The Matrix Inverse
Section 2.6 & 2.7 (**Hartman**)

Invertible Matrices and the Inverse of A:
Let A and X be $n \times n$ matrices where AX = I = XA. Then:

1. A is invertible.
2. X is the inverse of A, denoted by A^{-1}.

Example #1: Let $A = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$. Find a matrix X such that $AX = I$.

$$[A|I] = \left(\begin{array}{cc|cc} 2 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 \end{array}\right) \xrightarrow{-2R_2+R_1 \to R_2} \left(\begin{array}{cc|cc} 2 & 1 & 1 & 0 \\ 0 & -1 & 1 & -2 \end{array}\right) \xrightarrow{R_1+R_2 \to R_1} \left(\begin{array}{cc|cc} 2 & 0 & 2 & -2 \\ 0 & -1 & 1 & -2 \end{array}\right) \xrightarrow[-R_2 \to R_2]{\frac{1}{2}R_1 \to R_1} \underbrace{\left(\begin{array}{cc|cc} 1 & 0 & 1 & -1 \\ 0 & 1 & -1 & 2 \end{array}\right)}_{[I \,|\, A^{-1}]}$$

$$X = A^{-1} = \begin{pmatrix} 1 & -1 \\ -1 & 2 \end{pmatrix}$$

Uniqueness of Solutions to $AX = I_n$:
Let A be an $n \times n$ matrix and let X be a matrix where $AX = I_n$. Then X is unique; it is the only matrix that satisfies this equation.

Fact: Not all matrices are invertible.

Procedure:

- Let A be an $n \times n$ matrix.
- To find A^{-1}, put the augmented matrix $[\ \mathrm{A}\ \ \mathrm{I}_n]$ into reduced row echelon form.
- If the result is of the form $[\mathrm{I}_n\ \ \mathrm{X}]$, then $\mathrm{A}^{-1} = \mathrm{X}$. If not, then A is not invertible.

Example #2: Is the matrix $A = \begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix}$ invertible?

$$\left(\begin{array}{cc|cc} 1 & 2 & 1 & 0 \\ 2 & 4 & 0 & 1 \end{array}\right) \xrightarrow{R_2 - 2R_1 \to R_2} \left(\begin{array}{cc|cc} 1 & 2 & 1 & 0 \\ 0 & 0 & -2 & 1 \end{array}\right) \xrightarrow{-\frac{1}{2}R_2 \to R_2} \left(\begin{array}{cc|cc} 1 & 2 & 1 & 0 \\ 0 & 0 & 1 & -1/2 \end{array}\right) \xrightarrow{R_1 - R_2 \to R_1} \underbrace{\left(\begin{array}{cc|cc} 1 & 2 & 0 & 0 \\ 0 & 0 & 1 & -1/2 \end{array}\right)}_{\neq I}$$

A is not invertible

The Inverse of a 2×2 Matrix:

Let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. A is invertible if and only if $ad - bc \neq 0$.

If $ad - bc \neq 0$, then

$$A^{-1} = \frac{1}{ad - bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}.$$

Example #3: Find the inverse of $A = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$, if it exists.

$2(1) - 1(1) = 1 \neq 0$

$A^{-1} = \frac{1}{1}\begin{pmatrix} 1 & -1 \\ -1 & 2 \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ -1 & 2 \end{pmatrix}$

Example #4: Find the inverse of $A = \begin{pmatrix} 1 & 1 & -1 \\ 1 & -1 & 1 \\ 1 & 2 & 3 \end{pmatrix}$, if it exists.

$$\left(\begin{array}{ccc|ccc} 1 & 1 & -1 & 1 & 0 & 0 \\ 1 & -1 & 1 & 0 & 1 & 0 \\ 1 & 2 & 3 & 0 & 0 & 1 \end{array}\right) \begin{array}{l} R_2 - R_1 \to R_2 \\ R_3 - R_1 \to R_3 \end{array} \left(\begin{array}{ccc|ccc} 1 & 1 & -1 & 1 & 0 & 0 \\ 0 & -2 & 2 & -1 & 1 & 0 \\ 0 & 1 & 4 & -1 & 0 & 1 \end{array}\right) 2R_3 + R_2 \to R_3 \left(\begin{array}{ccc|ccc} 1 & 1 & -1 & 1 & 0 & 0 \\ 0 & -2 & 2 & -1 & 1 & 0 \\ 0 & 0 & 10 & -3 & 1 & 2 \end{array}\right) \tfrac{1}{10}R_3 \to R_3 \left(\begin{array}{ccc|ccc} 1 & 1 & -1 & 1 & 0 & 0 \\ 0 & -2 & 2 & -1 & 1 & 0 \\ 0 & 0 & 1 & \frac{-3}{10} & \frac{1}{10} & \frac{1}{5} \end{array}\right)$$

$$\begin{array}{l} R_2 - 2R_3 \to R_2 \\ R_1 + R_3 \to R_1 \end{array} \left(\begin{array}{ccc|ccc} 1 & 1 & 0 & 7/10 & 1/10 & 1/5 \\ 0 & -2 & 0 & -2/5 & 4/5 & -2/5 \\ 0 & 0 & 1 & -3/10 & 1/10 & 1/5 \end{array}\right) \tfrac{-1}{2}R_2 \to R_2 \left(\begin{array}{ccc|ccc} 1 & 1 & 0 & 7/10 & 1/10 & 1/5 \\ 0 & 1 & 0 & 1/5 & -2/5 & 1/5 \\ 0 & 0 & 1 & -3/10 & 1/10 & 1/5 \end{array}\right) R_1 - R_2 \to R_1 \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 1/2 & 1/2 & 0 \\ 0 & 1 & 0 & 1/5 & -2/5 & 1/5 \\ 0 & 0 & 1 & -3/10 & 1/10 & 1/5 \end{array}\right)$$

$$A^{-1} = \begin{pmatrix} 1/2 & 1/2 & 0 \\ 1/5 & -2/5 & 1/5 \\ -3/10 & 1/10 & 1/5 \end{pmatrix}$$

Invertible Matrix Theorem:
Let A be an $n \times n$. The following statements are equivalent.

(a) A is invertible.

(b) There exists a matrix B such that BA = I.

(c) There exists a matrix C such that AC = I.

(d) The reduced row echelon form of A is I.

(e) The equation $A\vec{x} = \vec{b}$ has exactly one solution for every $n \times 1$ vector $\vec{b}$ (namely, $\vec{x} = A^{-1}\vec{b}$).

(f) The equation $A\vec{x} = \vec{0}$ has exactly one solution (namely, $\vec{x} = \vec{0}$).

NOTE: If A is not invertible, then $A\vec{x} = \vec{b}$ has either infinitely many solutions or no solution.

Properties of Invertible Matrices:
Let A and B be $n \times n$ invertible matrices. Then:

1. AB is invertible; $(\mathrm{AB})^{-1} = \mathrm{B}^{-1}\mathrm{A}^{-1}$.

2. $(\mathrm{A}^{-1})^{-1} = \mathrm{A}$.

3. kA is invertible for any nonzero scalar n; $(k\mathrm{A})^{-1} = \dfrac{1}{k}\,\mathrm{A}^{-1}$

4. If A is a diagonal matrix, with diagonal entries $d_1, d_2, \cdots, d_n$, where none of the diagonal entries are 0, then A^{-1} exists and is a diagonal matrix. Furthermore, the diagonal entries of A^{-1} are $1/d_1, 1/d_2, , 1/d_n$.

5. If a product AB is not invertible, then A or B is not invertible.

6. If A or B are not invertible, then AB is not invertible.