Absolute convergence, Ratio Test
Section 10.5

Definition: A series $\sum_{n=1}^{\infty} a_n$ **converges absolutely** if the series $\sum_{n=1}^{\infty} |a_n|$ converges.

Example #1: The series $\sum_{n=1}^{\infty} \frac{\sin(n)}{n^2}$ converges absolutely (and hence converges).

$|\sin n| \leq 1$

$a_n = \frac{\sin n}{n^2}$ $\quad |a_n| = \left|\frac{\sin n}{n^2}\right| \leq \frac{1}{n^2}$

$\sum_{n=1}^{\infty} |a_n| = \sum_{n=1}^{\infty} \left|\frac{\sin n}{n^2}\right| \leq \sum_{n=1}^{\infty} \frac{1}{n^2}$ Converges

This implies $\sum_{n=1}^{\infty} a_n$ also converges

Notice that we could not use comparison tests directly on this series because it has negative terms.

Idea: Absolute convergence is "stronger" than regular convergence.

Theorem: If $\sum_{n=1}^{\infty} |a_n|$ converges, then $\sum_{n=1}^{\infty} a_n$ converges; i.e., an absolutely convergent series converges.

The opposite is not true: If $\sum_{n=1}^{\infty} a_n$ converges, that does not mean $\sum_{n=1}^{\infty} |a_n|$ converges; i.e., a series that converges does not necessarily converge absolutely.

Definition: If a series converges, but does not converge absolutely, we say that the series **converges conditionally;** i.e., a series converges conditionally if $\sum_{n=1}^{\infty} a_n$ converges but $\sum_{n=1}^{\infty} |a_n|$ diverges.

Ratio Test Let $\sum_{n=1}^{\infty} a_n$ be any series and suppose that

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = L.$$

- If $L < 1$, the series converges absolutely (and hence, converges).
- If $L > 1$, the series diverges.
- If $L = 1$, the test is inconclusive.

Remember that the Integral Test and Comparison Tests were only for series of positive terms. This test we can (try to) use for any series.

Example #2: Determine whether each of the following series converge or diverge.

(a) $\sum_{n=1}^{\infty} \frac{10^n}{n^{10}} = a_n$

$$\lim_{n\to\infty}\left|\frac{\cancel{10^n}}{\cancel{n^{10}}} \cdot \frac{\cancel{(n+1)^{10}}}{\cancel{10^{n+1}}}\right| = \left|10\left(\frac{n}{n+1}\right)^{10}\right|$$

$$\lim_{n\to\infty} 10\left(\frac{n}{n+1}\right)^{10} = 10\lim_{n\to\infty}\left(\frac{n}{n+1}\right)^{10} = 10 > 1$$

$\sum_{n=1}^{\infty} a_n$ diverges

(b) $\sum_{n=0}^{\infty} \frac{(-2)^n}{(n+1)!} = a_n$

$$\lim_{n\to\infty}\left|\frac{(-2)^{n+1}}{(n+2)!} \cdot \frac{(n+1)!}{(-2)^n}\right| = \left|\frac{-2}{n+2}\right|$$

$$\lim_{n\to\infty} \frac{2}{n+2} = 0 < 1$$

$\sum_{n=1}^{\infty} a_n$ converges

(c) $\displaystyle\sum_{n=2}^{\infty} \frac{2}{(n-1)^2} = a_n$

$$\lim_{n\to\infty} \left| \frac{2}{n^2} \cdot \frac{(n-1)^2}{2} \right| = \lim_{n\to\infty} \left(\frac{n-1}{n} \right)^2 = 1$$

Test is inconclusive (ratio)

Since $\displaystyle\sum_{n=1}^{\infty} a_{n+1}$ converges, $\displaystyle\sum_{n=2}^{\infty} a_n$ also converges

$\displaystyle\sum_{n=2}^{\infty} \frac{1}{n^2}$

When is the ratio test useful?

Often effective when terms of a series contain

- factorials of expressions involving n $(n+k)!$
- expressions raised to a power involving n $(k)^n$

Extra Practice

Determine whether the following series converge or diverge using any method.

(a) $\sum_{n=1}^{\infty} \frac{2+(-1)^n}{1.25^n}$ $= \frac{2}{1.25^n} + \frac{(-1)^n}{1.25^n} = 2\sum_{n=1}^{\infty}\left(\frac{1}{1.25}\right)^n + \sum_{n=1}^{\infty}\left(\frac{-1}{1.25}\right)^n$

(b) $\sum_{n=1}^{\infty} (-1)^n \left(1 - \frac{3}{n}\right)^n$ $(1+\frac{x}{n})^n = e^x$

$\sum_{n=1}^{\infty} (-1)^n e^{-3} \rightarrow \begin{cases} -1/e^3 \text{ when odd} \\ 1/e^3 \text{ when even} \end{cases}$

DNE

Diverges (nth term)

p series
geometric
nth term
comparison/ratio
alternating

(c) $\sum_{n=1}^{\infty} \frac{n^3}{3^n}$ (ratio) converges