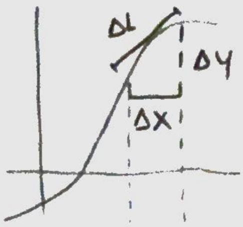
Arc Length and Surface Area
Sections 6.3 & 6.4

Arc Length:



$\Delta L^2 = \Delta x^2 + \Delta y^2$

$\Delta L = \sqrt{\Delta x^2 + \Delta y^2} = \sqrt{1 + \left(\frac{\Delta y}{\Delta x}\right)^2} \cdot \Delta x$

Definition If f' is continuous on $[a, b]$, then the **arc length** of the curve $y = f(x)$ from $(a, f(a))$ to $(b, f(b))$ is

$$L = \int_a^b \sqrt{1 + [f'(x)]^2}\, dx$$

The same is true if we want to calculate the arc length of $x = g(y)$ (assuming g' is continuous); integrate with respect to y instead.

Example #1: Find the arc length of $y = \frac{1}{3}(x^2 + 2)^{3/2}$ from $x = 0$ to $x = 3$.

$f(x) = \frac{1}{3}(x^2+2)^{3/2}$

$f'(x) = \frac{1}{2}(x^2+2)^{1/2} \cdot 2x = x\sqrt{x^2+2}$ continuous on $[0,3]$

$L = \int_0^3 \sqrt{1+(x\sqrt{x^2+2})^2}\, dx = \int_0^3 \sqrt{1+x^4+2x^2}\, dx = \int_0^3 \sqrt{(x^2+1)^2}\, dx = \int_0^3 x^2+1\, dx$

$\frac{x^3}{3} + x \Big|_0^3 = \frac{27}{3} + 3 - 0 = 12$

Dealing with discontinuities in $\frac{dy}{dx}$

$y = \left(\frac{x}{2}\right)^{2/3}$ on $[0,2]$

$y' = \frac{2}{3}\left(\frac{x}{2}\right)^{-1/3} \cdot \frac{1}{2} = \frac{1}{3}\left(\frac{2}{x}\right)^{1/3}$

$f'(x)$ undefined at $x=0$

Write the equation in terms of y

$x = 2y^{3/2}$

$x' = 3y^{1/2}$ continuous on $[0,1]$

$L = \int_0^1 \sqrt{1+(3\sqrt{y})^2}\, dy = \int_0^1 \sqrt{1+9y}\, dy = \frac{2}{27}(1+9y)^{3/2}\Big|_0^1$

$\frac{2}{27}(10)^{3/2} - \frac{2}{27}$

Example #2: Find the arc length of

$$x = \frac{y^3}{3} + \frac{1}{4y}$$

from $y = 1$ to $y = 3$.

$f'(y) = y^2 - \frac{1}{4y^2}$

$L = \int_1^3 \sqrt{1 + (y^2 - \frac{1}{4y^2})^2}\, dy = \frac{53}{6}$

Example #3: As a steady wind blows a kite east, its height above the ground as it moves horizontally from $x = 0$ to $x = 80$ feet is given by

$$4y = 150 - \frac{1}{40}(x - 50)^2$$

(also in feet). Find the distance traveled by the kite. *You may use Maple.*

$f(x) = \frac{75}{2} - \frac{1}{160}(x - 50)^2$

$f'(x) = \frac{-1}{80}(x - 50)$

$\int_0^{80} \sqrt{1 + \frac{1}{6400}(x - 50)^2}\, dx = 83.78$

Surface Area

Idea: We rotate a curve about a line. If we take a thin strip of the generated solid, it will look like a really thin cylinder (i.e., a ring). Hence its surface area will be $2\pi rh = 2\pi rL$.

Surface $= 2\pi r h$

$\Delta S = 2\pi y \, dL$ $y =$ avg height

$S = \int_a^b 2\pi y \sqrt{1+\left(\frac{dy}{dx}\right)^2}\, dx$

Definition If the function $f(x) \geq 0$ is continuously differentiable on $[a, b]$, the **area of the surface** generated by revolving the graph of $y = f(x)$ about the x-axis is

$$S = \int_a^b 2\pi f(x)\sqrt{1 + [f'(x)]^2}\, dx$$

Similarly, we can rotate $g(y) > 0$ about the y-axis and integrate with respect to y.

Example #4: Find the area of the surface generated by revolving $y = x/2$ from $x = 0$ to $x = 4$ about the x-axis.

$f(x) = \frac{x}{2}$

$f'(x) = \frac{1}{2}$ continuous on $[0,4]$

$S = \int_0^4 2\pi \frac{x}{2} \sqrt{1+\left(\frac{1}{2}\right)^2}\, dx = \frac{\pi\sqrt{5}}{2}\int_0^4 x\, dx$

$\frac{16\pi\sqrt{5}}{4} = 4\pi\sqrt{5}$ units^2