Matrices With Complex and Defective Eigenvalues

Section 4.3 (**Hartman**)

In Section 4.1 we only considered the problems with distinct, real eigenvalues. As eigenvalues occur as roots of a polynomial we can also have real repeated roots and complex roots. Then we will have defective eigenvalues and complex eigenvalues, respectively.

Example #1: Find eigenvalues and eigenvectors of $A = \begin{pmatrix} 1 & 4 \\ 0 & 1 \end{pmatrix}$.

$\det(A - \lambda I) = \begin{vmatrix} 1-\lambda & 4 \\ 0 & 1-\lambda \end{vmatrix} = (1-\lambda)(1-\lambda) = 0$

$\lambda = 1$

Two eigenvalues

$\lambda = 1$

$\left(\begin{array}{cc|c} 0 & 4 & 0 \\ 0 & 0 & 0 \end{array}\right) \to \left(\begin{array}{cc|c} 0 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right)$

$x_2 = 0$ x_1 is free

$\vec{x} = x_1 \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ $x_1 \neq 0$

In the cases we considered earlier, for a 2×2 matrix we can find two "independent" eigenvectors, but this is not the case for the matrix in the above example. For that matrix the eigenvalue 1 is said to be *defective*.

Example #2: Find eigenvalues and eigenvectors of $A = \begin{pmatrix} 2 & 3 \\ -3 & 2 \end{pmatrix}$.

$\det(A - \lambda I) = \begin{vmatrix} 2-\lambda & 3 \\ -3 & 2-\lambda \end{vmatrix} = (2-\lambda)^2 + 9 = 0$

$\lambda^2 - 4\lambda + 13 = 0$

$\lambda = \dfrac{4 \pm \sqrt{16 - 4(13)}}{2} = 2 \pm 3i$

We obtained complex eigenvalues. The process of finding eigenvectors will be the same:

$\lambda = 2+3i$

$$\left(\begin{array}{cc|c} -3i & 3 & 0 \\ -3 & -3i & 0 \end{array}\right) \xrightarrow{\frac{i}{3}R_1 \to R_1} \left(\begin{array}{cc|c} 1 & i & 0 \\ -3 & -3i & 0 \end{array}\right) \xrightarrow{R_2+3R_1 \to R_2} \left(\begin{array}{cc|c} 1 & i & 0 \\ 0 & 0 & 0 \end{array}\right)$$

$x_1 = -ix_2$ $\quad x_2$ is free

$\vec{x} = x_2 \begin{pmatrix} -i \\ 1 \end{pmatrix}$ $\quad x_2 \neq 0$

$\lambda = 2-3i$

$\vec{x} = x^2 \begin{pmatrix} i \\ 1 \end{pmatrix}$ $\quad x_2 \neq 0$

Eigenvectors are conjugates when eigenvalues are also conjugates

Complex Eigenvalues/Eigenvectors of a Real Matrix:
Let A be an $n \times n$ matrix with real entries. If λ is a complex eigenvalue for A with eigenvector $\vec{x}$ then $\bar{\lambda}$ is also an eigenvalue for A with eigenvector $\bar{\vec{x}}$.

Example #3: Find eigenvalues and eigenvectors of $A = \begin{pmatrix} 8 & -5 & 0 \\ 4 & 0 & 0 \\ 6 & -12 & -2 \end{pmatrix}$.