Forced Spring-Mass Systems
Section 3.5

Additional material on resonance is available on Moodle.

We will not be discussing beats.

Vocabulary:

- Transient solution
- Steady-state solution (or forced response)
- Pure resonance
- Practical resonance
- Resonance frequency

Main Idea: Recall that we have been studying *free motion* of a spring-mass system. The spring itself, gravity, and (perhaps) damping were the only forces acting on the mass; the system was *free* of all other external forces. These systems resulted in homogeneous differential equations.

Now we will consider **forced motion**. Here an external force $f(t)$ is also acting on our system. The right-hand side of our equation is now $f(t)$, so these systems result in nonhomogeneous differential equations. For example, the spring may be attached to something that is moving. Driving over a bumpy road is an example of this scenario; a shock absorber (i.e. spring) is attached to both the car (mass) and a tire that will be moving up and down on a bumpy road.

$mx''+bx'+kx=f(t)$

$f(t)\neq 0$

Example #1: Suppose that you have a 1 kg mass attached to a 4 N/m spring and the system has a damping coefficient of 2 kg/s. The spring is attached to a base that is oscillating and applying a force of $\cos(2t)$ Newtons (where t is in seconds) to the spring/mass. If the mass starts at the equilibrium position with an initial downward velocity of 1/2 m/s, find the displacement of the mass from equilibrium at time t. $m=1,\ k=4,\ d=2,\ f(t)=\cos(2t)$

Solution: This situation can be modeled by the initial-value problem

$$x''+2x'+4x=\cos(2t), \qquad x(0)=0,\ x'(0)=-\frac{1}{2}.$$

The associated homogeneous differential equation

$$x_h''+2x_h'+4x_h=0$$

has the characteristic equation

$$r^2+2r+4=0 \qquad \to \qquad r=\frac{-2\pm\sqrt{4-16}}{2}=-1\pm i\sqrt{3},$$

meaning that the general homogenous solution is

$$x_h(t)=c_1e^{-t}\cos(\sqrt{3}t)+c_2e^{-t}\sin(\sqrt{3}t).$$

A particular solution will have the form

$$x_p(t) = A\cos(2t) + B\sin(2t).$$

Notice that each term of $x_p(t)$ is linearly independent from each term in $x_h(t)$. Taking the derivative(s) of this function and plugging these into the differential equation, we are able to see that

$$x_p'' + 2x_p' + 4x_p = -2A\sin(2t) + 4B\cos(2t) = \cos(2t),$$

which implies that $A = 0$ and $B = \dfrac{1}{4}$. Therefore the general solution to the differential equation is

$$x(t) = x_h(t) + x_p(t) = c_1e^{-t}\cos(\sqrt{3}t) + c_2e^{-t}\sin(\sqrt{3}t) + \frac{\sin(2t)}{4}.$$

Finally, we can apply the initial conditions to see that

$$x(t) = \underbrace{-\frac{e^{-t}\sin(\sqrt{3}t)}{\sqrt{3}}}_{\text{Transient}} + \underbrace{\frac{\sin(2t)}{4}}_{\text{Steady-static}}.$$

You should fill in the details yourself for practice, verifying each step.

Definitions: Consider the general solution form of this example:

$$x(t) = e^{-at}(c_1\cos(\beta t) + c_2\sin(\beta t)) + A\cos(2t) + B\sin(2t),\ \alpha > 0$$

The homogeneous solution $x_h(t) = e^{-at}(c_1\cos(bt) + c_2\sin(bt))$ is called the **transient solution** because it is only temporary: $\lim_{t\to\infty} x_h(t) = 0$. *Homogeneous solutions to all damped spring-mass systems ($b \neq 0$) are transient.*

The particular solution $x_p(t) = A\cos(2t) + B\sin(2t)$ is called the **steady-state solution** (or forced response) because this is the part of the solution that exists in the long term.

Pure Resonance

In an undamped system ($b = 0$), **pure resonance** occurs when the natural frequency $\omega_0 = \sqrt{\frac{k}{m}}$ is equal to the frequency of the forcing function.

$$mx'' + kx = F_0 \cos\left(\sqrt{\frac{k}{m}}t\right) \text{ or } F_0 \sin\left(\sqrt{\frac{k}{m}}t\right)$$

where F_0 is some constant.

Example #2: Consider the spring-mass system modeled by the following differential equation.

$$x'' + 4x = \cos(2t),\ \ x(0) = 0,\ x'(0) = -\frac{1}{2}$$

Find the solution to this initial-value problem and analyze the resulting motion.
Notice that this is the same system as Example #1 except there is no damping.

$r^2 + 4 = 0$

$r = \pm 2i$ $\quad \alpha = 0$, $\beta = 2$

$x_h = c_1\cos(2t) + c_2\sin(2t)$

$x_p = At\cos(2t) + Bt\sin(2t)$

$x_p' = A\cos(2t) - 2At\sin(2t) + B\sin(2t) + 2Bt\cos(2t)$

$x_p'' = -2A\sin(2t) - 2A\sin(2t) - 4At\cos(2t) + 2B\cos(2t) + 2B\cos(2t) - 4Bt\sin(2t)$

$-4A\sin(2t) - \cancel{4At\cos(2t)} + 4B\cos(2t) - \cancel{4Bt\sin(2t)} + \cancel{4At\cos(2t)} + \cancel{4Bt\sin(2t)} = \cos(2t)$

$-4A = 0 \quad 4B = 1$

$A = 0 \quad B = \frac{1}{4}$

$x_p = \frac{1}{4}t\sin(2t)$

$x(t) = c_1\cos(2t) + c_2\sin(2t) + \frac{1}{4}t\sin(2t)$

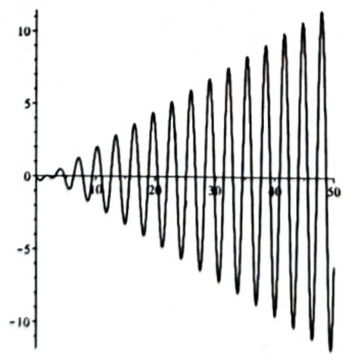
$x'(t) = 2c_1\sin(2t) + 2c_2\cos(2t) + \frac{1}{4}\sin(2t) + \frac{1}{2}t\cos(2t)$

$0 = c_1 \quad -\frac{1}{2} = 2c_2 \quad c_2 = -\frac{1}{4}$

$x(t) = -\frac{1}{4}\sin(2t) + \frac{1}{4}t\sin(2t)$

$\quad = \frac{1}{4}\sin(2t)(t-1)$

Graph of $x(t)$ on next page



Notice that here, the entire solution $(x_h + x_p)$ is a steady-state solution. Since there is no damping, x_h is NOT transient (i.e. $\lim_{t\to\infty} x_h(t) \neq 0$).

Practical Resonance

In damped systems ($b \neq 0$) with a periodic forcing function, the amplitude **will not** continually increase with time (as we saw with pure resonance in Example #2). First, as always, we consider the homogeneous solution. Since there is damping here, the solution will look like one of the following (where $a, r < 0$):

All are transient

As $t \to \infty$, $x_h \to 0$

$$\begin{cases} c_1e^{at} + c_2e^{rt} \\ c_1e^{at} + c_2te^{at} \\ c_1e^{at}\cos(\beta t) + c_2e^{at}\sin(\beta t) \end{cases}$$

Therefore, there is *no possibility of the homogeneous and particular solution being linearly dependent* since none of these solution are periodic (see Example #1 for an example of this). Unlike the pure resonance case, we will **not** need to multiply our 'guess' solution by t here.

Definitions: The **resonance frequency** is the frequency of the periodic forcing function that makes the long-term amplitude of the mass as large as possible. For large values of t, only the steady-state (particular) solution will have a noticeable effect on the amplitude (since the homogeneous solution is transient). We call this situation (of maximizing the amplitude) **practical resonance** because in real-life (practical) applications, there is always some damping.

Note: This only happens for certain values of m, b, and k; there is not always a maximum.

Goal: Find γ so that a particular solution of

$$mx'' + bx' + kx = F_0\cos(\gamma t) \text{ or } F_0\sin(\gamma t)$$

has the largest possible amplitude.

Find r such that $R(r)$ is maximized
This r is called the resonant frequency

$$R^2(r) = \frac{F_0^2}{(-mr^2+k)^2+(br)^2}$$

Find critical points
$(R^2(r))' = 0$, solve for r
Find which critical points will give max r

If $0 < 2mk - b^2$, then there is a practical resonance
If $0 > 2mk - b^2$, then there is no resonance

Example #3: Consider the spring-mass system

$$25x''(t) + 25x'(t) + 14x(t) = 67\cos(\gamma t).$$

(a) Determine the resonant frequency of the system.

(b) Determine the maximum size of the amplitude.

Maple:

dsolve

$$x(t) = x_h + \frac{\overbrace{(-1675\gamma^2+938)}^{A}\cos\gamma + \overbrace{1675\gamma}^{B}\sin\gamma}{625\gamma^4 - 75\gamma^2 + 196}$$

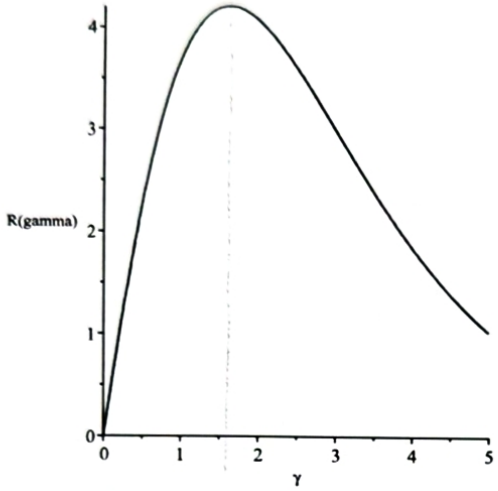
$R^2(\gamma) = A^2 + B^2$

solve $(R^2)' = 0$

$\gamma = 0, \pm\frac{\sqrt{6}}{10}$ must be ≥ 0

a. $\frac{\sqrt{6}}{10}$

b. $R\left(\frac{\sqrt{6}}{10}\right)$



Example #4: Consider the forced spring-mass system given by

$$mx''(t) + \overset{b}{\cancel{\beta}}x'(t) + kx(t) = \sin(\gamma t)$$

where $\overset{b}{\cancel{\beta}} \neq 0$. The graph above is of the amplitude (of a particular solution) versus γ.

(a) Find the value of γ at which practical resonance occurs. $3/2$

(b) Find the period of the corresponding steady-state solution. $4\pi/3$

(c) Sketch the corresponding steady-state solution with as much detail as you can.

$R \approx 4.2$