shown with a contrapositive proof.
Let n be an integer. Consider A: n^2 is a multiple of 5 and B: n is a multiple of 5. Clearly state $\neg B$ and $\neg A$. Rewrite $\neg B$ in terms of the cases from 146, and then argue that each case implies $\neg A$. You may use the cases from 146 without rewriting them.

Definition 150. Proof by contradiction. Another way to prove that a statement P is true is to pretend for a minute that $\neg P$ is true and argue to a false statement, conclude that $\neg P$ is false, and thus establish that P is true. It may be easiest to think of this as a proof by the process of elimination: we know that $P \vee \neg P$ is true, and we are eliminating $\neg P$.

Note 151. When using proof by contradiction that P is true, we will write "Pretend for a minute that $\neg P$ is true" and find a chain of implications resulting in a statement that is false. Sometimes it is difficult to put your finger on what specific statement is false, but you realize that two or more statements are true, but cannot be true at the same time. That is the nature of a contradiction. If you can put your finger on a specific statement that is false, that is better.

Guided proof 152. Let L be the statement "There is a largest integer." Prove that L is false.
Pretend for a minute that L is true. Write n for the largest integer. Consider $n + 1$.

Thus, L is false.

Prove 153. Show that $\sqrt{5}$ is irrational, following the proof that $\sqrt{2}$ is irrational. Start by pretending for a minute that $\sqrt{5}$ is rational and argue to a false statement or a contradiction.

Your name: ______________________________

Special words in mathematics

A short guide to how to use certain words

Overview

Using the right words in the right situation shows that you understand the logical structure of what you are writing. It also makes it clear to the reader what you mean.

Definition 154. Let. The word "Let" has two main uses in mathematics, both of them in proofs.

a. The word "Let" is used to introduce a new variable or other object and give it a specific value. This is often used in proofs where you need to show the existence of some object, but is also used in many other contexts.

- Let $f(x) = \sin(x) + \cos(x)$.
- Let $x = \frac{-b+\sqrt{b^2-4ac}}{2a}$.
- Let $n = \frac{1}{a} + 1$, rounded up to the next integer.

The word "set" can also be used here in place of "let."

b. The word "Let" can also be used to introduce a new variable having a particular property:

- Let n be even.
- Let $a > 0$.
- Let $r \in \mathbb{Q}$.
- Let $0 < x < 1$.

These statements cause the variable to take on a specific value. We don't know the specific value, only that it has the property we make it have. This is very useful when writing a generic proof that is supposed to work for all values of the variable having the property.

A very important point is that whenever you use the word "Let", you change the value of the variable. So for example, if you start a proof by saying "Let $a > 0$," then a becomes a specific real number. Based on this a, you may construct other variables like b which depend on a. For example, $b = 1/a$. Later in the same proof, if you say, "Let $a \geq 1$," then this changes the value of a, and any variable that depends on a will lose its connection. Instead, you may want to think of $a \geq 1$ as a case to consider and use the word "suppose."

Definition 155. Suppose. The word "Suppose" has two main uses in mathematics.

a. The word "Suppose" can be used to introduce cases in a proof, for example, to restrict consideration of an already–introduced variable to a smaller range of variables. Using "suppose" this way does not introduce a new variable or change the value of the variable.

- Let $a > 0$. Case 1. Suppose $a \geq 1$. ... Case 2. Suppose $0 < a < 1$. ...
- Let $x \in [3, 7]$. Case 1. Suppose that $x < 4$. ... Case 2. Suppose that $x \geq 4$.

b. The word "Suppose" is also used to introduce a logical statement at the beginning of a theorem or proof.

- Suppose that the function f is continuous on the interval $[a, b]$.
- Suppose that n is an odd integer.

Definition 156. Assume. The word "Assume" is most often used to introduce a proof by contradiction. Because it is helpful to know that a proof by contradiction is coming, it is helpful to use familiar wording. You can say things like:

a. Assume that $\sqrt{2}$ is rational.

b. Assume for the sake of contradiction that $\sqrt{2}$ is rational.

c. Pretend for a minute that $\sqrt{2}$ is rational. (Recommended in this class, but unconventional outside this class.)

Remark 157. The word "any" is ambiguous and it is best to avoid using it. Sometimes it means "for all" and sometimes it means "there exists" or "for some" and sometimes you just can't tell. Consider these examples:

a. Let $f(x) = \sin(4x)$ for any real number x. "any" means: ______________

b. Is it true that $\sin(x) = x$ for any value of x? "any" means: ______________

c. Is it true that $\sin^2(x) + \cos^2(x) = 1$ for any x? "any" means: ______________

d. Let a be a real number. Suppose that $n > a$ for any non-negative integer n. This would be true for all / for some (circle one choice) non-negative integers n if $a = -5$. But it would be true for **some** non-negative integer n if $a = 10$. The meaning is ambiguous.

Example 158. A badly told story. Amanda was a sophomore in college. One day after class, she went to study in the park. She walked past a family at a picnic table and headed toward a shady tree. Barney said, "This next test is going to be really hard!" Amanda told Barney to relax.

Who is Barney?!? We haven't been introduced. Does he know Amanda? Were they walking together? Were they meeting to study?

Writing a proof is a bit like tellling a story. It's important to introduce the variables you use. Don't let a variable barge in without introduction like Barney did. Make sure to relate a new variable to existing variables the first time it enters the story.

Exercise 159. Rewrite the story about Amanda and Barney.

Example 160. Another story. Rex, Bill, and Ted went for a walk. Bill said, "Rex, I see a cat over there. Please chase it away." Rex barked at the cat, the cat ran away, Bill squeaked in delight, and Ted whinnied in amusement.

In this story, names were introduced, but the names did not suggest that Rex, Bill, and Ted are animals, and we the reader is left to try to figure out what types of animals. What is Rex? _____________ Bill? _____________ Ted? _____________ When you introduce variables, tell what type of thing each variable represents, whether an integer, rational number, real number, vector, function, set, matrix, etc.

Exercise 161. Rewrite the story about Rex, Bill, and Ted.

Example 162. Fill in the blanks in this proof with the appropriate words and/or definitions.

Show that for all real numbers $a > 0$, there is an integer n with $\frac{1}{n^2} < a$.

Let $a > 0$.

Case 1. _____________ that $a \geq 1$. ____________________ Then $\frac{1}{n^2} = \frac{1}{4} < 1 \leq a$, and so $\frac{1}{n^2} < a$.

Case 2. _____________ that $a < 1$. Let n be the next integer larger than $\frac{1}{\sqrt{a}}$. Then $n > \frac{1}{\sqrt{a}}$. Squaring both sides, $n^2 > \frac{1}{a}$. Taking reciprocals, $\frac{1}{n^2} < a$, as desired.

In each case, we have shown the existence of an integer n with the desired property. Thus, _____________ $a > 0$, there is an integer n with $\frac{1}{n^2} < a$.

Your name: ______________________________

Integer-valued functions

Overview

It is often helpful to replace a real number with a nearby integer. There are standard functions from the real numbers to the integers to do this, and this activity helps you learn some of their properties and uses.

Exercise 163. Think of a function f with the following properties: First, $f : \mathbb{R} \to \mathbb{Z}$, meaning that the input to f is a real number, and the output from f will always be an integer. Second, f takes on the following values:

$$\begin{array}{rcrcrcr} f(0.5) & = & 0 & \qquad & f(-3.2) & = & -4 \\ f(0.9) & = & 0 & & f(-10) & = & -10 \\ f(1) & = & 1 & & f(-9.5) & = & -10 \\ f(1.1) & = & 1 & & f(18.2) & = & 18 \end{array}$$

Humans have an amazing ability to generalize from examples like this. Describe what f does to a generic input number x:

We will need the letter f for other functions. Common notation in mathematics for $f(x)$ is $\lfloor x \rfloor$. In programming languages, people write floor(x). In what way does the word floor describe what f does?

Exercise 164. Think of a function g with the following properties: First, $g : \mathbb{R} \to \mathbb{Z}$. Second, g takes on the following values:

$$\begin{array}{rcrcrcr} g(0.5) & = & 1 & \qquad & g(-3.2) & = & -3 \\ g(0.9) & = & 1 & & g(-10) & = & -10 \\ g(1) & = & 1 & & g(-9.5) & = & -9 \\ g(1.1) & = & 2 & & g(18.2) & = & 19 \end{array}$$

Describe what g does to a generic input number x:

Common mathematical notation for $g(x)$ is $\lceil x \rceil$. In programming languages, people write ceil(x), where ceil is short for ceiling. In what way does the word ceiling describe what g does?

Exercise 165. For what values of x do we have ceil(x) = floor(x)+1?

Are there any values of x for which the equality does not hold?

Exercise 166. What exactly is it about the definition of floor(x) that makes $x -$ floor(x) < 1?

Exercise 167. The inequalities listed below might be true for all x, or they might fail for some values of x. If one sometimes fails, give a specific example of x where it fails, called a *counterexample*, and calculate the quantities in the inequality to explain the counterexample. If an inequality appears to be true for all x, prove it this way: "Let x be a real number. Consider two cases. Case 1: Suppose x is an integer. Case 2: Suppose x is not an integer." Show that the inequality is correct in both cases.

a. $x \le \text{ceil}(x)$

b. $x < \text{ceil}(x)$

c. $0 \le \text{ceil}(x) - x < 1$ (check both inequalities)

Exercise 168. Let b be a real number. Let $n = \text{ceil}(b) + 1$. Show that $n > b$; cite a previous result or give a reason for each step.

$$\begin{array}{rcll} n & = & \text{ceil}(b) + 1 & \text{given} \\ & > & \text{ceil}(b) & \text{because } \underline{\hspace{6cm}} \\ & \ge & b & \text{because } \underline{\hspace{6cm}} \end{array}$$

Question 169. Let x be a real number. What is special about the integer floor$(x+1)$?

Question 170. Solve the equality $\frac{1}{x} = 0.07$ for x.

Question 171. Let $x > 0$. Solve the inequality $\frac{1}{x} < 0.07$ for x.

Question 172. Find the smallest integer $n > 0$ for which $\frac{1}{n} < 0.07$.

Question 173. Find the smallest integer $n > 0$ for which $\frac{1}{n} < 0.004$.

Question 174. Find the smallest integer $n > 0$ for which $\frac{1}{n} < 0.00009$.

Question 175. Let $\varepsilon > 0$. Describe a procedure for finding the smallest integer $n > 0$ for which $\frac{1}{n} < \varepsilon$.

Your name: ______________________________

Construction of an object with a property

Overview

Many proofs in mathematics require us to show that an object having a certain property exists. In many cases, you can tell exactly how to make, or construct, the desired object.

Prove 176. Let $x > 0$ be a real number.

a. Construct a real number a with $0 < a < x$. That is, write a formula to compute a in terms of x, then check that $0 < a < x$.

b. When $x = 0.0006$, what number does your formula produce for a?

c. Construct real numbers a and b with $0 < a < b < x$.

d. When $x = 0.0006$, what numbers does your formula produce for a and b?

e. Find two very different ways to construct a real number c satisfying $x < c$.

Prove 177. Given a real number x, describe a procedure that results in one integer n with $n > x$.

What result does your procedure give for $x = 13.1$? $x = 18$? $x = -22.2$?

Prove 178. **a.** Find the smallest integer n with $n^2 > 17$.

b. Find the smallest integer n with $n^2 > 177$.

c. Describe a procedure for finding the smallest integer n with $n^2 > 1777$.

d. Given an integer $k > 1$, describe a short procedure to find the smallest integer n with $n^2 > k$.

e. Describe a procedure for finding the largest integer n with $n^2 \leq 1777$.

Prove 179. Given a real number a, construct an integer n with $a < n \leq a + 1$. You may wish to consider two cases: Case 1, suppose a is an integer. Case 2, suppose a is not an integer.

What value of n does your procedure give when $a = 13.1$? $a = 18$? $a = -22.2$?

Prove 180. Let a and b be real numbers with $a < b$. Show that there exists a real number c with $a < c < b$. Describe how to construct c and then work with inequalities to show that it works.

Prove 181. Let $a > 0$. Show that there exists an integer $n > 0$ with $\frac{1}{n} < a$. Describe how to construct n and then show that it works.

Prove 182. Suppose that $x = 4.32\overline{764}$, meaning that the decimal expansion has 764 repeating forever. Show that x is a rational number. Start by explaining what needs to exist and what properties it needs. **Hint:** Look at $100000x - 100x$.

Prove 183. Suppose that x is a real number with a repeating decimal expansion (of d repeating digits starting after the pth decimal place). Show that x is a rational number.

Prove 184. Let a and b be real numbers with $b - a > 1$. Show that there exists an integer n with $a < n < b$. Describe how to construct n and then show that it works.

Prove 185. Let a and b be real numbers with $a < b$. Show that there exists a rational number r with $a < r < b$. Describe how to construct r and then show that it works.

Your name: ______________________________

Introduction to set theory

This activity introduces sets, ways to write them, and the relations between them.

Overview

Many important ideas in mathematics are expressed using sets, and many proofs come down to dealing with sets in the right way. This activity was co-authored by Johanson Berlie.

Definition 186. Set. A set is a well-defined collection of distinct objects. These objects are called **elements** or **members** of the set.

Example 187. Consider all students registered for at least one credit hour at this university this semester. The objects are students, and there is a clear criterion for deciding which students we have in mind, so the collection is well defined. It's OK that we don't have a list of the students, we can still talk about the set.

Example 188. Consider all the days last spring when it was somewhat gloomy. Here, the objects are days that were somewhat gloomy. Since 'somewhat gloomy' does not have a precise meaning, this collection is not a set.

Exercise 189. Which of the following are sets? Consider whether the set is well defined and explain your thinking in each case. If the set is small enough, list out its elements.

a. Your Facebook friends right now

b. Your high school friends on high school graduation day

c. All the stars in the Milky Way galaxy right now

d. All the small stars in the Milky Way galaxy right now

Remember to explain your thinking.

e. The ten most important characters in the Harry Potter books

f. The days in a year with exactly 20mm of rainfall

g. The days in 2020 in Bowling Green, Ohio with less than 20mm of rainfall

h. English letters that are vowels

i. Planets in our solar system

Remember to explain your thinking and list the elements of small sets.

j. Construct your own example or non-example of a set and explain why it is or is not a set.

Definition 190. Elements of a set. If A is a set and an object x is an element of A, we write $x \in A$. The symbol $\in$ looks a bit like a letter E and stands for "is an element of." Thus $x \in A$ should be read as "x is an element of A." If x is not an element of A, we write $x \notin A$ and say that x is not an element of A. Sets are usually denoted by capital letters and their elements by lower case letters.

Definition 191. Tabular form. We represent a set in **tabular form** by listing out its elements, separated by commas and enclosed in curly braces $\{ \}$. The order in which we list the elements does not matter, only which objects are elements of the set. If the set has too many elements to list, establish a pattern and use $\ldots$.

Example 192. If the set C consists of the primary colors, we could write $C = \{\text{red, blue, yellow}\}$ or $C = \{\text{yellow, red, blue}\}$, because the order in which we list the elements doesn't matter.

Remark 193. For convenience or from lack of attention, sometimes an element is repeated when listing in tabular form; this does not change the actual elements of a set. Thus, the tabular forms $\{a, b\}$ and $\{a, b, b, a\}$ both refer to the same set.

Example 194. The set of Fibonacci numbers can be written $F = \{1, 1, 2, 3, 5, 8, 13, \ldots\}$ or as $F = \{1, 2, 3, 5, 8, 13, \ldots\}$. The first way is how people usually write the Fibonacci numbers, the second way recognizes that since a set is a collection of distinct elements, listing 1 twice doesn't change the set. Be careful with the idea of establishing a pattern: If someone doesn't know what the Fibonacci numbers are, they might not be able to tell you the next element of the set.

Exercise 195. List out the next five elements of the set of Fibonacci numbers.

Definition 196. Set-builder form. We represent a set in **set-builder form** by stating the properties which its elements must satisfy.

Example 197. If the set C consists of the primary colors, then we can write $C = \{x : x \text{ is a primary color}\}$. In this notation we wrote x as a temporary name for an element of the set, and wrote the condition that x needs to satisfy after the colon character :. Sometimes people use a vertical line | instead of a colon.

Exercise 198.

a. Express the set $A = \{x : x \text{ is a "home row" character on a keyboard}\}$ in tabular form.

b. Express the set $B = \{A, L, G, E, B, R\}$ in set-builder form.

c. Express the set $P = \{2, 3, 5, 7, 11, \ldots, 97\}$ in set-builder form.

d. Express the set $T = \{x : x \text{ is a power of } 2\}$ in tabular form.

e. Suppose that $R = \{x | x \text{ is a zero of } f(x) = 5x^3 - 2x^2 + 7x - 1\}$. Suppose that $u \in R$. Write the equation that we know that u satisfies.

f. Suppose that $M = \{m : m \text{ is a multiple of } 7\ \}$. Let $k \in M$. Without using the words "multiple" or "divides" or "divisible," what do we know about k?

Definition 199. Empty set. A set which contains no elements is called the **null set** or **empty set**. We denote it by the symbol $\varnothing$.

Example 200. The set of real-valued solutions of the equation $x^2 + 1 = 0$ is empty, since there is no real number that solves the equation.

Definition 201. Singleton set. A set which contains only a single element is called a **singleton set**.

Example 202. The set of mountains on earth with height over 29,000 feet is a singleton set, since Mt. Everest is the only element of the set.

Exercise 203. For the following questions, identify the set and describe it using the definitions above. That's two things to do.

a. The set of all mountains in the state of Ohio above 2000 feet in height. This might require an internet search.

b. If Jill has classes on Mondays, Wednesdays and Fridays and has work on Wednesdays and Saturdays. Consider the set of days on which Jill has both work and classes.

c. Suppose we draw two straight lines in the plane and consider the intersection of the two lines, that is, the set of all points that are on both lines. Can this set be empty? If so, draw a picture. Can this set be a singleton? If so, draw a picture. Can this set be anything else? If so, draw a picture and describe it.

Definition 204. Subset of a set. Let A and B be sets. If every element of A is also an element of B, then A is said to be a **subset** of B and we write $A \subseteq B$.

Remark 205. If A is a set, then $A \subseteq A$, because every element of A is also a member of A.

Remark 206. The notation $\subseteq$ is much like the inequality symbol $\leq$ for real numbers. We know that $3 \leq 5$ and writing $x \leq 5$ means that x could be any number up to and including 5. It is always true that $x \leq x$ when x is a real number. When you see the statement $A \subseteq B$, think that A is a subset of B, and possibly equal to B.

Remark 207. The null set is a subset of every set: Suppose A is a set. Then it is true that every element of $\varnothing$ is a member of A. People sometimes say this is "vacuously true," because $\varnothing$ doesn't have any elements to bother with. We write $\varnothing \subseteq A$.

Definition 208. Not a subset. When A is not a subset of B, we write $A \not\subseteq B$. This happens when there is an element of A that is not an element of B.

Example 209. If $A = \{\text{green, yellow, red, black, dog, cat, mouse}\}$ and $B = \{\text{dog, cat}\}$ then $B \subseteq A$. However, $A \not\subseteq B$ because, for example, mouse $\in A$ but mouse $\notin B$.

Exercise 210. **a.** If A is the set of all cars manufactured by a Japanese car company and B is the set of all Toyota sedans, then what is the relationship between A and B?

b. Let M be the set of people you have communicated with on social media in the last week, and let C be the set of people you are taking a class with now. Is $M \subseteq C$? If not, give the first name of one person in M but not in C. Is $C \subseteq M$? If not, name one person in C but not in M.

c. Let V be the set of people who voted in the last US presidential election, and let C be the set of US citizens at the time. Is $V \subseteq C$? Under what condition would we have $V \nsubseteq C$?

Definition 211. Proper subset. If every element in a set A is also a member of a set B, and yet B contains at least one element that is not in A, then A is called a **proper subset** of B and we write $A \subset B$. Sometimes people write $A \subsetneq B$. When you want to show that $A \subset B$, you need to check that $A \subseteq B$ and that A and B are not equal.

Remark 212. The notation $\subset$ is much like the strict inequality symbol $<$ for real numbers. We know that $3 < 5$ is true and writing $x < 5$ means that x could be any number up to but not including 5. It is never true that $x < x$ when x is a real number. When you see the statement $A \subset B$, think that A is a subset of B but not equal to B. That means that B has an element that A does not have.

Exercise 213. If A = { green, yellow, red, black, dog, cat, mouse} and B = {dog, cat} then $B \subset A$. Put your finger on the two reasons why this is true. Be specific.

Exercise 214. If A is a set, explain why $A \subset A$ is not true, by talking about elements of A. We could write $A \not\subset A$.

Exercise 215. **a.** Suppose that L is the set of US citizens who voted legally in the last presidential election and C is the set of US citizens at that time.
Explain in the context of voting why $L \subseteq C$ is true.

Explain in the context of voting why $L \subset C$ is true.

b. Consider a class at the University. Let R be the set of students who are registered for the class at the time of the final exam, and let F be the set of people who take the final exam.

What needs to happen at the final exam to make $R = F$? Talk about students, not sets.

What two things need to happen to make $F \subset R$?

What two things need to happen to make $R \subset F$?

Which of the three possibilities do you think is most likely to happen?

Least likely? Why?

Your name: ______________________________

Set subsets and equality

This activity works on showing that one set is a subset of another, and showing equality between two sets.

Exercise 216. Let T be the set of all multiples of 3, and let S be the set of all multiples of 6.
List at least 5 elements of T:
List at least 5 elements of S:
Determine which of the following set inclusions is true. If not true, list at least one element which serves as a counterexample.

a. $T \subseteq S$

b. $S \subseteq T$

If one of the following set inclusions is true, list five elements that account for the strict set inclusion.

a. $T \subset S$

b. $S \subset T$

Exercise 217. Let C denote the set of composite numbers and let E denote the set of even numbers. Determine which of the following set inclusions is true. If not true, list up to five elements which serve as counterexamples.

a. $C \subseteq E$

b. $E \subseteq C$

Exercise 218. Let $\mathbb{R}$ denote the real numbers, $\mathbb{Z}$ denote the integers, and $\mathbb{Q}$ denote the rational numbers. Express the most informative set inclusions between these sets. You do not need to prove them.

Problem 219. Let $\mathbb{Q}$ denote the rational numbers. Let $A = \{x \in \mathbb{R} : x \text{ solves } x^2 = a \text{ where } a \text{ is an integer and } a \geq 0\}$. List out at least 5 elements of A: ____________________
Show that $A \not\subseteq \mathbb{Q}$ by giving one concrete example and telling how it relates to A and to $\mathbb{Q}$.

Problem 220. Continuing the previous problem, show that $\mathbb{Q} \not\subseteq A$ by giving one concrete example and telling how it relates to A and $\mathbb{Q}$.

Guided proof 221. When showing that $A \subseteq B$, you need to show that every element of A is also an element of B. Here is how to do it. Let $x \in A$. Use this fact and the definitions of A and B to show that $x \in B$. Since you made no further assumptions about x, this shows that every element of A is also an element of B. Note: Your proof must start with "Let $x \in A$." and must get to "Thus $x \in B$." before generalizing. Note: Often B will have a specific membership requirement, and you need to show that x exactly meets the requirement.

Guided proof 222. When showing that $A \subset B$, you need to show that $A \subseteq B$ and you need to show that there is an element of B that is not in A. If you can construct such an element, do that, because that should make the proof clearer to the reader.

Guided proof 223. Let $\mathbb{Z}$ denote the integers and $\mathbb{Q}$ denote the rational numbers. Recall that $\mathbb{Q} = \{x : x = \frac{p}{q}$ for some integers p and q with $q \neq 0\}$. Show that $\mathbb{Z} \subset \mathbb{Q}$ following the model.

a. Let $m \in \mathbb{Z}$. m meets the definition to be in $\mathbb{Q}$ because we can write it as _____________ where $p =$ _______ and $q =$ _______. Thus $m \in \mathbb{Q}$. We made no further assumption about $m \in \mathbb{Z}$. Thus for all $m \in Z$, we know that $m \in \mathbb{Q}$. Thus $\mathbb{Z} \subseteq \mathbb{Q}$.

b. To see that $\mathbb{Z} \subset \mathbb{Q}$, note that _____________ is in $\mathbb{Q}$ but is not in $\mathbb{Z}$. (One concrete example.)

Prove 224. Let $\mathbb{R}$ denote the real numbers and $\mathbb{C}$ denote the complex numbers. Recall that $\mathbb{C} = \{a + ib : a, b \in \mathbb{R}\}$ where $i = \sqrt{-1}$, which is not a real number. Show that $\mathbb{R} \subset \mathbb{C}$ following the model in the previous problem.

a.

b.

Prove 225. Let $2\mathbb{Z} = \{m \in \mathbb{Z} :$ there exists $j \in \mathbb{Z}$ such that $m = 2j\}$ and $6\mathbb{Z} = \{m \in \mathbb{Z} :$ there exists $j \in \mathbb{Z}$ such that $m = 6j\}$. Show that $6\mathbb{Z} \subset 2\mathbb{Z}$, following the model in 220.

Definition 226. Intervals of real numbers.. Let a and b be real numbers with $a \leq b$. We define the following four types of intervals:

a. $(a, b) = \{x \in \mathbb{R} : a < x < b\}$. We say that both endpoints are open.

b. $[a, b) = \{x \in \mathbb{R} : a \leq x < b\}$. The compound inequality $a \leq x \leq b$ means $a \leq x$ and $x \leq b$.

c. $(a, b] = \{x \in \mathbb{R} : a < x \leq b\}$

d. $[a, b] = \{x \in \mathbb{R} : a \leq x \leq b\}$. We say that both endpoints are closed.

Exercise 227. Sketch the following intervals on separate number lines. To indicate open endpoints, draw an open circle like $\circ$. To indicate closed endpoints, draw a closed circle like $\bullet$.

a. $(2, 9)$

b. $[2, 7)$

c. $(3, 5]$

d. $[0, 1]$

Note 228. Inequalities between real numbers have the *transitivity* property: If $a \leq b$ and $b \leq c$, then we can conclude that $a \leq c$. Similar inequalities are true with $\geq, <$, and $>$.

Exercise 229. Using standard interval notation, show that $(4, 9] \subset [2, 9]$. Begin with "Let $x \in (4, 9]$." then rewrite this as a compound inequality, then rewrite as two separate inequalities. Note when you use transitivity. Make sure to write that $2 \leq x \leq 9$. Make sure to show that $(4, 9]$ is a proper subset of $[2, 9]$.

Definition 230. Set equality. For sets A and B, we say that A equals B when the elements of A are exactly the same as the elements of B. We write $A = B$ when A and B are equal sets.

Example 231. The set consisting of all colors of a rainbow and the set consisting of colors of white light observed through a prism are equal sets.

Guided proof 232. One way to show that $A = B$ is to *show inclusion both ways.* That means to show that $A \subseteq B$ and $B \subseteq A$. Here is how you do it.

1. To show $A \subseteq B$: Let $x \in A$. Use this fact and the definitions of A and B to show that $x \in B$. Having made no further assumption about $x \in A$, you can conclude that $A \subseteq B$.

2. To show $B \subseteq A$: ____________________. Use this fact and the definitions of A and B to show that _____________. Having made no further assumption about _____________, you can conclude that _____________.

Exercise 233. Let $A = \{x : x \text{ solves } ax = b \text{ where } a \text{ and } b \text{ are integers and } a \neq 0\}$. Let $\mathbb{Q} = \{x : x \text{ is a rational number}\}$. Show that $A = \mathbb{Q}$ by showing set inclusion both ways.

a. Let $x \in A$. Then there exist a and b such that _____________ and $a \neq 0$. Dividing through by a, $x =$ _____________ where a and b are integers and a is not zero. Thus, $x \in \mathbb{Q}$. Since x was arbitrary, $A \subseteq \mathbb{Q}$.

b. Let $x \in \mathbb{Q}$.

Exercise 234. Let $A = \{x : (x - 3)^2 - 1 < 3\}$ and let $B = (1, 5)$. Show that $A = B$ by showing set inclusion both ways.

Exercise 235. Let $A = \{x : -x^2 + 5x + 14 \geq 0\}$ and let $B = [-2, 7]$. Show that $A = B$ by showing set inclusion both ways.

Problem 236. Let $2\mathbb{Z} = \{m \in \mathbb{Z} : \text{there exists } j \in \mathbb{Z} \text{ such that } m = 2j\}$. Let $3\mathbb{Z} = \{m \in \mathbb{Z} : \text{there exists } j \in \mathbb{Z} \text{ such that } m = 3j\}$, and similarly with other sets like $5\mathbb{Z}$ and $15\mathbb{Z}$. Show that $2\mathbb{Z} \cap 3\mathbb{Z} = 6\mathbb{Z}$ by showing set inclusion both ways.

Problem 237. Write out all elements in $6\mathbb{Z} \cap 8\mathbb{Z} \cap \{1, 2, 3, \ldots, 100\}$.

Exercise 238. Let $\mathbb{R}^3$ denote the set of all 3-dimensional vectors and let $S = \{v : v = t_1\langle 1, 0, 0\rangle + t_2\langle 1, 1, 0\rangle + t_3\langle 1, 0, 1\rangle$ where t_1, t_2, t_3 are real numbers $\}$.

a. Show that $S \subseteq \mathbb{R}^3$.

b. Show that $\mathbb{R}^3 \subseteq S$. Now, *you* determine what values of t_1, t_2, and t_3 will work to make v have the form of an element of S.

Your name: ______________________________

Quantifiers and nested quantifiers

Overview

Many statements that we want to prove are supposed to be true **for all** objects having a certain property. Other times, we want to show that **there exists** an object having a property. 'For all' and 'there exists' are called quantifiers. Sometimes the two occur together, as in a statement that 'for every object A, there exists an object B having a certain relationship to A.' Then we say that the quantifiers are *nested*. In this activity you will work with quantifiers and nested quantifiers.

Definition 239. For all. The phrase *for all* means that what follows is supposed to be true for all values of the indicated variable, and will probably require a generic proof to cover all possibilities. Alternative words are "for every" or "for each." Sometimes people write "for any" but please avoid that because it can be ambiguous. The notation $\forall$ is often used to represent "for all". The format is "for all (introduce a variable and optionally put restrictions on the variable), we have (property satisfied by the variable)." In this course, use the words "we have" except when "for all" is followed by "there exists."

Definition 240. There exists. The phrase *there exists* claims that an object with a certain property can be shown to exist. Often, you prove existence by constructing the object that is needed, but occasionally the proof works differently. The notation $\exists$ is often used to represent "there exists." The format is "there exists (introduce a variable and optionally put restrictions on the variable) such that (property satisfied by the variable)." In this course, write out the words "such that."

Exercise 241. Rewrite English sentences symbolically, and rewrite symbolic statements as English sentences. The first ones are done for you. Pay close attention to the format for writing the statements.

a. Every prime number is greater than 1. Solution: $\forall$ prime n, we have $n > 1$.

b. There is a real number x for which $x^2 = 2$. Solution: $\exists\ x \in \mathbb{R}$ such that $x^2 = 2$. Note: The set $\mathbb{R}$ is called the *universe* for the variable x.

c. $\forall\ x \in A$, we have $x^2 = x$. Rewrite in English:

d. The equation $x = \sin(x)$ has an integer-valued solution. Rewrite symbolically:

Exercise 242. Write the following statements symbolically:

a. For every a, there is a b for which $b^2 = a$

b. For every b, there is an a for which $b^2 = a$

c. For every a and every b, we have $b^2 = a$

d. There exists an a and there exists a b such that $b^2 = a$

Exercise 243. For each of the statements in the previous problem, decide whether it is true or false if the universe for both a and b is the set of non–negative integers. If false, give specific numbers as counterexamples using "Let …"

a.

b.

c.

d.

Remark 244. Now you will write proofs that involve nested quantifiers. For each, first write the statement using symbols $\forall$ and $\exists$. Note the order in which the quantifiers occur, because that will have a big impact on the structure of the proof. To prove a "for all" statement, use "Let …" to introduce a generic instance of the variable that satisfies whatever restriction is imposed. If your proof works for a generic value of the variable, then it will work for all variables satisfying the restriction. To prove a "there exists" statement, construct the required variable in terms of other variables, also using "Let …" to start the construction. It can help you organize your work if you indent under each "Let" statement.

Prove 245. Show that for all odd integers m and n, there exists an integer p such that $m + n = 2p$.
In symbols: $\forall$ ____________ integers m and n, ________________________
Let m and n be odd integers. (m and n satisfy the condition but are otherwise arbitrary)
 By the definition of odd, ____
 Thus, $m + n =$ ____
 Let $p =$ ____________ (this constructs p)
  Then p is an ______________ and $m + n = 2p$ as desired.
Because m and n were arbitrary odd numbers, we have shown that for all odd m and n, there exists ...

Prove 246. Show that for every rational number $r \neq 0$, there exists a rational number s such that $rs = 1$.
Follow the model.
In symbols:
Let r be a rational number with $r \neq 0$.
 By the definition of rational numbers, ____

 Let $s =$ ____________.

  Then s is rational and $rs =$ ____________ $= 1$.
Because ...

Prove 247. Show that for every real number y there is a value of x for which $y = 2x - 5$.
In symbols:

Prove 248. (Calculus required.) Show that for every continuous function $f : \mathbb{R} \to \mathbb{R}$, there exists a function F with $F' = f$ and $F(0) = 0$. Is there more than one possible choice for F?
In symbols:

Prove 249. (Linear algebra required.) Suppose that the 3 by 3 matrix A is invertible. (The matrix A is given; you don't need to say "Let A be a matrix" or construct A.) Show that for all 3-dimensional vectors b, the equation $Ax = b$ has a solution x, which is also a 3-dimensional vector. Is there more than one possible value for x?
In symbols:

Exercise 250.

a. Show that for every integer $p > 0$, there is an integer n with $2^n > p$. You don't need the smallest possible n, just one that works.

b. Show that there exists an integer n such that for all integers p with $p \leq 0$, we have $2^n > p$. First define n, then show that it works for all p by starting with "Let $p \leq 0$ be an integer."

c. Suppose that $2^n > p$. Show that for all $m > n$, we have $2^m > p$.

d. Fill in the blanks to document what you have shown about p, n, and m in (a), (b), and (c).

$\forall p$________, $\exists n$________ such that $\forall m$________, we have ______________

Exercise 251.

a. Suppose that $n^2 > \frac{1}{a}$. Cite a property of inequalities to show that $n^2 + 10 > \frac{1}{a}$.

b. Show that for every real number $a > 0$, there exists an integer n with $\frac{1}{n^2+10} < a$.

Let $a > 0$. Let $n = ceil(\frac{1}{\sqrt{a}})$. Then $n \geq \frac{1}{\sqrt{a}}$, (now you continue)

c. Suppose that $\frac{1}{n^2+10} < a$. Show that for all $m > n$, we have $\frac{1}{m^2+10} < a$.

d. As in the previous question, use three quantifiers to express what you have shown about a, n, and m in (b) and (c).

As a consequence, you have shown that $\lim_{n\to\infty} \frac{1}{n^2+10} = 0$.

Exercise 252.

a. Let $a > 0$. Solve the inequality $0 < \frac{1}{x^2-100} < a$ for a value of x where $x > 10$.

b. Show that for every real number $a > 0$, there is an integer n with $0 < \frac{1}{n^2-100} < a$.

c. Suppose that $\frac{1}{n^2-100} < a$. Show that for all $m > n$, we have $\frac{1}{m^2-100} < a$.

d. Use three quantifiers to express what you have shown about a, n, and m in (b) and (c).

Exercise 253. Show that there exists a number a such that for all $x \in \mathbb{R}$, $5\sin(x) + 7\cos(3x) < a$. Note that in this problem, you first construct a and then you show a "for all" statement.

In symbols:

Exercise 254. Show that there exists an integer a for which, for all integers b, $ab = b$.

In symbols:

Remark 255. The negation of a logical statement P is denoted by $\neg P$. It is true when P is false and false when P is true. When the logical statement begins with a quantifier, we can think through the result of negation. In what follows, $Q(x)$ is a logical statement whose truth value depends on the value of x. For example, $Q(x)$ could be the statement "$x^2 = x$."

1. Consider $\neg\forall\ x \in A$, we have $Q(x)$. This means that it is not true that all x values "work," so there must be an x value that does not work, that is, $\exists\ x \in A$ such that $\neg Q(x)$.

2. Consider $\neg\exists\ x \in A$ such that $Q(x)$. Try as we might, we cannot find a value of x that "works," so it must be that all values of x fail to work. That is, $\forall\ x \in A$, we have $\neg Q(x)$.

Note that in both cases, negating the quantifier can be done quite mechanically: negation turns $\forall$ into $\exists$ and it turns $\exists$ into $\forall$. Also note that in both cases we keep the restriction $x \in A$ and instead negate the property $Q(x)$ that x is supposed to have. When negating nested quantifiers, after flipping the first quantifier, the negation applies to the next quantifier, which then flips, and so on.

Exercise 256. Negate the following statements, getting rid of every "not" (except for part (c) where you will need "not equal").

a. $\exists$ integer k such that $k^2 < k$

b. $\exists\ x \geq 0$ such that $\cos(x) > e^x$

c. $\forall\ y \in \mathbb{R}$, $\exists\ x \in \mathbb{R}$ such that $x^2 = y$

d. $\forall\ a > 0$, $\exists$ integer n such that $\forall\ m > n$, $\sin(m) < a$

e. $\forall\ a > 0$, $\exists\ d > 0$ such that $\forall\ x \in (a-d, a+d)$, we have $|\cos(x) - \cos(a)| < a$

Exercise 257. Write the following statements symbolically. Introduce new notation as you need it. The first one is done for you.

a. Every state has a city named Springfield. Use variables s and c.

Solution: $\forall$ state s, $\exists$ city c in s such that NameOf(c) = Springfield.

b. Every bridge has a weight limit. Use variables b and w.

c. There is an integer that is larger than every other integer. Use variables m and n.

d. Every broken clock is right twice a day. Use variables c, t_1, t_2.

e. Every married couple can find a tax deduction. Use variables m and d.

f. Between every two locations in the US, there is a shortest driving route. Use variables L_1, L_2, r, and R.

Solution: $\forall$ locations L_1 and L_2 in the US, $\exists$ route r such that r starts at L_1 and ends at L_2 and $\forall$ route R such that R starts at L_1 and ends at L_2, length$(r) \leq$ length(R).

Why is the second $\forall$ needed?

Your name: ______________________________

Mathematical Induction

Proving that a claim is true for all $n = 1, 2, 3, \ldots$ by building on previous results.

Overview

One important task in mathematics is to find regular patterns and prove that they hold. The main method we use for this is mathematical induction. This activity was co-authored by Ying-Ju Chen.

Theorem 258. Mathematical induction
For each integer $n = 1, 2, 3, \ldots$, let $P(n)$ denote a true/false logical statement involving n.

(i) (The basis step) Prove that $P(1)$ is true.

(ii) (The inductive step) For each $n = 1, 2, 3, \ldots$, suppose that $P(n)$ is true, and use $P(n)$ to prove that $P(n+1)$ is true.

From the above two steps, we can conclude that $P(n)$ is true for all $n = 1, 2, 3, \ldots$.

Note 259. Proving the inductive step is usually done as a "rewrite" proof, where you start with the left hand side of what you want to show and rewrite until you come to the desired right hand side. Often, some quantity in the statement $P(n+1)$ can be written in terms of a similar quantity in the statement $P(n)$ plus a new part. You will always use the fact that $P(n)$ is true.

Question 260. With induction proofs, $P(n)$ is always a true/false logical statement. A student working on an induction problem wrote $P(n+1) = P(n) + \frac{1}{n^2}$. How can you tell that this must be wrong?

Guided proof 261. Use mathematical induction to show that 3^n is odd for all $n = 1, 2, 3, \ldots$.

a. State $P(n)$: $P(n)$ is that "__________________________"

b. Basis step: $P(1)$ is that "____________________." This is true because ____________________.

c. State $P(n+1)$: $P(n+1)$ is that "__________________________"

d. Inductive step: Let $n \geq 1$. suppose that $P(n)$ is true. Show that $P(n+1)$ is true. You may use facts you have already proven about odd numbers.

Because $P(n)$ is true, ____________________. Now $3^{n+1} =$ _____________, which is odd because ____. Thus $P(n+1)$ is true. Since $n \geq 1$ was arbitrary, by mathematical induction, $P(n)$ is true for all $n \geq 1$.

Exercise 262. Fill in the table using your powers of pattern recognition.

k	1	2	3	4	5	6	7	$\cdots$	n	$n+1$
kth odd integer	1	3	5	7				$\cdots$		
sum of first k odd integers	1	4						$\cdots$		

Column n of the table contains a conjecture about the sum of the first n odd integers. In the next problem, you will use mathematical induction to prove it.

Guided proof 263. Prove the conjecture in 259 using mathematical induction.

a. State $P(n)$: $P(n)$ is that "the sum of the first n odd integers equals ____________"

b. Basis step: $P(1)$ is that the sum of the 1st odd integer is 1^2. This is true because the sum is 1 and because $1^2 = 1$.

c. Write out $P(n+1)$: $P(n+1)$ is that "the sum of the first ____________________________________"

d. Inductive step: Let $n \geq 1$. suppose that $P(n)$ is true, and use that to show that $P(n+1)$ is true.

the sum of the first $n+1$ odd integers

= the sum of the first n odd integers plus ____________

= __________________ + __________________ since $P(n)$ is true

= __________________ by algebra

Thus $P(n+1)$ is true. Since ____________ was arbitrary, by ________________________ we conclude that the sum of the first n odd integers equals ____________ for all $n = 1, 2, 3, \ldots$.

Notation 264. Summation notation for $a_1 + a_2 + a_3 + \cdots + a_n$ is $\displaystyle\sum_{k=1}^{n} a_k$. For example, $\displaystyle\sum_{k=1}^{n} k = 1 + 2 + \cdots + n$.

Example 265. Use summation notation and the formula for the nth odd integer in 259 to rewrite $P(n)$ in 260: $P(n)$ is that __________________ = ______

Exercise 266. Fill in the blanks to practice splitting off the last term of a sum.

a. $\displaystyle\sum_{k=1}^{n+1} k^2 = \left(\sum_{k=1}^{n} k^2\right) + \_\_\_\_\_\_\_\_\_\_\_\_$ **b.** $\displaystyle\sum_{k=1}^{n+1} \frac{1}{k^3} = \left(\sum_{k=1}^{n} \frac{1}{k^3}\right) + \_\_\_\_\_\_\_\_\_\_\_\_$

Show 267. Show that $\sum_{k=1}^{n} 4k - 3 = n(2n-1)$ for all $n = 1, 2, 3, \ldots$.

a. State $P(n)$: $P(n)$ is that

b. Basis step: $P(1)$ is that This is true because: __________________

c. State $P(n+1)$: $P(n+1)$ is that

Suggestion: Use algebra to simplify the right hand side.

d. Inductive step: Let $n \geq 1$. suppose that $P(n)$ is true, and use that to show that $P(n+1)$ is true.

$$\begin{aligned}
\sum_{k=1}^{n+1} 4k - 3 &= \sum_{k=1}^{n} __________ + __________ \\
&= __________ + __________ \qquad \text{since } P(n) \text{ is true} \\
&= \\
&=
\end{aligned}$$

Thus, __________________. Since ...

Stop. Compare your proofs with the other people in your group before you move on.

Show 268. Use mathematical induction to show that $\sum_{k=1}^{n} 5^k = \frac{5}{4}(5^n - 1)$ for all integers $n = 1, 2, 3, \ldots$.

a. State $P(n)$:

b. Basis step:

c. State $P(n+1)$:

Suggestion: Multiply out the right hand side.

d. Inductive step: Let $n \geq 1$. suppose that $P(n)$ is true, and use that to show that $P(n+1)$ is true.

Note 269. The basis step need not use $n = 1$, for example, it can use $n = -3, n = 0$, or $n = 100$.

Show 270. Use mathematical induction to show that $2n + 1 < 2^n$ for all integers n with $n \geq 4$.

a. State $P(n)$:

b. Basis step: $P(4)$ is that: Check: ____________________

c. State $P(n+1)$:

d. Inductive step: Let $n \geq 4$. suppose that $P(n)$ is true, and use that to show that $P(n+1)$ is true.

$$
\begin{aligned}
2(n+1)+1 = \underline{\hspace{3cm}} &= \underline{\hspace{3cm}} \\
&< \underline{\hspace{3cm}} \qquad \text{since } P(n) \text{ is true} \\
&< 2^n + 2^n \qquad \text{since } \underline{\hspace{2cm}} \\
&= 2^{n+1}.
\end{aligned}
$$

Thus, ____________________. Since ...

Show 271. Use mathematical induction to show that $5^n > 2^n + 3^n$ for all integers n with $n \geq 2$. Use the same format as above.

a.

b.

c.

d.

Show 272. Use induction to prove Bernoulli's inequality: For all $x \in \mathbb{R}$, if $1+x > 0$, then $(1+x)^n \geq 1+nx$ for all $n = 0, 1, 2, \ldots$. Use the same format as above.
Let x be such that $1 + x > 0$.

Where did you use the assumption that $1 + x > 0$?

Show 273. Use induction to prove that $\frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \frac{1}{3\cdot 4} + \cdots + \frac{1}{n\cdot(n+1)} = \frac{n}{n+1}$ for all positive integers n. Start by writing $P(n)$ using summation notation.

Note 274. It is possible to show that the statement in 270 is true without using mathematical induction, but using an algebraic technique. How?

Show 275. For each $n \in \mathbb{Z}^+$, let $P(n)$ denote the statement "$n^2 + 5n + 1$ is an even integer."

a. State $P(n + 1)$

b. Suppose that $P(n)$ is true, and use that to prove that $P(n + 1)$ is true.

c. For which n is $P(n)$ actually true?

d. What is moral of this exercise?

Show 276. Use induction to prove that $n^3 - n$ is a multiple of 6 for all integers $n = 0, 1, 2, \ldots$. Do the induction step as a "rewrite" proof.

Show 277. Use induction to prove that $11^n - 4^n$ is a multiple of 7 for all $n = 0, 1, 2, \ldots$. Do the induction step as a "rewrite" proof. **Hint:** Use the equation $11^n - 4^n = 7k$ once.

Example 278. Write the following numbers as multiples of 5 plus a remainder from 0 to 4.

$37 = 5 \cdot$ _______ $+$ _______
$38 = 5 \cdot$ _______ $+$ _______
$39 = 5 \cdot$ _______ $+$ _______
$40 = 5 \cdot$ _______ $+$ _______
$41 = 5 \cdot$ _______ $+$ _______

Prove 279. Let $k > 0$ be an integer. For each integer n, let $P(n)$ be the statement: "There exist integers q and r with $0 \leq r < k$ such that $n = kq + r$." Use mathematical induction to show that $P(n)$ is true for all integers n.

a. Show that $P(0)$ is true.

b. Let n be an integer. Suppose that $P(n)$ is true, so that $n = kq + r$ for some integers q and r, with $0 \leq r < k$. Show that there exist integers q' and r' with $0 \leq r' < k$ so that $n + 1 = kq' + r'$, and thus conclude that $P(n + 1)$ is true. Note that you will define q' and r' in terms of q and r. It is helpful to do this with two cases, depending on the value of r:

Case 1. Suppose $0 \leq r < k - 1$.

Case 2. Suppose $r = k - 1$.

c. Suppose that $P(n)$ is true and show that $P(n - 1)$ is true. It is helpful to do this with two cases.

Use steps b and c and the idea of mathematical induction to conclude the proof that $P(n)$ is true for all n.

Show 280. Use mathematical induction to prove that $\sum_{k=1}^{n} k = \frac{1}{2}n(n+1)$ for all integers $n = 1, 2, 3, \ldots$.

Show 281. Use mathematical induction to prove that $\sum_{k=1}^{n} k^2 = \frac{1}{6}n(n+1)(2n+1)$ for all integers $n = 1, 2, 3, \ldots$.

Show 282. Let r be a real number not equal to 1. Use induction to prove that $\sum_{k=0}^{n} r^k = \frac{1-r^{n+1}}{1-r}$ for all integers $n = 0, 1, 2, \ldots$. Note where you use the assumption on r.

Show 283. Use mathematical induction to show that $n! > 3^n$ for all $n = 7, 8, 9, \ldots$.

Show 284. For $n = 2, 3, \ldots$, let $P(n)$ be the statement "n is prime or n can be written as the product of prime factors less than n. Supposing that $P(2), P(3), \ldots, P(n)$ are true, show that $P(n+1)$ is true, and thus conclude that $P(n)$ is true for all $n = 2, 3, \ldots$. This is called *strong induction*.

Show 285. Prove that $1^2 - 2^2 + 3^2 - 4^2 + 5^2 + \cdots - (2n)^2 + (2n+1)^2 = (n+1)(2n+1)$ for all $n = 0, 1, 2, \ldots$. Start by writing $P(n)$ using summation notation, starting from $k = 0$.

Your name: ______________________________

Union and intersection of sets

We introduce the union and intersection of sets and learn how to prove statements about them.

Note: The notation $x \in A$ is usually read "x is an element of A." The symbol $\in$ looks like the letter E because it stands for "element". As far as I can tell, the E is not there to mean "x exists in A". Being an element is the point, not existing.

Definition 286. Union of sets. The *union* of sets A and B is a new set, consisting of all elements which belong to A or to B or to both. We denote this new set by $A \cup B$. The logical statement "$x \in A \cup B$" is true when "$x \in A$ or $x \in B$" is true.

Definition 287. Intersection of sets. The *intersection* of sets A and B is a new set, consisting of all elements which belong to both A and B. We denote this new set by $A \cap B$. The logical statement "$x \in A \cap B$" is true when "$x \in A$ and $x \in B$" is true.

Example 288. Let C be the set of Computer Science majors and M be the set of Mathematics majors. Then $C \cap M$ is the set of students double majoring in Computer Science and Mathematics (a powerful combination!) while $C \cup M$ is the set of students majoring in one, the other, or both majors.

Example 289. Find $[1, 5] \cup (3, 7)$. Draw a diagram on a number line to illustrate.

Example 290. Find $[1, 5] \cap (3, 7)$. Draw a diagram on a number line to illustrate.

Problem 291. Let $3\mathbb{Z} = \{m \in \mathbb{Z} : \text{there exists } j \in \mathbb{Z} \text{ such that } m = 3j\}$. Let $5\mathbb{Z} = \{m \in \mathbb{Z} : \text{there exists } j \in \mathbb{Z} \text{ such that } m = 5j\}$, and similarly with other sets like $7\mathbb{Z}$ and $12\mathbb{Z}$.

a. Find $3\mathbb{Z} \cap 5\mathbb{Z}$ and write it in the most convenient form you can.

b. Find $3\mathbb{Z} \cup 5\mathbb{Z}$ and write it in the most convenient form you can.

Remark 292. Remember that to show $S \subseteq T$, you need to show that for all x in S, we have x in T. In symbols, $\forall x \in S, x \in T$. To do a proof like that, you need to start with "Let $x \in S$." and you need to end with something like "Thus $x \in T$. Since $x \in S$ was arbitrary, $S \subseteq T$."

Prove 293. Suppose that $A \cup B \subseteq C$.

a. Show that $A \subseteq C$ following the model.

Let $x \in A$. Then ____________ by definition of union. Thus ____________ since $A \cup B \in C$.

Since ____________ was arbitrary, ____________.

b. Show that $B \subseteq C$. Use good form.

Remark 294. For sets A, B, and C, to show that $A \cap B \subseteq C$, you need to start with "Let $x \in A \cap B$." This tells you that $x \in A$ and $x \in B$. Using those two pieces of information, you need to show that $x \in C$. You will usually need to use both pieces of information.

Guided proof 295. Show that $[1, 5] \cap (3, 7) \subseteq (3, 5]$.
Let $x \in [1, 5] \cap (3, 7)$. Then $x \in$ ____________ and $x \in$ ____________ by definition of _____________.
Thus, $1 \leq x \leq 5$ and __________________.
Thus, ______ $<$ ______ $\leq$ ______
Thus, $x \in (3, 5]$. Since $x \in [1, 5] \cap (3, 7)$ was arbitrary, we conclude that $[1, 5] \cap (3, 7) \subseteq (3, 5]$.

Prove 296. Suppose that $A \subseteq C$ and $B \subseteq C$. Show that $A \cap B \subseteq C$ following the model above.
Let ____________

Thus ____________. Since ____________ was arbitrary, ____________________________________.

Prove 297. Show that $2\mathbb{Z} \cap 7\mathbb{Z} \subseteq 14\mathbb{Z}$. Use good form. You do not need to prove any results about multiples or divisibility, simply use what you know is true from the definition of these sets.
Let __________________

Thus ____________. Since __________________ was arbitrary, ______________________________.

Remark 298. To show that $A \cup B \subseteq C$, you start with "Let $x \in A \cup B$." This tells you that $x \in A$ or $x \in B$. This is not all that much to work with, since maybe only one of these is true, and you don't know which one. No matter which one is true, you need to show that $x \in C$. You should do this by considering two cases. Case 1, when $x \in A$. Case 2, when $x \in B$. In both cases, show that $x \in C$ and then no matter which case applies, you get the result you need. In the end, you show that $A \subseteq C$ and $B \subseteq C$.

Guided proof 299. Suppose that $A \subseteq C$ and $B \subseteq C$. Show that $A \cup B \subseteq C$.
Let $x \in A \cup B$. Then $x \in A$ or $x \in B$ or both, by _____________________________.
Case 1. Suppose that $x \in A$. Then $x \in C$ because ____________.
Case 2. Suppose that $x \in B$. Then ____________ because ____________.
In both cases, ________. Since ____________ was arbitrary, we conclude that ____________

Prove 300. Show that $[1, 5] \cup (3, 7) \subseteq [1, 7)$. Use good form. You will use transitivity for inequalities.

Prove 301. Suppose that $B \subseteq A$. Show that $A \cup B \subseteq A$. Use good form.

Prove 302. Suppose that $A \cup B \subseteq A$. Show that $B \subseteq A$. Use good form.

Remark 303. To show that $A \subseteq B \cap C$, start with "Let $x \in A$ and show that $x \in B$ and $x \in C$, so you can conclude that $x \in B \cap C$. Make sure to write that you have shown that $x \in B \cap C$ before you make a general conclusion.

Prove 304. Show that $14\mathbb{Z} \subseteq 2\mathbb{Z} \cap 7\mathbb{Z}$. Use transitivity and make a general conclusion at the end.
Let ______________

Thus ______________. Since _______ was arbitrary, ____.

Prove 305. Show that $(3, 5] \subseteq [1, 5] \cap (3, 7)$. Use good form and make a general conclusion at the end.

Remark 306. To show that $A \subseteq B \cup C$, start with "Let $x \in A$ and show that $x \in B$ or that $x \in C$. Often, some x values are in B and others are in C, and so you will want to introduce two cases that split the values of x into two groups. Unfortunately, the cases cannot be "Case 1: Suppose $x \in B$" and "Case 2: Suppose $x \in C$" because you do not yet know that those case cover all possibiliities. Instead, you may need to use cases like "Case 1: Suppose $x < 0$" and "Case 2: Suppose $x \geq 0$." The details will be different for each problem. Each case must end with "Thus $x \in B \cup C$."

Prove 307. Show that $[1, 7) \subseteq [1, 5] \cup (3, 7)$. Use good form and make a general conclusion at the end.

Prove 308. Show that $(3, 8) \cup [6, 9] = (3, 9]$. There are two steps. Use inequalities and transitivity, not statements like $(3, 8) \subseteq (3, 9]$.

a. Step 1. Show that $(3, 8) \cup [6, 9] \subseteq (3, 9]$.

b. Step 2. Show that $(3, 9] \subseteq (3, 8) \cup [6, 9]$.

Prove 309. Show that $(3, 8) \cap [6, 9] = [6, 8)$. There are two steps.

a. Step 1.

b. Step 2.

Your name: ______________________________

Freethrow percentage

A few thought problems to work on.

Overview

The first question is inspired by a problem from the book "Reading, Writing, and Proving: A Closer Look at Mathematics" by Daepp and Gorkin.

Exercise 310. Frieda Freethrow plays for the Bowling Green State University women's basketball team. Early in the season, she had made 6 out of 10 freethrow attempts in games, giving her a 60% freethrow percentage. That was not good enough for her, so she practiced freethrows and eventually, later in the season, got her freethrow percentage up to 80% to that point in the season. The question is, was there a point in the season when her freethrow percentage was exactly 75%?

Explore this question in two ways: look at possible sequences of making or missing freethrows and the resulting freethrow percentages to see if you can find a way that she could have avoided hitting exactly 75%, and look for ways to prove that she must have hit 75% at some point. There is no single way to formulate this question, so be creative in finding ways to look at how the freethrow percentage can change over the course of the season so you can make your best argument.

Exercise 311. Freddie Freethrow plays for the BGSU men's basketball team. His season started out well, he made 8 of his first 10 freethrows, giving him an 80% freethrow percentage. Freddie neglected practicing freethrows, and by some point later in the season, his freethrow percentage had dropped to 60%. Was there a point in the season when Freddie's freethrow percentage was exactly 75%?

Again, look at possible sequences of freethrows to see if 75% is always hit or can be skipped over and make your best argument.

Your name: ________________________________

Infinite unions and intersections

Overview

We are working with infinitely many sets of real numbers. These exercises will give you practice with sets and teach you things about the real numbers as well.

Definition 312. Union. Let $A_1, A_2, \ldots$ be sets. The *union* of $A_1, A_2, \ldots$ is a new set consisting of all elements that are in A_n for some $n = 1, 2, 3, \ldots$. The union can be written in open form as $A_1 \cup A_2 \cup \cdots$ or in closed form as $\bigcup_{n=1}^{\infty} A_n$. Note that there is no set A_∞.

Definition 313. Intersection. Let $A_1, A_2, \ldots$ be sets. The *intersection* of $A_1, A_2, \ldots$ is a new set consisting of all elements which are in A_n for all $n = 1, 2, 3, \ldots$. The intersection can be written in open form as $A_1 \cap A_2 \cap \cdots$ or in closed form as $\bigcap_{n=1}^{\infty} A_n$.

Problem 314. Use quantifiers to express what it means that $x \in \bigcup_{n=1}^{\infty} A_n$.
Solution: $\exists\ n \geq 1$ such that $x \in A_n$. In words, there is at least one n for which x is in A_n. It does not take much to be in the union.

Problem 315. Use quantifiers to express what it means that $x \in \bigcap_{n=1}^{\infty} A_n$.

Remark 316. To show $A \subseteq \bigcap_{n=1}^{\infty} B_n$, you need to show that for all $x \in A$, we have $x \in \bigcap_{n=1}^{\infty} B_n$. That is, for all $x \in A$ and all $n = 1, 2, 3, \ldots$, we have $x \in B_n$. To show this, you need a generic proof that works for all x and all n, so start with "Let $x \in A$ and let $n \geq 1$," and then show that $x \in B_n$.

Exercise 317. To show $A \subseteq \bigcup_{n=1}^{\infty} B_n$, you need to show that $\forall\ x \in A$, we have $x \in \bigcup_{n=1}^{\infty} B_n$. That is, $\forall$ $x \in A$, $\exists\ n \geq 1$ such that $x \in B_n$. Think ahead to your proof.
How will the variable x be introduced?

How will the variable n be introduced?

Remark 318. To show $\bigcap_{n=1}^{\infty} A_n \subseteq B$: Let $x \in \bigcap_{n=1}^{\infty} A_n$. Then you know that $x \in A_n$ for all n, which is a lot of information about x. Use this information to show that $x \in B$. Exactly how that will work depends on the problem.

Remark 319. To show $\bigcup_{n=1}^{\infty} A_n \subseteq B$: Let $x \in \bigcup_{n=1}^{\infty} A_n$. Then you know that $x \in A_n$ for some n, but you don't know which n, so that is not very informative. There are infinitely many cases, one for each possible value of n. The best you can do is to start with, "Let $n \geq 1$. Suppose $x \in A_n$." and work forward from there to show that $x \in B$. In the end, this is the same as showing that $A_n \subseteq B$ for all n.

Problem 320. Let $A = \displaystyle\bigcup_{n=1}^{\infty} [n, n+1)$.

a. Write A in open form, listing out the first five sets in the union:

$A =$ __________ $\cup$ __________ $\cup$ __________ $\cup$ __________ $\cup$ __________ $\cup \cdots$

b. Let $B = [1, \infty)$. Explain intuitively why $A = B$.

c. Show that $A \subseteq B$.

Let $x \in A$. Then $x \in [n, n+1)$ for ____________________.

Thus, x satisfies the following inequalities: ___________________________

Thus, $x \in B$. Since $x \in A$ was arbitrary, $A \subseteq B$.

d. Show that $B \subseteq A$.

Let $x \in B$. You need to show that there exists an n for which $x \in [n, n+1)$. You need to construct the value of n, starting with x.

Thus, $x \in [n, n+1)$, and so $x \in A$. Since $x \in B$ was arbitrary, $B \subseteq A$.

Problem 321. Let $A = \displaystyle\bigcup_{n=1}^{\infty} [-n, n]$.

a. Write A in open form, listing out the first five sets in the union.

b. Figure out what single interval A is equal to and call the new interval B. $B =$ ____________________. Explain why $A = B$.

c. Show that $A \subseteq B$. Use good form.

d. Show that $B \subseteq A$. Use good form.

Problem 322. Let $A = \displaystyle\bigcap_{n=1}^{\infty} (-n, n)$.

a. Write A in open form, listing out the first five sets in the intersection.

b. Figure out what single interval A is equal to and call the interval B. $B =$ _____________.

c. Show that $A \subseteq B$.

d. Show that $B \subseteq A$.

Let $x \in B$. You need to show that $x \in (-n, n)$ for all n. How do you do that?

Problem 323.

a. Show that $(5, 6] \subseteq [5, 6]$.

b. Let n be an integer greater than or equal to 1. Show that $[5 + \frac{1}{n}, 6] \subseteq (5, 6]$

Problem 324. Let $A = \bigcup_{n=1}^{\infty} [\frac{1}{n}, 1]$. List out the first five sets in this union, as you did above. Draw a picture of them above a number line. Make a conjecture about what interval A is equal to, call the new interval B, then show that $A = B$ by showing containment both ways. You will need to use this property of real numbers: if $x > 0$, then there exists a positive integer n with $0 < \frac{1}{n} < x$.

Problem 325. Suppose that $x \leq 5 + \frac{1}{n}$ for all $n = 1, 2, 3, \ldots$. Show that $x \leq 5$. **Hint:** Consider different types of proof including direct, contrapositive, contradiction, etc.

Problem 326. Let $A = \bigcap_{n=1}^{\infty} [0, 1 + \frac{1}{n}]$. List out the first five sets in this intersection, as you did above. Draw a picture of them above a number line. Make a conjecture about what interval A is equal to, call the new set B, then prove that $A = B$ by showing containment both ways.

Problem 327. Let $a < b$. Show that $\bigcup_{n=1}^{\infty}[a, b - \frac{1}{n}] = [a, b)$. (If $b - \frac{1}{n} < a$, the interval is empty.) Draw pictures, then show set inclusion both ways.

Problem 328. Let $a < b$. Show that $\bigcap_{n=1}^{\infty}(a - \frac{1}{n}, b + \frac{1}{n}) = [a, b]$. Draw pictures, then show set inclusion both ways.

Problem 329. Let $A = \bigcup_{r \in \mathbb{Q}}(r - \frac{1}{10}, r + \frac{1}{10})$. Here, $\mathbb{Q}$ is the set of all rational numbers. Think of a simpler way to describe the set A, then prove your conjecture by showing set containment both ways.

Problem 330. Let $A = \bigcup_{n=0}^{\infty} [n, n^2]$.

a. List out the first five sets in this union, as you did above. Draw them on a number line if it helps.

b. Make a conjecture about how you can write A as a union of three simpler sets, and call the new union B.

c. Show that $A \subseteq B$. Let $x \in A$. Then $x \in [n, n^2]$ for some $n = 0, 1, 2, \ldots$. You want to show that $x \in B$. You can do this with three cases, depending on whether $n = 0$, $n = 1$, or $n > 1$. Each case needs to end with $x \in B$.

Case 1. $n = 0$

Case 2. $n = 1$

Case 3.

d. Show that $B \subseteq A$. Let $x \in B$. There are three cases, and in each one, you will need to construct n so that $x \in [n, n^2]$. Each case needs to end with $x \in A$.

Problem 331. Let $A = \bigcup_{k \in \mathbb{Z}} (k, k+1)$.

1. Draw out some of the intervals on a number line.

2. Make a conjecture about what set A is. You do not need to prove the conjecture.

Problem 332. For $n = 2, 3, 4, \ldots$, let $A_n = \{2n, 3n, 4n, \ldots\}$.

a. Write out the first five of the A_n.

b. Let $B = \bigcup_{n=2}^{\infty} A_n$. Describe the set B in simpler terms, perhaps by writing out the smallest 10 elements of B, then describe B in a sentence.

c. Describe the integers larger than 1 that are not in B.

Problem 333. Use quantifiers to write down what it means that $x \in \bigcup_{n=1}^{\infty} A_n$.
Use quantifiers to express what it means that $x \notin \bigcup_{n=1}^{\infty} A_n$ by negating quantifiers and rewriting until the expression is as simple as possible.

Problem 334. Use quantifiers to write down what it means that $x \in \bigcap_{n=1}^{\infty} A_n$.
Use quantifiers to express what it means that $x \notin \bigcap_{n=1}^{\infty} A_n$ by negating quantifiers and rewriting until the expression is as simple as possible.

Definition 335. Set complement. If A is a set, then A^c is all elements under consideration that are not in A. For example, if $A = [0, 8)$, then $A^c = (-\infty, 0) \cup [8, \infty)$.

Problem 336. de Morgan's law. Show that $(\bigcup_{n=1}^{\infty} A_n)^c = \bigcap_{n=1}^{\infty} A_n^c$ by writing logical expressions for x being in the set on the left side and for the right side. Start by writing a logical expression that means the same thing as $x \in (\bigcup_{n=1}^{\infty} A_n)^c$ and work with it until it is a logical expression for $x \in \bigcap_{n=1}^{\infty} A_n^c$. When you write the proof this way, you do not need to show containment both ways to show that the two sets are equal.

$x \in (\bigcup_{n=1}^{\infty} A_n)^c$ means $\neg(\exists \text{integer } n \text{ such that } x \in A_n)$, which means ...

Your name: ______________________________

Deriving properties of inequalities

We can define the < relation for real numbers and establish its properties.

Overview

In this activity, we back up to the point after the real numbers have been constructed, but before subtraction and inequalities have been defined. We define the < relation and prove a number of useful properties that it satisfies. Since the > relation is so similar, we will not define it or show its properties.

Remark 337. Most of us first learned numbers by counting, using 1, 2, 3, . . . , which we will call *positive integers*. Later, we learned about addition of positive integers and multiplication of positive integers. Both operations give back positive integers; we say that the set of positive integers is *closed* under addition and multiplication. Later, we learned about zero, negative numbers, rational numbers, and real numbers. It is not always made clear, but the negative integers can be constructed from the positive integers, the rationals from the integers, and the reals from the rationals. In this activity, we assume that the real numbers have been constructed and have been shown to have their usual algebraic properties, and work from there to prove some basic (and very familiar) facts.

Note 338. Let $\mathbb{R}$ denote the set of real numbers, and denote addition and multiplication of real numbers in the usual ways. **Addition** has these properties: commutativity ($a + b = b + a$), associativity ($a + (b + c) = (a + b) + c$), additive identity (there exists a unique real number called 0 for which $a + 0 = a$ for all $a \in \mathbb{R}$), and additive inverse (for each number a in $\mathbb{R}$, there exists a unique real number $-a$ for which $a + (-a) = 0$). **Multiplication** has these properties: commutativity ($ab = ba$), associativity ($a(bc) = (ab)c$), multiplicative identity (there exists a unique real number called 1, with $1 \neq 0$, such that $a \cdot 1 = a$ for all a in $\mathbb{R}$), multiplicative inverse (for each a in $\mathbb{R}$ with $a \neq 0$, there exists a unique number called a^{-1} for which $a \cdot a^{-1} = 1$. **Addition and multiplication** are related by the distributive property: $((a + b)c = ac + bc)$.

Note 339. In this activity, subtraction is not defined, so be careful not to use it!

Show 340. Let a be a real number. Justify each line in the following proof to show that $0 \cdot a = 0$.

$$\begin{aligned}
a + (-a) &= 0 \\
1 \cdot a + (-a) &= 0 \\
(0 + 1) \cdot a + (-a) &= 0 \\
(0 \cdot a + 1 \cdot a) + (-a) &= 0 \\
(0 \cdot a + a) + (-a) &= 0 \\
0 \cdot a + (a + (-a)) &= 0 \\
0 \cdot a + 0 &= 0 \\
0 \cdot a &= 0
\end{aligned}$$

Show 341. People sometimes ask if the additive inverse $(-a)$ is the same as the product $(-1)\cdot a$, where (-1) is the additive inverse of 1. It's true, and here is how you show it; fill in steps and write the justifications at the right side of each line.

$$\begin{aligned}
a + (-1) \cdot a &= 1 \cdot a + (-1) \cdot a \\
&= (1 + (-1)) \cdot a \\
&= \\
&= 0,
\end{aligned}$$

This shows that $(-1) \cdot a$ is the additive inverse of a, because that number is unique.

Show 342. Let $a \in \mathbb{R}$. The statement $-(-a) = a$ is just a statement about additive inverses. Prove that it is true.

Show 343. You might think that it is obvious that $(-1)(-1) = 1$, where (-1) is the additive inverse of 1, but this takes a few steps. Fill in steps and write justifications.

$$\begin{aligned}(-1) + (-1)(-1) &= (-1)(1) + (-1)(-1) \\ &= (-1)(1 + (-1)) \\ &= \\ &= 0\end{aligned}$$

Why does this show that $(-1)(-1)$ is the additive inverse of -1?

Show 344. The additive inverse of a sum works out nicely. Let a and b be real numbers and think about the additive inverse of $a + b$. Write justifications to the right of each statement.

$$\begin{aligned}-(a+b) &= (-1)(a+b) \\ &= (-1)(a) + (-1)(b) \\ &= (-a) + (-b)\end{aligned}$$

Definition 345. Positive real numbers. By construction, the real numbers have a subset $\mathbb{R}^+$, called the *positive real numbers,* for which:

a. If $a, b \in \mathbb{R}^+$, then $a + b \in \mathbb{R}^+$. ($\mathbb{R}^+$ is closed under addition.)

b. If $a, b \in \mathbb{R}^+$, then $a \cdot b \in \mathbb{R}^+$. ($\mathbb{R}^+$ is closed under multiplication.)

c. For every real number a, either $a \in \mathbb{R}^+$ or $(-a) \in \mathbb{R}^+$ or $a = 0$. Exactly one of the three happens.

Note that the positive real numbers are exactly analogous to the positive integers that you learned first. We can't use interval notation to write what $\mathbb{R}^+$ is, because intervals are defined in terms of inequalities, and we have not defined inequalities yet!

Exercise 346. Under each property in 340, write a sentence that states it in plain English

Show 347. Let $a \in \mathbb{R}$ and suppose that $a \neq 0$. Show that $a \cdot a \in \mathbb{R}^+$, justifying each step, citing previous definitions or results by number. **Hint:** Use a proof by cases, using the two remaining cases in 340c. For future reference, this gives us a new way to show that a number is in $\mathbb{R}^+$.

Case 1. Suppose that $a \in$ ______.

Case 2. Suppose that $(-a) \in$ ______.

Show 348. Show that $1 \in \mathbb{R}^+$. Justify each step.

Show 349. Show that $(-1) \notin \mathbb{R}^+$. **Hint:** Pretend for a minute that $(-1) \in \mathbb{R}^+$ and use 340c.

Definition 350. Less than. Let a and b be real numbers. If $b + (-a) \in \mathbb{R}^+$, we write that $a < b$ and say that a is *less than* b.

Note 351. All of the following problems rely on Definition 345, so you will use it over and over. Note that $>$ has not been defined yet, so be careful not to use it.

Show 352. Show that $-1 < 0$.

Show 353. Show that $1 < 1$ is not true. Thus, the $<$ relation is not reflexive. Justify each step, citing previous results by number.

Show 354. Show that $0 < 1$.

Show that $1 < 0$ is not true. Thus, the $<$ relation is not symmetric.

Show 355. Show that the $<$ relation on $\mathbb{R}$ is transitive. Follow good form by first letting a, b, c be real numbers and supposing that $a < b$ and $b < c$, then showing $a < c$. Justify each step by number. In this proof, you are likely to use the fact that $(-b) + b = 0$, which is the additive inverse property.

Show 356. Let $a, b \in \mathbb{R}$ and suppose that $a < b$. Show that $-b < -a$. Justify each step.

Show 357. Let $a, b, c \in \mathbb{R}$. Suppose that $a < b$. Show that $a + c < b + c$. Justify each step.

Show 358. Let $a, b, c, d \in \mathbb{R}$. Suppose that $a < b$ and $c < d$. Show that $a + c < b + d$. Justify each step.

Show 359. Let a, b, c be real numbers. Suppose that $a < b$ and $0 < c$. Show that $ac < bc$.

Show 360. Let a, b, c be real numbers. Suppose that $a < b$ and $c < 0$. Show that $bc < ac$.

Show 361. Let $a, b \in \mathbb{R}$ and suppose that $0 < a$ and $b < 0$. Use a previous result to show that $ab < 0$.

Show 362. Let $a \in \mathbb{R}$ and suppose that $0 < a$. Show that $0 < a^{-1}$. Here a^{-1} is the multiplicative inverse of a. **Hint:** This one take a bit more effort than the previous ones. Note that division has not been defined yet, so just use addition and multiplication.

Show 363. Let $a, b \in \mathbb{R}$ and suppose that $0 < a$ and $a < b$. Show that $b^{-1} < a^{-1}$.

Your name: ______________________________

Construction of the real numbers

Construction of the real numbers using Dedekind cuts

Overview

Many things can be defined and written about that don't actually exist; unicorns, little green men from Mars, and others may come to mind. To this point in your mathematical career, you have worked with real numbers and used many of their properties, but how do we know that they really exist? The mathematical answer is that we *construct* them from simpler numbers and show that they have the right properties.

Note 364. Natural numbers. At some point in your life you learned the natural numbers 1, 2, 3, ... Then you learned to add and multiply, and these operations have the familiar algebraic properties like commutativity, associativity, distributivity. Their biggest claim to fame: No matter how high you count, there is always a next number. Note, however, that we are not *defining* the natural numbers.

Definition 365. Zero. Subtraction problems like $9 - 5$ have answers that are natural numbers, but subtraction problems like $3 - 3$ and $12 - 12$ call for a new number. You can define 0 as $3 - 3$ or as $12 - 12$; there are many ways to write this new number.

Definition 366. Negative numbers. Subtraction problems like $5 - 9$ and $8 - 12$ need another set of new numbers to be defined. You can define -4 to be $5 - 9$ but also $8 - 12$. Anytime you want to work with -4, you can substitute in $5 - 9$ instead. Or $8 - 12$.

Exercise 367. With negative numbers, we say that $a - b = c - d$ if $a + d = c + b$. Check that this is the case for $5 - 9$ and $8 - 12$. It's important to check, and you only need to work with natural numbers.

Definition 368. Integers. The natural numbers, zero, and the negative numbers make up the integers. Each integer can be written as $a - b$ where a and b are natural numbers. The integers are closed under addition and multiplication, and these operations have the usual algebraic properties like commutativity and associativity, additive inverses, additive identity, and multiplicative identity.

Definition 369. Rational numbers. Division problems like $15 \div 5$ have answers that are integers, but problems like $5 \div 15$ need yet more new numbers to be defined. For some reason people decided to write the new number as $\frac{5}{15}$ but we could have chosen some other notation like $(5, 15)$ or $5\#15$. At any rate, these new "rational" numbers are made up of two integers, the second of which needs to be non-zero. The rules for rational numbers are worth noting in some detail:

a. Rational numbers $\frac{a}{b}$ and $\frac{c}{d}$ are *equal* if $ad = bc$, which is a matter of integer multiplication.

b. We say that $\frac{a}{b} < \frac{c}{d}$ if $ad < bc$, which again comes down to integers

c. The sum of $\frac{a}{b}$ and $\frac{c}{d}$ is the rational number $\frac{ad+bc}{bd}$. Note that you only need to use integer arithmetic to find the sum of two rational numbers.

d. The product of $\frac{a}{b}$ and $\frac{c}{d}$ is the rational number $\frac{ac}{bd}$.

It's important to note that everything about these new rational numbers is defined in terms of integers. There are multiple ways to define each rational number; $\frac{3}{12} = \frac{1}{4}$ for example. The rational numbers can be shown to have the usual algebraic properties, always because the integers have the property. Bonus: non-zero rational numbers have multiplicative inverses. The set of all rational numbers is denoted by $\mathbb{Q}$.

Exercise 370. Use Definition 364 for each part.

a. Check that $\frac{1}{3} = \frac{5}{15}$ using the definition.

b. Check that $\frac{1}{4} < \frac{2}{7}$ using the definition.

c. Add $\frac{1}{3} + \frac{1}{4}$ using the definition.

d. Show that $\frac{a}{b} + \frac{c}{d} = \frac{c}{d} + \frac{a}{b}$ using a rewrite proof. Proofs of other algebraic properties are similar.

Definition 371. Complex numbers. Once the real numbers have been defined, we can define the complex numbers by letting $i = \sqrt{-1}$ and then thinking about numbers of the form $a + ib$ where a and b are real numbers.

Remark 372. To construct the integers, the rational numbers, and the complex numbers, you put together two numbers of a simpler sort. As it happens, constructing the real numbers is harder. The basic idea is this: to refer to a real number like π, think of the rational numbers 3, 3.1, 3.14, 3.141, 3.1415, and all other rational numbers less than π. Then π is the "top" of this set of rational numbers. This is how we can use rational numbers to get our hands on real numbers like π that are not rational. In fact, we will literally *define* real numbers to be sets of rational numbers like this.

Remark 373. Here are a few examples of the sets we'll be using. After each set, describe it in words.

a. $A = \{q \in \mathbb{Q} : q < 0\}$

b. $B = \{q \in \mathbb{Q} : q \leq 7\}$

c. $C = \{q \in \mathbb{Q} : q^3 < 5\}$

d. $D = \{q \in \mathbb{Q} : q < 0 \text{ or } q^2 < 2\}$

It's OK to use words like "cube root of 5" in your description, even though that is not a rational number, and so has not been constructed yet.

Definition 374. Closed below. A set A of rational numbers is said to be *cl4osed below* if for all $q \in A$, for all $p \in \mathbb{Q}$ with $p < q$, we have $p \in A$. In words, if A contains the rational number q, then it contains every rational number less than q as well.

Exercise 375. Is the set $\{\frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \ldots\}$ closed below? Why or why not?

Remark 376. Note that "closed below" is unusual in that it has two "for all" quantifiers. To prove that a set A is closed below, follow this format: "Let $q \in A$. Let $p \in \mathbb{Q}$ such that $p < q$. Show that $p \in A$."

Exercise 377. Show that each of the following sets is closed below, following Remark 371.

a. $A = \{q \in \mathbb{Q} : q < 0\}$. **Hint:** Use transitivity.

b. $B = \{q \in \mathbb{Q} : q \leq 7\}$

c. $C = \{q \in \mathbb{Q} : q^3 < 5\}$. Do not write $\sqrt[3]{5}$, just use rational numbers and integers.

d. $D = \{q \in \mathbb{Q} : q < 0 \text{ or } q^2 < 2\}$. **Hint:** Consider two cases, $p < 0$ and $p \geq 0$. Do not write $\sqrt{2}$.

Remark 378. Sets C and D illustrate how we can use sets of rational numbers to point to a number that we know is irrational, in this case $\sqrt[3]{5}$ and $\sqrt{2}$. The irrational number we have in mind is at the "top" of the set, just like 0 is at the "top" of the set A.

Definition 379. Dedekind cut. A set $A \subseteq \mathbb{Q}$ is called a *Dedekind cut* if all of the following happen:

i. A is closed below

ii. A has no greatest element, meaning that for all $p \in A$, there is a number $q \in A$ with $p < q$. Keep in mind that q has to be a rational number.

iii. There exists an integer m with $m \in A$

iv. There exists an integer n with $n \notin A$

Exercise 380. Check each of the sets below to see if it is a Dedekind cut. You have already shown that they are closed below. If the set is a Dedekind cut, show that. If not, explain why not.

a. $A = \{q \in \mathbb{Q} : q < 0\}$. Let $m =$ ______________. Let $n =$ ______________. To show (ii), let $p \in A$. Let $q =$ ______________.

b. $B = \{q \in \mathbb{Q} : q \leq 7\}$

c. $C = \{q \in \mathbb{Q} : q^3 < 5\}$. Let $m =$ ______________. Let $n =$ ______________. To show (ii), given $p \in C$, I suggest you consider two cases. When $p < 1$, let $q = 1$, then $q^3 < 5$. When $p \geq 1$, let $q = p + c$ where $c = \frac{5-p^3}{100}$ when $p \geq 1$. Calculate $(p+c)^3$ and use the fact that $c < 1$, so that $c^2 < c$ and $c^3 < c$. Also keep in mind that $p < 2$.

d. $D = \{q \in \mathbb{Q} : q < 0 \text{ or } q^2 < 2\}$.

Exercise 381. Let A and B be Dedekind cuts. Define a new set C by $C = \{z :$ there exist $a \in A$ and $b \in B$ such that $z = a + b\}$. Show that C is a Dedekind cut by checking all four requirements in 374.

1. C is closed below. Let $d \in C$, and let $c \in \mathbb{Q}$ with $c < d$. We need to show that $c \in C$, so we need to write it as the sum of an element of A and an element of B. Because __________, there exist $a \in A$ and $b \in B$ such that $d = a + b$. Let $p = a - (d - c)/2$ and $q = b - (d - c)/2$. It's clear that p and q are _____________ numbers and that $p < a$ and that _____________. Since $p < a$ and A is closed below, we know that $p \in A$. Since $q < b$ and ___________________, we know that _____________. Finally, $p + q =$ ____ $= c$, and so _____________.

2. C has no greatest element. Let $p \in C$. Show that there is a number $q \in C$ with $p < q$.

3. There exists an integer m with $m \in C$. **Hint:** Use $m_A \in A$ and $m_B \in B$ from 374(c).

4. There exists an integer n with $n \notin C$. **Hint:** Use $n_A \notin A$ and $n_B \notin B$. Then $a < n_A$ for all $a \in A$ and $b < n_B$ for all $b \in B$. Then $a + b < n_A + n_B$ for all $a \in A$ and all $b \in B$. Now ... why does that mean that $n_A + n_B$ is not in C?

Definition 382. Real numbers. We will refer to each Dedekind cut as a "real number." The set of real numbers will be written as $\mathbb{R}$.

Remark 383. Yes, you read that right. A real number is being defined as nothing more, and nothing less, than a Dedekind cut, which is a set of rational numbers. This is simply one way to use rational numbers to describe and work with real numbers. It was easier to define 0 as $3 - 3$ or to define fractions as things you get from pairs of integers. You'll get used to it.

Definition 384. Equality of real numbers. If A and B are real numbers, we say that A and B are equal if $A \subseteq B$ and $B \subseteq A$. We write $A = B$.

Definition 385. Less than for real numbers. If A and B are real numbers we say that A is less than B if $A \subset B$. We write $A < B$.

Definition 386. The real number zero. Let $\mathbf{0} = \{q \in \mathbb{Q} : q < 0\}$. Note that 0 is in bold.

Definition 387. The real number one. Let $\mathbf{1} = \{q \in \mathbb{Q} : q < 1\}$.

Definition 388. Addition of real numbers. Let A and B be real numbers. The sum of A and B is the real number $C = \{z :$ there exist $a \in A$ and $b \in B$ such that $z = a + b\}$. This set is a Dedekind cut as explained in 376. We write $A \oplus B$ for the sum to make clear that it is a new type of addition.

Prove 389. Show that $\mathbf{0} \oplus \mathbf{1} = \mathbf{1}$ by showing set inclusion both ways.

a. Show $\mathbf{0} \oplus \mathbf{1} \subseteq \mathbf{1}$. Let $c \in \mathbf{0} \oplus \mathbf{1}$. Then $c = a + b$ for some $a \in \mathbf{0}$ and $b \in \mathbf{1}$. Thus $a < 0$ and $b < 1$. Thus, _____________ and so $c \in \mathbf{1}$.

b. Show $\mathbf{1} \subseteq \mathbf{0} \oplus \mathbf{1}$. Let $c \in \mathbf{1}$. Then ______________. Write $c = \frac{c-1}{2} + \frac{c+1}{2}$ and check that this means that $c \in \mathbf{0} \oplus \mathbf{1}$.

Prove 390. Commutativity. Let A and B be real numbers. Show that $A \oplus B = B \oplus A$. Instead of showing set inclusion both ways, do this as a rewrite proof:

$$\begin{aligned} A \oplus B &= \{a + b : a \in A, b \in B\} \\ &= \{b + a : \\ &= \end{aligned}$$

Prove 391. Associativity. Let A, B, and C be real numbers. Show that $A \oplus (B \oplus C) = (A \oplus B) \oplus C$. Do this as a rewrite proof, abbreviating the definition of the sum.

$$\begin{aligned} A \oplus (B \oplus C) &= A \oplus \{b + c : b \in B, c \in C\} \\ &= \{a + (b + c) : \\ &= \end{aligned}$$

Prove 392. Additive identity. Let A be a real number. Show that $A \oplus \mathbf{0} = A$.

a. Let $p \in A \oplus \mathbf{0}$. Then $p = a + b$ where $a \in$ ______________ and b ______________. **Hint:** You will use the fact that A is closed below.

b. Let $p \in A$. We need to write p as the sum of an element a of A and a rational number b less than zero. That means that a will be greater than p. Fortunately, such a number exists because A has no greatest element.

Let $a \in A$ such that $p < a$. Let $b = p - a$. Then ...

Definition 393. Additive inverse.. Let A be a real number. Define a new set $-A$ by

$$-A = \{b - c : b, c \in Q, b < 0, c \notin A\}$$

Think of b as being very close to 0, so the elements of $(-A)$ are like the negatives of the rational numbers bigger than the elements of A. It may be hard to understand this set intuitively, and so it may be easier just to work with the definition and not try to get the intuition straight.

Guided proof 394. Show that $-A$ is a Dedekind cut.

i. A is closed below. Let $q \in -A$. Let $p \in \mathbb{Q}$ with $p < q$. Since $q \in -A$, we can write $q = b - c$ where $b < 0$ and $c \notin A$. Check that $p = b + (p - q) - c$ by substituting in for q. Thus, we have written $p = b' - c'$ where $b' < 0$ and $c' \notin A$, by setting $b' =$______________ and $c' =$______________. This tells us that $p \in -A$, so $-A$ is closed below.

ii. A has no greatest element. Let $p \in -A$. Then we can write $p = b - c$ where $b < 0$ and $c \notin A$. Let $q = (b/2) - c$. Check that $q \in -A$ and that $p < q$.

iii. There exists an integer m with $m \in -A$. Check that $-1 - n_A$ where $n_A \notin A$ from 374(d) will work.

iv. Challenge: There exists an integer n with $n \notin -A$. Probably $-m_A + 1$ from 374(c) will work.

Prove 395. Additive inverse property. Let A be a real number. Show that $A \oplus (-A) = \mathbf{0}$.

a. Let $z \in A \oplus (-A)$. Then $z = p + q$ where $p \in A$ and $q = b - c$ where $b < 0$ and $c \notin A$. That is, $z = p + b - c$. Since $c \notin A$, we know that $c > p$, because __________________________. Then $z < 0$ because __________________________. Thus $z \in \mathbf{0}$, and so $A \oplus (-A) \subseteq \mathbf{0}$.

b. Let $z \in \mathbf{0}$. Then z is a rational number and $z < 0$. We need to write $z = p + q$ where $p \in A$ and $q \in (-A)$, and q needs to be written as $b - c$ where $b < 0$ and $c \notin A$. If z is close to 0, then p and c will need to be close to the "top" of the set A. The next result will show that there exists $p \in A$ and $c \notin A$ with $p - c = z/2$. Also let $b = z/2$. Then $z = z/2 + z/2 = p - c + b$ as required. So $z \in A \oplus (-A)$ and thus $\mathbf{0} \subseteq A \oplus (-A)$.

Guided proof 396. Let A be a real number. Let $c > 0$ be a rational number. Then there exists $a \in A$ and $b \notin A$ such that $b - a = c$.

By the definition of a Dedekind cut, there are integers $m \in A$ and $n \notin A$. Consider the numbers $m + kc$ for $k = 0, 1, 2, \ldots$. Draw a picture of these on a number line below. When $k = 0$, $m + kc \in A$. When $k > (n-m)/c$, $m+kc > n$ and so $m+kc \notin A$. Thus, for some value of k, $m+kc \in A$ but $m+(k+1)c \notin A$. Let $a = m + kc$ and $b = m + (k+1)c$.

Challenge 397. $\mathbf{0}$ is unique. That is, if there is another real number Z for which $A \oplus Z = A$ for all real numbers A, then $Z = \mathbf{0}$.

Challenge 398. Given a real number A, the additive inverse $-A$ is unique.

Definition 399. Multiplication of positive real numbers. Let A and B be real numbers with $\mathbf{0} < A$ and $\mathbf{0} < B$. The product of A and B is the real number $\{z : z \in \mathbb{Q} \text{ and } z \leq 0 \text{ or there exists } a \in A \text{ with } a > 0 \text{ and } b \in B \text{ with } b > 0 \text{ such that } z = ab\}$. We write $A \otimes B$ to denote the product.

Remark 400. It is easiest to define multiplication of positive real numbers. The definition is very much like the definition of the sum, only a bit more complicated because of the need to include the negative rational numbers.

Challenge 401. Show that $A \otimes B$ is a Dedekind cut.

Prove 402. Multiplicative identity for positive real numbers. Let A be a real number with $0 < A$. Then $A \otimes \mathbf{1} = A$.

Prove 403. Distributivity for positive real numbers. Let A, B, and C be real numbers with $0 < A$, $0 < B$, and $0 < C$. Show that $A \otimes (B \oplus C) = (A \otimes B) \oplus (A \otimes C)$.

Definition 404. Multiplicative inverse of positive real numbers. Let A be a real number with $0 < A$. Define A^{-1} to be $\{z : z \in \mathbb{Q} \text{ and } z \leq 0 \text{ or } 1/z \in A\}$.

Prove 405. Show that A^{-1} is a Dedekind cut.

Prove 406. Show that $A \otimes A^{-1} = \mathbf{1}$

Prove 407. Let $D = \{q \in \mathbb{Q} : q < 0 \text{ or } q^2 < 2\}$ and let $E = \{q \in \mathbb{Q} : q < 2\}$. Show that $D \otimes D = E$. This confirms that D corresponds to the square root of 2.

Prove 408. Let A and B be real numbers. Show that $A \otimes B = B \otimes A$.

Prove 409. Let A, B, and C be real numbers. Show that $A \otimes (B \otimes C) = (A \otimes B) \otimes C$.

Prove 410. Prove that the multiplicative identity **1** is unique

Prove 411. Prove that the multiplicative inverse A^{-1} is unique

Definition 412. Multiplication of non-positive real numbers.. Let A and B be real numbers.

- **a.** If $0 < A$ and $B < 0$, then $A \times B = A \times (-B)$
- **b.** If $A < 0$ and $0 < B$, then $A \times B = (-A) \times B$
- **c.** If $A < 0$ and $B < 0$, then $A \times B = (-A) \times (-B)$

Prove 413. Show that the properties of multiplication extend to multiplication of non-positive numbers.

Definition 414. Least upper bound. Let S be a collection of real numbers. A number B is the least upper bound of S if $A \leq B$ for all $A \in S$, and if for all other upper bounds C of S, we have $B \leq C$.

Prove 415. Given a set S of real numbers, the number $B = \cup_{A \in S} A$ is the least upper bound of S. Thus, every collection of real numbers has a least upper bound.

Your name: ______________________________

Union, intersection, Venn diagram, complement, difference

Overview

An introduction to set unions and intersections, Venn diagrams to represent sets, and set complements and differences. Co-authored by Johanson Berlie.

Definition 416. Disjoint sets. If two sets A and B have no elements in common, we say that they are **disjoint** sets.

Example 417. The set consisting of all World War II veterans and the set of all millennials are disjoint sets.

Definition 418. Comparable sets. Two sets A and B are **comparable** if $A \subseteq B$ or $B \subseteq A$. If this is not the case, then they are said to be **not comparable**.

Example 419. The sets A = {General Motors, Toyota, Ford, Renault} and B = {Tesla, Fisker} are not comparable. If we define a set C = {Ford, Toyota}, then A and C are comparable, since $C \subseteq A$.

Exercise 420. For the following questions, compare the sets in the context of the definitions above. Are they equal? Are they disjoint? Are they comparable? Is one a subset of the other? In every case, explain why.

a. The set A of US citizens and the set B of people in the US right now.

b. Think of the set A of US citizens and the set J of citizens of Japan. Do they have any elements in common, or are they disjoint? This might require an internet search.

c. The set $A = \{⋇, ⊠, ◀, ⊞\}$ and the set $B = \{⊞, ◀, ⋇, ⊠\}$

d. The set of humans that have been to more than one planet and the set of humans that have been to Pluto.

e. The set G consisting of the members of the Green Bay Packers on the first play of a game and the set M consisting of the members of the Manchester United soccer team in play during a game.

f. The set J consisting of all employees of the Department of Justice and the set F consisting of all federal employees.

g. Suppose you know that $A \subseteq B$ and also that $B \subseteq A$. What more can you say about A and B now? Why?

h. Is it possible for the universal set to be disjoint from any of its subsets? Explain why or why not.

Definition 421. Union of sets. The **union** of sets A and B is a new set, consisting of all elements which belong to A or to B or to both. We denote this by $A \cup B$.

Definition 422. Intersection of sets. The **intersection** of sets A and B is a new set, consisting of all elements which belong to both A and B. We denote this by $A \cap B$.

Example 423. Let C be the set of Computer Science majors and M be the set of Mathematics majors. Then $C \cap M$ is the set of students double majoring in Computer Science and Mathematics (a powerful combination!) while $C \cup M$ is the set of students majoring in one, the other, or both majors.

Exercise 424. Answer the following, expressing the answers as sets where possible.

a. If $P = \{red, blue, yellow\}$, $S = \{purple, green, orange\}$, $M = \{red, green, blue\}$, find:

b. $P \cup S$

c. $P \cap M$

d. $P \cup S \cup M$

e. $P \cap S \cap M$

f. What is $A \cap A$? What about $A \cup A$?

g. Is it always true that $A \cup B = B \cup A$? Explain.

h. If $A \subseteq B$ then what is $A \cup B$?

i. Is $A \cap B$ a subset of A? Is $A \cap B$ a subset of B? Explain.

j. What is $\varnothing \cap A$? Does it depend on the set A? Explain why or why not.

k. What is $\varnothing \cup A$? Does it depend on the set A? Explain why or why not.

l. If $A \cup B = \varnothing$, can we say anything about A or B? Is there something special about them? Explain.

m. If A and B are disjoint, what can we say about $A \cap B$?

n. If $A \subseteq B$ and B and C are disjoint, then what about A and C?

o. If U denotes the universal set, is it true that $U \cap A = A \cap \varnothing$? Explain.

Definition 425. Universal set. If all of the sets under discussion are subsets of a fixed set, this set is called the **universal set** or **universe of discourse** and denoted by **U**. Sometimes there is more than one possibility for U.

Example 426. If we were studying the citizenship of people around the world, the universal set would consist of all the people on earth. Citizens of the US would be one interesting set, citizens of Canada would be another. Do these two sets have any elements in common? If so, how could that happen?

Example 427. If we were studying binary stars, the universal set would be all the stars in the universe.

Exercise 428.

a. When thinking of prime numbers, what is a good choice for the universal set?

b. When solving linear equations like $5x + 7 = 3$, what a good choice for the universal set?

c. When solving quadratic equations like $x^2 + 5x + 8 = 0$, what is the universal set?

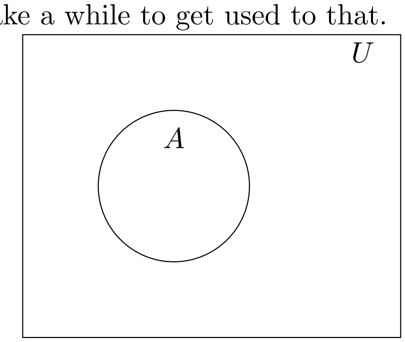
d. Is it possible for the universal set to be empty? Explain why or why not.

e. If two sets A and B are subsets of a given universal set, U, is it possible that $A = U$ or $B = U$? Explain why or why not.

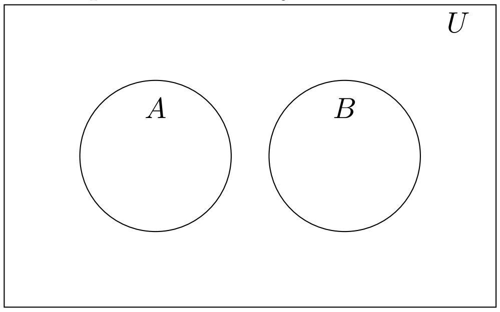
Definition 429. Venn diagram. A Venn diagram (also called **primary diagram**, **set diagram** or **logic diagram**) is a picture that shows all possible logical relations between a finite collection of sets. These diagrams depict elements as points in the plane, and sets as regions inside closed curves.

Remark 430. This definition might seem complicated, but for our purposes we will think of Venn diagrams as circles that represent sets. The following examples will help make this clear.

Example 431. To represent a single set using a Venn diagram, we draw a single circle (representing the set) inside a rectangle (representing the universal set). We usually write the label for a given set inside the curve that represents it. Note that the set A could be empty but we still draw it with a circle; it may take a while to get used to that.



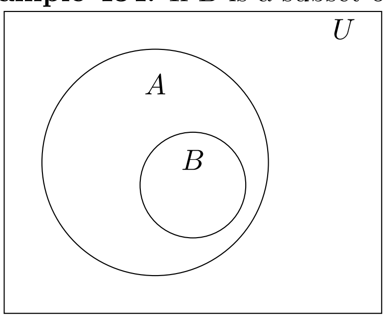
Example 432. The representation in the above example can be extended to any finite number of sets. If want to represent two disjoint sets, A and B, we can represent them as below.



The circles do not overlap, to indicate that the sets are disjoint. Note that A or B or both could actually be empty sets, a bit like looking down into empty paper bags.

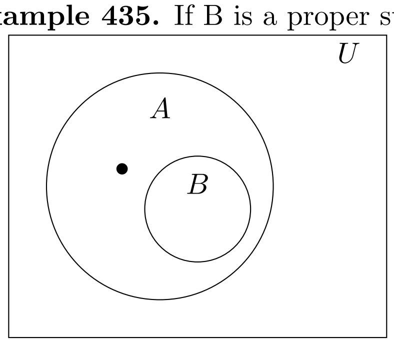
Remark 433. The power of Venn diagrams is that they make it easy to understand set relations and operations. Let's look at a few examples.

Example 434. If B is a subset of A, that is, if $B \subseteq A$ we can represent this as below.



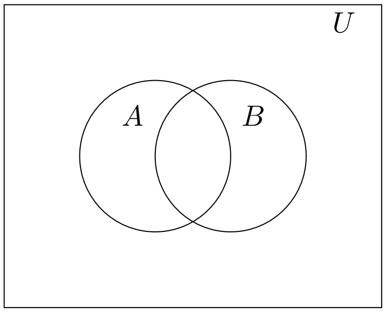
Note that it is possible that $B = A$; the blank areas in the diagram may or may not contain points; they may be empty.

Example 435. If B is a proper subset of A, we can represent this with a dot outside B, but within A.

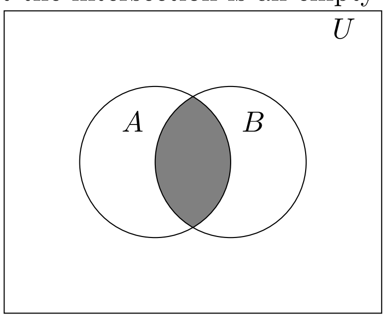


Now it's clear that every element of B is also an element of A, but that there is an element of A that is not an element of B. It could be that B is empty.

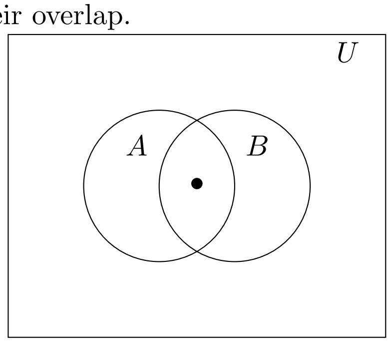
Example 436. The generic picture of two sets A and B shows them overlapping, but not completely:



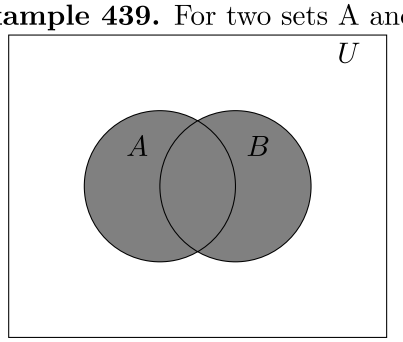
Example 437. To indicate the intersection of A and B, we shade $A \cap B$ as below, even if it is possible that the intersection is an empty set:



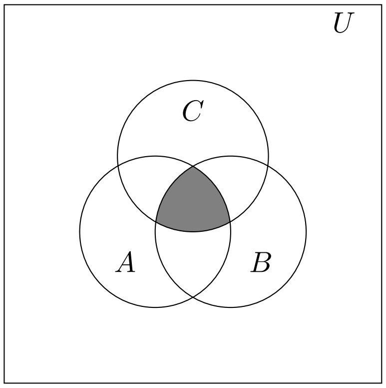
Example 438. If we know that A and B, are not disjoint, we indicate this by a dot inside the region of their overlap.



Example 439. For two sets A and B, we represent $A \cup B$ as below:



Example 440. For three sets A, B and C, we draw them to allow for all possible intersections. We represent $A \cap B \cap C$ as below, even though it may actually be an empty set:



Exercise 441. For the following questions, represent your answers as Venn diagrams, where possible. If a Venn diagram is not possible, explain why.

a. Make a Venn diagram for sets A and B, showing that A and B are disjoint, and shade $A \cup B$.

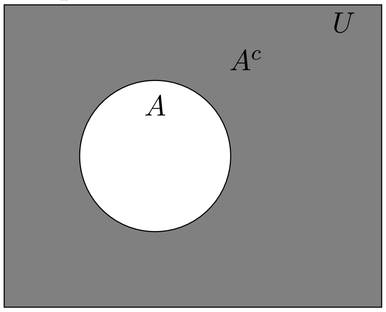
b. Make a Venn diagram for sets A, B, and C, showing that B and C are disjoint. Shade the set $(A \cap B) \cup C$.

c. Make a Venn diagram for sets A, B, and C, showing that $A \cap B \cap C$ is empty, and shading $(A \cap B) \cup (A \cap C) \cup (B \cap C)$.

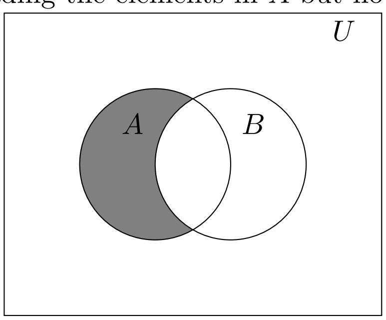
d. Make a Venn diagram for sets A, B, and C, showing that A and B are not disjoint and that $C \subset B$ but C is disjoint from A.

e. Make a Venn diagram for sets A, B, and C, showing that $A \cap B \subset C$, but $C \subset A \cup B$.

Definition 442. Set complement. The **complement** of a set A is the set of elements that do not belong to A. Therefore, it is every element of the universal set U that is not an element of A. We denote the complement of A as A^c or A'. Visually, we represent this as:



Definition 443. Set difference. The **difference** of sets A and B is the set of elements which belong to A but not to B. We denote this as $A - B$ or $A \setminus B$. Visually, we represent this by drawing A and B and shading the elements in A but not in B.



Exercise 444. For the following questions, start by drawing a Venn diagram for sets A and B, then shade the desired set or sets. For more than one set, use different shading for each one.

a. $(A \cap B)^c$, given that A and B are not disjoint.

b. $(A^c \cup B^c)$, given that A and B are not disjoint.

c. A^c and $A^c \setminus B$.

d. Supposing that $B \subset A$, shade in $A \setminus B$.

Your name: ______________________________

Operations on sets

This activity works with set identities and relates them to logic.

Overview

Sets are absolutely fundamental to mathematics. This chapter focuses on building up set identities, relationships between sets that are always true.

Problem 445. Let A and B be sets. Show that $(A \cup B)^c = A^c \cap B^c$ by showing set inclusion both ways. The first part is done for you. This is one of de Morgan's laws. Draw a really nice Venn diagram to illustrate.

- Let $x \in (A \cup B)^c$. Then $x \notin A \cup B$. So $x \notin A$ and $x \notin B$. That means that $x \in A^c$ and $x \in B^c$, and so $x \in A^c \cap B^c$. Since x was arbitrary, $(A \cup B)^c \subseteq A^c \cap B^c$.
- Let $x \in A^c \cap B^c$.

Problem 446. Let A and B be sets. Show that $(A \cap B)^c = A^c \cup B^c$ by showing set inclusion both ways. This is the other one of de Morgan's laws. Draw a really nice Venn diagram to illustrate.

Problem 447. Let A and B be sets. Let P be the logical statement $x \in A$, and let Q be the logical statement $x \in B$. Use P and Q and logic symbols ($\wedge$ for *and*, $\vee$ for *or*, $\neg$ for *not*) to translate statements about sets into logic statements:

1. $x \in A \cup B$ is ____________________

2. $x \in A^c$ is ____________________

3. $x \in B^c$ is ____________________

4. $x \in A^c \cap B^c$ is ____________________

5. $x \in (A \cup B)^c$ is ____________________

In the white space above and to the right, make a truth table for P, Q, and each of the other logical statements in the previous problem to establish that $x \in (A \cup B)^c$ is logically equivalent to $x \in A^c \cap B^c$. Compare the truth values in the columns corresponding to $x \in (A \cup B)^c$ to the Venn diagram you made above. Explain how they agree.

Problem 448. Let A and B be sets. Follow the previous exercise to use a truth table to show that $x \in (A \cap B)^c$ is logically equivalent to $x \in A^c \cup B^c$. Compare the truth table to the Venn diagram again.

Problem 449. Let D, E, and F be sets. Use one of de Morgan's laws that you showed above to establish that $(D \cup E \cup F)^c = D^c \cap E^c \cap F^c$. This proof works by rewriting, not by showing inclusion both ways. **Hint:** Let $A = D \cup E$ and $B = F$.

Problem 450. Let D, E, and F be sets. Use one of de Morgan's laws to show that $(D \cap E \cap F)^c = D^c \cup E^c \cup F^c$.

Problem 451. Let A, B, and C be sets. Use logical statements P, Q, and R and a truth table to show that $x \in A \cup (B \cap C)$ is logically equivalent to $x \in (A \cup B) \cap (A \cup C)$. Be sure to define P, Q, and R at the beginning.

Problem 452. Let A, B, and C be sets. Show that $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$ by showing inclusion both ways. When you encounter a union, use a proof by cases. For example, if you know that $x \in A \cup B$, one case is that $x \in A$, the other is that x is not in A, but $x \in B$. Organize your writing carefully to make the steps of this argument really clear.

Definition 453. Set difference. Let A and B be sets. The *set difference* $A \backslash B$ is the set $A \cap B^c$, which is all points that are in A but not in B. Draw a Venn diagram to illustrate this definition.

Definition 454. Symmetric difference. Let A and B be sets. The *symmetric difference* of A and B is the set $A \triangle B = (A \backslash B) \cup (B \backslash A)$. Draw a Venn diagram to illustrate this definition.

Problem 455. Consider again the logical statements from 442. Write a logical statement that is equivalent to $x \in A \triangle B$. Make a truth table with 4 rows, labeled 1, 2, 3, 4, and three columns, one for $x \in A$, one for $x \in B$, and the third for $x \in A \triangle B$. Draw a Venn diagram and label the regions in it 1, 2, 3, 4 so that they correspond to the truth table.

Problem 456. Let A and B be sets. Show that $A \triangle B = B \triangle A$ by showing set inclusion both ways. Draw a nice Venn diagram to illustrate.

Problem 457. Let A, B, and C be sets. Show that $(A \triangle B) \triangle C = A \triangle (B \triangle C)$ in three ways.

1. Draw separate Venn diagrams for the two sets.

2. Show set inclusion both ways.

3. Convert inclusion in $A \triangle B$, $B \triangle C$, and other sets to logical statements and use a truth table to show the equality.

Your name: ______________________________

Review exercises

Definition 458. Cross product. Let $\vec{a}$ and $\vec{b}$ be 3–dimensional vectors. The cross product of $\vec{a}$ and $\vec{b}$ is a new vector given by $\langle a_2b_3 - a_3b_2, a_3b_1 - a_1b_3, a_1b_2 - a_2b_1\rangle$. The cross product is denoted $\vec{a} \times \vec{b}$.

Prove 459. Show that the cross product is distributive over vector addition, that is, that $\vec{a} \times (\vec{b} \oplus \vec{c}) = \vec{a} \times \vec{b} \oplus \vec{a} \times \vec{c}$. Take small steps and justify each step.

Prove 460. Show that the cross product is anti-symmetric: for all 3–dimensional vectors $\vec{a}$ and $\vec{b}$, we have $\vec{a} \times \vec{b} = -\vec{b} \times \vec{a}$.

Prove 461. Define $\ln x = \int_1^x \frac{1}{t} dt$ for all real numbers $x > 0$. Show that for all real numbers x and y with $0 < x < y$, we have $\ln x < \ln y$. Working backward, rewrite $\ln x$ and $\ln y$ in terms of integrals, and then write an equality that relates integrals over different intervals.

Prove 462. Let n be an integer and suppose that n^2 is a multiple of 7. Show that n is a multiple of 7. Use the Division Algorithm to do this by cases, and be crystal clear about the structure of the proof.

Prove 463. Show that $\sqrt{7}$ is irrational.

Prove 464. For each $n = 1, 2, 3, \ldots$, let $a_n = \frac{n^2+3}{n^2+1}$. Show that for all $\varepsilon > 0$, there exists an integer n with $a_n - 1 < \varepsilon$. Be careful to recognize the two quantifiers in the statement and use appropriate proof techniques for each one. **Hint:** Rewrite a_n to look like 1 plus something small.

Problem 465. Suppose that $x \leq 5 + \frac{1}{n}$ for all $n = 1, 2, 3, \ldots$. Show that $x \leq 5$. **Hint:** Consider different types of proof including direct, contrapositive, contradiction, etc.

Prove 466. Show that for all $x \in \mathbb{R}$, there exists an integer $n \geq 1$ such that $x \in (-n, n)$. Use good form for proofs with nested quantifiers, and be sure to cover both positive and negative values of x. If you have trouble getting started, do scratchwork with $x = 4.2, x = -13.1, x = 0$.

Problem 467. Let $E = \{m \in \mathbb{Z} : \text{there exists } j \in \mathbb{Z} \text{ such that } m = 2j\}$. Let $O = \{m \in \mathbb{Z} : \text{there exists } k \in \mathbb{Z} \text{ such that } m = 2k + 1\}$. Show that $E \cap O = \emptyset$ by letting $m \in E \cap O$ and showing that this leads to a contradiction.

Problem 468. Continuing the previous problem, show that $E \cup O = \mathbb{Z}$ by showing set inclusion both ways.

Problem 469. Using standard interval notation, show that $[2, 6) \cap [3, 8) = [3, 6)$ by showing set inclusion both ways. As above, write compound inequalities, then individual inequalities, then compound inequalities again. Use a number line to illustrate.

Problem 470. Show that $[2, 6) \cup [3, 8) = [2, 8)$ by showing set inclusion both ways.

Prove 471. Show that $[2, 5] \cup (4, 7) = [2, 7)$.

Problem 472. Let $A = \bigcup_{n=1}^{\infty} (\frac{1}{n}, 1)$ and let $B = (0, 1)$. Show that $A = B$ by showing containment both ways.

Problem 473. Let $A = \bigcap_{n=1}^{\infty} [0, 1 + \frac{1}{n}]$ and $B = [0, 1]$. Prove that $A = B$ by showing containment both ways.

Your name: ______________________________

The Pigeonhole Principle

Surprising results can come from simple counting.

Overview

The Pigeonhole Principle says that if we have more pigeons than pigeonholes to put them in, then at least one pigeonhole must contain more than one pigeon. This idea was stated by Johann Peter Gustav Lejeune Dirichlet in 1834 and so is sometimes called Dirichlet's Box Principle.

Problem 474. If N pigeons are placed in n pigeonholes and $N > n$, then one of the pigeonholes must contain two or more pigeons. **Hint:** Prove by contradiction. That is, pretend for a minute that none of the pigeonholes has more than one pigeon. That leads to a contradiction.

Note 475. In each problem below, identify the "pigeonholes" and the rule you use to put each "pigeon" into a "pigeonhole." The first one is done for you. Then write a nice solution of the problem.

Problem 476. A bag contains M&M's in six different colors: Brown, Yellow, Green, Red, Orange and Blue. How many M&M's do you need to take out of the bag in order to have at least two of the same color? How many do you need to take out of the bag if you want to have three of the same color?
Pigeonholes: The colors Brown, Yellow, Green, Red, Orange, and Blue.
Rule: Put each M&M in a pigeonhole according to its color.

Problem 477. Prove that no matter how we choose 51 distinct natural numbers from $\{1, 2, 3, \ldots, 100\}$, at least two of them must be consecutive.
Pigeonholes: The sets $\{1, 2\}, \{3, 4\}, \ldots, \{99, 100\}$.
Rule:

Problem 478. Prove that given ten integers we can choose two of them such that their difference is divisible by nine.
Pigeonholes:
Rule: Calculate the remainder when dividing by 9.

Problem 479. Prove that if six distinct numbers are selected from the set $\{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$ then some set of two of them add up to eleven.
Pigeonholes:
Rule:

Problem 480. A classroom floor is painted white and black. Is it always possible to find two points of the same color, exactly one foot apart? **Hint:** Think about an equilateral triangle.
Pigeonholes:
Rule:

Problem 481. If you have even more pigeons, you sometimes need more than two pigeons in each pigeonhole. Suppose we need to place N items into n boxes and $N > n$. Then at least one box must contain at least $\lceil \frac{N}{n} \rceil$ items, which is $\frac{N}{n}$ rounded up to the nearest integer. **Hint:** Let N_i be the number of items in box i, where $1 \leq i \leq n$. Then $N = N_1 + N_2 + \ldots + N_n$. Assume for the sake of contradiction that $N_i \leq \lceil \frac{N}{n} \rceil - 1$, for all $i = 1, 2, \ldots, n$. Prove the fact that $\lceil \frac{N}{n} \rceil < \frac{N}{n} + 1$ and use it to reach a contradiction. **Suggestion:** Write $N = qn + r$ using the Division Algorithm.

Problem 482. The human head contains fewer than 150,000 hairs. Show that there are at least 50 people who all have the same number of hairs on their heads in New York City. You can assume that New York City has a population greater than 8,000,000 people.
Pigeonholes:
Rule:

Homework problems, week 12

Due on (put date here).

Write up solutions of each of the problems below. They are designed to be straightforward problems. The goal is to come as close to perfection in your solutions as you can.

- Do not take shortcuts.
- If you need to show that something is true for all n, or for all x, y, start the proof with "Let ..."
- If you need cases, explain what the cases are and why they cover all the possibilities.
- If you are doing a proof by contradiction, start that part by saying "Assume ..."
- If you are doing a proof by contrapositive, tell what P and Q are, and that you will be showing that $\neg Q$ implies $\neg P$.
- Take small steps in each proof, and explain each step.
- Follow good form.
- If your proof started with "Let ..." it will probably end by saying "We made no further assumption ..."

Here are the problems to do. You can write them in your notebook or on separate paper.

1. Show that if n is an integer and $7n$ is odd, then n is odd. **Hint:** Be clear what facts you are using about even and odd numbers.

2. Without consulting your book or your notes, prove that $\sqrt{2}$ is irrational. I mean it. Do this from memory. You should be able to write a very nice proof, with no missing steps.

3. Let x and y be real numbers, and suppose that the product xy is irrational. Show that either x or y (or both) must be irrational. **Hint:** You can do this. Be patient, think about it.

4. Let $A = \{2k + 1 : k \in \mathbb{Z}\}$ and let $B = \{2m - 11 : m \in \mathbb{Z}\}$. Show that $A = B$ by showing containment both ways. **Hint:** Use good form!

5. Let $A = \{(x, y) \in \mathbb{R}^2 : y = 5x/7 - 2/7\}$ and $B = \{(x, y) \in \mathbb{R}^2 : 5x - 7y = 2\}$. Show that $A = B$ by showing containment both ways.

6. Let $A = \{m \in \mathbb{Z} : m = 15k \text{ for some } k \in \mathbb{Z}\}$, let $B = \{m \in \mathbb{Z} : m = 35j \text{ for some } j \in \mathbb{Z}\}$, and let $C = \{m \in \mathbb{Z} : m = 105n \text{ for some } n \in \mathbb{Z}\}$. Show that $A \cap B = C$ by showing containment both ways. One direction is easier than the other. Label one of them "the easy direction" and the other "the hard direction". **Hint:** Yes, we worked on a problem just like this in class. Don't go back and find it, work through this one on your own. **Another hint:** In the hard direction, you should come to something like $3k = 7j$ where j and k are integers. You will need to conclude that j is a multiple of 3. If you are up for the challenge, show this using the division algorithm. Don't use any ideas about prime factorization.

Your name: ______________________________

Set theory practice

More practice working with sets

Overview

Write really detailed proofs with crystal clear logic. In particular, when showing that $A \subseteq B$, start with "Let $x \in A$," show that x is in B, and then say, "Since $x \in A$ was arbitrary, $A \subseteq B$."

Show 483. Let A, B, and C be sets. Suppose that $A \subseteq B$ and $B \subseteq C$. Show that $A \subseteq C$. **Note:** This *shows* transitivity but it does not *use* transitivity.

Show 484. Let A, B, and C be sets. Suppose that $A \subset B$ and $B \subset C$. Show that $A \subset C$. **Note:** Here we have strict set inclusion, so you will need to show that A is not equal to C.

Show 485. Let A, B, and C be sets. Show that $C \subseteq A$ and $C \subseteq B$ if and only if $C \subseteq A \cap B$. **Note:** "If and only if" means there are two things to show:

1. Suppose that $C \subseteq A$ and $C \subseteq B$. Show that $C \subseteq A \cap B$.

2. Suppose that $C \subseteq A \cap B$. Show that $C \subseteq A$ and $C \subseteq B$.

Show 486. Let A and B be sets. Show that $A \cap B = B$ if and only if $B \subseteq A$.

Show 487. Let $A_1, A_2, A_3, \ldots$ and $B_1, B_2, B_3, \ldots$ be sets. Suppose that $A_n \subseteq B_n$ for all $n = 1, 2, 3, \ldots$. Show that $\bigcap_{n=1}^{\infty} A_n \subseteq \bigcap_{n=1}^{\infty} B_n$

Show 488. Let $A_1, A_2, A_3, \ldots$ and $B_1, B_2, B_3, \ldots$ be sets. Suppose that $A_n \subseteq B_n$ for all $n = 1, 2, 3, \ldots$. Show that $\bigcup_{n=1}^{\infty} A_n \subseteq \bigcup_{n=1}^{\infty} B_n$

Show 489. Let A and $B_1, B_2, B_3, \ldots$ be sets. Suppose that $A \subseteq \bigcap_{n=1}^{\infty} B_n$. Show that $A \subseteq B_n$ for all $n = 1, 2, 3, \ldots$. Start the proof with “Let $n \geq 1$ be an integer” and be sure to end the proof by generalizing over n. The second step in the proof is “Let $x \in A$.”

Your name: ____________________________

The power set and the Cartesian product

Useful constructions with sets.

Overview

The power set is our first example of thinking hard about collections of sets. The Cartesian product is used often when you want ordered pairs or ordered triples of numbers or other objects.

Problem 490. Write out the members of the following power sets. It may be helpful to do #3, #4, then #2, #1, and finally #5.

1. $S = \emptyset$. $\mathcal{P}(S) =$

2. $S = \{1\}$. $\mathcal{P}(S) =$

3. $S = \{1, 2\}$. $\mathcal{P}(S) =$

4. $S = \{1, 2, 3\}$. $\mathcal{P}(S) =$

5. $S = \{1, 2, 3, 4\}$. $\mathcal{P}(S) =$

6. $S = \{1, 2, 3, 4, 5\}$. $\mathcal{P}(S) =$

Question 491. If S has n elements, how many members will $\mathcal{P}(S)$ have? Explain as well as you can.

Problem 492. Write the appropriate symbol between the entities, or mark the statement as true or false. Give an explanation for anything that is not obvious enough.

1. $1 \qquad \mathcal{P}(\{1, 2, 3\})$
2. $[3, 10] \qquad \mathbb{Z}$
3. $[3, 10] \qquad \mathbb{R}$
4. $\mathbb{Q} \qquad \mathbb{R}$
5. $\mathbb{Q} \qquad \mathcal{P}(\mathbb{R})$
6. $[3, 10] \qquad \mathcal{P}(\mathbb{R})$
7. $\mathbb{N} \qquad \mathbb{R}$
8. $\emptyset \qquad \mathbb{R}$
9. $\emptyset \qquad \mathcal{P}(\mathbb{R})$
10. $\{\emptyset\} \subseteq A$?
11. $\emptyset \subset \mathcal{P}(A)$?

Question 493. Suppose that S is a set. Then $\mathcal{P}(S)$ is also a set, but if we let $A \in \mathcal{P}(S)$, then A is also a set. Explain how this can be. What is the relationship between A and S?

Problem 494. Let I be a set, and for each i in I, let B_i be a set. Show that $\mathcal{P}\left(\bigcap_{i\in I} B_i\right) = \bigcap_{i\in I} \mathcal{P}(B_i)$. Let A be an element of the set on the left-hand side. Notice that A is a set. Argue that it is an element of the set on the right-hand side. Let A be an element of the set on the right-hand side ...

Problem 495.

1. Sketch the Cartesian product $A = [1, 3] \times [2, 5]$.

2. Sketch the Cartesian product $B = [2, 4] \times [1, 3]$.

3. Sketch the intersection $A \cap B$.

4. It seems that $A \cap B$ is also a Cartesian product. Identify the sets whose product is $A \cap B$.

5. What is $([1, 3] \cap [2, 4]) \times ([2, 5] \cap [1, 3])$?

Your name: ______________________________

Relations

Often treated as the little brother to functions, relations have unsuspected depth.

Overview

You are already familiar with a number of relations, including $<, \leq, =, \geq$, and $>$ for real numbers, plus $\subset$, $\subseteq$, and $=$ for sets. Many other relations can be defined. The most useful ones are called equivalence relations; they are analogous to equality for numbers and for sets. They partition the space into equivalence classes, which are very useful in a number of ways.

Definition 496. Relation. A *relation* on a set X is a subset S of $X \times X$.

Notation 497. Suppose that S is a relation on a set X. That is, suppose that S is a subset of $X \times X$, which means that S is a set of points of the form (x, y), where $x \in X$ and $y \in X$. Rather than write $(x, y) \in S$, we usually write $x \sim y$. How to read this out loud? There is no perfect solution. I would suggest that you read it as "x tilde y" because $\sim$ is the tilde that appears above the n in some Spanish words.

Problem 498. You are going to write out the subsets of $\{1, 2, 3, 4\}$ and then draw arrows between them to indicate the proper subset relation. You might want to lay the sets out in a nice order to make the arrows easy to draw and to read. What is the set X on which this relation is defined?

Definition 499. Reflexive. A relation $\sim$ is *reflexive* if $x \sim x$ for all x in X.

Definition 500. Symmetric. A relation $\sim$ is *symmetric* if $x \sim y$ implies $y \sim x$.

Definition 501. Transitive. A relation $\sim$ is *transitive* if $x \sim y$ and $y \sim z$ implies $x \sim z$.

Note 502. Sometimes people have a hard time remembering the words reflexivity, symmetry, and transitivity. Notice that they are in alphabetical order, and that they involve 1, 2, or 3 objects at a time, respectively.

Definition 503. Equivalence relation. A relation $\sim$ is called an *equivalence relation* if it is reflexive, symmetric, and transitive. Note that equality is an equivalence relation on the set of real numbers.

Definition 504. Equivalence class. Suppose that $\sim$ is an equivalence relation. Fix x in X. The set of all elements y for which $x \sim y$ is called the *equivalence class containing x*.

Problem 505. Let $X = \mathbb{Z}^+$ and say that $x \sim y$ if y is divisible by x. People often write $x|y$ for this relation and say that x divides y.

1. Check whether this relation is reflexive. If so, prove that it is, starting with "Let $x \in \mathbb{Z}^+$." If not, give a counterexample.

2. Check whether this relation is symmetric. If so, prove that it is, starting with "Let $x, y \in \mathbb{Z}^+$ and suppose that $x \sim y$." If not, give a counterexample.

3. Check whether this relation is transitive. If so, prove that it is, starting with "Let $x, y, z \in \mathbb{Z}^+$ and suppose that $x \sim y$ and $y \sim z$." If not, give a counterexample.

4. Thinking of the relation as a set of ordered pairs, write out ten different ordered pairs satisfying the relation, and graph them on the xy plane.

Problem 506. Consider all cities in the US that have population over 30,000. For each of the following relations, determine whether they are reflexive, symmetric, and/or transitive. Provide a counterexample for any property that fails to hold. If all three hold, the relation is an equivalence relation. In that case, identify the equivalence classes and tell how many such classes there are.

1. Say that $x \sim y$ if the names of cities x and y start with the same letter.

2. Say that $x \sim y$ if x and y are in the same state.

3. Say that $x \sim y$ if cities x and y are within 50 miles of each other.

Problem 507. Let X be the set of all English words. Say that $x \sim y$ if the letters in x and y appear on the same number keys on a cell phone, in the same order. For example, BAR $\sim$ CAP.

1. Check whether this relation is reflexive.

2. Check whether this relation is symmetric.

3. Check whether this relation is transitive.

4. If all three properties hold, describe the equivalence classes, and tell what the equivalence class of BAR is.

Problem 508. Let $X = \mathbb{Z}$. Say that $x \sim y$ if x and y has the same remainder as y when they are divided by 3. Then, for example, $13 \sim 19$ and $9 \sim 27$.

1. Show that this relation is reflexive, symmetric, and transitive. Do this in general, starting with "Let."

2. Describe all elements of the equivalence class containing 0.

3. Describe the other equivalence classes. How many are there?

Problem 509. Consider the set X of all non–zero 3–dimensional vectors. For $\vec{a}$ and $\vec{b}$ in X, say that $\vec{a} \sim \vec{b}$ if there exists a constant c for which $\vec{a} = c\vec{b}$.

1. Show that this relation is reflexive, starting with "Let." Tell what c is.

2. Show that this relation is symmetric. You will need two values of c.

3. Show that this relation is transitive. Here there will be three values of c.

4. This is an equivalence relation. Describe the equivalence classes. The collection of all equivalence classes is called *projective space*.

5. Could you use the angle between lines to define a distance between equivalence classes? What would the maximum distance be?

Problem 510. Consider the set of all English words. Say that $x \sim y$ if one can be obtained from the other by changing exactly one letter. For example, BAT $\sim$ CAT but BAT $\nsim$ CAR. Check whether this relation is reflexive, symmetric, and/or transitive. Provide counterexamples if necessary.

Problem 511. Consider the set of all functions on the real line. That is, consider the set of all $f : \mathbb{R} \to \mathbb{R}$. Say that $f \sim g$ if f and g are equal except at a finite number of points. For example, if $f(x) = x^2$ and $g(x) = \begin{cases} x^2, & x \neq 0 \\ 5, & x = 0 \end{cases}$, then $f \sim g$. Show that this is an equivalence relation. How can you describe the equivalence classes?

Note 512. Problems 10.1 and 10.3 from Daepp and Gorkin[1] are particularly good at this stage in the course.

[1]Reading, Writing, and Proving: A Closer Look at Mathematics, 2011, by Ulrich Daepp and Pamela Gorkin

Your name: ______________________________

Absolute value and related functions

A careful development of the properties of the absolute value function.

Overview

The absolute value function is easy to understand for numbers like 9 and -13, but it's harder to show its properties because our intuition works so hard to see all variables as having positive values. In this activity, we will not use the standard notation for the absolute value function and will have to keep our intuition at bay. We will instead rely completely on the definition. **When you use a property of inequalities, cite it by number.**

Definition 513. Absolute value. The function $f : \mathbb{R} \to \mathbb{R}$ defined by

$$f(x) = \begin{cases} x, & \text{if } x \geq 0 \\ -x, & \text{if } x < 0 \end{cases}$$

is called the *absolute value* function.

Notation 514. In this activity, do not use the standard notation for absolute value, not even once. Every time you work with the absolute value function, use and cite the definition.

Show 515. Show that $f(ab) = f(a)f(b)$ for all real numbers a and b. Follow the model.

Let a and b be real numbers. There are four cases.

1. Suppose that $a \geq 0$ and $b \geq 0$. Then $ab \geq 0$ so $f(ab) = ab$ and $f(a) = a$ and $f(b) = b$, so $f(ab) = ab = f(a)f(b)$.

2. Suppose that $a \geq 0$ and $b < 0$.

3. Suppose that $a < 0$ and $b \geq 0$.

4. Suppose that $a < 0$ and $b < 0$.

In each case, we see that ____________________________. We made no further assumptions about ______________, thus ____.

Show 516. Following the model above, show that $f(f(a)) = f(a)$ for all real numbers a.

Show 517. Show that $f(-a) = f(a)$ for all real numbers a.

Show 518. Show that $f(a - b) = f(b - a)$ for all real numbers a and b.

Show 519. Show that $f(a) \geq 0$ for all real numbers a.

Show 520. Follow the model in **??** to show that $f(a+b) \leq f(a)+f(b)$ for all real numbers a and b. When a and b have different signs, consider two cases, $a + b \geq 0$ and $a + b < 0$. You will probably want to show that if $b < 0$, then $b < -b$. Make a good, solid argument using transitivity of $<$.

Show 521. Show that for real numbers a and b, $f(a) \leq b$ if and only if $-b \leq a \leq b$. Remember that an "if and only if" proof has two directions. In both directions, you will have to consider two cases, $a \geq 0$ and $a < 0$. Note that the statement $-b \leq a \leq b$ is equivalent to $(-b \leq a$ and $a \leq b)$.

Show 522. Show that for all real numbers a and b, $f(a) \geq b$ if and only if ($a \geq b$ or $a \leq -b$).

Show 523. Show that for all real numbers a and b, $f(a) \leq f(a-b) + f(b)$. **Hint:** Look at $f((a-b)+b)$.

Show 524. Show that for all real numbers a and b, $f(a) - f(b) \leq f(a-b)$ and also $f(b) - f(a) \leq f(b-a)$.

Show 525. Show that for all real numbers a and b, $f(a-b) \geq f(f(a) - f(b))$.

Definition 526. Minimum function. The function $h : [0, \infty) \to \mathbb{R}$ defined by

$$h(x) = \begin{cases} x, & \text{if } x \leq 1 \\ 1, & \text{if } x > 1 \end{cases}$$

can be called the minimum function.

Show 527. Show that for all real numbers a, $h(a) = 0$ if and only if $a = 0$.

Show 528. Show that if $a \leq b$, then $h(a) \leq h(b)$.

Show 529. Show that if $h(a) < h(b)$, then $a < b$.

Show 530. Show that for all real numbers a and b, $h(a + b) \leq h(a) + h(b)$. **Hint:** Use a proof by cases. But what are the cases?

Your name: ______________________________

Functions

One-to-one, onto and bijective functions

Overview

You are comfortable working with functions already. There are various ways of describing functions. Here you will learn the formal definition of a function.

Definition 531. Function. Let X and Y be sets. A function f from X to Y is a relation from X to Y that satisfies:

1. for each $x \in X$ there is a $y \in Y$ such that $(x, y) \in f$, and
2. if $(x, y) \in f$ and $(x, z) \in f$, then $y = z$.

The set X is called the domain of f and the set Y is called the codomain of f.

Notation 532. We write $f : X \to Y$ to describe a function f from X to Y and we write $f(x) = y$ instead of $(x, y) \in f$.

Definition 533. Injective Functions. Let X and Y be sets and let $f : X \to Y$ be a function. The function f is said to be injective or one-to-one if whenever $x_1, x_2 \in X$ are such that $x_1 \neq x_2$ then $f(x_1) \neq f(x_2)$.

Definition 534. Surjective Functions. Let X and Y be sets and let $f : X \to Y$ be a function. The function f is said to be surjective or onto if for each $y \in Y$ there exists an $x \in X$ such that $f(x) = y$.

Definition 535. Bijective Functions. Let X and Y be sets and let $f : X \to Y$ be a function. The function f is said to be bijective if it is both injective and surjective.

Example 536. Let $X = \{Monday, \diamondsuit, \sqrt{\pi}, purple\}$ and $Y = \{\alpha, \heartsuit, fun\}$ be sets and define the relation f from X to Y by $f = \{(Monday, fun), (\diamondsuit, \alpha), (\sqrt{\pi}, fun), (purple, fun)\}$. Draw a diagram to illustrate the relation. Is the relation f a function? Prove your answer by using the definition.

Problem 537. Let $X = \{Cleveland, Chicago, LosAngeles, Miami\}$ be a set of American cities and let $Y = \{Cavaliers, Heat, Lakers, Bulls, Clippers\}$ be a set of NBA teams.

a) Provide an example of a relation that is a function from X to Y and draw a diagram to illustrate your example.

b) Provide an example of a relation from X to Y that is not a function and draw a diagram to illustrate your example.

c) Provide an example of an injective function from X to Y. Draw a diagram to illustrate your example.

d) Show that there are no surjective functions from X to Y.

Note 538. The next problem states an equivalent definition for injectivity. This definition is very useful when proving that a function is injective.

Problem 539. Let X and Y be two sets and let $f : X \to Y$ be a function. Then f is injective if and only if for all $x_1, x_2 \in X$ such that $f(x_1) = f(x_2)$ we have $x_1 = x_2$.

Problem 540. Let $f : \mathbb{N} \to \mathbb{N}$ be a function defined by $f(n) = 2n + 1$. Show that the function f is injective but not surjective.

Hint: Use the previous problem to prove injectivity. In order to prove that f is not surjective you need to find an $m \in \mathbb{N}$ which cannot be written as $2n + 1$.

Problem 541. Let $f : \mathbb{N} \to \mathbb{Z}$ be a function defined by

$$f(n) = \begin{cases} \frac{n}{2}, & \text{if } n \text{ is even} \\ \frac{-(n+1)}{2}, & \text{if } n \text{ is odd} \end{cases}$$

Show that f is a bijective function.

Hint: To show that f is injective let $n_1, n_2 \in \mathbb{N}$ be such that $f(n_1) = f(n_2)$ and look at all the possible cases according to the parity of n_1 and n_2.

To prove the surjectivity let $m \in \mathbb{Z}$. Consider the two possible cases; one when $m \geq 0$ and the other one when $m < 0$. Then in each case find an $n \in \mathbb{N}$ such that $f(n) = m$.

Your name: ______________________________

Square roots of prime numbers are irrational

This is a classic example of proof by contradiction.

Overview

Most students are familiar with the fact that $\sqrt{2}$ is irrational, but few can prove it. Having read a proof of this fact in your textbook or online, the starting point is to re-create the proof from memory, then to move on to showing that $\sqrt{3}$ is irrational. The proof is similar and yet different.

Definition 542. Irrational. A real number is said to be *irrational* if it cannot be written as the quotient of two integers.

Note 543. If you are new to proof by contradiction, you might prefer to start the next proof by writing "Let's pretend for a minute that $\sqrt{2}$ can be written as $\frac{p}{q}$ where p and q are integers." This makes it extra clear that you don't particularly believe that $\sqrt{2}$ is rational, you are just exploring what would happen if that were true. When you arrive at a contradiction, you realize it's time to stop pretending; $\sqrt{2}$ must be irrational.

Prove 544. Prove that $\sqrt{2}$ is irrational by contradiction. The proof begins with "Assume for the sake of contradiction that $\sqrt{2}$ can be written as $\frac{p}{q}$ where p and q are integers." Argue to a contradiction.

Show 545. A key step in the proof is that if n is an integer and n^2 is even, then n is even. You may have already shown this, using a proof by contradiction (which begins, "Assume for the sake of contradiction that n is odd.") or a proof by contrapositive (which begins, "Let us show the contrapositive, that if n is odd, then n^2 is odd.") or a proof by cases (which begins, "There are two possibilities for n, that n is even or that n is odd.") Whichever one you have already seen, choose a different one and write the proof here.

Show 546. Mimic the proof that $\sqrt{2}$ is irrational to show that $\sqrt{3}$ is irrational.

Show 547. A key step in the proof that $\sqrt{3}$ is irrational is the fact for an integer n that: if n^2 is a multiple of 3, then n is a multiple of 3. Prove this by contradiction, starting with "Assume for the sake of contradiction that n is not a multiple of 3".

Show 548. Now write a proof by contrapositive that if n^2 is a multiple of 3, then n is a multiple of 3. Clearly state what the contrapositive is, then prove it.

Show 549. Now write a proof by cases.

Show 550. Suppose that n is an integer and that n^2 is a multiple of 5. Show that n is a multiple of 5. Make the logic of your proof crystal clear.

Show 551. Show that $\sqrt{5}$ is irrational. Now that you are getting good at proofs like this, try to write a picture perfect proof.

Show 552. Suppose that n is an integer and that n^2 is a multiple of p, where p is a prime number. Make a good start on showing that n is a multiple of p.

Show 553. Suppose that p is a prime number. Show that $\sqrt{p}$ is irrational, assuming that the previous result is true.

Show 554. Suppose that n is odd and suppose that $n^3 - n$ is a multiple of 24. Show that $(n+2)^3 - (n+2)$ is also a multiple of 24.

Show 555. Check that when $n = 1$, $n^3 - n$ is a multiple of 24. Use this together with the previous problem to conclude that $n^3 - n$ is a multiple of 24 for additional values of n. What values of n can your argument cover?

Show 556. Write an argument that will cover all other odd values of n

Anonymous!

Class survey

I would like your feedback to improve the course. Many thanks in advance!

1. In class, we work through activities without much "lecture." How does this work for you?

2. What is going well in the class, so that we should not change it?

3. Is there anything we should change about the class to help you learn better?

4. If there are specific things you are able to do in other classes because of taking this class, please list them.

5. What else could we do to make you unstoppable in your other math courses?

6. Do you look forward to coming to class? Why or why not?

7. You have been asked to read the textbook, and we have checked your notes to make sure this is happening. Is this working well for you? Why or why not?

Would you recommend that other faculty do the same in their courses? **yes** **no**
Please explain.

8. In what way(s) have you changed how you work with the textbook in other courses that you are taking? Do you read them more? Differently? Please explain.

9. Make any other comments you like here or on the back of the sheet.

Read Chapter #1

The idea is to read Chapter 1 of the textbook by Daniel Solow. The assignment is to read it in a particular way. It may take 3 hours to get it done, but you will learn something in those three hours, and you will start to develop a very important skill.

Get a copy of Chapter 1, "The Truth of it All." Get out your notebook or some paper or write directly on a blank PDF. You will need to turn in your notes so I can have a look at them. Go somewhere quiet, where you won't be interrupted for a while. Silence your phone so you aren't disturbed. Don't listen to music that will distract you, and make sure there is no video going where you can see it or hear it.

Put the notebook or paper right in front of you. Put the textbook itself a bit farther away. Make note of the time that you start reading in your notebook, maybe in the left margin. Read the first paragraph of the chapter, then write one or two sentences in your notes which capture the main idea(s) of the paragraph.

Continue reading and writing a sentence summarizing each paragraph. I believe that if you are not writing, you are probably not thinking as hard as you need to. Read slowly. If you run into a word you don't know, look it up. If you really don't know it, write the definition in your notebook. It is OK to spend 15 minutes on each page of the book. Really. It is not a goal of the course to learn how to read faster. The goal is to learn how to get more out of the time you spend reading. If you stop to take a break, note the time that you stopped and the time you start again so you can calculate the total time.

When you read Section 1.1, comment on the goals he lays out. Do you have the same goals? Different? How?

When you read Section 1.2, there are a few vocabulary words in bold, be sure to learn those. There is a very important example about when you might call your friend a liar. Personally, I don't think this is about your friend being a liar; if you study hard and don't get a good grade, then your friend was just wrong, not trying to deceive you. Still, the point is that when trying to prove that A implies B, the only way that things can go wrong is a situation in which A is true but B is false. Read and reflect on this example multiple times. Write a sentence or two to demonstrate that you understand this example. Use the additional examples he gives to understand better. If you are left with questions, please write them in your notes, perhaps highlight them, so that I can respond to them when I read your notes.

Examples 1, 2, 3, and 4 illustrate how to do some of the exercises.

Note that solutions of the exercises marked with W are available online at http://higheredbcs.wiley.com/legacy/college/solow/1118164024/sm/sm.pdf Resist the urge to turn your brain off and just read the solutions. That is not what they are there for!

Please do the following exercises and write solutions in your notebook:

1.2

1.4

1.7

1.8 "Not lose" is the same as "win" in this problem.

1.12 This is all about avoiding a situation in which A is true and B is false.

1.16 Answer the question yourself, then compare your answer with the online solution.

1.18 I suggest that you try various choices for n and x on a calculator and see what you find. Write out clearly what A and B are, and how you have made A true but B false. For part (b), look for agreement on 5 decimal places, not just 3, and explain how you found an answer to part (b). After doing 1.18, check your answer to 1.16.

At the end, tally up how much time you have spent on reading this chapter. Write this number at the beginning of your notes in the upper left corner.

Read Chapter #2, The Forward-Backward Method

Read Chapter 2 in the book by Daniel Solow, about the forward-backward method for finding a proof that statement A implies statement B.

As you read the chapter, write short summaries of each paragraph, so that your notes provide a short version of the ideas in the chapter. Specific things to do are listed below to help you check them off.

1. Keep track of the time it takes you to read the chapter.
2. Read Proposition 1 and make sure you understand what it is saying. Apparently the area of a right triangle is not always equal to the hypotenuse squared divided by 4.
3. When reading Section 2.1, recall our work on vector sums and dot product, where we were able to write down where we need to start, leave a lot of space, and then write down where we need to end. At first, Solow is working up from the bottom, from statement B back to B1 back to B2

   For working backward, he introduces the idea of a "key question," which is a way to put your finger on what is needed to know that B is true. The key question needs to be specific enough to be helpful to the problem at hand, but general enough that it make sense to someone who is not immersed in the details of the problem. It may help to think of formulating the key question as an internet search, since most people have experience with writing searches that are not too specific and not too general at the same time.
4. When starting Section 2.2, use some space in your notes to write out A and A1 at the top, leave 10 lines of space, and write out B2, B1, and B at the bottom. Fill in steps as you read the section.
5. When reading Section 2.3, use the numbering of Table 2.1 to list out the statements that are made in each of the four proofs of Proposition 1, in the order that they are made. If a statement is not actually made, don't write the corresponding label. This will help to illustrate the order in which the proofs are written and what steps are left out.
6. At the end of the chapter there is an illustration of a maze. Work through the maze from A to B and count how many dead ends there are on the way to B. Work through the maze from B to A and count how many dead ends there are. Note that some of the dead ends are different, depending which way you are going.
7. Do exercise 2.5.
8. Do exercise 2.7.
9. Do exercise 2.11.
10. Do exercise 2.14a and 2.15b.
11. Do exercise 2.19.
12. Do exercise 2.24.
13. Do exercise 2.30.

14. Do exercise 2.37, by writing out the steps in order from A to B.

15. At the end, tally up how much time you have spent on this chapter. Write this number in the upper left corner of your notes.

Read Chapter #3, On Definitions and Mathematical Terminology

Read Chapter 3 in the book by Daniel Solow. This chapter is about definitions and how to use them in the forward and backward process. Fortunately, this does not seem to be a complicated chapter, so work through it and make sure you get the message. Follow these instructions carefully.

1. Set yourself up in a place where you won't be disturbed. Read slowly, and write notes in your own words that reflect your understanding of the material. You do not need to paraphrase each paragraph, but I would encourage you to write down things that you learn, things that surprise you, or things that you need to remember.
2. When reading Section 3.1, pay close attention to "if and only if" because it comes up often in proofs, and you need to know how to show an "if and only if" statement.
3. In Section 3.1, there is a page and a half on overlapping notation. This is a good discussion.
4. In Section 3.2, read the analysis of proof for Proposition 3 and then write down the steps A, A1, B2, B1 in order to make your own proof of the result. Explain why each step in the proof can be taken.
5. Make yourself a glossary (vocabulary list) for the terms converse, inverse, contrapositive, proposition, theorem, lemma, corollary, axiom, so you learn them well. In Section 3.3, the author says that a proposition is a true statement that you are trying to prove. I would emphasize that we don't call it a proposition until we know that it can be proven, and maybe we are trying to understand the proof. Something we are simply trying to prove should be a called a conjecture; it might not turn out to be true!
6. Do Exercise 3.2. You do not need to write proofs, only pose the key question, answer it abstractly, and rephrase your answer in terms used in the problem.
7. Do Exercise 3.5 parts c and d. This will be a bit challenging because it requires you to look up two new definitions and interpret them. That's an excellent skill, so practice it.
8. Do Exercise 3.12.
9. Do Exercise 3.15.
10. Do Exercise 3.19. Note that definitions are not the kind of previous knowledge that we are looking for here. Look for the two previous implications that are used here.
11. Do Exercise 3.21.
12. Do Exercise 3.27.
13. At the end, tally up how much time you have spent on this chapter. Write this number in your notes.

Reading assignment #4

Due on Tuesday, September 26. 20 points

Read Chapter 4 in the book by Daniel Solow. It is about showing that there is an "object" with a "certain property" such that "something happens." We have already done a number of proofs of this general form.

From the class survey, I am reminded that people like to take notes in different ways. Do what works for you, but make sure that your notes show that you read each section and that you found and understood the main messages there.

Also, from the class survey, people would like to work on something related to the reading, so we'll start with something from this reading on Tuesday. Good idea!

Specific requirements

- Read Section 4.1 and take notes. Then, look back through the Even and Odd activity and the Vector Sum and Dot Product activity and list by number all of the exercises that are of the form "Show that there is an object with a certain property such that something happens." I've posted previous activities on the Syllabus section on Canvas. Note: Showing that n is even means showing that there exists an integer k for which $n = 2k$.

- The existence part of the Division Algorithm is of the form described in this chapter. Write it out following the general pattern that there is an "object' with a "certain property" such that "something that happens," in that order. Hint: The last thing to write is $n = qk + r$.

- In Section 4.3, Proposition 5 assumes that m is even. Suppose instead that m is odd, and show that $m^2 + n^2 - 1$ is a multiple of 4.

- Do exercise 4.2.

- Do exercise 4.9. In each case, explain how you found the object.

- Do exercise 4.11. There are two objects getting constructed here, k and x. Where do their values come from? Under what condition could you produce an additional rational root?

- Do exercise 4.13. This is an excellent project with several parts. Work hard on it.

- Do exercise 4.16.

- Do exercise 4.22.

- At the end, tally up how much time you have spent on this chapter. Write this number in your notebook. Bring your notebook to class and turn it in for grading.

General comments

Set yourself up in a place where you won't be disturbed. Read slowly, and write notes in your own words that reflect your understanding of the material.

Reading assignment #5

Due on Tuesday, October 3. 20 points

Read Chapter 5 in the book by Daniel Solow. It is about showing that something happens "for all" objects with a certain property. We have already done a number of proofs of this general form.

Pay particular attention to the beginning of the chapter, about set theory. We will soon start doing set theory activities in class.

Specific requirements

- Read Section 5.1 slowly and make sure to think through every sentence. There is a lot of new content in just a few pages. Set theory is super important, and this is a very nice introduction to certain aspects of it that we will spend a lot of time with. Make sure that your notes reflect the time you spend and your understanding of the material.

- Read Section 5.2. It has an extended discussion of using the forward-backward method to do a proof. After you have read it, write out the statements in order and using the labels **A, A1, A2, . . . , B6, B5, . . . B2** so that it is clear that you understand exactly how the proof works. I think this will help make it clearer to you, also. Because only half of the proof is being done here, start with the definitions of sets S and T, which is part **A** of the proof, write **A1:** as "Let x be an element of S." and end with **B2:** S is a subset of T. In **A1:**, the word "Let" means that a specific object is being brought into existence, with a specific property, for you to work with.

- Do exercise 5.2.

- Do exercise 5.6.

- Do exercise 5.7.

- Do exercise 5.14. Following the chapter, first identify the objects, the certain property, and the something that happens in the for-all statements. Then do a nice job explaining what is right or wrong about a, b, c, d, and e.

- At the end, tally up how much time you have spent on this chapter. Write this number in your notebook. Bring your notebook to class and turn it in for grading.

General comments

Set yourself up in a place where you won't be disturbed. Read slowly, and write notes in your own words that reflect your understanding of the material.

Reading assignment Chapter 7

Due on Tuesday, October 17. 20 points

Read Chapter 7 in the book by Daniel Solow. This chapter is about nested quantifiers. We have actually been working with these from day 1 of the course, but now you will write things out more explicitly.

Here is an example. When you proved that the sum of two even numbers is even, you proved that:

- For all integers m and n, there exists an integer k such that $m + n = 2k$.

In order to prove this, you used the choose method (described in Chapter 5) to fix particular values of m and n and then wrote a model proof which worked for all m and n. As part of that model proof, you constructed the new integer k, which follows the construction method described in Chapter 4. There are many theorems following the general pattern "For all objects a, there exists an object b for which something depending on a and b happens.' Note that in the example, a is the pair m, n and b is the integer k. Also, note that b pretty much always depends on a.

There are also theorems following the pattern "There exists an object a such that for all objects b, something depending on a and b happens. These work differently. Here you have to do the construction of a in a way that it will work for all b simultaneously, then you fix an object b and write a model proof that will work for all b. The difference here is the order in which the nested quantifiers occur. Note that here b does not depend on a, and a cannot be chosen for any one particular b but needs to work for all b.

Specific requirements

- As you read Section 7.1, outline how you would use the "construction" and the "choose" methods to prove S1, S2, S3, and S4, following the model above.
- Similarly, when reading Section 7.2, outline how you would show that a function is onto.
- Do exercise 7.2.
- Do exercise 7.4. Instead of doing it exactly as stated, instead create five examples of (x, y) pairs that satisfy the criteria in part a and part b, and then answer part c. This is why we don't fuss too much about the general pattern "there exists a such that there exists b for which hsomething depending on a and b happens."
- Do exercise 7.7. Instead of doing it exactly as stated, follow the model at the beginning of this assignment to explain how to use the construction method and the choose method to do a, b, and c.
- Show that for all $a > 0$, there exists an integer $n > 0$ such that $\frac{1}{n} < a$. While doing so, mention which part uses the "construction" method and which part uses the "choose" method.
- Do exercise 7.18. I suggest that for every z in $\mathbb{R}$, you show that there exists an x in $\mathbb{R}$ for which $f(g(x)) = z$. That leaves the letter y available, and you'll want to use it.
- Do exercise 7.19. Fortunately, you can construct x.
- At the end, tally up how much time you have spent on this chapter. Write this number in your notebook. Bring your notebook to class and turn it in for grading.

General comments

Set yourself up in a place where you won't be disturbed. Read slowly, and write notes in your own words that reflect your understanding of the material.

Reading assignment Chapter 8

Due on Tuesday, October 24. 20 points

Read Chapter 8 in the book by Daniel Solow. This chapter is about negation of logical statements, especially of statements containing quantifiers. This just takes some practice and you'll be good at it. The key text is steps 1, 2, 3 at the top of page 95.

Specific requirements

- Write the NOT of this statement: "For all $x \in \mathbb{R}$, $\ln(x) < 14$.
- Write the NOT of this statement: "For all $a \in [2,5]$, there exists $b \in [2,5]$ such that $a < b$.
- Write the NOT of this statement: "There exists $a \in A$ such that for all $b \in B$, $f(b) < a$.
- Write the NOT of this statement: "For all $\varepsilon > 0$, there exists $\delta > 0$ such that for all x with $|x-a| < \delta$, $|f(x) - L| < \varepsilon$.
- Write the NOT of this statement: "For all $a \in \mathbb{R}$, for all $\varepsilon > 0$, there exists $\delta > 0$ such that for all x with $|x - a| < \delta$, $|f(x) - f(a)| < \varepsilon$.
- Do exercise 8.2. Note that taking the NOT of "For all a and for all b, something happens" becomes "There exists a and there exists b such that, NOT something happens."
- Do exercise 8.3. For part (a), note that "There is no integer n with ..." is how you write English for the logical statement "NOT (there exists an integer n with ...)". So it's easy to negate. For all parts, note that you are just going to put NOT in front of the bold word, and then write the NOT of what comes after "if and only if".
- Do exercise 8.7. Clearly identify the logical statements A and B in each case, and then clearly state NOT B and NOT A. Leave out statements such as k is an integer; these are like the fabric of reality, not to be negated in these exercises. Make clear what you will work forward from and what you will work backward from. Note that you do not have to do the proofs, but if you see how to do them, you might as well do them.
- Is this statement true? "For all $a \in [2,5]$, there exists $b \in [2,5]$ such that $a < b$. If so, explain. If not, find a counterexample.
- Is this statement true? "For all $a \in (2,5)$, there exists $b \in (2,5)$ such that $a < b$. If so, explain. If not, find a counterexample.
- At the end, tally up how much time you have spent on this chapter. Write this number in your notebook. Bring your notebook to class and turn it in for grading.

General comments

Set yourself up in a place where you won't be disturbed. Read slowly, and write notes in your own words that reflect your understanding of the material.

Reading assignment Chapter 9

Due on Tuesday, October 31. 20 points

Read Chapter 9 in the book by Daniel Solow. It is about proof by contradiction. We have seen a few examples of this in class, in this form: If you want to prove that the logical statement P is true, pretend for a minute that $\neg P$ is true, and make a series of logical deductions that lead to a statement you know is false. Then you know that $\neg P$ is false, and so P is true.

Chapter 9 is mostly about proving implications like $P \to Q$. Recall that in a direct proof, you suppose that P is true and make a series of logical deductions to show that Q is true. We have discussed the contrapositive method in class, where you suppose that $\neg Q$ is true and make a series of deductions to show that $\neg P$ is true. In both cases, you are trying to show that it cannot happen that P is true and $\neg Q$ is true at the same time. One way to look at proof by contradiction is that you pretend for a minute that $P \wedge \neg Q$ is true and make a series of logical deductions that lead to a false statement, so you know that $P \wedge \neg Q$ is false. The beauty of this method is that you have two statements to work forward from: P and $\neg Q$. The downside is that you can't work backward; you are trying to argue toward a false statement, and you don't know for sure what that is.

Note that in doing a proof of $P \to Q$ by contradiction, you will need to negate $\neg Q$, and when Q has quantifiers you will need to be extra careful.

Specific requirements

- Write a defintion of what a "contradiction" is from your reading of the chapter.
- In Section 9.4, please completely rewrite the proof of proposition 14 in your own words and with your own structure. The next two pages have an analysis of proof, but instead of reading that, work through the proof on your own and make sense of it. Be patient, get it done.
- Do exercise 9.2. In every case, explicitly write out P and Q and then P and $\neg Q$ to answer the question.
- Prove the result in exercise 9.3. Identify P and Q and $\neg Q$ and work from P and $\neg Q$ to arrive at a false statement. Ignore (a) and (b).
- Do exercise 9.7 in this way. Identify P and Q and describe how you would use the "construct" and "choose" methods to do a direct proof. Then, write out $\neg Q$ and describe how you would do a proof by contradiction.
- Do exercise 9.11 as a proof by contradiction.
- Do exercise 9.15 as a proof by contradiction.
- Do exercise 9.23 by once again identifying P, Q, and $\neg Q$ and then reading the proof.
- At the end, tally up how much time you have spent on this chapter. Write this number in your notebook. Bring your notebook to class and turn it in for grading.

General comments

Set yourself up in a place where you won't be disturbed. Read slowly, and write notes in your own words that reflect your understanding of the material.

Reading assignment Chapter 11

Due on Tuesday, November 21. 20 points

Read Chapter 11 in the book by Daniel Solow. It is about showing uniqueness. We have seen an example already in the Division Algorithm, when you showed that given an integer $k > 0$ and an integer n, there are unique integers q and r with $0 \leq r < k$ so that $n = kq + r$. You used the Direct Uniqueness Method: you supposed that you could also write $n = kq_2 + r_2$ with $0 \leq r_2 < k$ and showed that $q = q_2$ and $r = r_2$. There is also an Indirect Uniqueness Method, where you would pretend for a minute that $q \neq q_2$ or that $r \neq r_2$ and argue to a false statement, so that you know this is fantasyland.

Specific requirements

- Do exercise 11.2. The notation may be confusing. In part (a), your goal is to show that x^* equals y^*. How could you do that? It might be easier to call the numbers x_1^* and x_2^*. In part (b), the function f is given and fixed. You might want to think of the problem as showing that $G_1 = G_2$. In part (c), p and q play the role of a.
- Do exercise 11.5. The answer to part (c) is "specialization" which was covered in Chapter 6, which we did not read. Please explain what specialization means in the context of this question. It's not a hard concept.
- Do exercise 11.6. Draw a relevant picture. Rewrite the proof so that each step is on a different line, and give a justification for each step. Explain which uniqueness method is being used.
- Do exercise 11.7. Draw a relevant picture. Rewrite the proof so that each step is on a different line, and give a justification for each step. Explain which uniqueness method is being used.
- Do exercise 11.9.
- Do exercise 11.11.
- At the end, tally up how much time you have spent on this chapter. Write this number in your notebook. Bring your notebook to class and turn it in for grading.

General comments

Set yourself up in a place where you won't be disturbed. Read slowly, and write notes in your own words that reflect your understanding of the material.

Read Chapter #12, Mathematical induction

Read Chapter 12 in the book by Daniel Solow. This chapter is about induction. It is well written and will hopefully help you understand induction much better. Pay special attention to the introduction of strong induction in sections 12.2 and 12.3.

Specific requirements

- Write useful notes as you read the chapter, and turn those in.
- Do exercise 12.2, and please do a good job on it.
- Do exercise 12.6.
- Do exercise 12.10.
- Do exercise 12.21. In addition to doing what the book asks, rewrite the proof and justify each step of the proof. That is, give a reason that each step of the proof is true, especially with the string of equalities and inequalities. This is going to take some work. Roll up your sleeves and get it done.
- Do exercise 12.22. This will also take some real work. In addition to answering the questions in the problem, please answer this question:

  d. Could we use $n = 1$ as the base case, and save ourselves the trouble of checking the base case for $n = 2$?
- Do exercise 12.23. Part (a) is asking about the sentence "Let x_0 be a real number." which is called the Choose Method in Chapter 5.

  The point x_* is called a *fixed point* of the function, and the inequality involving α means that the fixed point is *attractive*. The point of the result is that if you apply the function f over and over again, the values converge quickly to x_*.

  Before you do 12.23, you might enjoy playing this little game. On a calculator, calculate the square root of a number like 20, then take the square root again, and again, and again, and see what happens. Then start with a number like 0.02 and take the square root again and again and again. You could also do this with cosine, or with sine, or with exp. These functions may or may not have a fixed point, and the fixed points may all be different.

  If you took Math 3370, Differential Equations, you may have seen the result that Picard iteration has an attractive fixed point, and that is how you show existence of solutions of differential equations.
- At the end, tally up how much time you have spent on this chapter. Write this number in your notebook.

General comments

Set yourself up in a place where you won't be disturbed. Read slowly, and write notes in your own words that reflect your understanding of the material.

Reading assignment #1

Due on the second day of class. 10 points

The idea is to read Chapter 1 of the textbook by Daepp and Gorkin[2]. The assignment is to read it in a particular way. It may take 3 hours to get it done, but you will learn something in those three hours, and you will start to develop a very important skill.

Get a copy of Chapter 1, "The How, When, and Why of Mathematics." Get out your notebook or some paper. Go somewhere quiet, where you won't be interrupted for a while. Turn off your phone so you aren't disturbed. Don't listen to music that will distract you, and make sure there is no TV or youtube on where you can see it or hear it.

Put the notebook or paper right in front of you. Put the textbook itself a bit farther away. Make note of the time that you start reading in your notebook, maybe in the left margin. Read the first paragraph of the chapter, then write one or more sentences in your notes which capture the main idea(s) of the paragraph.

Read the second paragraph, about Geogre Pólya's list of guidelines. Look up the list in the Appendix. Consider writing them in your notebook, or abbreviated versions of them.

Continue to write a sentence summarizing each paragraph. I believe that if you are not writing, you are probably not thinking as hard as you need to. Read slowly. If you run into a word you don't know, google it or look it up in a dictionary. If you really don't know it, write the definition in your notebook. It is OK to spend 15 minutes on each page of the book. Really. It is not a goal of the course to learn how to read faster. The goal is to learn how to get more out of the time you spend reading. If you stop to take a break, note the time that you stopped and the time you start again.

Read Exercise 1.1 and the text that walks you through Pólya's guidelines. Use your notebook to try to solve the puzzle yourself. I've printed the alphabet twice for you. That should save you a little time.

A second example starts on page 3 of the textbook. As you read it, draw diagrams in your notebook. Yes, there are diagrams printed in the textbook, but you will think harder about the diagram and understand more if you draw your own.

Example 1.2 asks a question. Read the question and see if you can answer it on your own, without reading further in the book.

On page 7, you will see that solutions of the exercises are provided. Resist the urge to turn your brain off and just read the solutions. That is not what they are there for!

Read through each of the problems that begin on page 8 in the book. Figure out what each problem is asking for and write that in your notes. If you can solve the problem, do that. If not, that's OK.

Problems 1.1 to 1.8 look nice.

You might start Problem 1.9 by trying some possible values for n.

Problem 1.10 doesn't interest me. Does it interest you?

Can you draw the region described in Problem 1.11?

Problem 1.12 is good. Would it help to make a graph?

Problem 1.13 seems silly. Do you like it anyway?

[2] Reading, Writing, and Proving: A Closer Look at Mathematics, 2011, by Ulrich Daepp and Pamela Gorkin

Read the Tips on Doing Homework. At the end, tally up how much time you have spent on reading this chapter. Write this number in your notebook and remember the number when you come to class.

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Reading assignment, Chapter 2

Due on Wednesday, September 2.

Read Chapter 2 of the book by Daepp and Gorkin. As with Chapter 1,

1. Read somewhere quiet, minimizing distractions from phones and friends

2. Note the times that you start and stop reading, and add up the minutes

3. Read with a pencil in your hand and your notebook open in front of you

4. Write a sentence to summarize each paragraph, re-draw diagrams, work out examples and exercises on your own

5. Look up words you don't know, and write down ones you really don't know

6. Read slowly. You are not reading a comic book or a newspaper. It is not a goal of this class for you to learn how to read faster. The goal is to learn how to get more out of the time you spend reading, and to learn to concentrate for longer periods of time.

7. At the end, tally up how much time you have spent on reading this chapter. Write this number in your notebook and remember the number when you come to class.

You will read about "statements." Focus on the ones about mathematical things, and don't worry too much about interpreting the ones that are non-mathematical.

Note that on page 14, there is a statement about the color of the cover of the book. Books from Springer used to be plain yellow, but the authors must not have realized that someone would put a big blue bar on the cover of this edition of the book. Just imagine that the book cover is all yellow.

Fill out every truth table that is suggested in the chapter. Truth tables are an excellent way to get great clarity about complicated combinations of statements. The idea is to consider every possible combination of True and False for the basic statements. For example, if there are two statements, P and Q, there will be four rows in the table, running through the four possible combinations of True and False for P and Q. On page 21, there is a truth table for three statements, P, Q, and R. It has eight rows.

The most important use of truth tables is to tell when two complicated combinations of logical expressions are, in fact, the same.

For me, the hardest thing about truth tables is making columns for implications like $P \to Q$. Here is the best way I know to think about them. Each row of the truth table for P and Q covers one combination of truth values for P and Q. Some of these combinations are consistent with the implication that P implies Q. For example, when P is True and Q is True, this is consistent with $P \to Q$, so we put T in the $P \to Q$ column. The row in which P is True and Q is False, however, is inconsistent with the implication $P \to Q$, so we put F in that row. The cases in which P is False are a bit different, but they are also consistent with $P \to Q$, since $P \to Q$ only has anything to say about P and Q when P is True. So we put T in those rows too.

Problems 1 to 7 are good, so please do those. Rather than working on problems 9-21, I would much prefer that you spend your time making the truth tables I describe below.

1. Make a truth table for $\neg(P \vee Q)$ and $\neg P \wedge \neg Q$.

2. Make a big truth table for $P, Q, R,\ P \wedge (Q \vee R),\ P \vee (Q \wedge R),\ (P \wedge Q) \vee (P \wedge R)$, and $(P \vee Q) \wedge (P \vee R)$. Which of these are equal? How can you rememeber that?

Reading assignment, Chapter 3

Read Chapter 3 of the textbook by Daepp and Gorkin. As with Chapters 1 and 2,

1. Read somewhere quiet, minimizing distractions from phones and friends

2. Note the time that you start and stop reading, and add up the minutes

3. Read with a pencil in your hand and your notebook open in front of you

4. Write a sentence to summarize each paragraph, re–draw diagrams, work out examples and exercises on your own

5. Look up words you don't know, and write down ones you really don't know

6. Read slowly. You are not reading a comic book or a newspaper. It is not a goal of this class for you to learn how to read faster. The goal is to learn how to get more out of the time you spend reading, and to learn to concentrate for longer periods of time.

7. At the end, tally up how much time you have spent on reading this chapter. Write this number in your notebook and remember the number when you come to class.

Theorem 3.1 lists three properties of logical statements. Please make truth tables for each of them to check that they are tautologies. Also add de Morgan's laws from Theorem 2.9. Then you'll have the whole set. Having de Morgan's laws handy should make Exercise 3.2 easier.

How can you remember the distributive property?

The contrapositive is really important. See if you can explain it just by thinking about $P \to Q$ and $\neg Q \to \neg P$, without using truth tables.

Theorem 3.3 is proven using the contrapositive. This is a very useful method of proof. Please note that it differs from proof by contradiction.

Read about the converse, and make sure never to confuse an implication with its converse.

Problems 2, 3, 4, 9, 14, 16, 18, 5, 6, 8, 19, 15 are good to work on, in that order. Work through at least half of these problems.

Chapters 1 to 5 are mostly there to help develop proof techniques. After Chapter 5, we will spend more of our time on definitions, examples, theorems, and proofs. Use your time now to develop basic logic and proof techniques that will help you for the rest of the semester and beyond!

Reading assignment, Chapter 4

Due on Friday, September 18.

Read and understand Chapter 4 of the textbook by Daepp and Gorkin. As with previous chapters,

1. Read somewhere quiet, minimizing distractions from phones and friends

2. Note the time that you start and stop reading, and add up the minutes

3. Read with a pencil in your hand and your notebook open in front of you

4. Write a sentence to summarize each paragraph, re–draw diagrams, work out examples and exercises on your own

5. Look up words you don't know, and write down ones you really don't know

6. Read slowly. You are not reading a comic book or a newspaper. It is not a goal of this class for you to learn how to read faster. The goal is to learn how to get more out of the time you spend reading, and to learn to concentrate for longer periods of time.

7. At the end, tally up how much time you have spent on reading this chapter. Write this number in your notebook and remember the number when you come to class.

This is a very important chapter, one with real substance. Hopefully you will feel that way when you read it, and will enjoy it more as a result. It is a little bit about set theory, but mostly about quantifiers.

This chapter has a large number of very dense expressions involving quantifiers, implications, and logical operators. Slow way down when you run into one of them. Pick them apart in your mind and then write them down so they are crystal clear. Every symbol is important. It's a bit like when you're reading someone your credit card number or you're giving your phone number to someone you really want to call you. Every symbol is important.

Exercises 4.1, 4.2, 4.3, and 4.6 are all useful to do. 4.2(a) is harder than they make out, because you not only want to write that a solution x exists, but that if y is also a solution, then $x = y$. The discussion that begins at the bottom of page 36 is very important, negating statements with quantifiers.

There are 20 problems. The more of them you do, the better, of course, but you may not be able to work through all of them. **Please at least do problems # 1–7 and 20.** Read # 11. Does this joke work on your friends?

Pay attention to the phrase "only if" which appears in Problem 12. It is often used in a way that can be confusing. Compare these two statements for example, in which R means Race and P means prize:

1. I will race if there is a prize offered. $P \to R$. This is the most common way that people use the word "if." The prize will make me race.

2. I will race only if there is a prize offered. $R \to P$. People say this sort of thing pretty often too, but it's a bit less clear unless you think about it carefully. Part of the problem is the time order in which things happen, because the racing comes *after* the prize is offered. "If you see me racing, you can be sure that there was a prize offered. (But offering a prize is no guarantee that I will race.)"

Reading assignment, Chapter 5

Due Friday, September 25.

Read and understand Chapter 5 of the textbook by Daepp and Gorkin. As with previous chapters,

1. Read somewhere quiet, minimizing distractions from phones and friends
2. Note the time that you start and stop reading, and add up the minutes
3. Read with a pencil in your hand and your notebook open in front of you
4. Write a sentence to summarize each paragraph, re–draw diagrams, work out examples and exercises on your own
5. Look up words you don't know, and write down ones you really don't know
6. Read slowly. You are not reading a comic book or a newspaper. It is not a goal of this class for you to learn how to read faster. The goal is to learn how to get more out of the time you spend reading, and to learn to concentrate for longer periods of time.
7. At the end, tally up how much time you have spent on reading this chapter. Write this number in your notebook and remember the number when you come to class.

This chapter walks you through a number of types of proofs and gives examples of each. **Rewrite these proofs in your notes, in your own words as much as possible, so that you make them yours.** By the end of reading the chapter, you should **know** the proof that the square root of 2 is irrational and you should know the other proofs as well.

It might help, in your notes, to make a list of proof techniques from the chapter and from previous chapters. What chapter talked about proof by contrapositive? Is that in Chapter 5? What about truth tables? You can prove things with those. What kinds of things?

Read and understand Problem 1. It is important.

Read the other problems, find the ones that are easy, and do them. This may seem like a strange assignment, but I really mean it. Think about each problem (if you can get through all of them), and make sure that if a problem is easy, that you recognize that and write out the solution. Don't worry if a problem looks hard but turns out to be easy. That happens all the time. But hopefully you will spot a number of them that really are easy, and do them. We will go over these problems in class the following week.

Reading assignment, Chapter 6

Due on Wednesday, October 7

Read and understand Chapter 6 of the textbook by Daepp and Gorkin. As with previous chapters,

1. Read somewhere quiet, minimizing distractions from phones and friends
2. Note the time that you start and stop reading, and add up the minutes
3. Read with a pencil in your hand and your notebook open in front of you
4. Write a sentence to summarize each paragraph, re–draw diagrams, work out examples and exercises on your own
5. Look up words you don't know, and write down ones you really don't know
6. Read slowly. You are not reading a comic book or a newspaper. It is not a goal of this class for you to learn how to read faster. The goal is to learn how to get more out of the time you spend reading, and to learn to concentrate for longer periods of time.
7. At the end, tally up how much time you have spent on reading this chapter. Write this number in your notebook and remember the number when you come to class.

Many proofs in other classes involve showing that two sets are equal, or that one set is a subset of another, or that one set is not a subset of another. Work really hard on this chapter and it will pay you dividends for a long, long time.

Make sure to add value when you take notes. Write new thoughts, new questions, new comments.

This chapter introduces sets, subsets, equality of sets, and how to tell what the members of a set are. As you read, take time to write out at leart 5 members of each set that is introduced. Note that A being a subset of B is the same as the logical implication $x \in A$ implies $x \in B$. There is a tight connection between statements in set theory and logical statements. Here is another: Set A being equal to set B is the same as the logical implications $x \in A$ if and only if $x \in B$.

There are many examples in this chapter. Work through them by rewriting them and adding useful steps in your notes.

On page 64, intersections, unions, and complements of sets are introduced. As you read about them, explain in your notes how these relate logical statements such as $x \in A$ and $x \in B$ to $x \in A \cap B$.

You may enjoy reading about the paradoxes on page 67. Give them a try. Even if they are not your cup of tea, try to see what the issue is.

Problems 1 − 9 are essential. Do them.

Problem 10 is a good thought problem. Think about it and write your answer.

Starting with Problem 11, there are things for you to prove. I would be happy to see you do many of these by yourself.

Reading assignment, Chapter 7

Due Monday, November 2.

Read and understand Chapter 7 of the textbook by Daepp and Gorkin called "Operations on sets."

This is a short chapter, all about working with sets. You can approach these problems in a number of ways. Often it helps to draw a nice Venn diagram and get the right intuitive idea for what is being claimed, but don't stop there. You can also just focus on letting $x \in A$ or whatever and working with that, without thinking about Venn diagrams.

Most of the chapter is devoted to one example, showing that, if A, B, and C are sets, then $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$. The book suggests working forward from one side, and backward from the other, just as people sometimes build a bridge by starting at each bank of a river and meeting in the middle.

It also suggests breaking into cases at some point. For example, if $x \in A \cup (B \cap C)$, you can consider the case $x \in A$, which is great because then it's pretty clear that $x \in (A \cup B) \cap (A \cup C)$. But you also need to consider the case $x \notin A$, so that $x \in B \cap C$. But that's helpful, because then $x \in B$ and $x \in C$, and pretty soon it is clear that $x \in (A \cup B) \cap (A \cup C)$.

Do problem 7.1, parts a, c, d, e, f. Take your time and use really good form so that the proof is crystal clear. Notice that part (c) (statement 18 in the theorem) is an "if and only if" statement, so it has two parts. It's going to look something like this:

1. Suppose that $A \subseteq B$. We want to show that $(X \backslash B) \subseteq (X \backslash A)$. Let $x \in X \backslash B$. Then $x \notin B$. (More steps here.) Thus, $x \in X \backslash A$, and so $(X \backslash B) \subseteq (X \backslash A)$.

2. Suppose that $(X \backslash B) \subseteq (X \backslash A)$. We want to show that $A \subseteq B$. Let $x \in A$. (More steps here.) Thus, $x \in B$.

Do problem 7.4.

Do problem 7.6.

I guess that these problems are a bit dull, but it really is helpful to be good at proving things about sets.

As with the previous chapters,

1. Read somewhere quiet, minimizing distractions from phones and friends

2. Note the time that you start and stop reading, and add up the minutes

3. Read with a pencil in your hand and your notebook open in front of you

4. Write a sentence to summarize each paragraph, re-draw diagrams, work out examples and exercises on your own

5. Look up words you don't know, and write down ones you really don't know

6. Read slowly. You are not reading a comic book or a newspaper. It is not a goal of this class for you to learn how to read faster. The goal is to learn how to get more out of the time you spend reading, and to learn to concentrate for longer periods of time.

7. At the end, tally up how much time you have spent on reading this chapter. Write this number in your notebook and remember the number when you come to class.

Reading assignment, Chapter 8

Make a good effort by Monday, November 9; due on Friday, November 13.

Read and understand Chapter 8 of the textbook by Daepp and Gorkin, called "More on operations on sets."

This chapter is a challenge. You will really need to use all the reading skills you have been practicing when you read this chapter. The ideas are harder, and some are really hard, but not impossible. Just slow yourself down and write things out in lots of detail.

Example 8.2(a) would be a great one to write out concrete fractions with different values of p and q to understand the sets A_q and then the union of these sets. For Example 8.2(b), do the same to understand what the sets B_i are, and then what their intersection is. No shortcuts! Write out elements for each set.

Exercise 8.3 is also good.

In the middle of page 82 the phrase "collection of subsets of X" appears. This is a very new, very difficult concept; do not underestimate how tricky it can be, but patiently think about it and keep coming back to it. For example, $\mathcal{A}$ might be all intervals of the form $[k, k+1]$ and you might want to take the union of all such intervals, or the intersection.

Exercise 8.4 is excellent. Draw pictures until everything is crystal clear. Exercise 8.5 is also excellent.

Rewrite the proofs of Examples 8.6 and 8.7 to make them your own. Really.

Exercises 8.9 and 8.10 are also excellent. Do them on your own, then compare to the solutions in the book.

Do problems 1, 2, and 3.

Here is a challenge problem. Let $a < b$. Show that $\bigcup_{n=1}^{\infty}[a, b - \frac{1}{n}] = [a, b)$. Draw pictures, then show set inclusion both ways.

Here is another challenge problem. Let $a < b$. Show that $\bigcap_{n=1}^{\infty}[a, b + \frac{1}{n}) = [a, b]$. Draw pictures, then show set inclusion both ways.

As with the previous chapters,

1. Read somewhere quiet, minimizing distractions from phones and friends
2. Note the time that you start and stop reading, and add up the minutes
3. Read with a pencil in your hand and your notebook open in front of you
4. Write a sentence to summarize each paragraph, re-draw diagrams, work out examples and exercises on your own
5. Look up words you don't know, and write down ones you really don't know
6. Read slowly. You are not reading a comic book or a newspaper. It is not a goal of this class for you to learn how to read faster. The goal is to learn how to get more out of the time you spend reading, and to learn to concentrate for longer periods of time.
7. At the end, tally up how much time you have spent on reading this chapter. Write this number in your notebook and remember the number when you come to class.

Reading assignment, Chapter 9

Due in the tenth week of class.

Read and understand Chapter 9 of the textbook by Daepp and Gorkin, called "The Power Set and the Cartesian Product." This is the last chapter on plain set theory. It should stretch your mind in a few new directions. Prepare to move slowly and think carefully.

When A is a set, the power set of A is the collection of all subsets of A. Read Example 9.1 and do Exercise 9.3 and then **do Problem 9.1.** Work through Exercise 9.2 and then **do Problem 9.2.** Problem 9.2 is hard, but excellent for you. Take it very slowly. Work through Exercise 9.4 and then **do Problem 9.5. Do Problem 9.8.**

Do Problem 9.11. For 9.11, you have already seen the power set of a set containing 2 elements and 3 elements. **Hint:** When you are making a subset of a set A, for each element of A, you have to decide whether it goes in or out of the subset. There are two choices (in or out) each time. If the hint doesn't help you, write out the power set of $\{1, 2, 3, 4\}$, then read the hint again. Hopefully you don't have to write out the power set of $\{1, 2, 3, 4, 5\}$!

You are already very familiar with one Cartesian product: making ordered pairs (x, y) of real numbers is the Cartesian product $\mathbb{R} \times \mathbb{R}$, which you know better as the xy plane. Every problem involving Cartesian products of sets containing real numbers can be depicted as points in the xy plane. Make a graph in every case. This will help your intuition. When there are only finitely many points, like with $\{0, 1\} \times \{2, 3\}$, also list out all of the (x, y) pairs.

Answer these questions: Who is the Cartesian product named after? Why, exactly?

Work through Exercise 9.5 a, b, e.

For Theorem 9.7, draw A and C as intervals on the x axis and draw B and D as intervals on the y axis, then draw out the sets in the statement of the theorem on two separate sets of axes. Make sure you are crystal clear about what these sets are, and you will be close to mastering Cartesian products.

Do Problem 9.12. It connects Cartesian products to things you learned in geometry.

Do Problem 9.17a. Notice that this is an "if and only if" proof, and it has three set equalities to show. Suppose that $A \times B = C \times D$ and show that $A = C$ and $B = D$ by showing containment each way. Here is one part of the argument: Let $x \in A$. Also let $y \in B$. Then $(x, y) \in A \times B = C \times D$, and so $x \in C$. Thus $A \subseteq C$. After that part is done, suppose that $A = C$ and $B = D$ and argue that $A \times B = C \times D$.

Think about Problem 9.19.

As with the previous chapters,

1. Read somewhere quiet, minimizing distractions from phones and friends
2. Note the time that you start and stop reading, and add up the minutes
3. Read with a pencil in your hand and your notebook open in front of you
4. Write a sentence to summarize each paragraph, re-draw diagrams, work out examples and exercises on your own
5. Look up words you don't know, and write down ones you really don't know
6. Read slowly.

7. At the end, tally up how much time you have spent on reading this chapter. Write this number in your notebook and remember the number when you come to class.

Reading assignment, Chapter 10

Due in the eleventh week of class.

Read and understand Chapter 10 of the textbook by Daepp and Gorkin, called "Relations."

The main definition for Chapter 10 appears at the end of Chapter 9, on page 93. Here is the deal. A *relation* S from a set X to a set Y is a subset of $X \times Y$. If $Y = X$, we say the relation is a relation on X. At the beginning of Chapter 10, we see that we are going to be only working with relations on a set X.

Suppose that S is a relation on a set X. That is, suppose that S is a subset of $X \times X$, which means that S is a set of points of the form (x, y), where $x \in X$ and $y \in X$. Rather than write $(x, y) \in S$, we usually write $x \sim y$. How to read this out loud? There is no perfect solution. I would suggest that you read it as "x tilde y" (because $\sim$ is the tilde that appears above the n in some Spanish words).

Suppose that $X = \mathbb{R}$ and let $S = \{(x, y) : x \leq y\}$. Then $x \sim y$ means that $(x, y) \in S$, which means that $x \leq y$. In this way, we see that $\leq$ is a relation on $\mathbb{R}$. **Write out the set S corresponding to the relations $<$, $\leq$, $=$, $\geq$, and $>$. Then also sketch these as regions in the xy plane.**

Note that relations are between two elements. Thus, "divisible by 4" is not a relation. However, if $X = \mathbb{Z}^+$, you could say that $x \sim y$ if y is divisible by x, and then you would have a relation. People often write $x|y$ for this relation and say that x divides y. Call this relation S. **Write out at least ten of the ordered pairs in S, using at least five different values of x.**

Read Exercises 10.1 and 10.2.

Read the definitions of reflexive, symmetric, and transitive. A relation that satisfies all three is called an equivalence relation. This is where most of the action is with relations. **Do Problem 10.2. You should start every part of the problem by writing down examples.** For example, for (a), the example $3 < 3$ will tell you whether the relation is reflexive, $3 < 5$ and $5 < 3$ will tell you about symmetry, and $3 < 5, 5 < 7$, and $3 < 7$ will get you started on transitivity.

Read Example 10.3, then **do Problem 10.3.** Use examples to check reflexivity, symmetry, and transitivity.

Equivalence relations are very important, as are equivalence classes. An equivalence relation is like the equality relation (=), but applied to other contexts. Here is an example that is useful. Think of the integers, $\mathbb{Z}$. Say that $x \sim y$ if x and y have the same remainder when you divide by 2. Then $6 \sim 22$ and $31 \sim 7$. This relation is reflexive, because $x \sim x$. It is symmetric because if $x \sim y$ then $y \sim x$. And it is transitive because if $x \sim y$ and $y \sim z$, then $x \sim z$. Now we can say that 6 is equivalent to 22, and 31 is equivalent to 7, according to this definition of equivalence. The equivalence class that contains 6 and 22 is all even numbers, and the equivalence class containing 31 and 7 is all odd numbers. Let this sink into your mind, and you will start to see that it makes for a useful way to organize things, when an equivalence relation is available.

Do Problem 10.1 Start by writing out examples for the pairs (x, y) and (w, z). Think about lines and circles in the plane.

1. Read somewhere quiet, minimizing distractions from phones and friends
2. Note the time that you start and stop reading, and add up the minutes
3. Read with a pencil in your hand and your notebook open in front of you
4. Write a sentence to summarize each paragraph, re-draw diagrams, work out examples and exercises on your own
5. Look up words you don't know, and write down ones you really don't know

6. Read slowly.

7. Tally up how much time you have spent on reading this chapter.

Reading assignment, Chapter 18

Reading due on November 30, notes and exercises due on December 2.

Read and understand Chapter 18 of the textbook by Daepp and Gorkin, called "Mathematical Induction," up to the statement, but not the proof, of Theorem 18.6.

Mathematical induction and recursion play an important role especially in discrete mathematics. Prepare to move slowly and think carefully. To understand the proof of Theorem 18.1, you will need **Well-ordering principle of the natural numbers**: Every nonempty subset of the natural numbers contains a minimum.

Read Theorem 18.1 and then **do Problem 18.1** and **Problem 18.3**. Follow the steps in Theorem 18.1, defining the assertion $P(n)$ for the problem first. You will need the condition "$P(n)$ is true" to show the induction step. Work through Exercises 18.3 to 18.5 and then **do Problem 18.9** without going back to Exercise 18.5. You can do it!

Recursion is a very useful tool to define functions, sequences and sets. Before you move to Theorem 18.6, read the definition of n factorial for $n \in \mathbb{N}$. Write out $3!, 4!$ and $5!$. As an exercise, simplify $\frac{6!}{2!4!}$. More generally, simplify $\frac{n!}{m!(n-m)!}$ where n and m are two positive integers with $n \geq m$. These fractions are called *binomial coefficients* and are useful in probability.

Here is another example of using recursion: Let $n \in \mathbb{Z}^+$. Consider the function $S(n) = S(n-1) + n$ with $S(0) = 0$. Write out $S(1), S(2), S(3)$ and $S(4)$. Can you figure out what this function does for us? Together with Problem 18.1, you should be able to see the connection between induction and recursion.

Theorem 18.6 shows the existence and uniqueness of a recursive function $g : N \to X$ given a function $f : X \to X$ and $a \in X$, where X is a nonempty set. The function g satisfies

(i) The base step: $g(0) = a$, and

(ii) The recursive step: $g(n+1) = f(g(n))$ for all $n \in \mathbb{N}$.

The proof of Theorem 18.6 is too long for us to read this semester. You can come back it later.

As with the previous chapters,

1. Read somewhere quiet, minimizing distractions from phones and friends

2. Note the time that you start and stop reading, and add up the minutes

3. Read with a pencil in your hand and your notebook open in front of you

4. Write a sentence to summarize each paragraph, re-draw diagrams, work out examples and exercises on your own

5. Look up words you don't know, and write down ones you really don't know

6. Read slowly.

7. At the end, tally up how much time you have spent on reading this chapter. Write this number in your notebook and remember the number when you come to class.