# Separable First Order Differential Equations

## Section 2.1 (Noonburg)

**Definition:** A differential equation of the form

$$\frac{dy}{dx} = g(x)f(y)$$

is said to be **separable**.

Solving:

$$\frac{dy}{dx} = g(x)f(y) \Rightarrow \int \frac{1}{f(y)}\,dy = \int g(x)\,dx$$

**Example #1:** Find a general solution for the differential equation $(1-x)y' - y = 0$.

$$\frac{dy}{dx}(1-x) = y \qquad \frac{dy}{dx} = \frac{y}{1-x}$$

$$\int \frac{1}{y}\,dy = \int \frac{1}{1-x}\,dx = -\int \frac{1}{x-1}\,dx$$

$$\ln|y| = -\ln|x-1| + C$$

$$\ln|y| = -\ln|x-1| + \ln C_0 = \ln\left(\frac{C_0}{|x-1|}\right)$$

$$y = \pm\frac{C_0}{|x-1|} \Rightarrow y = \frac{C_0}{x-1}$$

**Example #2:** Find the solution to the initial value problem $\dfrac{dy}{dx} = \dfrac{4-2x}{3y^2-5}$, $y(0) = 1$.

$$\int 3y^2 - 5\,dy = \int 4 - 2x\,dx$$

$$y^3 - 5y = 4x - x^2 + C$$

$$1^3 - 5(1) = 4(0) - 0^2 + C$$

$$C = -4$$

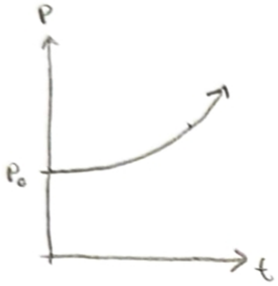
$$y^3 - 5y = 4x - x^2 - 4$$

**Exponential Growth/Decay**

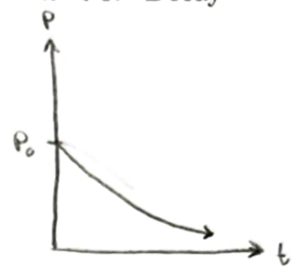
*Assumption: The rate at which the population is changing is proportional to the current population.*

$$\frac{dP}{dt} = kP \quad \cancel{(\text{with } k > 0)}$$

$k > 0$: Growth




$k < 0$: Decay




**Example #3:** A culture initially has $P_0$ amount of bacteria and after an hour, the number of bacteria present is measured to be $\frac{3}{2}P_0$. Find the time necessary for the amount of bacteria to triple.

$P(0) = P_0$

$P(1) = \frac{3}{2}P_0$

$\frac{dP}{dt} = kP$

$\int \frac{1}{P}\,dP = \int k\,dt$

$\ln|P| = kt + C_0$

$P = Ce^{kt}$

$P_0 = Ce^0$

$P_0 = C$

$P = P_0 e^{kt}$

$\frac{3}{2}P_0 = P_0 e^k$

$e^k = \frac{3}{2}$

$k = \ln\left(\frac{3}{2}\right)$

$P = P_0 e^{\ln\left(\frac{3}{2}\right)t} = P_0\left(\frac{3}{2}\right)^t$

$3P_0 = P_0\left(\frac{3}{2}\right)^t$

$3 = \left(\frac{3}{2}\right)^t$

$t = \log_{1.5} 3 = 2.71$ hrs

$\downarrow$

$\frac{\ln 3}{\ln 1.5}$

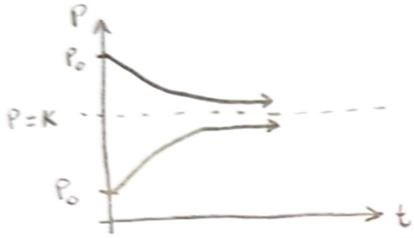
***Real world constraints*** (Logistic population growth model):
*When the population is high there won't be enough resources. This will cause population to thin out.*

$$\frac{dP}{dt} = aP - bP^2$$

We can factor and write as

$$\frac{dP}{dt} = rP\left(1 - \frac{P}{K}\right) \quad \text{(with } r > 0 \text{ a constant)}$$

*Graph:*




*Solution:*

$$\int \frac{1}{P(1-\frac{P}{K})}\, dP = \int r\, dt$$

$$\int \frac{1}{P} + \frac{1/K}{1-P/K}\, dP = rt + C_0$$

$$\ln P - \ln\left|1-\frac{P}{K}\right| = rt + C_0$$

$$\ln\left|\frac{P}{1-P/K}\right| = rt + C_0$$

$$\frac{P}{1-P/K} = Ce^{rt}$$

$$P = \left(1-\frac{P}{K}\right)Ce^{rt} = Ce^{rt} - \frac{P}{K}Ce^{rt}$$

$$P\left(1+\frac{Ce^{rt}}{K}\right) = Ce^{rt}$$

$$P = \frac{Ce^{rt}}{1+\frac{Ce^{rt}}{K}}$$

$$P(0) = P_0$$

$$P_0 = \frac{C}{1+\frac{C}{K}}$$

$$P_0\left(1+\frac{C}{K}\right) = C$$

$$P_0 + \frac{P_0 C}{K} = C$$

$$P_0 = C\left(1-\frac{P_0}{K}\right)$$

$$C = \frac{P_0}{1-\frac{P_0}{K}}$$

$$P(t) = \frac{(1-\frac{P_0}{K})e^{rt}}{1+\frac{(1-\frac{P_0}{K})e^{rt}}{K}} = \frac{P_0 K}{P_0 + (K-P_0)e^{-rt}}$$

**Falling Body Problems**

$$\sum_k F_k = ma = m\frac{dv}{dt}$$

**Example #4:** Suppose that you are on top of a 30 meter tall building and you toss a 1.5 kg ball upwards with a speed of 20 m/sec. Assume that air resistance has a magnitude of 0.8 times the speed of the ball. How long is the ball in the air?

$\sum_k F_k = -mg - 0.8v = m\frac{dv}{dt}$

$\frac{dv}{dt} = -g - 0.8\frac{v}{m} = -(9.8 + \frac{0.8}{1.5}v)$

$\int \frac{1}{9.8 + \frac{0.8}{1.5}v}\,dv = \int -dt$

$\frac{1.5}{0.8}\ln|9.8 + \frac{0.8}{1.5}v| = -t + C_0$

$\ln|9.8 + \frac{0.8}{1.5}v| = -\frac{0.8}{1.5}t + C_1$

$9.8 + \frac{0.8}{1.5}v = Ce^{-0.8t/1.5}$

$v = \frac{1.5(Ce^{-0.8t/1.5} - 9.8)}{0.8}$

$v(0) = 20$

$20 = \frac{1.5}{0.8}(C - 9.8)$

$38.375 = 1.875C$

$C = \frac{307}{15}$

$v(t) = \frac{1.5}{0.8}(\frac{307}{15}e^{-0.8t/1.5} - 9.8)$

$v(t) = 1.875(\frac{307}{15}e^{-8t/15} - 9.8) = s'(t)$

$s(0) = 30$

$s(t) = -71.953e^{-0.533t} - 18.375t + 101.953$

$s(t) = 0 \quad t = 5.319\,s$