Basics

- A square matrix $A \in \mathbb{R}^{n,n}$ is *symmetric* if it is equal to its transpose: $A = A^\top$, that is: $A_{ij} = A_{ji}$, $1 \leq i,j \leq n$.
- Elements above the diagonal in a symmetric matrix are thus identical to corresponding elements below the diagonal.
- Symmetric matrices are ubiquitous in engineering applications. They arise, for instance, in the description of graphs with undirected weighted edges between the nodes, in geometric distance arrays (between, say, cities), in defining the Hessian of a nonlinear function, in describing the covariances of random vectors, etc.
- The following is an example of a 3×3 symmetric matrix:

$$A = \begin{bmatrix} 4 & 3/2 & 2 \\ 3/2 & 2 & 5/2 \\ 2 & 5/2 & 2 \end{bmatrix}.$$

- The set of symmetric $n \times n$ matrices is is a subspace of $\mathbb{R}^{n,n}$, and it is denoted with $\mathbb{S}^n$.

Basics

Example 1 (Sample covariance matrix)

- *Given m points $x^{(1)}, \dots, x^{(m)}$ in $\mathbb{R}^n$, we define the sample covariance matrix to be the $n \times n$ symmetric matrix*

$$\Sigma \doteq \frac{1}{m} \sum_{i=1}^{m} (x^{(i)} - \hat{x})(x^{(i)} - \hat{x})^\top,$$

where $\hat{x} \in \mathbb{R}^n$ is the sample average of the points: $\hat{x} \doteq \frac{1}{m} \sum_{i=1}^{m} x^{(i)}$.

- *The covariance matrix Σ is obviously a symmetric matrix. This matrix arises when computing the sample variance of the scalar products $s_i \doteq w^\top x^{(i)}$, $i = 1, \dots, m$, where $w \in \mathbb{R}^n$ is a given vector:*

$$\sigma^2 = \sum_{i=1}^{m} (w^\top x^{(i)} - \hat{s})^2 = \sum_{i=1}^{m} (w^\top (x^{(i)} - \hat{x}))^2 = w^\top \Sigma w.$$

Basics

Example 2 (Portfolio variance)

- *For n financial assets, we can define a vector $r \in \mathbb{R}^n$ whose components r_k are the rate of returns of the k-th asset, $k = 1, \dots, n$.*
- *Assume now that we have observed m samples of historical returns $r^{(i)}$, $i = 1, \dots, m$. The sample average over that history of return is $\hat{r} = (1/m)(r^{(1)} + \dots + r^{(m)})$, and the sample covariance matrix has (i, j) component given by*

$$\Sigma_{ij} = \frac{1}{m} \sum_{t=1}^{m} (r_i^{(t)} - \hat{r}_i)(r_j^{(t)} - \hat{r}_j), \ \ 1 \leq i, \ \ j \leq n.$$

- *If $w \in \mathbb{R}^n$ represents a portfolio "mix," that is $w_k \geq 0$ is the fraction of the total wealth invested in asset k, then the return of such a portfolio is given by $\rho = r^\top w$.*
- *The sample average of the portfolio return is $\hat{r}^\top w$, while the sample variance is given by $w^\top \Sigma w$.*

Basics

Example 3 (Hessian matrix)

- *The Hessian of a twice differentiable function* $f : \mathbb{R}^n \to \mathbb{R}$ *at a point* $x \in \operatorname{dom} f$ *is the matrix containing the second derivatives of the function at that point. That is, the Hessian is the matrix with elements given by*

$$H_{ij} = \frac{\partial^2 f(x)}{\partial x_i \partial x_j}, \quad 1 \leq i, \ \ j \leq n.$$

- *The Hessian of* f *at* x *is often denoted as* $\nabla^2 f(x)$.
- *Since the second-derivative is independent of the order in which derivatives are taken, it follows that* $H_{ij} = H_{ji}$ *for every pair* (i, j), *thus the Hessian is always a symmetric matrix.*

Basics

Quadratic functions

- Consider the quadratic function (a polynomial function is said to be quadratic if the maximum degree of its monomials is equal to two)

$$q(x) = x_1^2 + 2x_1x_2 + 3x_2^2 + 4x_1 + 5x_2 + 6.$$

- The Hessian of q at x is given by

$$H = \left[\frac{\partial^2 q(x)}{\partial x_i \partial x_j}\right]_{1 \le i,j \le 2} = \begin{bmatrix} \frac{\partial q}{\partial x_1^2} & \frac{\partial^2 q}{\partial x_1 \partial x_2} \\ \frac{\partial^2 q}{\partial x_2 \partial x_1} & \frac{\partial q}{\partial x_2^2} \end{bmatrix} = \begin{bmatrix} 2 & 2 \\ 2 & 6 \end{bmatrix}.$$

- The monomials in $q(x)$ of degree two can also be written compactly as

$$x_1^2 + 2x_1x_2 + 3x_2^2 = \frac{1}{2}x^\top Hx.$$

- Any quadratic function can be written as the sum of a quadratic term involving the Hessian, and an affine term:

$$q(x) = \frac{1}{2}x^\top Hx + c^\top x + d, \quad c^\top = [4 \;\; 5], \; d = 6.$$

The spectral theorem

Any symmetric matrix is orthogonally similar to a diagonal matrix. This is stated in the following so-called *spectral theorem* for symmetric matrices.

Theorem 1 (Spectral Theorem)

Let $A \in \mathbb{R}^{n,n}$ be symmetric, let $\lambda_i \in \mathbb{R}$, $i = 1, \ldots, n$, be the eigenvalues of A (counting multiplicities). Then, there exist a set of orthonormal vectors $u_i \in \mathbb{R}^n$, $i = 1, \ldots, n$, such that $Au_i = \lambda_i u_i$. Equivalently, there exist an orthogonal matrix $U = [u_1 \cdots u_n]$ (i.e., $UU^\top = U^\top U = I_n$) such that

$$A = U\Lambda U^\top = \sum_{i=1}^{n} \lambda_i u_i u_i^\top, \quad \Lambda = \mathrm{diag}\,(\lambda_1, \ldots, \lambda_n).$$

Variational characterization of eigenvalues

- Since the eigenvalues of $A \in \mathbb{S}^n$ are real, we can arrange them in decreasing order:

$$\lambda_{\max}(A) = \lambda_1(A) \geq \lambda_2(A) \geq \cdots \geq \lambda_n(A) = \lambda_{\min}(A).$$

- The extreme eigenvalues can be related to the minimum and the maximum attained by the quadratic form induced by A over the unit Euclidean sphere.
- For $x \neq 0$ the ratio

$$\frac{x^\top A x}{x^\top x}$$

is called a *Rayleigh quotient*.

Variational characterization of eigenvalues

Theorem 2 (Rayleigh quotients)

Given $A \in \mathbb{S}^n$, it holds that

$$\lambda_{\min}(A) \leq \frac{x^\top A x}{x^\top x} \leq \lambda_{\max}(A), \quad \forall x \neq 0.$$

Moreover,

$$\begin{aligned} \lambda_{\max}(A) &= \max_{x:\, \|x\|_2=1} x^\top A x \\ \lambda_{\min}(A) &= \min_{x:\, \|x\|_2=1} x^\top A x, \end{aligned}$$

and the maximum and minimum are attained for $x = u_1$ and for $x = u_n$, respectively, where u_1 (resp. u_n) is the unit-norm eigenvector of A associated with its largest (resp. smallest) eigenvalue of A.

Matrix gain

- Given a matrix $A \in \mathbb{R}^{m,n}$, let us consider the linear function associated to A, which maps input vectors $x \in \mathbb{R}^n$ to output vectors $y \in \mathbb{R}^m$:

$$y = Ax.$$

- Given a vector norm, the matrix *gain*, or operator norm, is defined as the maximum value of the ratio $\|Ax\|/\|x\|$ between the size (norm) of the output and the of the input.
- In particular, the gain with respect to the Euclidean norm is defined as

$$\|A\|_2 = \max_{x \neq 0} \frac{\|Ax\|_2}{\|x\|_2},$$

  and it is often referred to as the *spectral* norm of A.
- The square of the input-output ratio in the Euclidean norm is

$$\frac{\|Ax\|_2^2}{\|x\|_2^2} = \frac{x^\top (A^\top A) x}{x^\top x}$$

Matrix gain

- This quantity is upper and lower bounded by the maximum and by the minimum eigenvalue of the symmetric matrix $A^\top A \in \mathbb{S}^n$, respectively:

$$\lambda_{\min}(A^\top A) \leq \frac{\|Ax\|_2^2}{\|x\|_2^2} \leq \lambda_{\max}(A^\top A)$$

- The upper and lower bounds are actually attained when x is equal to an eigenvector of $A^\top A$ corresponding respectively to the maximum and to minimum eigenvalue of $A^\top A$. Therefore,

$$\|A\|_2 = \max_{x \neq 0} \frac{\|Ax\|_2}{\|x\|_2} = \sqrt{\lambda_{\max}(A^\top A)},$$

where this maximum gain is obtained for x along the direction of eigenvector u_1 of $A^\top A$, and

$$\min_{x \neq 0} \frac{\|Ax\|_2}{\|x\|_2} = \sqrt{\lambda_{\min}(A^\top A)},$$

where this minimum gain is obtained for x along the direction of eigenvector u_n of $A^\top A$.

Positive-semidefinite matrices

- A symmetric matrix $A \in \mathbb{S}^n$ is said to be *positive semidefinite* (PSD) if the associated quadratic form is nonegative, i.e.,

$$\underline{x^\top A x \geq} 0, \quad \forall x \in \mathbb{R}^n.$$

$x^T y$

- If, moreover,

$$x^\top A x > 0, \quad \forall 0 \neq x \in \mathbb{R}^n,$$

  then A is said to be *positive definite*. To denote a symmetric positive semidefinite (resp. positive definite) matrix, we use the notation $A \succeq 0$ (resp. $A \succ 0$).

- We say that A is negative semidefinite, written $A \preceq 0$, if $-A \succeq 0$, and likewise A is negative definite, written $A \prec 0$, if $-A \succ 0$.

- It is immediate to see that a positive semidefinite matrix is actually positive definite if and only if it is invertible.

- It holds that

$$\begin{cases} A \succeq 0 \quad \Leftrightarrow \quad \lambda_i(A) \geq 0, \; i = 1, \ldots, n \\ A \succ 0 \quad \Leftrightarrow \quad \lambda_i(A) > 0, \; i = 1, \ldots, n. \end{cases}$$

Congruence transformations

Corollary 1

For any matrix $A \in \mathbb{R}^{m,n}$ it holds that:

1. $A^\top A \succeq 0$, *and* $AA^\top \succeq 0$;
2. $A^\top A \succ 0$ *if and only if A is full-column rank, i.e.,* $\operatorname{rank} A = n$;
3. $AA^\top \succ 0$ *if and only if A is full-row rank, i.e.,* $\operatorname{rank} A = m$.

Matrix square-root and Cholesky decomposition

- Let $A \in \mathbb{S}^n$. Then

$$A \succeq 0 \quad \Leftrightarrow \quad \exists\, B \succeq 0 : \; A = B^2$$
$$A \succ 0 \quad \Leftrightarrow \quad \exists\, B \succ 0 : \; A = B^2.$$

- Matrix $B = A^{1/2}$ is called the *matrix square-root* of A.
- Any $A \succeq 0$ admits the spectral factorization $A = U\Lambda U^\top$, with U orthogonal and $\Lambda = \mathrm{diag}\,(\lambda_1, \ldots, \lambda_n)$, $\lambda_i \geq 0$, $i = 1, \ldots, n$. Defining $\Lambda^{1/2} = \mathrm{diag}\left(\sqrt{\lambda_1}, \ldots, \sqrt{\lambda_1}\right)$ and $B = U\Lambda^{1/2}U^\top$:

$$A \succeq 0 \quad \Leftrightarrow \quad \exists\, B : \; A = B^\top B$$
$$A \succ 0 \quad \Leftrightarrow \quad \exists\, B \text{ nonsingular} : \; A = B^\top B.$$

- A is positive definite if and only if it is congruent to the identity.

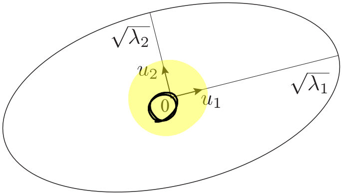
Positive-definite matrices and ellipsoids

- Positive-definite matrices are intimately related to geometrical objects called *ellipsoids*.
- A full-dimensional, bounded ellipsoid with center in the origin can indeed be defined as the set

$$\mathcal{E} = \{x \in \mathbb{R}^n : \; x^\top P^{-1} x \le 1\}, \quad P \succ 0.$$

- The eigenvalues λ_i and eigenvectors u_i of P define the orientation and shape of the ellipsoid: u_i the directions of the semi axes of the ellipsoid, while their lengths are given by $\sqrt{\lambda_i}$.
- Using the Cholesky decomposition $P^{-1} = A^\top A$, the previous definition of ellipsoid $\mathcal{E}$ is also equivalent to: $\mathcal{E} = \{x \in \mathbb{R}^n : \; \|Ax\|_2 \le 1\}$.

$x^\top A^\top A x$



The PSD cone and partial order

- The set of positive semidefinite matrices $\mathbb{S}^n_+$ is a *convex cone*.
- First, it is a *convex* set, since it satisfies the defining property of convex sets (more on this later!), that is for any two matrices $A_1, A_2 \in \mathbb{S}^n_+$ and any $\theta \in [0,1]$, it holds that
$$x^\top(\theta A_1 + (1-\theta)A_2)x = \theta x^\top A_1 x + (1-\theta)x^\top A_2 x \geq 0, \ \forall x,$$
hence $\theta A_1 + (1-\theta)A_2 \in \mathbb{S}^n_+$.
- Moreover, for any $A \succeq 0$ and any $\alpha \geq 0$, we have that $\alpha A \succeq 0$, which says that $\mathbb{S}^n_+$ is a *cone*.
- The relation "$\succeq$" defines a partial order on the cone of PSD matrices. That is, we say that $A \succeq B$ if $A - B \succeq 0$ and, similarly, $A \succ B$ if $A - B \succ 0$. This is a *partial* order, since not any two symmetric matrices may be put in a $\preceq$ or $\succeq$ relation.