Autonomous Differential Equations
Section 2.7 (Noonburg)

Definition: A differential equation of the form

$$\frac{dx}{dt} = f(x)$$

is called ***autonomous***.

- An autonomous first-order equation is always separable. We can always solve them, at least implicitly. Depending on our goals, getting qualitative information about solutions may be enough.

- A constant function $x(t) \equiv r$ such that $f(r) = 0$, is called ***an equilibrium solution*** of the autonomous differential equation.

Example #1: Find the equilibrium solution(s) for $\dfrac{dx}{dt} = x + 1$.

$f(x) = x + 1 = 0$

$x = -1$

- In autonomous equations, the slopes depend only on x. Thus the slope field completely determines once slopes along any vertical line are plotted.
This line is called the ***phase line***.

Example #2: Consider the autonomous DE $\dfrac{dx}{dt} = x(x-1)^2(x-2)$.

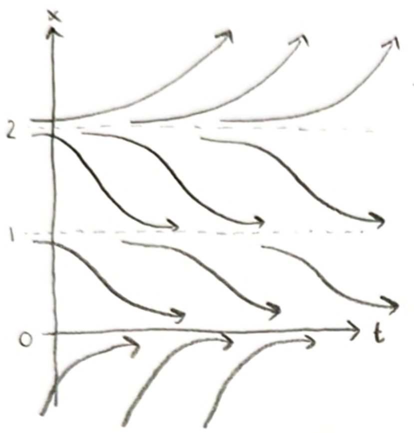
(a) Draw a phase line.

$f(x) = x(x-1)^2(x-2) = 0$

$x = 0, 1, 2$

		0		1		2	
$\frac{dx}{dt}$	+		−		−		+
x	−		+		+		+
$(x-1)^2$	+		+		+		+
$(x-2)$	−		−		−		+

(b) Sketch possible solution curves.



(c) Depending on the result in part (b), determine the behaviour of the solutions as $t \to \infty$.

This depends on where we start

$x(t_0) = x_0$

If $x_0 > 2$, $x(t) \to \infty$

$x_0 = 2$, $x(t) = 2$

$1 \leq x_0 < 2$, $x(t) \to 1$

$x_0 < 1$, $x(t) \to 0$

NOTE: Solutions near $x = 0$ behaves very differently than the solutions near $x = 1$ or $x = 2$.

- We call $x = 2$ ***an unstable equilibrium*** or ***a source***. If a solution starts even a tiny bit away from $x = 2$, the solution will "run away" from the equilibrium $x = 2$.

- $x = 0$ is ***an asymptotically stable equilibrium*** or ***a sink***. If a solution starts "close" to $x = 0$, it stays "close" to $x = 0$ AND the solution will approach $x = 0$ as $t \to \infty$.

- $x = 1$ is neither a source nor a sink. Solutions approach $x = 1$ from one side and escape on the other. We call this ***a semi-stable equilibrium*** or ***a node***.

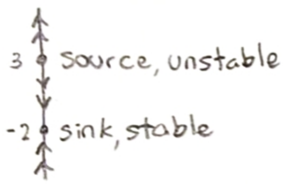
Example #3: Consider the autonomous differential equation, $\dfrac{dx}{dt} = e^x(x+2)(x-3)$.

Draw a phase line for the DE and label each equilibrium point as a sink, source or node.

$f(x) = e^x(x+2)(x-3) = 0$

$x = -2, 3$

	$x<-2$	$-2<x<3$	$x>3$
$\frac{dx}{dt}$	+	−	+
e^x	+	+	+
$x+2$	−	+	+
$x-3$	−	−	+



If $x(1) = 0$, what happens to $x(t)$ as $t \to \infty$? $x \to -2$

If $x(1) = 4$, will there be a point such that $x(t) = 0$? No, $x \to \infty$