Integral Test
Section 10.3

An improper integral (of Type 1) and an infinite series are similar. In fact, they are so similar that they do the same thing!

Integral Test: Let $\{a_n\}$ be a positive sequence and define the function $f(x)$ so that $f(n) = a_n$ for all positive integers. If there exists a positive integer N such that f is a continuous, positive, decreasing function for all $x \geq N$ (i.e., for all large x), then the series $\sum_{n=N}^{\infty} a_n$ and the integral $\int_N^{\infty} f(x)\,dx$ do the same thing (i.e., both converge or both diverge).

NOTE: If they both converge, they do **NOT** both converge to the same value. In other words, the integral test can tell you that a series converges, but CANNOT tell you what the series converges to.

$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{1}{1^2} + \frac{1}{2^2} + \cdots$ $\quad f(n) = \frac{1}{n^2}$

$f(x) = \frac{1}{x^2}$

So all the partial sums are less than $\frac{1}{1^2} + \int_1^{\infty} \frac{1}{x^2}\,dx = 1 + 1 = 2$

$\sum_{n=1}^{\infty} \frac{1}{n^2} < 2$ so the series converges

Example #1: p-series $\sum_{n=1}^{\infty} \frac{1}{n^p}$

$p<0$:

$\rightarrow \lim_{n\to\infty} \frac{1}{n^p} = \infty$

$p=0$

$\rightarrow \lim_{n\to\infty} \frac{1}{n^0} = 1$

In both cases

$\lim_{n\to\infty} a_n \neq 0$

$\rightarrow$ The series diverges by the nth term test

$p>0$:

$f(x) = \frac{1}{x^p}$ positive, continuous, decreasing on $[1, \infty)$

$p=1$:

$\rightarrow \sum_{n=1}^{\infty} \frac{1}{x}$ diverges (harmonic series)

$$\int_1^{\infty} \frac{1}{x^p}\,dx = \lim_{n\to\infty} \int_1^{n} \frac{1}{x^p}\,dx = \lim_{n\to\infty} \frac{1}{1-p}\left(\frac{1}{n^{p-1}} - 1\right)$$

$p>1$:

$\frac{1}{n^{p-1}} \Rightarrow 0$ so $\int_1^{\infty} \frac{1}{x^p}\,dx$ converges

$\rightarrow \sum_{n=1}^{\infty} \frac{1}{n^p}$ also converges

$0<p<1$:

$\frac{1}{n^{p-1}} \Rightarrow \infty$ so $\int_1^{\infty} \frac{1}{x^p}\,dx$ diverges

$\rightarrow \sum_{n=1}^{\infty} \frac{1}{n^p}$ also diverges

Diverges: $p \leq 1$

Converges: $p>1$

Observations:

The p-series $\sum_{n=1}^{\infty} \frac{1}{n^p}$ converges if $p>1$ and diverges if $p \leq 1$

Example #2: $\displaystyle\sum_{n=1}^{\infty} ne^{-n^2}$

$f(x) = xe^{-x^2}$ $[1, \infty)$, continuous

$= \dfrac{x}{e^{x^2}}$

$$\int_1^\infty xe^{-x^2}\,dx = \lim_{n\to\infty}\int_1^n xe^{-x^2}\,dx \qquad u=-x^2 \quad du=-2x\,dx$$

$$= \lim_{n\to\infty} -\frac{1}{2}\int_1^n e^u\,du = \lim_{n\to\infty} \frac{-1}{2}e^{-x^2}\Big|_1^n$$

$$= \lim_{n\to\infty} \frac{-1}{2}\left(\underset{\overset{\downarrow}{0}}{e^{-n^2}} - e^{-1}\right) = \frac{1}{2e} \text{ converges}$$

$\displaystyle\sum_{n=1}^{\infty} ne^{-n^2}$ also converges

Example #3: $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n\ln(n)}$

$f(x) = \dfrac{1}{x\ln x}$ $[2, \infty)$ continuous
decreasing

$$\int_2^\infty \frac{1}{x\ln x}\,dx = \lim_{n\to\infty}\int_2^n \frac{1}{x\ln x}\,dx \qquad u=\ln x \quad du=\frac{1}{x}\,dx$$

$$\lim_{n\to\infty}\int_2^n \frac{1}{u}\,du = \lim_{n\to\infty} \ln|\ln x|\Big|_2^n$$

$$\lim_{n\to\infty} \underset{\overset{\downarrow}{\infty}}{\ln|\ln n|} - \ln|\ln 2| = \infty \text{ diverges}$$

$\displaystyle\sum_{n=2}^{\infty} \frac{1}{n\ln n}$ also diverges