EE 127AT/227AT 9/12/2018

Instructors: L. El Ghaoui, A. Bayen

GSI: P. Hidalgo-Gonzalez

Discussion # 3 Solutions

Exercise 1 (Maximum singular value) Prove $\max_{\|u\|_2=1} \|Au\|_2 = \sigma_1(A)$, where $\sigma_1(A)$ is the maximum singular value of A.

Solution 1 Any matrix A has a singular value decomposition, $A = USV^\top$. U and V are orthogonal matrices, meaning: $U^\top U = I$ and $V^\top V = I$. We replace A with its decomposition in the objective function:

$$\begin{aligned}
\max_{\|u\|_2=1} \|Au\|_2 &= \max_{\|u\|_2=1} \sqrt{(Au)^\top (Au)} \\
&= \max_{\|u\|_2=1} \sqrt{(USV^\top u)^\top (USV^\top u)} \\
&= \max_{\|u\|_2=1} \sqrt{u^\top VS^\top U^\top USV^\top u} \\
&= \max_{\|y\|_2=1} \sqrt{y^\top \mathrm{diag}(\sigma_1^2, \cdots, \sigma_r^2, 0, \cdots, 0) y}
\end{aligned}$$

where $y = V^\top u$, and as we proved in discussion, $||y||_2^2 = y^\top y = u^\top u = ||u||_2^2$.

$$\begin{aligned}
&= \max_{\|y\|_2=1} \sqrt{\sigma_1^2 y_1^2 + \cdots + \sigma_r^2 y_r^2 + 0 \cdots + 0} \\
&= \sqrt{\sigma_1^2} \\
&= \sigma_1 \\
&\square
\end{aligned}$$

Exercise 2 (Frobenius norm and Least Squares) Let $A \in \mathbb{R}^{m \times n}$, consider the optimization problem given by $\min_X \|AX - I_m\|_F$, where the variable is $X \in \mathbb{R}^{n \times m}$, I_m is the $m \times m$ identity matrix, and $\| \cdot \|_F$ is the Frobenius norm.

1. Show that the problem can be reduced to a number of ordinary least squares problems. How do you recover X?

2. Show that when A is full column rank, then the optimal solution is unique, and given by $X^* = (A^T A)^{-1} A^T$.

Solution 2

Reminder: $||A||_F = \sqrt{\sum_i \sum_j a_{ij}^2} = \sqrt{\mathrm{tr}(A^\top A)}$, where a_{ij} are the components of the matrix A.

1.

$$\begin{aligned}
\arg\min_X \|AX - I_m\|_F &= \arg\min_X \|AX - I_m\|_F^2 \\
&= \arg\min_{x_i} \|A[x_1 \cdots x_m] - [e_1 \cdots e_m]\|_F^2 \\
&= \arg\min_{x_i} \|[Ax_1 - e_1 \cdots Ax_m - e_m]\|_F^2 \\
&= \arg\min_{x_i} \|[Ax_1 - e_1 \cdots Ax_m - e_m]\|_F^2 \\
&= \arg\min_{x_i} \|Ax_1 - e_1\|_2^2 + \cdots + \|Ax_m - e_m\|_2^2 \\
&= \arg\min_{x_i} \sum_{i=1}^{m} \|Ax_i - e_i\|_2^2
\end{aligned}$$

Each x_i is a column of the original matrix X, thus $X^* = [x_1^* \cdots x_m^*]$.

2.

$$\begin{aligned}
\min_{x_i} \sum_{i=1}^{m} \|Ax_i - e_i\|_2^2 &= \min_{x_i} (Ax_i - e_i)^\top (Ax_i - e_i) \\
&= \min_{x_i} x_i^\top A^\top A x_i - x_i^\top A^\top e_i - e_i^\top A x_i + e_i^\top e_i \\
&= \min_{x_i} x_i^\top A^\top A x_i - 2e_i^\top A x_i + 1
\end{aligned}$$

We know that least-squares are convex problems, so we can take the gradient of the objective function and make it equal to the zero. From there we can find optimal x_i^*.

$$\begin{aligned}
\nabla f(x_i) &= 0 \\
x_i^\top (A^\top A + (A^\top A)^\top) - 2e_i^\top A &= 0 \\
x_i^\top (A^\top A + A^\top A) - 2e_i^\top A &= 0 \\
2x_i^\top A^\top A - 2e_i^\top A &= 0 \\
x_i^* &= (A^\top A)^{-1} A^\top e_i
\end{aligned}$$

We used the fact that $A^\top A$ is positive definite when A is full column rank (proved in discussion). This implies $A^\top A$ is invertible. We notice that x_i^* corresponds to the i-th column of the matrix $(A^\top A)^{-1}A^\top$ (because $(A^\top A)^{-1}A^\top$ is being multiplied by e_i).

$$\therefore X^* = (A^\top A)^{-1}A^\top$$

Exercise 3 (Null space) Let $A \in \mathbb{R}^{n\times m}$, prove that $\mathcal{N}(AA^T) = \mathcal{N}(A^T)$.

Solution 3

Let $v \in \mathcal{N}(A^\top)$

$$\begin{aligned} &\rightarrow A^\top v = 0 \\ &AA^\top v = A0 = 0 \\ &AA^\top v = 0 \\ &\rightarrow v \in \mathcal{N}(A^\top A) \\ \rightarrow \mathcal{N}(A^\top) &\subseteq \mathcal{N}(A^\top A) \end{aligned}$$

(Which is true because we proved that any element v in $\mathcal{N}(A^\top)$ will also belong to $\mathcal{N}(A^\top A)$).

Let $v \in \mathcal{N}(A^\top A)$

$$\begin{aligned} \rightarrow A^\top Av &= 0 \\ v^\top AA^\top v &= v^\top 0 \\ v^\top AA^\top v &= 0 \\ (A^\top v)^\top(A^\top v) &= 0 \\ ||A^\top v||_2^2 &= 0 \\ &\rightarrow v \in \mathcal{N}(A^\top) \\ \rightarrow \mathcal{N}(A^\top A) &\subseteq \mathcal{N}(A^\top) \end{aligned}$$

$$\therefore \mathcal{N}(A^\top A) = \mathcal{N}(A^\top)$$

Exercise 4 (Frobenius norm and trace) Let $A \in \mathbb{S}_+^n$, be a symmetric, positive semidefinite matrix. Show that trace A and Frobenius norm, $\|A\|_F$, depend only on its eigenvalues, and express both in terms of the vector of eigenvalues.

Solution 4

$A^\top = A \to A = VDV^\top$, $V^\top V = I$.

$$\begin{aligned}
\operatorname{tr}(A) &= \operatorname{tr}(VDV^\top) \\
&= \operatorname{tr}(DV^\top V) \\
&= \operatorname{tr}(D) \\
&= \lambda_1 + \cdots + \lambda_n
\end{aligned}$$

A is semipositive definite, so we know $\lambda_i \geq 0 \quad \forall i$. Thus, $\operatorname{tr}(A) = ||\lambda||_1$. To get this results we used the cyclic property of the trace.

$$\begin{aligned}
||A||_F &= \sqrt{\operatorname{tr}(A^\top A)} \\
&= \sqrt{\operatorname{tr}((VDV^\top)^\top VDV^\top)} \\
&= \sqrt{\operatorname{tr}(VD^\top V^\top VDV^\top)} \\
&= \sqrt{\operatorname{tr}(VD^2V^\top)} \\
&= \sqrt{\operatorname{tr}(D^2V^\top V)} \\
&= \sqrt{\operatorname{tr}(D^2)} \\
&= \sqrt{\sum_{i=1}^{n} \lambda_i^2} \\
&= ||\lambda||_2
\end{aligned}$$