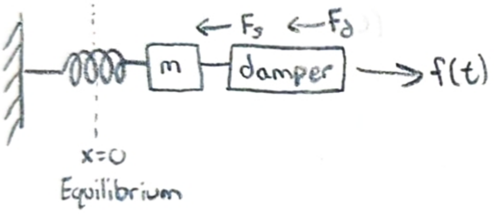
The Spring-mass Equation
Section 3.3 (Noonburg)

In this section we will consider the simple spring-mass system which is one of the primary applications of second order linear DEs with constant coefficients.

Consider a mass attached to a spring and a damper. Let $x(t)$ denote the position of the mass at time t.
$x = 0$ is the ***equilibrium position***, where the object comes to rest. Also we know $x'(t)$ is the velocity and $x''(t)$ is the acceleration of the object.

Picture:



Let;
m = mass of the object

F_s = the restoring force due to the spring
Hooke's Law
$-kx$
Spring constant >0

F_d = the force due to damping in the system
Resists motion
$-bv = -bx'$
Damping constant ≥ 0

$f(t)$ = external forces

Then the Newton's second law, $F = ma$ implies;

$$F = mx''$$

$$F_s + F_d + f(t) = mx''$$

$$-kx - bx' + f(t) = mx''$$

$$x'' + \frac{b}{m}x' + \frac{k}{m}x = \frac{f(t)}{m}$$

Second order linear DE with constant coefficients

The unforced spring-mass system

The equation for an unforced spring-mass system (that is when $f(t) = 0$) is given by

$$mx'' + bx' + kx = 0.$$

This is a homogeneous linear DE and we already learned the method of solving.

- **System with no damping:** The simplest case to consider is an unforced system with no damping is present. That is when $b = 0$. So we have,

$$mx'' + kx = 0.$$

$mr^2 + k = 0$

$r = \pm\sqrt{\frac{-k}{m}} = \pm\sqrt{\frac{k}{m}}\, i$

$x = C_1 \cos\sqrt{\frac{k}{m}}\, t + C_2 \sin\sqrt{\frac{k}{m}}\, t$

$= R\sin\left(\underbrace{\sqrt{\frac{k}{m}}}_{\omega_0}\, t + \phi\right)$, $R = \sqrt{C_1^2 + C_2^2}$, $\phi = \tan^{-1}\left(\frac{C_1}{C_2}\right)$

ω_0 = natural frequency

ϕ = phase angle

R = amplitude

$\frac{2\pi}{\omega_0}$ = period

* When $b \neq 0$,

$x(t) = Re^{\alpha t}\sin\left(\sqrt{\frac{k}{m}}\, t + \phi\right)$

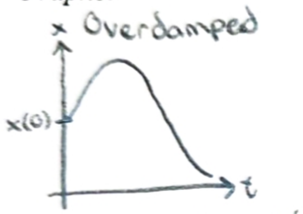
- **System with damping:** When damping is present, we are looking at the equation,

$$mx'' + bx' + kx = 0.$$

Looking at the characteristic polynomial, $mr^2 + br + k$, we have $r = \dfrac{-b \pm \sqrt{b^2 - 4mk}}{2m}$.
There are three possible solutions depending on the discriminant.

a) $b^2 - 4mk > 0$: ***overdamped***
b) $b^2 - 4mk = 0$: ***critically damped***
c) $b^2 - 4mk < 0$: ***underdamped***

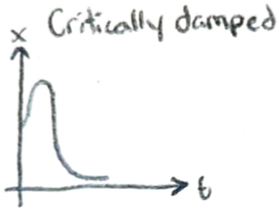
Graphs:



$x(t) = c_1e^{r_1t} + c_2e^{r_2t}$

$r_1, r_2 < 0$

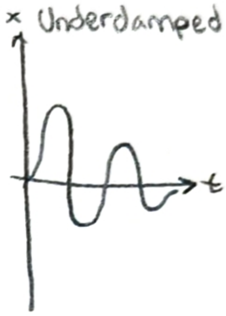
$t \to \infty, x \to 0$



$x(t) = c_1e^{rt} + c_2te^{rt}$

$r < 0$

$t \to \infty, x \to 0$



$x(t) = c_1e^{\alpha t}\cos\beta t + c_2e^{\alpha t}\sin\beta t$

$\alpha < 0$

$t \to \infty, x \to 0$

Quasi-period

Example #1: A 1 kg object is suspended on a spring with spring constant 4 N/m. The system is submerged in water, causing it to have a large damping constant 5 $N\ s/m$.

(a) Find the equation for $x(t)$.

$m = 1$ $\quad mx'' + bx' + kx = 0$
$k = 4$
$b = 5$ $\quad x'' + 5x' + 4x = 0$

$r^2 + 5r + 4 = 0$

$r = -4, -1$

$x(t) = c_1e^{-t} + c_2e^{-4t}$ Overdamped

(b) If the object is lifted up 1 m and let go, solve the IVP.

$x(t) = c_1e^{-t} + c_2e^{-4t}$ $\quad x(0) = 1$

$x'(0) = 0$

$1 = c_1 + c_2$

$x'(t) = -c_1e^{-t} - 4c_2e^{-4t}$

$0 = -c_1 - 4c_2$

$c_1 = 4/3$

$c_2 = -1/3$

$x(t) = \frac{4}{3}e^{-t} - \frac{1}{3}e^{-4}$

(c) If the object is lifted up 1 m and given a downward velocity 5 m/s, solve the IVP.

$x(0) = 1 \quad x'(0) = -5$

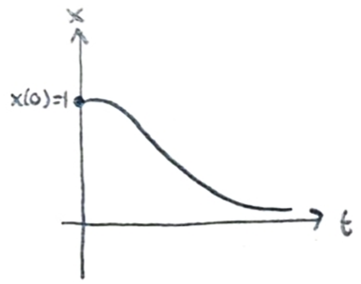
$1 = c_1 + c_2$

$-5 = -c_1 - 4c_2$

$c_1 = -1/3$

$c_2 = 4/3$

$x(t) = -\frac{1}{3}e^{-t} + \frac{4}{3}e^{-4t}$



Section 3.3 continued (Noonburg)

Example #2: A 4 kg mass is attached to a spring and damper in the absence of an external force. The spring is known to compress 2 m when acted on by a force of 400 N, while the damper is known to exert 32 N of force on an object moving at a speed of 4 m/s.

a) Write down the equation to model the motion of the mass.

$mx'' + bx' + kx = 0$

$4x'' + 8x' + 200x = 0$

b) Classify the system as undamped, underdamped, critically damped or overdamped.

$b^2 - 4mk = 8^2 - 4(4)(200) = 64 - 3200 = -3136 < 0$

Underdamped

c) Find the general solution for $x(t)$.

$4r^2 + 8r + 200 = 0$

$r = \frac{-8 \pm \sqrt{3136}i}{8} = -1 \pm 7i \quad \alpha = -1, \ \beta = 7$

$x(t) = c_1 e^{-t}\cos(7t) + c_2 e^{-t}\sin(7t)$

$= \sqrt{c_1^2 + c_2^2}\, e^{-t}\sin(7t + \tan^{-1}(\frac{c_1}{c_2}))$

d) Find the period or quasi-period of the motion as appropriate.

Quasi-period $= \frac{2\pi}{\omega_0}$

$\frac{2\pi}{7}$

e) Sketch the motion of the mass if $x(0) > 0$ and $x'(0) > 0$. Label important physical quantities.