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Discussion #1 Solutions

Exercise 1 (Subpaces and dimensions) Consider the set $\mathcal{S}$ of points such that

$$x_1 + 2x_2 + 3x_3 = 0, \quad 3x_1 + 2x_2 + x_3 = 0.$$

Show that $\mathcal{S}$ is a subspace. Determine its dimension, and find a basis for it.

Solution 1 The set $\mathcal{S}$ is a subspace, as can be checked directly: if $x, y \in \mathcal{S}$, then for every $\lambda, \mu \in \mathbb{R}$, we have $\lambda x + \mu y \in \mathcal{S}$. To find the dimension, we solve the equation and find that any solution to the equations is of the form $x_1 = x_3$, $x_2 = -2x_3$, where x_3 is free. Hence the dimension of $\mathcal{S}$ is 1, and a basis for S is the vector $(1, -2, 1)$.

Exercise 2 (Direct sum) Find $k \in \mathbb{R}$ such that $\mathbb{R}^3 = S \oplus T$, with $S = span\left\{\begin{bmatrix} -3 \\ 4 \\ 1 \end{bmatrix}\right\}$ and $T = \{(x, y, z) \in \mathbb{R}^3 : kx + y - z = 0\}$

Solution 2 (Direct sum) $S \oplus T \rightarrow S + T = \mathbb{R}^3$ and $S \cap T = \{\vec{0}\}$

To ensure those two conditions (span $\mathbb{R}^3$ and only the zero vector in the intersection) we need a basis for T. For this we write in vector form the conditions to belong to T. We know that $z = kx + y$, then

$$\begin{gathered}
T = \{(x, y, z) : (x, y, z) = (x, y, kx + y) \text{ with } x, y \text{ free}\} \\
T = \{(x, y, z) : (x, y, z) = (x, 0, kx) + (0, y, y) \text{ with } x, y \text{ free}\} \\
T = \{(x, y, z) : (x, y, z) = (1, 0, k)x + (0, 1, 1)y \text{ with } x, y \text{ free}\} \\
T = \text{span}\{(1, 0, k), (0, 1, 1)\}
\end{gathered}$$

$S \cap T = \{\vec{0}\} \rightarrow$ vectors in basis of T and S are linearly independent. To enforce linear independence, we form a matrix using the vectors in the basis as columns. We then perform Gaussian elimination and find conditions over k so the three columns have a pivot.

$$\begin{bmatrix} -3 & 0 & 1 \\ 4 & 1 & 0 \\ 1 & 1 & k \end{bmatrix} \sim \begin{bmatrix} 0 & 3 & 1+k \\ 0 & -3 & -4k \\ 1 & 1 & k \end{bmatrix} \sim \begin{bmatrix} 0 & 0 & 1-k \\ 0 & -3 & -4k \\ 1 & 1 & k \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & k \\ 0 & -3 & -4k \\ 0 & 0 & 1-k \end{bmatrix}$$

To force a pivot in the third column, $1 - k \neq 0$. Thus $k \neq 1$.

Exercise 3 (Extrema of inner product over a ball) Let $y \in \mathbb{R}^n$ be a given non-null vector, and let $\mathcal{X} = \{x \in \mathbb{R}^n : \|x\|_2 \leq r\}$, where r is some given positive number.

1. Determine the optimal value p_1^* and the optimal set (i.e., the set of *all* optimal solutions) of the problem $\min_{x\in\mathcal{X}} |y^\top x|$.
2. Determine the optimal value p_2^* and the optimal set of the problem $\max_{x\in\mathcal{X}} |y^\top x|$.
3. Determine the optimal value p_3^* and the optimal set of the problem $\min_{x\in\mathcal{X}} y^\top x$.
4. Determine the optimal value p_4^* and the optimal set of the problem $\max_{x\in\mathcal{X}} y^\top x$.

Solution 3 (Extrema of inner product over a ball) The problems are related to the extrema of the inner product $y^\top x$ (or its absolute value) over an Euclidean ball of radius r. From the Cauchy-Schwartz inequality we know that, for fixed norm of x, the amplitude of $y^\top x$ is maximized when $x = \alpha y$, while $y^\top x = 0$ if x is orthogonal to y. Let us denote with $V \in \mathbb{R}^{n,n-1}$ a matrix containing by columns an orthonormal basis for the subspace orthogonal to y.

1. Clearly, the minimum value of $\min_{x\in\mathcal{X}} |y^\top x|$ is $p_1^* = 0$. This value is attained either by $x = 0$, or by any vector $x \in \mathcal{X}_r$ orthogonal to y. The optimal set is therefore the set

   $$\mathcal{X}_{\text{opt}} = \{x : x = Vz, \ \|z\|_2 \leq r\}$$

   (note that we used the fact that the ℓ_2 norm is invariant by orthogonal transformations, i.e., $\|x\|_2 = \|Vz\|_2 = \|z\|_2$, for matrices V with orthonormal columns).
2. The optimal value $\max_{x\in\mathcal{X}} |y^\top x|$ is attained for any $x = \alpha y$ with $\|x\|_2 = r$, thus for $|\alpha| = r/\|y\|_2$, for which we have $p_2^* = r\|y\|_2$. The optimal set contains two points:

   $$\mathcal{X}_{\text{opt}} = \{x : x = \alpha y, \ \alpha = \pm\frac{r}{\|y\|_2}\}.$$

3. For problem $\min_{x\in\mathcal{X}} y^\top x$ we have $p_3^* = -r\|y\|_2$, which is attained at the unique optimal point $x^* = -\frac{r}{\|y\|_2}y$.
4. For problem $\max_{x\in\mathcal{X}} y^\top x$ we have $p_4^* = r\|y\|_2$, which is attained at the unique optimal point $x^* = \frac{r}{\|y\|_2}y$.

Exercise 4 (Inner product) Let $x, y \in \mathbb{R}^n$. Under which condition on $\alpha \in \mathbb{R}^n$ does the function

$$f(x, y) = \sum_{k=1}^{n} \alpha_k x_k y_k$$

define an inner product on $\mathbb{R}^n$?

Solution 4 The axioms for inner product are all satisfied for any $\alpha \in \mathbb{R}^n$, except the conditions

$$f(x,x) \geq 0;$$
$$f(x,x) = 0 \text{ if and only if } x = 0.$$

These properties hold if and only if $\alpha_k > 0$, $k = 1, \ldots, n$. Indeed, if the latter is true, then the above two conditions hold. Conversely, if if there exist k such that $\alpha_k \leq 0$, setting $x = e_k$ (the k-th unit vector in $\mathbb{R}^n$) produces $f(e_k, e_k) \leq 0$; this contradicts one of the two above conditions.