Your name: ______________________________

Quiz on even and odd and three–dimensional vectors

20 points

Work hard to write really nice proofs.

Definition 1. Even. An integer n is *even* if there exists an integer k for which $n = 2k$.

Definition 2. Odd. An integer n is *odd* if there exists an integer k for which $n = 2k + 1$.

Definition 3. Three–dimensional vector. A three–dimensional vector is an ordered triple $\langle a_1, a_2, a_3 \rangle$, where a_1, a_2, and a_3 are real numbers.

Definition 4. Sum of 3–dimensional vectors. The sum of 3–dimensional vectors $\langle a_1, a_2, a_3 \rangle$ and $\langle b_1, b_2, b_3 \rangle$ is the 3–dimensional vector $\langle a_1 + b_1, a_2 + b_2, a_3 + b_3 \rangle$. We write $\vec{a} \oplus \vec{b}$, using a new symbol so we don't confuse addition of vectors with addition of real numbers.

Definition 5. Scalar product for 3–dimensional vectors. Let c be a real number and let $\vec{a} = \langle a_1, a_2, a_3 \rangle$ be a 3–dimensional vector. The *scalar product* of c and $\vec{a}$ is defined as:

$$c\vec{a} = \langle ca_1, ca_2, ca_3 \rangle.$$

Show. Show that the product of two odd numbers is odd.

Show. On the back of this sheet, show that the scalar product is distributive over vector addition. That is, show that $c(\vec{a} \oplus \vec{b}) = c\vec{a} \oplus c\vec{b}$.

Your name: ______________________________

Quiz on even and odd and three–dimensional vectors

40 points

Work hard to write really nice proofs.

Definition 1. Even. An integer n is *even* if there exists an integer k for which $n = 2k$.

Definition 2. Odd. An integer n is *odd* if there exists an integer k for which $n = 2k + 1$.

Definition 3. Three–dimensional vector. A three–dimensional vector is an ordered triple $\langle a_1, a_2, a_3 \rangle$, where a_1, a_2, and a_3 are real numbers.

Definition 4. Sum of 3–dimensional vectors. The sum of 3–dimensional vectors $\langle a_1, a_2, a_3 \rangle$ and $\langle b_1, b_2, b_3 \rangle$ is the 3–dimensional vector $\langle a_1 + b_1, a_2 + b_2, a_3 + b_3 \rangle$. We write $\vec{a} \oplus \vec{b}$, using a new symbol so we don't confuse addition of vectors with addition of real numbers.

Definition 5. Dot product of 3–dimensional vectors. The dot product of 3–dimensional vectors $\langle a_1, a_2, a_3 \rangle$ and $\langle b_1, b_2, b_3 \rangle$ is the real number $a_1b_1 + a_2b_2 + a_3b_3$. The dot product of vectors $\vec{a}$ and $\vec{b}$ is denoted $\vec{a} \bullet \vec{b}$.

Show. Show that the sum of an even number and an odd number is odd.

Show. On the back of this sheet, show that the dot product is distributive over vector addition. That is, show that $\vec{a} \bullet (\vec{b} \oplus \vec{c}) = \vec{a} \bullet \vec{b} + \vec{a} \bullet \vec{c}$.

Your name: ____________________________

Quiz on even and odd and three–dimensional vectors

Write really nice proofs. Use good form. Start in the right place, take small steps, justify each step, and generalize appropriately.

Definition 1. Even. An integer n is *even* if there exists an integer k for which $n = 2k$.

Definition 2. Odd. An integer n is *odd* if there exists an integer k for which $n = 2k + 1$.

Show. Show that the difference of two odd numbers is even.

Definition 3. Three–dimensional vector. A three–dimensional vector is an ordered triple $\langle a_1, a_2, a_3\rangle$, where a_1, a_2, and a_3 are real numbers.

Definition 4. Sum of 3–dimensional vectors. The sum of 3–dimensional vectors $\langle a_1, a_2, a_3\rangle$ and $\langle b_1, b_2, b_3\rangle$ is the 3–dimensional vector $\langle a_1 + b_1, a_2 + b_2, a_3 + b_3\rangle$. We write $\vec{a} \oplus \vec{b}$, using a new symbol so we don't confuse addition of vectors with addition of real numbers.

Definition 5. 1234 product of 3–dimensional vectors. The 1234 product of 3–dimensional vectors $\langle a_1, a_2, a_3\rangle$ and $\langle b_1, b_2, b_3\rangle$ is the real number $1a_1b_1 + 2a_2b_2 + 3a_3b_3 + 4a_1b_3$. The 1234 product of vectors $\vec{a}$ and $\vec{b}$ is denoted $\vec{a} \star \vec{b}$.

Show. Show that the 1234 product is distributive over vector addition. That is, show that $\vec{a} \star (\vec{b} \oplus \vec{c}) = \vec{a} \star \vec{b} + \vec{a} \star \vec{c}$ for all vectors $\vec{a}, \vec{b}$, and $\vec{c}$.

Your name: ______________________________

Quiz on a new operation with 3–dimensional vectors

20 points

Definition Sum of 3–dimensional vectors. The sum of 3–dimensional vectors $\langle a_1, a_2, a_3 \rangle$ and $\langle b_1, b_2, b_3 \rangle$ is the 3–dimensional vector $\langle a_1 + b_1, a_2 + b_2, a_3 + b_3 \rangle$.

Definition The *twist product* of 3–dimensional vectors $\langle a_1, a_2, a_3 \rangle$ and $\langle b_1, b_2, b_3 \rangle$ is the 3–dimensional vector $\langle a_1 b_3, a_2 b_2, a_3 b_1 \rangle$. It is denoted $\vec{a} * \vec{b}$.

Example. For example, $\langle 1, 3, 6 \rangle * \langle 2, 7, 10 \rangle = \langle 1 \cdot 10, 3 \cdot 7, 6 \cdot 2 \rangle = \langle 10, 21, 12 \rangle$.

Show. Show that the twist product is distributive over vector addition. That is, show that $(\vec{a} \oplus \vec{b}) * \vec{c} = \vec{a} * \vec{c} \oplus \vec{b} * \vec{c}$. Start with "Let …," take one step at a time, write the justification for the step, and make a general conclusion.

Show. Prove or disprove: "The twist product is commutative."

Show. On the other side of this piece of paper, show that for all 3–dimensional vectors $\vec{a}$ and $\vec{b}$ and real numbers c, $\vec{a} * (c\vec{b}) = (c\vec{a}) * \vec{b} = c(\vec{a} * \vec{b})$. Use parentheses *every* time three things are multiplied together.

Your name: ______________________________

Quiz on a new operation with 3–dimensional vectors

20 points

Definition Sum of 3–dimensional vectors. The sum of 3–dimensional vectors $\langle a_1, a_2, a_3\rangle$ and $\langle b_1, b_2, b_3\rangle$ is the 3–dimensional vector $\langle a_1 + b_1, a_2 + b_2, a_3 + b_3\rangle$.

Definition Duplicate product. The *duplicate product* of 3–dimensional vectors $\langle a_1, a_2, a_3\rangle$ and $\langle b_1, b_2, b_3\rangle$ is the 3–dimensional vector $\langle a_1 b_1, a_2 b_3, a_3 b_3\rangle$. (That is not a typo, b_3 is used twice. That is why it is called the duplicate product.) It is denoted $\vec{a} * \vec{b}$.

Example. For example, $\langle 1, 3, 6\rangle * \langle 5, 2, 4\rangle = \langle 1 \cdot 5, 3 \cdot 4, 6 \cdot 4\rangle = \langle 5, 12, 24\rangle$.

Show. Show that the duplicate product is distributive over vector addition. That is, show that $(\vec{a} \oplus \vec{b}) * \vec{c} = \vec{a} * \vec{c} \oplus \vec{b} * \vec{c}$. Start with "Let ...," take one step at a time, write the justification for the step, and make a general conclusion.

Show. Prove or disprove: "The duplicate product is commutative." (You use a proof to prove, a counterexample to disprove.)

Show. On the other side of this piece of paper, show that for all 3–dimensional vectors $\vec{a}$ and $\vec{b}$ and real numbers c, $\vec{a} * (c\vec{b}) = (c\vec{a}) * \vec{b} = c(\vec{a} * \vec{b})$. Use parentheses *every* time three things are multiplied together.

Your name: ____________________________

Quiz on a new operation with 3–dimensional vectors

20 points

Definition Sum of 3–dimensional vectors. The sum of 3–dimensional vectors $\langle a_1, a_2, a_3\rangle$ and $\langle b_1, b_2, b_3\rangle$ is the 3–dimensional vector $\langle a_1 + b_1, a_2 + b_2, a_3 + b_3\rangle$.

Definition The *twist product* of 3–dimensional vectors $\langle a_1, a_2, a_3\rangle$ and $\langle b_1, b_2, b_3\rangle$ is the 3–dimensional vector $\langle a_1b_3, a_2b_2, a_3b_1\rangle$. It is denoted $\vec{a} * \vec{b}$.

Example. For example, $\langle 1, 3, 6\rangle * \langle 2, 7, 10\rangle = \langle 1 \cdot 10, 3 \cdot 7, 6 \cdot 2\rangle = \langle 10, 21, 12\rangle$.

Show. Show that the twist product is distributive over vector addition. That is, show that $(\vec{a} \oplus \vec{b}) * \vec{c} = \vec{a} * \vec{c} \oplus \vec{b} * \vec{c}$. Start with "Let ...," take one step at a time, write the justification for the step, and make a general conclusion.

Show. Prove or disprove: "The twist product is commutative."

Show. On the other side of this piece of paper, show that for all 3–dimensional vectors $\vec{a}$ and $\vec{b}$ and real numbers c, $\vec{a} * (c\vec{b}) = (c\vec{a}) * \vec{b} = c(\vec{a} * \vec{b})$. Use parentheses *every* time three things are multiplied together.

Your name: ____________________________

Quiz on sum and dot product of 3–dimensional vectors

10 points

Definition 3–dimensional vector. A three–dimensional vector is an ordered triple $\langle a_1, a_2, a_3 \rangle$, where a_1, a_2, and a_3 are real numbers.

Definition Equality of 3–dimensional vectors. 3–dimensional vectors $\langle a_1, a_2, a_3 \rangle$ and $\langle b_1, b_2, b_3 \rangle$ are equal if $a_1 = b_1, a_2 = b_2$, and $a_3 = b_3$. The order of the numbers is important.

Definition Sum of 3–dimensional vectors. The sum of 3–dimensional vectors $\langle a_1, a_2, a_3 \rangle$ and $\langle b_1, b_2, b_3 \rangle$ is the 3–dimensional vector $\langle a_1 + b_1, a_2 + b_2, a_3 + b_3 \rangle$. We write $\vec{a} \oplus \vec{b}$, using a new symbol so we don't confuse addition of vectors with addition of real numbers.

Show. Show that addition of 3–dimensional vectors is commutative. Start with "Let," take one step at a time, write the justification for the step, and make a general conclusion.

Definition Dot product of 3–dimensional vectors. The dot product of 3–dimensional vectors $\langle a_1, a_2, a_3 \rangle$ and $\langle b_1, b_2, b_3 \rangle$ is the real number $a_1 b_1 + a_2 b_2 + a_3 b_3$.

Show. On the other side of this sheet of paper, show that the dot product is distributive over vector addition. That is, show that $(\vec{a} \oplus \vec{b}) \bullet \vec{c} = \vec{a} \bullet \vec{c} + \vec{b} \bullet \vec{c}$. Start with "Let ...," take one step at a time, write the justification for the step, and make a general conclusion. Please also explain why one addition symbol is $\oplus$ and the other is $+$.

Possible questions for the quiz over the Division Algorithm

The problems may be chosen from the ones below or from new problems related to that activity.

I suggest that you write out solutions for each of these before the quiz, and that you try to do them without consulting your notes. Rediscover the arguments, and you will own them. Then, some hours later, write them again on a fresh sheet of paper. This is the best way to learn them.

I will be happy to look at your practice solutions in office hours or just before or after class.

1. Let k be an integer. Show that exactly one of the integers $k, k+1, k+2, k+3$ is a multiple of 4.

2. Let n be even, so that $n = 2k$ for some integer k. Use the Division Algorithm to write k as $2j$ or $2j+1$. For each case, show that exactly one of the numbers n and $n+2$ is a multiple of 4. This will also require the use of the Division Algorithm.

3. Let n be an integer. Use the Division Algorithm to write $n = 10q + r$. In an efficient way, write n^2 as a multiple of 10 plus a remainder and tell what final digits are possible when n^2 is written in base 10.

4. Let n be an integer. Use the Division Algorithm to write $n = 10q + r$. In an efficient way, write n^4 as a multiple of 10 plus a remainder and tell what final digits are possible when n^2 is written in base 10.

5. Let $n > 0$ and $k > 0$ be integers. Argue that there exist integers q and r such that $n = qk + r$ and $0 \leq r < k$. You can phrase the argument in terms of dealing out n cards to k people, or in terms of starting with n and subtracting k repeatedly.

6. Let $n > 0$ and $k > 0$ be integers. Suppose that there exist integers q and r for which $n = qk + r$ and $0 \leq r < k$, and at the same time that there exist integers q_2 and r_2 for which $n = q_2 k + r_2$ and $0 \leq r_2 < k$. Show that $q = q_2$ and $r = r_2$, including deriving any new inequalities that you need. This shows that there is at most one way to write $n = qk + r$ with $0 \leq r < k$.

7. Let n be an integer and suppose that $n = 3m + 1$ for some integer m. Use the uniqueness part of the Division Algorithm to argue that n cannot be written as $n = 3k$ where k is an integer. Thus, n is not a multiple of 3.

8. Let n be an integer and suppose that n^2 is a multiple of 3. Use the Division Algorithm to write n as $3m$, $3m+1$, or $3m+2$, and then use the Division Algorithm to rule out the last two cases. Make clear which part of the Division Algorithm you use in each part.

9. Let n be even, so that $n = 2k$ for some integer k. Use the Division Algorithm to write k as $2j$ or $2j+1$. For each case, show that exactly one of the numbers n and $n+2$ is a multiple of 4. This will also require the use of the Division Algorithm.

Your name: ______________________________

Quiz on things related to the Division Algorithm

10 points

1. Let $n > 0$ and $k > 0$ be integers. Suppose that there exist integers q and r for which $n = qk + r$ and $0 \leq r < k$, and at the same time that there exist integers a and b for which $n = ak + b$ and $0 \leq b < k$. Show that $q = a$ and $r = b$. This shows that there is at most one way to write $n = qk + r$ with $0 \leq r < k$.

2. Let k be an integer. Show that exactly one of the integers $k, k+1, k+2, k+3$ is a multiple of 4. You can use the back of this sheet of paper.

Your name: ______________________________

Quiz on the Division Algorithm

40 points

1. Let n and k be integers and suppose that $k > 0$. Suppose that there exist integers q and r for which $n = qk + r$ and $0 \leq r < k$, and at the same time that there exist integers q_2 and r_2 for which $n = q_2k + r_2$ and $0 \leq r_2 < k$. Show that $q = q_2$ and $r = r_2$, including deriving any new inequalities that you need. Bonus points for doing this without dividing by k. This shows that there is at most one way to write n as $qk + r$ with $0 \leq r < k$.

2. Write your solution to this problem on the back of the page. Let n be even, so that $n = 2k$ for some integer k. Use the Division Algorithm to write k as $2j$ or $2j + 1$. For each case, show that exactly one of the numbers n and $n + 2$ is a multiple of 4. This will also require the use of the Division Algorithm.

Your name: ______________________________

Quiz on the Division Algorithm

40 points

1. Let n and k be integers and suppose that $k > 0$. Suppose that there exist integers q and r for which $n = qk + r$ and $0 \leq r < k$, and at the same time that there exist integers q_2 and r_2 for which $n = q_2 k + r_2$ and $0 \leq r_2 < k$. Show that $q = q_2$ and $r = r_2$. Bonus points for doing this without dividing by k. This shows that there is at most one way to write n as $qk + r$ with $0 \leq r < k$.

2. Write your solution to this problem on the back of the page. Let n be an integer and suppose that n^2 is a multiple of 3. Use the Division Algorithm to write n as $3m$, $3m + 1$, or $3m + 2$, and then use the Division Algorithm to rule out the last two cases. Make clear which part of the Division Algorithm you use in each part.

Possible questions for the quiz over Contrapositive, Elimination, and Contradiction

The problems may be chosen from the ones below or from new problems related to that activity.

I suggest that you write out solutions for each of these before the quiz, and that you try to do them without consulting your notes. Rediscover the arguments, and you will own them. Then, some hours later, write them again on a fresh sheet of paper. This is the best way to learn them.

I will be happy to look at your practice solutions in office hours or just before or after class.

1. Let n be an integer. Show that if n^2 is a multiple of 7, then n is a multiple of 7 by using the process of elimination. Carefully and explicitly use the Division Algorithm to generate 7 cases, one of which must be true. You might want to name these $P_0, P_1, \ldots, P_6$. Then show that six of the cases are false, so that the remaining case must be true. To save time, calculate $(7k + r)^2$ just once, and then substitute in different values of r. Make the overall logic of the argument crystal clear.

2. Show that $\sqrt{7}$ is irrational. Follow the model that $\sqrt{2}$ is irrational, by pretending for a minute that $\sqrt{7}$ is rational and making logical deductions that lead to a statement known to be false.

3. Show that there are infinitely many prime numbers. The proof in class was a fill-in-the-blank proof. Here, you will need to know the proof and understand every part. In particular, explain clearly why n is not a multiple of p_1, why n is not a multiple of p_2, etc.

4. Let k and j be integers and suppose that $2k = 3j$.

a) Use the process of elimination to show that j is a multiple of 2.

b) Use the process of elimination to show that k is a multiple of 3.

5. Let n be an integer and suppose that n^2 is a multiple of 6. Show that n is a multiple of 6.

6. Show that $\sqrt{6}$ is irrational, using the previous result.

7. Show that $2\sqrt{2}$ is irrational. Thus, $\sqrt{8}$ is irrational.

Your name: ______________________________

Quiz on Implication, Contrapositive, and Contradiction

40 points

1. Let k and j be integers and suppose that $2k = 3j$. Use the process of elimination to show that k is a multiple of 3.

2. On the back of this page, show that there are infinitely many prime numbers. In the proof, explain clearly why the integer n that you define is not a multiple of p_1, why n is not a multiple of p_2, etc. Also explain clearly what statement you know is false and how you know it is false.

Your name: ______________________________

Quiz on Division Algorithm, Contrapositive, Elimination, and Contradiction

1. The goal is to show that if an integer n is a multiple of 3 and also a multiple of 5, then it is a multiple of 15.

We can write $n = 3i = 5j$ for integers i and j.

Use three cases and eliminate two of them to show that j is a multiple of 3. Be very clear about where the three cases come from and about your logic.

2. Suppose that n is an integer and that n^2 is a multiple of 15. Use results we showed in class and the result above to show that n is a multiple of 15. Explain clearly.

3. Use a proof by contradiction to show that $\sqrt{15}$ is irrational.

Possible questions for the quiz over nested quantifiers and negation of quantifiers

Quiz on Thursday, November 16

This will be a 40–point quiz. The problems may be chosen from the ones below or from new problems related to that activity or from Chapter 8. I will be happy to look at your practice solutions in office hours or just before or after class.

1. I saw this quote in the news today: "There is no one here that doesn't know that I'm not an angel." Please rewrite this with quantifiers and with as few "nots" as possible.

2. Suppose you want to prove the statement P : "for every integer $n > 0$, there are prime numbers p and q with $q = p + 2$. (The numbers p and q are called twin primes, like 5 and 7 or 11 and 13.) No one in the world right now knows whether the statement is true or false, but people are trying!

a. Suppose you want to show that P is true. What kind of proof would you need? Use the words construct and choose. Write an outline of the proof, starting where it needs to start, ending where it would need to end.

b. Suppose you want to show that P is false. Negate P. Probably best to end with $q \neq p + 2$. Explain what it would take to prove $\neg P$, using the words construct and choose. Write an outline of the proof, starting where it needs to start, ending where it would need to end.

3. For each of the following statements, if the statement is true, prove it with good form. If it is not true, negate it and prove that the negation is true. It would be a good idea to be clear about "choose" and "construct". If you need to work backwards, or do some scratchwork, fine. If you need to do a construction, be clear when that happens.

a. For all real numbers b, there exists an integer n such that $n \leq b < n + 1$.

b. For all real numbers $b > 0$, there exists an integer n such that $n^2 \leq b < (n + 1)^2$.

c. For all real numbers b, there exists an integer n such that $n^2 \leq b$.

d. For all real numbers $a > 0$, there exists an integer n such that $\frac{1}{\sqrt{n}} < a$.

e. For all real numbers $a > 0$, there exists an integer n such that for all $m > n$, $\frac{1}{\sqrt{m}} < a$. Approach this carefully using the words "construct" and "choose" to make sure you do what you need to do.

4. Define $\ln x = \int_0^x \frac{1}{t} dt$ for all real numbers $x > 0$. For all real numbers x and y with $0 < x < y$, show that $\ln x < \ln y$.

5. Show that for all real numbers $y > 0$, there is a real number x such that $x^3 + x + 1 > y$.

6. Show that for all real numbers $y > 0$, there exists a real number x such that $x^3 + x + 1 = y$. Rather than use a construction, use a graphical argument.

7. Show that for all integers $m < n$, $2^m < 2^n$.

8. Consider the expression "not all who wander are lost". Rewrite it by pushing the "not" past the quantifier. How does the result differ from the statement "all who wander are not lost"?

Your name: ______________________________

Quiz on nested quantifiers and negation of quantifiers

40 points

1. Show that for all real numbers $a > 0$, there exists an integer n such that for all $m > n$, $\frac{1}{\sqrt{m}} < a$. Since there are three quantifiers, you will use the word "let" three times. Be clear what is happening each time!

2. Rewrite the statement in Problem 1 using quantifiers.

   Now, negate the statement and push the "not" past all of the quantifiers.

3. On the back of the page, show that for all real numbers x and y with $0 < x < y$, we have $x^2 < y^2$. It's not enough to say that the result is obvious or just look at examples, you need to find a way to break this into smaller steps that you can verify. Make note of each time you use one of the assumptions in the proof.

   Extra credit for writing two distinct proofs. For example, for a second proof, you could use integrals.

Your name: ______________________________

Quiz on even and odd

15 points

Work hard to write really nice proofs.

1. Show that the product of two odd numbers is odd.

2. Show that for all integers n, the quantity $n^2 + 10n + 21$ is either odd or is a multiple of 4.

3. The numbers $0, 1, 4, 9, 16, 25, \ldots$ are called perfect squares. The differences between consecutive perfect squares are $1, 3, 5, 7, 9, \ldots$. Show that the difference between consecutive perfect squares is always an odd number.

Your name: ______________________________

Quiz on irrationality of square roots of odd primes

20 points

Show. Show that if n is an integer and n^2 is a multiple of 47, then n is a multiple of 47. You choose the type of proof you want to do. Whatever you do, there are too many cases to check one by one, so organize your thoughts efficiently.

Show. On the other side of this sheet of paper, show that $\sqrt{47}$ is irrational. I recommend a proof by contradiction.

Your name: ______________________________

Quiz on infinite set operations

5 points

1. Let $B = \bigcup_{n=0}^{\infty} [n, n^2]$. List out the first five or more sets in this union. Draw them on a number line if it helps.

Let $C = \{0\} \cup \{1\} \cup [2, \infty)$. Show that $B = C$ by showing containment in both directions. You will need to use three cases in each direction to deal with $0, 1$, and the rest.

Your name: ______________________________

Quiz on some problems from Chapter 5 of Daepp and Gorkin

15 points

1. Let x and y be real numbers. Use the triangle inequality to show that $||x| - |y|| \leq |x - y|$.

2. Prove or refute the following conjecture: There are no positive integers x and y such that $x^2 - y^2 = 10$. You can use the back of the sheet if you like.

Your name: ______________________________

Quiz on inequalities

5 points

Definition Positive real numbers. By construction, the real numbers have a subset $\mathbb{R}^+$, called the *positive real numbers,* for which:

a. If $a, b \in \mathbb{R}^+$, then $a + b \in \mathbb{R}^+$. ($\mathbb{R}^+$ is closed under addition.)

b. If $a, b \in \mathbb{R}^+$, then $a \cdot b \in \mathbb{R}^+$. ($\mathbb{R}^+$ is closed under multiplication.)

c. For every real number a, either $a \in \mathbb{R}^+$ or $(-a) \in \mathbb{R}^+$ or $a = 0$. Exactly one of the three happens.

Definition Less than. Let a and b be real numbers. We write that $a < b$ if $b - a \in \mathbb{R}^+$.

Show. Let a, b, c be real numbers. Suppose that $a < b$ and $0 < c$. Show that $ac < bc$. Take very small steps and be careful to cite justifications for every single step.

Your name: ____________________________

Quiz on induction

15 points

For each problem below, clearly state $P(1), P(k)$, and $P(k+1)$ as logical statements with double quotes around them. When proving that $P(k)$ being true implies that $P(k+1)$ is true, do not write down $P(k+1)$ as if it were true, but rather start with one side and work with it until it turns into the other side.

Show. Use induction to show that for $n > 0$, 8 divides $5^n + 2(3^{n-1}) + 1$. **Hint:** As in other proofs of divisibility, add and subtract to be able to use $P(n)$ to simplify $P(n+1)$.

Show. On the back of this piece of paper, use induction to show that for all $n \geq 1$, we have that $1(1!) + 2(2!) + \cdots + n(n!) = (n+1)! - 1$.

Review questions for the final quiz

Taken from a variety of sources without attribution.

Overview

The final quiz will consist of approximately 6 questions and will be worth approximately 100 points, which is just under 20% of the total points for the course. I will try to make sure that it can be done by a prepared student in 2 hours. The best way to prepare is to work out problems on the review sheet and on the handouts that we have had in class.

Key things to review are all in-class activities about set theory, constructions, and induction, plus the review exercises. Note that some of these questions have already appeared in review questions or activities.

Induction problems

1. Show that $1 + 2 + 2^2 + 2^3 + \cdots + 2^n = 2^{n+1} - 1$ for all $n \geq 0$.

2. Show that $\sum_{j=0}^{n} r^j = \frac{r^{n+1}-1}{r-1}$ for all $n \geq 0$ and all real numbers $r \neq 1$. (There is an easier formula when $r = 1$!)

3. Show that $2^n < n!$ for all $n \geq 4$.

4. Show that $3^n < n!$ for all $n \geq 7$.

5. Show that $n! < n^n$ for all $n > 1$.

6. Show that $1 \cdot 2 + 2 \cdot 3 + 3 \cdot 4 + \cdots + n(n+1) = \frac{n(n+1)(n+2)}{3}$ for all $n \geq 1$.

7. Show that $1 + \frac{1}{4} + \frac{1}{9} + \cdots + \frac{1}{n^2} < 2 - \frac{1}{n}$ for all $n \geq 2$.

8. Show that $n^5 - n$ is a multiple of 5 for all n.

9. Show that $n^2 - 1$ is a multiple of 8 for all odd n. **Hint:** You could try showing $P(1)$ and then show that $P(n)$ implies $P(n+2)$.

10. Use the product rule to show that, for every integer $n \geq 1$, the derivative of x^n is nx^{n-1}.

11. Show that $1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{2^n} \geq 1 + \frac{n}{2}$. This is too hard for the final quiz, but you may enjoy working on it. Notice that increasing n by 1 will double the number of terms, unlike most of the other problems you have worked on. This result shows that the harmonic series diverges.

12. Draw n lines in the plane such that no two lines are parallel and no three lines go through a common point. Show that this divides the plane into $\frac{n^2+n+2}{2}$ regions. **Hint:** How many regions does the $n+1$st line add?

Set theory

1. Show that $[2,5) \cap (3,7) = (3,5)$ by showing inclusion both ways. Start by letting $x \in [2,5) \cap (3,7)$, so that $x \in [2,5)$ and $x \in (3,7)$, then write it as $2 \leq x < 5$ and $3 < x < 7$. There are four inequalities here. Soon enough you can conclude that $x \in (3,5)$. Then show containment the other way as well.

2. Show that $\bigcup_{n=1}^{\infty} [3, 5 - \frac{1}{n}] = [3,5)$ by showing inclusion in both directions.

3. Show that $\bigcap_{n=1}^{\infty} [2 - \frac{1}{n}, 8] = [2,8]$ by showing inclusion in both directions.

4. Show that $A \cup B = A$ if and only if $B \subseteq A$.

5. Show that $A \subseteq C$ and $B \subseteq C$ if and only if $A \cup B \subseteq C$.

6. Let $A = \bigcup_{n=1}^{\infty} H_n$. Write an outline of a proof that $A = B$, where you show set inclusion both ways. For example, to show $A \subseteq B$, start with "Let $x \in A$, then $x \in H_n$ for some n. Now we need to show that $x \in B$. Since $x \in A$ was arbitrary, $A \subseteq B$. For the other direction, there will be a construction.

7. Let $A = \bigcap_{n=1}^{\infty} H_n$. Write an outline of a proof that $A = B$, where you show set inclusion both ways. Will you need a construction in this proof?

8. Let $A = \displaystyle\bigcup_{r \in \mathbb{Q}} (r - \frac{1}{10}, r + \frac{1}{10})$. Here, $\mathbb{Q}$ is the set of all rational numbers. Show that $A = \mathbb{R}$.

Constructions

1. Let $x < 6$. Show that there exists an integer n such that $x + \frac{1}{n} < 6$.

2. Let $x > 0$. Show that there exists an integer n such that $n \leq x < n^2$.

3. Let p be a rational number with $p < 0$. Construct a rational number q with $p < q < 0$.

Other problems

1. Show that if n is odd, then $n^2 + 2n - 7$ is a multiple of 4.

2. Let a, b, and c be integers. Suppose that b is a multiple of a or c is a multiple of a. Show that bc is a multiple of a.

3. Let a and b be integers. Suppose that a is a multiple of b and that b is a multiple of a. Show that $a = \pm b$.

4. Suppose that n is an integer and n^2 is a multiple of 5. Show that n is a multiple of 5. Try to do this without looking back at your notes!

5. Suppose that n is an odd integer. Show that $n^3 - 25n$ is a multiple of 24. This is similar to something we did in class. See if you can do it that way. Can you do it by induction instead? Work with $P(n)$ and $P(n+2)$. Which way is easier?

6. Show that an integer n cannot be both even and odd. What kind of proof did you use?

7. Suppose that m and n are integers and that $3m = 7n$. Show that n is a multiple of 3. Do this by writing n as $3q + r$ for different possible values of r.

8. Give a complete proof that $\sqrt{3}$ is irrational.

Your name: ______________________________

Final quiz

100 points, 20 for each problem

Write your name only on this page. Write your solutions on the blank sheets of paper, but do not write your name on them. When you are done, I will staple this cover sheet to your solutions.
Use good form in every problem. That is, be very clear about every step of the proof, especially things like “Let $n \geq 1$.” and generalizing. This is an extremely important part of this quiz.
If you are totally stuck, you can ask for a hint, but it may cost you some points. Good luck!

1. a) Use mathematical induction to show that $9^n - 8n - 1$ is a multiple of 64 for each integer $n \geq 1$.

b) Explain how you have shown that $P(n)$ being true implies that $P(n+1)$ is true **for all** $n \geq 1$.

2. Suppose that $x \leq 7 + \frac{1}{\sqrt{n}}$ for all integers $n \geq 1$. Show that $x \leq 7$.

3. a) Suppose that $A_n \subseteq B$ for all $n \geq 1$. Show that $\bigcup_{n=1}^{\infty} A_n \subseteq B$.

b) Suppose that $\bigcup_{n=1}^{\infty} A_n \subseteq B$. Show that for all $n \geq 1$, we have $A_n \subseteq B$.

4. Show that $\bigcap_{n=1}^{\infty} (3, 7 + \dfrac{1}{\sqrt{n}}) = (3, 7]$.

5. Show that for all odd integers $n \geq 1$, the expression $n^3 - 25n$ is a multiple of 24. Actually, we already know this is true, because we showed that $n^3 - n$ is a multiple of 24, and we can write $n^3 - 25n = n^3 - n - 24n$. But here I want you to write a new proof that does not use the result that $n^3 - n$ is a multiple of 24, and does not simply repeat that proof and then subtract $24n$.

I recommend that you either try factoring $n^3 - 25n$ and use an approach like we did in class, or else try induction. Be patient, you can do this, but it will take some time either way.

Your name: ______________________________

Final quiz

40 points

Use good form in every problem. Be very clear about every step of the proof, especially things like "Let $n \geq 1$." and generalizing. This is an important part of this quiz.
If you are stuck, you can ask for a hint. Good luck!

1. Let $A = \{x : 3x + 2 < 7\}$. Let $B = \{u : 6u < 16\}$.

a) Show that $A \subseteq B$. Make sure to use the membership requirement for A and check the membership requirement for B.

b) Show that $A \subset B$.

2. Use mathematical induction to show that $7^n - 1$ is a multiple of 6 for all $n = 0, 1, 2, \ldots$. Use good form.

3. Give a complete proof that $\sqrt{7}$ is irrational. Prove all statements that are the sort of thing that we prove in Math 3280.

4. Suppose that $A \subset B$ and $B \subseteq C$.

a) Show that $A \subseteq C$.

b) Show that $A \subset C$.

5. Use the Division Algorithm to show that 14 is not a multiple of 6.

6. Let a and b be rational numbers with $a < b$. Construct an irrational number t with $a < t < b$.

7. Suppose that $x \leq 5 + 1/n$ for all $n = 1, 2, 3, ...$ Show that $x \leq 5$ using a proof contrapositive.