Method of Undetermined Coefficients
Section 3.4 (Noonburg)

In this section we will describe a method to find the particular solution, x_p of a non-homogeneous equation. There are two methods, namely; *Method of undetermined coefficients* and *Variation of parameters*. But we will not cover the method of variation of parameters in this course.
Again, for simplicity, we only focus on second order non-homogeneous linear DEs.

Recall: $x_h + x_p$ makes a general solution of the non-homogeneous linear DE. (See Section 3.1)

The method of undetermined coefficients only works if;

- the coefficients are all constants.
- for $ax'' + bx' + cx = f(t)$, where a, b, c are constants, the function $f(t)$ is
  - an exponential function
    e.g. $ax'' + bx' + cx = 2e^{-3t}$
  - a linear combination of sines and cosines
    e.g. $ax'' + bx' + cx = 5\sin(2t)$
  - a polynomial
    e.g. $ax'' + bx' + cx = 2t^3 - t + 1$
  - or an algebraic combination of such functions
    e.g. $ax'' + bx' + cx = 2t^3 - t + \sin(3t)$

Procedure to determine x_p:

Looking at the $f(t)$, guess the form of the x_p as give in the table below:

Sample $f(t)$	x_p used
$2e^{-3t}$	Ae^{-3t}
$5\sin(2t)$	$A\sin(2t) + B\cos(2t)$
$2t^3 - t + 1$	$At^3 + Bt^2 + Ct + D$

If $f(t) = 2t^2e^{-3t}$, then $x_p = (At^2 + Bt + C)e^{-3t}$

NOTE: If x_p is a solution of the associated homogeneous equation, then use $t^k x_p$ where k is the smallest positive integer which distinguishes x_p from x_h.
e.g. Suppose $x_h = 5e^{-3t} + e^t$ and the $f(t) = 2e^{-3t}$. Then you must use the guess as $x_p = Ate^{-3t}$.

After finding the suitable guess for x_p, the function x_p and its derivatives are substituted to the DE and then solve for the unknown constants of x_p by matching the coefficients of $f(t)$.

Example #1: Find the solution of the IVP

$$2x'' + 5x' - 3x = 3t^2 - 4t + 1, \quad x(0) = 1, \ x'(0) = 0.$$

$2x''+5x'-3x=0$

$2r^2+5r-3=0$

$r = 1/2, -3$

$x_h = c_1e^{1/2t}+c_2e^{-3t}$

$f(t)=3t^2-4t+1$

$x_p = At^2+Bt+C$

$x_p' = 2At+B$

$x_p'' = 2A$

$4A+10At+5B-3At^2-3Bt-3C=3t^2-4t+1$

$-3At^2+(10A-3B)t+(4A+5B-3C)=3t^2-4t+1$

$-3A=3 \quad A=-1$

$10A-3B=-4 \quad -10-3B=-4 \quad B=-2$

$4A+5B-3C=1 \quad -4-10-3C=1 \quad C=-5$

$x_p = -t^2-2t-5$

$x(t)=c_1e^{1/2t}+c_2e^{-3t}-t_2-2t-5$

$1=c_1+c_2-5 \quad 6=c_1+c_2$

$x'(t)=\frac{1}{2}c_1e^{1/2t}-3c_2e^{-3t}-2t-2$

$0=\frac{1}{2}c_1-3c_2-2 \quad 2=\frac{1}{2}c_1-3c_2$

$c_1 = 40/7 \quad c_2 = 2/7$

$x(t)=\frac{40}{7}e^{1/2t}+\frac{2}{7}e^{-3t}-t^2-2t-5$

Example #2: Find the particular solution, x_p of the IVP. (HW: Solve the IVP.)

$$x'' + x' + x = 3\sin(2t), \quad x(0) = 1, \; x'(0) = 0.$$

$r^2 + r + 1 = 0$

$r = \frac{-1 \pm \sqrt{1-4}}{2} = \frac{-1}{2} \pm \frac{\sqrt{3}i}{2}$ $\quad \alpha = -1/2$

$\beta = \sqrt{3}/2$

$x_h = c_1 e^{-1/2 t} \cos\left(\frac{\sqrt{3}}{2} t\right) + c_2 e^{-1/2 t} \sin\left(\frac{\sqrt{3}}{2} t\right)$

$f(t) = 3\sin(2t)$

$x_p = A\sin(2t) + B\cos(2t)$

$x_p' = 2A\cos(2t) - 2B\sin(2t)$

$x_p'' = -4A\sin(2t) - 4B\cos(2t)$

$-4A\sin(2t) - 4B\cos(2t) + 2A\cos(2t) - 2B\sin(2t) + A\sin(2t) + B\cos(2t) = 3\sin(2t)$

$\sin(2t)(-4A - 2B + A) + \cos(2t)(-4B + 2A + B) = 3\sin(2t)$

$(-3A - 2B)\sin(2t) + (2A - 3B)\cos(2t) = 3\sin(2t)$

$-3A - 2B = 3$ $\quad 2A - 3B = 0$

$\frac{-9}{2}B - 2B = 3$ $\quad 2A = 3B$

$A = \frac{3}{2}B$

$\frac{-13}{2}B = 3$

$B = \frac{-6}{13}$ $\quad A = \frac{-9}{13}$

$x_p = \frac{-9}{13}\sin(2t) - \frac{6}{13}\cos(2t)$

Example #2: Find the particular solution, x_p of the IVP. (HW: Solve the IVP.)

$$x'' + 4x' + 4x = 2e^{-2t}, \quad x(0) = 1, \; x'(0) = 1.$$

$r^2 + 4r + 4 = 0$

$r = -2$

$x_h = c_1 e^{-2t} + c_2 t e^{-2t}$

$x_p = At^2 e^{-2t}$

$x_p' = 2Ate^{-2t} - 2At^2 e^{-2t} = 2Ae^{-2t}(t - t^2)$

$x_p'' = 2A\left((1-2t)e^{-2t} + (t-t^2)(-2e^{-2t})\right) = 2Ae^{-2t}(1 - 4t + 2t^2)$

$2Ae^{-2t}(2t^2 - 4t + 1) + 8Ae^{-2t}(t - t^2) + 4At^2 e^{-2t} = 2e^{-2t}$

$2A(2t^2 - 4t + 1) + 8A(t - t^2) + 4At^2 = 2$

$2At^2 \cancel{- 4At} + A + \cancel{4At} \cancel{- 4At^2} + \cancel{2At^2} = 1$

$A = 1$

$x_p = t^2 e^{-2t}$

Important observation:

Example #3: Consider $x'' + 4x' + 4x = 2t^2e^{-2t}$.

From Example #2, we have $x_h = C_1e^{-2t} + C_2te^{-2t}$ as the homogeneous solution. Here I need to find the particular solution according to $2t^2e^{-2t}$.

Suppose I assumed $x_p = (At^2 + Bt + C)e^{-2t}$. This contains terms in x_h.

So if I modified it to $x_p = (At^3 + Bt^2 + Ct)e^{-2t}$, still it contains terms in x_h.

But if I used $x_p = (At^4 + Bt^3 + Ct^2)e^{-2t}$, then it is different from x_h.

So this will be the correct guess for my x_p.

Example #4: Consider $x'' + 2x' + 10x = te^{-t}\sin(3t) + t$.
Given that $x_h = C_1e^{-t}\cos(3t) + C_2e^{-t}\sin(3t)$. (*Check:* HW). Find the structure of the particular solution, x_p.

$x_{p_1} = (At^2+Bt)e^{-t}\sin(3t)+(Ct^2+Dt)e^{-t}\cos(3t)$

$x_{p_2} = Et+F$

$x_p = (At^2+Bt)e^{-t}\sin(3t)+(Ct^2+Dt)e^{-t}\cos(3t)+Et+F$