Vector Solutions to Linear Systems
Section 2.4 (**Hartman**)

In this section we consider equations of the type $A\vec{x} = \vec{b}$, where we know the matrix A and the vector $\vec{b}$. We will want to find what vector $\vec{x}$ satisfies this equation. According to the matrix A one can determine the dimension of $\vec{x}$.
The process utilizes the methods we learned in Sections 1.2 and 1.3. However, the way we express the solution would be different.

Notation: The n- dimensional vector x is denoted by $\vec{x}$ and defined as;

$$\vec{x} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$$

Example #1: Solve the linear system $A\vec{x} = \vec{0}$ and write the solution in vector form, where

$$A = \begin{pmatrix} 2 & -3 \\ -2 & 3 \end{pmatrix}$$

$$\underbrace{\begin{pmatrix} 2 & -3 \\ -2 & 3 \end{pmatrix}}_{2\times 2} \underbrace{\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}}_{2\times 1} = \underbrace{\begin{pmatrix} 0 \\ 0 \end{pmatrix}}_{2\times 1}$$

$$\begin{array}{cc} x_1 & x_2 \end{array}$$
$$\left(\begin{array}{cc|c} 2 & -3 & 0 \\ -2 & 3 & 0 \end{array}\right) \xrightarrow{\text{reduce}} \left(\begin{array}{cc|c} 1 & -3/2 & 0 \\ 0 & 0 & 0 \end{array}\right)$$

$x_1 - \frac{3}{2}x_2 = 0$

$x_1 = \frac{3}{2}x_2 \quad x_2$ is free

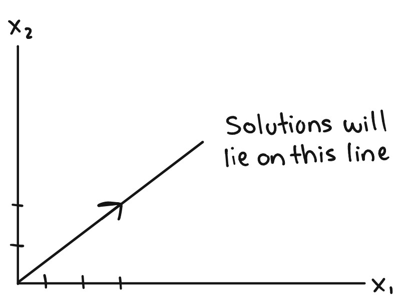
Infinitely many solutions

$$\vec{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 3/2\, x_2 \\ x_2 \end{pmatrix} = x_2 \begin{pmatrix} 3/2 \\ 1 \end{pmatrix} = x_2 \vec{v}$$

Graph:

$x_2 = 2$

$x_2\vec{v} = \begin{pmatrix} 3 \\ 2 \end{pmatrix}$



Example #2: Solve the linear system $A\vec{x} = \vec{b}$ and write the solution in vector form, where

$$A = \begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix} \qquad \vec{b} = \begin{pmatrix} 3 \\ 6 \end{pmatrix}$$

$$\left(\begin{array}{cc|c} 1 & 2 & 3 \\ 2 & 4 & 6 \end{array}\right) \xrightarrow{\text{reduce}} \left(\begin{array}{cc|c} 1 & 2 & 3 \\ 0 & 0 & 0 \end{array}\right)$$

$x_1 + 2x_2 = 3$

$x_1 = 3 - 2x_2$ x_2 is free

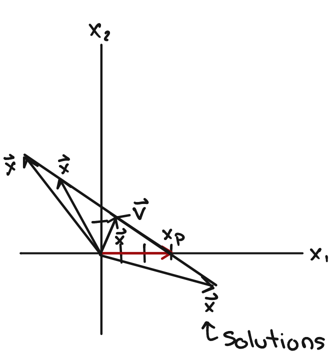
$$\vec{x} = \begin{pmatrix} 3-2x_2 \\ x_2 \end{pmatrix} = \begin{pmatrix} 3 \\ 0 \end{pmatrix} + x_2 \begin{pmatrix} -2 \\ 1 \end{pmatrix} = \vec{x}_p + x_2 \vec{v}$$

particular solution

Homogeneous solution

Infinitely many solutions

Graph:



A system of linear equations is *homogeneous* if the constants in each equation are zero. A homogeneous system of equations can be written in vector form as $A\vec{x} = \vec{0}$.

All homogeneous linear systems are consistent!

Solutions of Consistent Systems:
Let $A\vec{x} = \vec{b}$ be a consistent system of linear equations.

- If $A\vec{x} = \vec{0}$ has exactly one solution, then $A\vec{x} = \vec{b}$ has exactly one solution.
- If $A\vec{x} = \vec{0}$ has infinite solutions, then $A\vec{x} = \vec{b}$ has infinite solutions.

If $A\vec{x} = \vec{b}$ is inconsistent, the solution $A\vec{x} = \vec{0}$ does not have any effect on changing that

Example #3: Rewrite the linear system

$$\begin{array}{rcrcrcrcrcr} x_1 & + & 2x_2 & - & 3x_3 & + & 2x_4 & + & 7x_5 & = & 2 \\ 3x_1 & + & 4x_2 & + & 5x_3 & + & 2x_4 & + & 3x_5 & = & -4 \end{array}$$

as a matrix–vector equation, solve the system using vector notation, and give the solution to the related homogeneous equations.

$$\left(\begin{array}{ccccc|c} 1 & 2 & -3 & 2 & 7 & 2 \\ 3 & 4 & 5 & 2 & 3 & -4 \end{array}\right) \xrightarrow{\text{reduce}} \left(\begin{array}{ccccc|c} 1 & 0 & 11 & -2 & -11 & -8 \\ 0 & 1 & -7 & 2 & 9 & 5 \end{array}\right)$$

$x_1 + 11x_3 - 2x_4 - 11x_5 = -8$ $\quad$ $x_1 = -8 - 11x_3 + 2x_4 + 11x_5$

$x_2 - 7x_3 + 2x_4 + 9x_5 = 5$ $\quad$ $x_2 = 5 + 7x_3 - 2x_4 - 9x_5$ $\quad$ x_3, x_4, x_5 are free

$$\vec{x} = \begin{pmatrix} -8 - 11x_3 + 2x_4 + 11x_5 \\ 5 + 7x_3 - 2x_4 - 9x_5 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} = \underbrace{\begin{pmatrix} -8 \\ 5 \\ 0 \\ 0 \\ 0 \end{pmatrix}}_{\vec{x}_p} + \underbrace{x_3 \begin{pmatrix} -11 \\ 7 \\ 1 \\ 0 \\ 0 \end{pmatrix} + x_4 \begin{pmatrix} 2 \\ -2 \\ 0 \\ 1 \\ 0 \end{pmatrix} + x_5 \begin{pmatrix} 11 \\ -9 \\ 0 \\ 0 \\ 1 \end{pmatrix}}_{\substack{\text{homogeneous solution} \\ x_3\vec{u} + x_4\vec{v} + x_5\vec{w}}}$$