Definitions and Terminology
Section 1.1 (Noonburg)

Definition: A ***differential equation*** is any equation involving an unknown function and one or more of its derivatives.

- Ordinary DE's (ODE): The unknown function in the equation is a function of only one variable.
  e.g. $\frac{dy}{dx} + 5 = x^2$

- Partial DE's (PDE): The unknown function in the equation depends on more than one independent variable.
  e.g. $\frac{\partial^2}{\partial x^2}u(x,y) + \frac{\partial^2}{\partial y^2}u(x,y) = 0$

- ***Order*** of a differential equation is the order of the highest derivative of the unknown function.

- An ODE is ***linear*** if we can write it so that;
  (i) No y is touching another y or any derivative of y.
  (ii) All derivatives of y, and y are raised to the first power only.
  (iii) All derivatives of y, or y are not inside a function. (e.g. $\cos(y)$ makes it non-linear)

- An n-th order DE of the form:

$$f_n(x)y^{(n)} + f_{n-1}(x)y^{(n-1)} + \cdots + f_1(x)y^{(1)} + f_0(x)y = g(x)$$

  is said to be ***homogeneous*** if $g(x) = 0$. If $g(x) \neq 0$ it is said to be ***non-homogeneous***.
  Nonlinear $\Rightarrow$ non-homogeneous

Example #1: Classify the following DE's by order, linearity and homogeneity:

(a) $\frac{dy}{dx} + 5 = x^2$

1st order
Linear
Non-homogeneous

(b) $\frac{\partial^2}{\partial x^2}u(x,y) + \frac{\partial^2}{\partial y^2}u(x,y) = 0$

2nd order
Linear
Homogeneous

(c) $-3y''' + xy' + 8 = y$

3rd order
Linear
Non-homogeneous

Definition: A function $y = \phi(x)$ is ***solution*** to an $n-th$ order differential equation, if ϕ has n continuous derivatives, makes the equation an identity over some interval of the independent variable.

Example #2: Show that $\phi(x) = \frac{6}{5} + Ce^{-20x}$ is a solution of $y' + 20y = 24$, for any constant C.

$y = \frac{6}{5} + Ce^{-20x}$

$y' = 0 - 20Ce^{-20x}$

$-20Ce^{-20x} + 20\left(\frac{6}{5} + Ce^{-20x}\right) = 24$

$24 = 24 \checkmark$

Defined on $(-\infty, \infty)$

NOTE:

- Explicit solution: Solution is written as a function of the independent variable.
- Implicit solution: Only defines a relation between the unknown function and the independent variable.

  e.g. $y = Ae^{2x}$ is an explicit solution. $\ln|y| - 5 = 3x^2$ is an implicit solution.

Extra Practice: For each equation, show that the given function is a solution and determine the interval on which the solution exists.

(a) $y'' + 4y = 0$; $y(x) = \sin(2x) + \cos(2x)$

$y' = 2\cos(2x) - 2\sin(2x)$

$y'' = -4\sin(2x) - 4\cos(2x)$

$-4\sin(2x) - 4\cos(2x) + 4(\sin(2x) + \cos(2x)) = 0$

$0 = 0 \checkmark$

Defined on $(-\infty, \infty)$

(b) $t^2x'' + 3tx' = -x; \quad x(t) = 1/t$ $\quad t \neq 0$

$= t^{-1}$

$x' = -t^{-2}$

$x'' = 2t^{-3}$

$\frac{2}{t} + 3\left(\frac{-1}{t}\right) = \frac{-1}{t}$

$\frac{-1}{t} = \frac{-1}{t}$ ✓

Defined on $(-\infty, 0) \cup (0, \infty)$

(c) $x' = (t+2)/x; \quad x(t) = \sqrt{t^2 + 4t + 1}$ $= (t^2+4t+1)^{1/2}$

$x' = \frac{1}{2}(t^2+4t+1)^{-1/2}(2t+4)$

$\frac{t+2}{\sqrt{t^2+4t+1}} = \frac{t+2}{\sqrt{t^2+4t+1}}$ ✓

$t^2+4t+1 > 0$

$t = \frac{-4 \pm \sqrt{16-4}}{2} = \frac{-4 \pm 2\sqrt{3}}{2} = -2 \pm \sqrt{3}$

$\alpha = -2+\sqrt{3} \quad \beta = -2-\sqrt{3}$

$(t-\alpha)(t-\beta) > 0$

$-4 \quad \beta \quad -2 \quad \alpha \quad 2$

$(-4-\alpha)(-4-\beta) > 0$ ✓

$(-2-\alpha)(-2-\beta) < 0$ ✗

$(2-\alpha)(2-\beta) > 0$ ✓

Defined on $(-\infty, -2-\sqrt{3}) \cup (-2+\sqrt{3}, \infty)$