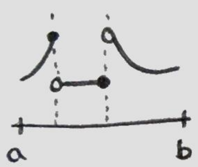
Definite Integrals
Section 5.3

$$\begin{aligned}\int_a^b f(x)\,dx &= \lim_{n\to\infty} R_n = \lim_{n\to\infty} \sum_{k=1}^{n} f(x_k)\Delta x \\ &= \lim_{n\to\infty} \sum_{k=1}^{n} f\left(a + k\frac{b-a}{n}\right)\left(\frac{b-a}{n}\right) \\ &= \lim_{||P||\to 0} \sum_{k=1}^{n} f(c_k)\Delta x_k \\ &= \mathtt{int}(f(x), x = a..b) \qquad \text{in Maple}\end{aligned}$$

Integrability Theorem If a function f is continuous over the interval $[a, b]$ or has at most finitely many jump discontinuities in $[a, b]$, then $\int_a^b f(x)\,dx$ exists, and we say that f is **integrable** over $[a, b]$. *For now we are assuming that a and b are finite.*



Power Rule of Integration

$$\int_a^b x^n\,dx = \left.\frac{x^{n+1}}{n+1}\right|_a^b = \frac{b^{n+1}}{n+1} - \frac{a^{n+1}}{n+1}$$

Other Definite Integration Rules Suppose that f and g are integrable over $[a, b]$. Then

1. $\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$
2. $\int_a^a f(x)\,dx = 0$
3. $\int_a^b kf(x)\,dx = k\int_a^b f(x)\,dx$
4. $\int_a^b (f(x) \pm g(x))\,dx = \int_a^b f(x)\,dx \pm \int_a^b g(x)\,dx$
5. $\int_a^b f(x)\,dx + \int_b^c f(x)\,dx = \int_a^c f(x)\,dx$

$\int_0^3 (x^4+2x)\,dx = \int_0^3 x^4 + 2\int_0^3 x = \left(\frac{x^5}{5}+x^2\right)\Big|_0^3 = \frac{3^5}{5}+3^2$

$\int_1^2 \left(\frac{1}{x^5}-2\right)dx = \int_1^2 x^{-5} - \int_1^2 2 = \frac{-1}{4x^4} - 2x\Big|_1^2 = \left(\frac{-1}{4(2)^4}-4\right)-\left(\frac{-1}{4}-2\right)$

Example #1: Suppose that f and g are integrable and

$$\int_1^2 f(x)\,dx = -4, \qquad \int_1^5 f(x)\,dx = 6, \qquad \int_1^5 g(x)\,dx = 8.$$

Find

(a) $\int_2^2 g(x)\,dx = 0$

(b) $\int_5^1 g(x)\,dx = -\int_1^5 g(x)\,dx = -8$

(c) $\int_1^2 3f(x)\,dx = 3\int_1^2 f(x)\,dx = -12$

(d) $\int_2^5 f(x)\,dx = \int_1^5 f(x)\,dx - \int_1^2 f(x)\,dx = 10$

(e) $\int_1^5 [4f(x) - g(x)]\,dx = 4\int_1^5 f(x)\,dx - \int_1^5 g(x)\,dx = 16$

Example #2: Evaluate the following definite integrals.

(a) $\int_0^{1/2} x^2\,dx = \frac{x^3}{3}\Big|_0^{1/2} = \frac{(\frac{1}{2})^3}{3} = \frac{1}{24}$

(b) $\int_\pi^{2\pi} \sqrt{\theta}\,d\theta = \frac{\theta^{3/2}}{3/2}\Big|_\pi^{2\pi} = \frac{2(2\pi)^{3/2}}{3} - \frac{2\pi^{3/2}}{3} = \frac{2\pi^{3/2}}{3}$

$\theta^{1/2}$

(c) $\int_3^1 \left(\frac{2}{z^3} - 4z\right)\,dz = 2\left(\frac{-1}{2z^2} - z^2\right)\Big|_3^1 = 2\left(\frac{-1}{2}-1\right)-2\left(\frac{-1}{18}-9\right) = (-1-2)-\left(\frac{-1}{9}-18\right)$

$2(z^{-3}-2z)$

$= 15 + \frac{1}{9}$

$\hookrightarrow 16 - \frac{8}{9}$?

> **Definition:** If f is integrable on $[a, b]$, then its **average value** on $[a, b]$, which is also called its **mean**, is
>
> $$\text{av}(f) = \frac{1}{b-a}\int_a^b f(x)\,dx.$$

$\int_a^b 1\,dx = x$