Hyperbolic Functions
Section 7.3

Inverse Trigonometric Functions: Ratio → Angle

Example #1:

(a) $\arcsin\left(\frac{1}{2}\right) = \frac{\pi}{6}$

(b) $\tan^{-1}(1) = \frac{\pi}{4}$

Derivatives

$$\frac{\mathrm{d}}{\mathrm{d}\,x}\sin^{-1}(x) = \frac{1}{\sqrt{1-x^2}} \qquad \frac{\mathrm{d}}{\mathrm{d}\,x}\cos^{-1}(x) = -\frac{1}{\sqrt{1-x^2}}$$

$$\frac{\mathrm{d}}{\mathrm{d}\,x}\tan^{-1}(x) = \frac{1}{1+x^2} \qquad \frac{\mathrm{d}}{\mathrm{d}\,x}\cot^{-1}(x) = -\frac{1}{1+x^2}$$

$$\frac{\mathrm{d}}{\mathrm{d}\,x}\sec^{-1}(x) = \frac{1}{|x|\sqrt{x^2-1}} \qquad \frac{\mathrm{d}}{\mathrm{d}\,x}\csc^{-1}(x) = -\frac{1}{|x|\sqrt{x^2-1}}$$

Example #2:

(a) $\displaystyle\int \frac{2}{\sqrt{4-x^2}}\,dx$

$2\displaystyle\int \frac{1}{\sqrt{4(1-\frac{x^2}{4})}}\,dx$

$2\displaystyle\int \frac{1}{2\sqrt{1-(\frac{x}{2})^2}}\,dx$

$u = \frac{x}{2} \quad du = \frac{1}{2}\,dx$

$2\displaystyle\int \frac{1}{\sqrt{1-u^2}}\,du = 2\sin^{-1}u + C$

$2\sin^{-1}\frac{x}{2} + C$

(b) $\displaystyle\int_0^{\pi/20} \frac{1}{2+50y^2}\,dy$

$\displaystyle\int_0^{\pi/20} \frac{1}{2(1+(5y)^2)}\,dy$

$u = 5y \quad du = 5\,dy$

$\frac{1}{10}\displaystyle\int_0^{\pi/4} \frac{1}{1+u^2}\,du$

$\frac{1}{10}\tan^{-1}u\Big|_0^{\pi/4} = \frac{\tan^{-1}\frac{\pi}{4}}{10}$

Hyperbolic functions (Nothing to do with geometry, trigonometry, or angles!)

$$\sinh(x) = \frac{e^x - e^{-x}}{2} \qquad \cosh(x) = \frac{e^x + e^{-x}}{2}$$

Derivatives:

$$\frac{\mathrm{d}}{\mathrm{d}\,x}\sinh(x) = \cosh(x) \qquad \frac{\mathrm{d}}{\mathrm{d}\,x}\cosh(x) = \sinh(x)$$

$$\frac{\mathrm{d}}{\mathrm{d}\,x}\tanh(x) = \operatorname{sech}^2(x) \qquad \frac{\mathrm{d}}{\mathrm{d}\,x}\coth(x) = -\operatorname{csch}^2(x)$$

$$\frac{\mathrm{d}}{\mathrm{d}\,x}\operatorname{sech}(x) = -\operatorname{sech}(x)\tanh(x) \qquad \frac{\mathrm{d}}{\mathrm{d}\,x}\operatorname{csch}(x) = -\operatorname{csch}(x)\coth(x)$$

Integrals:

$$\int \sinh u \; du = \cosh u + C$$

$$\int \cosh u \; du = \sinh u + C$$

$$\int \operatorname{sech}^2 u \; du = \tanh u + C$$

$$\int \operatorname{sech} u \tanh u \; du = \operatorname{sech} u + C$$

Identities:

- $\cosh^2 x - \sinh^2 x = 1$
- $\cancel{\tanh^{x}} = 1 - \operatorname{sech}^2 x = \tanh^2 x$
- $\sinh 2x = 2 \sinh x \cosh x$
- $\cosh^2 x = \dfrac{\cosh 2x + 1}{2}$
- $\sinh^2 x = \dfrac{\cosh 2x - 1}{2}$

Example #3:

(a) Show that

$$\cosh^2(x) - \sinh^2(x) = 1.$$

$\left(\frac{e^x + e^{-x}}{2}\right)^2 - \left(\frac{e^x - e^{-x}}{2}\right)^2 = 1$

$\frac{1}{4}\left(e^{2x} + 2 + e^{-2x} - (e^{2x} - 2 + e^{-2x})\right) = 1$

$\frac{1}{4}(4) = 1$

(b) Find the derivative of $y = \sinh\theta + \ln(\operatorname{sech}\theta)$.

$\frac{dy}{d\theta} = \cosh\theta - \frac{1}{\operatorname{sech}\theta}(\operatorname{sech}\theta\tanh\theta)$

$\cosh\theta - \tanh\theta$

(c) Evaluate $\int 6\cosh\left(\frac{x}{2} - \ln 2\right)\,dx$

$6\int \cosh\left(\frac{x}{2} - \ln 2\right)dx$

$u = \frac{x}{2} - \ln 2 \quad du = \frac{1}{2}dx$

$12\int \cosh(u)\,du$

$12\sinh(u) + C$

$12\sinh\left(\frac{x}{2} - \ln 2\right) + C$

There are inverse hyperbolic functions similar to inverse trigonometric functions (*so don't be surprised if you see these appear in Maple*). It turns out that they are equal to logarithmic functions. You will always be able to evaluate these integrals in terms of logarithms (coming in Chapter 8).