Properties of the Determinant
Section 3.4 (**Hartman**)

In this section we learn some of the properties of the determinant, which will allow us to compute determinants more easily.

Co-factor Expansion Along Any Row or Column:
Let A be an $n \times n$ matrix. The determinant of A can be computed using co-factor expansion along any row of A.

Example #1: Consider $A = \begin{pmatrix} 1 & -2 & 4 \\ -1 & 0 & 0 \\ 5 & -3 & 6 \end{pmatrix}$. Find the $\det(A)$ using the co-factor expansion along;

(a) row 1:

$1(0)+2(-6)+4(3)$

0

(b) row 2:

$1(-12+12)$

0

The Determinant of Triangular Matrices:
The determinant of a triangular matrix is the product of its diagonal elements.

Example #2: Find the determinant of $A = \begin{pmatrix} 1 & -2 & 4 & 5 \\ 0 & -1 & 5 & 7 \\ 0 & 0 & 3 & -9 \\ 0 & 0 & 0 & 1 \end{pmatrix}$.

$1(-1)(3)(1) = -3$

Question: If we perform an elementary row operation on a matrix, how the determinant of the new matrix will be changed compared to the determinant of the original matrix?

Example #3: Consider $A = \begin{pmatrix} 1 & 4 \\ -2 & 3 \end{pmatrix}$.

(a) Find $\det(A)$.

$3+8=11$

(b) Let B be the matrix obtained using the row operation "$2 \times R_2 \to R_2$". Find $\det(B)$.

$\begin{pmatrix} 1 & 4 \\ -4 & 6 \end{pmatrix}$

$6+16=22$

$2\det(A)$

(c) Let C be the matrix obtained using the row operation "$R_1 \leftrightarrow R_2$". Find $\det(C)$.

$\begin{pmatrix} -2 & 3 \\ 1 & 4 \end{pmatrix}$

$-8-3=-11$

$-\det(A)$

(d) Let D be the matrix obtained using the row operation "$2R_1 + R_2 \to R_2$". Find $\det(D)$.

$\begin{pmatrix} 1 & 4 \\ 0 & 11 \end{pmatrix}$

$11-0=11$

$\det(A)$

(e) List your observations.

$\det(B)=2\det(A)$

$\det(C)=-\det(A)$

$\det(D)=\det(A)$

The Determinant and Elementary Row Operations:
Let A be an $n \times n$ matrix and let B be formed by performing one elementary row operation on A.

1. If B is formed from A by adding a scalar multiple of one row to another, then $\det(B) = \det(A)$.
2. If B is formed from A by multiplying one row of A by a scalar k, then $\det(B) = k \cdot \det(A)$.
3. If B is formed from A by interchanging two rows of A, then $\det(B) = -\det(A)$.

Example #4: The matrix B was formed by A using the following elementary row operations, though not necessarily in this order. Find $\det(A)$.

$$B = \begin{pmatrix} 1 & 2 & 3 \\ 0 & 4 & 5 \\ 0 & 0 & 6 \end{pmatrix} \qquad 2R_1 \to R_1, \qquad 1/3R_3 \to R_3, \qquad R_1 \leftrightarrow R_2, \qquad 6R_1 + R_2 \to R_2.$$

2det(A)   2/3 det(A)   -2/3 det(A)   -2/3 det(A)

det(B) = 24

$24 = -\frac{2}{3}\det(A)$

det(A) = -36

Determinant Properties:
Let A and B be $n \times n$ matrices and let k be a scalar. Then:

1. $\det(kA) = k^n \cdot \det(A)$
2. $\det(A^T) = \det(A)$
3. $\det(AB) = \det(A)\det(B)$
4. If A is invertible, then $\det(A^{-1}) = \dfrac{1}{\det(A)}$
5. A matrix A is invertible if and only if $\det(A) \neq 0$. (Invertible Matrix Theorem.)