Using Matrices to Solve Systems of Equations
Section 1.2 (**Hartman**)

In a linear equation, the most important information is the coefficients and the constants. The names of the variables really do not matter. Thus we can identify and define a system of linear equations using their coefficients and the constants.

Consider the system of equations;

$$\begin{array}{rcrcrcr} 2x & + & y & - & z & = & 7 \\ x & & & + & z & = & -1 \\ -x & + & 2y & + & 2z & = & 1 \end{array}$$

We can represent this system using a matrix as below:

$$\left(\begin{array}{rrr|r} 2 & 1 & -1 & 7 \\ 1 & 0 & 1 & -1 \\ -1 & 2 & 2 & 1 \end{array}\right)$$

This matrix is called ***augmented matrix***. To solve the system we apply necessary row operations to the augmented matrix and convert left-hand side of the augmented matrix into an identity matrix.

$$\begin{array}{c} \begin{array}{cccc} x & y & z & \end{array} \\ \left(\begin{array}{rrr|r} 2 & 1 & -1 & 7 \\ 1 & 0 & 1 & -1 \\ -1 & 2 & 2 & 1 \end{array}\right) \end{array} \rightarrow \text{ row operations } \rightarrow \left(\begin{array}{ccc|c} 1 & 0 & 0 & A \\ 0 & 1 & 0 & B \\ 0 & 0 & 1 & C \end{array}\right)$$

Then we can identify the solution as $x = A,\ y = B,\ z = C$.

NOTE:

- We use the notation R_i to denote the i-th row of the matrix.
- In each step of the procedure you must clearly state the row operations used, in terms of R_i.

  e. g.: "multiply 1 st row by '1/2' and add it to the 3 rd row replaces the 3 rd row";

$$\frac{1}{2}R_1 + R_3 \rightarrow R_3$$

- In a step, it is possible to apply more than one row operation. But do this if they do not depend on each other (to avoid errors).
- There is no right way to do this, but one can observe the values and apply the row operations correctly so that it will not add unnecessary work.

Example #1: Find a solution to the following system of linear equations by simultaneously manipulating the equations and the corresponding augmented matrices:

$$\begin{array}{rcrcrcr} 2x & + & y & - & z & = & 7 \\ x & & & + & z & = & -1 \\ -x & + & 2y & + & 2z & = & 1 \end{array}$$

$$\left(\begin{array}{ccc|c} 2 & 1 & -1 & 7 \\ 1 & 0 & 1 & -1 \\ -1 & 2 & 2 & 1 \end{array}\right) \xrightarrow{R_2+R_3 \to R_2} \left(\begin{array}{ccc|c} 2 & 1 & -1 & 7 \\ 0 & 2 & 3 & 0 \\ -1 & 2 & 2 & 1 \end{array}\right) \xrightarrow{2R_3+R_1 \to R_3} \left(\begin{array}{ccc|c} 2 & 1 & -1 & 7 \\ 0 & 2 & 3 & 0 \\ 0 & 5 & 3 & 9 \end{array}\right) \xrightarrow{\frac{1}{5}R_3 \to R_3} \left(\begin{array}{ccc|c} 2 & 1 & -1 & 7 \\ 0 & 2 & 3 & 0 \\ 0 & 1 & 3/5 & 9/5 \end{array}\right) \xrightarrow{-2R_3+R_2 \to R_3} \left(\begin{array}{ccc|c} 2 & 1 & -1 & 7 \\ 0 & 2 & 3 & 0 \\ 0 & 0 & 9/5 & -18/5 \end{array}\right)$$

$$\xrightarrow{\frac{5}{9}R_3 \to R_3} \left(\begin{array}{ccc|c} 2 & 1 & -1 & 7 \\ 0 & 2 & 3 & 0 \\ 0 & 0 & 1 & -2 \end{array}\right) \xrightarrow[R_1+R_3 \to R_1]{R_2-3R_3 \to R_2} \left(\begin{array}{ccc|c} 2 & 1 & 0 & 5 \\ 0 & 2 & 0 & 6 \\ 0 & 0 & 1 & -2 \end{array}\right) \xrightarrow{\frac{1}{2}R_2 \to R_2} \left(\begin{array}{ccc|c} 2 & 1 & 0 & 5 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & -2 \end{array}\right) \xrightarrow{R_1-R_2 \to R_1} \left(\begin{array}{ccc|c} 2 & 0 & 0 & 2 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & -2 \end{array}\right) \xrightarrow{\frac{1}{2}R_1 \to R_1} \left(\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & -2 \end{array}\right)$$

$x=1$
$y=3$
$z=-2$

Example #2: Solve the system by simultaneously manipulating the equations and the corresponding augmented matrices:

$$\begin{array}{rcrcrcl} x & + & y & + & z & = & 0 \\ 2x & + & 2y & + & z & = & 0 \\ -x & + & y & - & 2z & = & 2 \end{array}$$

$$\left(\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 2 & 2 & 1 & 0 \\ -1 & 1 & -2 & 2 \end{array}\right) \xrightarrow[R_3+R_1 \to R_1]{R_2-2R_1 \to R_2} \left(\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 2 & -1 & 2 \end{array}\right) \xrightarrow{R_2 \leftrightarrow R_3} \left(\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & 2 & -1 & 2 \\ 0 & 0 & -1 & 0 \end{array}\right) \xrightarrow[-R_3 \to R_3]{\frac{1}{2}R_2 \to R_2} \left(\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & 1 & -1/2 & 1 \\ 0 & 0 & 1 & 0 \end{array}\right) \xrightarrow{R_2+\frac{1}{2}R_3 \to R_2}$$

$$\left(\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{array}\right) \xrightarrow{R_1-R_2 \to R_1} \left(\begin{array}{ccc|c} 1 & 0 & 1 & -1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{array}\right) \xrightarrow{R_1-R_3 \to R_1} \left(\begin{array}{ccc|c} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{array}\right)$$

$x=-1$
$y=1$
$z=0$