Homogeneous Linear Equations with Constant Coefficients
Section 3.2 (Noonburg)

We will now discuss how the general solution x_h can always be found. For simplicity, we will only study second order linear DEs with constant coefficients.

Consider the second order homogeneous linear equation;

$$ax''(t) + bx'(t) + cx(t) = 0,$$

where $a \neq 0, b$ and c are constants.

$x(t) = e^{rt}$ is a reasonable guess for a solution. With $x(t) = e^{rt}$, $x'(t) = re^{rt}$ and $x''(t) = r^2e^{rt}$, we get

$$ar^2e^{rt} + bre^{rt} + ce^{rt} = e^{rt}(ar^2 + br + c) = 0.$$

Thus, r is a root of the quadratic polynomial $P(r) = ar^2 + br + c$. This polynomial is called the *characteristic polynomial* of the DE.

Here, $r = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$. The format of the general solution depends on the value of $b^2 - 4ac$.

- When $b^2 - 4ac > 0$:

  In this situation we will have two distinct real roots r_1, r_2. Thus, $x_1(t) = e^{r_1 t}$ and $x_2(t) = e^{r_2 t}$ are solutions of the DE.
  Further, $W(x_1, x_2) = (r_2 - r_1)e^{(r_1+r_2)t}$

  Check:
  $x_1'(t) = r_1 e^{r_1 t}$
  $x_2'(t) = r_2 e^{r_2 t}$
  $W(x_1, x_2) = e^{r_1 t}(r_2 e^{r_2 t}) - e^{r_2 t}(r_1 e^{r_1 t}) = (r_2 - r_1)e^{(r_1+r_2)t}$ ✓

  And this is never zero. $r_1 \neq r_2$
  Thus, the general solution is $x(t) = Ae^{r_1 t} + Be^{r_2 t}$, where A and B are constants.

- When $b^2 - 4ac = 0$:

In this situation we will have a single real root $\bar{r}$. Thus, $x_1(t) = e^{\bar{r}t}$ is a solution of the DE.

$\frac{-b}{2a} \Rightarrow 2a\bar{r} + b = 0$

In this case $x_2(t) = te^{\bar{r}t}$ is also a solution.

Check:

$x_2'(t) = e^{\bar{r}t} + \bar{r}te^{\bar{r}t} = e^{\bar{r}t}(1+\bar{r}t)$

$x_2''(t) = \bar{r}e^{\bar{r}t}(1+\bar{r}t) + \bar{r}e^{\bar{r}t} = \bar{r}e^{\bar{r}t}(2+\bar{r}t)$

$a\bar{r}e^{\bar{r}t}(2+\bar{r}t) + be^{\bar{r}t}(1+\bar{r}t) + cte^{\bar{r}t} = 0$

$e^{\bar{r}t}(a\bar{r}(2+\bar{r}t) + b(1+\bar{r}t) + ct) = 0$

$e^{\bar{r}t}(t\underbrace{(a\bar{r}^2+b\bar{r}+c)}_{0} + \underbrace{2a\bar{r}+b}_{0}) = 0$

$0 = 0$ ✓

Further, $W(x_1, x_2) = e^{2\bar{r}t}$ (check: HW). And this is never zero.

Thus, the general solution is $x(t) = Ae^{\bar{r}t} + Bte^{\bar{r}t}$, where A and B are constants.

- When $b^2 - 4ac < 0$:

In this situation we will have two complex roots given by $r_1 = \alpha + \beta i,\ r_2 = \alpha - \beta i$. Thus, the two complex functions $z_1(t) = e^{(\alpha+\beta i)t}$ and $z_2(t) = e^{(\alpha-\beta i)t}$ are solutions of the DE.

Note that,

$$z_1(t) = e^{\alpha t}(\cos(\beta t) + i\sin(\beta t)),\quad z_2(t) = e^{\alpha t}(\cos(\beta t) - i\sin(\beta t)),$$

and

$$\frac{z_1 + z_2}{2} = e^{\alpha t}(\cos(\beta t)),\quad \frac{z_1 - z_2}{2i} = e^{\alpha t}(\sin(\beta t))$$

Since linear combinations of solutions are again solutions of the DE, now we have two distinct solutions which are real (no more complex involved!).

Set $x_1(t) = e^{\alpha t}(\cos(\beta t)),\quad x_2(t) = e^{\alpha t}(\sin(\beta t))$.

Further, $W(x_1, x_2) = \beta e^{2\alpha t}$ and this is never zero (check: HW).

Thus, the general solution is $x(t) = Ae^{\alpha t}(\cos(\beta t)) + Be^{\alpha t}(\sin(\beta t))$, where $\underset{C_1}{A}$ and $\underset{C_2}{B}$ are constants.

Example #1: Solve the IVP $x'' + 2x' + 5x = 0$, $x(0) = 1$, $x'(0) = 3$.

$x(t) = e^{rt}$

$x' = re^{rt}$

$x'' = r^2 e^{rt}$

$r^2 + 2r + 5 = 0$

$r = \frac{-2 \pm \sqrt{2^2 - 4(5)}}{2} = -1 \pm 2i$

$z_1(t) = -1 + 2i$, $z_2(t) = -1 - 2i$

$\alpha = -1$, $\beta = 2$

$x(t) = c_1 e^{-t} \cos(2t) + c_2 e^{-t} \sin(2t)$

$x'(t) = -c_1 e^{-t} \cos(2t) - 2c_1 e^{-t} \sin(2t) - e^{-t} \sin(2t) + 2e^{-t} \cos(2t)$

$1 = c_1 e^0 \cos(0) + c_2 e^0 \sin(0)$

$c_1 = 1$

$3 = c_1 e^0 \cos(0) + 2e^0 \cos(0)$

$c_2 = 2$

$x(t) = e^{-t} \cos(2t) + 2e^{-t} \sin(2t)$