What you might say on the first day of class

This class is designed to help you develop key skills in math that will help you succeed in higher level, more abstract courses. My goal is that by working hard in this course, you will become unstoppable in your future math courses.

There are two main components to the course: learning how to read a math book on your own, and practicing the basic steps in mathematics, where we start with examples and non–examples, make a definition, examine more more examples, make a conjecture, see if we can find a counterexample to the conjecture, make another conjecture, and if we can find a proof, call the result a theorem.

The simplest way to describe this course is that it is the "proof course." A better way to describe it is that it focuses on the core ideas of mathematics, over and over again, in different settings. The core idea of mathematics is to make definitions of mathematical objects, consider examples and non–examples of those definitions to make sure we understand them well, make conjectures about what we think might be true, and then try to prove those things. It is all about understanding mathematical ideas and how they fit together. Proofs are a part of that, but they are only a part. This course will not have the same feel as algebra and calculus courses, which are heavier on calculations and don't have quite as many different ideas or different types of examples.

You will work on activities in class, then hand them in at the end of class. I will read over them and make comments, then give them back to you at the beginning of the next class. There are often multiple correct ways to do a problem. It's not question of "what I want" as the teacher, but what works?

There is no need to rush through the activities. Take your time, think about what you're doing. Work with your group to make sure you all understand everything along the way. You do not need to finish the activity; I always try to add extra material at the end so that no group runs out of things to do.

Most people learn by doing. Me talking at the board is not the same as you learning. Most of your time in the course will be spent working on activities as a group while I go from group to group, seeing how you are doing, answering questions, and making suggestions.

Outside of class, you will be reading the textbook, taking notes on what you read, and solving some exercises. You will bring your notes to class and they will be read over quickly to make sure that you are doing the reading and thinking. Each chapter will also have a short reading quiz. I will not lecture over the material in the book; that is one of the keys to you learning how to read a book on your own.

There will be a final exam, but rather than mid-term exams, there will be quizzes over the material in activities, once you have had a chance to get good at it.

Your name: ______________________________

Even and odd

Our first example of definitions, examples, theorems, and proofs

Overview

Definitions are important to read and understand by looking at examples. Many proofs are little more than working with the definitions and rewriting things. With a bit of practice, these become very routine. This activity has you work through two definitions, a few examples, and then some proofs. Everything relies on the definitions, so keep coming back to them. We will use a similar model many times during the semester.

Definition 1. Even. An integer n is *even* if there exists an integer k for which $n = 2k$.

Definition 2. Odd. An integer n is *odd* if there exists an integer k for which $n = 2k + 1$.

Example 3. Check that 12 meets the definition to be even. Find the value of k and write $12 = 2k$.

Example 4. Does -9 meet the definition to be odd? Find the value of k and write $-9 = 2k + 1$.

Example 5. Does 0 meet the definition to be even? Find the value of k.

Example 6. Does 1.73 meet the definition to be odd? Explain.

Note 7. Suppose that m is an integer. Then $2m + 1$ is an integer and it is odd because it meets the definition to be odd. Also, $2m + 2$ is even because it is an integer and can be rewritten as $2(m + 1)$, which is of the form $2k$ where $k = m + 1$, which is an integer.

Show 8. Suppose that m is an integer. Show that $2m+6$ is even by rewriting it until it meets the definition to be even.

Show 9. Suppose that m is an integer. Show that $4m+9$ is odd by rewriting it until it meets the definition to be odd.

Stop. Compare your answers to the questions above with the other people in your group before you move on. Resolve any differences in your answers.

Show 10. Suppose that m is even. Then $m = 2k$ for some integer k. Show that $m + 8$ is even by rewriting it as 2 times an integer.

Note 11. You know perfectly well that the sum of two even numbers is even. The next item guides you through a proof of this fact, using the definitions of even and odd above.

Guided proof 12. Suppose that m and n are even. Fill in the blanks to show that $m + n$ is even.

1. There exist integers j and k such that __________ and __________.

2. Thus, $m + n =$ __________.

3. This number is even because ____.

4. We saw that if m and n are even, then $m + n$ is even. We made no further assumption about m and n. Thus, the sum of any two even numbers is even.

Guided proof 13. Suppose that m is even and n is odd. Fill in the blanks to show that mn is even. Use the previous exercise as a model.

1. There exist ___________________________ such that $m =$__________ and $n =$__________.

2. Thus, $mn =$____.

3. This number is even because ____________________________.

4. We saw that ____. We made no ____. Thus, ____________________________________.

Prove 14. Let m and n be odd. Follow the examples above as a model to show that mn is odd by rewriting mn until it meets the definition to be odd. Good form is critically important in proofs.

1.

2.

3.

4.

Prove 15. Let m and n be odd. Follow the examples above as a model to show that $m + n$ is even by rewriting it. Good form is critically important in proofs.

1.

2.

3.

4.

Group work 16. Steps 1 and 3 in your proofs connect to the definitions of even and odd. Write a sentence telling what happens in step 1 and in step 3. Make sure that every member of your group has gotten to this step. Compare your sentence to those written by the other people in your group, then work together to write the best version of the sentence that your group can.

Prove 17. Let m be odd. Follow the models above to prove that m^2 is odd. Use good form.

Prove 18. Let m be even. Follow the models above to prove that m^2 is even. Use good form.

Question 19. What does it mean that an integer is a multiple of 4? Give your own definition analogous to the definitions of even and odd.

Prove 20. Let m be even. Show that $m^2 + 2m + 4$ is a multiple of 4.

Question 21. Let m be even. Can $m^2 + 2m + 4$ be a multiple of 8? Explain.

Challenges. Here are some statements that are harder to prove, because they require a bit more than simply restating the definitions. See if you can make a good argument for them.

Prove 22. If m is an integer, then either m is even or m is odd. **Hint**: 0 is even. If n is even, then $n + 1$ is odd. If n is odd, then $n + 1$ is even. This tells us something about all positive integers. If n is even, then $-n$ is even.

Prove 23. If m is an integer and m^2 is odd, then m is odd. **Hint**: There are two cases to check, the case in which m is even and the case in which m is odd.

Your name: ______________________________

Syllabus for Math (insert course number here)

This syllabus is an assignment for you to read and respond to. Please read it carefully, fill in answers, put your name on it, and turn it in on the second day of class.

Course description. There are two main goals for the course:

1. Improving your ability to work with definitions, examples, counterexamples, claims, and proofs.

2. Improving your ability to read a mathematics textbook on your own.

My hope is that your new abilities in these two areas will make you unstoppable in your mathematics classes. We will spend most of class time on #1. I will design activities for us to do together in class for this purpose. Most of your time outside of class will be spent on #2. Being able to read mathematics on your own is a fantastic skill. Be sure to set aside quiet time to read the textbook.

Q: How comfortable are you already with the "definition, example, theorem, proof" sequence in mathematics classes?

Q: What kinds of experiences have you had in the past with proofs?

Professor and contact information.

Q: Do you check your university email regularly?

Schedule.

Office hours. You are welcome to visit me in my office, which is room XXX in the mathematics building. The best way to arrange a time to meet is to send an email listing a few times that would work for you. I will reply with one that works for me as well.

(Sample question to get students to find my office) Q: If you take the elevator to the fourth floor, do you turn right or left to get to my office?

(Sample question to get students to find my office) Q: What are the two flyers on my door

about?

Textbook. (A suggested textbook; reading assignments are given in these materials for this book.) The textbook for the course is *Reading, writing and proving, A closer look at mathematics*, second edition, by Ulrich Daepp and Pamela Gorkin. 2011. The textbook is very good, but not perfect. You can learn from it, especially if you take time to read it.
Q: Have you ever had success reading a mathematics textbook and really learning from it? If so, please tell what book, what course, and what made it work. If not, please tell me what you think prevented you from being able to read the book.

Q: Do you have a hard copy of the textbook that you can read? Have you been able to get a PDF file of the first chapter from the library?

Graduate assistant.
Q: Do you have any interest in going to graduate school? Please explain.

Coursework. Here are the main things that you will be doing:

1. Written work on in–class activities.

2. Taking notes on each chapter in the textbook in your notebook. Bring your notebook to class so that the graduate assistant can read through it in class and give it back.

3. Occasional quizzes (very much like the work you'll already be doing in class) instead of one or two big exams

4. A final exam (which should be very similar to what we have been doing all semester long)

Q: Do you have any questions or concerns about the coursework?

Grading. My general plan is this. Many things you do during the semester will have a point value attached to them. The number of points will indicate their relative importance to your grade. In–class work and homework will count for a larger share than in most courses, while quizzes and exams will count for a lower share. I will announce the relative percentages at least two weeks before the first exam.

Q: Do you have any questions or concerns about the grading?

Attendance. Attendance and class participation will be vitally important. Class time is the best time to make attempts and get immediate feedback. If you cannot attend a class, notify me as soon as possible by email or phone, before class if possible. Don't even imagine that you can miss a class without letting me know. I don't particularly need to know **why**, but I do need to know.

Q: What is the most likely reason that you will miss class? I'm just curious.

Background questions.

1. What mathematics courses are you taking this semester? It's OK to just list the numbers, like Math 3410.

2. What mathematics courses have you already taken here?

3. Including this one, how many semesters until you graduate?

4. Please let me know anything you think I should know about you. I'll read it all. Sometimes people like to tell about their hobbies, movies they like, where they're from, etc.

Your name: ______________________________

Sum and dot product of 3–dimensional vectors

We define a new mathematical object and do some work with it.

Overview

In Calculus III and Linear Algebra, one defines vectors and works with them. They have a geometric interpretation, but here we will simply give an algebraic definition and work with their algebraic properties. This activity illustrates proofs in which all that is needed is the definition and some rewriting. Notice how we often use the same definition twice in one proof, once to "unpack" and the second time to "re–pack."

Definition 24. 3–dimensional vector. A three–dimensional vector is an ordered triple $\langle a_1, a_2, a_3\rangle$, where a_1, a_2, and a_3 are real numbers.

Notation 25. A 3–dimensional vector $\langle a_1, a_2, a_3\rangle$ is often denoted by a single letter with an arrow over the top, like this $\vec{a}$. When it is written like $\langle a_1, a_2, a_3\rangle$ it is said to be in *open form.*

Definition 26. Equality of 3–dimensional vectors. 3–dimensional vectors $\langle a_1, a_2, a_3\rangle$ and $\langle b_1, b_2, b_3\rangle$ are equal if $a_1 = b_1, a_2 = b_2$, and $a_3 = b_3$. The order of the numbers is important.

Definition 27. Sum of 3–dimensional vectors. The sum of 3–dimensional vectors $\langle a_1, a_2, a_3\rangle$ and $\langle b_1, b_2, b_3\rangle$ is the 3–dimensional vector $\langle a_1 + b_1, a_2 + b_2, a_3 + b_3\rangle$. We write $\vec{a} \oplus \vec{b}$, using a new symbol so we don't confuse addition of vectors with addition of real numbers.

Example 28. Is $\langle 3, 9, 12\rangle$ a 3–dimensional vector? Explain. Is it equal to $\langle 12, 3, 9\rangle$? Explain.

Example 29. Is $\langle \sqrt{3}, \sqrt[3]{9}, \sqrt[4]{-12}\rangle$ a 3–dimensional vector? Explain.

Example 30. Is $\langle 8, 13.35321, \pi, -7\rangle$ a 3–dimensional vector? Explain.

Example 31. Is $\langle 1, \pm 7, \heartsuit\rangle$ a 3–dimensional vector? Explain.

Example 32. Is $\langle 3 + 9 + 12\rangle$ a 3–dimensional vector? Explain.

Example 33. Is $\langle \begin{bmatrix} 6 & 0 \\ 2 & 5 \end{bmatrix}, -4, 7\rangle$ a 3–dimensional vector?

Example 34. Let x be a real number. Is $\langle \frac{14}{3}, 2-7x, \sqrt{16}\rangle$ a 3–dimensional vector? Explain.

Stop. Compare your answers the the questions above with the members of your group. Make sure you agree on everything.

Example 35. Calculate the sum of $\vec{a} = \langle 12, -5, 3\rangle$ and $\vec{b} = \langle 6, 4, -11\rangle$. Start by writing $\vec{a} \oplus \vec{b} = \ldots$ and write the vectors in open form next.

Show 36. We are going to show that addition of 3–dimensional vectors is commutative. We will do this by rewriting. Follow the model.
Let $\vec{a}$ and $\vec{b}$ be 3–dimensional vectors. Then,

$$\begin{aligned}
\vec{a} \oplus \vec{b} &= \langle \qquad , \qquad , \qquad \rangle \oplus \langle \qquad , \qquad , \qquad \rangle \\
&= \langle \qquad\qquad , \qquad\qquad , \qquad\qquad \rangle \\
&= \langle \qquad\qquad , \qquad\qquad , \qquad\qquad \rangle \\
&= \langle \qquad , \qquad , \qquad \rangle \oplus \langle \qquad , \qquad , \qquad \rangle \\
&= \vec{b} \oplus \vec{a}
\end{aligned}$$

We have seen that $\vec{a} \oplus \vec{b} = \vec{b} \oplus \vec{a}$. We made no further assumption about $\vec{a}$ and $\vec{b}$. Thus, for all 3–dimensional vectors $\vec{a}$ and $\vec{b}$, we know that $\vec{a} \oplus \vec{b} = \vec{b} \oplus \vec{a}$. Thus, additional of 3–dimensional vectors is commutative.

Show 37. Go back to each line of the proof above and give a reason for the equality on that line at the very right side of the line. The first one is "Write in open form." Somewhere in the middle you will use the fact that addition of real numbers is commutative. Thus, at the heart of it, commutativity of vector addition comes from commutativity of addition of real numbers.

Show 38. You are going to show that addition of 3–dimensional vectors is associative. Let $\vec{a}, \vec{b}$, and $\vec{c}$ be 3–dimensional vectors. Start with $(\vec{a} \oplus \vec{b}) \oplus \vec{c}$ and rewrite it until it becomes $\vec{a} \oplus (\vec{b} \oplus \vec{c})$ following the model of the previous proof, starting with the word "Let". Also write explanations for each step. Conclude, following the model, that you have shown that vector addition is associative. This last part is very important.

Stop. Compare your argument to the rest of the members of your group. Make sure that you agree on absolutely every step and every justification.

Definition 39. Scalar product for 3–dimensional vectors. Let c be a real number and let $\vec{a} = \langle a_1, a_2, a_3 \rangle$ be a 3–dimensional vector. The *scalar product* of c and $\vec{a}$ is defined as:

$$c\vec{a} = \langle ca_1, ca_2, ca_3 \rangle.$$

Example 40. Let $c = 3$ and $\vec{a} = \langle 7, -4, \sqrt{2} \rangle$. Calculate $c\vec{a}$, starting by writing $c\vec{a} = \ldots$.

Example 41. Calculate $(2 + \sqrt{3})\langle 9, 4, 1 \rangle$.

Show 42. You are going to show that the scalar product is distributive over vector addition. First use the word "Let" to settle on one real number c and two 3–dimensional vectors, $\vec{a}$ and $\vec{b}$. Then write $c(\vec{a} \oplus \vec{b})$ and rewrite it until it equals $c\vec{a} \oplus c\vec{b}$. Provide a reason for each step, as in the proofs above. At the end, follow the model to conclude that you have shown distributivity in general.

Show 43. You are going to show that the scalar product is distributive over real number addition. Start with "Let." Write $(c + d)\vec{a}$ and rewrite it until it equals $c\vec{a} \oplus d\vec{a}$. Provide justifications for each step. At the end, follow the model to conclude that this shows distributivity in general.

Stop. Check over what everyone in your group has done, and make sure that you are in complete agreement.

Definition 44. Zero vector. The vector $\langle 0, 0, 0\rangle$ is a special 3–dimensional vector, called the *zero vector*. We denote it by $\vec{0}$.

Definition 45. Additive inverse. Let $\vec{a}$ be a 3–dimensional vector, with open form $\langle a_1, a_2, a_3\rangle$. Define a new vector by $-\vec{a} = \langle -a_1, -a_2, -a_3\rangle$. It is called the *additive inverse* of $\vec{a}$.

Show 46. Let $\vec{a}$ be a 3–dimensional vector. Show that $\vec{a} \oplus \vec{0} = \vec{a}$. It's not very exciting. Make a general conclusion.

Show 47. Let $\vec{a}$ be a 3–dimensional vector, and let $-\vec{a}$ be its additive inverse. Show that $\vec{a} \oplus (-\vec{a}) = \vec{0}$. This is also not very exciting. Make a general conclusion.

Definition 48. Dot product of 3–dimensional vectors. The dot product of 3–dimensional vectors $\langle a_1, a_2, a_3\rangle$ and $\langle b_1, b_2, b_3\rangle$ is the real number $a_1b_1 + a_2b_2 + a_3b_3$.

Notation 49. The dot product of 3–dimensional vectors $\vec{a}$ and $\vec{b}$ is denoted $\vec{a} \bullet \vec{b}$.

Example 50. Calculate the dot product of $\vec{a} = \langle 12, -5, 3\rangle$ and $\vec{b} = \langle 6, 4, -11\rangle$ Do this by writing

$$\begin{aligned} \vec{a} \bullet \vec{b} &= \langle a_1, a_2, a_3\rangle \bullet \langle b_1, b_2, b_3\rangle \\ &= a_1 b_1 + a_2 b_2 + a_3 b_3 \end{aligned}$$

and then substituting in the numbers. This makes the calculation just a matter of rewriting.

Show 51. You will show that the dot product is commutative, just as multiplication of real numbers is commutative. This time, you write the first line, "Let $\vec{a}$ and $\vec{b}$ be ...." Follow the models from previous examples, and be sure to make a general conclusion.

Show 52. You will show that the dot product is distributive over vector addition. That is, you want to show that $(\vec{a} \oplus \vec{b}) \bullet \vec{c} = \vec{a} \bullet \vec{c} + \vec{b} \bullet \vec{c}$. Start with "Let ...". Please explain why one addition symbol is $\oplus$ and the other is $+$.

Example 53. Calculate $\vec{a} \bullet \vec{0}$. Is this a general result?

Show 54. Let $\vec{a}$ and $\vec{b}$ be 3–dimensional vectors and let c be a real number. Show in general that $c(\vec{a} \bullet \vec{b}) = (c\vec{a}) \bullet \vec{b} = \vec{a} \bullet (c\vec{b})$. Since there are two equalities to show, you might want to think about how you will go about it.

Your name: ______________________________

Practicing creativity

You can get better at generating creative ideas

Overview

Many proofs require a spark of genius, an idea that might not come to you right away. It's OK to try the first thing you think of, but when it does not seem to be working out, deliberately look for a large number of possible things to try, and then choose which one looks the most promising.

Note 55. These problems ask you to work with your group to generate ideas. After the groups have had a chance, we'll summarize the ideas from the whole class. Try to think of good ideas to share with the class! The point is to generate ideas, not just to solve the problems.

Group work 56. Problem 1.9 in the book by Daepp and Gorkin[1] asks you to show that if n is odd, then $n^3 - n$ is a multiple of 24. With your group, take some time to try to think of 5 different things you could possibly do to approach this problem. Think about $n^3 - n$. Think about n being odd. Think about the number 24. Think about how to show that something is a multiple of 24. Look at examples. If you already know how to do this problem, please don't tell the members of your group, but rather think of other things that might be good to try. Maybe there is more than one way to solve this problem.

1.

2.

3.

4.

5.

Group work 57. Let n be an integer. Could it be that n is both even and odd? Go back to the definitions. They don't say that it can't happen. How can we be sure that it can't happen? Here again, work with your group to think of a number of ways to possibly convince yourself that a number cannot be both even and odd.

1.

2.

[1] Reading, Writing, and Proving: A Closer Look at Mathematics, 2011, by Ulrich Daepp and Pamela Gorkin

3.

4.

5.

Note 58. At some point you may find yourself wondering just how integers are defined. This might help: 0 is an integer. Each integer has a unique "next" integer. For 0, the next integer is 1, and there is no integer between them. Similarly, each integer has a unique "previous" integer. For 0, the previous integer is -1.

Group work 59. Let n be an integer. Does n need to be either even or odd? Could it be neither? How can we be sure that each integer meets the definition to be odd or meets the definition to be even? Work with your group to try to think of a number of ways to see that each integer has to be even or odd.

1.

2.

3.

4.

5.

Group work 60. Let n be an integer. Suppose we know that n^2 is odd. In every case that you check, you will find that n is odd. Is this always the case? Can we be sure that n will be odd? Think of a number of possible ways to approach this.

1.

2.

3.

Your name: ______________________________

Practicing creativity part 2

You can get better at generating creative ideas

Overview

Many proofs require a spark of genius, an idea that might not come to you right away. It's OK to try the first thing you think of, but when it does not seem to be working out, deliberately look for a large number of possible things to try, and then choose which one looks the most promising.

Note 61. Give these problems a try on your own. If you get stuck, deliberately stop and look for other ways to do the problem. Then share these ideas with the people in your group.

Group work 62. Let n be an integer. Show that $4n^2 + 2n + 12$ is a multiple of 2.

Group work 63. Let n be an integer. Show that $n^2 + 5n + 6$ is even. There are a few different ways to do this problem. Can you find one way to do it? More than one? Be patient and specifically look for different ways to look at it.

Group work 64. Another problem is needed here.

Group work 65. Provide counterexamples to each of the following claims.

1. Every odd number is divisible by 3.

2. The difference between two different prime numbers is always 2 or more.

3. Every line in the plane defines a function.

4. If $x^2 = 16$, then $x = 4$.

5. x^2 is always larger than x.

Your name: ______________________________

Quantifier assessment

This is not part of your grade in the course, but it will be checked for correctness.

A. Write the following statements symbolically:

1. For every a, there is a b for which $b^2 = a$

2. For every b, there is an a for which $b^2 = a$

3. For every a and every b, it is the case that $b^2 = a$

4. There exists an a and there exists a b such that $b^2 = a$

B. Which of the statements in the previous problem are true if the universe for both a and b is the set of non–negative integers? If not true, explain why not.

1.

2.

3.

4.

C. Negate the statements from problem A.

1.

2.

3.

4.

D. Write the following statements symbolically:

1. Every rose has a thorn.

2. Every married couple with a child gets a tax deduction.

Your name: ______________________________

The Division Algorithm

Dividing integers with remainders will form the basis for several things we want to prove

Overview

We would like to distribute n objects evenly among k people and find out how many are left over. We will investigate a procedure for doing this. Procedures that are guaranteed to work are called *algorithms* after the 9th century Persion mathematician al-Khwarizmi, who worked on procedures for arithmetic. The division algorithm itself dates to Euclid's *Elements* from around 300 BC.

Example 66. You are the dealer in a card game that has 37 cards. (It's not a standard deck of cards.) Everyone needs to end up with the same number of cards. Dealing one card to each leaves 32. Write down the numbers 37, 32, and continue until you cannot deal out any more cards evenly. Let r denote the number of cards left at the end, and let q denote the number of times you subtracted 5, which is also the number of cards that each person got. You see that $37 - 5q = r$, which you can rewrite as $37 = 5q + r$. Fill in q and r.

Example 67. Now you're playing a card game that you have not played before, and you haven't taken the time to count how many cards are in the deck. You are the dealer again, and there are 5 people who need cards. Let n denote the number of cards in the deck. Repeat the procedure from the previous example until you can no longer deal out cards evenly. Again, let r denote the number of cards you have left at the end and q denote the number of cards that each person got. What do we know for sure about the possible values of r? What do we know for sure about the possible values of q? Write the relationship between n, 5, q, and r analogous to $37 - 5q = r$ and the expression analogous to $37 = 5q + r$. The last expression accounts for where all of the n cards have gone; some are dealt out, some are in your hands. Write out a sentence that explains this.

Example 68. Continuing to divide by 5, complete this sentence: Given an integer $n \geq 0$, there exist integers q and r where (list properties of q here) ________________________ and (list properties of r

here) __________________________, such that (write the relationship between n, q, and r analogous to $37 = 5q + r$) __________________________.

Example 69. Now let's divide by k, where $k > 0$ is an integer. Rewrite Example 66, replacing 37 by n and replacing 5 by k, and tell what you can about the possible values of q and r and the final relationship between n, k, q, and r. Your goal is to write a general statement about how to divide n objects among k people, keeping track of the possible values of the remainder r.

Stop. Compare your work to the others in your group. Work to get the best possible phrasing of 69.

Question 70. What happens when $k = 0$?

Note 71. You dealt n cards to k people. Each one got q cards and you had r cards left over. Could the numbers have worked out any other way? In 69, you have shown that when you divide n objects among k people, you can write $n = qk + r$, where q and r are integers and r can take values 0 to $k - 1$. Suppose that, using some other procedure, someone else was able to write $n = ak + b$ where a and b are integers and b can take values 0 to $k - 1$. This would also be a way to divide n objects evenly among k people. **Must these two ways in fact be the same?** Our goal is to show that they are, that $a = q$ and that $b = r$. This is called *uniqueness*. It will take a few steps.

Show 72. Suppose that r and b are integers for which $0 \le r < k$ and $0 \le b < k$. Carefully combine these inequalities to show that $-k < r - b < k$. Your goal is to provide a crystal clear argument with no extra steps. You can add inequalities that run the same direction; for example, if $a < b$ and $c < d$, then $a + c < b + d$.

Show 73. Suppose that n and k are integers, that $k > 0$, and that q, r, a, and b are integers such that $n = qk + r$ and $n = ak + b$ and that r and b take values between 0 and $k - 1$. Set $qk + r$ equal to $ak + b$ and argue that $q = a$ and $r = b$. Work on scratch paper and then write a final argument here. Take the time to write a clear argument that your whole group likes. If you use a proof by contradiction, say so. If you use a proof by cases, say so.

Theorem 74. The Division Algorithm. Let n and k be integers greater than 0. There are two parts to the theorem.

1. Existence. There exist unique integers q and r for which $n = qk + r$ and for which $0 \leq r < k$.

2. Uniqueness. The numbers q and r are unique; $n = qk + r$ is the only way to write n as a multiple of k plus a remainder from $0, 1, 2, \ldots, k - 1$.

The number q is called the *quotient* and r is called the *remainder*. You have proven this theorem above. Identify where the existence part was proven, and where the uniqueness part was proven.

Prove 75. Let n be an integer greater than 0. Use the existence part of the division algorithm to show that n must be even or n must be odd. That is, it must be at least one of them.

Prove 76. Let n be an integer greater than 0. Use the division algorithm to divide n by 2 and use the uniqueness part to conclude that n cannot be both even and odd.

Note 77. Now we will divide negative numbers by positive numbers, with remainder.

Example 78. Start with -37, add 5 to get -32, and add 5 repeatedly, writing down the numbers you come to, until you reach a number between 0 and 4. Count the number of 5's that you added to write $-37+5k = r$, where you fill in k and r. Rewrite this as $-37 = 5q+r$ and note the sign of q. This represents division of a negative number with remainder. How does it differ from division of a positive number with remainder?

Prove 79. Let n be an integer less than 0. Let k be an integer greater than 0. Add k to n repeatedly until you reach a number between 0 and $k-1$. Can you be sure that you will ever get all the way to 0? Can you be sure that you don't jump over the numbers $0, 1, \dots, k-1$ and keep adding k forever? Use the result to argue that you can write $n + pk = r$ for some integer p, and rearrange it to read $n = qk + r$. What do you know about the possible values of q and r?

Prove 80. Let n be an integer less than 0 and let k be an integer greater than 0. Argue that there exist *unique* integers q and r for which $n = qk + r$ and $0 \leq r < k$. This is like 73. What changes are needed, if any?

Show 81. Suppose that n is an integer and $n = 3m + 1$ where m is an integer. Show that n cannot be a multiple of 3.

Show 82. Let n be an integer and suppose that n^2 is a multiple of 3. We would like to conclude that n is a multiple of 3. Use the division algorithm to divide n by 3, which will give three cases, one for each possible remainder. Consider each case. What can you conclude about n being a multiple of 3?

Show 83. Suppose n is an integer. Can n^2 be of the form $3m + 2$ where m is an integer? Examine the cases in the previous question carefully.

Show 84. Let k be an integer and consider the numbers k, $k + 1$, and $k + 2$. Show that exactly one of these is a multiple of 3. A proof by cases might be a good idea. Note that there are two things to show: that one of k, $k + 1$, and $k + 2$ is a multiple of 3, and that the other two are not.

Show 85. Let k be even and consider the numbers k and $k + 2$. Show that exactly one of these is a multiple of 4.

Show 86. Let n be an odd integer. Show that $n^3 - n$ is a multiple of 24.

Your name: ____________________________

Quiz on sum and dot product of 3–dimensional vectors

10 points

Definition. 3–dimensional vector. A three–dimensional vector is an ordered triple $\langle a_1, a_2, a_3\rangle$, where a_1, a_2, and a_3 are real numbers.

Definition. Equality of 3–dimensional vectors. 3–dimensional vectors $\langle a_1, a_2, a_3\rangle$ and $\langle b_1, b_2, b_3\rangle$ are equal if $a_1 = b_1, a_2 = b_2$, and $a_3 = b_3$. The order of the numbers is important.

Definition. Sum of 3–dimensional vectors. The sum of 3–dimensional vectors $\langle a_1, a_2, a_3\rangle$ and $\langle b_1, b_2, b_3\rangle$ is the 3–dimensional vector $\langle a_1 + b_1, a_2 + b_2, a_3 + b_3\rangle$. We write $\vec{a} \oplus \vec{b}$, using a new symbol so we don't confuse addition of vectors with addition of real numbers.

Show. Show that addition of 3–dimensional vectors is commutative. Start with "Let," take one step at a time, write the justification for the step, and make a general conclusion.

Definition. Dot product of 3–dimensional vectors. The dot product of 3–dimensional vectors $\langle a_1, a_2, a_3\rangle$ and $\langle b_1, b_2, b_3\rangle$ is the real number $a_1b_1 + a_2b_2 + a_3b_3$.

Show. On the other side of this sheet of paper, show that the dot product is distributive over vector addition. That is, show that $(\vec{a} \oplus \vec{b}) \bullet \vec{c} = \vec{a} \bullet \vec{c} + \vec{b} \bullet \vec{c}$. Start with "Let . . . ," take one step at a time, write the justification for the step, and make a general conclusion. Please also explain why one addition symbol is $\oplus$ and the other is $+$.

Anonymous!

Class survey

I would like your feedback to improve the course and learn from it. Many thanks in advance!

1. You have been asked to read the textbook, and we have checked your notes to make sure this is happening. Would you recommend that other faculty do the same in their courses? **yes** **no**
Please explain.

Would you recommend that other students read their textbooks the same way in other courses? **yes** **no**
Please explain.

2. To what extent do you minimize distractions when you are reading the textbook? Do you turn off your cell phone? Do you study alone? I'm curious how hard you try to minimize distractions which will break your chain of thought, and whether it helps you understand the book more.

3. In what way(s) have you changed how you work with the textbook in other courses that you are taking? Do you read them more? Differently?

4. In class, we work through activities without much "lecture." How does this work for you?

5. Is there anything we should change about the class?

6. What is going well in the class, so that we should not change it?

7. Is this class making good progress on making you unstoppable in your other math courses?

8. What else could we do to make you unstoppable in your other math courses?

9. Do you look forward to coming to class? Why or why not?

Your name: ______________________________

Square roots of prime numbers are irrational

This is a classic example of proof by contradiction.

Overview

Most students are familiar with the fact that $\sqrt{2}$ is irrational, but few can prove it. Having read a proof of this fact in your textbook or online, the starting point is to re-create the proof from memory, then to move on to showing that $\sqrt{3}$ is irrational. The proof is similar and yet different.

Definition 87. Irrational. A real number is said to be *irrational* if it cannot be written as the quotient of two integers.

Prove 88. Prove that $\sqrt{2}$ is irrational by contradiction. The proof begins with "Assume that $\sqrt{2}$ can be written as $\frac{p}{q}$ where p and q are integers." Argue to a contradiction.

Note 89. If you are new to proof by contradiction, you might prefer to start the previous proof by writing "Let's pretend for a minute that $\sqrt{2}$ can be written as $\frac{p}{q}$ where p and q are integers." This makes it extra clear that you don't really believe that $\sqrt{2}$ is rational, you are just exploring what would happen if that were true. When you arrive at a contradiction, you realize it's time to stop pretending; $\sqrt{2}$ must be irrational.

Show 90. A key step in the proof is that if n is an integer and n^2 is even, then n is even. You may have already shown this, using a proof by contradiction (which begins, "Assume that n is odd.") or a proof by contrapositive (which begins, "Let us show the contrapositive, that if n is not odd, then n^2 is not odd.") or a proof by cases (which begins, "There are two possibilities for n, that n is even or that n is odd.) Whichever one you used, choose another and write the proof here.

Show 91. Mimic the proof that $\sqrt{2}$ is irrational to show that $\sqrt{3}$ is irrational. Work with the members of your group to figure out how to do this.

Show 92. A key step in the proof that $\sqrt{3}$ is irrational is the fact that if n is an integer and n^2 is a multiple of 3, then n is also a multiple of 3. This is not as straightforward as with multiples of 2, but it can still be done by contradiction, by contrapositive, or by cases. Think about these possibilities and choose the one that seems to you to be the best approach.

Your name: ______________________________

Quiz on irrationality of square roots of odd primes

10 points

Show. Show that if n is an integer and n^2 is a multiple of 7, then n is a multiple of 7. I recommend using a proof by cases.

Show. On the other side of this sheet of paper, show that $\sqrt{7}$ is irrational. I recommend a proof by contradiction.

Possible questions for the quiz over the Division Algorithm

This will be a 10–point quiz. I will choose two of the following four problems for the quiz. I strongly suggest that you write out solutions for each of these before Thursday, and that you try to do them without consulting your notes. Rediscover the arguments, and you will own them.

Then, some hours later, write them again on a fresh sheet of paper. This is the best way to learn them.

I will be happy to look at your practice solutions before class on Thursday. If you have questions, you may ask them by email.

1. Let $n > 0$ and $k > 0$ be integers. Argue that there exist integers q and r such that $n = qk + r$ and $0 \leq r < k$. You can phrase the argument in terms of dealing out n cards to k people, or in terms of starting with n and subtracting k repeatedly.

2. Let $n > 0$ and $k > 0$ be integers. Suppose that there exist integers q and r for which $n = qk + r$ and $0 \leq r < k$, and at the same time that there exist integers a and b for which $n = ak + b$ and $0 \leq b < k$. Show that $q = a$ and $r = b$. This shows that there is at most one way to write $n = qk + r$ with $0 \leq r < k$.

3. Let k be an integer. Show that exactly one of the integers $k, k + 1, k + 2, k + 3$ is a multiple of 4. **Note:** I may ask instead for you to show that exactly one of the numbers $k, k+1, k+2$ is a multiple of 3, or that exactly one of the numbers $k, k + 1, k + 2, k + 3, k + 4$ is a multiple of 5. The argument is basically the same in every case.

4. Let n be an odd integer. Show that $n^3 - n$ is a multiple of 24.

Your name: ______________________________

Quiz on things related to the Division Algorithm

10 points

1. Let $n > 0$ and $k > 0$ be integers. Suppose that there exist integers q and r for which $n = qk + r$ and $0 \leq r < k$, and at the same time that there exist integers a and b for which $n = ak + b$ and $0 \leq b < k$. Show that $q = a$ and $r = b$. This shows that there is at most one way to write $n = qk + r$ with $0 \leq r < k$.

2. Let k be an integer. Show that exactly one of the integers $k, k+1, k+2, k+3$ is a multiple of 4. You can use the back of this sheet of paper.

Your name: ______________________________

Quiz on even and odd

15 points

Work hard to write really nice proofs.

1. Show that the product of two odd numbers is odd.

2. Show that for all integers n, the quantity $n^2 + 10n + 21$ is either odd or is a multiple of 4.

3. The numbers $0, 1, 4, 9, 16, 25, \ldots$ are called perfect squares. The differences between consecutive perfect squares are $1, 3, 5, 7, 9, \ldots$. Show that the difference between consecutive perfect squares is always an odd number.

Your name: ______________________________

Operations on sets

This activity works with set identities and relates them to logic.

Overview

Sets are absolutely fundamental to mathematics. This chapter focuses on building up set identities, relationships between sets that are always true.

Problem 93. Let A and B be sets. Show that $(A \cup B)^c = A^c \cap B^c$ by showing set inclusion both ways. This is one of de Morgan's laws. Draw a really nice Venn diagram to illustrate.

Problem 94. Let A and B be sets. Show that $(A \cap B)^c = A^c \cup B^c$ by showing set inclusion both ways. This is the other one of de Morgan's laws. Draw a really nice Venn diagram to illustrate.

Problem 95. Let A and B be sets. Let P be the logical statement $x \in A$, and let Q be the logical statement $x \in B$. Use P and Q and logic symbols to fill in the simplest expressions:

1. $x \in A \cup B$ is ____________

2. $x \in (A \cup B)^c$ is $\neg(P \wedge Q)$

3. $x \in A^c$ is ____________

4. $x \in B^c$ is ____________

5. $x \in A^c \cap B^c$ is ____________

Make a truth table for P, Q, and each of the other logical statements here to establish that $x \in (A \cup B)^c$ is logically equivalent to $x \in A^c \cap B^c$. Compare the truth values in the columns corresponding to $x \in (A \cup B)^c$ to the Venn diagram you made above. Explain how they agree.

Problem 96. Let A and B be sets. Use the approach of the previous exercise to show that $x \in (A \cap B)^c$ is logically equivalent to $x \in A^c \cup B^c$. Compare the truth table to the Venn diagram again.

Problem 97. Let D, E, and F be sets. Use one of de Morgan's laws that you showed above to establish that $(D \cup E \cup F)^c = D^c \cap E^c \cap F^c$. This proof works by rewriting, not by showing inclusion both ways. **Hint:** Let $A = D \cup E$ and $B = F$.

Problem 98. Let D, E, and F be sets. Use one of de Morgan's laws to show that $(D \cap E \cap F)^c = D^c \cup E^c \cup F^c$.

Problem 99. Let A, B, and C be sets. Use a proof by cases to show that $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$. Remember to show inclusion both ways. Organize your writing carefully to make the steps of this argument really clear.

Problem 100. Let A, B, and C be sets. Use logical statements P, Q, and R and a truth table to show that $x \in A \cup (B \cap C)$ is logically equivalent to $x \in (A \cup B) \cap (A \cup C)$. Be sure to define P, Q, and R at the beginning.

Definition 101. Symmetric difference. Let A and B be sets. The *symmetric difference* of A and B is the set $A \triangle B = (A \backslash B) \cup (B \backslash A)$.

Problem 102. Let A and B be sets. Show that $A \triangle B = B \triangle A$ by showing set inclusion both ways. Draw a nice Venn diagram to illustrate.

Problem 103. Consider again the logical statements from 95. Write a logical statement that is equivalent to $x \in A \triangle B$. Make a truth table with 4 rows, labeled 1, 2, 3, 4, and three columns, one for $x \in A$, one for $x \in B$, and the third for $x \in A \triangle B$. Draw a Venn diagram and label the regions in it 1, 2, 3, 4 so that they correspond to the truth table.

Problem 104. Let A, B, and C be sets. Show that $(A \triangle B) \triangle C = A \triangle (B \triangle C)$ in these ways.

1. Draw separate Venn diagrams for the two sets.

2. Show set inclusion both ways.

3. Convert inclusion in $A \triangle B$, $B \triangle C$, and other sets to logical statements and use a truth table to show the equality.

Your name: ______________________________

Infinite unions, intersections, and a few other things

This activity is a prequel to working with infinite unions and intersections.

Overview

Infinite unions and intersections take a bit of getting used to. Fortunately, we can understand them with quantifiers, and we can understand intervals with inequalities.

Definition 105. Union. Let $A_1, A_2, \ldots$ be sets, with universe X. The *union* of $A_1, A_2, \ldots$, which is denoted $\bigcup_{n=1}^{\infty} A_n$, is all elements of X which are in A_n for some $n = 1, 2, 3, \ldots$.

Definition 106. Intersection. Let $A_1, A_2, \ldots$ be sets, with universe X. The *intersection* of $A_1, A_2, \ldots$, which is denoted $\bigcap_{n=1}^{\infty} A_n$, is all elements of X which are in A_n for all $n = 1, 2, 3, \ldots$.

Problem 107. Use quantifiers to express what it means that $x \in \bigcup_{n=1}^{\infty} A_n$.
Solution: $\exists n, x \in A_n$. In words, there is at least one n for which x is in A_n.

Problem 108. Work with quantifiers to express what it means that $x \notin \bigcup_{n=1}^{\infty} A_n$.
Hint. Negate the previous expression and use rules of quantifiers to rewrite it, then rewrite again using complements.

Problem 109. Use quantifiers to express what it means that $x \in \bigcap_{n=1}^{\infty} A_n$.

Problem 110. Work with quantifiers to express what it means that $x \notin \bigcap_{n=1}^{\infty} A_n$.
Hint. Negate the previous expression, then rewrite again using complements.

Stop. Go back to each of the four preceding problems and write a sentence explaining the logic of the expression that you wrote down.

Problem 111. de Morgan's law. Show that $\left(\bigcup_{i\in I} A_i\right)^c = \bigcap_{i\in I} A_i^c$ by writing logical expressions for x being in the set on the left side and for the right side. Note that here the union is over sets A_i where the index i comes from an index set I, but the logic is the same as in the previous problems. Start by writing a logical expression that means the same thing as $x \in \left(\bigcup_{i\in I} A_i\right)^c$ and work with it until it is a logical expression for $x \in \bigcap_{i\in I} A_i^c$. When you write the proof this way, you do not need to show containment both ways to show that the two sets are equal.

Problem 112. Show that $(3,\infty) \subset [3,\infty)$; these are both intervals on the real number line. Remember that when you show $\subset$ there are two things to show: containment, and that there is an element of one set that is not an element of the other. Solve this problem by letting $x \in (3,\infty)$ and writing that information as the logical statement "$x > 3$ is true".

Problem 113. Show that $[2,5) \cap (3,7) \subseteq (3,5)$ using inequalities. Start by letting $x \in [2,5) \cap (3,7)$.

Problem 114. Let $x > 0$. Show that there exists an integer n such that $0 < \frac{1}{n} < x$. **Hint:** Look at $\frac{1}{x}$ and round up. **Another hint:** Suppose that $x = 0.31$. What value of n works?

Problem 115. Let n be an integer greater than 0. Show that $[\frac{1}{n}, 1] \subseteq (0,1] \subseteq [0,1]$ by working with inequalities. Then show that $[\frac{1}{n}, 1] \subset (0,1] \subset [0,1]$ by looking at individual points.

Your name: ______________________________

Infinite operations on sets

Unions, intersections, and complements of infinitely many sets

Overview

We are working with sets of real numbers. These exercises will give you practice with sets and teach you things about the real numbers as well.

Problem 116. Let $A = \bigcup_{n=1}^{\infty} [\frac{1}{n}, 1]$. List out the first five sets in this union. Draw a picture of them above a number line. Make a conjecture about what interval A is equal to, call the new set B, then show that $A = B$ by showing containment both ways. You will need to use this property of real numbers: if $x > 0$, then there exists a positive integer n with $0 < \frac{1}{n} < x$.

Problem 117. Let $B = \bigcup_{n=0}^{\infty} [n, n^2]$. List out the first five or more sets in this union. Draw them on a number line if it helps. Make a conjecture about how you can write B in a simpler way, call the new set C, then prove that $B = C$ by showing containment in both directions.

Problem 118. For $n = 2, 3, 4, \ldots$, let $C_n = \{2n, 3n, 4n, \ldots\}$.

1. Write out the first five of the C_n.

2. Let $D = \bigcup_{n=2}^{\infty} C_n$. Describe the set D in simpler terms, perhaps by writing out the smallest 10 elements of D.

3. What is $\mathbb{N} \backslash D$? Remember that $\mathbb{N} = \{0, 1, 2, 3, \ldots\}$.

Problem 119. Let I be a set, and for each i in I, let A_i be a set, all subsets of the same universe X. Show de Morgan's law: $\left(\bigcup_{i \in I} A_i\right)^c = \bigcap_{i \in I} A_i^c$ by showing set containment in both directions. I hope you will find that it is actually easier to do this for a collection of sets than for two sets.

Problem 120. Let $E = \bigcap_{n=1}^{\infty} [0, 1 + \frac{1}{n}]$. List out the first five sets in this union. Draw a picture of them above a number line. Make a conjecture about what interval E is equal to, call the new set F, then prove that $E = F$ by showing containment both ways.

Hint: You may want to show that $E \subseteq F$ by showing the logically equivalent statement that $F^c \subseteq E^c$. This is the same as the contrapositive: suppose that $y \notin F$, then show that $y \notin E$. You may find it useful to keep in mind that if $x > 0$, then there exists an integer n for which $0 < \frac{1}{n} < x$.

Problem 121. Let $F = \bigcup_{r\in\mathbb{Q}}(r - \frac{1}{10}, r + \frac{1}{10})$. Here, $\mathbb{Q}$ is the set of all rational numbers. Make a conjecture about a simpler way to describe the set F, then prove your conjecture by showing set containment both ways.

Problem 122. Let $G = \bigcup_{k\in\mathbb{Z}}(k, k+1)$.

1. Draw out some of the intervals here.

2. Make a conjecture about what set G is.

3. Use one of de Morgan's laws to re–express G^c as an intersection. Does that help?

4. What is easier to describe, or to think of, G or G^c? Give the simplest description.

Problem 123. Let $a < b$. Show that $\bigcup_{n=1}^{\infty}[a, b - \frac{1}{n}] = [a, b)$. Draw pictures, then show set inclusion both ways. Is there a problem if $b - \frac{1}{n} < a$?

Problem 124. Let $a < b$. Show that $\bigcap_{n=1}^{\infty}[a, b + \frac{1}{n}) = [a, b]$. Draw pictures, then show set inclusion both ways.

Your name: ______________________________

Quiz on infinite set operations

5 points

1. Let $B = \bigcup_{n=0}^{\infty} [n, n^2]$. List out the first five or more sets in this union. Draw them on a number line if it helps.

Let $C = \{0\} \cup \{1\} \cup [2, \infty)$. Show that $B = C$ by showing containment in both directions. You will need to use three cases in each direction to deal with $0, 1$, and the rest.

Your name: ______________________________

The power set and the Cartesian product

Useful constructions with sets.

Overview

The power set is our first example of thinking hard about collections of sets. The Cartesian product is used often when you want ordered pairs or ordered triples of numbers or other objects.

Problem 125. Write out the members of the following power sets. It may be helpful to do #3, #4, then #2, #1, and finally #5.

1. $S = \emptyset$. $\mathcal{P}(S) =$

2. $S = \{1\}$. $\mathcal{P}(S) =$

3. $S = \{1, 2\}$. $\mathcal{P}(S) =$

4. $S = \{1, 2, 3\}$. $\mathcal{P}(S) =$

5. $S = \{1, 2, 3, 4\}$. $\mathcal{P}(S) =$

6. $S = \{1, 2, 3, 4, 5\}$. $\mathcal{P}(S) =$

Question 126. If S has n elements, how many members will $\mathcal{P}(S)$ have? Explain as well as you can.

Problem 127. Write the appropriate symbol between the entities, or mark the statement as true or false. Give an explanation for anything that is not obvious enough.

1. $1 \qquad \mathcal{P}(\{1, 2, 3\})$
2. $[3, 10] \qquad \mathbb{Z}$
3. $[3, 10] \qquad \mathbb{R}$
4. $\mathbb{Q} \qquad \mathbb{R}$
5. $\mathbb{Q} \qquad \mathcal{P}(\mathbb{R})$
6. $[3, 10] \qquad \mathcal{P}(\mathbb{R})$
7. $\mathbb{N} \qquad \mathbb{R}$
8. $\emptyset \qquad \mathbb{R}$
9. $\emptyset \qquad \mathcal{P}(\mathbb{R})$
10. $\{\emptyset\} \subseteq A$?
11. $\emptyset \subset \mathcal{P}(A)$?

Question 128. Suppose that S is a set. Then $\mathcal{P}(S)$ is also a set, but if we let $A \in \mathcal{P}(S)$, then A is also a set. Explain how this can be. What is the relationship between A and S?

Problem 129. Let I be a set, and for each i in I, let B_i be a set. Show that $\mathcal{P}\left(\bigcap_{i\in I} B_i\right) = \bigcap_{i\in I} \mathcal{P}(B_i)$. Let A be an element of the set on the left-hand side. Notice that A is a set. Argue that it is an element of the set on the right-hand side. Let A be an element of the set on the right-hand side ...

Problem 130.

1. Sketch the Cartesian product $A = [1,3] \times [2,5]$.

2. Sketch the Cartesian product $B = [2,4] \times [1,3]$.

3. Sketch the intersection $A \cap B$.

4. It seems that $A \cap B$ is also a Cartesian product. Identify the sets whose product is $A \cap B$.

5. What is $([1,3] \cap [2,4]) \times ([2,5] \cap [1,3])$?

Your name: ______________________________

Quiz on a new operation with 3–dimensional vectors

15 points

Definition. Sum of 3–dimensional vectors. The sum of 3–dimensional vectors $\langle a_1, a_2, a_3\rangle$ and $\langle b_1, b_2, b_3\rangle$ is the 3–dimensional vector $\langle a_1 + b_1, a_2 + b_2, a_3 + b_3\rangle$.

Definition. The *twist product* of 3–dimensional vectors $\langle a_1, a_2, a_3\rangle$ and $\langle b_1, b_2, b_3\rangle$ is the 3–dimensional vector $\langle a_1b_3, a_2b_2, a_3b_1\rangle$. It is denoted $\vec{a} * \vec{b}$.

Example. For example, $\langle 1, 3, 5\rangle * \langle 2, 7, 10\rangle = \langle 1 \cdot 10, 3 \cdot 7, 5 \cdot 2\rangle = \langle 10, 21, 10\rangle$.

Show. Show that the twist product is distributive over vector addition. That is, show that $(\vec{a} \oplus \vec{b}) * \vec{c} = \vec{a} * \vec{c} \oplus \vec{b} * \vec{c}$. Start with "Let . . . ," take one step at a time, write the justification for the step, and make a general conclusion.

Show. Prove or disprove: "The twist product is commutative."

Show. On the other side of this piece of paper, show that for all 3–dimensional vectors $\vec{a}$ and $\vec{b}$ and real numbers c, $\vec{a} * (c\vec{b}) = (c\vec{a}) * \vec{b} = c(\vec{a} * \vec{b})$. Use parentheses *every* time three things are multiplied together.

Your name: ______________________________

Relations

Often treated as the little brother to functions, relations have unsuspected depth.

Overview

You are already familiar with a number of relations, including $<, \leq, =, \geq$, and $>$ for real numbers, plus $\subset$, $\subseteq$, and $=$ for sets. Many other relations can be defined. The most useful ones are called equivalence relations; they are analogous to equality for numbers and for sets. They partition the space into equivalence classes, which are very useful in a number of ways.

Definition 131. Relation. A *relation* on a set X is a subset S of $X \times X$.

Notation 132. Suppose that S is a relation on a set X. That is, suppose that S is a subset of $X \times X$, which means that S is a set of points of the form (x, y), where $x \in X$ and $y \in X$. Rather than write $(x, y) \in S$, we usually write $x \sim y$. How to read this out loud? There is no perfect solution. I would suggest that you read it as "x tilde y" because $\sim$ is the tilde that appears above the n in some Spanish words.

Problem 133. You are going to write out the subsets of $\{1, 2, 3, 4\}$ and then draw arrows between them to indicate the proper subset relation. You might want to lay the sets out in a nice order to make the arrows easy to draw and to read. What is the set X on which this relation is defined?

Definition 134. Reflexive. A relation $\sim$ is *reflexive* if $x \sim x$ for all x in X.

Definition 135. Symmetric. A relation $\sim$ is *symmetric* if $x \sim y$ implies $y \sim x$.

Definition 136. Transitive. A relation $\sim$ is *transitive* if $x \sim y$ and $y \sim z$ implies $x \sim z$.

Note 137. Sometimes people have a hard time remembering the words reflexivity, symmetry, and transitivity. Notice that they are in alphabetical order, and that they involve 1, 2, or 3 objects at a time, respectively.

Definition 138. Equivalence relation. A relation $\sim$ is called an *equivalence relation* if it is reflexive, symmetric, and transitive. Note that equality is an equivalence relation on the set of real numbers.

Definition 139. Equivalence class. Suppose that $\sim$ is an equivalence relation. Fix x in X. The set of all elements y for which $x \sim y$ is called the *equivalence class containing* x.

Problem 140. Let $X = \mathbb{Z}^+$ and say that $x \sim y$ if y is divisible by x. People often write $x|y$ for this relation and say that x divides y.

1. Check whether this relation is reflexive. If so, prove that it is, starting with "Let $x \in \mathbb{Z}^+$." If not, give a counterexample.

2. Check whether this relation is symmetric. If so, prove that it is, starting with "Let $x, y \in \mathbb{Z}^+$ and suppose that $x \sim y$." If not, give a counterexample.

3. Check whether this relation is transitive. If so, prove that it is, starting with "Let $x, y, z \in \mathbb{Z}^+$ and suppose that $x \sim y$ and $y \sim z$." If not, give a counterexample.

4. Thinking of the relation as a set of ordered pairs, write out ten different ordered pairs satisfying the relation, and graph them on the xy plane.

Problem 141. Consider all cities in the US that have population over 30,000. For each of the following relations, determine whether they are reflexive, symmetric, and/or transitive. Provide a counterexample for any property that fails to hold. If all three hold, the relation is an equivalence relation. In that case, identify the equivalence classes and tell how many such classes there are.

1. Say that $x \sim y$ if the names of cities x and y start with the same letter.

2. Say that $x \sim y$ if x and y are in the same state.

3. Say that $x \sim y$ if cities x and y are within 50 miles of each other.

Problem 142. Let X be the set of all English words. Say that $x \sim y$ if the letters in x and y appear on the same number keys on a cell phone, in the same order. For example, BAR $\sim$ CAP.

1. Check whether this relation is reflexive.

2. Check whether this relation is symmetric.

3. Check whether this relation is transitive.

4. If all three properties hold, describe the equivalence classes, and tell what the equivalence class of BAR is.

Problem 143. Let $X = \mathbb{Z}$. Say that $x \sim y$ if x and y has the same remainder as y when they are divided by 3. Then, for example, $13 \sim 19$ and $9 \sim 27$.

1. Show that this relation is reflexive, symmetric, and transitive. Do this in general, starting with "Let."

2. Describe all elements of the equivalence class containing 0.

3. Describe the other equivalence classes. How many are there?

Problem 144. Consider the set X of all non–zero 3–dimensional vectors. For $\vec{a}$ and $\vec{b}$ in X, say that $\vec{a} \sim \vec{b}$ if there exists a constant c for which $\vec{a} = c\vec{b}$.

1. Show that this relation is reflexive, starting with "Let." Tell what c is.

2. Show that this relation is symmetric. You will need two values of c.

3. Show that this relation is transitive. Here there will be three values of c.

4. This is an equivalence relation. Describe the equivalence classes. The collection of all equivalence classes is called *projective space*.

5. Could you use the angle between lines to define a distance between equivalence classes? What would the maximum distance be?

Problem 145. Consider the set of all English words. Say that $x \sim y$ if one can be obtained from the other by changing exactly one letter. For example, BAT $\sim$ CAT but BAT $\nsim$ CAR. Check whether this relation is reflexive, symmetric, and/or transitive. Provide counterexamples if necessary.

Problem 146. Consider the set of all functions on the real line. That is, consider the set of all $f : \mathbb{R} \to \mathbb{R}$. Say that $f \sim g$ if f and g are equal except at a finite number of points. For example, if $f(x) = x^2$ and $g(x) = \begin{cases} x^2, & x \neq 0 \\ 5, & x = 0 \end{cases}$, then $f \sim g$. Show that this is an equivalence relation. How can you describe the equivalence classes?

Note 147. Problems 10.1 and 10.3 from Daepp and Gorkin[2] are particularly good at this stage in the course.

[2] Reading, Writing, and Proving: A Closer Look at Mathematics, 2011, by Ulrich Daepp and Pamela Gorkin

Homework problems, week 12

Due on (put date here).

Write up solutions of each of the problems below. They are designed to be straightforward problems. The goal is to come as close to perfection in your solutions as you can.

- Do not take shortcuts.
- If you need to show that something is true for all n, or for all x, y, start the proof with "Let ..."
- If you need cases, explain what the cases are and why they cover all the possibilities.
- If you are doing a proof by contradiction, start that part by saying "Assume ..."
- If you are doing a proof by contrapositive, tell what P and Q are, and that you will be showing that $\neg Q$ implies $\neg P$.
- Take small steps in each proof, and explain each step.
- Follow good form.
- If your proof started with "Let ..." it will probably end by saying "We made no further assumption ..."

Here are the problems to do. You can write them in your notebook or on separate paper.

1. Show that if n is an integer and $7n$ is odd, then n is odd. **Hint:** Be clear what facts you are using about even and odd numbers.

2. Without consulting your book or your notes, prove that $\sqrt{2}$ is irrational. I mean it. Do this from memory. You should be able to write a very nice proof, with no missing steps.

3. Let x and y be real numbers, and suppose that the product xy is irrational. Show that either x or y (or both) must be irrational. **Hint:** You can do this. Be patient, think about it.

4. Let $A = \{2k + 1 : k \in \mathbb{Z}\}$ and let $B = \{2m - 11 : m \in \mathbb{Z}\}$. Show that $A = B$ by showing containment both ways. **Hint:** Use good form!

5. Let $A = \{(x, y) \in \mathbb{R}^2 : y = 5x/7 - 2/7\}$ and $B = \{(x, y) \in \mathbb{R}^2 : 5x - 7y = 2\}$. Show that $A = B$ by showing containment both ways.

6. Let $A = \{m \in \mathbb{Z} : m = 15k \text{ for some } k \in \mathbb{Z}\}$, let $B = \{m \in \mathbb{Z} : m = 35j \text{ for some } j \in \mathbb{Z}\}$, and let $C = \{m \in \mathbb{Z} : m = 105n \text{ for some } n \in \mathbb{Z}\}$. Show that $A \cap B = C$ by showing containment both ways. One direction is easier than the other. Label one of them "the easy direction" and the other "the hard direction". **Hint:** Yes, we worked on a problem just like this in class. Don't go back and find it, work through this one on your own. **Another hint:** In the hard direction, you should come to something like $3k = 7j$ where j and k are integers. You will need to conclude that j is a multiple of 3. If you are up for the challenge, show this using the division algorithm. Don't use any ideas about prime factorization.

Your name: ________________________________

Inequalities

We can define the $<$ relation for real numbers and establish its properties.

Overview

Most students at your level take the real numbers as things that simply exist and have a number of properties such as commutativity. In fact, the real numbers can be *constructed* from the rational numbers, and the rational numbers from the integers, and the integers from the positive integers. In this activity, we back up to the point that the real numbers have been constructed, but before inequalities have been defined. We define the $<$ relation and prove a number of useful properties that it satisfies. Since the $>$ relation is so similar, we will not define it or show its properties.

Note 148. Let $\mathbb{R}$ denote the set of real numbers, and denote addition and multiplication of real numbers in the usual ways. **Addition** has these properties: commutativity ($a + b = b + a$), associativity ($a + (b + c) = (a + b) + c$), additive identity (there exists a unique real number called 0 for which $a + 0 = a$ for all $a \in \mathbb{R}$), and additive inverse (for each number a in $\mathbb{R}$, there exists a unique real number $-a$ for which $a + (-a) = 0$). **Multiplication** has these properties: commutativity ($ab = ba$), associativity ($a(bc) = (ab)c$), multiplicative identity (there exists a unique real number called 1, with $1 \neq 0$, such that $a \cdot 1 = a$ for all a in $\mathbb{R}$), multiplicative inverse (for each a in $\mathbb{R}$ with $a \neq 0$, there exists a unique number called a^{-1} for which $a \cdot a^{-1} = 1$. **Addition and multiplication** are related by the distributive property: $((a + b)c = ac + bc)$.

Definition 149. Subtraction. Let a and b be real numbers. The difference of a and b, denoted $a - b$, is the real number $a + (-b)$, where $(-b)$ denotes the additive inverse of b.

Note 150. At the end of this activity, you will see how to establish the following useful properties regarding additive inverses and subtraction:

a. $a \cdot 0 = 0$ for all real numbers a.

b. The additive inverse $(-a)$ is equal to $(-1) \cdot a$, where (-1) is the additive inverse of 1

c. $(-1)(-1) = 1$.

d. The additive inverse of $a + b$ is $(-a) + (-b)$. Using subtraction notation, $-(a + b) = -a - b$.

e. $-(-a) = a$.

You can use subtraction as usual in this activity, but if you would like to avoid subtraction notation and just use additive inverses, that is worth attempting.

Definition 151. Positive real numbers. By construction, the real numbers have a subset $\mathbb{R}^+$, called the *positive real numbers,* for which:

a. If $a, b \in \mathbb{R}^+$, then $a + b \in \mathbb{R}^+$. ($\mathbb{R}^+$ is closed under addition.)

b. If $a, b \in \mathbb{R}^+$, then $a \cdot b \in \mathbb{R}^+$. ($\mathbb{R}^+$ is closed under multiplication.)

c. For every real number a, either $a \in \mathbb{R}^+$ or $(-a) \in \mathbb{R}^+$ or $a = 0$. Exactly one of the three happens.

Under each property above, write a sentence that states it in plain English. Think of $\mathbb{R}^+$ as being the positive, non–zero numbers. We can't use interval notation to write what $\mathbb{R}^+$ is, because intervals are defined in terms of inequalities, and we have not defined inequalities yet!

Show 152. Let $a \in \mathbb{R}$ and suppose that $a \neq 0$. Show that $a \cdot a \in \mathbb{R}^+$. When you use properties from 150 or 151, cite them by number. **Hint:** There are two cases left in 151c.

Show 153. Show that $1 \in \mathbb{R}^+$. Be careful to cite any previous properties that you use.

Show 154. Show that $(-1) \notin \mathbb{R}^+$. **Hint:** Assume that $(-1) \in \mathbb{R}^+$ and use 151a.

Definition 155. Less than. Let a and b be real numbers. We write that $a < b$ if $b - a \in \mathbb{R}^+$.

Note 156. All of the following problems rely on Definition 155, so you will use it over and over. Note that $>$ has not been defined yet, so be careful not to use it.

Show 157. Show that $-1 < 0$. **Hint:** use 150d.

Show 158. Show that $1 < 1$ is not true. Thus, the $<$ relation is not reflexive. When you use properties from 150 or 151, cite them by number.

Show 159. Show that $0 < 1$ but that $1 < 0$ is not true. Thus, the $<$ relation is not symmetric.

Show 160. Show that the $<$ relation on $\mathbb{R}$ is transitive. Follow good form by first letting a, b, c be real numbers and supposing that $a < b$ and $b < c$. When you use properties from 150 or 151, cite them by number. You may enjoy ticking off the properties of the real numbers that you use. For example, in this proof, you are likely to use the fact that $(-b) + b = 0$, which is the additive inverse property.

Show 161. Let $a, b \in \mathbb{R}$ and suppose that $a < b$. Show that $-b < -a$. When you use properties from 150 or 151, cite them by number.

Show 162. Let $a, b, c \in \mathbb{R}$. Suppose that $a < b$. Show that $a + c < b + c$. When you use properties from 150 or 151, cite them by number.

Show 163. Let $a, b, c, d \in \mathbb{R}$. Suppose that $a < b$ and $c < d$. Show that $a + c < b + d$. When you use properties from 150 or 151, cite them by number.

Show 164. Let a, b, c be real numbers. Suppose that $a < b$ and $0 < c$. Show that $ac < bc$.

Show 165. Let a, b, c be real numbers. Suppose that $a < b$ and $c < 0$. Show that $bc < ac$.

Show 166. Let $a, b \in \mathbb{R}$ and suppose that $0 < a$ and $b < 0$. Use a previous result to show that $ab < 0$.

Show 167. Let $a \in \mathbb{R}$ and suppose that $0 < a$. Show that $0 < a^{-1}$. Here a^{-1} is the multiplicative inverse of a. **Hint:** This one take a bit more effort than the previous ones. Note that division has not been defined yet, so just use addition, subtraction, and multiplication.

Show 168. Let $a, b \in \mathbb{R}$ and suppose that $0 < a$ and $a < b$. Show that $b^{-1} < a^{-1}$.

Note 169. Below, you are asked to prove basic properties of additive inverses and subtraction.

Show 170. Let a be a real number. Show that $a \cdot 0 = 0$.

Show 171. People sometimes ask if the additive inverse $(-a)$ is the same as $(-1) \cdot a$, where (-1) is the additive inverse of 1. It's true, and here is how you show it; you should fill in steps and write the justifications at the right side of each line.

$$\begin{aligned} a + (-1) \cdot a &= 1 \cdot a + (-1) \cdot a \\ &= (1 + (-1)) \cdot a \\ &= \\ &= 0, \end{aligned}$$

This shows that $(-1) \cdot a$ is the additive inverse of a.

Show 172. You might think that it is obvious that $(-1)(-1) = 1$, where (-1) is the additive inverse of 1, but this takes a few steps beyond the properties of the real numbers in 148. Write justifications and complete the following steps to show it.

$$\begin{aligned} (-1) + (-1)(-1) &= (-1)(1) + (-1)(-1) \\ &= (-1)(1 + (-1)) \\ &= \\ &= 0, \end{aligned}$$

which shows that $(-1)(-1)$ is the additive inverse of -1, which is 1.

Show 173. The additive inverse of a sum works out nicely. Let a and b be real numbers and think about the additive inverse of $a+b$. Write justifications to the right of each statement.

$$\begin{aligned} -(a+b) &= (-1)(a+b) \\ &= (-1)(a)+(-1)(b) \\ &= (-a)+(-b) \end{aligned}$$

Show 174. Let $a \in \mathbb{R}$. The statement $-(-a) = a$ is just a statement about additive inverses. Prove that it is true.

Your name: ______________________________

Mathematical Induction

Proving that a claim is true for all n

Overview

One important task in mathematics is to find and distinguish regular patterns or sequences. The main method we use to prove certain propositions involving positive integers or about sequences is mathematical induction. Induction is also a very useful tool in computer science since a feature of many programs is repetition of a sequence of statements.

Theorem 175. Mathematical induction
For an integer n, let $P(n)$ denote an assertion.

(i) (The basis step) Prove $P(1)$ is true.

(ii) (The inductive step) for any positive integer k, assume that $P(k)$ is true, then prove $P(k+1)$ is true.

From the above two steps, we have $P(n)$ is true for all positive integers n.

Example 176. Write out the sum of the first n positive odd integers with $n = 1, 2, 3, 4, 5$, respectively and make a conjecture about the formula for the sum for any given positive integer n. For example, when $n = 3$, the sum is $1 + 3 + 5$.

n	1	2	3	4	5
sum					

Guided proof 177. Prove the conjecture in 176 using mathematical induction. Let $P(n)$ be the assertion $1 + 3 + 5 + \cdots + (2n-1) =$ ______.

1. Show that $P(n)$ is true when $n = 1$:

2. Write out what $P(k+1)$ is:

3. The inductive step: Let $k \in \mathbb{Z}^+$. Assume $P(k)$ is true, check $P(k+1)$:

By ________________________________ we conclude that $1 + 3 + 5 + \cdots + (2n-1) =$ ______ for all positive integers n.

Show 178. Show that 3^n is odd for all $n \in \mathbb{N}$.

0. State $P(n)$:

1. Basis step:

2. Inductive step:

Notation 179. The standard notation for the sum of $a_1, a_2, \ldots, a_n$ is $a_1 + a_2 + a_3 + \cdots + a_n = \sum_{k=1}^{n} a_k$ and the notation for the product of $a_1, a_2, \ldots, a_n$ is $a_1 \cdot a_2 \cdot a_3 \cdots \cdot a_n = \prod_{k=1}^{n} a_k$.

Example 180. Rewrite the conjecture in Example 176 by using the standard notation for the sum.

Show 181. Show that $\sum_{j=1}^{n}(4j-3) = n(2n-1)$ for all positive integers n.

0. State $P(n)$:
1. Basis step:
2. Inductive step:

Show 182. Use mathematical induction to show that $\sum_{j=1}^{n} 5^j = \frac{5}{4}(5^n - 1)$ for all positive integers n.

0. State $P(n)$:
1. Basis step:
2. Inductive step:

Stop. Compare your proofs to the problems above with the other people in your group before you move on.

Note 183. In the basis step, it needs not always begin with $n = 1$, for example, it can begin with $n = -3, n = 0, n = 100$.

Show 184. Use mathematical induction to show that $2n + 1 < 2^n$ for all integers n with $n \geq 4$.

0. State $P(n)$:
1. Basis step:
2. Inductive step:

Show 185. Use mathematical induction to show that $5^n > 2^n + 3^n$ for all integers n with $n \geq 2$.

Show 186. Use induction to prove Bernoulli's inequality: For $x \in \mathbb{R}$, if $1 + x > 0$, then $(1 + x)^n \geq 1 + nx$ for all $n = 0, 1, 2, \ldots$.

Show 187. Use induction to prove that $\frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \frac{1}{3\cdot 4} + \cdots + \frac{1}{n\cdot(n+1)} = \frac{n}{n+1}$ for all positive integers n.

Note 188. In Problem 187, we should be able to show the statement is true without using mathematical induction. How?

Show 189. For each $n \in \mathbb{Z}^+$, let $P(n)$ denote the assertion "$n^2 + 5n + 1$ is an even integer."

(a) Prove that $P(k + 1)$ is true whenever $P(k)$ is true.

(b) For which k is $P(k)$ actually true? What is moral of this exercise?

Show 190. Use induction to prove that 6 divides $n^3 - n$ whenever n is a natural number.

Show 191. Use induction to prove that $11^n - 4^n$ is divisible by 7 when n is a natural number.

Show 192. Prove that $1^2 - 2^2 + 3^2 - 4^2 + 5^2 + \cdots - (2n)^2 + (2n+1)^2 = (n+1)(2n+1)$ for all natural numbers n. **Hint** It would be helpful to find out $P(0)$ and $P(1)$ first.

Note 193. We also can use mathematical induction to show some propositions about sets.

Show 194. Prove that if $A_1, A_2, \ldots, A_n$ and $B_1, B_2, \ldots, B_n$ are sets such that $A_j \subseteq B_j$ for $j = 1, 2, \ldots, n$, then $\bigcap_{j=1}^{n} A_j \subseteq \bigcap_{j=1}^{n} B_j$. **Hint** In the initial step, show that if $A_1 \subseteq B_1$ and $A_2 \subseteq B_2$, then $A_1 \cap A_2 \subseteq B_1 \cap B_2$.

Show 195. Prove that if $A_1, A_2, \ldots, A_n$ and B are sets, then $(A_1 \cup A_2 \cup \cdots \cup A_n) \cap B = (A_1 \cap B) \cup (A_2 \cap B) \cdots \cup (A_n \cap B)$ for any given positive integer n. **Hint** In the initial step, we have to show Distributive Property of Intersection over Union of sets: $(A_1 \cup A_2) \cap B = (A_1 \cap B) \cup (A_2 \cap B)$. You will use this property in the inductive step.

Note 196. In 194 and 195, instead of showing $P(1)$ is true, we show $P(2)$ is true in the initial step. But why? Explain!

Your name: ______________________________

Quiz on inequalities

5 points

Definition. Positive real numbers. By construction, the real numbers have a subset $\mathbb{R}^+$, called the *positive real numbers,* for which:

a. If $a, b \in \mathbb{R}^+$, then $a + b \in \mathbb{R}^+$. ($\mathbb{R}^+$ is closed under addition.)

b. If $a, b \in \mathbb{R}^+$, then $a \cdot b \in \mathbb{R}^+$. ($\mathbb{R}^+$ is closed under multiplication.)

c. For every real number a, either $a \in \mathbb{R}^+$ or $(-a) \in \mathbb{R}^+$ or $a = 0$. Exactly one of the three happens.

Definition. Less than. Let a and b be real numbers. We write that $a < b$ if $b - a \in \mathbb{R}^+$.

Show. Let a, b, c be real numbers. Suppose that $a < b$ and $0 < c$. Show that $ac < bc$. Take very small steps and be careful to cite justifications for every single step.

Your name: ______________________________

Quiz on induction

15 points

For each problem below, clearly state $P(1), P(k)$, and $P(k+1)$ as logical statements with double quotes around them. When proving that $P(k)$ being true implies that $P(k+1)$ is true, do not write down $P(k+1)$ as if it were true, but rather start with one side and work with it until it turns into the other side.

Show. Use induction to show that for $n > 0$, 8 divides $5^n + 2(3^{n-1}) + 1$. **Hint:** As in other proofs of divisibility, add and subtract to be able to use $P(n)$ to simplify $P(n+1)$.

Show. On the back of this piece of paper, use induction to show that for all $n \geq 1$, we have that $1(1!) + 2(2!) + \cdots + n(n!) = (n+1)! - 1$.

Your name: ______________________________

Absolute value and related functions

A careful development of the properties of the absolute value function.

Overview

The absolute value function is easy to understand for numbers like 9 and -13, but it's harder to show its properties because our intuition works so hard to see all variables as having positive values. In this activity, we will not use the standard notation for the absolute value function and will have to keep our intuition at bay. We will instead rely completely on the definition. **When you use a property of inequalities, cite it by number.**

Definition 197. Absolute value. The function $f : \mathbb{R} \to \mathbb{R}$ defined by

$$f(x) = \begin{cases} x, & \text{if } x \geq 0 \\ -x, & \text{if } x < 0 \end{cases}$$

is called the *absolute value* function.

Notation 198. In this activity, do not use the standard notation for absolute value, not even once. Every time you work with the absolute value function, use and cite the definition.

Show 199. Show that $f(ab) = f(a)f(b)$ for all real numbers a and b. Follow the model.

Let a and b be real numbers. There are four cases.

1. Suppose that $a \geq 0$ and $b \geq 0$. Then $ab \geq 0$ so $f(ab) = ab$ and $f(a) = a$ and $f(b) = b$, so $f(ab) = ab = f(a)f(b)$.

2. Suppose that $a \geq 0$ and $b < 0$.

3. Suppose that $a < 0$ and $b \geq 0$.

4. Suppose that $a < 0$ and $b < 0$.

In each case, we see that ____________________________. We made no further assumptions about ______________, thus ____.

Show 200. Following the model above, show that $f(f(a)) = f(a)$ for all real numbers a.

Show 201. Show that $f(-a) = f(a)$ for all real numbers a.

Show 202. Show that $f(a - b) = f(b - a)$ for all real numbers a and b.

Show 203. Show that $f(a) \geq 0$ for all real numbers a.

Show 204. Follow the model in 199 to show that $f(a + b) \leq f(a) + f(b)$ for all real numbers a and b. When a and b have different signs, consider two cases, $a + b \geq 0$ and $a + b < 0$. You will probably want to show that if $b < 0$, then $b < -b$. Make a good, solid argument using transitivity of $<$.

Show 205. Show that for real numbers a and b, $f(a) \leq b$ if and only if $-b \leq a \leq b$. Remember that an "if and only if" proof has two directions. In both directions, you will have to consider two cases, $a \geq 0$ and $a < 0$. Note that the statement $-b \leq a \leq b$ is equivalent to $(-b \leq a$ and $a \leq b)$.

Show 206. Show that for all real numbers a and b, $f(a) \geq b$ if and only if ($a \geq b$ or $a \leq -b$).

Show 207. Show that for all real numbers a and b, $f(a) \leq f(a-b) + f(b)$. **Hint:** Look at $f((a-b)+b)$.

Show 208. Show that for all real numbers a and b, $f(a) - f(b) \leq f(a-b)$ and also $f(b) - f(a) \leq f(b-a)$.

Show 209. Show that for all real numbers a and b, $f(a-b) \geq f(f(a) - f(b))$.

Definition 210. Minimum function. The function $h : [0, \infty) \to \mathbb{R}$ defined by

$$h(x) = \begin{cases} x, & \text{if } x \leq 1 \\ 1, & \text{if } x > 1 \end{cases}$$

can be called the minimum function.

Show 211. Show that for all real numbers a, $h(a) = 0$ if and only if $a = 0$.

Show 212. Show that if $a \leq b$, then $h(a) \leq h(b)$.

Show 213. Show that if $h(a) < h(b)$, then $a < b$.

Show 214. Show that for all real numbers a and b, $h(a + b) \leq h(a) + h(b)$. **Hint:** Use a proof by cases. But what are the cases?

Your name: ______________________________

Functions

One-to-one, onto and bijective functions

Overview

You are comfortable working with functions already. There are various ways of describing functions. Here you will learn the formal definition of a function.

Definition 215. Function. Let X and Y be sets. A function f from X to Y is a relation from X to Y that satisfies:

1. for each $x \in X$ there is a $y \in Y$ such that $(x, y) \in f$, and
2. if $(x, y) \in f$ and $(x, z) \in f$, then $y = z$.

The set X is called the domain of f and the set Y is called the codomain of f.

Notation 216. We write $f : X \to Y$ to describe a function f from X to Y and we write $f(x) = y$ instead of $(x, y) \in f$.

Definition 217. Injective Functions. Let X and Y be sets and let $f : X \to Y$ be a function. The function f is said to be injective or one-to-one if whenever $x_1, x_2 \in X$ are such that $x_1 \neq x_2$ then $f(x_1) \neq f(x_2)$.

Definition 218. Surjective Functions. Let X and Y be sets and let $f : X \to Y$ be a function. The function f is said to be surjective or onto if for each $y \in Y$ there exists an $x \in X$ such that $f(x) = y$.

Definition 219. Bijective Functions. Let X and Y be sets and let $f : X \to Y$ be a function. The function f is said to be bijective if it is both injective and surjective.

Example 220. Let $X = \{Monday, \diamondsuit, \sqrt{\pi}, purple\}$ and $Y = \{\alpha, \heartsuit, fun\}$ be sets and define the relation f from X to Y by $f = \{(Monday, fun), (\diamondsuit, \alpha), (\sqrt{\pi}, fun), (purple, fun)\}$. Draw a diagram to illustrate the relation. Is the relation f a function? Prove your answer by using the definition.

Problem 221. Let $X = \{Cleveland, Chicago, LosAngeles, Miami\}$ be a set of American cities and let $Y = \{Cavaliers, Heat, Lakers, Bulls, Clippers\}$ be a set of NBA teams.

a) Provide an example of a relation that is a function from X to Y and draw a diagram to illustrate your example.

b) Provide an example of a relation from X to Y that is not a function and draw a diagram to illustrate your example.

c) Provide an example of an injective function from X to Y. Draw a diagram to illustrate your example.

d) Show that there are no surjective functions from X to Y.

Note 222. The next problem states an equivalent definition for injectivity. This definition is very useful when proving that a function is injective.

Problem 223. Let X and Y be two sets and let $f : X \to Y$ be a function. Then f is injective if and only if for all $x_1, x_2 \in X$ such that $f(x_1) = f(x_2)$ we have $x_1 = x_2$.

Problem 224. Let $f : \mathbb{N} \to \mathbb{N}$ be a function defined by $f(n) = 2n + 1$. Show that the function f is injective but not surjective.

Hint: Use the previous problem to prove injectivity. In order to prove that f is not surjective you need to find an $m \in \mathbb{N}$ which cannot be written as $2n + 1$.

Problem 225. Let $f : \mathbb{N} \to \mathbb{Z}$ be a function defined by

$$f(n) = \begin{cases} \frac{n}{2}, & \text{if } n \text{ is even} \\ \frac{-(n+1)}{2}, & \text{if } n \text{ is odd} \end{cases}$$

Show that f is a bijective function.

Hint: To show that f is injective let $n_1, n_2 \in \mathbb{N}$ be such that $f(n_1) = f(n_2)$ and look at all the possible cases according to the parity of n_1 and n_2.

To prove the surjectivity let $m \in \mathbb{Z}$. Consider the two possible cases; one when $m \geq 0$ and the other one when $m < 0$. Then in each case find an $n \in \mathbb{N}$ such that $f(n) = m$.

Your name: ______________________________

Pigeonhole Principle

It was stated by Johann Peter Gustav Lejeune Dirichlet in 1834 and it is sometimes called Dirichlet's Box Principle.

Overview

The principle simply states that if we have more pigeons than holes then at least one hole must contain more than one pigeon. The next problem asks you to prove this principle.

Problem 226. If $n + 1$ pigeons or more are placed in n holes then one of the holes must contain two or more pigeons. **Hint:** Prove by induction on n.

Problem 227. The human head contains less than $200,000$ hairs. Name three cities in which at least two people have the same number of hairs.

Problem 228. A bag contains M&M's in six different colors: Brown, Yellow, Green, Red, Orange and Blue. How many M&M's do you need to take out of the bag in order to have at least two of the same color? How many do you need to take out of the bag if you want to have three of the same color?

Problem 229. A classroom floor is painted white and black. Is it always possible to find two points of the same color exactly one inch apart?

Problem 230. Prove that no matter how we choose 51 natural numbers from $\{1, 2, 3, \ldots, 100\}$, at least two of them must be consecutive. **Hint:** Consider the pigeonholes $\{1, 2\}, \{3, 4\}, \ldots, \{99, 100\}$.

Problem 231. Prove that given any ten natural numbers we can choose two of them such that their difference is divisible by nine. **Hint:** Consider the remainders when dividing by nine.

Problem 232. Prove that if six integers are selected at random from the set $\{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$ then at least two of them add up to eleven.

Review questions for the final quiz

Taken from a variety of sources without attribution.

Overview

The final quiz will consist of approximately 6 questions and will be worth approximately 30 points. I will try to make sure that it can be done by a prepared student in 2 hours. The best way to prepare is to work out problems on the review sheet and on the handouts that we have had in class.

Induction problems

1. Show that $1 + 2 + 2^2 + 2^3 + \cdots + 2^n = 2^{n+1} - 1$ for all $n \geq 0$.

2. Show that $\sum_{j=0}^{n} r^j = \frac{r^{n+1}-1}{r-1}$ for all n and all real numbers $r \neq 1$. (There is an easier formula when $r = 1$!)

3. Show that $2^n < n!$ for all $n \geq 4$.

4. Show that $3^n < n!$ for all $n \geq 7$.

5. Show that $n! < n^n$ for all $n > 1$.

6. Show that $1 \cdot 2 + 2 \cdot 3 + 3 \cdot 4 + \cdots + n(n+1) = \frac{n(n+1)(n+2)}{3}$ for all $n \geq 1$.

7. Show that $1 + \frac{1}{4} + \frac{1}{9} + \cdots + \frac{1}{n^2} < 2 - \frac{1}{n}$ for all $n \geq 2$.

8. Show that $n^5 - n$ is a multiple of 5 for all n.

9. Show that $n^2 - 1$ is a multiple of 8 for all odd n. **Hint:** You could try showing $P(1)$ and then show that $P(n)$ implies $P(n+2)$.

10. Draw n lines in the plane such that no two lines are parallel and no three lines go through a common point. Show that this divides the plane into $\frac{n^2+n+2}{2}$ regions. **Hint:** How many regions does the $n+1$st line add?

11. Show that $1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{2^n} \geq 1 + \frac{n}{2}$. This is too hard for the final quiz, but you may enjoy working on it. Notice that increasing n by 1 will double the number of terms, unlike most of the other problems you have worked on. This result shows that the harmonic series diverges.

12. Show that 5^n is odd for all $n \geq 0$.

13. Use the product rule to show that, for every integer $n \geq 1$, the derivative of x^n is nx^{n-1}.

Sample problems

1. Show that if n is odd, then $n^2 + 2n - 7$ is a multiple of 4.

2. Let a, b, and c be integers. Suppose that b is a multiple of a or c is a multiple of a. Show that bc is a multiple of a.

3. Let a and b be integers. Suppose that a is a multiple of b and that b is a multiple of a. Show that $a = \pm b$.

4. Suppose that n is an integer and n^2 is a multiple of 5. Show that n is a multiple of 5. Try to do this without looking back at your notes!

5. Suppose that n is an odd integer. Show that $n^3 - 25n$ is a multiple of 24. This is similar to something we did in class. See if you can do it that way. Can you do it by induction instead? Work with $P(n)$ and $P(n+2)$. Which way is easier?

6. Show that an integer n cannot be both even and odd. What kind of proof did you use?

7. We used the Division Algorithm, number 73, a lot once we proved it. It has both an existence part and a uniqueness part. Please review the statements and arguments for both:

Suppose that n and k are positive integers. Argue that there **exist** integers q and r with $0 \leq r < k$ such that $n = kq + r$.

Suppose that $n = kq + r$ where q and r and integers and $0 \leq r < k$ and also that $n = ka + b$ where a and b are integers and $0 \leq b < k$. Show that $q = a$ and $r = b$.

8. Suppose that m and n are integers and that $3m = 7n$. Show that n is a multiple of 3. Do this by writing n as $3q + r$ for different possible values of r.

9. Give a complete proof that $\sqrt{3}$ is irrational.

10. Show that $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$ by showing containment in both directions and using cases. Then show it again using logical statements such as $P =$ "$x \in A$" and a truth table. Try to do this without looking back at Problems 124 and 125.

11. Now that we have practiced inequalities, re–do Problem 130.

12. (Problem 133). Let I be a set, and for each i in I, let A_i be a set, all subsets of the same universe X. Show de Morgan's law: $\left(\bigcup_{i \in I} A_i\right)^c = \bigcap_{i \in I} A_i^c$ by showing set containment in both directions.

13. Show that $[2,5)\cap(3,7) = (3,5)$ by showing inclusion both ways. Start by letting $x \in [2,5)\cap(3,7)$, so that $x \in [2,5)$ and $x \in (3,7)$, then write it as $2 \le x < 5$ and $3 < x < 7$. There are four inequalities here. Soon enough you can conclude that $x \in (3,5)$. Then show containment the other way as well.

14. Let $x > 0$. Show that there exists an integer n such that $0 < \frac{1}{n} < x$.

15. Show that $\bigcup_{n=1}^{\infty}[\frac{1}{n}, 1] = (0,1]$ by showing inclusion in both directions. This should be easier now that we have practiced inequalities.

16. Do Problem 137, about unions.

17. Do Problem 138, about intersections.

18. This is a challenge involving two–dimensional sets and the Cartesian product. Let $T = \{(x,y) \in \mathbb{R}^2 : x \ge 0, y \ge 0, x+y \le 1\}$. Let $U = \bigcup_{0\le a\le 1}[0,a]\times[0,1-a]$. Show that $T = U$ by showing containment both directions.

19. Do problem 169, about relations.

20. Do problem 170, about relations.

21. Finish problems 225, 226, 227, 229, 230, 231 about absolute value.

Your name: ______________________________

Final quiz

30 points

Please use one side of a sheet of paper for each problem. If you are totally stuck, you can ask for a hint, but it may cost you a little something. Good luck!

1. a) State the definition of odd.

b) Show that the product of two odd numbers is odd. Do your best to write a picture–perfect proof.

c) Let $P(n)$ be the statement "The product of n odd numbers is odd." You may want to write this as "$k_1 k_2 \cdots k_n$ is odd." Show that $P(n)$ is true for all $n = 1, 2, \ldots$.

2. Answer this question on the next page.

a) Let n be an integer. Explain how you know that you can write $n = 5q + r$ where q and r are integers and $0 \leq r < 5$.

b) Let m be an integer and suppose that $5m = 7n$. Show that n is a multiple of 5. Don't use prime factorization.

3. Let X be the set of all ordered pairs of integers. Then X contains things like $(1, 3), (-4, 11)$, and $(9, 7)$. We say that $(a, b) \sim (p, q)$ if $aq = bp$.

a) Is $(6, 10) \sim (9, 15)$?

b) Show that $\sim$ is reflexive.

c) Show that $\sim$ is symmetric.

d) Show that $\sim$ is transitive.

e) Describe all members of the equivalence class containing $(6, 10)$.

4. Answer this question on the next page. Show that $\bigcup_{n=1}^{\infty} [2, 5 - \frac{1}{n}] = [2, 5)$. Draw pictures, then show set inclusion both ways.

Definition. 176. Positive real numbers. By construction, the real numbers have a subset $\mathbb{R}^+$, called the *positive real numbers,* for which:

a. If $a, b \in \mathbb{R}^+$, then $a + b \in \mathbb{R}^+$. ($\mathbb{R}^+$ is closed under addition.)

b. If $a, b \in \mathbb{R}^+$, then $a \cdot b \in \mathbb{R}^+$. ($\mathbb{R}^+$ is closed under multiplication.)

c. For every real number a, either $a \in \mathbb{R}^+$ or $(-a) \in \mathbb{R}^+$ or $a = 0$. Exactly one of the three happens.

5. Let $a \in \mathbb{R}$ with $a \neq 0$. Show that $a \cdot a \in \mathbb{R}^+$. Be extremely clear about every step that you take.

Definition. Absolute value. The function $f : \mathbb{R} \to \mathbb{R}$ defined by

$$f(x) = \begin{cases} x, & \text{if } x \geq 0 \\ -x, & \text{if } x < 0 \end{cases}$$

is called the *absolute value* function.

6. Answer this question on the next page. Show that $f(-a) = f(a)$ for all real numbers a.

Reading assignment #1

Due on the second day of class.

The idea is to read Chapter 1 of the textbook by Daepp and Gorkin[3]. The assignment is to read it in a particular way. It may take 3 hours to get it done, but you will learn something in those three hours, and you will start to develop a very important skill.

Get a copy of Chapter 1, "The How, When, and Why of Mathematics." Get out your notebook or some paper. Go somewhere quiet, where you won't be interrupted for a while. Turn off your phone so you aren't disturbed. Don't listen to music that will distract you, and make sure there is no TV or youtube on where you can see it or hear it.

Put the notebook or paper right in front of you. Put the textbook itself a bit farther away. Make note of the time that you start reading in your notebook, maybe in the left margin. Read the first paragraph of the chapter, then write one or more sentences in your notes which capture the main idea(s) of the paragraph.

Read the second paragraph, about Geogre Pólya's list of guidelines. Look up the list in the Appendix. Consider writing them in your notebook, or abbreviated versions of them.

Continue to write a sentence summarizing each paragraph. I believe that if you are not writing, you are probably not thinking as hard as you need to. Read slowly. If you run into a word you don't know, google it or look it up in a dictionary. If you really don't know it, write the definition in your notebook. It is OK to spend 15 minutes on each page of the book. Really. It is not a goal of the course to learn how to read faster. The goal is to learn how to get more out of the time you spend reading. If you stop to take a break, note the time that you stopped and the time you start again.

Read Exercise 1.1 and the text that walks you through Pólya's guidelines. Use your notebook to try to solve the puzzle yourself. I've printed the alphabet twice at the bottom of this page. If you cut off the bottom version, you can slide it along the top one and easily keep track of how the letters correspond. That should save you a little time.

A second example starts on page 3 of the textbook. As you read it, draw diagrams in your notebook. Yes, there are diagrams printed in the textbook, but you will think harder about the diagram and understand more if you draw your own.

Example 1.2 asks a question. Read the question and see if you can answer it on your own, without reading further in the book.

On page 7, you will see that solutions of the exercises are provided. Resist the urge to turn your brain off and just read the solutions. That is not what they are there for!

Work through some of the problems that begin on page 7 in the book. You do not need to do all of them, but you should at least understand most of them and attempt a few of them.

You may be able to do problem 1.9 because we have done similar things in class. Give it a try. Problem 1.10 doesn't interest me. Does it interest you?

Problem 1.12 is good. Will it help to make a graph? Problem 1.13 seems silly. Do you like it anyway?

Read the Tips on Doing Homework. At the end, tally up how much time you have spent on reading this chapter. Write this number in your notebook and remember the number when you come to class.

`ABCDEFGHIJKLMNOPQRSTUVWXYZABCDEFGHIJKLMNOPQRSTUVWXYZ`

`ABCDEFGHIJKLMNOPQRSTUVWXYZABCDEFGHIJKLMNOPQRSTUVWXYZ`

[3]Reading, Writing, and Proving: A Closer Look at Mathematics, 2011, by Ulrich Daepp and Pamela Gorkin

Reading assignment, Chapter 2

Due in the second week of class.

Read Chapter 2 of the book by Daepp and Gorkin. As with Chapter 1,

1. Read somewhere quiet, minimizing distractions from phones and friends

2. Note the time that you start and stop reading, and add up the minutes

3. Read with a pencil in your hand and your notebook open in front of you

4. Write a sentence to summarize each paragraph, re-draw diagrams, work out examples and exercises on your own

5. Look up words you don't know, and write down ones you really don't know

6. Read slowly. You are not reading a comic book or a newspaper. It is not a goal of this class for you to learn how to read faster. The goal is to learn how to get more out of the time you spend reading, and to learn to concentrate for longer periods of time.

7. At the end, tally up how much time you have spent on reading this chapter. Write this number in your notebook and remember the number when you come to class.

You will read about "statements." Focus on the ones about mathematical things, and don't worry too much about interpreting the ones that are non-mathematical.

Note that on page 14, there is a statement about the color of the cover of the book. Books from Springer are always yellow, but the authors must not have realized that someone would put a big blue bar on the cover of this edition of the book. Just imagine that the book cover is all yellow.

Fill out every truth table that is suggested in the chapter. Truth tables are an excellent way to get great clarity about complicated combinations of statements. The idea is to consider every possible combination of True and False for the basic statements. For example, if there are two statements, P and Q, there will be four rows in the table, running through the four possible combinations of True and False for P and Q. On page 21, there is a truth table for three statements, P, Q, and R. It has eight rows.

The most important use of truth tables is to tell when two complicated combinations of logical expressions are, in fact, the same.

For me, the hardest thing about truth tables is making columns for implications like $P \to Q$. Here is the best way I know to think about them. Each row of the truth table for P and Q covers one combination of truth values for P and Q. Some of these combinations are consistent with the implication P implies Q. For example, when P is True and Q is True, this is consistent with $P \to Q$, so we put T in the $P \to Q$ column. The row in which P is True and Q is False, however, is inconsistent with the implication $P \to Q$, so we put F in that row. The cases in which P is False are a bit different, but they are also consistent with $P \to Q$, since $P \to Q$ only has anything to say about P and Q when P is True. So we put T in those rows too.

Problems 1 to 8 are good. Rather than working on problems 9-21, I would much prefer that you spend your time making some truth tables. Think of this as a specific assignment.

1. Do Problem 3 and also make a truth table for $\neg(P \vee Q)$ and $\neg P \wedge \neg Q$.

2. Make a big truth table for $P, Q, R,\ P \wedge (Q \vee R),\ P \vee (Q \wedge R),\ (P \wedge Q) \vee (P \wedge R)$, and $(P \vee Q) \wedge (P \vee R)$. Which of these are equal? How can you rememeber that?

Reading assignment, Chapter 3

Due in the second week of classes.

Read Chapter 3 of the textbook by Daepp and Gorkin. As with Chapters 1 and 2,

1. Read somewhere quiet, minimizing distractions from phones and friends
2. Note the time that you start and stop reading, and add up the minutes
3. Read with a pencil in your hand and your notebook open in front of you
4. Write a sentence to summarize each paragraph, re–draw diagrams, work out examples and exercises on your own
5. Look up words you don't know, and write down ones you really don't know
6. Read slowly. You are not reading a comic book or a newspaper. It is not a goal of this class for you to learn how to read faster. The goal is to learn how to get more out of the time you spend reading, and to learn to concentrate for longer periods of time.
7. At the end, tally up how much time you have spent on reading this chapter. Write this number in your notebook and remember the number when you come to class.

Theorem 3.1 lists three properties of logical statements. I would suggest that you write them down in your notebook (it won't take long) and add de Morgan's laws from Theorem 2.9. Then you'll have the whole set. Which of these are hard to remember?

Having de Morgan's laws handy should make Exercise 3.2 easier.

The contrapositive is really important. See if you can explain it just by thinking about $P \to Q$ and $\neg Q \to \neg P$, without using truth tables.

Theorem 3.1 is proven using the contrapositive. This is one way to prove it, but there are other ways.

Read about the converse, and make sure never to confuse an implication with its converse.

Problems 2, 3, 4, 5, 6, 8, 9, 14, 15, 16, 18, 19 are good to work on.

Chapters 1 to 5 are mostly there to help develop proof techniques. After Chapter 5, we will spend more of our time on definitions, examples, theorems, and proofs. Use your time now to develop basic logic and proof techniques that will help you for the rest of the semester and beyond!

Reading assignment, Chapter 4

Due in the third week of classes.

Read and understand Chapter 4 of the textbook by Daepp and Gorkin. As with previous chapters,

1. Read somewhere quiet, minimizing distractions from phones and friends
2. Note the time that you start and stop reading, and add up the minutes
3. Read with a pencil in your hand and your notebook open in front of you
4. Write a sentence to summarize each paragraph, re–draw diagrams, work out examples and exercises on your own
5. Look up words you don't know, and write down ones you really don't know
6. Read slowly. You are not reading a comic book or a newspaper. It is not a goal of this class for you to learn how to read faster. The goal is to learn how to get more out of the time you spend reading, and to learn to concentrate for longer periods of time.
7. At the end, tally up how much time you have spent on reading this chapter. Write this number in your notebook and remember the number when you come to class.

This is a very important chapter, and one with real substance. Hopefully you will feel that way when you read it, and will enjoy it more as a result.

This chapter has a large number of very dense expressions involving quantifiers, implications, and logical operators. Slow way down when you run into one of them. Pick them apart in your mind and then write them down so they are crystal clear. Every symbol is important. It's a bit like when you're reading someone your credit card number or you're giving your phone number to someone you really want to call you. Every symbol is important.

Exercises 4.1, 4.2, 4.3, and 4.6 are all useful to do. The discussion that begins at the bottom of page 36 is very important, negating statements with quantifiers.

There are 20 problems. The more of them you do, the better, of course, but you may not be able to work through all of them. **Please at least do problems # 1–7, and 20.** Read # 11. Does this joke work on your friends?

People have asked about grading, or about a rubric that Ying-Ju Chen is using when she reads the notebook. She'll be assigning numeric values between 0 and 5 for each chapter. The bulk of the points go toward the notes on the chapter itself. This is to emphasize that reading and taking notes is the primary concern. Less than half of the points go toward attempting the exercises, with more emphasis on attempting than on getting them all the way right. Ying-Ju writes as many helpful comments as she can on each notebook, but there is only so much time, and sometimes things that are incorrect do not get marked as incorrect. Even so, I think that having you take notes and having Ying-Ju read them every class is working very well.

Pay attention to the phrase "only if." It is often used in a way that can be confusing. Compare these two statements for example, in which R means Race and P means prize:

1. I will race if there is a prize offered. $P \to R$. This is the most common way that people use the word "if." The prize will make me race.
2. I will race only if there is a prize offered. $R \to P$. People say this sort of thing pretty often too, but it's a bit less clear unless you think about it carefully. Part of the problem is the time order in which things happen, because the racing comes *after* the prize is offered. "If you see me racing, you can be sure that there was a prize offered. (But offering a prize is no guarantee that I will race.)"

Reading assignment, Chapter 5

Due in the fourth week of classes.

Read and understand Chapter 5 of the textbook by Daepp and Gorkin. As with previous chapters,

1. Read somewhere quiet, minimizing distractions from phones and friends

2. Note the time that you start and stop reading, and add up the minutes

3. Read with a pencil in your hand and your notebook open in front of you

4. Write a sentence to summarize each paragraph, re–draw diagrams, work out examples and exercises on your own

5. Look up words you don't know, and write down ones you really don't know

6. Read slowly. You are not reading a comic book or a newspaper. It is not a goal of this class for you to learn how to read faster. The goal is to learn how to get more out of the time you spend reading, and to learn to concentrate for longer periods of time.

7. At the end, tally up how much time you have spent on reading this chapter. Write this number in your notebook and remember the number when you come to class.

This chapter walks you through a number of types of proofs and gives examples of each. **Rewrite these proofs in your notes, in your own words as much as possible, so that you make them yours.** By the end of reading the chapter, you should **know** the proof that the square root of 2 is irrational and you should know the other proofs as well.

It might help, in your notes, to make a list of proof techniques from the chapter and from previous chapters. What chapter talked about proof by contrapositive? Is that in Chapter 5? What about truth tables? You can prove things with those. What kinds of things?

Read and understand Problem 1. It is important.

Read the other problems, find the ones that are easy, and do them. This may seem like a strange assignment, but I really mean it. Think about each problem (if you can get through all of them), and make sure that if a problem is easy, that you recognize that and write out the solution. Don't worry if a problem looks hard but turns out to be easy. That happens all the time. But hopefully you will spot a number of them that really are easy, and do them. We will go over these problems in class next week.

Your name: ______________________________

Quiz on some problems from Chapter 5 of Daepp and Gorkin

15 points

1. Let x and y be real numbers. Use the triangle inequality to show that $||x| - |y|| \leq |x - y|$.

2. Prove or refute the following conjecture: There are no positive integers x and y such that $x^2 - y^2 = 10$. You can use the back of the sheet if you like.

Reading assignment, Chapter 6

Due in the fifth week of class.

Read and understand Chapter 6 of the textbook by Daepp and Gorkin. As with previous chapters,

1. Read somewhere quiet, minimizing distractions from phones and friends

2. Note the time that you start and stop reading, and add up the minutes

3. Read with a pencil in your hand and your notebook open in front of you

4. Write a sentence to summarize each paragraph, re–draw diagrams, work out examples and exercises on your own

5. Look up words you don't know, and write down ones you really don't know

6. Read slowly. You are not reading a comic book or a newspaper. It is not a goal of this class for you to learn how to read faster. The goal is to learn how to get more out of the time you spend reading, and to learn to concentrate for longer periods of time.

7. At the end, tally up how much time you have spent on reading this chapter. Write this number in your notebook and remember the number when you come to class.

This chapter introduces sets, subsets, equality of sets, and how to tell what the members of a set are. As you read, take time to write out several members of each set that is introduced. Note that A being a subset of B is the same as the logical implication $x \in A$ implies $x \in B$. There is a tight connection between statements in set theory and logical statements. Here is another: Set A being equal to set B is the same as the logical implications $x \in A$ if and only if $x \in B$.

There are many examples in this chapter. Work through them by rewriting them and adding useful steps in your notes.

On page 64, intersections, unions, and complements of sets are introduced. As you read about them, explain in your notes how these relate logical statements such as $x \in A$ and $x \in B$ to $x \in A \cap B$.

You may enjoy reading about the paradoxes on page 67. Give them a try. Even if they are not your cup of tea, try to see what the issue is.

Problems 1 – 9 are essential. Do them.

Problem 10 is a good thought problem. Think about it.

Starting with Problem 11, there are things for you to prove. I would be happy to see you do some of these by yourself. We will do these problems in class, but I'd like us to move through them fairly quickly, so have a look at them before class.

Reading assignment, Chapter 7

Due in the seventh week of class.

Read and understand Chapter 7 of the textbook by Daepp and Gorkin called "Operations on sets."

This is a short chapter, all about working with sets. You can approach these problems in a number of ways. Often it helps to draw a nice Venn diagram and get the right intuitive idea for what is being claimed, but don't stop there. You can also just focus on letting $x \in A$ or whatever and working with that, without thinking about Venn diagrams.

Most of the chapter is devoted to one example, showing that, if A, B, and C are sets, then $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$. The book suggests working forward from one side, and backward from the other, just as people sometimes build a bridge by starting at each bank of a river and meeting in the middle.

It also suggests breaking into cases at some point. For example, if $x \in A \cup (B \cap C)$, you can consider the case $x \in A$, which is great because then it's pretty clear that $x \in (A \cup B) \cap (A \cup C)$. But you also need to consider the case $x \notin A$, so that $x \in B \cap C$. But that's helpful, because then $x \in B$ and $x \in C$, and pretty soon it is clear that $x \in (A \cup B) \cap (A \cup C)$.

Do problem 7.1, all six parts. Take your time and use really good form so that the proof is crystal clear. Notice that part (c) (statement 18 in the theorem) is an "if and only if" statement, so it has two parts. It's going to look something like this:

1. Suppose that $A \subseteq B$. We want to show that $(X \backslash B) \subseteq (X \backslash A)$. Let $x \in X \backslash B$. Then $x \notin B$. (More steps here.) Thus, $x \in X \backslash A$, and so $(X \backslash B) \subseteq (X \backslash A)$.
2. Suppose that $(X \backslash B) \subseteq (X \backslash A)$. We want to show that $A \subseteq B$. Let $x \in A$. (More steps here.) Thus, $x \in B$.

Do problem 7.4.

Do problem 7.6.

I guess that these problems are a bit dull, but it really is helpful to be good at proving things about sets. As the course goes on, I suspect that we will run into sets often, and these basic skills will pay off again and again.

As with the previous chapters,

1. Read somewhere quiet, minimizing distractions from phones and friends
2. Note the time that you start and stop reading, and add up the minutes
3. Read with a pencil in your hand and your notebook open in front of you
4. Write a sentence to summarize each paragraph, re-draw diagrams, work out examples and exercises on your own
5. Look up words you don't know, and write down ones you really don't know
6. Read slowly. You are not reading a comic book or a newspaper. It is not a goal of this class for you to learn how to read faster. The goal is to learn how to get more out of the time you spend reading, and to learn to concentrate for longer periods of time.
7. At the end, tally up how much time you have spent on reading this chapter. Write this number in your notebook and remember the number when you come to class.

Reading assignment, Chapter 8

Due in the eighth week of class.

Read and understand Chapter 8 of the textbook by Daepp and Gorkin, called "More on operations on sets."

This chapter is a challenge. You will really need to use all the reading skills you have been practicing when you read this chapter. The ideas are harder, and some are really hard, but not impossible. Just slow yourself down and write things out in lots of detail.

Example 8.2(a) would be a great one to write out concrete fractions with different values of p and q to understand the sets A_q and then the union of these sets. For Example 8.2(b), do the same to understand what the sets B_i are, and then what their intersection is. No shortcuts! Write out elements for each set.

Exercise 8.3 is also good.

In the middle of page 82 the phrase "collection of subsets of X" appears. This is a very new, very difficult concept; do not underestimate how tricky it can be, but patiently think about it and keep coming back to it. For example, $\mathcal{A}$ might be all intervals of the form $[k, k+1]$ and you might want to take the union of all such intervals, or the intersection.

Exercise 8.4 is excellent. Draw pictures until everything is crystal clear. Exercise 8.5 is also excellent.

Rewrite the proofs of Examples 8.6 and 8.7 to make them your own. Really.

Exercises 8.9 and 8.10 are also excellent. Do them on your own, then compare to the solutions in the book.

Do problems 1, 2, and 3.

Here is a challenge problem. Let $a < b$. Show that $\bigcup_{n=1}^{\infty}[a, b - \frac{1}{n}] = [a, b)$. Draw pictures, then show set inclusion both ways.

Here is another challenge problem. Let $a < b$. Show that $\bigcap_{n=1}^{\infty}[a, b + \frac{1}{n}) = [a, b]$. Draw pictures, then show set inclusion both ways.

As with the previous chapters,

1. Read somewhere quiet, minimizing distractions from phones and friends
2. Note the time that you start and stop reading, and add up the minutes
3. Read with a pencil in your hand and your notebook open in front of you
4. Write a sentence to summarize each paragraph, re-draw diagrams, work out examples and exercises on your own
5. Look up words you don't know, and write down ones you really don't know
6. Read slowly. You are not reading a comic book or a newspaper. It is not a goal of this class for you to learn how to read faster. The goal is to learn how to get more out of the time you spend reading, and to learn to concentrate for longer periods of time.
7. At the end, tally up how much time you have spent on reading this chapter. Write this number in your notebook and remember the number when you come to class.

Reading assignment, Chapter 9

Due in the tenth week of class.

Read and understand Chapter 9 of the textbook by Daepp and Gorkin, called "The Power Set and the Cartesian Product." This is the last chapter on plain set theory. It should stretch your mind in a few new directions. Prepare to move slowly and think carefully.

When A is a set, the power set of A is the collection of all subsets of A. Read Example 9.1 and do Exercise 9.3 and then **do Problem 9.1.** Work through Exercise 9.2 and then **do Problem 9.2.** Problem 9.2 is hard, but excellent for you. Take it very slowly. Work through Exercise 9.4 and then **do Problem 9.5. Do Problem 9.8.**

Do Problem 9.11. For 9.11, you have already seen the power set of a set containing 2 elements and 3 elements. **Hint:** When you are making a subset of a set A, for each element of A, you have to decide whether it goes in or out of the subset. There are two choices (in or out) each time. If the hint doesn't help you, write out the power set of $\{1, 2, 3, 4\}$, then read the hint again. Hopefully you don't have to write out the power set of $\{1, 2, 3, 4, 5\}$!

You are already very familiar with one Cartesian product: making ordered pairs (x, y) of real numbers is the Cartesian product $\mathbb{R}\times\mathbb{R}$, which you know better as the xy plane. Every problem involving Cartesian products of sets containing real numbers can be depicted as points in the xy plane. Make a graph in every case. This will help your intuition. When there are only finitely many points, like with $\{0, 1\} \times \{2, 3\}$, also list out all of the (x, y) pairs.

Answer these questions: Who is the Cartesian product named after? Why, exactly?

Work through Exercise 9.5 a, b, e.

For Theorem 9.7, draw A and C as intervals on the x axis and draw B and D as intervals on the y axis, then draw out the sets in the statement of the theorem on two separate sets of axes. Make sure you are crystal clear about what these sets are, and you will be close to mastering Cartesian products.

Do Problem 9.12. It connects Cartesian products to things you learned in geometry.

Do Problem 9.17a. Notice that this is an "if and only if" proof, and it has three set equalities to show. Suppose that $A \times B = C \times D$ and show that $A = C$ and $B = D$ by showing containment each way. Here is one part of the argument: Let $x \in A$. Also let $y \in B$. Then $(x, y) \in A \times B = C \times D$, and so $x \in C$. Thus $A \subseteq C$. After that part is done, suppose that $A = C$ and $B = D$ and argue that $A \times B = C \times D$.

Think about Problem 9.19.

As with the previous chapters,

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4. Write a sentence to summarize each paragraph, re-draw diagrams, work out examples and exercises on your own
5. Look up words you don't know, and write down ones you really don't know
6. Read slowly.
7. At the end, tally up how much time you have spent on reading this chapter. Write this number in your notebook and remember the number when you come to class.

Reading assignment, Chapter 10

Due in the eleventh week of class.

Read and understand Chapter 10 of the textbook by Daepp and Gorkin, called "Relations."

The main definition for Chapter 10 appears at the end of Chapter 9, on page 93. Here is the deal. A *relation* S from a set X to a set Y is a subset of $X \times Y$. If $Y = X$, we say the relation is a relation on X. At the beginning of Chapter 10, we see that we are going to be only working with relations on a set X.

Suppose that S is a relation on a set X. That is, suppose that S is a subset of $X \times X$, which means that S is a set of points of the form (x, y), where $x \in X$ and $y \in X$. Rather than write $(x, y) \in S$, we usually write $x \sim y$. How to read this out loud? There is no perfect solution. I would suggest that you read it as "x tilde y" (because $\sim$ is the tilde that appears above the n in some Spanish words).

Suppose that $X = \mathbb{R}$ and let $S = \{(x, y) : x \leq y\}$. Then $x \sim y$ means that $(x, y) \in S$, which means that $x \leq y$. In this way, we see that $\leq$ is a relation on $\mathbb{R}$. **Write out the set S corresponding to the relations $<$, $\leq$, $=$, $\geq$, and $>$. Then also sketch these as regions in the xy plane.**

Note that relations are between two elements. Thus, "divisible by 4" is not a relation. However, if $X = \mathbb{Z}^+$, you could say that $x \sim y$ if y is divisible by x, and then you would have a relation. People often write $x|y$ for this relation and say that x divides y. Call this relation S. **Write out at least ten of the ordered pairs in S, using at least five different values of x.**

Read Exercises 10.1 and 10.2.

Read the definitions of reflexive, symmetric, and transitive. A relation that satisfies all three is called an equivalence relation. This is where most of the action is with relations. **Do Problem 10.2.** **You should start every part of the problem by writing down examples.** For example, for (a), the example $3 < 3$ will tell you whether the relation is reflexive, $3 < 5$ and $5 < 3$ will tell you about symmetry, and $3 < 5, 5 < 7$, and $3 < 7$ will get you started on transitivity.

Read Example 10.3, then **do Problem 10.3.** Use examples to check reflexivity, symmetry, and transitivity.

Equivalence relations are very important, as are equivalence classes. An equivalence relation is like the equality relation ($=$), but applied to other contexts. Here is an example that is useful. Think of the integers, $\mathbb{Z}$. Say that $x \sim y$ if x and y have the same remainder when you divide by 2. Then $6 \sim 22$ and $31 \sim 7$. This relation is reflexive, because $x \sim x$. It is symmetric because if $x \sim y$ then $y \sim x$. And it is transitive because if $x \sim y$ and $y \sim z$, then $x \sim z$. Now we can say that 6 is equivalent to 22, and 31 is equivalent to 7, according to this definition of equivalence. The equivalence class that contains 6 and 22 is all even numbers, and the equivalence class containing 31 and 7 is all odd numbers. Let this sink into your mind, and you will start to see that it makes for a useful way to organize things, when an equivalence relation is available.

Do Problem 10.1 Start by writing out examples for the pairs (x, y) and (w, z). Think about lines and circles in the plane.

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6. Read slowly.
7. Tally up how much time you have spent on reading this chapter.

Reading assignment, Chapter 18

Due in the thirteenth week of class.

Read and understand Chapter 18 of the textbook by Daepp and Gorkin, called "Mathematical Induction."

Mathematical induction and recursion play an important role especially in discrete mathematics. Prepare to move slowly and think carefully. To understand the proof of Theorem 18.1, you will need **Well-ordering principle of the natural numbers**: Every nonempty subset of the natural numbers contains a minimum. Read Theorem 18.1 and then **do Problem 18.1** and **Problem 18.3**. Follow the steps in Theorem 18.1, define the assertion $P(n)$ for the problem first. You will need the condition "$P(n)$ is true" to show the induction step. Work through Exercise 18.3 to Exercise 18.5 and then **do Problem 18.9** without going back Exercise 18.5. You can do it!!

Recursion is a very useful tool to define functions, sequences and sets. Before you move to Theorem 18.6, read the definition of n factorial for $n \in \mathbb{N}$. Write out $3!, 4!$ and $5!$, then try $\frac{6!}{2!4!}$. More general, simplify $\frac{n!}{m!(n-m)!}$ where n and m are two positive integers with $n \geq m$. Here is another simple example: Let $n \in \mathbb{Z}^+$. Consider the function $S(n) = S(n-1) + n$ with $S(0) = 0$. Write out $S(1), S(2)$ and $S(3)$. Can you figure out what this function does for us? With Problem 18.1, you should be able to see the connection between induction and recursion.

Theorem 18.6 shows the existence and uniqueness of a recursive function $g : N \to X$ given a function $f : X \to X$ and $a \in X$, where X is a nonempty set. The function g satisfies

(i) The base step: $g(0) = a$, and

(ii) The recursive step: $g(n+1) = f(g(n))$ for all $n \in \mathbb{N}$.

The proof of Theorem 18.6 is long and hard. Be patient! You may not get the idea of the proof at beginning, try to write outlines of the proof. You can come back it later.

The keys to a successful recursive solution is to identify the base case and make sure the recursive step is making progress toward the solution. Do Exercise 18.7 and then **do Problem 18.10** and **Problem 18.11**.

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