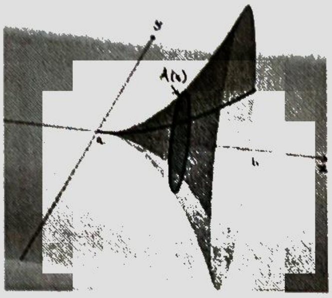
# Volumes Using Cross-Sections
## Section 6.1

Idea: We integrate a curve (one-dimensional) to get area (two-dimensional). What should we integrate to get volume (three-dimensional)?

The **volume** of a solid of integrable cross-section area $A(x)$ from $x = a$ to $x = b$:

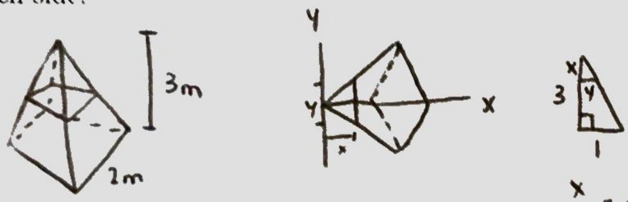
must be continuous on [a,b]

$$V = \int_a^b A(x)\, dx$$




1. Sketch the solid and a typical cross-section
2. Find a formula for A(x)
3. Find the limits of integration
4. Integrate A(x) to find the volume

**Example #1:** Find the volume of a pyramid that is 3 m high and has a square base that is 2 m on each side.




$\frac{x}{y} = \frac{3}{1}$

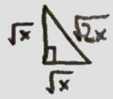
$x = 3y \quad y = \frac{x}{3}$

$A(x) = 4y^2 = 4\left(\frac{x}{3}\right)^2 = \frac{4}{9}x^2$

$\int_0^3 \frac{4}{9}x^2\, dx = \frac{4}{9} \cdot \frac{x^3}{3} = \left.\frac{4x^3}{27}\right|_0^3$

$\frac{4(3)^3}{27} - \frac{4(0)^3}{27} = 4$

**Example #2:** Find the volume of the solid that lies between the planes perpendicular to the $x$-axis at $x = 0$ and $x = 4$ and whose cross-sections (perpendicular to the $x$-axis) are squares whose diagonals run from the parabola $y = -\sqrt{x}$ to the parabola $y = \sqrt{x}$.


$A(x) = 2x$

$\int_0^4 2x\,dx = x^2\Big|_0^4$

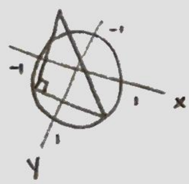
$16 - 0 = 16$


$A(x) = \pi x$

$\int_0^4 \pi x\,dx = \frac{\pi x^2}{2}\Big|_0^4$

$8\pi$

**Example #3:** Find the volume of the solid whose base is the disk $x^2 + y^2 \leq 1$ and whose cross-sections (perpendicular to the $y$-axis) are isosceles right triangles with one leg in the disk.




$x = \pm\sqrt{1-y^2}$

$x > 0:\ x = \sqrt{1-y^2}$

$A(y) = \frac{1}{2}\left(2\sqrt{1-y^2}\right)^2 = 2(1-y^2)$

$\int_{-1}^{1} 2(1-y^2)\,dy = \frac{8}{3}$

# Disk/Washer Method

Idea: We generate a solid by rotating a "planar region" about a line. The cross-sections perpendicular to that line will be circles so the area of each cross section will be $\pi r^2$.

**Disk Method:** Suppose that we rotate a planar region about the $x$-axis. Let $r(x)$ be the distance from the $x$-axis to the boundary of the planar region. Then the volume of the solid generated by this rotation is

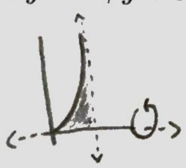
$$V = \int_a^b \pi[r(x)]^2\,dx.$$

A similar formula hold for rotation about the $y$-axis.

Volume by washers for rotation about x-axis: $V=\int_a^b \pi(R(x)^2 - r(x)^2)\,dx$ (outer - inner radius)

**Examples:** Find the volume of the solid generated by revolving the region bounded by

1. $y = x^3$, $y = 0$, and $x = 2$ about the $x$-**axis**.



$r = x^3$

$\int_0^2 \pi(x^3)^2\,dx = \int_0^2 \pi x^6\,dx$

$\frac{\pi x^7}{7}\Big|_0^2 = \frac{128\pi}{7}$

2. $y = 2x$, $y = 1$, and $x = 0$ about the $y$-**axis**.

$x = \frac{y}{2} = r$

$\int_0^1 \pi(\frac{y}{2})^2\,dy = \int_0^1 \pi \frac{y^2}{4}\,dy$

$\frac{\pi}{4}\cdot\frac{y^3}{3}\Big|_0^1 = \frac{y^3\pi}{12}\Big|_0^1 = \frac{\pi}{12}$

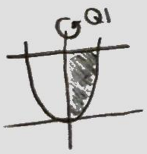
3. $y = 2x$, $y = 1$, and $x = 0$ about the $x$-**axis**.

$r = 2x$
$R = 1$

$$\int_0^{1/2} \pi(1-(2x)^2)\,dx = \int_0^{1/2} \pi - 4\pi x^2\,dx$$

$$\pi x - \frac{4\pi}{3}x^3 \Big|_0^{1/2} = \frac{\pi}{3}$$

4. $y = x^2$, $y = 4$, and $x = 0$ in the first quadrant about the $y$-**axis.**

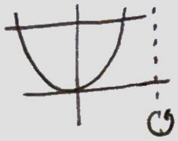



$r = \sqrt{y}$

$$\frac{\pi}{4}\int_0^4 y\,dy = \frac{\pi}{4}\cdot\frac{y^2}{2}\Big|_0^4 = \frac{\pi y^2}{8}\Big|_0^4 = 2\pi$$

$8\pi$?

5. $y = x^2$, $y = 4$, and $x = 0$ in the first quadrant **about the line** $x = 5$.



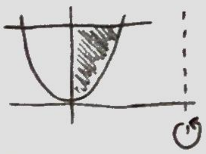
R = 5
r = 5 - $\sqrt{y}$

$$\pi\int_0^2 5^2-(5-\sqrt{y})^2\,dy = \pi\int_0^2 25\,dy - \pi\int_0^2 25 - \cancel{10}y^{1/2}+y\,dy$$

$$\cancel{[25\pi y]_0^2} - \left[\pi(\cancel{25y}+\tfrac{20}{3}y^{3/2}+\tfrac{-y^2}{2}\right]_0^2 = -2\pi+\frac{40\pi\sqrt{2}}{3}$$

$$\pi\int_0^2 -y+10y^{1/2}\,dy = -\frac{y^2}{2}+\frac{20}{3}y^{3/2}\Big|_0^2\pi = -2\pi+\frac{40\sqrt{2}\pi}{3}$$

6. $y = x^2$, $y = 4$, and $x = 0$ in the first quadrant **about the line** $y = 6$.



R = 6
r = 6 - $\sqrt{y}$

$$\pi\int_0^2 6^2-(6-y)^2\,dy = \pi\int_0^2 36-36+12\sqrt{y}-y\,dy$$

$$\pi\int_0^2 12y^{1/2}-y\,dy = \pi\left[8y^{3/2}-\frac{y^2}{2}\right]_0^2$$

$16\sqrt{2}\,\pi - 2\pi$