Properties of Eigenvalues and Eigenvectors
Section 4.2 (**Hartman**)

Properties:
Let A be an $n \times n$ invertible matrix. The following are true:

1. If A is triangular, then the diagonal elements of A are the eigenvalues of A.

   Example #1: Find eigenvalues of $B = \begin{pmatrix} 1 & -2 & 4 \\ 0 & 2 & 3 \\ 0 & 0 & -1 \end{pmatrix}$.

   $\det(B-\lambda I) = \begin{vmatrix} 1-\lambda & -2 & 4 \\ 0 & 2-\lambda & 3 \\ 0 & 0 & -1-\lambda \end{vmatrix} = (1-\lambda)(2-\lambda)(-1-\lambda) = 0$

   $\lambda = \pm 1, 2$

2. The product of the eigenvalues of A is equal to $\det(A)$.

3. If λ is an eigenvalue of A with eigenvector $\vec{x}$, then $\dfrac{1}{\lambda}$ is an eigenvalue of A^{-1} with eigenvector $\vec{x}$.

   Proof: Consider an invertible matrix A with eigenvalue λ and eigenvector $\vec{x}$. Then, by definition,
   $$A\vec{x} = \lambda\vec{x}.$$

   Multiply both sides by A^{-1}. Then;

   $$A^{-1}A\vec{x} = A^{-1}\lambda\vec{x}$$
   $$I\vec{x} = \lambda A^{-1}\vec{x}$$
   $$\frac{1}{\lambda}\vec{x} = A^{-1}\vec{x}$$

4. If λ is an eigenvalue of A then λ is an eigenvalue of A^T.

   Proof: Recall $(A+B)^T = A^T + B^T$ and $\det(A) = \det(A^T)$.
   Using the characteristic polynomial of A^T, we have

   $$\det(A^T - \lambda I) = \det(A^T - \lambda I^T) = \det(A - \lambda I)^T = \det(A - \lambda I)$$

   The characteristic polynomial of A^T is the same as that for A. Therefore they have the same eigenvalues.