Riemann Sums
Section 5.2

<u>Idea:</u> We will estimate the "area under a curve" using rectangles of small width.

For simplicity, here we will take all of our rectangles to have equal width so if my interval is $[a, b]$ and I want to use n rectangles, then the width of each rectangle is

$$\Delta x = \frac{b-a}{n} \text{ for all } k = 1...n$$

and the endpoints of the rectangles are

$$x_0 = a \qquad \text{and} \qquad x_k = x_{k-1} + \Delta x_k = a + k\Delta x \text{ for all } k = 1...n.$$

<u>Methods:</u> Here a few common ways to choose the "height" of our rectangles.

1. Left-hand endpoint
$$L_n = \Delta x \sum_{k=0}^{n-1} f(x_k)$$
2. Right-hand endpoint
$$R_n = \Delta x \sum_{k=1}^{n} f(x_k)$$
3. Midpoint
$$M_n = \Delta x \sum_{k=0}^{n-1} f\left(\frac{x_k + x_{k+1}}{2}\right)$$

Taking the limit as $n \to \infty$ of any of the three methods above will give the same answer, which is equivalent to the "area".

Example #1: Find L_4, R_4, and M_4 for the function $f(x) = \cos(x)$ over the interval $[0, \pi]$.

$\Delta x = \frac{\pi - 0}{4} = \frac{\pi}{4}$

$x_0 = 0$ $\qquad L_4 = \frac{\pi}{4} \sum_{k=0}^{3} \cos x_k = \frac{\pi}{4}$

$x_1 = \frac{\pi}{4}$ $\qquad R_4 = \frac{\pi}{4} \sum_{k=1}^{4} \cos x_k =$

$x_2 = \frac{\pi}{2}$

$x_3 = \frac{3\pi}{4}$ $\qquad M_4 = \frac{\pi}{4} \sum_{k=0}^{3} \cos\left(\frac{x_k + x_{k+1}}{2}\right) =$

$x_4 = \pi$

Example #2: Find a formula for the Reimann sum of $f(x) = x + x^2$ over $[0, 1]$ using n subintervals of equal width and right-hand endpoints. Then take the limit to find the exact area under the curve.

> You do not need to have these formulas memorized, but you do need to be able to use them.
>
> $$\sum_{k=1}^{n} k = \frac{n(n+1)}{2}$$
> $$\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}$$
> $$\sum_{k=1}^{n} k^3 = \left(\frac{n(n+1)}{2}\right)^2$$

$$f(x) = x + x^2 \quad [0, 1]$$

$$\Delta x = \frac{1-0}{n} = \frac{1}{n}$$

$$\sum_{k=1}^{n} f(x)\cdot\frac{1}{n} = \frac{1}{n}\sum_{k=1}^{n} (k/n)+(k/n)^2$$

$$\frac{1}{n^2}\sum_{k=1}^{n} k + \frac{1}{n^3}\sum_{k=1}^{n} k^2$$

$$\frac{1}{n^2}\cancel{\sum_{k=1}^{n}}\frac{n(n+1)}{2} + \frac{1}{n^3}\cancel{\sum_{k=1}^{n}}\frac{n(n+1)(2n+1)}{6}$$

$$\frac{n^2+n}{2n^2} + \frac{2n^2+3n+1}{6n^2} = \frac{5n^2+6n+1}{6n^2} = S_n$$

$$\lim_{n\to\infty} S_n = \lim_{n\to\infty}\frac{5n^2+6n+1}{6n^2} = \frac{5}{6}$$

These are not the only ways to do this. We can choose any point we want inside each rectangle to evaluate $f(x)$ at; it doesn't even have to be the same for all rectangles. The rectangles also do not need to have the same width.

General Case: ***Riemann Sums*** A **partition** of the interval $[a, b]$ is a set of points $P = \{x_0, x_1, x_2, \ldots, x_n\}$ satisfying

$$a = x_0 < x_1 < x_2 < \ldots < x_{n-1} < x_n = b.$$

The **norm** of a partition P is denoted $\|P\|$ and is equal to

$$\|P\| = \max_k \Delta x_k$$

where $\Delta x_k = x_k - x_{k-1}$.

The **Riemann sum for f on the interval** $[a, b]$ is

$$\sum_{k=1}^{n} f(c_k)\Delta x_k$$

where $x_{k-1} \leq c_k \leq x_k$.

We obtain the "area" between the function f and the x-axis (*which may be negative since area under the x-axis is considered negative*) from $x = a$ to $x = b$ by taking the limit as the norm of the partition goes to zero.

Example #3: Find the norm of the partition:

$$P = \{-1.2, -0.5, 0.3, 0.9, 1.4, 1.9\}$$

$\Delta x_k = -.5 + 1.2 = .7$

$\boxed{.3 + .5 = .8} = \|P\|$

$.9 - .3 = .6$

$1.4 - .9 = .5$

$1.9 - 1.4 = .5$