Existence and Uniqueness of Solutions
Section 2.4 (Noonburg)

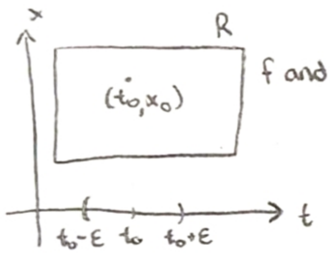
When solving real-world situations using DE models, we should be confident that there is a solution to our IVP ***and*** that it is unique!

Existence and Uniqueness Theorem:

Consider the differential equation $\dfrac{dx}{dt} = f(t, x)$. If f is defined and continuous everywhere inside the rectangle $\mathbf{R} = \{(t, x)|a \le t \le b,\ c \le x \le d\}$ in the tx-plane, containing the initial data point (t_0, x_0), then there exists a solution $x = \phi(t)$ passing through the point (t_0, x_0), and this solution is continuous on an interval $I = (t_0 - \varepsilon, t_0 + \varepsilon)$ for some $\varepsilon > 0$.

If $\dfrac{\partial f}{\partial x}$ is continuous in $\mathbf{R}$, the solution is unique.

Picture:

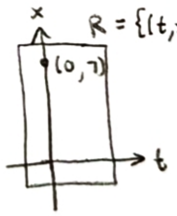


f and $\frac{\partial f}{\partial x}$ are continuous in R

Example #1: Consider the IVP $x' = 3x \quad x(0) = 7$.
$(t_0, x_0) = (0, 7)$

(a) Is it guaranteed to have a unique solution?

$R = \{(t,x)| -\infty < t < \infty, -\infty < x < \infty\}$



$f(t,x) = 3x$ Continuous and defined for all x, t

$\frac{\partial f}{\partial x} = 3$ Continuous and defined in R

We are garanteed to have a unique solution on some t-interval around $t=0$

(b) Are there any initial conditions $x(t_0) = x_0$ that do not guarantee a unique solution?

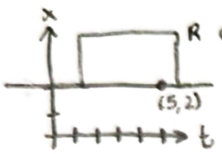
No, R covers every value for $x(t_0) = x_0$
Every point in the t-x plane has a solution curve passing through it and two solutions cannot intersect

Example #2: What does the existence and uniqueness theorem tells us about the following IVP?

$$x' = t\sqrt{x-2}, \quad x(5) = 2$$

$(t_0, x_0) = (5,2)$

$f(t,x) = t\sqrt{x-2}$ $x \geq 2$ Continuous and defined for $[2, \infty) = x$



R does not satisfy theorem

We cannot enclose the initial point (5,2) inside a rectangle, so we cannot define a rectangle R

f does not satisfy the continuity hypothesis of the theorem

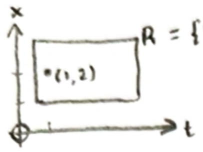
The existance and uniqueness theorem cannot give any information about the solution to this IVP

This does not imply the IVP does not have a solution

Example #3: Does the IVP $tx' + x = \cos t$, $x(1) = 2$, have a unique solution?

$x' = \frac{\cos t}{t} - \frac{x}{t}$ $\quad t \neq 0$ $\quad (t_0, x_0) = (1, 2)$

Defined and continuous for $(-\infty, 0) \cup (0, \infty) = t$



$R = \{(t,x) \mid 0 < t < \infty, -\infty < x < \infty\}$

We can draw a rectangle R such that (1,2) is in R and f is continuous and defined in R

The IVP has a solution

$\frac{\partial f}{\partial x} = -\frac{1}{t}$ $\quad t \neq 0$ Continuous and defined in R

The IVP has a unique solution