Eigenvalues and Eigenvectors
Section 4.1 (**Hartman**)

Eigenvalues and Eigenvectors:
Let A be an $n \times n$ matrix, $\vec{x}$ a nonzero $n \times 1$ column vector and λ a scalar. If

$$A\vec{x} = \lambda\vec{x}$$

then $\vec{x}$ is an *eigenvector* of A and λ is an *eigenvalue* of A.

Example #1: Let $A = \begin{pmatrix} 1 & 4 \\ 2 & 3 \end{pmatrix}$ and $\vec{x} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$. Then

$$A\vec{x} = \underset{A}{\begin{pmatrix} 1 & 4 \\ 2 & 3 \end{pmatrix}} \underset{\vec{x}}{\begin{pmatrix} 1 \\ 1 \end{pmatrix}} = \begin{pmatrix} 5 \\ 5 \end{pmatrix} = \underset{\lambda}{5} \underset{\vec{x}}{\begin{pmatrix} 1 \\ 1 \end{pmatrix}} = 5\vec{x}.$$

In this section we will learn; for a given square matrix A, how to find a **nonzero vector** $\vec{x}$ and a scalar λ such that $A\vec{x} = \lambda\vec{x}$.

Consider $A\vec{x} = \lambda\vec{x}$. Then $A\vec{x} - \lambda\vec{x} = \vec{0}$.

We can now factor out $\vec{x}$. But note that $A - \lambda$ doesn't make any sense. So we use the identity matrix in order for this to be logical as below:

$$(A \overset{B}{-} \lambda I)\vec{x} = \vec{0}$$

This is a matrix equation of the type we solved in Section 2.4.

Facts:

- If the matrix $(A - \lambda I)$ is invertible, then the only solution to $(A - \lambda I)\vec{x} = \vec{0}$ is $\vec{x} = \vec{0}$.
- Therefore, in order to have other solutions, we need $(A - \lambda I)$ not invertible.
- Recall that non-invertible matrices all have determinant of 0.
- Therefore, if we want to find eigenvalues λ and eigenvectors $\vec{x}$, we need $\det(A - \lambda I) = 0$.

Thus, in order to find eigenvalues, we solve the equation $\det(A - \lambda I) = 0$ for λ, for a given matrix A.

Example #2: Let $A = \begin{pmatrix} 1 & 4 \\ 2 & 3 \end{pmatrix}$.

(a) Find the eigenvalues of A.

$A - \lambda I$

$\begin{pmatrix} 1 & 4 \\ 2 & 3 \end{pmatrix} - \lambda \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1-\lambda & 4 \\ 2 & 3-\lambda \end{pmatrix}$

$$\begin{aligned} \det(A - \lambda I) &= (1-\lambda)(3-\lambda) - 8 \\ &= \lambda^2 - 4\lambda - 5 \\ &= (\lambda - 5)(\lambda + 1) \end{aligned}$$

$\boxed{\lambda = -1, 5}$

(b) Find eigenvectors of A.

$A\vec{x} = 5\vec{x}, \lambda = 5$

$A\vec{x} - 5\vec{x} = 0$

$(A - 5I)\vec{x} = 0$

$\left(\begin{pmatrix} 1 & 4 \\ 2 & 3 \end{pmatrix} - \begin{pmatrix} 5 & 0 \\ 0 & 5 \end{pmatrix} \right) \vec{x} = 0$

$\begin{pmatrix} -4 & 4 \\ 2 & -2 \end{pmatrix} \vec{x} = 0$

$\left(\begin{array}{cc|c} -4 & 4 & 0 \\ 2 & -2 & 0 \end{array} \right) \rightarrow \left(\begin{array}{cc|c} 1 & -1 & 0 \\ 0 & 0 & 0 \end{array} \right)$

$x_1 = x_2 \quad x_2$ is free

$\vec{x} = \begin{pmatrix} x_2 \\ x_2 \end{pmatrix} = \boxed{x_2 \begin{pmatrix} 1 \\ 1 \end{pmatrix} \quad x_2 \neq 0}$

$A\vec{x} = -\vec{x}, \lambda = -1$

$A\vec{x} + \vec{x} = 0$

$(A + I)\vec{x} = 0$

$\begin{pmatrix} 2 & 4 \\ 2 & 4 \end{pmatrix} \vec{x} = 0$

$\left(\begin{array}{cc|c} 2 & 4 & 0 \\ 2 & 4 & 0 \end{array} \right) \rightarrow \left(\begin{array}{cc|c} 1 & 2 & 0 \\ 0 & 0 & 0 \end{array} \right)$

$x_1 = -2x_2 \quad x_2$ is free

$\vec{x} = \begin{pmatrix} -2x_2 \\ x_2 \end{pmatrix} = \boxed{x_2 \begin{pmatrix} -2 \\ 1 \end{pmatrix} \quad x_2 \neq 0}$

Characteristic Polynomial:
Let A be an $n \times n$ matrix. The characteristic polynomial of A is the n th degree polynomial $p(\lambda) = \det(A - \lambda I)$.

Finding Eigenvalues and Eigenvectors:

Let A be an $n \times n$ matrix.

1. To find the eigenvalues of A, compute $p(\lambda)$, set it equal to 0, then solve for λ.
2. To find the eigenvectors of A, solve the homogeneous system $(A - \lambda I)\vec{x} = \vec{0}$ for each eigenvalue.

Eigenvectors are not unique

Eigenvectors of different eigenvalues are linearly independent

Example #3: Let $A = \begin{pmatrix} 2 & -1 & 1 \\ 0 & 1 & 6 \\ 0 & 3 & 4 \end{pmatrix}$. Find the eigenvalues of A, and for each eigenvalue, give one eigenvector.

$$\det(A - \lambda I) = \begin{vmatrix} 2-\lambda & -1 & 1 \\ 0 & 1-\lambda & 6 \\ 0 & 3 & 4-\lambda \end{vmatrix} = (2-\lambda)\big((1-\lambda)(4-\lambda)-18\big) = (2-\lambda)(\lambda-7)(\lambda+2)$$

$$\boxed{\lambda = \pm 2, 7}$$

$\lambda = 2$

$$\left(\begin{array}{ccc|c} 0 & -1 & 1 & 0 \\ 0 & -1 & 6 & 0 \\ 0 & 3 & 2 & 0 \end{array}\right) \rightarrow \left(\begin{array}{ccc|c} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

$x_2 = x_3 = 0$ x_1 is free

$$\boxed{\vec{x} = x_1\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \quad x_1 \neq 0}$$

$\lambda = -2$

$$\left(\begin{array}{ccc|c} 4 & -1 & 1 & 0 \\ 0 & 3 & 6 & 0 \\ 0 & 3 & 6 & 0 \end{array}\right) \rightarrow \left(\begin{array}{ccc|c} 1 & 0 & 3/4 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

$x_1 = \frac{3}{4}x_3$ x_3 is free

$x_2 = -2x_3$

$$\boxed{\vec{x} = x_3\begin{pmatrix} 3/4 \\ -2 \\ 1 \end{pmatrix} \quad x_3 \neq 0}$$

$\lambda = 7$

$$\left(\begin{array}{ccc|c} -5 & -1 & 1 & 0 \\ 0 & -6 & 6 & 0 \\ 0 & 3 & -3 & 0 \end{array}\right) \rightarrow \left(\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

$x_1 = 0$ x_3 is free

$x_2 = x_3$

$$\boxed{\vec{x} = x_3\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} \quad x_3 \neq 0}$$

$$B = \begin{pmatrix} | & | & | \\ v_1 & v_2 & v_3 \\ | & | & | \end{pmatrix}$$

When $\det(B) \neq 0$, B is invertible

Then $B\vec{x} = \vec{0}$ has a unique solution, $\vec{x} = \vec{0}$

If $a\vec{v}_1 + b\vec{v}_2 + c\vec{v}_3 = \vec{0}$, then $a = b = c = 0$ and the values are linearly independent