Trigonometric Integrals
Section 8.3

Idea: We can solve more complicated trigonometric integrals using u-substitution along with trigonometric identities.

$\int \cos^2 x =$
$\frac{1}{2}\left(x + \frac{\sin 2x}{2}\right) + C$

Identities:

$$\sin^2(x) + \cos^2(x) = 1$$
$$\sin^2(x) = \frac{1 - \cos(2x)}{2}$$
$$\cos^2(x) = \frac{1 + \cos(2x)}{2}$$

$\int \sin(mx) =$
$\frac{-\cos(mx)}{m} + C$

$\int \cos(mx) =$
$\frac{\sin(mx)}{m} + C$

Example #1: $\int \sin^3(x)\cos^2(x)\,dx$

$\int \sin^2 x \cos^2 x \sin x \, dx$ $\quad \frac{d}{dx}\cos x = -\sin x \Rightarrow -d\cos x = \sin x \, dx$

$u = \cos x$

$du = -\sin x \, dx$

$\int (1 - \cos^2 x)\cos^2 x \sin x \, dx$

$-\int (1 - u^2) u^2 \, du$

$-\int u^2 - u^4 \, du$

$-\frac{u^3}{3} + \frac{u^5}{5} + C$ $\quad -\frac{\cos^3 x}{3} + \frac{\cos^5 x}{5} + C$

Example #2: $\int \sin^2(x)\cos^3(x)\,dx$

$\int \sin^2 x \cos^2 x \cos x \, dx$

$u = \sin x$

$du = \cos x \, dx$

$\int \sin^2 x (1 - \sin^2 x)\cos x \, dx$

$\int u^2 (1 - u^2) \, du$

$\frac{u^3}{3} - \frac{u^5}{5} + C$

$\frac{\sin^3 x}{3} - \frac{\sin^5 x}{5} + C$

Example #3: $\int \sin^2(x)\cos^2(x)\,dx$

$\int \left(\frac{1-\cos 2x}{2}\right)\left(\frac{1+\cos 2x}{2}\right) dx$

$\frac{1}{4}\int 1-\cos^2(2x)\, dx$

$\frac{1}{8}\int 1-\cos(4x)\, dx$

$\frac{1}{8}\left(x-\frac{\sin 4x}{4}\right)+C$

Summary: To evaluate integrals of the form $\int \sin^m(x)\cos^n(x)\,dx$

1. **m is odd:** Write $m = 2k+1$.Then, $u=\cos x$

$$\sin^m(x) = \sin^{2k+1}(x) = (\sin^2(x))^k \sin(x) = (1-\cos^2(x))\sin(x).$$

Next combine $\sin(x)\ dx$ which equals to $-d(\cos(x))$ and solve!

2. **n is odd:** Write $n = 2k+1$. Then, $u=\sin x$

$$\cos^n(x) = \cos^{2k+1}(x) = (\cos^2(x))^k \cos(x) = (1-\sin^2(x))\cos(x).$$

Next combine $\cos(x)\ dx$ which equals to $d(\sin(x))$ and solve!

3. **If both m, n even:** Substitute,

$$\sin^2(x) = \frac{1-\cos(2x)}{2}, \qquad \cos^2(x) = \frac{1+\cos(2x)}{2}$$

and simplify!

Notes:
(1) We will only be covering powers of sine and cosine. Your textbook also discusses powers of tangent and secant.

(2) Maple will frequently solve these problems a different way, so the answers it displays may look very different (though they are equivalent, which can be shown using trigonometric identities).

Extra practice:

1. $\displaystyle\int \sin^5(2\theta)\cos^4(2\theta)\ d\theta$

$\int \sin^4(2\theta)\cos^4(2\theta)\sin(2\theta)\,d\theta$

$u = \cos 2\theta$

$du = -2\sin 2\theta\ d\theta$

$\int (1-\cos^2(2\theta))^2 \cos^4 2\theta \sin 2\theta\ d\theta$

$\int (1-u^2)^2 u^4\,du$

$-\frac{1}{2}\int u^4 - 2u^6 + u^8\,du$

$-\frac{1}{2}\left(\frac{u^5}{5} - \frac{2u^7}{7} + \frac{u^9}{9}\right) + C$

$-\frac{1}{2}\left(\frac{\cos^5(2\theta)}{5} - \frac{2\cos^7(2\theta)}{7} + \frac{\cos^9(2\theta)}{9}\right) + C$

2. $\displaystyle\int \sin^3(x)\cos^3(x)\ dx$

$\int \sin^2 x \cos^2 x\ \sin x\ \cos x\ dx$

$u = \sin x$
$du = \cos x\ dx$

$\int \sin^3 x(1-\sin^2 x)\cos x\ dx$

$\int u^3(1-u^2)\,du$

$\int u^3 - u^5\,du$

$\frac{u^4}{4} - \frac{u^6}{6} + C$

$\frac{\sin^4 x}{4} - \frac{\sin^6 x}{6} + C$