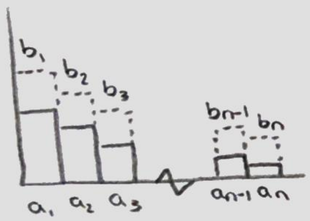
Comparison Tests
Section 10.4

I. Direct Comparison Test

Idea: Suppose that $0 \le a_n \le b_n$ for all n. Then $0 \le \sum_{n=1}^{\infty} a_n \le \sum_{n=1}^{\infty} b_n$.

If the total area Σb_n is finite, then so is the total area Σa_n of the shorter rectangles

Picture!



Direct Comparison Test Let $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ be two series with $\underbrace{0 \le a_n \le b_n}_{\text{All positive terms}}$ for all n. Then

- if $\sum_{n=1}^{\infty} b_n$ converges, $\sum_{n=1}^{\infty} a_n$ also converges.
- if $\sum_{n=1}^{\infty} a_n$ diverges, then $\sum_{n=1}^{\infty} b_n$ also diverges.

Example #1: Determine whether each of the following series converge or diverge.

(a) $\sum_{n=1}^{\infty} \frac{\sin^2(n)}{n^2}$ $0<|\sin n| \leq 1$

$0<\sin^2 n \leq 1$

$\frac{\sin^2 n}{n^2} \leq \frac{1}{n^2}$

Since $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges, then $\sum_{n=1}^{\infty} \frac{\sin^2 n}{n^2}$ also converges

(b) $\sum_{n=2}^{\infty} \frac{n}{\sqrt{n^3-1}} > \frac{n}{\sqrt{n^3}} = \frac{1}{\sqrt{n}}$

Since $\sum_{n=2}^{\infty} \frac{1}{\sqrt{n}}$ diverges, then $\sum_{n=2}^{\infty} \frac{n}{\sqrt{n^3-1}}$ also diverges

(c) $\sum_{n=2}^{\infty} \frac{n}{\sqrt{n^3+1}} \leq \frac{n}{\sqrt{n^3}} = \frac{1}{\sqrt{n}}$

Since $\sum_{n=2}^{\infty} \frac{1}{\sqrt{n}}$ diverges, then $\sum_{n=2}^{\infty} \frac{n}{\sqrt{n^3+1}}$ also diverges

II. Limit Comparison Test

Idea: If two sequences are "essential" the same (for large n), then the series do the same thing!

Limit Comparison Test Suppose that $a_n > 0$ and $b_n > 0$ for all $n \geq N$ (i.e., for all "large" n).

(I) If $\lim_{n\to\infty} \frac{a_n}{b_n} = c$ (where $c \neq 0$ or ∞), then $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ do the same thing.

(II) If $\lim_{n\to\infty} \frac{a_n}{b_n} = 0$ and $\sum_{n=1}^{\infty} b_n$ converges, then $\sum_{n=1}^{\infty} a_n$ converges.

(III) If $\lim_{n\to\infty} \frac{a_n}{b_n} = \infty$ and $\sum_{n=1}^{\infty} b_n$ diverges, then $\sum_{n=1}^{\infty} a_n$ diverges.

Choose dominating terms in numerator and denominator for b_n

Example #1 (c): $\displaystyle\sum_{n=2}^{\infty} \frac{n}{\sqrt{n^3+1}}$

$a_n = \frac{n}{\sqrt{n^3+1}} \quad b_n = \frac{n}{\sqrt{n^3}} = \frac{1}{\sqrt{n}}$

$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{n}{\sqrt{n^3+1}} \cdot \frac{\sqrt{n}}{1} = \lim_{n\to\infty} \frac{\sqrt{n^3}}{\sqrt{n^3+1}} = 1$

$\sum_{n=2}^{\infty} \frac{1}{\sqrt{n}}$ diverges, so $\sum_{n=2}^{\infty} \frac{n}{\sqrt{n^3+1}}$ diverges

Hint: Find another sequence of the same "power" of n so that the limit is a nonzero finite number. Remember, you know geometric series and p-series so these are good candidates.

Example #2: Determine whether each of the following series converge or diverge.

(a) $\displaystyle\sum_{n=1}^{\infty} \frac{3n^2+n}{n^3+2n^2+1} = a_n$

$b_n = \frac{n^2}{n^3} = \frac{1}{n}$ ← Ignore constants

$\lim_{n\to\infty} \frac{a_n}{b_n} = \frac{3n^2+n}{n^3+2n^2+1} \cdot \frac{n}{1} = \frac{3n^3+n^2}{n^3+2n^2+1}$

$\lim_{n\to\infty} \frac{3n^3+n^2}{n^3+2n^2+1} = 3 \neq 0 \text{ or } \infty$

Since $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges, $\sum_{n=1}^{\infty} \frac{3n^2+n}{n^3+2n^2+1}$ also diverges

(b) $\displaystyle\sum_{n=1}^{\infty} \sqrt{\frac{n+4}{n^4+4}} = a_n$

$b_n = \sqrt{\frac{n}{n^4}} = \frac{1}{\sqrt{n^3}}$

$\lim_{n\to\infty} \frac{a_n}{b_n} = \sqrt{\frac{n+4}{n^4+4}} \cdot \sqrt{n^3} = \sqrt{\frac{n^4+4n^3}{n^4+4}}$

$\lim_{n\to\infty} \sqrt{\frac{n^4+4n^3}{n^4+4}} = 1 \neq 0 \text{ or } \infty$

Since $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n^3}}$ converges, $\sum_{n=1}^{\infty} \sqrt{\frac{n+4}{n^4+4}}$ also converges

Extra Practice

1. $\displaystyle\sum_{n=2}^{\infty} \frac{n^{7/2} - \sqrt{n^2-1}}{n^2+n} = a_n$

$b_n = \frac{n^{7/2}}{n^2} = n^{3/2}$

$\lim\limits_{n\to\infty} \frac{a_n}{b_n} = \frac{n^{7/2} - \sqrt{n^2-1}}{n^2+n} \cdot \frac{1}{n^{3/2}} = \frac{n^{7/2} - \sqrt{n^2-1}}{n^{7/2}+n^{5/2}}$

$\lim\limits_{n\to\infty} \frac{n^{7/2} - \sqrt{n^2-1}}{n^{7/2}+n^{5/2}} = 1 \neq 0 \text{ or } \infty$

Since $\sum\limits_{n=2}^{\infty} b_n$ diverges, $\sum\limits_{n=2}^{\infty} a_n$ also diverges
(by limit comparison)

2. $\displaystyle\sum_{n=0}^{\infty} \frac{n}{n^2+1} = a_n$

$b_n = \frac{n}{n^2} = \frac{1}{n}$

$\lim\limits_{n\to\infty} = \frac{n}{n^2+1} \cdot n = \frac{n^2}{n^2+1}$

$\lim\limits_{n\to\infty} \frac{n^2}{n^2+1} = 1 \neq 0 \text{ or } \infty$

Since $\sum\limits_{n=0}^{\infty} b_n$ diverges, $\sum\limits_{n=0}^{\infty} a_n$ also diverges
(by limit comparison)

$0 + \sum\limits_{n=1}^{\infty} \frac{n}{n^2+1} = a_n \leq \frac{n}{n^2} = \frac{1}{n} \div \frac{1}{2n}$

$\lim\limits_{n\to\infty} \frac{1}{n}$ = diverges, so $\sum\limits_{n=1}^{\infty} a_n$ also converges
(by direct comparison)

3. $\displaystyle\sum_{n=1}^{\infty} \frac{2^n + 3^n}{3^n + 4^n} = a_n$

$b_n = \left(\frac{3}{4}\right)^n \quad |r| = \frac{3}{4} < 1$

$\lim\limits_{n\to\infty} \frac{a_n}{b_n} = \frac{2^n+3^n}{3^n+4^n} \cdot \left(\frac{4}{3}\right)^n = \frac{12^n+8^n}{12^n+9^n}$

$\lim\limits_{n\to\infty} \frac{12^n+8^n}{12^n+9^n} = 1 \neq 0 \text{ or } \infty$

Since $\sum\limits_{n=1}^{\infty} b_n$ converges, $\sum\limits_{n=1}^{\infty} a_n$ also converges
(by limit comparison)