

After the exam starts, please write your student ID (or name) on **EVERY PAGE**.

There are **4** questions for a total of **13** parts. You may consult your sheet of notes. Calculators, phones, computers, and other electronic devices are not permitted. There are **17** pages on the exam. **Notify a proctor immediately if a page is missing.** You may, without proof, use theorems and lemmas that were proven in the notes and/or in lecture, unless we explicitly ask for a derivation. However, you must clearly state what theorem or lemma you are using and where/how you are using it.

Please write legibly if you want full credit on all problems.

**You have 75 minutes.**

PRINT and SIGN Your Name: \_\_\_\_\_,  
(last) (first) (signature)

PRINT Your Student ID: \_\_\_\_\_

Person before you: \_\_\_\_\_,  
(name) (SID)

Person behind you: \_\_\_\_\_,  
(name) (SID)

Person to your left: \_\_\_\_\_,  
(name) (SID)

Person to your right: \_\_\_\_\_,  
(name) (SID)

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Do not turn this page until your instructor tells you to do so.
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Extra page. If you want the work on this page to be graded, mention it on the problem's main page.

# 1 Finding the Centroid (3 parts, 20 points)

Let  $\mathbf{x}_1, \dots, \mathbf{x}_n \in \mathbb{R}^d$ . We consider computing the centroid of this dataset. Consider the loss function

$$\mathcal{L}(\mathbf{w}) := \frac{1}{2n} \sum_{i=1}^n \|\mathbf{x}_i - \mathbf{w}\|_2^2.$$

(a) (5 points) First, we compute the gradient of the loss function. **Show that**

$$\nabla_{\mathbf{w}} \mathcal{L}(\mathbf{w}) = \mathbf{w} - \bar{\mathbf{x}},$$

where  $\bar{\mathbf{x}} := \frac{1}{n} \sum_{i=1}^n \mathbf{x}_i$ .

(b) (5 points) **Show that the minimizer of the loss function is given by  $\bar{\mathbf{x}}$ , i.e.  $\arg \min_{\mathbf{w} \in \mathbb{R}^d} \mathcal{L}(\mathbf{w}) = \bar{\mathbf{x}}$ .**  
Make sure to justify your answer.

- (c) (10 points) Suppose  $\mathbf{x}_1, \dots, \mathbf{x}_n$  are identically and independently distributed according to a normal distribution with mean  $\mathbf{x}_*$  and diagonal covariance, i.e.  $\mathbf{x}_i \sim \mathcal{N}(\mathbf{x}_*, \sigma^2 \mathbf{I}_d)$  for  $i = 1, \dots, n$ .  
**Calculate**  $\mathbb{E}[\|\bar{\mathbf{x}} - \mathbf{x}_*\|_2^2]$ .

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## 2 A Spectral View of Linear Regression (5 parts, 25 points)

Assume we are given training data in the form of the matrix  $\mathbf{X} \in \mathbb{R}^{n \times d}$  where the rows are the  $d$ -dimensional feature vectors  $\mathbf{x}_i$  and  $\mathbf{y} \in \mathbb{R}^n$  which is the vector of the corresponding target values. We do not assume that  $\mathbf{X}$  is full rank, and take its rank to be  $r$ . Note that  $d \leq n$ .

Recall that the compact singular value decomposition is  $\mathbf{X} = \mathbf{U}\mathbf{\Sigma}\mathbf{V}^\top$  where  $\mathbf{U} \in \mathbb{R}^{n \times d}$ ,  $\mathbf{V} \in \mathbb{R}^{d \times d}$ , and  $\mathbf{\Sigma} = \text{diag}(\sigma_1, \dots, \sigma_d)$ . We denote the  $n$ -dimensional column vectors of  $\mathbf{U}$  by  $\mathbf{u}_i$  and the  $d$ -dimensional column vectors of  $\mathbf{V}$  by  $\mathbf{v}_i$ . Furthermore, let  $\sigma_1 \geq \sigma_2 \geq \dots \geq \sigma_d$ .

In this problem, we consider the result of two different linear regression techniques: ridge regression and applying ordinary least squares after using PCA to reduce the feature dimension from  $d$  to  $k$  (PCA-OLS). In particular, we compare the predicted value  $\hat{y}$  of a new datapoint  $\mathbf{x}$  by writing an expression of the form:

$$\hat{y}(\mathbf{x}) = \mathbf{x}^\top \mathbf{w} = \mathbf{x}^\top \sum_{i=1}^d \rho(\sigma_i) \mathbf{v}_i \mathbf{u}_i^\top \mathbf{y}. \quad (1)$$

In the following questions you will find the form of the spectral function  $\rho(\sigma)$  for ridge regression and PCA-OLS.

(a) (5 points) Recall that the ridge regression optimizer is defined (for  $\lambda > 0$ ) as

$$\mathbf{w}_{\text{ridge}} = \arg \min_{\mathbf{w} \in \mathbb{R}^d} \|\mathbf{X}\mathbf{w} - \mathbf{y}\|_2^2 + \lambda \|\mathbf{w}\|_2^2.$$

**Show that the closed-form solution for  $\mathbf{w}_{\text{ridge}}$  has the form**

$$\mathbf{w}_{\text{ridge}} = \mathbf{V} \text{diag}(\rho_\lambda(\sigma_1), \dots, \rho_\lambda(\sigma_d)) \mathbf{U}^\top \mathbf{y},$$

**and find the ridge-regression spectral function  $\rho_\lambda$ .**

- (b) (5 points) Using the expression for  $\mathbf{w}_{\text{ridge}}$  from the previous part, **write down the ridge regression predictor function in the form of (1).**

- (c) (5 points) The ordinary least squares problem on the reduced  $k$ -dimensional PCA feature space (PCA-OLS) can be written as

$$\tilde{\mathbf{w}}_{\text{PCA}} = \arg \min_{\mathbf{w} \in \mathbb{R}^d} \|\mathbf{X}\mathbf{V}_k \mathbf{w} - \mathbf{y}\|^2$$

where the columns of  $\mathbf{V}_k$  consist of the first  $k$  right singular vectors of  $\mathbf{X}$ . This expression embeds the raw feature vectors onto the top  $k$  principle components by the transformation  $\mathbf{V}_k^\top \mathbf{x}_i$ . Assume the PCA dimension is less than the rank of the data matrix,  $k \leq r$ .

**Write down the expression for the optimizer  $\tilde{\mathbf{w}}_{\text{PCA}} \in \mathbb{R}^k$  in terms of  $\mathbf{U}$ ,  $\mathbf{y}$  and the singular values of  $\mathbf{X}$ .**

Hint:  $k \leq r$  implies that the matrix of PCA embedded data matrix  $\mathbf{X}\mathbf{V}_k$  is full rank.

- (d) (5 points) Now, use the expression for  $\tilde{\mathbf{w}}_{\text{PCA}}$  from the previous part to **write down the predictor function in the form of (1)**. In doing so, you should **define the form of the PCA-OLS spectral function  $\rho_k$** .

- (e) (5 points) The ridge regression regularization parameter  $\lambda$  and the PCA dimension  $k$  are both hyperparameters that affect the resulting model and predictions. In practice, we would tune these parameters based on the dataset we were given. **Briefly describe a principled method for choosing  $\lambda$** .



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### 3 Classification (3 parts, 25 points)

- (a) (5 points) The plots below show labeled data  $\{\mathbf{x}_i\}_{i=1}^n$ , where  $\mathbf{x}_i \in \mathbb{R}^2$ . For each plot, points corresponding to  $y_i = -1$  are denoted by an O, and points corresponding to  $y_i = +1$  are denoted by an X. The origin is labeled as the point  $(0, 0)$ . Now, consider classifiers of the form

$$\phi_{\mathbf{w}}(\mathbf{x}) = \begin{cases} +1, & \mathbf{w}^\top \mathbf{x} \geq 0 \\ -1, & \mathbf{w}^\top \mathbf{x} < 0 \end{cases}$$

where  $\mathbf{w} \in \mathbb{R}^2$ .

**For each of the five plots, determine if the data can be perfectly classified by a classifier of this form.**

- **If so, draw the decision boundary of the classifier on the plot.**
- **If not, write “not separable” in the appropriate cell in the following table.**

Plot	Separable?
1	
2	
3	
4	
5	

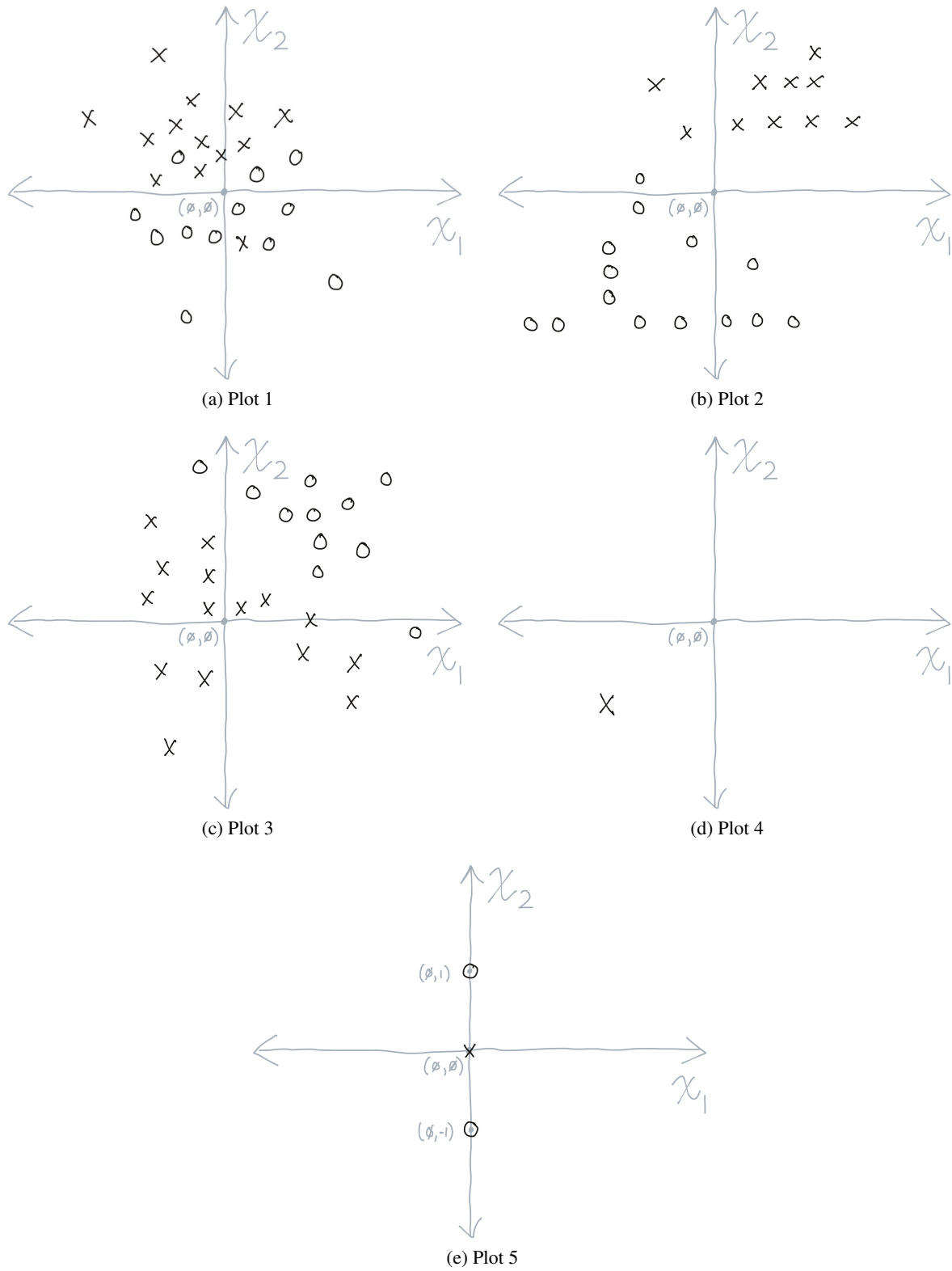


Figure 1: Problem 3(a)

(b) (10 points) Consider the data shown in Figure 2.

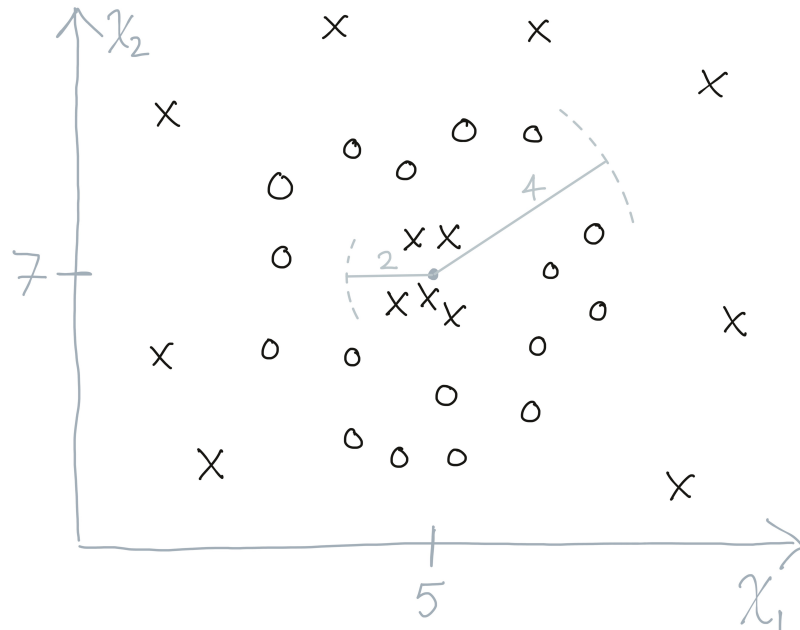


Figure 2: Problem 3(b)

Again, points corresponding to  $y_i = -1$  are denoted by an O, and points corresponding to  $y_i = +1$  are denoted by an X. Note that the O points (and only the O points) are contained between two circles of radii 2 and 4, both centered at the point (5, 7). This data can not be perfectly classified by a classifier described in the previous problem. However, we can make a nonlinear transformation of the data to make it easier to classify. Specifically, we seek a transformation  $\varphi(\mathbf{x}) : \mathbb{R}^2 \rightarrow \mathbb{R}$  such that each transformed point  $z_i = \varphi(\mathbf{x}_i)$  can be perfectly classified by a classifier  $h_b : \mathbb{R} \rightarrow \{-1, +1\}$  of the form

$$h_b(z) = \begin{cases} +1, & z \geq b \\ -1, & z < b \end{cases}.$$

- (i) **Give such a transformation  $\varphi(\mathbf{x})$ .** (You should not need to estimate exact locations of points.)
- (ii) **Plot the (nonlinear) decision boundary on the original plot** (Figure 2).
- (iii) **Plot the transformed data and the decision boundary in the transformed space  $\mathbb{R}$ , i.e. on a number line (you should have a tick mark for 0).** This plot should be qualitative to illustrate the situation; you do not need to find an explicit  $b$  for the decision boundary, nor do you need to exactly plot every transformed point.

- (c) (10 points) Now consider classifying two data points  $x_1 = (a, b)$ ,  $x_2 = (-a, -b)$ , with labels  $y_1 = +1$  and  $y_2 = -1$ , respectively, shown in Figure 3.

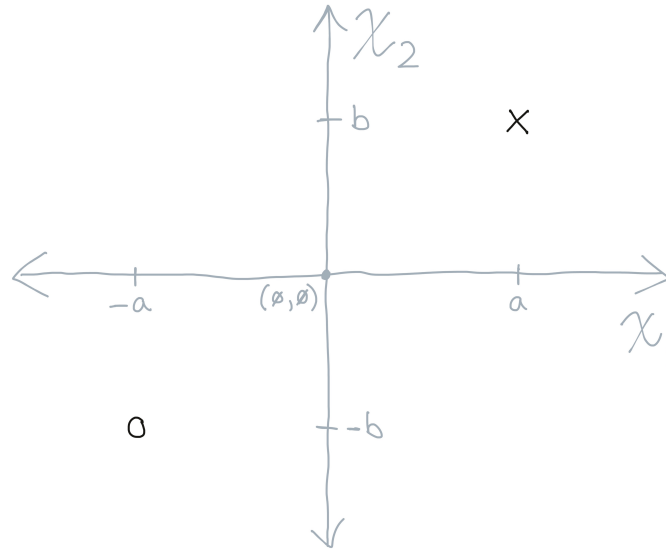


Figure 3: Problem 3(c)

**For this data, calculate the form of the maximum margin separating hyperplane which goes through the origin. Make sure you justify your answer mathematically.** Recall that for linear classifiers, the maximum margin is defined as:

$$\max_{\mathbf{w} \in \mathbb{R}^d} \min_{1 \leq i \leq n} \left( \frac{\mathbf{w}^\top \mathbf{x}_i}{\|\mathbf{w}\|_2} y_i \right)$$

Extra page. If you want the work on this page to be graded, mention it on the problem's main page.

## 4 Checking Kernels (2 parts, 10 points)

Recall that for a function  $k$  to be a valid kernel, it must be symmetric in its arguments and its Gram matrices must be positive semi-definite. More precisely, for every sample  $\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_n \in \mathbb{R}^d$ , the Gram matrix

$$\mathbf{K} = \begin{bmatrix} k(\mathbf{x}_1, \mathbf{x}_1) & \cdots & k(\mathbf{x}_1, \mathbf{x}_n) \\ \vdots & k(\mathbf{x}_i, \mathbf{x}_j) & \vdots \\ k(\mathbf{x}_n, \mathbf{x}_1) & \cdots & k(\mathbf{x}_n, \mathbf{x}_n) \end{bmatrix}$$

must be positive semi-definite. Also, recall that a matrix is positive semi-definite if it is symmetric and all its eigenvalues are non-negative.

- (a) (5 points) **Give an example of two positive semi-definite matrices  $A_1$  and  $A_2$  in  $\mathbb{R}^{2 \times 2}$  such that  $A_1 - A_2$  is not positive semi-definite.**

As a consequence, **show that the function  $k$  defined by  $k(\mathbf{x}_i, \mathbf{x}_j) = k_1(\mathbf{x}_i, \mathbf{x}_j) - k_2(\mathbf{x}_i, \mathbf{x}_j)$  is not necessarily a kernel even when  $k_1$  and  $k_2$  are valid kernels.**

- (b) (5 points) **Show that the function  $k$  defined by  $k(\mathbf{x}_i, \mathbf{x}_j) = \|\mathbf{x}_i - \mathbf{x}_j\|_2^2$  is not a valid kernel.**

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