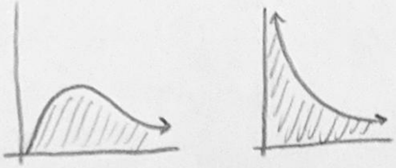
Improper Integrals
Section 8.8

Question: An integral is "area." What happens when the "height" or "width" is infinite?

e. g.



Type I: Infinite domain (i.e., infinite "width")

- If $f(x)$ is continuous on $[a,\infty)$, then $$\int_a^\infty f(x)\,dx = \lim_{b\to\infty}\int_a^b f(x)\,dx.$$
- If $f(x)$ is continuous on $(-\infty,b]$, then $$\int_{-\infty}^b f(x)\,dx = \lim_{a\to-\infty}\int_a^b f(x)\,dx.$$
- If $f(x)$ is continuous on $(-\infty,\infty)$, then $$\int_{-\infty}^\infty f(x)\,dx = \int_{-\infty}^c f(x)\,dx + \int_c^\infty f(x)\,dx$$ where c is any real number.

Example #1: $\int_1^\infty e^{-x}\,dx$ $= \lim_{n\to\infty}\int_1^n e^{-x}\,dx$

$$\lim_{n\to\infty} -e^{-x}\Big|_1^n = \lim_{n\to\infty}\left[\underset{\downarrow\,0}{\frac{-1}{e^n}} + \frac{1}{e^1}\right]$$

$$\frac{1}{e}$$

Definition: (True for both Type I and Type II)

- If the limit of the integral exists and is finite, we say that the improper integral **converges** and the limit is the value of the improper integral.
- If the limit does not exist or is infinite, we say that the improper integral **diverges**.

Example #2: $\int_{-\infty}^{-1} xe^x\,dx = \lim_{n\to-\infty} \int_n^{-1} xe^x\,dx$

$u = x \quad dv = e^x\,dx$

$du = dx \quad v = e^x$

$\lim_{n\to\infty} xe^x - \int e^x\,dx \Big|_n^{-1} = \lim_{n\to\infty} xe^x - e^x \Big|_n^{-1}$

$\lim_{n\to-\infty} \left[-e^{-1} - e^{-1} - (\underset{\downarrow 0}{ne^n} - \underset{\downarrow 0}{e^n})\right] = \frac{-2}{e}$ Converges

Example #3: For what values of p $\int_1^\infty \frac{1}{x^p}\,dx$ converges?

$p = 1$

$\int_1^\infty \frac{1}{x}\,dx = \lim_{n\to\infty} \int_1^n \frac{1}{x}\,dx = \lim_{n\to\infty} \ln|x| \Big|_1^n$

$\lim_{n\to\infty} [\underset{\downarrow \infty}{\ln|n|} - \underset{\downarrow 0}{\ln|1|}] = \infty$ diverges

$p \neq 1$

$\int_1^\infty \frac{1}{x^p}\,dx = \lim_{n\to\infty} \int_1^n \frac{1}{x^p}\,dx = \lim_{n\to\infty} \frac{x^{-p+1}}{1-p} \Big|_1^n$

$\lim_{n\to\infty} \left[\frac{n^{-p+1}}{1-p} - \frac{1}{1-p}\right] = \lim_{n\to\infty} \frac{1}{1-p}\left(\frac{1}{n^{p-1}} - 1\right)$

$p-1 > 0$ when $p > 1 \Rightarrow$ Converges to $\frac{1}{p-1}$

$p-1 < 0$ when $p < 1 \Rightarrow$ Diverges

Type II: Discontinuity in domain (i.e., infinite "height")

- If $f(x)$ is continuous on $(a, b]$ and discontinuous at a, then

$$\int_a^b f(x)\,dx = \lim_{c\to a^+} \int_c^b f(x)\,dx.$$

- If $f(x)$ is continuous on $[a, b)$ and discontinuous at b, then

$$\int_a^b f(x)\,dx = \lim_{c\to b^-} \int_a^c f(x)\,dx.$$

- If $f(x)$ is discontinuous at c, where $a < c < b$ and continuous on $[a, c) \cup (c, b]$, then

$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx.$$

Example #4: $\displaystyle\int_{-1}^{1} \frac{1}{x^2}\,dx$

$$\int_{-1}^{0} \frac{1}{x^2}\,dx + \int_0^1 \frac{1}{x^2}\,dx$$

$$\lim_{b\to 0^-} \int_{-1}^{b} \frac{1}{x^2}\,dx + \lim_{a\to 0^+} \int_a^1 \frac{1}{x^2}\,dx$$

$$\lim_{b\to 0^-} \left[\frac{-1}{x}\right]_{-1}^{b} + \lim_{a\to 0^+} \left[\frac{-1}{x}\right]_a^1$$

$$\lim_{b\to 0^-} \Big[\underset{\downarrow\,\infty}{\frac{-1}{b}} - 1\Big] + \lim_{a\to 0^+} \Big[-1 + \underset{\downarrow\,\infty}{\frac{1}{a}}\Big] = \infty \text{ diverges}$$

Example #5: $\int_0^2 \frac{1}{\sqrt{x}}\, dx$

$\lim_{n\to 0^+} \int_n^2 \frac{1}{\sqrt{x}}\, dx$

$\lim_{n\to 0^+} 2\sqrt{x}\Big|_n^2$

$\lim_{n\to 0^+} [2\sqrt{2} - 2\underbrace{\sqrt{n}}_{\downarrow\, 0}] = 2\sqrt{2}$ converges

Example #6: $\int_1^\infty \frac{\ln(x)}{x}\, dx$

$\lim_{n\to\infty} \int_1^n \frac{\ln x}{x}\, dx$

$u = \ln x \quad du = \frac{1}{x}\, dx$

$\lim_{n\to\infty} \int_1^n u\, du = \lim_{n\to\infty} \frac{1}{2}u^2\Big|_1^n$

$\lim_{n\to\infty} \frac{1}{2}\ln^2 x\Big|_1^n$

$\lim_{n\to\infty} \frac{1}{2}[\underbrace{\ln^2 n}_{\downarrow\, \infty} - \underbrace{\ln^2 1}_{\downarrow\, 0}] = \infty$ diverges

Tests for Convergence and Divergence

> **Theorem: Direct Comparison Test**
>
> Let f and g be continuous on $[a, \infty)$ with $0 \le f(x) \le g(x)$ for all $x \ge a$. Then
>
> - If $\displaystyle\int_a^\infty g(x)\,dx$ converges, then $\displaystyle\int_a^\infty f(x)\,dx$ also converges.
> - If $\displaystyle\int_a^\infty f(x)\,dx$ diverges, then $\displaystyle\int_a^\infty g(x)\,dx$ also diverges.

Example #7: Test the following integrals for convergence:

(a) $\displaystyle\int_1^\infty \frac{1}{\sqrt{x^2 - \cancel{20}}}\,dx$ 0.2 $\displaystyle > \int_1^\infty \frac{1}{x}\,dx = \lim_{n\to\infty}\int_1^n \frac{1}{x}\,dx$

$\displaystyle = \lim_{n\to\infty} \ln|x|\Big|_1^n = \lim_{n\to\infty} \underset{\infty}{\underset{\downarrow}{\ln|n|}} - \underset{0}{\underset{\downarrow}{\ln 1}} = \infty$ diverges

Then $\displaystyle\int_1^\infty \frac{1}{\sqrt{x^2-0.2}}\,dx$ diverges

(b) $\displaystyle\int_0^{\pi/2} \frac{\cos x}{\sqrt{x}}\,dx$ $\quad |\cos x| \le 1$

$0 \le \cos x \le 1$ on $[0, \frac{\pi}{2}]$

$\displaystyle \le \int_0^{\pi/2} \frac{1}{\sqrt{x}}\,dx = \int_0^{\pi/2} x^{-1/2}\,dx$

$\displaystyle = \lim_{a\to 0^+}\int_a^{\pi/2} x^{-1/2}\,dx = \lim_{a\to 0^+} 2x^{1/2}\Big|_a^{\pi/2} = \lim_{a\to 0^+} 2\sqrt{\frac{\pi}{2}} - \underset{0}{\underset{\downarrow}{2\sqrt{a}}} = 2\sqrt{\frac{\pi}{2}}$ converges

Then $\displaystyle\int_0^{\pi/2} \frac{\cos x}{\sqrt{x}}\,dx$ converges

Theorem: Limit Comparison Test

Let f and g be positive, continuous on $[a, \infty)$. If

$$\lim_{x\to\infty} \frac{f(x)}{g(x)} = L \quad \textit{such that} \quad 0 < L < \infty,$$ positive, finite limit

then $\int_a^\infty f(x)\,dx$ and $\int_a^\infty g(x)\,dx$ either both converge or both diverge.

Example #8: Test the convergence of $\int_1^\infty \frac{1-e^{-x}}{x}\,dx$

$\int_1^\infty \frac{1}{x}\,dx$

$f(x) = \frac{1-e^{-x}}{x}$

$g(x) = \frac{1}{x}$

$\lim_{x\to\infty} \frac{f(x)}{g(x)} = \lim_{x\to\infty} 1-\underbrace{e^{-x}}_{\to 0} = 1$ positive, finite

Since $\int_1^\infty \frac{1}{x}$ diverges, $\int_1^\infty \frac{1-e^{-x}}{x}$ also diverges

Ex #9: $\int_1^\infty \frac{1}{1+x^2}\,dx$