Substitution Method of Integration
Section 5.5

Definition: The **indefinite integral** of the function $f(x)$ with respect to x is the set of all antiderivatives of $f(x)$. This is denoted as

$$\int f(x)\,dx = F(x) + C$$

where $F(x)$ is any antiderivative of $f(x)$ (because all antiderivatives differ only by a constant).

Example #1:

(a) $\int \sin(x)\,dx$

$-\cos x + C$

(b) $\int \left(\frac{\sec^2(x)}{3}\right) dx$

$\frac{1}{3}\tan x + C$

The Substitution Rule If $u = g(x)$ is a differentiable function whose range is an interval I, and f is continuous on I, then

$$\int f(g(x))g'(x)\,dx = \int f(u)\,du.$$

Idea: The substitution method is to used to "undo" the chain rule.

Example #2:

(a) $\int \sqrt{x^2+1} \cdot 2x\,dx$

$u = x^2+1$

$du = 2x\,dx$

$\int u^{1/2}\,du = \frac{2}{3}u^{3/2} + C$

$\frac{2}{3}(x^2+1)^{3/2} + C$

(b) $\int 5\sec^2(5x+2)\,dx$

$u = 5x+2$

$du = 5\,dx$

$\int \sec^2 u\,du = \tan u + C$

$\tan(5x+2) + C$

(c) $\int xe^{x^2/2}\,dx$

$u = \frac{x^2}{2}$

$du = \frac{1}{2}2x\,dx = x\,dx$

$\int e^u\,du = e^u + C$

$e^{x^2/2} + C$

(d) $\int (3t-5)^{10}\,dt$

$u = 3t-5$

$du = 3\,dt \Rightarrow \frac{1}{3}du = dt$

$\frac{1}{3}\int u^{10}du = \frac{1}{3}\cdot\frac{u^{11}}{11} + C$

$\frac{(3t-5)^{11}}{33} + C$

Sometimes the derivative du may not be there precisely. We can (sometimes) use algebra to get an expression involving just u. Never mix x's and u's (or whatever your variable(s) are)!

Integrals of trigonometric functions

$$\int \sin(x)\,dx = -\cos(x) + C$$

$$\int \cos(x)\,dx = \sin(x) + C$$

$$\int \tan(x)\,dx = \ln|\sec(x)| + C$$

$$-\ln|\cos x| + C$$

$$\int \cot(x)\,dx = \ln|\sin(x)| + C$$

$$\int \sec(x)\,dx = \ln|\sec(x) + \tan(x)| + C$$

$$\int \csc(x)\,dx = -\ln|\csc(x) + \cot(x)| + C$$

$\int \tan x\,dx = \int \frac{\sin x}{\cos x}\,dx$

$u = \cos x$

$du = -\sin x\,dx \Rightarrow -du = \sin x\,dx$

$\int \frac{-1}{u}\,du = -\ln|u| + C$

$-\ln|\cos x| + C = \ln|\cos x|^{-1} + C = \ln|\sec x| + C$

$\int \sec x\,dx = \int \frac{\sec x(\sec x + \tan x)}{\sec x + \tan x}\,dx$

$\int \frac{\sec^2 x + \sec x \tan x}{\sec x + \tan x}\,dx$

$u = \tan x + \sec x$

$du = \sec^2 x + \sec x \tan x\,dx$

$\int \frac{1}{u}\,du = \ln|u| + C$

$\ln|\tan x + \sec x| + C$

Example #3: Evaluate the following integrals.

(a) $\int \tan(5x)\,dx$

$u = 5x$

$du = 5\,dx \Rightarrow \frac{1}{5}du = dx$

$\frac{1}{5}\int \tan u\,du = \frac{1}{5}\ln|\sec u| + C$

$\frac{1}{5}\ln|\sec(5x)| + C$

(b) $\displaystyle\int \frac{t}{t^2+1}\,dt$

$u = t^2+1$

$du = 2t\,dt \Rightarrow \frac{1}{2}du = t\,dt$

$\frac{1}{2}\int \frac{1}{u}\,du = \frac{1}{2}\ln|u| + C$

$\frac{1}{2}\ln|t^2+1| + C$

(c) $\displaystyle\int z^3\sqrt{z^2+1}\,dz$

$u = z^2+1$

$du = 2z\,dz \Rightarrow \frac{1}{2}du = z\,dz$

$\frac{1}{2}\int (u-1)u^{1/2}\,du = \frac{1}{2}\int u^{3/2} - u^{1/2}\,du$

$\frac{1}{2}\left(\frac{2}{5}u^{5/2} - \frac{2}{3}u^{3/2}\right) + C$

$\frac{1}{5}(z^2+1)^{5/2} - \frac{1}{3}(z^2+1)^{5/2} + C$

(d) $\displaystyle\int \frac{(1+\sqrt{x})^2}{\sqrt{x}}\,dx = \int (1+x)^2\, x^{-1/2}\,dx$

$u = 1 + x^{1/2}$

$du = \frac{1}{2}x^{-1/2}\,dx \Rightarrow 2du = x^{-1/2}\,dx$

$2\int u^2\,du = 2\left(\frac{u^3}{3}\right) + C$

$\frac{2}{3}(1+\sqrt{x})^3 + C$

What is u?

A function inside the integral that is the derivative of another function inside the integral

Substitution method with definite integrals

Idea: The "bounds" of an integral are in terms of the variable following the "d". They must be changed appropriately when you change the variable of integration.

Substitution in definite integrals If g' is continuous on the interval $[a, b]$ and f is continuous on the range of $g(x) = u$, then

$$\int_a^b f(g(x))g'(x)\,dx = \int_{g(a)}^{g(b)} f(u)\,du.$$

You MUST either (1) change the bounds or (2) rewrite your antiderivative in terms of the original variable before plugging in the original numbers. Never do both!

Example #4: Evaluate the following definite integrals.

(a) $\displaystyle\int_3^5 \frac{x}{(x^2-4)^3}\,dx$

$u = x^2 - 4$

$du = 2x\,dx \Rightarrow \frac{1}{2}du = x\,dx$

$\frac{1}{2}\int_5^{21} \frac{1}{u^3}du = \frac{1}{2}\left[\frac{-1}{2u^2}\right]_5^{21}$

$\frac{1}{2}\left(\frac{-1}{2(21)^2} + \frac{1}{2(5)^2}\right) = 9.43\times10^{-3}$

$\frac{1}{2}\int \frac{1}{u^3}du = \frac{1}{4}\left(\frac{-1}{u^2}\right) + C$

$\left.\frac{-1}{4(x^2-4)^2}\right|_3^5$

$\frac{-1}{4(5^2-4)^2} + \frac{1}{4(3^2-4)^2} = 9.43\times10^{-3}$

(b) $\displaystyle\int_0^{\pi/4} \sin^2(2x)\cos(2x)\,dx$

$u = \sin 2x$

$du = 2\cos 2x\,dx \Rightarrow \frac{1}{2}du = \cos 2x\,dx$

$\frac{1}{2}\int u^2\,du = \frac{1}{2}\left(\frac{u^3}{3}\right) + C$

$\left.\frac{\sin^3 2x}{6}\right|_0^{\pi/4}$

$\frac{\sin^3\frac{\pi}{2}}{6} - \frac{\sin^3 0}{6} = \frac{1}{6}$