10.1 Sequences

1. $\{a_n\} = a_1, a_2, a_3, \ldots, a_n, \ldots$

The sequence $2, 4, 6, 8, \ldots$

sends 1 to $a_1 = 2$

2 to $a_2 = 4$

3 to $a_3 = 6$ and so on

2. a. $a_1 = \frac{1}{1!} = 1$ b. $b_1 = \frac{2^1}{2(1)+1} = \frac{2}{3}$ c. $a_1 = 1$

$a_2 = \frac{1}{2!} = \frac{1}{2}$ $b_2 = \frac{2^2}{2(2)+1} = \frac{4}{5}$ $a_2 = 1 + \frac{1}{2^1} = \frac{3}{2}$

$a_3 = \frac{1}{3!} = \frac{1}{6}$ $b_3 = \frac{2^3}{2(3)+1} = \frac{8}{7}$ $a_3 = \frac{3}{2} + \frac{1}{2^2} = \frac{7}{4}$

Fibonacci numbers: $a_{n+1} = a_n - a_{n-1}$; $a_1 = 1$, $a_2 = 1$ for $n > 2$

Newton's method: $x_n = x_n - \frac{(\sin x_n - x_n^2)}{(\cos x_n - 2x_n)}$, $x_0 = 1$ for $x > 0$

3. a. $a_n = \sqrt{2^n}$ b. $a_n = \frac{n^2}{3^n}$ c. $a_n = (-1)^n$

* $\lim_{n\to\infty} \left(1 + \frac{x}{n}\right)^n = e^x$

L'Hopital's Rule

a. $\lim_{x\to 0} \frac{1-\cos x}{x+x^2} = \lim_{x\to 0} \frac{\sin x}{1+2x} = \frac{0}{1} = \boxed{0}$

b. $\lim_{x\to 0} \frac{x-\sin x}{x^2} = \lim_{x\to 0} \frac{1-\cos x}{2x} = \lim_{x\to 0} \frac{\sin x}{2} = \frac{0}{2} = \boxed{0}$

c. $\lim_{x\to 2} \frac{x+2}{x-2} = \lim_{x\to 2} \frac{1}{1} = \boxed{1}$

d. $\lim_{n\to\infty} \frac{n}{e^n} = \lim_{n\to\infty} \frac{1}{e^n} = \boxed{0}$

e. $\lim_{n\to\infty} n^{1/n}$ | $\lim_{n\to\infty} \ln a_n = \lim_{n\to\infty} \frac{1}{n} \ln n = \lim_{n\to\infty} \frac{1/n}{1} = 0$

$\ln a_n = \ln n^{1/n}$ $\ln a_n = \frac{1}{n} \ln n$

$\ln a_n \to 0 = e^{\ln a_n} \to e^0 = a_n \to \boxed{1}$

The sequence $\{-1, 1, -1, \ldots, (-1)^n\}$ diverges

If this converges, the sequence gets arbitrarily close to a finite number L. This cannot happen if they keep oscillating

$a_n = \frac{(-1)^n}{n}$ converges

4. a. $a_n = \cos(n\pi) = (-1)^n$ Diverges

$\rightarrow$ lim of the function = function of the limit

$\lim_{n\to\infty} \cos\left(\frac{2}{n}\right) = \cos\left(\lim_{n\to\infty} \frac{2}{n}\right) = \cos 0 = 1$

b. $a_n = \sqrt{\frac{2n}{n+1}}$ Converges to $\sqrt{2}$

c. $a_n = \frac{\ln(n+1)}{\sqrt{n}} = \lim_{n\to\infty} \frac{1/(n+1)}{1/2\, n^{-1/2}} = \lim_{n\to\infty} \frac{2\sqrt{n}}{n+1}$ Converges to 0

d. $a_n = \frac{(n+1)!}{(n+3)!}$ Converges to 0

e. $a_n = \left(\frac{n+1}{n-1}\right)^n$

$\ln a_n = \ln\left(\frac{n+1}{n-1}\right)^n = n\ln\left(\frac{n+1}{n-1}\right) = \ln\left(\frac{n+1}{n-1}\right) / \frac{1}{n} = \ln(n+1) - \ln(n-1) / \frac{1}{n}$

$\lim_{n\to\infty} \ln a_n = \lim_{n\to\infty} \ln(n+1) - \ln(n-1) / \frac{1}{n} = \lim_{n\to\infty} \frac{1}{n+1} - \frac{1}{n-1} / -\frac{1}{n^2} = \lim_{n\to\infty} \frac{-2}{n^2-1} / -\frac{1}{n^2}$

$\lim_{n\to\infty} \frac{2n^2}{n^2-1} = 2$ $\quad \ln a_n \rightarrow 2$ as $n \rightarrow \infty$

$a_n \rightarrow e^2$ Converges to e^2

5. $\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{n+(-1)^n}{2n} = \lim_{n\to\infty} \frac{n}{2n} + \lim_{n\to\infty} \frac{(-1)^n}{2n} = \lim_{n\to\infty} \frac{1}{2} + \frac{1}{2} \lim_{n\to\infty} \frac{(-1)^n}{n}$

$\frac{1}{2} + \frac{1}{2}(0) = \frac{1}{2}$ Converges to $\frac{1}{2}$