Linear First Order Differential Equations

Section 2.3 (Noonburg)

Goal: We want to solve linear, first-order, DE's of the form:

$$\frac{dx}{dt} + p(t)x(t) = q(t)$$

$x' + t^2 x = 0$
$x' = \sin t$
$x' + 2x = -5$
$\Big\}$ Seperable and linear

In general, these will not be separable equations.
However when $q(t) = 0$ (homogeneous) or $p(t) = 0$, it can be solved using separation.
or when both $q(t)$ and $p(t)$ are constants

Idea: It would be easier to solve if the left side of the DE is a derivative of some function. That would make the first order linear equation look more like a separable equation.

Recall Product Rule:

$$\frac{d}{dt}[\mu(t)x(t)] = \mu(t)x'(t) + \mu'(t)x(t)$$
$$= \mu\frac{dx}{dt} + x\frac{d\mu}{dt}$$

$$\frac{1}{\mu}\frac{d}{dt}(\mu \cdot x) = \frac{dx}{dt} + \frac{x}{\mu}\frac{d\mu}{dt}$$

$$p(t) = \frac{1}{\mu}\frac{d\mu}{dt}$$
$$q(t) = \frac{1}{\mu}\frac{d}{dt}(\mu \cdot x)$$

$$\frac{1}{\mu}d\mu = p(t)\,dt$$
$$\ln|\mu| = \int p(t)\,dt$$
$$\mu = e^{\int p(t)\,dt}$$

$$\mu \cdot q(t) = \frac{d}{dt}(\mu \cdot x)$$
$$\int \mu \cdot q(t)\,dt = \int d(\mu \cdot x)$$
$$\int \mu \cdot q(t)\,dt = \mu x$$

Integrating Factor:
When the linear equation is in the standard form $\frac{dx}{dt} + p(t)x(t) = q(t)$, the *integrating factor*, $\mu(t)$ is given by

$$\mu(t) = e^{\int p(t)dt}$$

STEPS to follow:

1. Get the linear equation in to the standard form: $\frac{dx}{dt} + p(t)x(t) = q(t)$
2. Find the integrating factor using, $\mu(t) = e^{\int p(t)dt}$.
3. Multiply **both** side of the equation by $\mu(t)$.
4. Check that the left side of the equation is now a derivative: $\frac{d}{dt}[\mu(t)x(t)]$
5. Integrate both sides (using separation).

Example #1: Solve $\frac{dx}{dt} - 2tx = t$

$p(t) = -2t$

$\mu = e^{\int -2t\,dt} = e^{-t^2}$

$x'e^{-t^2} - 2txe^{-t^2} = te^{-t^2}$

$\frac{d}{dt}(xe^{-t^2}) = te^{-t^2}$

$\int d(xe^{-t^2}) = \int te^{-t^2}dt$ $\quad u = -t^2 \quad -du = 2t\,dt \quad -\frac{1}{2}\int e^u du$

$xe^{-t^2} = -\frac{1}{2}e^{-t^2} + C$

$x = -\frac{1}{2} + Ce^{t^2}$

Example #2: Solve the IVP, $tx' = 5x + 6t^2$, $x(1) = 4$.

$tx' - 5x = 6t^2$

$x' - \frac{5x}{t} = 6t$

$p(t) = \frac{-5}{t}$

$\mu = e^{\int -5/t\,dt} = e^{-5\ln|t|} = t^{-5}$

$t^{-5}x' - \frac{5}{t^6}x = \frac{6}{t^4}$

$\frac{d}{dt}(t^{-5}x) = 6t^{-4}$

$\int d(t^{-5}x) = \int 6t^{-4}dt$

$t^{-5}x = -2t^{-3} + C$

$x = -2t^2 + Ct^5$

$4 = -2 + C \quad C = 6$

$x(t) = -2t^2 + 6t^5$

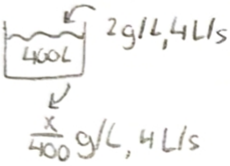
Applications: Mixing Problems

The rate at which the concentration changes is equal to its positive change minus the negative change.

$$\frac{dx}{dt} = \text{(rate in) - (rate out)}$$

Example #3: A tank initially contains 400 liters of water. Brine with a concentration 2 grams of salt per liter is pumped into the tank at a rate of 4 liters per second. The well-stirred mixture flows out of the tank at a rate of 4 liters per second.

(a) If you know that there is initially 200 g of salt in the tank, construct an IVP that models the amount of salt in the tank at any time, t.



x = amount of salt in the tank at time t

$\frac{dx}{dt} = 8 - \frac{x}{100}$ $\quad x(0) = 200$

(b) Solve the IVP.

$\frac{dx}{dt} + \frac{x}{100} = 8$

$p(t) = \frac{1}{100}$

$\mu = e^{\int 1/100\, dt} = e^{t/100}$

$x' e^{t/100} + \frac{x}{100} e^{t/100} = 8e^{t/100}$

$\frac{d}{dt}\left(xe^{t/100}\right) = 8e^{t/100}$

$xe^{t/100} = \int 8e^{t/100} dt = 800e^{t/100} + C$

$x = 800 + Ce^{-t/100}$

$200 = 800 + C$

$C = -600$

$x = 800 - 600e^{-t/100}$

$(t \to \infty, x \to 800)$

Example #4: A 120 gallon tank contains 90 lb of salt in 90 gallons of water initially. Brine with a concentration 2 lb/gal flows into the tank at a rate of 4 gal/min. The well-stirred mixture flows out at a rate of 3 gal/min. How much salt is in the tank when it is full?

2 lb/gal, 4 gal/min

x = amount of salt at time t

$V(t) = 90 + t \quad t \le 30$

$\frac{x}{90+t}$ lb/gal, 3 gal/min

$$\frac{dx}{dt} = 8 - \frac{3x}{90+t}$$

$$\frac{dx}{dt} + \frac{3x}{90+t} = 8$$

$$p(t) = \frac{3}{90+t}$$

$$\mu = e^{\int 3/90+t \, dt} = e^{3\ln|90+t|} = (90+t)^3$$

$$x'(90+t)^3 + 3x(90+t)^2 = 8(90+t)^3$$

$$(90+t)^3 x = \int 8(90+t)^3 \, dt$$

$$x(90+t)^3 = 2(90+t)^4 + C$$

$$x = 2(90+t) + C(90+t)^{-3} \qquad x(0) = 90$$

$$90 = 2(90) + C(90)^{-3}$$

$$1 = 2 + C(90)^{-4}$$

$$C = -90^4$$

$$x = 2(90+t) - 90^4(90-t)^{-3}$$

$$x(30) = 2(120) - 90^4(120)^{-3} = 202.03 \text{ lb}$$