Cylindrical and Spherical Coordinates

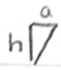
Write an equation for a sphere of radius 5 centered at the origin, (a, b, c)

$r^2 + z^2 = 25$

$$(x-a)^2+(y-b)^2+(z-c)^2=25 \qquad (r\cos\theta-a)^2+(r\sin\theta-b)^2+(z-c)=25$$

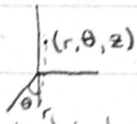
$$x^2-2ax+a^2+y^2-2by+b^2+(z-c)^2=25$$

$$r^2-2ar\cos\theta-2br\sin\theta+a^2+b^2+(z-c)^2=25$$

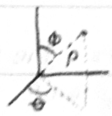


$\frac{z}{h}=\frac{r}{a}$ $\quad dV=dz\underbrace{r\,dr\,d\theta}_{dA}$

$z=\frac{hr}{a}$



Cylindrical

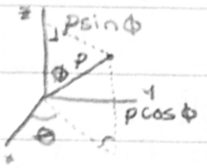


Spherical

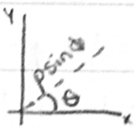
p = distance from the origin ≥ 0

ϕ = angle from the positive z-axis, $0 \leq \phi \leq \pi$

θ = polar angle, $0 \leq \theta \leq 2\pi$



$z = p\cos\phi$

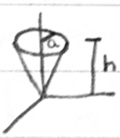


$x = p\sin\phi\cos\theta$

$y = p\sin\phi\sin\theta$

$p^2 = x^2+y^2+z^2 \qquad \tan\theta=\frac{y}{x} \qquad \cos\phi=\frac{z}{p}$

$dV = p^2\sin\phi\,dp\,d\theta\,d\phi$



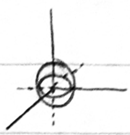
$\tan\phi=\frac{a}{h} \qquad \phi=\arctan\left(\frac{a}{h}\right)$

$0<\phi<\frac{\pi}{2}$

$$V=\iiint_D 1\,dV=\int_0^{2\pi}\int_0^{\tan^{-1}(\frac{a}{h})}\int_0^{h\sec\phi} p^2\sin\phi\,dp\,d\phi\,d\theta$$

$$=\int_0^{2\pi}\int_0^{\tan^{-1}(\frac{a}{h})} \frac{1}{3}h^3\sec^3\phi\sin\phi\,d\phi\,d\theta=\frac{1}{3}\pi a^2 h$$

Volume of a sphere of radius 2



$$V=\int_0^{\pi}\int_0^{2}\int_0^{2\pi} p^2\sin\phi\,d\theta\,dp\,d\phi=\int_0^{\pi}\int_0^{2} 2\pi p^2\sin\phi\,dp\,d\phi=\int_0^{\pi}\frac{16\pi}{3}\sin\phi\,d\phi$$

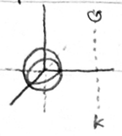
$$=\boxed{\frac{4}{3}\pi(2)^3} = \int_0^{\pi}\sin\phi\,d\phi\int_0^2 p^2\,dp\int_0^{2\pi}d\theta$$

A sphere of radius a with uniform density and mass M is rotated around an axis through its center. Find the moment of inertia.

$$I=\iiint_R \underbrace{\text{density}}_{\frac{M}{V}=\frac{3M}{4\pi a^3}}\cdot\underbrace{\text{distance}^2}_{x^2+y^2=p^2\sin^2\phi}\,dV=\text{density}\cdot\int_0^a\int_0^{\pi}\int_0^{2\pi} p^4\sin^3\phi\,d\theta\,d\phi\,dp=\text{density}\cdot\frac{8}{15}\pi a^5$$

$$=\boxed{\frac{2}{5}a^2M}$$

Sphere of radius a with uniform density, with the rotational axis k units from the center

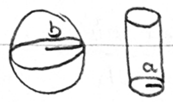


$$I = \frac{3M}{4\pi a^3}\iiint x^2+(y-k)^2\,dV = \frac{3M}{4\pi a^3}\Big(\underbrace{\iiint x^2+y^2\,dV}_{\frac{8}{15}\pi a^5} - \underbrace{\iiint 2ky\,dV}_{0} + \underbrace{\iiint k^2\,dV}_{V}\Big) = \boxed{\frac{2}{5}a^2M + k^2M}$$

Parallel axis theorem for spheres

Sphere, remove the "core"

Cylindrical hole through the sphere
Which coordinate system?



Set up an integral for volume in cylindrical and spherical coordinates, then compute

$x^2+y^2+z^2=b^2 \Rightarrow \rho=b \Rightarrow r^2+z^2=b^2 \qquad x^2+y^2=a^2 \Rightarrow \rho\sin\phi=a \Rightarrow r=a$

$$\int_0^{2\pi}\int_{-\sqrt{b^2-a^2}}^{\sqrt{b^2-a^2}}\int_a^{\sqrt{b^2-z^2}} r\,dr\,dz\,d\theta = \frac{1}{2}\int_0^{2\pi}\int_{-\sqrt{b^2-a^2}}^{\sqrt{b^2-a^2}} b^2-a^2-z^2\,dz\,d\theta = \int_0^{2\pi}(b^2-a^2)^{3/2}-\frac{1}{3}(b^2-a^2)^{3/2}\,d\theta$$

$$= \frac{2}{3}(b^2-a^2)^{3/2}\int_0^{2\pi}d\theta = \boxed{\frac{4\pi}{3}(b^2-a^2)^{3/2}}$$

Volume of a sphere when $a=0$
Volume of a cylinder when $b=0$

$$\int_0^{2\pi}\int_a^b\int_{-\sqrt{b^2-r^2}}^{\sqrt{b^2-r^2}} r\,dz\,dr\,d\theta = \int_0^{2\pi}\int_a^b 2r\sqrt{b^2-r^2}\,dr\,d\theta = -\frac{2}{3}\int_0^{2\pi}-(b^2-a^2)^{3/2}\,d\theta = \boxed{\frac{4\pi}{3}(b^2-a^2)^{3/2}}$$

$$\int_0^{2\pi}\int_{\arcsin(\frac{a}{b})}^{\pi-\arcsin(\frac{a}{b})}\int_{a\csc\phi}^{b} \rho^2\sin\phi\,d\rho\,d\phi\,d\theta = \int_0^{2\pi}-\frac{1}{3}b^3\cos\phi+\frac{1}{3}a^3\cot\phi\Big|_{\arcsin(a/b)}^{\pi-\arcsin(a/b)}\,d\theta = \boxed{\frac{4\pi}{3}(b^2-a^2)^{3/2}}$$