Elementary Row Operations and Gaussian Elimination
Section 1.3 (**Hartman**)

We already learned how we can use row operations to simplify matrices and solve systems of equations. In this section we will discuss more on the topic introducing an efficient technique.

Elementary Row Operations:

1. Add a scalar multiple of one row to another row, and replace the latter row with that sum.
2. Multiply one row by a nonzero scalar.
3. Swap the position of two rows

Row Echelon Form:
A matrix is in row echelon form if its entries satisfy the following conditions:

1. The first nonzero entry in each row is a 1 (called a pivot or leading 1).
2. Each pivot comes in a column to the right of the pivots in rows above it.
3. All rows of all 0's come at the bottom of the matrix.
4. If a column contains a pivot, then all other entries below the pivot in that column are 0.

A matrix that satisfies all these conditions and additionally has all entries above each pivot equal to zero is said to be in ***reduced row echelon form.***

e.g. $\begin{pmatrix} 1 & 4 & -1 & 2 \\ 0 & 1 & 3 & 0 \\ 0 & 0 & 1 & 3 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 6 & 4 & 2 \\ 0 & 0 & 1 & 5 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$ $\qquad$ $\begin{pmatrix} 1 & 0 & 0 & 41 \\ 0 & 1 & 0 & -9 \\ 0 & 0 & 1 & 3 \end{pmatrix}$

row echelon form $\qquad$ reduced row echelon form

Gaussian Elimination:
Gaussian elimination is the technique for finding the row echelon form of a matrix efficiently. It can be abbreviated to:

1. Create a pivot.

2. Use this pivot to put zeros underneath it.

3. Repeat the above steps until all possible rows have pivots.

Example #1: Use Gaussian elimination to put the given matrix into row echelon and then reduced row echelon form:

$$\begin{pmatrix} 1 & 2 & 1 & 3 \\ 2 & 1 & 1 & 1 \\ 3 & 3 & 2 & 1 \end{pmatrix} \begin{matrix} -2R_1+R_2 \to R_2 \\ -3R_1+R_3 \to R_3 \end{matrix} \begin{pmatrix} 1 & 2 & 1 & 3 \\ 0 & -3 & -1 & -5 \\ 0 & -3 & -1 & -3 \end{pmatrix} -\tfrac{1}{3}R_2 \to R_2 \begin{pmatrix} 1 & 2 & 1 & 3 \\ 0 & 1 & 1/3 & 5/3 \\ 0 & -3 & -1 & -3 \end{pmatrix} 3R_2+R_3 \to R_3 \begin{pmatrix} 1 & 2 & 1 & 3 \\ 0 & 1 & 1/3 & 5/3 \\ 0 & 0 & 0 & -3 \end{pmatrix} -\tfrac{1}{3}R_3 \to R_3$$

$$\begin{pmatrix} 1 & 2 & 1 & 3 \\ 0 & 1 & 1/3 & 5/3 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{matrix} -3R_3+R_1 \to R_1 \\ -5/3R_3+R_2 \to R_2 \end{matrix} \begin{pmatrix} 1 & 2 & 1 & 0 \\ 0 & 1 & 1/3 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} -2R_2+R_1 \to R_1 \begin{pmatrix} 1 & 0 & 1/3 & 0 \\ 0 & 1 & 1/3 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Example #2: Suppose the following matrix is an augmented matrix for a linear syste (x, y, z). Use Gaussian elimination to put the given matrix into row echelon form and use back substitution to find the solution.

$$\begin{pmatrix} 2 & 1 & -1 & 4 \\ 1 & -1 & 2 & 12 \\ 2 & 2 & -1 & 9 \end{pmatrix} \xrightarrow{\frac{1}{2}R_1 \to R_1} \begin{pmatrix} 1 & 1/2 & -1/2 & 2 \\ 1 & -1 & 2 & 12 \\ 2 & 2 & -1 & 9 \end{pmatrix} \xrightarrow[R_3 - 2R_1 \to R_3]{R_2 - R_1 \to R_2} \begin{pmatrix} 1 & 1/2 & -1/2 & 2 \\ 0 & -3/2 & 5/2 & 10 \\ 0 & 1 & 0 & 5 \end{pmatrix} \xrightarrow{-\frac{2}{3}R_2 \to R_2} \begin{pmatrix} 1 & 1/2 & -1/2 & 2 \\ 0 & 1 & -5/3 & -20/3 \\ 0 & 1 & 0 & 5 \end{pmatrix}$$

$$\xrightarrow{R_3 - R_2 \to R_3} \begin{pmatrix} 1 & 1/2 & -1/2 & 2 \\ 0 & 1 & -5/3 & -20/3 \\ 0 & 0 & 5/3 & 35/3 \end{pmatrix} \xrightarrow{\frac{3}{5}R_3 \to R_3} \left(\begin{array}{ccc|c} 1 & 1/2 & -1/2 & 2 \\ 0 & 1 & -5/3 & -20/3 \\ 0 & 0 & 1 & 7 \end{array}\right)$$

$$x + \frac{1}{2}y - \frac{1}{2}z = 2$$
$$y - \frac{5}{3}z = \frac{-20}{3}$$
$$\boxed{z = 7}$$
$$y - \frac{35}{3} = \frac{-20}{3} \quad \boxed{y = 5}$$
$$x + \frac{5}{2} - \frac{7}{2} = 2 \quad \boxed{x = 3}$$

NOTE: For a given matrix, the row echelon form is not unique. But the reduced row echelon form is unique.