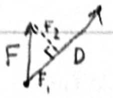
Dot and Cross Products MA 113

Dot product: $u \cdot w = \sum_{i=1}^{n} u_i w_i = ||u|| \, ||w|| \cos\theta$

$u \cdot u = \sum_{i=1}^{n} u_i^2 = ||u||^2$

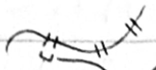
$\hat{u} \cdot \hat{u} = 1$

$\hat{u} \cdot \hat{w} = \cos\theta \in [-1, 1]$

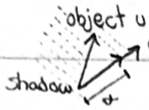



$F_2$ is not doing work

$W = FD = ||F_1|| \, ||D|| = ||F|| \cos\theta \, ||D|| = F \cdot D$



$dW = F \cdot ds$ ← arc length vector $\quad W = \int F \cdot ds$




Projection - The projection of u onto w is the component of u in the direction of w $\text{Proj}_w(u)$

$u = \underbrace{\alpha \frac{w}{||w||}}_{\text{Proj}} + \underbrace{w_\perp}_{\text{Orthogonal to } w}$

$w \cdot u = w \cdot \alpha \frac{w}{||w||} + \cancel{w \cdot w_\perp}^{\,0}$

$= \frac{\alpha}{||w||}(w \cdot w) = \alpha ||w||$

$\alpha = \frac{w \cdot u}{||w||} \qquad \text{Proj}_w(u) = \frac{w \cdot u}{||w||}\left(\frac{w}{||w||}\right)$

$\hat{u} = \hat{w} \quad \underbrace{\hat{u} \cdot \hat{w}}_{} = 1 \qquad \hat{u} \perp \hat{w} \quad \hat{u} \cdot \hat{w} = 0 \qquad \hat{u} = -\hat{w} \quad \hat{u} \cdot \hat{w} = -1$

Gives a fraction or percentage describing how alike they are

$\frac{u \cdot w}{||u|| \, ||w||} = \cos\theta$

Signals $s_1 = \langle a_1 + a_2 + \cdots + a_n \rangle \quad s_2 = \langle b_1 + b_2 + \cdots + b_n \rangle \quad n = 10^9$

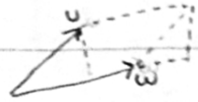
$\frac{s_1 \cdot s_2}{||s_1|| \, ||s_2||}$ can be used to determine how similar the signals are

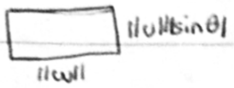
Cross product: $u \times w = (u_1\hat{x} + u_2\hat{y} + u_3\hat{z}) \times (w_1\hat{x} + w_2\hat{y} + w_3\hat{z})$ $\quad \hat{x} \cdot \hat{x} = 0 \quad \hat{x} \cdot \hat{z} \quad \hat{x} \cdot \hat{y} = \hat{z} \quad \ldots \quad = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ u_1 & u_2 & u_3 \\ w_1 & w_2 & w_3 \end{vmatrix} =$

$u = \langle u_1, u_2, u_3 \rangle \quad w = \langle w_1, w_2, w_3 \rangle$

$(u_2 w_3 - u_3 w_2)\hat{x} - (u_1 w_3 - u_3 w_1)\hat{y} + (u_1 w_2 - u_2 w_1)\hat{z}$

$\|u \times w\| = \|u\|\,\|w\|\,|\sin\theta| =$ area of parallelogram




$u = w$ or $u = -w \Rightarrow \theta = 0$ or $\pi \Rightarrow \sin\theta = 0$

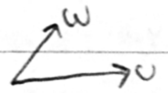
$u \times w = \vec{0} = \langle 0,0,0 \rangle$

$u \perp w \Rightarrow \theta = \pm \frac{\pi}{2}$

$\|u \times w\| = \|u\|\,\|w\|$

$\frac{\|u \times w\|}{\|u\|\,\|w\|} = |\sin\theta|$ Describes the extent to which the vectors are orthogonal

Direction - right hand rule




$u \times w \Rightarrow$ out of the page

$\vec{a} \cdot (\vec{b} + \vec{c}) = (\vec{a} \cdot \vec{b}) + (\vec{a} \cdot \vec{c})$ Distributive

$a \cdot b = b \cdot a$ Commutative

$a \times (b + c) = (a \times b) + (a \times c)$

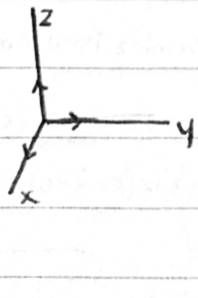
$a \times b = -(b \times a)$

$a \times (b \times c) \neq (a \times b) \times c \longrightarrow (a \times b) \times c + b \times (a \times c)$

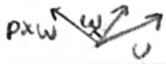
$a \times b = (\|a\|\,\|b\| \sin\theta)\,\hat{n} \leftarrow$ Determined by right hand rule $\theta \in [0, \pi]$

$\hat{x} \times (\hat{y} \times \hat{y}) = \hat{x} \times \vec{0} = \vec{0}$

$\underbrace{(\hat{x} \times \hat{y})}_{\|\hat{x}\|\,\|\hat{y}\| \sin\theta = 1} \times \hat{y} = \underbrace{\hat{z} \times \hat{y}}_{\|\hat{z}\|\,\|\hat{y}\| \sin\theta = 1} = -\hat{x}$



$\underbrace{(u\times w)}_{p}\times w$

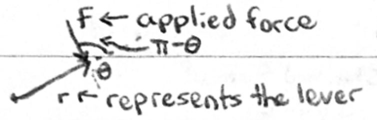


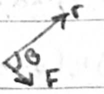

Lies in the same plane as $u$ and $w$

$\|p\| = \|u\|\|w\|\sin\theta \qquad \|p\times w\| = \|p\|\|w\|\sin\phi = \|u\|\|w\|^2 \underbrace{\sin\theta\sin\phi}_{[0,1]}$

The magnitude is somewhere between

$0$ and $\|u\|\|w\|^2$




$\|\tau\| = \|F_\perp\| r = \|F\|\sin\theta\|r\| = \|F\times r\|$

$\curvearrowleft \|F\|\underbrace{\sin(\pi-\theta)}_{\sin\theta}$

$\tau = r\times F$

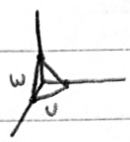
If the force $F=\langle 2,1\rangle$ is applied to the lever arm $r=\langle 4,0\rangle$, what is the magnitude and direction of the torque

$\tau = r\times F = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ 4 & 0 & 0 \\ 2 & 1 & 0 \end{vmatrix} = 0\hat{x} + 0\hat{y} + 4\hat{z} = \boxed{4\hat{z}}$

* FEQ Do either of the points $(\frac{1}{2},0,\frac{1}{2})$ or $(1,-1,2)$ lie in the same plane as the triangle with verticies $(1,0,0), (0,1,0), (0,0,1)$?

(Hard) If either point is in the same plane, is it inside the triangle?

1. Find the equation for the plane
2. See if points satisfy the equation




$u\times w$ will be normal to the plane

$u=\langle -1,1,0\rangle \quad w=\langle -1,0,1\rangle \quad u\times w = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{vmatrix} = \hat{x}+\hat{y}+\hat{z} = \langle 1,1,1\rangle$

$\langle x-1,y,z\rangle\cdot\langle 1,1,1\rangle = 0 \quad x+y+z=1$

$\frac{1}{2}+0+\frac{1}{2}=1 \checkmark \quad 1-1+2\neq 1$

$\boxed{(\frac{1}{2},0,\frac{1}{2}) \text{ is in the plane}}$