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Discussion #2 Solutions

Exercise 1 (Eigenvalues) Let $A \in \mathbb{R}^{n,n}$ and $B = A^2 + I$.

1. Prove that if λ is an eigenvalue of A then $\lambda^2 + 1$ is an eigenvalue of B.
2. Prove that if A has an eigenvalue decomposition, then B has one as well.

Solution 1

1. In the review we proved that if λ is an eigenvalue of A , then $\left|A - \lambda I\right| = 0$, where $\left|A - \lambda I\right|$ denotes the determinant of the matrix $A - \lambda I$. Thus, we want to prove $\left|B - (\lambda^2 + 1)I\right| = 0$.

$$\begin{aligned}
\left|B - (\lambda^2 + 1)I\right| &= \left|A^2 + I - (\lambda^2 + 1)I\right| \\
&= \left|A^2 + I - \lambda^2 I - I\right| \\
&= \left|A^2 - \lambda^2 I\right| \\
&= \left|(A - \lambda I)(A + \lambda I)\right| \\
&= \left|(A - \lambda I)\right| \left|(A + \lambda I)\right| \\
&= 0 \left|(A + \lambda I)\right| \\
&= 0
\end{aligned}$$

   Therefore, $(\lambda^2 + 1)$ is an eigenvalue of B.

2. A has an eigenvalue decomposition $\rightarrow A = VDV^{-1}$, where $D = \mathrm{diag}(\lambda_i)$ with λ_i eigenvalues of A, and V is a matrix with the eigenvectors of A as columns.

$$
\begin{aligned}
B &= A^2 + I \\
&= (VDV^{-1})^2 + I \\
&= VDV^{-1}VDV^{-1} + I \\
&= VDDV^{-1} + I \\
&= VD^2V^{-1} + I \\
&= VD^2V^{-1} + VV^{-1} \\
&= V(D^2 + I)V^{-1}
\end{aligned}
$$

Thus, B has the same eigenvectors as A (columns of the matrix V), and the diagonal matrix of B is $D^2 + I$, where D is the diagonal matrix with the eigenvalues of A. Therefore, the eigenvalues of B will be $\lambda_i^2 + 1$, with λ_i eigenvalue of A.

Exercise 2 (Eigenvectors of a symmetric matrix) Let $p, q \in \mathbb{R}^n$ be two linearly independent vectors, with unit norm ($\|p\|_2 = \|q\|_2 = 1$). Define the symmetric matrix $A \doteq pq^\top + qp^\top$. In your derivations, it may be useful to use the notation $c \doteq p^\top q$.

1. Show that $p + q$ and $p - q$ are eigenvectors of A, and determine the corresponding eigenvalues.
2. Determine the nullspace and rank of A.
3. Find an eigenvalue decomposition of A, in terms of p, q. *Hint:* use the previous two parts.
4. What is the answer to the previous part if p, q are not normalized?

Solution 2

1. We have
$$Ap = (cp + q), \quad Aq = p + cq,$$
from which we obtain
$$A(p - q) = (c - 1)(p - q), \quad A(p + q) = (c + 1)(p + q).$$
Thus $u_\pm := p \pm q$ is an (un-normalized) eigenvector of A, with eigenvalue $c \pm 1$.

2. The condition on $x \in \mathbb{R}^n$: $Ax = 0$, holds if and only if

$$0 = (q^\top x)p + (p^\top x)q = 0.$$

Since p, q are linearly independent, the above is equivalent to $p^\top x = q^\top x = 0$. The nullspace is the set of vectors orthogonal to p and q. The range is the span of p, q. The rank is thus 2.

3. Since the rank is 2, there is a total of two non-zero eigenvalues. Note that, since p, q are normalized, c is the cosine angle between p, q; $|c| < 1$ since p, q are independent. We have found two linearly independent eigenvectors $u_\pm = p \pm q$ that do not belong to the nullspace (since $|c| < 1$). We can complete this set with eigenvectors corresponding to the eigenvalue zero; simply choose an orthonormal basis for the nullspace.

   Then, the eigenvalue decomposition is

$$A = (c-1)v_- v_-^\top + (c+1)v_+ v_+^\top,$$

   where $v_\pm$ are the normalized vectors $v_\pm = u_\pm / \|u_\pm\|_2$. We have

$$v_\pm = \frac{1}{\sqrt{2(1 \pm c)}}(p \pm q),$$

   so that the eigenvalue decomposition amounts to the trivial identity

$$A = \frac{1}{2}\left((p+q)(p+q)^\top - (p-q)(p-q)^\top\right).$$

4. We can always scale the matrix: with $\bar{p} = p/\|p\|_2$, $\bar{q} = q/\|q\|_2$, we have

$$A = \|p\|_2 \|q\|_2 \left(\bar{p}\bar{q}^\top + \bar{q}\bar{p}^\top\right).$$

   The eigenvalues are scaled accordingly: with $c = (p/\|p\|_2)^\top (q/\|q\|_2)$,

$$\lambda_\pm = \|p\|_2 \|q\|_2 (c \pm 1) = p^\top q \pm \|p\|_2 \|q\|_2.$$

   Note that, since p, q are independent, none of these eigenvalues is zero; one is positive, the other negative. This is due to the Cauchy-Schwartz inequality, which says that $|p^\top q| \le \|p\|_2 \|q\|_2$, with equality if and only if p, q are linearly dependent.

   The corresponding (un-normalized) eigenvectors are $\bar{p} \pm \bar{q}$. The eigenvalue decomposition obtained before leads to

$$A = \frac{\|p\|_2 \|q\|_2}{2}\left((\bar{p}+\bar{q})(\bar{p}+\bar{q})^\top - (\bar{p}-\bar{q})(\bar{p}-\bar{q})^\top\right).$$

Exercise 3 (Maximum singular value) Prove $\max_{\|u\|_2=1} \|Au\|_2 = \sigma_1(A)$, where $\sigma_1(A)$ is the maximum singular value of A. This problem will be solved in discussion #3.