Integration by Partial Fraction Decomposition
Section 8.5

This section shows how to express a rational function as a sum of simpler fractions, called *partial fractions*, which are easily integrated.

Example #1:

The rational function $\dfrac{5x-3}{x^2-2x-3}$ can be written as;

$$\frac{5x-3}{x^2-2x-3} = \frac{2}{x+1} + \frac{3}{x-3}$$

Hence,

$$\int \frac{5x-3}{x^2-2x-3}\,dx = \int \frac{2}{x+1}\,dx + \int \frac{3}{x-3}\,dx =$$

To find the partial fractions of the rational function $\dfrac{f(x)}{g(x)}$ successfully, we need two things:

- We must know the factors of $g(x)$. (In practice, factors may be hard to find.)
- The degree of $f(x)$ must be less than the degree of $g(x)$.
  If $\deg f(x) \geq \deg g(x)$ then divide $f(x)$ by $g(x)$ and work with the remainder term.
  (you may use long division for that.)

e. g. $\dfrac{x^4+1}{x^2+1} =$

$$\begin{array}{r|l} & x^2-1 \\ \hline x^2+1 & x^4+1 \\ & -(x^4-x^2) \\ \hline & -x^2+1 \\ & -(-x^2-1) \\ \hline & 2 \end{array}$$

$$\frac{x^4+1}{x^2+1} = x^2-1+\frac{2}{x^2+1}$$

$$\int x^2-1+\frac{2}{x^2+1}\,dx$$

$$\frac{x^3}{3}-x+2\tan^{-1}(x)+C$$

Steps to follow:

1. Make the degree of the numerator **less than** the degree of the denominator, if it is not already in that form. Then we work with the remainder term.

   e.g. $\dfrac{x^3+2x^2+x-1}{x^2+2x} = x + \dfrac{x-1}{x^2+2x}$

2. Factor the denominator as much as possible.

   e.g. $\dfrac{x-1}{x^2+2x} = \dfrac{x-1}{x(x+2)}$

3. Write each piece on the denominator as its own fraction, introducing (undetermined) constants in the numerator. The polynomial in the numerator of each piece should be one degree less than the denominator.
   When there are repeating terms in the denominator (say repeating n times), then we repeat the partial fraction n times (see e.g. (ii)).

   e.g. (i) $\dfrac{x-1}{x(x+2)} = \dfrac{A}{x} + \dfrac{B}{x+2}$

   e.g. (ii) $\dfrac{x+4}{x(x^2+5)^3} = \dfrac{A}{x} + \dfrac{Bx+C}{(x^2+5)} + \dfrac{Dx+E}{(x^2+5)^2} + \dfrac{Fx+G}{(x^2+5)^3}$

4. Solve for the constants.

   - Set the original function equal to the sum of the partial fractions.
   - Clear the fractions.
   - Equate the coefficient of corresponding powers of x and solve for unknown constants.

   e.g. $\dfrac{x-1}{x(x+2)} = \dfrac{A}{x} + \dfrac{B}{x+2} = \dfrac{A(x+2)+Bx}{x(x+2)} = -\dfrac{1}{2x} + \dfrac{3}{2(x+2)}$

   $x-1 = A(x+2)+Bx$

   $x-1 = x(A+B)+2A$

   $-1 = 2A$

   $A = -1/2$

   $1 = A+B$

   $1 = \frac{-1}{2}+B$

   $B = 3/2$

Example #2: Evaluate $\int \frac{x^3+2x^2+x-1}{x^2+2x}\,dx$

$\int x - \frac{1}{2x} + \frac{3}{2(x+2)}\,dx$

$\frac{x^2}{2} - \frac{1}{2}\ln|x| + \frac{3}{2}\ln|x+2| + C$

Example #3: Evaluate $\int \frac{2x-1}{x^3-2x^2+x}\,dx$

$\frac{2x-1}{x(x-1)^2} = \frac{A}{x} + \frac{B}{x-1} + \frac{C}{(x-1)^2}$

$2x-1 = A(x-1)^2 + Bx(x-1) + Cx$

$2x-1 = Ax^2 - 2Ax + A + Bx^2 - Bx + Cx$

$2x-1 = (A+B)x^2 + (-2A-B+C)x + A$

$A=-1$ $\quad x^2: 0 = A+B$ $\quad x: 2 = -2A-B+C$

$0 = -1+B$ $\quad 2 = 2-1+C$

$B = 1$ $\quad C = 1$

$\int \frac{-1}{x} + \frac{1}{x-1} + \frac{1}{(x-1)^2}\,dx$

$-\ln|x| + \ln|x+1| - \frac{1}{x-1} + C$

Extra practice: Evaluate the the following:

(a) $\displaystyle\int \frac{5x-13}{x^2-5x+6}\,dx$

$\frac{5x-13}{(x-3)(x-2)} = \frac{A}{x-3} + \frac{B}{x-2}$

$\int \frac{2}{x-3} + \frac{3}{x-2}\,dx$

$2\ln|x-3| + 3\ln|x-2| + C$

$5x-13 = A(x-2) + B(x-3)$

$5x-13 = Ax - 2A + Bx - 3B$

$5x-13 = x(A+B) - 2A - 3B$

$A+B=5 \qquad 13=-2A-3B$

$A=5-B \qquad -13=-2(5-B)-3B$

$-13=-10+2B-3B$

$A=2 \qquad -3=-B \quad B=3$

(b) $\displaystyle\int_0^1 \frac{x^3}{x^2+2x+1}\,dx$

$\frac{x^3}{x^2+2x+1} = \frac{x^3}{(x+1)^2} = \frac{A}{x+1} + \frac{B}{(x+1)^2}$

$x^3 = A(x+1) + B$

$x^3 = Ax + A + B$

(c) Evaluate the following integral skipping step #4 (solving for the constants); these (un-calculated) constants will appear in your final answer.

$$\int \frac{-x^3+x^2-\sqrt{3}}{(x^3+4x)(x^2-4)(x^2-5x+6)}\,dx$$

$$x(x^2+4)(x+2)(x-2)(x-2)(x-3) \Rightarrow \frac{-x^3+x^2-\sqrt{3}}{x(x^2+4)(x+2)(x-3)(x-2)^2}$$

$$\int \frac{A}{x} + \frac{Bx+C}{x^2+4} + \frac{D}{x-2} + \frac{E}{(x-2)^2} + \frac{F}{x+2} + \frac{G}{x-3}\,dx$$

$$A\ln|x| + \int \frac{Bx}{x^2+4}\,dx + \int \frac{C}{x^2+4}\,dx + D\ln|x-2| + F\ln|x+2| + \int \frac{E}{(x-2)^2}\,dx + G\ln|x-3|$$

$\downarrow$ $\frac{B}{2}\ln|x^2+4|$

$\searrow$ $\frac{C}{2}\tan^{-1}\left(\frac{x}{2}\right)$

$\downarrow$ $\frac{-F}{x-2}$