Existence and Uniqueness of Solutions

Section 1.4 (**Hartman**)

So far, whenever we have solved a system of linear equations, we have always found exactly one solution. This is not always the case. We will find in this section that some systems do not have a solution, and others have more than one.

Consider the three systems of linear equations given below:

A: $\begin{array}{rcrcr} x & + & y & = & 2 \\ x & - & y & = & 0 \end{array}$ **B:** $\begin{array}{rcrcr} x & + & y & = & 1 \\ 2x & + & 2y & = & 2 \end{array}$ **C:** $\begin{array}{rcrcr} x & + & y & = & 1 \\ x & + & y & = & 2 \end{array}$

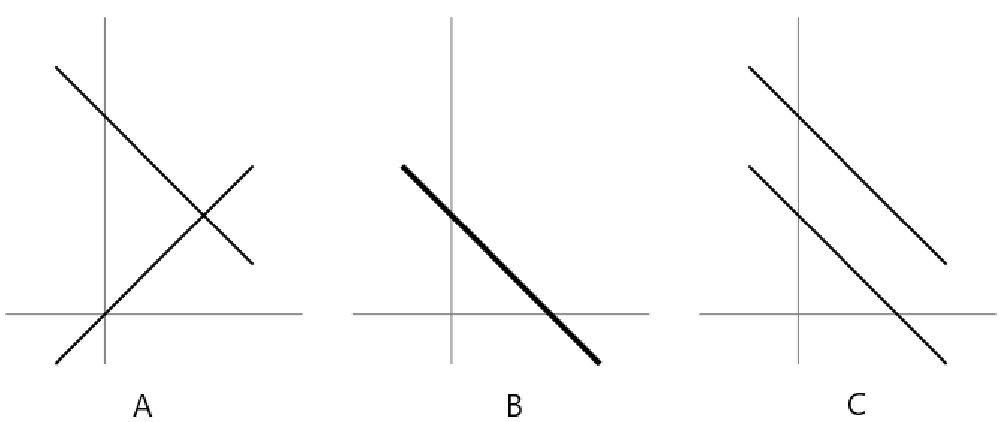
Each of these equations can be viewed as lines in the coordinate plane.

- System **A**:
  When two lines have different slopes, we know they intersect somewhere on the plane. In this example they intersect at $(1, 1)$. Since this is the only place the two lines intersect, this is the only solution.

- System **B**:
  Although we have two equations, they are essentially the same equation. Therefore, when we graph the two equations, they will coincide. In this case, we have an infinite solution set.

  We often write the solution as $y = 1 - x$ to demonstrate that x can be any real number, and y is determined once we pick a value for x. That is x is *independent* or *free*.

- System **C**:
  We can directly spot a problem with this system. When $x + y$ is equal to 1, it cannot be equal to 2 as well. This is a contradiction. We can visualize the system as two parallel lines and we know they never intersect. So, this linear system has no solution.



A system of linear equations is ***consistent*** if it has a solution (perhaps more than one). A linear system is ***inconsistent*** if it does not have a solution.

Facts:

- When we have n variables, we must have at least n independent equations to solve the system.
- We can use the row echelon form to check whether the system is consistent or not.

Example #1: Solve the following linear system:

$$\begin{array}{rcrcrcr} & & y & - & z & = & 3 \\ x & & & + & 2z & = & 2 \\ & - & 3y & + & 3z & = & -9 \end{array}$$

$$\left(\begin{array}{ccc|c} 0 & 1 & -1 & 3 \\ 1 & 0 & 2 & 2 \\ 0 & -3 & 3 & -9 \end{array}\right) \xrightarrow[-\frac{1}{3}R_3 \to R_3]{R_1 \leftrightarrow R_2} \left(\begin{array}{cccc} 1 & 0 & 2 & 2 \\ 0 & 1 & -1 & 3 \\ 0 & 1 & -1 & 3 \end{array}\right) \xrightarrow{R_3 - R_2 \to R_3} \left(\begin{array}{cccc} 1 & 0 & 2 & 2 \\ 0 & 1 & -1 & 3 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

$x + 2z = 2$ $x = 2 - 2z$
$y - z = 3$ $y = 3 + z$
z is free

This system has infinitely many solutions

Example #2: Solve the following linear system:

$$
\begin{array}{ccccccc}
x & + & y & + & z & = & 1 \\
x & + & 2y & + & z & = & 2 \\
2x & + & 3y & + & 2z & = & 0
\end{array}
$$

$$
\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 1 & 2 \\ 2 & 3 & 2 & 0 \end{pmatrix}
\xrightarrow[R_3 - 2R_1 \to R_3]{R_2 - R_1 \to R_2}
\begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & -2 \end{pmatrix}
\xrightarrow[R_1 - R_2 \to R_1]{R_3 - R_2 \to R_3}
\begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & -3 \end{pmatrix}
\Rightarrow
\left(\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right)
$$

$x + z = 0$

$y = 1$

$0 = -3$

No solution

- A system of linear equations is inconsistent if the reduced row echelon form of its corresponding augmented matrix has a **pivot in the last column**.