Linear Independence

Section 2.8 (**Hartman**)

Linear Combinations of Vectors:
A linear combination or superposition of vectors $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n$ is a sum of the form

$$c_1\vec{v}_1 + c_2\vec{v}_2 + \ldots + c_n\vec{v}_n$$

where $c_1, c_2, ..., c_n$ are constants.

Example #1: Consider the vectors given below. Show that at least one of these vectors can be constructed(written) as a linear combination of the other two.

(a) $\vec{v}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$, $\vec{v}_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$ $\vec{v}_3 = \begin{pmatrix} -1 \\ 0 \end{pmatrix}$.

$$\frac{-1}{2}\begin{pmatrix} 1 \\ 1 \end{pmatrix} - \frac{1}{2}\begin{pmatrix} 1 \\ -1 \end{pmatrix} = \begin{pmatrix} -1 \\ 0 \end{pmatrix}$$

$$\frac{-1}{2}\vec{v}_1 - \frac{1}{2}\vec{v}_2 = \vec{v}_3$$

$$\vec{v}_1 + \vec{v}_2 + 2\vec{v}_3 = \vec{0}$$

(b) $\vec{v}_1 = \begin{pmatrix} 3 \\ 4 \\ -5 \end{pmatrix}$, $\vec{v}_2 = \begin{pmatrix} 6 \\ -2 \\ -3 \end{pmatrix}$ $\vec{v}_3 = \begin{pmatrix} -9 \\ 38 \\ -20 \end{pmatrix}$.

Suppose $\vec{v}_3 = a\vec{v}_1 + b\vec{v}_2$

$$\begin{pmatrix} -9 \\ 38 \\ -20 \end{pmatrix} = a\begin{pmatrix} 3 \\ 4 \\ -5 \end{pmatrix} + b\begin{pmatrix} 6 \\ -2 \\ -3 \end{pmatrix}$$

$3a + 6b = -9$
$4a - 2b = 38$
$-5a - 3b = -20$

$a = 7$
$b = -5$

$$7\vec{v}_1 - 5\vec{v}_2 - \vec{v}_3 = \vec{0}$$

Linearly Dependent Vectors:
A set of vectors $\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n\}$ is ***linearly dependent*** if there are constants $c_1, c_2, \ldots, c_n$ so that $c_1\vec{v}_1 + c_2\vec{v}_2 + \ldots + c_n\vec{v}_n = \vec{0}$. with at least one of the $c_k \neq 0$.

Otherwise we say the vectors are *linearly independent.*

Recall Example #1. What can we say about the results?

In both examples, we can write one of the vectors as a linear combination of the other two vectors such that at least one coefficient is non-zero

$$\left.\begin{array}{l} \text{a. } \vec{v}_1 + \vec{v}_2 + 2\vec{v}_3 = \vec{0} \\ \text{b. } 7\vec{v}_1 - 5\vec{v}_2 - \vec{v}_3 = \vec{0} \end{array}\right\} \text{Linearly dependent}$$

Let's redo the problem in Example #1 - (a), using a different approach.

Determine if the vectors $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ with $\vec{v}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$, $\vec{v}_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$ $\vec{v}_3 = \begin{pmatrix} -1 \\ 0 \end{pmatrix}$ are linearly dependent.

$a\vec{v}_1 + b\vec{v}_2 + c\vec{v}_3 = \vec{0}$

$a\begin{pmatrix} 1 \\ 1 \end{pmatrix} + b\begin{pmatrix} 1 \\ -1 \end{pmatrix} + c\begin{pmatrix} -1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

$a+b-c=0$
$a-b=0$

$$\left(\begin{array}{ccc|c} 1 & 1 & -1 & 0 \\ 1 & -1 & 0 & 0 \end{array}\right) \xrightarrow{R_2 - R_1 \to R_2} \left(\begin{array}{ccc|c} 1 & 1 & -1 & 0 \\ 0 & -2 & 1 & 0 \end{array}\right) \xrightarrow{-\frac{1}{2}R_2 \to R_2} \left(\begin{array}{ccc|c} 1 & 1 & -1 & 0 \\ 0 & 1 & -1/2 & 0 \end{array}\right) \xrightarrow{R_1 - R_2 \to R_1} \left(\begin{array}{ccc|c} 1 & 0 & -1/2 & 0 \\ 0 & 1 & -1/2 & 0 \end{array}\right)$$

$a - \frac{1}{2}c = 0 \qquad a = 1/2\, c$ $\qquad$ c is free
$b - \frac{1}{2}c = 0 \qquad b = 1/2\, c$

$\frac{1}{2}c\vec{v}_1 + \frac{1}{2}c\vec{v}_2 + c\vec{v}_3 = 0$

Linearly dependent

- There are infinitely many non-trivial linear combinations that equal the zero vector when the vectors are linearly dependent.

Example #2: Determine the value(s) of the constant k for which the following vectors are linearly independent:

$$\vec{v}_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \ \vec{v}_2 = \begin{pmatrix} 2 \\ 3 \\ 2 \end{pmatrix} \ \vec{v}_3 = \begin{pmatrix} 1 \\ 0 \\ k \end{pmatrix}$$

$$\left(\begin{array}{ccc|c} 1 & 2 & 1 & 0 \\ 1 & 3 & 0 & 0 \\ 1 & 2 & k & 0 \end{array}\right) \xrightarrow[R_3 - R_1 \to R_3]{R_2 - R_1 \to R_2} \left(\begin{array}{ccc|c} 1 & 2 & 1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & k-1 & 0 \end{array}\right) \xrightarrow[k \neq 1]{\frac{1}{k-1} R_3 \to R_3} \left(\begin{array}{ccc|c} 1 & 2 & 1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right)$$

Linearly independent when $k \neq 1$

We can obtain the reduced row-echelon form equal to I_3 when $k-1 \neq 0$

Linearly dependent when $k=1$

$$\left(\begin{array}{ccc|c} 1 & 2 & 1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \xrightarrow{\text{reduce}} \begin{array}{c} \begin{array}{cccc} a & b & c & \ \end{array} \\ \left(\begin{array}{ccc|c} 1 & 0 & 3 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \end{array}$$

$a = -3c$
$b = c$ c is free

$-3c\vec{v}_1 + c\vec{v}_2 + c\vec{v}_3 = \vec{0}$

Definition: The ***rank*** of a matrix A is the number of pivots in the reduced row echelon form of A.

Linear Independence and Rank:

Given a set of n vectors $\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n\}$, we form the matrix

$$A = \begin{pmatrix} | & | & & | \\ v_1 & v_2 & \cdots & v_n \\ | & | & & | \end{pmatrix}.$$

The vectors $\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n\}$ are linearly independent when the rank of A is n and linearly dependent when the rank of A is less than n.

Linearly independent when rank $= n$

Linearly dependent when rank $< n$

Extra Practice: state whether or not the following vectors are linearly dependent:

(a) $\vec{v}_1 = \begin{pmatrix} 1 \\ 3 \\ -1 \end{pmatrix}$, $\vec{v}_2 = \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix}$ $\vec{v}_3 = \begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix}$

$$\left(\begin{array}{ccc|c} 1 & 2 & 2 & 0 \\ 3 & 1 & 1 & 0 \\ -1 & 1 & 2 & 0 \end{array}\right) \xrightarrow{\text{reduce}} \begin{array}{c} \begin{array}{ccc} a & b & c \end{array} \\ \left(\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right) \end{array}$$

$a = b = c = 0$

Linearly independent

(b) $\vec{v}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$, $\vec{v}_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$ $\vec{v}_3 = \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}$, $\vec{v}_4 = \begin{pmatrix} 1 \\ 3 \\ -2 \end{pmatrix}$

Linearly dependent

Not a square matrix

$$\left(\begin{array}{cccc|c} 1 & 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 3 & 0 \\ 0 & 0 & 2 & -2 & 0 \end{array}\right) \xrightarrow{\text{reduce}} \begin{array}{c} \begin{array}{cccc} a & b & c & d \end{array} \\ \left(\begin{array}{cccc|c} 1 & 0 & 0 & 2 & 0 \\ 0 & 1 & 0 & 4 & 0 \\ 0 & 0 & 1 & -1 & 0 \end{array}\right) \end{array}$$

$a = -2d$

$b = -4d$ d is free $-2d\vec{v}_1 - 4d\vec{v}_2 + d\vec{v}_3 + d\vec{v}_4 = \vec{0}$

$c = d$