Modelling with Differential Equations
Section 1.3 (Noonburg)

Idea: We can model data and make predictions using Mathematics. Derivatives measure change. So anything that changes can be modelled using DE.

e.g. when T measures temperature and t measures time, $\dfrac{dT}{dt}$ gives the rate of change of temperature.

Example #1: Let $C(t)$ be the concentration of a drug in the bloodstream. As the body eliminates the drug, $C(t)$ decreases at a rate that is proportional to the amount of the drug that is present at the time. Model this using DE.

$\dfrac{dC}{dt} = -kC(t)$ k = positive constant
t = time
$C(t)$ = concentration at time t

Example #2: Newton's Law of Cooling states: " The rate at which an object cools is proportional to the difference in temperature between the object and its surroundings." Model this using DE.

T = temperature of object at time t
T_s = temperature of surroundings
t = time
k = constant > 0

$\dfrac{dT}{dt} = -k(T - T_s)$

Falling body problems: In a free fall, Newton's second law can be used to write:

$$\textit{sum of forces} = \textit{mass} \times \textit{acceleration}$$

Since the acceleration is the rate of change of velocity (i. e. $a(t) = v'(t)$), this results in a DE model:

$$mv'(t) = mg - k(v(t))^p$$
$$\implies v'(t) = g - \frac{k}{m}(v(t))^p$$

where m is the mass, g is the gravity, k is the coefficient of friction and p is an exponent usually assumed to be 1.

Example #3: Assuming $p = 1$ and $g = 9.8\ ms^{-2}$, the equation $v' = g - \dfrac{k}{m}v$ has a solution

$$v(t) = \frac{mg}{k} + Ce^{-\frac{k}{m}t}.$$

A man drops from a high flying plane and falls for 5 seconds before opening his parachute. With the parachute closed, $\frac{k}{m} = 1s^{-1}$.

(a) Find the man's velocity when he opens his parachute. (Use $v(0) = 0$)

Maple:

dsolve($v'(t) = g - \frac{k}{m} \cdot v(t)$) $\rightarrow$ $v(t) = \frac{gm}{k} + e^{-kt/m} C_1$

$0 = g + e^{-0} C_1 \quad C_1 = -9.8 \qquad v(t) = 9.8(1 - e^{-t})$ for first 5 seconds

$v(5) = 9.8 - 9.8e^{-5} = 9.73$ m/s

(b) After the parachute opens, what must be the value of $\frac{k}{m}$ to get his terminal velocity down to $2.5ms^{-1}$? (Assume he has a very long way to fall.)

Terminal velocity: $v = \frac{mg}{k} = \frac{g}{k/m}$

$\frac{k}{m} = \frac{9.8}{2.5} = 3.92$ Hz

We will discuss more applications/ models in Chapter 2.