# Power Series
## Section 10.7

**Definition:** A **power series about** $x = a$ is a *function* and *series* of the form

$$F(x) = \sum_{n=0}^{\infty} c_n(x-a)^n = c_0 + c_1(x-a) + c_2(x-a)^2 + \cdots + c_n(x-a)^n + \cdots$$

We say that a power series of this form is **centered about** $a$. Here $c_0, c_1, \cdots$ are constants, which are the coefficients for each term.
Each power series will converge for some value(s) of $x$ and (most) will diverge for all other values of $x$.

**Example #1:** For what values of $x$ do the following power series converge?

(a) $\sum_{n=0}^{\infty} x^n = 1 + x + x^2 + \cdots + x^n$

Geometric power series $(r = x)$

Converges when $|r| = |x| < 1$ to the value $\frac{1}{1-x}$

Diverges when $|r| = |x| \geq 1$

(b) $\sum_{n=1}^{\infty} (-1)^n \frac{x^{2n-1}}{2n-1} = -x + \frac{x^3}{3} - \frac{x^5}{5}$

$$\frac{a_{n+1}}{a_n} = \left((-1)^{n+1} \frac{x^{2(n+1)-1}}{2(n+1)-1}\right)\left(\frac{2n-1}{(-1)^n x^{2n-1}}\right) = \left|\frac{x^2(2n-1)}{2n+1}\right|$$

$$\lim_{n\to\infty} \left(\frac{2n-1}{2n+1}\right) x^2 = x^2$$

The series converges when $x^2 < 1$ or $-1 < x < 1$
(ratio)

Check separately

$x = 1 \rightarrow \sum_{n=1}^{\infty} \frac{(-1)^n}{2n-1}$ converges (AST)

$x = -1 \rightarrow \sum_{n=1}^{\infty} \frac{-(-1)^n}{2n-1}$ converges (AST)

Converges when $-1 \leq x \leq 1$ and diverges elsewhere

**Definition:** The radius of convergence of $\sum_{n=0} c_n(x-a)^n$ is the number $R \geq 0$ such that the power series

1. Converges (absolutely) for $x$ with $|x-a| < R$.
2. Diverges for $x$ with $|x-a| > R$.

**We use Ratio Test to find the radius of convergence!**

Special Cases:

1. If $R=\infty$ the series converges (absolutely) for every $x$

2. If $R=0$ the series converges at $x=a$ and diverges elsewhere

**Example #2:** Find the radius of convergence of the power series $\sum_{n=1}^{\infty} \frac{([redacted]x-1)^n}{\sqrt{n}3^n}$.

$a_n = \frac{(x-1)^n}{\sqrt{n}\,3^n}$

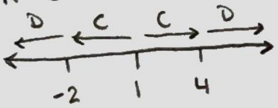
$\lim_{n\to\infty} \left| \frac{(x-1)^{n+1}}{\sqrt{n+1}\,3^{n+1}} \cdot \frac{\sqrt{n}\,\cancel{3^n}}{\cancel{(x-1)^n}} \right| = \left| \frac{(x-1)}{3} \cdot \sqrt{\frac{n}{n+1}} \right| = \frac{1}{3}\sqrt{\frac{n}{n+1}}\,|x-1|$

$\lim_{n\to\infty} \frac{1}{3}\sqrt{\frac{n}{n+1}}\,|x-1| = 1 \to \frac{1}{3}|x-1|$

The series converges when

$\frac{1}{3}|x-1| < 1 \to |x-1| < 3$

$R = 3$




$-3 < x-1 < 3$

$-2 < x < 4$

<u>Definition:</u> Given a power series $\sum_{n=0}^{\infty} c_n(x-a)^n$, the **interval of convergence** of the power series is the largest interval on which the series converges.

Ex. 2: $\sum_{n=1}^{\infty} \frac{(x-1)^n}{\sqrt{n}\,3^n}$

$x=-2$

$\sum_{n=1}^{\infty} \frac{(-3)^n}{\sqrt{n}\,3^n} = \frac{(-1)^n}{\sqrt{n}}$

Positive terms ✓ $\lim_{n\to\infty} \frac{1}{\sqrt{n}} = 0$ ✓

Nonincreasing ✓ Converges @ $x=2$ (AST)

$x=4$

$\sum_{n=1}^{\infty} \frac{3^n}{\sqrt{n}\,3^n} = \frac{1}{\sqrt{n}}$

Diverges @ $x=4$ (p series)

$\boxed{-2 \le x < 4}$

**Example #3:** If a power series centered at $x = 1$ converges at $x = -1$ and diverges at $x = 5$, then

- at $x = 0$ the series converges
- at $x = 2$ the series converges
- at $x = -4$ the series diverges
- at $x = 3$ the series (not enough information)

$2 < R < 4$

**Example #4:** Find the interval of convergence for $\sum_{n=1}^{\infty} \frac{5^n(x+2)^{2n}}{n}$. $= a_n$

$$\lim_{n\to\infty} \left| \frac{5^{\cancel{n}+1}(x+2)^{\cancel{2n}+2}}{n+1} \cdot \frac{n}{\cancel{5^n}\cancel{(x+2)^{2n}}} \right| = \left| \frac{5n(x+2)^2}{n+1} \right|$$

$$\lim_{n\to\infty} \frac{5n}{n+1}(x+2)^2 = 5(x+2)^2$$

The series converges when

$$5(x+2)^2 < 1$$

$$(x+2)^2 < \frac{1}{5}$$

$$x+2 < \sqrt{\frac{1}{5}} = \frac{1}{\sqrt{5}}$$

$$R = \frac{1}{\sqrt{5}}$$

$$\frac{-1}{\sqrt{5}} - 2 < x < \frac{1}{\sqrt{5}} - 2$$

$$x = \frac{-1}{\sqrt{5}} - 2$$

$$\sum_{n=1}^{\infty} \frac{5^n(\frac{-1}{\sqrt{5}})^{2n}}{n} = \frac{5^n(\frac{1}{5^n})}{n} = \frac{1}{n}$$

Diverges @ $x = \frac{-1}{\sqrt{5}} - 2$ (p series)

$$\sum_{n=1}^{\infty} \frac{5^n(\frac{1}{\sqrt{5}})^{2n}}{n} = \frac{1}{n}$$

Diverges

**Theorem:** Suppose that the power series $f(x) = \sum_{n=0}^{\infty} c_n(x-a)^n$ converges on the interval $I = (a - R, a + R)$. Then

$$f'(x) = \sum_{n=1}^{\infty} nc_n(x-a)^{n-1},$$

$$\int f(x)\,dx = \sum_{n=0}^{\infty} c_n \frac{(x-a)^{n+1}}{n+1} + C,$$

and both of these power series also converge on $I$.

**Example #5:** Let $f(x) = \sum_{n=0}^{\infty} \frac{x^n}{n!}$. Find $a_n = \underbrace{\frac{1}{n!}}_{c_n} \cdot x^n$

(a) the interval of convergence.

$\left| \frac{x^{n+1}}{(n+1)!} \cdot \frac{n!}{x^n} \right| = \frac{|x|}{n+1}$

$\lim_{n\to\infty} \frac{1}{n+1} \cdot |x| = 0$

$R = \infty$

Interval of convergence

$-\infty < x < \infty$

(b) the derivative $f'(x)$. $\sum_{n=1}^{\infty} \frac{n}{n!} \cdot x^{n-1} = \frac{1}{(n-1)!} \cdot x^{n-1}$

(c) the integral $\int f(x)\,dx$. $\sum_{n=1}^{\infty} \frac{1}{n!} \cdot \frac{x^{n+1}}{n+1} + C = \frac{x^{n+1}}{(n+1)!} + C$