Complex Numbers and Arithmetic

A1, A2, A3 (Hartman)

Definition: A ***complex number*** is a quantity of the form

$$z = a + bi$$

where a and b are real numbers and $i = \sqrt{-1}$. Here $Re(z) = a$ and $Im(z) = b$.

e.g. $z = 2 - 3i$ is a complex number with $Re(z) = 2$ and $Im(z) = -3$.

Arithmetic
Let $z = a \pm bi$ and $w = c \pm di$.

- Addition: $z + w = (a \pm c) + (b \pm d)i$.
- Multiplication: $zw = (a + bi)(c + di) = (ac - bd) + (ad + bc)i$.
- Division: $\dfrac{z}{w} = \dfrac{(a + bi)}{(c + di)} = \dfrac{(ac + bd)}{c^2 + d^2} + \dfrac{(bc - ad)}{c^2 + d^2}i.$

Conjugation: The conjugate of $z = a + bi$ is denoted by $\bar{z}$ and $\bar{z} = a - bi$.
$z\bar{z} = a^2 + b^2$

Properties:

- $\overline{(z \pm w)} = \overline{z} \pm \overline{w}$
- $\overline{(zw)} = \overline{z} \cdot \overline{w}$
- $\overline{\left(\dfrac{z}{w}\right)} = \dfrac{\overline{z}}{\overline{w}}$

Roots of Polynomials:
If α is a root of a polynomial, then $\bar{\alpha}$ is also a root.

Example #1: Find a cubic polynomial with roots $z = 2,\ z = -4i$.

$z = 2, -4i, 4i$

$(x-2)(x+4i)(x-4i) = (x-2)(x+16)$

$x^3 - 2x^2 + 16x - 32$

Euler's Identity: For any real number y we define

$$e^{iy} = \cos y + i \sin y$$

The exponential function for any complex number $z = x + iy$, we define

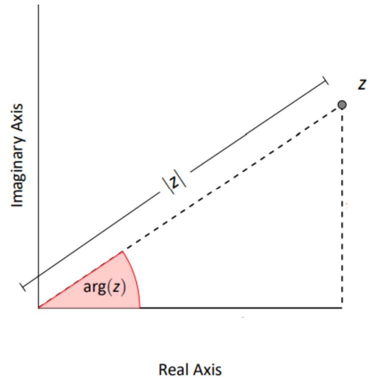
$$e^z = e^x \cos y + ie^x \sin y$$

$$= e^x + e^{iy} = e^x(\cos y + i\sin y)$$

Example #2: Compute e^{2+3i}.

$e^2 e^{3i} = e^2(\cos 3 + i\sin 3)$

$e^2 \cos 3 + ie^2 \sin 3 \approx -7.315 + 1.043i$



Modulus and Argument: If $z = x + iy$ is a complex number in rectangular form we define the modulus $|z|$ and argument $\arg(z)$ as

$$|z| = \sqrt{x^2 + y^2}$$

$$\arg(z) = \arctan(y/x).$$

NOTE: $|z^n| = |z|^n$ and $\arg(z^n) = n \arg(z)$.

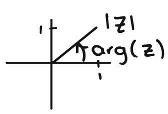
Example #3: Let $z = 1 + i$.

(a) Compute $|z|$ and $\arg(z)$.

$x = 1,\ y = 1$

$|z| = \sqrt{1^2 + 1^2} = \sqrt{2}$

$\arg(z) = \arctan(\frac{1}{1}) = \frac{\pi}{4}$

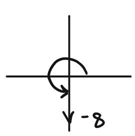


(b) Compute z^6.

$|z|^6 = \sqrt{2}^6 = 8$

$6\arg(z) = \frac{6\pi}{4} = \frac{3\pi}{2}$

$z^6 = -8i$

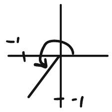


Example #3b: Let $z = -1 - i$

$x = -1,\ y = -1$

$|z| = \sqrt{(-1)^2 + (-1)^2} = \sqrt{2}$

$\arg(z) = \arctan(\frac{-1}{-1}) = \frac{5\pi}{4}$



Polar Form: For a complex number z, the polar form is given by

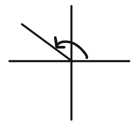
$$z = |z|e^{i\arg(z)}$$

Example #4: Convert $z = -2 + 2i$ to polar form.

$x = -2,\ y = 2$

$|z| = \sqrt{(-2)^2 + 2^2} = 2\sqrt{2}$

$\arg(z) = \arctan\left(\frac{2}{-2}\right) = \frac{3\pi}{4}$



$z = 2\sqrt{2}\,e^{\frac{3\pi}{4}i}$

Roots

Let n be a positive integer and z be a complex number; we will assume $z \neq 0$, so $|z| > 0$. We are interested in computing all solutions to $w^n = z$, the "n-th roots of z".

Procedure:

- We use the polar form $z = |z|e^{i\theta}$ and $w = |w|e^{i\alpha}$ where $\theta = \arg(z)$ and $\alpha = \arg(w)$.
- Then $w^n = z$ becomes $|w|^n e^{in\alpha} = |z|e^{i\theta}$, giving $|w|^n = |z|$ and $n\alpha = \theta$.
  So we have, $|w| = |z|/n$ and $\alpha = \theta/n$.
  Thus, $w = |z|^{1/n}e^{i\theta/n}$.
- The above is only one solution for w. To obtain all solutions we write $z = |z|e^{i\theta + 2k\pi i}$.
  Then $\alpha = \theta/n + 2k\pi/n$.
- The solutions w to the $w^n = z$ are of the form;

  $$w = |z|^{1/n}e^{i(\theta/n + 2k\pi/n)}$$

  Different choices for k give different solutions to w. (After a some value k, the solutions will repeat.)

Example #5: Find all solutions to $z^3 + 8 = 0$.

$z^3 = -8$

$x = -8, y = 0$

$|-8| = 8$

$\arg(-8) = \arctan\left(\frac{0}{-8}\right) = \pi$

$-8 = 8e^{\pi i + 2k\pi i}$

$z = 8^{1/3} e^{1/3(\pi i + 2k\pi i)} = 2e^{\frac{\pi i}{3}(1+2k)}$

$k=0 \quad z = 2e^{\frac{\pi i}{3}} = 2\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right) = 1 + i\sqrt{3}$

$1 \quad 2e^{\pi i} = 2(\cos\pi + i\sin\pi) = -2$

$2 \quad 2e^{\frac{5\pi i}{3}} = 2\left(\cos\frac{5\pi}{3} + i\sin\frac{5\pi}{3}\right) = 1 - i\sqrt{3}$

Example #6: Find all solutions to $z^4 = 2i$

$x = 0, y = 2$

$|2i| = 2$

$\arg(2i) = \arctan\left(\frac{2}{0}\right) = \frac{\pi}{2}$

$2i = 2e^{\frac{\pi}{2}i + 2k\pi i}$

$z = 2^{1/4} e^{\frac{\pi i}{8}(1+4k)}$

$k=0 \quad z = 2^{1/4} e^{\frac{\pi i}{8}} = 2^{1/4}\left(\cos\frac{\pi}{8} + i\sin\frac{\pi}{8}\right) = 1.099 + 0.455i$

$1 \quad 2^{1/4} e^{5\pi i/8} = 2^{1/4}\left(\cos\frac{5\pi}{8} + i\sin\frac{5\pi}{8}\right) = -0.455 + 1.099i$

$2 \quad 2^{1/4} e^{9\pi i/8} = 2^{1/4}\left(\cos\frac{9\pi}{8} + i\sin\frac{9\pi}{8}\right) = -1.099 - 0.455i$

$3 \quad 2^{1/4} e^{13\pi i/8} = 2^{1/4}\left(\cos\frac{13\pi}{8} + i\sin\frac{13\pi}{8}\right) = 0.455 - 1.099i$