General Theory of Homogeneous Linear Equations
Section 3.1 (Noonburg)

Goal: In this section we will study second order linear DEs. That is DEs of the form;

$$x'' + p(t)x' + q(t)x = f(t)$$

Here we will focus more on the structure of the solutions to such DEs and will learn the solving strategies in detail starting from Section 3.2.

Example #1: Consider $x'' + t^2x' + x + t = 0$. Determine whether this DE is linear. If it is linear, state whether it is homogeneous.

$x'' + t^2x' + x = -t$

$p(t) = t^2$

$q(t) = 1$ Linear, nonhomogeneous

$f(t) = -t$

$f(t) \neq 0$

Theorem: If x_h is a general solution of $x'' + p(t)x' + q(t)x = 0$ and x_p is any solution of $x''+p(t)x'+q(t)x = f(t)$, then the sum $x = x_h+x_p$ is a general solution of $x''+p(t)x'+q(t)x = f(t)$.

The solution x_p usually referred to as ***particular solution.***

NOTE: Homogeneous linear differential equations with constant coefficients are always solvable. We use this fact to find the solution to a DE with the above theorem.
Here we use an educated guess $x = e^{rt}$ to turn the DE to a simple algebra problem.

Example #2: Find all constants r such that $x(t) = e^{rt}$ satisfies the DE $x'' + 3x' + 2x = 0$.

$x' = re^{rt}$

$x'' = r^2e^{rt}$

$r^2e^{rt} + 3re^{rt} + 2e^{rt} = 0$

$e^{rt}(r^2 + 3r + 2) = 0$ $x(t) = e^{-t}, e^{-2t}$

$e^{rt}(r+1)(r+2) = 0$

$r = -1, -2$

Lemma: If $x_1(t)$ and $x_2(t)$ are solutions of $x'' + p(t)x' + q(t)x = 0$, then $Ax_1(t) + Bx_2(t)$ is also a solution for any constants A and B.

Caution: You must convince yourself that $x_1(t)$ and $x_2(t)$ are not constant multiples of each other.
Linearly independent

Example #3: Find a solution to $x'' + 3x' + 2x = 0$ using Example # 2.

$\left.\begin{array}{l} x_1(t) = e^{-t} \\ x_2(t) = e^{-2t} \end{array}\right\}$ Not constant multiples of each other

$x(t) = Ae^{-t} + Be^{-2t}$ is a solution to the DE

Definition: If $x_1(t)$ and $x_2(t)$ are solutions of $x'' + p(t)x' + q(t)x = 0$, then the determinant

$$W(x_1, x_2)(t) \equiv det \begin{pmatrix} x_1(t) & x_2(t) \\ x_1'(t) & x_2'(t) \end{pmatrix} \equiv x_1(t)x_2'(t) - x_1'(t)x_2(t)$$

is called the ***Wronskian*** of the functions x_1 and x_2.

Theorem: If $x_1(t)$ and $x_2(t)$ are any two solutions of the homogeneous DE, $x''+p(t)x'+q(t)x = 0$, and their Wronskian is non-zero for all t, then $Ax_1(t) + Bx_2(t)$ is a **general solution** of $x'' + p(t)x' + q(t)x = 0$.
Here $\{x_1(t), x_2(t)\}$ is called the ***fundamental solution set***.

Example #4: Find the general solution to $x'' + 3x' + 2x = 0$ using Example # 2 and 3.

$x_1(t) = e^{-t} \quad x_1'(t) = -e^{-t}$

$x_2(t) = e^{-2t} \quad x_2'(t) = -2e^{-t}$

$W(x_1, x_2)(t) = det\begin{pmatrix} e^{-t} & e^{-2t} \\ -e^{-t} & -2e^{-2t} \end{pmatrix} = e^{-t}(-2e^{-2t}) - e^{-2t}(-e^{-t})$

$= -2e^{-3t} + e^{-3t} = -e^{-3t} \neq 0$ for all t

$x(t) = Ae^{-t} + Be^{-2t}$ is the general solution of the DE

Example #5: Consider $2x'' + 5x' - 3x = 0$.
(a) Find the general solution.

$x = e^{rt}$
$x' = re^{rt}$
$x'' = r^2e^{rt}$

$2r^2e^{rt} + 5re^{rt} - 3e^{rt} = 0$

$e^{rt}(2r^2 + 5r - 3) = 0$
$e^{rt}(2r - 1)(r + 3) = 0$
$r = \frac{1}{2}, -3$
$x_1 = e^{1/2t}, \; x_2 = e^{-3t}$
$x_1' = \frac{1}{2}e^{1/2t}, \; x_2' = -3e^{-3t}$
$W(x_1, x_2) = e^{1/2t}(-3e^{-3t}) - e^{-3t}(\frac{1}{2}e^{1/2t}) = -3e^{-5/2t} - \frac{1}{2}e^{-5/2t} = -\frac{7}{2}e^{-5/2t} \neq 0$ for all t

$x(t) = Ae^{1/2t} + Be^{-3t}$ is the general solution of the DE

(b) Solve the IVP $2x'' + 5x' - 3x = 0, \quad x(0) = 1, \; x'(0) = 0$.

$1 = A + B \qquad 0 = \frac{1}{2}A - 3B$

$x'(t) = \frac{1}{2}Ae^{1/2t} - 3Be^{-3t}$

$0 = \frac{1}{2}A - 3(1 - A) = \frac{7}{2}A - 3$

$6 = 7A$

$A = \frac{6}{7}$

$B = \frac{1}{7}$

$x(t) = \frac{6}{7}e^{1/2t} + \frac{1}{7}e^{-3t}$

Existence and Uniqueness Theorem: For the linear equation

$$x'' + p(t)x' + q(t)x = f(t),$$

given two initial conditions of the form $x(t_0) = x_0$ and $x'(t_0) = v_0$, if the functions p, q and f are all continuous in some interval $t_1 < t < t_2$ containing t_0, then there exists a unique solution $x(t)$ that is continuous in the entire interval (t_1, t_2).

Example #6: Consider the IVP $2x'' + 5x' - 3x = 0,\quad x(0) = 1,\ x'(0) = 0$. Check the existence and the uniqueness of the solution.

$x'' + \frac{5}{2}x' - \frac{3}{2}x = 0$

$p(t) = \frac{5}{2}$, $q(t) = -\frac{3}{2}$, $f(t) = 0$ Continuous for all t

There exists a unique solution for this DE

Example #7: Consider $x'' + \tan(t)\ x' + \dfrac{1}{1-t}x = 0$. Assume that the initial conditions are given at $t = 0$. Determine the largest interval which guarantees the existence and continuity of the unique solution.

$p(t) = \tan(t)$, $t \neq$ odd multiples of $\frac{\pi}{2}$

$q(t) = \frac{1}{1-t}$, $t \neq 1$

$f(t) = 0$, Continuous for all t

$-\frac{\pi}{2} < t < 1$

To include $t=0$, we can choose the interval $t = (-\frac{\pi}{2}, 1)$ because $p, q,$ and f are defined in it

There exists a unique solution on $(-\frac{\pi}{2}, 1)$