Alternating Series
Section 10.6

Alternating Series Test: Consider the series $\sum_{n=1}^{\infty}(-1)^n u_n$ or $\sum_{n=1}^{\infty}(-1)^{n+1}u_n$. If

1. $\{u_n\}$ is a sequence of positive terms,
2. the sequence $\{u_n\}$ is eventually nonincreasing (i.e., $u_{n+1} \leq u_n$ for large n), and
3. $\lim_{n\to\infty} u_n = 0$,

then the series converges.

Inconclusive if any conditions fail

Example #1: Determine whether each of the following series converge or diverge.

(a) $\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$

$\{u_n\} = \frac{1}{n}$

Positive terms ✓

Nonincreasing ✓

$\lim_{n\to\infty} \frac{1}{n} = 0$ ✓

$\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$ converges

(b) $\sum_{n=1}^{\infty}(-1)^{n+1}\ln\left(1+\frac{1}{n}\right)$

$\{u_n\} = \ln(1+\frac{1}{n})$

Positive terms ✓

Nonincreasing ✓

$\lim_{n\to\infty} \ln(1+\frac{1}{n}) = \ln(1) = 0$ ✓

$\sum_{n=1}^{\infty}(-1)^{n+1}\ln(1+\frac{1}{n})$ converges

(c) $\sum_{n=0}^{\infty}(-1)^{n}\frac{n-2}{n+5} = a_n$

$\{u_n\} = \frac{n-2}{n+5}$

Positive terms X

Nonincreasing X

$\lim_{n\to\infty} \frac{n-2}{n+5} = 1$ X

AST conditions violated

$\lim_{n\to\infty} (-1)^n \frac{n-2}{n+5} \to \begin{cases} -1 \text{ when } n \text{ is odd} \\ 1 \text{ when } n \text{ is even} \end{cases}$

$\lim_{n\to\infty} a_n$ DNE

$\sum_{n=0}^{\infty} a_n$ diverges (nth term test)

Extra Practice

1. $\displaystyle\sum_{n=2}^{\infty}\left(\frac{n+1}{n+2}\right)^n = a_n$

$\ln a_n = n\ln\left(\frac{n+1}{n+2}\right) = n(\ln(n+1)-\ln(n+2))$

$\lim_{n\to\infty} = \frac{\ln(n+1)-\ln(n+2)}{1/n} = \frac{1/n+1 - 1/n+2}{-1/n^2}$

$\lim_{n\to\infty} \frac{1}{(n+1)(n+2)}\cdot -n^2 = \frac{-2n^2}{(n+1)(n+2)} = -2$

$a_n = 1/e^2 \neq 0$

$\sum_{n=2}^{\infty}\left(\frac{n+1}{n+2}\right)^n$ diverges (nth term)

2. $\displaystyle\sum_{n=1}^{\infty}\frac{(-1)^n(2n)!}{2^n n!} = a_n$

$\lim_{n\to\infty}\left|\frac{(-1)^{n+1}(2n+2)!}{2^{\cancel{n+1}}(n+1)\cancel{!}}\cdot\frac{\cancel{2^n}\,\cancel{n!}}{(-1)^n(2n)!}\right| = \frac{(2n+2)!}{2(2n)!(n+1)}$

$\lim_{n\to\infty}\frac{(2n+1)(2n+2)}{2(n+1)} = 2n+1 = \infty > 1$

$\sum_{n=1}^{\infty} a_n$ diverges (ratio)

3. $\displaystyle\sum_{n=1}^{\infty}\frac{\cos(n\pi)}{\sqrt{n}} = \frac{(-1)^n}{\sqrt{n}}$

$\{u_n\} = \frac{1}{\sqrt{n}}$

Positive terms ✓

Nonincreasing ✓

$\lim_{n\to\infty} u_n = 0$ ✓

$\sum_{n=1}^{\infty}\frac{\cos(n\pi)}{\sqrt{n}}$ converges