# The Determinant
## Section 3.3 (**Hartman**)

In this section we'll learn another operation on square matrices that returns a number, called the ***determinant***.

### The Determinant of $2 \times 2$ Matrices

Let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. The determinant of A, denoted by det(A) or $\begin{vmatrix} a & b \\ c & d \end{vmatrix}$ is equal to $ad - bc$.

***Recall:*** We saw that a $2 \times 2$ matrix A has inverse $\dfrac{1}{ad-bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$.

We can rephrase the above statement now: If det(A) $\neq 0$, then $A^{-1} = \dfrac{1}{det(A)}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$.

**Example #1:** Find the determinant of the following matrices and state whether they are invertible.

(a) $\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$

det $= 4 - 6 = -2 \neq 0$

Invertible

(b) $\begin{pmatrix} 1 & -3 \\ -2 & 6 \end{pmatrix}$.

det $= 6 - 6 = 0$

Not invertible

"How do we compute the determinant of matrices that are not $2 \times 2$?" We first need to define some terms.

> **Matrix Minor, Co-factor**
>
> Let A be an $n \times n$ matrix. The $i, j$ minor of A, denoted $A_{i,j}$ , is the determinant of the $(n-1) \times (n-1)$ matrix formed by deleting the $i$ th row and $j$th column of A. The $i, j$-cofactor of A is the number
> $$c_{ij} = (-1)^{i+j} A_{i,j}.$$

**Example #2:** Given $A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix}$. Evaluate;

(a) $C_{1,2}$

$(-1)^{1+3} A_{1,2}$

$\begin{vmatrix} 4 & 6 \\ 7 & 9 \end{vmatrix} = 36-42$

$6$

(b) $C_{3,1}$

$(-1)^{3+1} A_{3,1}$

$\begin{vmatrix} 2 & 3 \\ 5 & 6 \end{vmatrix} = 12-15$

$-3$

**Cofactor Expansion**

Let A be an $n \times n$ matrix. The cofactor expansion of A along the $i$ th row is the sum

$$a_{i,1}C_{i,1} + a_{i,2}C_{i,2} + \ldots + a_{i,n}C_{i,n}.$$

The cofactor expansion of A down the jth column can be defined similarly.

**Example #3:** Find the cofactor expansion along the second row of $A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix}$.

$a_{2,1}C_{2,1} + a_{2,2}C_{2,2} + a_{2,3}C_{2,3}$

$4(-1)^3 \begin{vmatrix} 2 & 3 \\ 8 & 9 \end{vmatrix} = -4(18-24) = 24$

$5(-1)^4 \begin{vmatrix} 1 & 3 \\ 7 & 9 \end{vmatrix} = 5(9-21) = -60$

$6(-1)^5 \begin{vmatrix} 1 & 2 \\ 7 & 8 \end{vmatrix} = -6(8-14) = 36$

$24-60+36 = 0 = \det(A)$

**The Determinant**

The determinant of an $n \times n$ matrix A, denoted det(A) or $|A|$, is a number given by the following:

- if A is a $1 \times 1$ matrix A $= (a)$, then det(A) $= a$.
- if A is a $2 \times 2$ matrix $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, then det(A) $= ad - bc$.
- if A is an $n \times n$ matrix, where $n \geq 2$, then det(A) is the number found by taking the cofactor expansion along the first row of A. That is,

$$\det(A) = a_{1,1}C_{1,1} + a_{1,2}C_{1,2} + \ldots + a_{1,n}C_{1,n}.$$

**Example #4:** Find the determinant of;

(a) $\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix}$

$1(-1)^2\begin{vmatrix} 5 & 6 \\ 8 & 9 \end{vmatrix} = (45-48) = -3$

$2(-1)^3\begin{vmatrix} 4 & 6 \\ 7 & 9 \end{vmatrix} = -2(36-42) = 12$

$3(-1)^4\begin{vmatrix} 4 & 5 \\ 7 & 8 \end{vmatrix} = 3(32-35) = -9$

$\det(A) = -3+12-9 = 0$

(b) $\begin{pmatrix} 7 & -1 & 1 \\ 3 & 0 & 3 \\ 6 & 2 & -1 \end{pmatrix}$

$7(-1)^2\begin{vmatrix} 0 & 3 \\ 2 & -1 \end{vmatrix} = 7(-6) = -42$

$-1(-1)^3\begin{vmatrix} 3 & 3 \\ 6 & -1 \end{vmatrix} = -3-18 = -21$

$1(-1)^4\begin{vmatrix} 3 & 0 \\ 6 & 2 \end{vmatrix} = 6$

$\det(B) = -42-21+6 = -57$

(c) $\begin{pmatrix} 1 & 3 & 0 & 2 \\ 3 & 1 & -1 & 2 \\ 6 & 1 & 0 & -1 \\ 4 & -2 & 3 & 0 \end{pmatrix}$

$1(-1)^2 \begin{vmatrix} 1 & -1 & 2 \\ 1 & 0 & -1 \\ -2 & 3 & 0 \end{vmatrix} \Rightarrow 1(-1)^3 \begin{vmatrix} -1 & 2 \\ 3 & 0 \end{vmatrix} = -(-6) = 6 \Rightarrow 7$

$-1(-1)^5 \begin{vmatrix} 1 & -1 \\ -2 & 3 \end{vmatrix} = 3-2 = 1$

$3(-1)^3 \begin{vmatrix} 3 & -1 & 2 \\ 6 & 0 & -1 \\ 4 & 3 & 0 \end{vmatrix} \Rightarrow 6(-1)^3 \begin{vmatrix} -1 & 2 \\ 3 & 0 \end{vmatrix} = -6(-6) = -36 \Rightarrow 49$

$-1(-1)^5 \begin{vmatrix} 3 & -1 \\ 4 & 3 \end{vmatrix} = 9+4 = 13$

$0(-1)^4 \begin{vmatrix} 3 & 1 & 2 \\ 6 & 1 & -1 \\ 4 & -2 & 0 \end{vmatrix} = 0$

$2(-1)^5 = \begin{vmatrix} 3 & 1 & -1 \\ 6 & 1 & 0 \\ 4 & -2 & 3 \end{vmatrix} \Rightarrow 6(-1)^3 \begin{vmatrix} 1 & -1 \\ -2 & 3 \end{vmatrix} = -6(3-2) = -6 \Rightarrow 7$

$1(-1)^4 \begin{vmatrix} 3 & -1 \\ 4 & 3 \end{vmatrix} = 9+4 = 13$

$7 - 3(49) - 2(7) = -154$