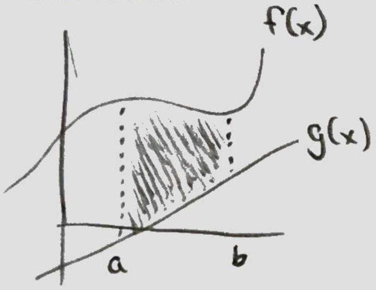
Area Between Curves
Section 5.6

Definition: If f and g are continuous with $f(x) \geq g(x)$ throughout $[a, b]$, then the **area of the region between the curves** $y = f(x)$ **and** $y = g(x)$ **from a to b** is given by;

$$A = \int_a^b [f(x) - g(x)]\,dx.$$

- When applying the definition it is usually helpful to graph the curves. It reveals which curve is the upper curve f and which is the lower curve g.
- If limits are not given, you may solve $f(x) = g(x)$ for x to find points of intersection. Then you can integrate the function $f - g$ between the intersections.
- Area between the curves are always positive. (If you ended up with a negative value, either you may have switched f and g or you should check your computations!)

General idea:



Example #1: Find the area of the region between $y = 3x^2 + 12$ and $y = 4x + 4$ over [-3,3].

$\int_{-3}^{3} 3x^2 + 12 - (4x + 4)\,dx$

$\int_{-3}^{3} 3x^2 - 4x + 8\,dx = x^3 - 2x^2 + 8x \Big|_{-3}^{3}$

$3^3 - 2(3)^2 + 8(3) - ((-3)^3 - 2(-3)^2 + 8(-3)) = 102$

Example #2: Find the area of the region enclosed by the graphs of $f(x) = x^3 - 10x$ and $g(x) = 6x$.

$f(x) = g(x)$

$x^3 - 10x = 6x$

$x^3 - 16x = 0$

$x(x^2 - 16) = 0$

$x = 0, \pm 4$

$\int_{-4}^{0} f(x) - g(x)\,dx + \int_{0}^{4} g(x) - f(x)\,dx$

$\int_{-4}^{0} x^3 - 10x - 6x\,dx + \int_{0}^{4} 6x - x^3 + 10x\,dx$

$\int_{-4}^{0} x^3 - 16x\,dx + \int_{0}^{4} -x^3 + 16x\,dx$

$\left[\frac{x^4}{4} - 8x^2\right]_{-4}^{0} + \left[8x^2 - \frac{x^4}{4}\right]_{0}^{4}$

$0 - \left(\frac{(-4)^4}{4} - 8(-4)^2\right) + 8(4)^2 - \frac{4^4}{4} - 0 = 128$

Example #3: Find the area of the region lying to the right of $y = \sqrt{x-2}$ and to the left of $y = x - 8$.

(Here it is easier to use integration with respect to y using $x = y^2 + 2$ and $x = y + 8$.)

$$\int_2^8 \sqrt{x-2}\,dx + \int_8^8 \sqrt{x-2} - x + 8\,dx$$

$$x - 8 = \sqrt{x-2}$$
$$x = 8$$

$$\int_0^3 y + 8 - y^2 - 2\,dy = \int_0^3 y - y^2 + 6\,dy$$

$$y^2 + 2 = y + 8$$
$$y^2 - y - 6 = 0$$
$$y = -\cancel{2}, 3$$

$$\frac{y^2}{2} - \frac{y^3}{3} + 6y \Big|_0^3$$

$$\frac{3^2}{2} - \frac{3^3}{3} + 6(3) - 0 = \frac{27}{2}$$