Tangent Approximation

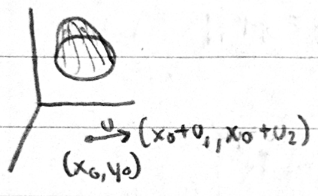
MA113

Taylor series

$$\sum_{n=0}^{\infty} \frac{f^{n}(x_0)}{n!}(x-x_0)^n$$

Taylor polynomial

$$f(x)=\underbrace{f(x_0)+\dots+\frac{1}{k!}f^{k}(x_0)(x-x_0)^k}_{k\text{th degree @ } x_0}+\underbrace{R_k(x)}_{\text{Remainder}}$$



$$F(t)=f(x_0+u_1,\ y_0+u_2)$$

$$F'(t)=\frac{\partial f}{\partial x}(x_0+u_1,\ y_0+u_2)u_1+\frac{\partial f}{\partial y}(x_0+u_1,\ y_0+u_2)u_2$$

$$\frac{dF}{dt}=\frac{dF}{\partial x}\frac{\partial x}{dt}+\frac{dF}{dy}\frac{\partial y}{\partial t}$$

Tangent plane approximation $F(t)\approx F(0)+F'(0)t=f(x_0,y_0)+\left(\frac{\partial f}{\partial x}(x_0,y_0)u_1+\frac{\partial f}{\partial y}(x_0,y_0)u_2\right)t$

$$F''(t)=\left(\frac{\partial^2 f}{\partial x^2}(x_0+u_1,\ y_0+u_2)u_1+\frac{\partial^2 x}{\partial xy}(x_0+u_1,\ y_0+u_2)u_2\right)u_1+\left(\frac{\partial^2 f}{\partial xy}(x_0+u_1,\ y_0+u_2)u_1+\frac{\partial^2 f}{\partial y^2}(x_0+u_2,\ y_0+u_2)u_2\right)u_2$$

$$F''(0)=\frac{\partial^2 f}{\partial x^2}(x_0,y_0)u_1^2+2\frac{\partial^2 f}{\partial xy}(x_0,y_0)u_1u_2+\frac{\partial^2 f}{\partial y^2}(x_0,y_0)u_2^2$$

$$F(t)\approx f(x_0,y_0)+F'(t)t+\tfrac{1}{2}F''(t)t^2+\dots$$

$f(x,y)=\ln(2x+y+1)$ Find a quadratic approximation near $(1,0)$

$f_x=\frac{2}{2x+y+1}$ $f_y=\frac{1}{2x+y+1}$ $f_{xx}=\frac{-4}{(2x+y+1)^2}$ $f_{yy}=\frac{-1}{(2x+y+1)^2}$ $f_{xy}=\frac{-2}{(2x+y+1)^2}$

$f(1,0)=\ln 3$ $f_x=\frac{2}{3}$ $f_y=\frac{1}{3}$ $f_{xx}=\frac{-4}{9}$ $f_{yy}=\frac{-1}{9}$ $f_{xy}=\frac{-2}{9}$

$$\ln(2x+y+1)\approx\ln 3+\tfrac{2}{3}(x-1)+\tfrac{1}{3}(y)+\tfrac{1}{2}\left(-\tfrac{4}{9}(x-1)^2-\tfrac{4}{9}(x-1)y-\tfrac{1}{9}(y)^2\right)$$

$$\approx\boxed{\ln 3+\frac{2(x-1)}{3}+\frac{y}{3}-\frac{4(x-1)^2}{18}-\frac{2y(x-1)}{9}-\frac{y^2}{18}}$$

$f(x,y)=x^2\sin y$ Find the first set of nonzero terms in a Taylor polynomal near $(0,\pi)$

$f(0,\pi)=0$ $f_x=2x\sin y$ $f_y=x^2\cos y$ $f_x(0,\pi)=f_y(0,\pi)=0$

$f_{xx}=2\sin y$ $f_{yy}=-x^2\sin y$ $f_{xy}=2x\cos y$ $f_{xx}=0$ $f_{yy}=0$ $f_{xy}=0$

$f_{xxx}=0$ $f_{xxy}=-2\cos y=2$ $f_{yyy}=-x^2\cos y=0$ $f_{yyx}=-2x\sin y=0$

$$x^2\sin y\approx\frac{1}{3!}(-2\cdot 6\Delta x^2\Delta y)=\frac{-6}{6}(x)^2(y-\pi)=\boxed{-x^2(y-\pi)}$$