Taylor and Maclaurin Series
Sections 10.8 & 10.9

Idea: Find a series representation of a function near a point.

Suppose we have a function $f(x)$. We want to find a power series $F(x)$ such that $F(a) = f(a)$, $F'(a) = f'(a)$, $F''(a) = f''(a)$, and so on. These two functions will then be "nearly identical" near $x = a$.

Let $F(x) = \sum_{n=0}^{\infty} c_n(x-a)^n$. We are looking for $\{c_n\}$ that make the above statements true.

$c_0 + c_1(x-a) + c_2(x-a)^2 + \cdots + c_n(x-a)^n$

$f'(x) = F'(x) = 0 + c_1 + c_2 \cdot 2(x-a) + c_3 \cdot 3(x-a)^2 + \cdots + nc_n(x-a)^{n-1}$

$f''(x) = F''(x) = 0 + 2c_2 + 3c_3 \cdot 2(x-a) + \cdots + nc_n(n-1)(x-a)^{n-2}$

$f'''(x) = F'''(x) = 0 + 6c_3 + \cdots + nc_n(n-1)(n-2)(x-a)^{n-3}$

$f^n(x) = F^n(x) = c_n n! +$ a sum of terms with $(x-a)$ as a factor

$f^n(a) = F^n(a) = c_n n! + 0$

$c_n = \dfrac{f^n(a)}{n!}$

Definition: If f is a function with derivatives of all orders in some interval containing a, then the **Taylor series generated by f at $x = a$** is

$$\sum_{n=0}^{\infty} \frac{f^n(a)(x-a)^n}{n!} = f(a) + f'(a)(x-a) + f''(a)\frac{(x-a)^2}{n!} + \cdots$$

The **Maclaurin series of f** is the Taylor series generated by f at $x = 0$.

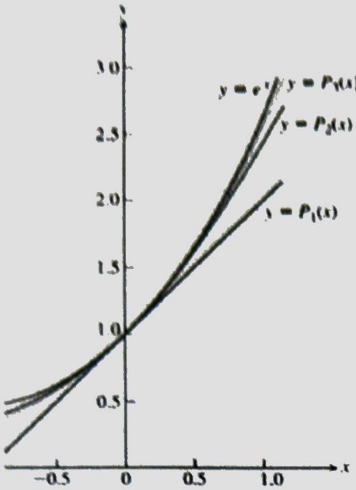

FIGURE 10.22 The graph of $f(x) = e^x$ and its Taylor polynomials

Example #1: Find the Taylor series of $f(x) = 2^x$ centered at $x = 1$.

$$\sum_{n=0}^{\infty} \frac{f^n(1)}{n!}(x-1)^n$$

$$\ln y = x\ln 2$$

$$\frac{1}{y}\frac{dy}{dx} = \ln 2 \qquad \frac{dy}{dx} = y\ln 2 = 2^x \ln 2$$

$$f'(x) = 2^x \ln 2$$
$$f''(x) = 2^x(\ln 2)^2$$
$$f'''(x) = 2^x(\ln 2)^3$$
$$f^n(x) = 2^x(\ln 2)^n$$

$x = 1$

$$f^n(1) = 2(\ln 2)^n$$

Taylor series

$$\sum_{n=0}^{\infty} \frac{2(\ln 2)^n(x-1)^n}{n!} =$$

Example #2: Find the Taylor series of $f(x) = \dfrac{1}{x^2}$ centered at $x = 2$ and its radius of convergence.

$$\sum_{n=0}^{\infty} f^n a \frac{(x-a)^n}{n!} = \frac{f^n(2)(x-2)^n}{n!}$$

$$f(x) = \frac{1}{x^2} = x^{-2}$$

$$f'(x) = -2x^{-3} = -1(2)x^{-3}$$

$$f''(x) = 6x^{-4} = (-1)^2(2)(3)x^{-4}$$

$$f'''(x) = -24x^{-3} = (-1)^3(2)(3)(4)x^{-5}$$

$$f^n(x) = (-1)^n(n+1)!\,x^{-(n+2)}$$

At $x = 2$:

$$f^n(2) = (-1)^n(n+1)!\,2^{-n-2}$$

The Taylor series is:

$$\sum_{n=0}^{\infty} \frac{f^n(2)}{n!}(x-2)^n = \frac{(-1)^n(n+1)!\,2^{-n-2}(x-2)^n}{n!} = \frac{(-1)^n(n+1)(x-2)^n}{2^{n+2}}$$

$$\frac{1}{2}\left(\frac{n+2}{n+1}\right)|x-2| \Rightarrow \frac{1}{2}|x-2|$$

$$\frac{1}{2}|x-2| < 1$$

$$|x-2| < 2$$

$$R = 2$$

Example #3: Find the Maclaurin series of $\sin(x)$ and its interval of convergence.

$$\sum_{n=0}^{\infty} \frac{f^n(0)x^n}{n!}$$

$f(x) = \sin x = 0$

$f'(x) = \cos x = 1$

$f''(x) = -\sin x = 0$

$f'''(x) = -\cos x = -1$

Values repeat in cycles of 4

$$\sum_{n=0}^{\infty} \frac{f^n(0)x^n}{n!} = f(0) + f'(0) + \frac{f''(0)x^2}{2!}$$
$$= 0 + x + 0 + \frac{-1x^3}{3!} + 0 + \frac{x^5}{5!} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$$
$$= \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}$$

$$\left|\frac{x^{2n+3}}{(2n+3)!} \cdot \frac{(2n+1)!}{x^{2n+1}}\right| = \left|\frac{x^2}{(2n+2)(2n+3)}\right|$$

$$\lim_{n\to\infty} \frac{1}{(2n+2)(2n+3)} \cdot x^2 = 0 < 1$$

Converges for any value of x

$R = \infty$

$-\infty < x < \infty$

$$\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}$$

<u>Definition:</u> Let f be a function whose derivatives of order $k = 1, 2, \ldots n$ exist in an interval containing a. The **Taylor polynomial of order** n generated by f at $x = a$ is the polynomial;

$$P_n(x) = f(a) + f^{(1)}(a)(x-a) + \frac{f^{(2)}(a)}{2!}(x-a)^2 + \cdots + \frac{f^{(k)}(a)}{k!}(x-a)^k + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n$$

In other words, it's the nth partial sum ($P_n = s_n$) of the Taylor series.

Example #4: Find the second Taylor polynomial (i.e., $P_2(x)$) of $f(x) = \cos(x)$ centered at $x = \pi$.

$f(x) = \cos x$

$P_2(x) = f(\pi) + f'(\pi)(x-\pi) + \frac{f''(\pi)}{2!}(x-\pi)^2 + \cdots$

$f(\pi) = \cos\pi = -1$

$f'(\pi) = -\sin\pi = 0$

$f''(\pi) = -\cos\pi = 1$

$f'''(\pi) = \sin\pi = 0$

$P_2(x) = -1 + 0 + \frac{1}{2}(x-\pi)^2$

Taylor's Formula and Remainder Estimation
If f has derivatives of all orders in an open interval I containing a, then for each positive integer n and for each x in I,

$$f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \ldots + \frac{f^{(n)}(a)}{n!}(x-a)^n + R_n(x) = P_n(x) + R_n(x)$$

where

$$R_n(x) = \frac{f^{(n+1)}(c)}{(n+1)!}(x-a)^{n+1} \quad \text{for some } c \text{ between } x \text{ and } a$$

and

$$|R_n(x)| \le \frac{M|x-a|^{n+1}}{(n+1)!} \quad \text{where } |f^{(n+1)}(t)| \le M \text{ for all } t \text{ between } x \text{ and } a.$$

Example #8: The approximation $e^x \approx 1 + x + x^2/2$ is used when x is small. Use the Remainder Theorem to estimate the error when $|x| < 0.1$.

$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots + \frac{x^n}{n!}$ Centered at 0

$R_2(x) = \frac{f^3(c)(x-0)^3}{3!} = \frac{f^3(c)x^3}{3!}$

$f(x) = e^x$

$f'(x) = e^x$

Since $|x| < 0.1$ and $0 < c < x$, $c < 0.1$

$f^3(c) = e^c < 3^{0.1}$

$|R_2(x)| = \frac{f^3(c)|x|^3}{3!} < \frac{3^{0.1}(0.1)^3}{3!} \approx 1.86 \times 10^{-4}$

Ex. 9 Find the Maclaurin series of e^{5x}

$\sum_{n=0}^{\infty} \frac{f^n(0)x^n}{n!}$

$f(x) = e^{5x} = 1$

$f'(x) = 5e^{5x} = 5$

$f''(x) = 25e^{5x} = 25$

$1 + 5x + \frac{25x^2}{2!} + \cdots + \frac{(5x)^n}{n!}$

$\sum_{n=0}^{\infty} \frac{(5x)^n}{n!}$

$\left|\frac{(5x)^{n+1}}{(n+1)!} \cdot \frac{n!}{(5x)^n}\right| = \left|\frac{5x}{n+1}\right| = \frac{5}{n+1}|x|$

$\lim_{n\to\infty} \frac{5}{n+1}|x| = 0$

Converges for any value of x

$R = \infty$

$-\infty < x < \infty$