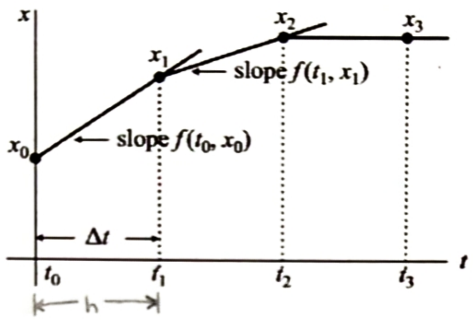
Numerical Methods
Section 2.6 (Noonburg)

In this section we will see how the solutions to an IVP can be numerically approximated. This idea is useful with the DEs or IVPs which cannot be solved easily.
We will only focus on Euler's method in this course, although there are other effective methods.

Euler's Method

On each interval $[t_i, t_{i+1}]$ we approximate $x(t)$ by the tangent line approximation at the left end point of the interval.



Procedure:
Consider an IVP $x' = f(t,x)$, $x(t_0) = x_0$.

At the point (t_0, x_0), the slope of the tangent line to the true solution $x(t)$ is given by $m_0 = f(t_0, x_0)$.

We can use this to approximate $x(t_1) = x(t_0 + h)$ where $h = \Delta t$ is the "step size".

$f(t_0, x_0) = \frac{x_1 - x_0}{t_1 - t_0} = \frac{x_1 - x_0}{h}$

$x_1 = x_0 + hf(t_0, x_0)$

This process can be iterated indefinitely. Let $t_n = t_0 + n \cdot h$. Then $x(t_n)$ is the approximated value of x_n. Then,

$$x_{n+1} = x_n + h \cdot f(t_n, x_n),$$

for $n = 0, 1, 2, \cdots$

Example #1: Use Euler's Method to approximate $x(2)$ with $h = 0.5$, for the solution to the IVP,

$x' = x^2 - t^2, \quad x(0) = 0.5.$

$f(t,x) = x^2 - t^2$ $\qquad x_{n+1} = x_n + hf(t_n, x_n)$

t_n	x_n	$f(t_n, x_n)$	x_{n+1}
0	0.5	0.25	0.625
0.5	0.625	0.141	0.695
1	0.695	-0.517	0.437
1.5	0.437	-2.059	-0.592
2	-0.592		

$x(2) \approx -0.592$

Example #2: Repeat the Example #1 with $h = 0.25$. What can you observe?

t_n	x_n	$f(t_n, x_n)$	x_{n+1}
0	0.5	0.25	0.563
0.25	0.563	0.254	0.627
0.5	0.627	0.143	0.662
0.75	0.662	-0.124	0.631
1	0.631	-0.601	0.481
1.25	0.481	-1.331	0.148
1.5	0.148	-2.228	-0.409
1.75	-0.409	-2.895	-1.133
2	-1.133		

$x(2) \approx -1.133$

More error with more iterations

Approximate values close to initial point

Example #3: Use Euler's Method to approximate $x(5)$ for the solution to the IVP,

$$x' = \sqrt{t + x^2}, \quad x(3) = 1$$

(a) with $h = 1$

t_n	x_n	$f(t_n, x_n)$	x_{n+1}
3	1	2	3
4	3	3.606	6.606
5	6.606		

$x(5) \approx 6.606$

(b) with $h = 0.5$

t_n	x_n	$f(t_n, x_n)$	x_{n+1}
3	1	2	2
3.5	2	2.739	3.369
4	3.369	3.918	5.328
4.5	5.328	5.735	8.196
5	8.196		

$x(5) \approx 8.196$

Other Numerical methods:

- Improved Euler Method
- Forth-order Runge-Kutta Method

Before we start throwing numerical methods at problems, we should be confident that there is a solution to our IVP and that it is unique! That is, we want one and ONLY one solution. This is where our work on the Existence and Uniqueness Theorem is very helpful!

Approximations have error! To have good approximations, we should:

(i) Select h small.
Linear approximations are bad the further away from the initial point we move.

(ii) For a fixed value of h , the solution is better if n is small.
More steps means more error.

(iii) Develop better methods.
The Improved Euler method typically gives approximations with less error. The Runge-Kutta method is more complicated, taking into account more information, is usually even better.