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# Discussion #4 Solutions

**Exercise 1 (Minimizing a quadratic function)** Consider the *unconstrained* optimization problem

$$p^* = \min_x \frac{1}{2} x^\top Q x - c^\top x$$

where $Q = Q^\top \in \mathbb{R}^{n,n}$, $Q \succeq 0$, and $c \in \mathbb{R}^n$ are given. The goal of this exercise is to determine the optimal value $p^*$ and the set of optimal solutions, $\mathcal{X}^{\text{opt}}$, in terms of $c$ and the eigenvalues and eigenvectors of the (symmetric) matrix $Q$.

1. Assume that $Q \succ 0$. Show that the optimal set is a singleton, and that $p^*$ is finite. Determine both in terms of $Q, c$.
2. Assume from now on that $Q$ is not invertible. Assume further that $Q$ is diagonal: $Q = \text{diag}(\lambda_1, \dots, \lambda_n)$, with $\lambda_1 \geq \dots \geq \lambda_r > \lambda_{r+1} = \dots = \lambda_n = 0$, where $r$ is the rank of $Q$ ($1 \leq r < n$). Solve the problem in that case.
3. Now we do not assume that $Q$ is diagonal anymore. Under what conditions (on $Q, c$) is the optimal value finite? Make sure to express your result in terms of $Q$ and $c$, as explicitly as possible.

**Solution 1**

1. When $Q \succ 0$, it admits a Cholesky decomposition $Q = R^\top R$, with $R$ upper-triangular and invertible. We can define the new variable $\bar{x} = Rx$, which leads to the problem

   $$\min_{\bar{x}} \frac{1}{2} \bar{x}^\top \bar{x} - \bar{c}^\top \bar{x},$$

   where $\bar{c} = (R^{-1})^\top c$. We can express the objective in the above problem as

   $$\frac{1}{2} \|\bar{x} - \bar{c}\|_2^2 - \|\bar{c}\|_2^2,$$

   from which it is clear that the unique minimizer is $\bar{x} = \bar{c}$. In terms of the $x$-variable, the unique solution is $x = R^{-1}\bar{c} = Q^{-1}c$.

   The same result can be obtained by invoking the fact that the minimizers of a convex differentiable function $f$ without any constraints, are characterized by the optimality condition $\nabla f(x) = 0$. In our case, we have $\nabla f(x) = Qx - c$.

2. The objective function writes

$$f(x) = \sum_{i=1}^{r} \left( \frac{1}{2}\lambda_i x_i^2 - c_i x_i \right) + \sum_{i=r+1}^{n} c_i x_i.$$

If any element $c_i$, $i = r+1, \ldots, n$, is non-zero, the optimal value is $-\infty$. Otherwise, that is, when $c$ is in the range of $Q$, the optimal value is obtained with $x_i = c_i/\lambda_i$, $i = 1, \ldots, r$, and the other variables $x_{r+1}, \ldots, x_n$ free. That value is

$$p^* = -\frac{1}{2} \sum_{i=1}^{r} \frac{c_i^2}{\lambda_i}.$$

3. We use the eigenvalue decomposition of $Q$: $Q = U\Lambda U^\top$, with $\Lambda = \mathrm{diag}\,(\lambda_1, \ldots, \lambda_n)$. The problem is more conveniently formulated in terms of the new variable $\bar{x} = U^\top x$:

$$p^* = \min_{\bar{x}} \ \frac{1}{2}\bar{x}^\top \Lambda \bar{x} - \bar{c}^\top \bar{x},$$

with $\bar{c} \doteq Uc$.

Assuming as before $\lambda_1 \geq \ldots \geq \lambda_r > \lambda_{r+1} = \ldots = \lambda_n = 0$, where $r$ is the rank of $Q$ $(1 \leq r < n)$, we are lead to the similar conclusions. In particular, the optimal value is finite if and only if the last $n - r$ components of $\bar{c}$ are zero. This means that $c$ must be in the range of $Q$, in order for the value to be finite.

**Exercise 2 (Schur complement. Bonus problem, only solved in discussion.)** Let $A \in \mathbb{R}^{p\times p}$, $C = C^\top \in \mathbb{R}^{q\times q}$, $C$ invertible, $B \in \mathbb{R}^{p\times q}$ and $p + q = n$. Let

$$M = \begin{bmatrix} A & B \\ B^\top & C \end{bmatrix}$$

1. Prove

$$M = \begin{bmatrix} A & B \\ B^\top & C \end{bmatrix} = \begin{bmatrix} I & BC^{-1} \\ 0 & I \end{bmatrix} \begin{bmatrix} A - BC^{-1}B^\top & 0 \\ 0 & C \end{bmatrix} \begin{bmatrix} I & BC^{-1} \\ 0 & I \end{bmatrix}^\top$$

2. Prove that $C \succ 0$ and $A - BC^{-1}B^\top \succ 0 \rightarrow \ M \succ 0$

Some extra solved problems:

**Exercise 3 (Orthogonal Matrix)** Consider the matrix

$$A = \begin{bmatrix} -1 & 4 & -6 \\ 2 & -2 & -6 \\ 2 & 4 & 3 \end{bmatrix}.$$

1. Show that the columns of $A = [a_1, a_2, a_3]$ are mutually orthogonal, that is, $a_i^T a_j = 0$ when $i \neq j$.

2. Show that we can write $A = BD$, with $B$ a matrix with mutually orthogonal columns, each having unit Euclidean norm; and $D$ a diagonal matrix with positive diagonal elements. *Hint:* for every $i = 1, 2, 3$, express $a_i$ as $d_i b_i$, with scalar $d_i > 0$ and $b_i$ a unit-norm vector.

3. Find a singular value decomposition of $A$. Is it unique?

**Solution 2**

1. The columns are easily shown to be mutually orthogonal.

2. We have $a_i = d_i b_i$, where $d_i = \|a_i\|_2$, $b_i = a_i / \|a_i\|_2$, $i = 1, 2, 3$. Thus $A = BD$, with $D = \mathrm{diag}(d_1, d_2, d_3)$, and $B = [b_1, b_2, b_3]$.

3. We note that $B$ is a square matrix with orthonormal columns, hence $B^T B = I$. In addition $D$ is diagonal and positive-definite. Hence an SVD of $A$ is $A = U\Sigma V^T$, with $U = B$, $\Sigma = D$, and $V = I$. As usual, there is no unicity: we can also choose $V = -I$, $U = -B$.

**Exercise 4** A matrix $A \in \mathbb{R}^{m,n}$ with rank $r$ has singular values $\sigma_1 > \sigma_2 > \ldots > \sigma_r > 0$. Prove that the spectral norm satisfies $||A||_2^2 = \sigma_1^2$

**Solution 3** Note that $||Ax||_2^2 = x^\top A^\top Ax$. Since $A^\top A$ is a positive semidefinite matrix, then it can be decomposed in a basis of orthonormal eigenvectors $\{v_i\}_{i=1}^n$ with associated eigenvalue $\lambda_i$. This entails that every vector $x$ can be decomposed as:

$$x = \sum_{i=1}^{n} \alpha_i v_i$$

Then:

$$\begin{aligned}
||Ax||_2^2 &= x^\top A^\top Ax \\
&= \langle x, A^\top Ax\rangle \\
&= \left\langle \sum_{i=1}^{n} \alpha_i v_i, A^\top A \sum_{i=1}^{n} \alpha_i v_i \right\rangle \\
&= \left\langle \sum_{i=1}^{n} \alpha_i v_i, \sum_{k=1}^{n} \alpha_k \lambda_k v_k \right\rangle \\
&= \sum_{k=1}^{n} \alpha_k \alpha_k \lambda_k \langle v_k, v_k\rangle \\
&= \sum_{k=1}^{n} \lambda_k |\alpha_k|^2
\end{aligned}$$

Note that the fourth line occurs due to the orthogonality of the basis. Then:

$$\begin{aligned}
||Ax||_2^2 &= \sum_{k=1}^{n} \lambda_k |\alpha_k|^2 \\
&\leq \lambda_{max} \sum_{k=1}^{n} |\alpha_k|^2 \\
&\leq \lambda_{max} ||x||_2^2
\end{aligned}$$

This implies that:

$$||A||_2 = \sup_{x \neq 0} \frac{||Ax||_2}{||x||_2} \leq \sqrt{\lambda_{max}}$$

By picking $x = v_{max}$, the eigenvector that has the maximum eigenvalue associated, we got that:

$$\frac{||Av_{max}||_2}{||v_{max}||_2} = \sqrt{\lambda_{max}} \frac{||v_{max}||_2}{||v_{max}||_2} = \sqrt{\lambda_{max}}$$

Then since the induced norm is the supremum of that quotient, it is clear that $||A||_{i,2} \geq \sqrt{\lambda_{max}}$. So from both inequalities it is straightforward that the supremum is attained at the eigenvector with the maximum eigenvalue, and hence:

$$||A||_2 = \sqrt{\lambda_{max}(A^\top A)} = \sigma_1(A)$$

the largest singular value of $A$. Squaring the equality, the result holds.

**Exercise 5 (Properties of dyad.)** Let $x, y \in \mathbb{R}^n$, both not identical to the zero vector, and $A = xy^\top \in \mathbb{R}^{n,n}$.

1. Determine an eigenvalue and an eigenvector of $A$.
2. We know that $A$ has rank one. Write a proof of this fact.
3. What is the dimension of $\mathcal{N}(A)$?
4. Compute a singular value decomposition of $A$ and write it in compact form.

**Solution 4**

1. Eigenvalue $\lambda = y^\top x$ and eigenvector $u = x$ work because, $Au = xy^\top x = u\lambda$.
2. $\mathcal{R}(A) = \{z \in \mathbb{R}^n \; : \; z = Av, v \in \mathbb{R}^n\}$. Since $Av = xy^\top v = \gamma x$ for $\gamma = y^\top v$, the range of $A$ is simply a line. Thus, there is only one linearly independent column in $A$.
3. The dimension of $\mathcal{N}(A) = n - \operatorname{rank} A = n - 1$ by the fundamental theorem of linear algebra.
4. Take $\sigma = \|x\|_2 \|y\|_2$, $u = x/\|x\|_2$, and $v = y/\|y\|_2$. Clearly, $A = \sigma uv^\top$. Moreover, $u^\top u = 1$, $v^\top v = 1$, $Av = xy^\top y/\|y\|_2 = \sigma u$, and $u^\top A = x^\top xy^\top/\|x\|_2 = \sigma v$. Thus, $\sigma, u, v$ is a SVD of $A$.