Numerical Integration
Section 8.7

There are (infinitely) many integrals that you still do not know how to solve (such as $\int \sin(x^2)\,dx$).

When we cannot find a workable anti-derivative for a function f, we can partition the interval of integration, replace f by closely fitting polynomial and approximate the definite integral of f.

There are also many applications for which you will only have data (i.e., the function value at certain points) rather than an actual function. This is why more accurate numerical approximations techniques are important.
In this section we study two such methods; the *Trapezoidal Rule* and *Simpson's Rule.*

Trapezoidal Approximation

To approximate $\int_a^b f(x)\,dx$ with n trapezoids, use;

$$T = \frac{\Delta x}{2}(y_0 + 2y_1 + 2y_2 + \cdots + 2y_{n-1} + y_n)$$

where $\Delta x = \dfrac{b-a}{n}$ and $y_k = f(x_k)$ for $x_k = a + k\Delta x$.

Example #1: Approximate $\int_1^3 (x^2+1)\,dx$ with 4 trapezoids of equal width.

1 1.5 2 2.5 3

x_0 $x_{n=4}$

$\Delta x = \frac{3-1}{4} = \frac{1}{2}$

$T = \frac{.5}{2}(y_0 + 2y_1 + 2y_2 + 2y_2 + y_4)$

$= \frac{1}{4}(2 + \frac{13}{2} + 10 + \frac{29}{2} + 10) = \frac{43}{4}$

$f(x) = x^2+4$

$y_k = f(x_k)$

$y_0 = f(1) = 2$

$y_1 = f(1.5) = \frac{13}{4}$

$y_2 = f(2) = 5$

$y_3 = f(2.5) = \frac{29}{4}$

$y_4 = f(3) = 10$

Example #2: The speed of a car known at a few times:

	t(min)	$v(t)$(mi/hr)
0	0	10
1/3	20	35
2/3	40	25
1	60	20

Assuming the car was traveling in a straight line, use the trapezoidal rule to estimate how far the car traveled in that hour.

$\Delta x = \frac{1-0}{3} = \frac{1}{3}$

$T = \frac{\Delta x}{2}(y_0 + 2y_1 + 2y_1 + y_3)$

$= \frac{1}{6}(10 + 2(35) + 2(25) + 20) = 25$ mi

Trapezoidal Error Estimate

If f'' is continuous and $|f''(x)| \leq M$ for all x in $[a, b]$, then the error E_T in the trapezoidal approximation of $\int_a^b f(x)\,dx$ using n trapezoids (or steps) satisfies

$$|E_T| \leq \frac{M(b-a)^3}{12n^2}.$$

Example #3: Find an upper bound for the error using the trapezoidal approximation in #1.

$\int_1^3 (x^2+1)\,dx$

$$|E_T| \leq \frac{2(2)^3}{12(4)^2} = \frac{16}{12(16} = \frac{1}{12}$$

$f(x) = x^2+1$

$|E_T| \leq \frac{1}{12} = .083$

$f'(x) = 2x$

$f''(x) = 2$

Check error

$|f''(x)| \leq 2$

$10.75 - 10.67 = .083$

$n = 4$

$[1, 3]$

$b - a = 2$

Simpson's Rule: (Approximation by parabolas)

To approximate $\int_a^b f(x)\,dx$, use;

$$S = \frac{\Delta x}{3}(y_0 + 4y_1 + 2y_2 + 4y_3 + \cdots + 2y_{n-2} + 4y_{n-1} + y_n)$$

where $\Delta x = \dfrac{b-a}{n}$ and $y_k = f(x_k)$ for $x_k = a + k\Delta x$.

The number n must be even, as each "parabola" covers two subintervals.

$$A = \int_{-h}^{h} Ax^2 + Bx + C\,dx = \frac{2Ah^3}{3} + 2Ch$$

$$y_0 = Ah^2 + Bh + C$$

$$y_1 = C$$

$$y_2 = Ax^2 + Bx + C$$

Example #4: Use Simpson's Rule with $n = 4$ to approximate $\int_0^4 3x^4\,dx$.

$$3\int_0^4 x^4\,dx$$

$$\Delta x = \frac{4-0}{4} = 1$$

$$f(x) = 3x^4$$

$$S = \frac{\Delta x}{3}(\overset{0}{\cancel{y_0}} + 4y_1 + 2y_2 + 4y_3 + y_4)$$

$$S = \frac{1}{3}(4(3)(1)^4 + 4(3)(2)^4 + 2(3)(3)^4 + 2(3)(4)^4)$$

$$= 616$$

actual $= 614.4$

Simpson's Rule Error Estimate

If $f^{(4)}$ is continuous and $|f^{(4)}(x)| \leq M$ for all x in $[a, b]$, then the error E_S in Simpson's Rule approximation of $\int_a^b f(x)\,dx$ using n steps satisfies

$$|E_S| \leq \frac{M(b-a)^5}{180n^4}.$$

Example #5: If we use Simpson's Rule with $n = 6$ to estimate the integral

$$\int_0^{\pi} \sin(2x)\,dx,$$

how close is the estimate to the actual answer? (Find the upper bound of the error.)

$f(x) = \sin 2x$

$f'(x) = 2\cos 2x$

$f''(x) = -4\sin 2x$

$f'''(x) = -8\cos 2x$

$f^{iv}(x) = 16\sin 2x$

$|f^{4}x| = 16|\underbrace{\sin 2x}_{\leq 1}|$

≤ 16

$M = 16$

$[a, b] = [0, \pi]$

$n = 6$

$|E_S| \leq \frac{16(\pi - 0)^5}{180(6)^4} = 0.021$