The Logarithm Defined as an Integral
Section 7.1

The **natural logarithm** is the function given by

$$\ln(x) = \int_1^x \frac{1}{t}\,dt \;;\; x > 0$$

The **number** e is the positive number that satisfies

$$\ln(e) = \int_1^e \frac{1}{t}\,dt = 1$$

If u is a differentiable function that is never zero, then

$$\int \frac{1}{u}\,du = \ln|u| + C.$$

Recall:
If u is any differentiable function of x, then

$$\frac{d}{dx}e^u = e^u\frac{du}{dx}$$

This will give us the indefinite integral of e^u as;

$$\int e^u\,du = e^u + C$$

e^x and $\ln(x)$ are <u>inverse functions</u> which means that;

- $e^{\ln x} = x$ for all $x > 0$
- $\ln(e^x) = x$ for all x

For any numbers $a > 0$ and x, the exponential function with base a is given by;

$$a^x = e^{x \ln a}$$

Then, $\frac{d}{dx}a^x = \frac{d}{dx} e^{x\ln a} = e^{x\ln a} \ln a$

In general, if $a > 0$ and u is any differentiable function of x, then

$$\frac{d}{dx}a^u = a^u . \ln a \, \frac{du}{dx}$$

This will give us the indefinite integral of a^u as;

$$\int a^u \, du = \frac{a^u}{\ln a} + C$$

Example #1: Evaluate the following integrals:

(a) $\int 7^{\cos(t)} \sin(t) \, dt$

$u = \cos t$

$du = -\sin t \, dt$

$-\int 7^u \, du$

$\frac{-7^u}{\ln 7} + C$

$\frac{-7^{\cos t}}{\ln 7} + C$

(b) $\int_1^2 \frac{2^{\ln x}}{x} \, dx$

$u = \ln x$

$du = \frac{1}{x} dx$

$\int_1^2 2^u \, du = \frac{2^u}{\ln 2} \Big|_1^2$

$\frac{2^{\ln x}}{\ln 2} \Big|_1^2 = \frac{1}{\ln 2}(2^{\ln 2} - 1)$

Logarithms with base a:

For any positive number $a \neq 1$, the **logarithm of x with base a**, denoted by $\log_a(x)$, is the inverse function of a^x.

We can use the following change of base formula when we solve derivatives/integrals involving $\log_a(x)$:

$$\log_a x = \frac{\ln x}{\ln a}$$

Proof:

$y = \log_a x$

$a^y = x$

$\ln a^y = \ln x$

$y \ln a = \ln x$

$y = \frac{\ln x}{\ln a}$

Example #2: Evaluate $\int_1^e \frac{2\log_5(x)}{x}\,dx$

$\int_1^e \frac{2\ln x}{x\ln 5} = \frac{2}{\ln 5}\int_1^e \frac{\ln x}{x}\,dx$

$u = \ln x$

$du = \frac{1}{x}$

$\int_1^e u\,du = \frac{u^2}{2}$

$\frac{2}{\ln 5}\left[\frac{\ln^2 x}{2}\right]_1^e$

$\frac{\ln^2 x}{\ln 5}\Big|_1^e = \frac{1}{\ln 5}$

Extra Practice

1. $\int x^{\pi-1}\,dx = \frac{x^{\pi}}{\pi} + C$

2. $\int \pi^{x-1}\,dx = \frac{\pi^{x-1}}{\ln \pi} + C$

3. $\int \frac{1}{x \log_2 x}\,dx = \int \frac{\ln 2}{x \ln x}\,dx$

$u = \ln x \quad du = \frac{1}{x}\,dx$

$\ln 2 \int \frac{1}{u}\,du = \ln 2 \ln|u| + C$

$\ln 2 \ln|\ln x| + C$

4. $\int \frac{\ln(\ln(x))}{x \ln(x)}\,dx$

$u = \ln x$

$du = \frac{1}{x}\,dx$

$\int \frac{\ln u}{u}\,du$

$v = \ln u$

$dv = \frac{1}{u}\,du$

$\int v\,dv = \frac{v^2}{2} + C = \frac{(\ln u)^2}{2} + C = \frac{(\ln(\ln x))^2}{2} + C$