Indeterminacy and Imperfect Information*

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Abstract

We study equilibrium determination in an environment where two types of agents have different information sets: Fully informed agents observe histories of all exogenous and endogenous variables. Less informed agents observe only a strict subset of the full information set and need to solve a dynamic signal extraction problem to gather information about the variables they do not directly observe. Both types of agents know the structure of the model and form expectations rationally. In this environment, we identify a new channel that generates equilibrium indeterminacy: Optimal information processing of the less informed agent introduces stable dynamics into the equation system that lead to self-fulling expectations. For parameter values that imply a unique equilibrium under full information, the limited information rational expectations equilibrium is indeterminate. We illustrate our framework with a monetary policy problem where an imperfectly informed central bank follows an interest rate rule.

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1 Introduction

Asymmetric information is a pervasive feature of economic environments. Even when agents are fully rational, their expectation formation and decision-making process are constrained by the fact that information may be imperfectly distributed in the economy. Asymmetric information is also a central issue for the conduct of monetary policy as policymakers regularly face uncertainty about the true state of the economy, for example because they receive data in real time that are subject to measurement error. In environments where information is perfect and symmetrically shared, the literature has shown that policy rules can cause indeterminacy, unless they respond to economic outcomes with sufficient strength. We study equilibrium determinacy in an asymmetric information setting, where policy is conducted based on policymakers' optimal estimates of economic outcomes rather than their true values.

We consider an economic environment with two types of agents, one who has full information about the state of economy while the other agent is imperfectly informed. We think of the two agents as a representative private-sector agent and a less informed policymaker. While the representative agent observes aggregate outcomes without error, the imperfect information of the policymaker can, for example, take the form of aggregate data subject to measurement error.[1]

A key assumption of our modelling framework is that both types of agents, the policymaker and the private sector, employ rational expectations, but based on different information sets. Private-sector behavior is characterized by a set of linear, expectational difference equations. On the other hand, the policymaker's behavior is characterized by an instrument rule, which responds to the policymaker's estimates of economic conditions.[2] Formally, we consider linear, stochastic equilibria with time-invariant decision rules and Gaussian shocks. In this case, the rational inference efforts of the policymaker are represented by a dynamic signal extraction problem as captured by the Kalman filter. The interaction of the two expectation formation processes is the source of the new mechanism underlying equilibrium multiplicity in our environment.

The central result of our paper is that indeterminacy is generic for a broad class of linear imperfect-information models that have unique equilibria under full information. As is standard in the literature, we consider only stationary equilibria. In our imperfect information setup, optimal information processing of the less informed agent introduces stable dynamics into the equation system that can transmit self-fulling expectational shocks (sunspots). The interaction of the two expectation processes generates an endogenous feedback mechanism in a similar vein to strategic complementarities or the application of ad-hoc behavior in the standard indeterminacy literature.

[1] Such a dichotomy is well-established in the learning literature. Where our work differs is that both agents have rational expectations and know the structure of the economy, although not necessarily its state.

[2] As a specific example, we consider the conduct of monetary policy with a Taylor-type interest-rate rule that responds to the policymaker's projection of current inflation rather than its actual value.

We characterize the outcomes of different equilibria as the result of non-fundamental disturbances, similar to the perfect-information literature on equilibrium determinacy in linear rational expectations models. Such sunspot or belief shocks are unrelated to fundamental shocks in the original economic setup and can be interpreted as self-fulfilling shifts in expectations, or beliefs, that cause fluctuations consistent with the concept of a linear, stationary equilibrium. When there is indeterminacy in the perfect-information case, there are no restrictions on the scale and direction of effects caused by belief shocks. In contrast, the potential effects of belief shocks are tightly bounded in our imperfect information environment. The bounds arise from the required consistency of expectations of the public and the policymaker and the assumption that we consider only environments that have a unique equilibrium under full information. While the rationality of expectations under both information sets places non-trivial restrictions on outcomes, they are not sufficient to rule out multiple equilibria. Moreover, the interplay of expectations based on different information sets results in equilibrium outcomes that are not certainty equivalent even though we only consider environments that are linear.

We illustrate key insights from our framework in a simple model of inflation determination, where monetary policy follows a Taylor-type interest-rate rule. The rule satisfies the Taylor principle, that is, it responds more than one-for-one to the policymaker's reading of current inflation, which guarantees determinacy under full information. Under imperfect information, however, the policymaker observes only a noisy signal of inflation, and the policy rule responds only to an optimal projection of current inflation. The sensitivity of the policy rate to movements in the endogenous inflation rate thus depends on the sensitivity of the policymaker's projection to the incoming signal. The more the central bank is successful at stabilizing inflation, for example by reacting more aggressively to projected inflation, the noisier will be the signal and the policymaker's projection will barely respond. By the same logic, non-fundamental shocks cannot become an arbitrarily large driver of inflation. Otherwise, the central bank's signal would become highly informative and the policy rate would respond with sufficient strength to actual inflation to re-establish determinacy.

The Taylor principle effectively ceases to hold in this environment and indeterminacy is generic. Nevertheless, the extent to which indeterminacy manifests itself is bounded. This variance bound on indeterminacy-induced fluctuations is a novel and important result of our setup. The bound arises when the central bank observes noisy signals of endogenous variables and it reflects the endogenous response of the signals' information content to monetary policy.

Our paper touches upon three strands in the literature. First, we contribute to the burgeoning literature on imperfect information in macroeconomic models. An important topic of the existing literature have been the implications of dispersed information among different members of the public and the resulting effects on their strategic interactions and the informational value of prices.

Key contributions by Nimark (2008*a*, 2008*b*, 2014), Angeletos & La'O (2013), and Acharya, Benhabib & Huo (2017) demonstrate that imperfect information has important implications for the amplification and propagation of economic shocks. In Angeletos & La'O (2013), aggregate fluctuations are driven by non-fundamental shocks, which are modelled explicitly as exogenous shocks to agents' beliefs, and the equilibrium is unique and always exists. In contrast, the equilibrium in Benhabib, Wang & Wen (2015) is unique without non-fundamental shocks. Their sentiment shocks arise endogenously and their existence and statistical properties depend on the model's primitives. Acharya et al. (2017) extend the notion of this sentiment equilibrium to allow for persistent sentiment processes. Our paper is similar to this work in the sense that the sunspot shocks arise endogenously, and that by adding imperfect information the unique equilibrium property fails to hold. We differ in that we explore forward-looking behavior and and consider hierarchy of information structure instead of dispersed information. In that respect, our framework is closer to Rondina & Walker (2017). More recently, Fajgelbaum, Schaal & Taschereau-Dumouchel (2017) have studied information externalities in the private sector, which lead to uncertainty traps, namely self-reinforcing periods of high uncertainty and low activity, which shares similarities with outcomes in our framework. Similarly, Angeletos, Iovino & La'O (2020) and Kozlowski, Veldkamp & Venkateswaran (*forthcoming*) model endogenous learning in the private sector which can be seen as complementary to our setup of endogenous learning by policymakers.[3]

A number of papers have also analyzed the effects of policy actions on the informational value of market signals and their interplay when policy itself responds to market information (Goodhart 1984, Bernanke & Woodford 1997, Morris & Shin 2018, Siemroth 2019). This literature has typically found a tension between a policymaker's desire to extract information from prices as policy seeks to influence market outcomes. In our setup, the degree to which indeterminacy gives rise to fluctuations driven by non-fundamental shocks depends on the outcomes targeted by policy. What is unique in our analysis is the interplay between noisy signals gleaned from endogenous variables and indeterminacy.[4]

Second, our research also makes a contribution to the literature on indeterminacy in linear rational expectations models by expanding the set of plausible economic mechanisms that can lead to multiple equilibria. A key element of the indeterminacy literature is the presence of a mechanism

[3]We focus on the case of an imperfectly informed policymaker while treating the private sector as fully informed. However, imperfect information of the private sector is an additional concern to be studied. For example, Nakamura & Steinsson (2018) and Jarociński & Karadi (2018) point to an information effect from policy changes on private-sector behavior although the quantitative relevance of this effect has recently been questioned by Bu, Rogers & Wu (2019) and Bauer & Swanson (2020). However, the key ingredient for our indeterminacy result is the policymaker's inability to respond directly to self-fulfilling belief shocks, which could otherwise be used to rule out indeterminate equilibria.

[4]Bernanke & Woodford (1997) consider implications for existence and determinacy of equilibria arising from an inflation-targeting central bank's use of private-sector forecasts in lieu of its own. A key difference of our work are the consequences for determinacy arising from the explicit consideration of an imperfectly informed central bank's signal extraction problem.

that validates self-fulfilling expectations. These could arise from what is often termed strategic complementarities, such as increasing returns to scale in production that are not internalized, as in the seminal contributions of Benhabib & Farmer (1994), Farmer & Guo (1994), and Schmitt-Grohe (1997). An alternative mechanism is the interplay between economic agents' forward-looking behavior and the reaction function of a policymaker, which Clarida, Gali & Gertler (2000) and Lubik & Schorfheide (2004) show to be a key feature of macroeconomic fluctuations.[5]

In contrast, our framework does not rely on these previously identified sources of indeterminacy but rather on the interaction of different expectation formation processes under asymmetric information. This also sets our framework apart from the broader imperfect information literature, which is largely concerned with the strategic interaction between agents in the private sector. Although our framework utilizes the root-counting formalism of the indeterminacy literature, where we build on the contributions of Lubik & Schorfheide (2003, 2004) and Farmer, Khramov & Nicolò (2015), the mechanism to get there is novel. Critically, while equilibria are described by solutions to linear difference systems, the roots of these dynamic systems depend on endogenous Kalman gains and are not invariant to the equilibrium outcomes. We show that the set of multiple equilibria, despite the generic pervasiveness of indeterminacy, is tightly circumscribed by internal consistency requirements for the interaction between the two expectation processes. Our paper thereby puts some caveats on the notion that sunspot shocks are unrestricted in their effects on macroeconomic outcomes.

Third, the example application in our paper speaks to the monetary policy literature concerned with the effects of interest-rate rules on determinacy. A well-known result from this literature is the Taylor principle, which requires that interest rate rules respond to endogenous variables with sufficient strength, to avoid multiple equilibria and ensure determinacy. Clarida et al. (2000) and Lubik & Schorfheide (2004) have pointed to a neglect of the Taylor rule as a possible factor behind the Great Inflation. However, their evidence is based on a full-information perspective that does not account for the uncertainties faced by the Federal Reserve in assessing the state of the economy in real time, as discussed by Orphanides (2001). Our results point to the critical role played by the central bank's (in)ability to observe inflation (and output gap) in real time when seeking to design rules that conform to the Taylor principle to avoid indeterminacy.

Similar to our framework, Orphanides (2003) models the economic consequences of an imperfectly informed central bank that responds to estimates of economic conditions generated by optimal signal extraction efforts. But in a fundamental difference to our framework, his model is purely backward-looking so that the issue of indeterminacy does not arise. Our paper also relates to Svensson & Woodford (2004) and Aoki (2006) who derive conditions for optimal policy when

[5]Ascari, Bonomolo & Lopes (2019) also point to sunspot-driven equilibria as a key source of fluctuations during the high-inflation era of the 1970s.

the policymaker is less informed than the public in forward-looking linear rational expectations models, but take determinacy as given.[6]

We proceed as follows. In the next section, we introduce and motivate our framework by means of a simple example, for which we can derive analytical results. This section also contains a preview of the full results. Section 3 contains the main body of the paper. We present a general linear rational expectations framework with asymmetric information sets and use results from general linear systems theory to prove properties of the resulting equilibria. It is here that we establish our central result that equilibrium indeterminacy is generic in this framework. This section also discusses how to solve and compute equilibria numerically, and we present a quantitative example based on a simple model of inflation determination in Section 4. Section 5 concludes and discusses further extensions of our framework.

2 A Simple Example

We motivate our framework and illustrate key results in the context of a simple example. We consider a rational expectations environment with two agents: a central bank and the private sector. The latter is represented by the (linearized) optimality conditions of a representative household. This gives rise to a Fisher equation that relates the nominal rate of interest, i_t, to the sum of the real interest rate, r_t, and expected inflation, $E_t\pi_{t+1}$:

$$i_t = r_t + E_t\pi_{t+1}\,, \tag{1}$$

where E_t is the rational expectations operator. For now, we assume that the central bank and the public share the same information and expectations. The nominal interest rate is the instrument of monetary policy, while policy is implemented via an interest-rate rule that relates i_t to economic conditions and determines inflation.

In this simple environment, the real side of the economy is determined independently of monetary policy. For exposition, it is thus sufficient to treat the real rate of interest as a given exogenous

[6] Applications following Svensson and Woodford to various economic issues are Carboni & Ellison (2011), Dotsey & Hornstein (2003), and Nimark (2008*b*). Evans & Honkapohja (2001) and Orphanides & Williams (2006, 2007) revisit the question of policymaking under imperfect information in an environment with learning. In a similar vein, Lubik & Matthes (2016) explore indeterminacy in a setting where the central bank employs constant-gain learning, however, without imposing that the policymaker employs rational expectations. Faust & Svensson (2002) and Mertens (2016*b*) study the implications for optimal policy of the opposite informational asymmetry, where the public does not perfectly share the policymaker's information set.

process. We assume that the real rate, r_t, follows a stable AR(1) with Gaussian shocks:[7]

$$r_t = \rho\, r_{t-1} + \varepsilon_t\,, \qquad |\rho| < 1\,, \qquad \varepsilon_t \sim N\left(0, \sigma_\varepsilon^2\right)\,. \tag{2}$$

The environment considered here corresponds to the textbook example of price-level determination with an interest-rate rule described, for example, by Woodford (2003) or Galì (2008).

2.1 Determinacy and Indeterminacy under Full Information

Environments such as the one laid out above can give rise to a multiplicity of equilibria also under full information. Suppose monetary policy observes the real rate and inflation, but sets the nominal rate only in relation to the exogenously given real rate,

$$i_t = r_t.$$

Combining this policy rule with the Fisher equation, restricts inflation expectations to be zero at all times, $E_t \pi_{t+1} = 0$, but leaves innovations to the inflation process unrestricted.

While the policy of setting the nominal rate equal to the real rate is consistent with zero inflation, there are infinitely many alternative outcomes, with arbitrary shocks to inflation that are equally valid. Notably, all of these equilibria restrict inflation to be a martingale difference sequence, and thus non-explosive. As is common in linear rational expectations (RE) models, we consider all equilibria to be valid that satisfy the expectational difference equations characterizing behavior of public and central bank and that do not lead to explosive outcomes.[8] A more formal definition is provided in Section 3.

Formally, as in Sims (2002), Lubik & Schorfheide (2003), or Farmer et al. (2015), we define the endogenous forecast error in inflation, η_t, which can be split into a component correlated with the fundamental shocks in the economy, ε_t, and a non-fundamental component, b_t:

$$\begin{aligned} \eta_t &\equiv \pi_t - E_{t-1}\pi_t \\ &= \gamma_\varepsilon \varepsilon_t + \gamma_b b_t\,, \qquad \text{with} \quad b_t \sim N(0,1) \end{aligned} \tag{3}$$

Without loss of generality, we can standardize b_t, so that the coefficient γ_b captures the volatility

[7]The assumed normality of the real-rate shocks, ε_t, facilitates the representation of the central bank's inference problem further below.

[8]As argued by Cochrane (2011), explosive inflation outcomes need not be of concern in an economy where real outcomes are determined independently from inflation. However, in a more general setup, where nominal frictions lead to real costs, explosive inflation is a concern for keeping real outcomes stable; this case arises, for example, when price adjustments are costly as in Calvo (1983) or Rotemberg (1996).

of the non-fundamental component of η_t.[9] The non-fundamental shocks b_t act like sunspot shocks, but can also be interpreted as "belief shocks" (Lubik & Schorfheide 2003). In the example, the equilibrium process for inflation is given by $\pi_t = \eta_t = \gamma_\varepsilon \varepsilon_t + \gamma_b b_t$. Equilibrium indeterminacy manifests itself in two forms: first, the presence of non-fundamental shocks in the inflation process; and second, the shock loadings γ_ε and γ_ε can take arbitrary values.

So far, we have considered the case where the interest rate only responds to an exogenous input variable (r_t). This gives rise to indeterminacy as noted already by Sargent & Wallace (1975). However, it is straightforward to establish a unique equilibrium with an augmented rule that features an appropriate response to the endogenous inflation outcome. Consider a rule of the form:

$$i_t = r_t + \phi\, \pi_t$$

which, combined with the Fisher equation (1), yields an AR(1) equilibrium process for inflation:

$$\pi_{t+1} = \phi\, \pi_t + \eta_{t+1} \tag{4}$$

When the policy coefficient ϕ is sufficiently large, namely $|\phi| > 1$, the only stable outcome for this process is zero inflation (and thus also $\eta_t = 0$) at all times. Otherwise, any sequence of η_t generates a valid equilibrium path for inflation. The condition $|\phi| > 1$ for equilibrium uniqueness is known as the Taylor principle, as discussed, among others, by Woodford (2003).

When $|\phi| < 1$, the endogenous forecast errors are not restricted, and can be represented as in (3), for arbitrary values of γ_ε and γ_b, so that inflation can also be affected by non-fundamental shocks. Lubik & Schorfheide (2003) and Farmer et al. (2015) refer to these b_t shocks as "belief shocks", since they can be interpreted as non-fundamental shifts in expectations. In our example, the equilibrium process for inflation in equation (4) implies $(E_t - E_{t-1})\pi_{t+1} = \phi\, \eta_t$, which provides a direct link from the non-fundamental (and fundamental) shocks driving η_t to changes in inflation expectations.

The determinacy requirement of a sufficiently strong response to inflation also applies in the case when monetary policy responds only to inflation, without tracking the real rate. For example, consider the following Taylor-type rule: $i_t = \phi\, \pi_t$. Combining policy rule and Fisher equation yields, $\pi_{t+1} = \phi\, \pi_t - r_t + \eta_{t+1}$. If $|\phi| > 1$, the only stable equilibrium has $\pi_t = r_t/(\phi - \rho)$.[10]. In contrast, η_t is unrestricted for $|\phi| < 1$. Henceforth, we assume that $|\phi| > 1$, i.e. the policy rule satisfies the Taylor principle (at least under full information).

[9] As in the case of ε_t, the assumed normality of b_t is important later on in the context of the central bank's signal extraction problem under imperfect information.

[10] To see this, note that this case implies $(\pi_{t+1} - \bar{g}\, r_{t+1}) = \phi\, (\pi_t - \bar{g}\, r_t) + (\eta_{t+1} - \bar{g}\, \varepsilon_{t+1})$ with $\bar{g} = 1/(\phi - \rho)$.

2.2 Asymmetric Information in the Simple Example

We now turn to introducing the key aspect of our framework into the simple example, namely the assumption that the policymaker cannot perfectly observe inflation, at least not in real time. Suppose that the central bank observes only a noisy signal of inflation:

$$Z_t = \pi_t + \nu_t \,, \qquad \nu_t \sim N\left(0, \sigma_\nu^2\right) \,, \tag{5}$$

while the public continues to observe π_t, r_t, and i_t. Lacking knowledge of π_t, the policymaker is assumed to form conditional expectations $\pi_{t|t}$, and use these in the Taylor rule:

$$i_t = \phi\, \pi_{t|t} \,, \qquad \text{where} \quad \pi_{t|t} \equiv E\left(\pi_t | Z^t\right) \,, \qquad |\phi| > 1 \,, \tag{6}$$

where $Z^t = \{Z_t, Z_{t-1}, \ldots\}$ denotes the entire history of signals observed by the policymaker. We continue to denote public sector expectations with the operator E_t.[11] The public and the central bank are assumed to know the structure of the economy and all its parameters. However, the central bank is limited to observing only Z_t instead of the individual realizations of all shocks.

Given the simple structure of the model, observing the history of i_t, and thus also $\pi_{t|t}$, allows the public to perfectly infer the history of central bank signals Z^t.[12] The law of iterated expectations then implies that the central bank's projections of future inflation are identical to its projection of the public's current expectations about future inflation: $\pi_{t+1|t} = E\left(E_t \pi_{t+1} | Z^t\right)$. The evolution of central bank projections is then characterized by the same dynamic system as in the full-information case:

$$\pi_{t+1|t} = \phi\, \pi_{t|t} - r_{t|t} \,, \qquad \text{and} \quad r_{t+1|t} = \rho\, r_{t|t}, \tag{7}$$

Since we continue to consider only stable equilibria, the evolution of central bank projections must be stable as well, and with $|\phi| > 1$ the only stable solution for (7) is

$$\pi_{t|t} = \bar{g}\, r_{t|t} \,, \qquad \text{with} \quad \bar{g} = \frac{1}{\phi - \rho} \tag{8}$$

where $\bar{g}$ is identical to the coefficient known from the corresponding full-information case.

We refer to (8) as the *projection condition* as it restricts the joint behavior of central bank projections for inflation and the natural rate. We consider the projection condition as an equilibrium

[11] Formally, let $S_t = \begin{bmatrix} r_t & \pi_t & i_t \end{bmatrix}'$, and E_t is shorthand for expectations taken conditional on the history of the public's information: $E_t \pi_{t+1} = E\left(\pi_{t+1} | S^t\right)$.

[12] As described in Section 3, our general framework assumes the central bank's information set to be nested inside the public's information set also in cases where the observed policy path for i_t may not reveal S^t by itself.

condition that has to hold as an internal consistency requirement. Section 3 extends the concept to the more general case.

We consider equilibria that are characterized by linear, time-invariant decisions rules where all shocks, including the belief shocks, b_t, are normally distributed. Under these conditions, and as discussed more formally in Section 3, the policymaker's expectations $\pi_{t|t}$ are identical to linear projections. The steady-state Kalman filter provides a recursive representation of the central bank's inflation projection based on

$$\pi_{t|t} = (1-\kappa_\pi)\pi_{t|t-1} + \kappa_\pi\left(\pi_t + \nu_t\right) , \qquad \text{with} \quad \kappa_\pi = \frac{\mathrm{Cov}\left(\pi_t, \pi_t + \nu_t \mid Z^{t-1}\right)}{\mathrm{Var}\left(\pi_t + \nu_t \mid Z^{t-1}\right)} , \tag{9}$$

where the Kalman gain κ_π depends on the second moments of inflation. The projection condition (8) implies that $\pi_{t|t-1} = \rho\,\pi_{t-1|t-1}$ so that we can obtain a recursive law of motion for $\pi_{t|t}$ from (9), which is driven by the noisy inflation signal.

2.3 Generic Indeterminacy with Optimal Projections

The imperfect information setup leads to a stark implication, namely that the equilibrium is generically indeterminate, even if the Taylor principle holds in the corresponding full-information case. The result follows from the endogenous attenuation of the incoming inflation signal when constructing the optimal projection. The policy rule (6) depends on projected inflation, which in turn is given by the projection equation (9), so that the sensitivity of the policy response to actual inflation depends on $\phi \cdot \kappa_\pi$, which itself is determined by the dynamics of inflation in equilibrium.

The generic indeterminacy can be seen most directly when $\rho = 0$, so that the real rate is not serially correlated. The Kalman filter then reduces to a static signal extraction problem with $\pi_{t|t} = \kappa_\pi\left(\pi_t + \nu_t\right)$ and the policy rule becomes $i_t = \phi\,\kappa_\pi\left(\pi_t + \nu_t\right)$. In this case, determinacy solely depends on $\phi\,\kappa_\pi$.[13] In the appendix, we show that $|\phi \cdot \kappa_\pi| < 1$ for any ϕ (except for a case discussed below, when r_t could be perfectly inferred by the central bank).[14] Consequently, the equilibrium is generically indeterminate in the simple example even if the Taylor principle nominally holds. Intuitively, for a given κ_π a unique equilibrium might be obtained by setting ϕ sufficiently high. However, higher ϕ feeds back into stronger inflation stabilization that reduces the signal-to-noise ratio of the central bank's signal. Optimal signal extraction therefore leads to a lower κ_π.

[13] More generally, the central bank's filtering adds $\pi_{t|t}$ as an endogenous state variable, which depends on lagged inflation outcomes. In this case, the determinacy conditions are not solely characterized by the rule's sensitivity to contemporaneous inflation, $\phi \cdot \kappa_\pi$. Section 3 derives the general conditions and shows numerical results for a simple example model.

[14] Checking whether $\phi \cdot \kappa_\pi$ is inside or outside the unit circle is an application of the root-counting criteria established, among others, by Blanchard & Kahn (1980).

In the simple example it is even straightforward to construct an equilibrium, where r_t is perfectly revealed, and $|\phi \cdot \kappa_\pi| > 1$, which suggest uniqueness under the standard root-counting criteria.[15] Yet, the potential presence of alternative equilibria with $|\phi \cdot \kappa_\pi| < 1$ indicates that the usual criteria are not sufficient to guarantee determinacy because of the endogeneity of κ_π. Section 3 extends the determinacy discussion to the more general case and we apply the solution techniques presented there to our example in section 4.[16]

A crucial insight of our framework is that certainty equivalence does not hold despite the linear Gaussian setting. Importantly, the Kalman gain κ_π depends on equilibrium outcomes for the second moments of inflation and the signal, which in turn depend on the Taylor-rule coefficient ϕ.

2.4 Determinacy without Optimal Projections

In the model with imperfect information, indeterminacy arises generically from the interplay between the effects of policy on inflation and the policymaker's use of optimal projections. An alternative policy would be for the policymaker to respond directly to the noisy signal Z_t, rather than an optimal projection based on the signal. This alternative setup obviates the role of the endogenous Kalman gain κ_π. Consider the policy rule:

$$i_t = \phi \left(\pi_t + \nu_t\right) . \tag{10}$$

When $|\phi| > 1$, policy responds by more than one-for-one to actual inflation; the Taylor principle applies and results in a unique equilibrium, given by $\pi_t = \bar{g}\, r_t - \nu_t$. While the equilibrium is unique, inflation is subject to additional fluctuations, at least compared to the full-information case, due to the measurement error ν_t, which should be undesirable. In fact, the projection condition (8) guarantees that any of the multiple equilibria obtained under the projection-based Taylor rule in (6) imply a (weakly) lower variance of inflation than under the determinate policy rule in (10):[17]

$$\operatorname{Var}\left(\pi_t\right) \leq \bar{g}^2 \ \operatorname{Var}\left(r_t\right) + \operatorname{Var}\left(\nu_t\right) \tag{11}$$

[15] We discuss this scenario in the context of the simple model in the appendix.

[16] As noted in section 4, solutions with $|\phi \cdot \kappa_\pi| < 1$ require a minimum level of noise variance, σ_ν^2, when the signal consists of inflation plus noise. Otherwise, the projection condition can only be satisfied by the solution with $|\phi \cdot \kappa_\pi| > 1$ that perfectly reveals r_t. In Lubik, Matthes & Mertens (2019), we show that meaningful multiplicity of equilibria can easily arise in a standard New Keynesian model where the measurement error process is calibrated to match revisions in US macroeconomic data.

[17] The projection condition (8) implies $\operatorname{Var}(\pi_{t|t}) = \bar{g}^2 \operatorname{Var}(r_t)$, and the optimality of projections leads to $\operatorname{Var}(\pi_{t|t}) \leq \bar{g}^2 \operatorname{Var}(r_{t|t})$. Moreover, the definition of the signal implies $\pi_t - \pi_{t|t} = -(\nu_t - \nu_{t|t})$, since $Z_{t|t} = Z_t$, so that $\operatorname{Var}(\pi_t - \pi_{t|t}) = \operatorname{Var}(\nu_t - \nu_{t|t})$. We obtain $\operatorname{Var}(\pi_t) = \operatorname{Var}(\pi_{t|t}) + \operatorname{Var}(\pi_t - \pi_{t|t}) \leq \bar{g}^2 \operatorname{Var}(r_t) + \operatorname{Var}(\nu_t)$. Furthermore, recall that $\operatorname{Var}(\nu_t) = \sigma_\nu^2$ and, in the iid case, $\operatorname{Var}(r_t) = \sigma_\varepsilon^2$.

2.5 The Set of Equilibria in the iid Case

The variance bound in (11) points to an important and novel aspect of the indeterminacy mechanism in our framework with imperfect information. Since the variance of inflation remains bounded in all equilibria, the potential role for belief shocks in driving inflation outcomes must also be bounded. This bound is a consequence of the projection condition placed onto the central bank's projections of inflation and real rate, and does not arise in full-information models. Given the projection-based policy rule (6), and an iid real rate, the equilibrium process for inflation is:

$$\begin{aligned} \pi_{t+1} &= \phi\,\kappa_\pi(\pi_t+\nu_t) - r_t + \eta_{t+1} \\ &= -\left(r_t - r_{t|t}\right) + \eta_{t+1} \qquad\qquad \text{with} \quad \eta_{t+1} = \gamma_\varepsilon\varepsilon_{t+1} + \gamma_\nu\nu_{t+1} + \gamma_b b_{t+1} \end{aligned} \tag{12}$$

where the last line follows from the projection condition (8), and γ_ε, γ_ν, and γ_b are restricted to ensure that (8) holds.[18] The projection condition requires:

$$\phi \cdot \mathrm{Cov}\left(\pi_t, \pi_t+\nu_t \,|\, Z^{t-1}\right) = \mathrm{Cov}\left(r_t, \pi_t+\nu_t \,|\, Z^{t-1}\right), \tag{13}$$

which imposes one non-linear restriction on three coefficients.[19] The restriction in (13) interacts with the Kalman filter's Riccati equation, so that the solution is non-linear. We generally resort to a numerical method for solving for the shock loadings, which we describe in more detail in Section 3.

To illustrate the belief shock bound with a closed-form solution, let us assume that the central bank not only receives the noisy inflation signal considered thus far, but also knows the real rate, r_t. In sum, the central bank observes the measurement vector $\boldsymbol{Z}_t = [\pi_t+\nu_t, r_t]'$. In this case, it follows that $r_{t|t} = r_t$ and the projection condition requires $\pi_{t|t} = \bar{g}\, r_t$, so that inflation dynamics reduce to $\pi_{t+1} = \bar{g}\, r_t + \eta_{t+1}$. The projection condition (8) then requires $\eta_{t|t} = \bar{g}\,\varepsilon_{t|t}$, and with $\varepsilon_{t|t} = \varepsilon_t$, we can conclude that $\gamma_\varepsilon = \bar{g}$. Furthermore, we have $\pi_t - \pi_{t|t} = \gamma_\nu\,\nu_t + \gamma_b\, b_t$. Since $E\left(\pi_t - \pi_{t|t}\,\middle|\boldsymbol{Z}^t\right) = 0$, we obtain the following restriction on γ_ν and γ_b:

$$\begin{aligned} &\mathrm{Cov}\left(\gamma_\nu\,\nu_t + \gamma_b\, b_t\,,\ (1+\gamma_\nu)\,\nu_t + \gamma_b\, b_t\right) = \gamma_\nu(1+\gamma_\nu)\sigma_\nu^2 + \gamma_b^2 = 0 \\ &\qquad\qquad\qquad\qquad \Rightarrow \quad \gamma_\nu = -\frac{1}{2} \pm \sqrt{\frac{1}{4} - \frac{\gamma_b^2}{\sigma_\nu^2}}\,. \end{aligned} \tag{14}$$

Real solutions to (14) require $|\gamma_b| < 0.5\,\sigma_\nu$ leading to a continuum of solutions with $\gamma_\nu \in [-1, 0]$.

[18] The set of fundamental shocks includes the measurement error, ν_t, as it affects the central bank's information set, and thus other endogenous outcomes, as part of the economy's structure.

[19] The covariance restriction boils down to $\phi\left(\mathrm{Var}\left(\pi_t\,|\,Z^{t-1}\right) + \gamma_\nu\sigma_\nu^2\right) = \gamma_\varepsilon\,\sigma_\varepsilon^2$ with $\mathrm{Var}\left(\pi_t\,|\,Z^{t-1}\right) = \mathrm{Var}\left(r_t\,|\,Z^t\right) + \gamma_\varepsilon^2\sigma_\varepsilon^2 + \gamma_\nu^2\sigma_\nu^2 + \gamma_b^2$.

In sum, there is a unique solution for the shock loading $\gamma_\varepsilon = \bar{g}$, but multiple solutions for γ_ν and γ_b. The belief-shock loading, γ_b, has a bound that is proportional to the amount of measurement noise.

3 Multiple Equilibria in Imperfect Information Systems

We lay out a general class of expectational linear difference systems that feature conditional expectations of two types of agents with possibly different information sets. After reviewing rational expectations equilibria (REE) under full information, we turn to the imperfect information case, where one of the agents is strictly less informed than the other.

3.1 A Linear Rational Expectations System with Two Information Sets

We consider a generic model economy with multiple agents that is described by the following system of linear expectational difference equations:

$$E_t \boldsymbol{S}_{t+1} + \hat{\boldsymbol{J}} \boldsymbol{S}_{t+1|t} = \boldsymbol{A} \boldsymbol{S}_t + \hat{\boldsymbol{A}} \boldsymbol{S}_{t|t} + \boldsymbol{A}_i \, \boldsymbol{i}_t \,, \tag{15}$$

$$\boldsymbol{i}_t = \boldsymbol{\Phi}_i \boldsymbol{i}_{t-1} + \boldsymbol{\Phi}_J \boldsymbol{S}_{t+1|t} + \boldsymbol{\Phi}_A \boldsymbol{S}_{t|t} \,, \tag{16}$$

$$\boldsymbol{S}_t = \begin{bmatrix} \boldsymbol{X}_t' & \boldsymbol{Y}_t' \end{bmatrix}' . \tag{17}$$

$\boldsymbol{X}_t$ and $\boldsymbol{Y}_t$ are, respectively, vectors of backward- and forward-looking variables, while $\boldsymbol{i}_t$ denotes a vector of policy instruments that are under the control of one of the agents in the model.[20] There are N_x backward- and N_y forward-looking variables and N_i policy instruments. The backward-looking variables are characterized by exogenous forecast errors $\boldsymbol{\varepsilon}_t$:

$$\boldsymbol{X}_t - E_{t-1} \boldsymbol{X}_t = \boldsymbol{B}_{x\varepsilon} \, \boldsymbol{\varepsilon}_t \,, \qquad \boldsymbol{\varepsilon}_t \sim N(\boldsymbol{0}, \boldsymbol{I}) \,,$$

where the number of independent shocks N_ε can be smaller than the number of backward-looking variables N_x, while $\boldsymbol{B}_{x\varepsilon}$ is assumed to have full rank. Furthermore, we assume that the initial value of the backward-looking variables $\boldsymbol{X}_0$ is exogenously given. In contrast, as in Sims (2002), forecast errors for the forward-looking variables

$$\boldsymbol{\eta}_t \equiv \boldsymbol{Y}_t - E_{t-1} \boldsymbol{Y}_t,$$

[20] Throughout this paper we denote vectors and matrices with bold letters. Usage of lower- and uppercase letters does *not* distinguish between matrices and vectors, however. In most applications, $\boldsymbol{i}_t$ is likely to be a scalar, but nothing in our framework hinges on this assumption. In our context, keeping the policy instrument separate from $\boldsymbol{X}_t$ and $\boldsymbol{Y}_t$ is convenient since we assume that $\boldsymbol{i}_t$ is perfectly known and observable to all agents.

are endogenous and remain to be determined as part of the model's rational expectations solution.[21]

The system of linear difference equations (15) - (16) captures the interdependent decision making of two types of agents, potentially derived from (linearized) optimally conditions of households, firms, or policymakers. All agents form rational expectations but they might condition on different information sets. Specifically, one type of agent has access to full information about the state of the economy, for instance a representative agent for the private sector, whose decisions are represented by (15) and also depend on the setting of a policy instrument $\boldsymbol{i}_t$ chosen by the other agent.

The second agent is an imperfectly informed policymaker, who sets $\boldsymbol{i}_t$ according to the rule given in (16). By definition, the policymaker knows the current value and history of the instrument choices, but lacks full knowledge of the state of the economy. Moreover, all variables entering the policy rule (16) are expressed as expectations conditional on the policymaker's information set, denoted $\boldsymbol{S}_{t+1|t}$ and $\boldsymbol{S}_{t|t}$. The policymaker forms rational expectations based on an information set that is characterized by the observed history of a vector of N_z signals, denoted $\boldsymbol{Z}_t$, and knowledge of all model parameters. We assume common knowledge about the structure of the economy and all model parameters. For any variable $\boldsymbol{V}_t$, and any lead or lag h, $E_t\boldsymbol{V}_{t+h}$ denotes expectations based on full information, whereas:

$$\boldsymbol{V}_{t+h|t} \equiv E(\boldsymbol{V}_{t+h}|\boldsymbol{Z}^t) \qquad\qquad \boldsymbol{Z}^t = \{\boldsymbol{Z}_t, \boldsymbol{Z}_{t-1}, \boldsymbol{Z}_{t-2}, \ldots\} \tag{18}$$

denotes conditional expectations under the policymaker's information set. For further use, we introduce the following notation for innovations $\tilde{\boldsymbol{V}}_t$ and residuals $\boldsymbol{V}_t^*$:

$$\tilde{\boldsymbol{V}}_t \equiv \boldsymbol{V}_t - \boldsymbol{V}_{t|t-1} \ , \qquad\qquad \boldsymbol{V}_t^* \equiv \boldsymbol{V}_t - \boldsymbol{V}_{t|t} = \tilde{\boldsymbol{V}}_t - \tilde{\boldsymbol{V}}_{t|t} \ .$$

Henceforth we will use the term "shocks" in reference to martingale difference sequences defined relative to the full information set, and the term "innovations" when referring to martingale difference sequences with respect to the imperfect information set.

The less informed agent observes a subset of model variables in a measurement vector with N_z elements, each of which is a linear combination of backward- and forward-looking variables:

$$\boldsymbol{Z}_t = \boldsymbol{H}\boldsymbol{S}_t = \boldsymbol{H}_x\boldsymbol{X}_t + \boldsymbol{H}_y\boldsymbol{Y}_t \,. \tag{19}$$

The measurement vector can also be affected by "measurement errors", which are disturbances to the measurement equation that would otherwise be absent from the full-information version of the model. Such measurement errors are assumed to have been lumped into the vector of backward-

[21] In principle, there are N_y endogenous forecast errors, but their variance-covariance matrix need not have full rank.

looking variables, $\boldsymbol{X}_t$. By construction, policymaker actions $\boldsymbol{i}_t$ are spanned by the history of observed signals, such that trivially $\boldsymbol{i}_t = \boldsymbol{i}_{t|t}$. $\boldsymbol{i}_t$ merely reflects information contained in $\boldsymbol{Z}^t$ and thus need not be added to the description of the measurement vector.

In the next step, we briefly describe the equilibrium and solution to the full-information counterpart of our framework, which is standard in the literature and rests on certainty equivalence and homogeneity of expectations. In our imperfect-information framework, the interactions between different expectation formation processes require alternative solution procedures and lead to multiple equilibrium outcomes, that abandon certainty equivalence.

3.2 Full Information Equilibrium

The general framework nests the case of full information, where $\boldsymbol{S}_{t+h|t} = E_t\boldsymbol{S}_{t+h}\ \forall h \geq 0$. This holds, for example, when $\boldsymbol{H} = \boldsymbol{I}$ such that $\boldsymbol{Z}_t = \boldsymbol{S}_t$. The full-information system can easily be solved using standards methods such as those found in King & Watson (1998), Klein (2000) or Sims (2002). We stack all variables, including the policy control, into a vector $\boldsymbol{\mathcal{S}}_t$ that is partitioned into a vector of $N_i + N_x$ backward-looking variables, $\boldsymbol{\mathcal{X}}_t$, and a vector of $N_y + N_i$ forward-looking variables, $\boldsymbol{\mathcal{Y}}_t$.[22]

$$\boldsymbol{\mathcal{S}}_t = \begin{bmatrix} \boldsymbol{\mathcal{X}}_t \\ \boldsymbol{\mathcal{Y}}_t \end{bmatrix}, \qquad \text{where} \quad \boldsymbol{\mathcal{X}}_t = \begin{bmatrix} \boldsymbol{i}_{t-1} \\ \boldsymbol{X}_t \end{bmatrix} \qquad \boldsymbol{\mathcal{Y}}_t = \begin{bmatrix} \boldsymbol{Y}_t\ , \\ \boldsymbol{i}_t \end{bmatrix}. \tag{20}$$

Using $\boldsymbol{\mathcal{S}}_t' = \begin{bmatrix} \boldsymbol{i}_{t-1}' & \boldsymbol{S}_t' & \boldsymbol{i}_t' \end{bmatrix}$, the dynamics of the system under full information are then characterized by the following expectational difference equation:

$$\underbrace{\begin{bmatrix} \boldsymbol{I} & 0 & 0 \\ 0 & \boldsymbol{I} + \hat{\boldsymbol{J}} & 0 \\ 0 & \boldsymbol{\Phi}_J & 0 \end{bmatrix}}_{\boldsymbol{\mathcal{J}}} E_t\boldsymbol{\mathcal{S}}_{t+1} = \underbrace{\begin{bmatrix} 0 & 0 & \boldsymbol{I} \\ 0 & \boldsymbol{A} + \hat{\boldsymbol{A}} & \boldsymbol{A}_i \\ -\boldsymbol{\Phi}_i & -\boldsymbol{\Phi}_H & \boldsymbol{I} \end{bmatrix}}_{\boldsymbol{\mathcal{A}}} \boldsymbol{\mathcal{S}}_t \tag{21}$$

Existence and uniqueness of a solution to this system depend on the properties of the matrices $\boldsymbol{\mathcal{J}}$ and $\boldsymbol{\mathcal{A}}$. Throughout this paper we focus on environments where a unique full-information solution exists and therefore impose the following assumption known from Klein (2000) and King & Watson (1998).[23]

[22] The presence of the lagged policy control among the backward-looking variables in $\boldsymbol{\mathcal{X}}_t$ serves to handle the case of interest-rate smoothing, $\boldsymbol{\Phi}_i \neq 0$, and can otherwise be omitted.

[23] This is akin to a root-counting criterion where the number of explosive eigenvalues matches the number of forward-looking variables. Sims (2002) offers a slight generalization of this criterion. In the case of simple monetary policy models, the root-counting condition is satisfied by requiring that the central bank's interest-rate rule satisfies the Taylor principle, that is, by responding more than one-to-one to fluctuations in inflation.

ASSUMPTION 1 (Unique full-information solution) *The set of generalized eigenvalues of the system matrices $\boldsymbol{\mathcal{J}}$ and $\boldsymbol{\mathcal{A}}$ as defined in* (21) *has $N_i + N_x$ roots inside the unit circle and $N_y + N_i$ roots outside the unit circle. Moreover, there is some complex number z such that $|\boldsymbol{\mathcal{J}}\, z - \boldsymbol{\mathcal{A}}| \neq 0$.*

The solution can, for instance, be computed using the numerical methods of Klein (2000), and has the following form:

$$E_t \boldsymbol{\mathcal{X}}_{t+1} = \boldsymbol{\mathcal{P}}\, \boldsymbol{\mathcal{X}}_t\,, \qquad \boldsymbol{\mathcal{Y}}_t = \boldsymbol{\mathcal{G}}\, \boldsymbol{\mathcal{X}}_t\,, \qquad \boldsymbol{\mathcal{G}} = \begin{bmatrix} \boldsymbol{\mathcal{G}}_{yi} & \boldsymbol{\mathcal{G}}_{yx} \\ \boldsymbol{\mathcal{G}}_{ii} & \boldsymbol{\mathcal{G}}_{ix} \end{bmatrix}, \tag{22}$$

where $\boldsymbol{\mathcal{P}}$ is a stable matrix with all eigenvalues inside the unit circle. In the full-information version of our framework certainty equivalence holds; that is, the decision-rule coefficients $\boldsymbol{\mathcal{P}}$ and $\boldsymbol{\mathcal{G}}$ do not depend on the shock variances encoded in $\boldsymbol{B}_{x\varepsilon}$ nor on the measurement loadings $\boldsymbol{H}$.[24] Equilibrium dynamics in the full-information case are then summarized by:

$$\boldsymbol{\mathcal{S}}_{t+1} = \bar{\boldsymbol{\mathcal{T}}} \boldsymbol{\mathcal{S}}_t + \bar{\boldsymbol{\mathcal{H}}} \boldsymbol{\varepsilon}_{t+1}\,, \qquad \bar{\boldsymbol{\mathcal{T}}} = \begin{bmatrix} \boldsymbol{\mathcal{P}} & 0 \\ \boldsymbol{\mathcal{G}}\boldsymbol{\mathcal{P}} & 0 \end{bmatrix}, \qquad \bar{\boldsymbol{\mathcal{H}}} = \begin{bmatrix} 0 \\ \boldsymbol{I} \\ \boldsymbol{\mathcal{G}}_{yx} \\ \boldsymbol{\mathcal{G}}_{ix} \end{bmatrix} \boldsymbol{B}_{x\varepsilon}\,, \tag{23}$$

where $\bar{\boldsymbol{\mathcal{T}}}$ is stable because $\boldsymbol{\mathcal{P}}$ is, and the endogenous forecast errors are given by $\boldsymbol{\eta}_t = \boldsymbol{\mathcal{G}}_{yx}\, \boldsymbol{B}_{x\varepsilon}\, \boldsymbol{\varepsilon}_t$. One defining feature of this equilibrium is that non-fundamental disturbances, such as sunspot or belief shocks, do not affect equilibrium outcomes; as discussed, for example, by Lubik & Schorfheide (2003). Hence, the equilibrium is determinate. This is no longer the case under imperfect information.

3.3 Expectation Formation under Imperfect Information

We now turn to the imperfect information case, where agents' expectation formation processes conditional on different information sets interact. While our general framework assumes that all agents form expectations rationally, we show that the less-informed agent's expectations can be represented by a Kalman filter, which implies a linear relationship between projected innovations of unobserved variables and observed innovations in the signal received from the fully informed agent. It is the endogeneity of the filtering, which depends on the evolution of the economy, that

[24] The imperfect information setup could include measurement errors as part of the vector of backward-looking variables, $\boldsymbol{X}_t$. They would affect endogenous variables of the system only via $\boldsymbol{H}$, which does not play a role in the full-information solution. Conceptually, there is no harm including measurement errors in $\boldsymbol{X}_t$, and the corresponding columns of $\boldsymbol{\mathcal{G}}_{yx}$ are equal to zero in this case.

gives rise to equilibrium multiplicity and a break-down of certainty equivalence under imperfect information.

We proceed by making a sequence of assumptions that allow us to map the general framework under asymmetric information into a standard linear rational expectations setting that can be solved and studied with familiar methods. To ensure that a Kalman filter delivers an exact representation of the true conditional expectations, we are interested in linear equilibria, driven by normally distributed disturbances. Therefore, we make the following assumption:

ASSUMPTION 2 (Jointly normal forecast errors) *The endogenous forecast errors are a linear combination of the N_ε exogenous errors, $\boldsymbol{\varepsilon}_t$, and N_y so-called belief shocks, $\boldsymbol{b}_t$, that are mean zero and uncorrelated with $\boldsymbol{\varepsilon}_t$:*

$$\boldsymbol{\eta}_t = \boldsymbol{\Gamma}_\varepsilon \boldsymbol{\varepsilon}_t + \boldsymbol{\Gamma}_b \boldsymbol{b}_t$$

Moreover, exogenous shocks and belief shocks are generated from a joint standard normal distribution,

$$\boldsymbol{w}_t \equiv \begin{bmatrix} \boldsymbol{\varepsilon}_t \\ \boldsymbol{b}_t \end{bmatrix} \sim N\left(\begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} \boldsymbol{I} & 0 \\ 0 & \boldsymbol{I} \end{bmatrix} \right)$$

The matrix of belief shock loadings $\boldsymbol{\Gamma}_b$ need not have full rank so that linear combinations of endogenous and exogenous forecast errors might be perfectly correlated.

As a corollary of Assumption 2, exogenous and endogenous forecast errors are joint normally distributed as well:

$$\begin{bmatrix} \boldsymbol{\varepsilon}_t \\ \boldsymbol{\eta}_t \end{bmatrix} \sim N\left(\begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} \boldsymbol{I} & \boldsymbol{\Gamma}_\varepsilon' \\ \boldsymbol{\Gamma}_\varepsilon & \boldsymbol{\Omega}_\eta \end{bmatrix} \right), \qquad \text{with} \quad \boldsymbol{\Omega}_\eta = \boldsymbol{\Gamma}_b \boldsymbol{\Gamma}_b' + \boldsymbol{\Gamma}_\varepsilon \boldsymbol{\Gamma}_\varepsilon' . \tag{24}$$

In this specification of the endogenous forecast error we follow Lubik & Schorfheide (2003) and Farmer et al. (2015). In a determinate equilibrium, $\boldsymbol{\Gamma}_b$ is the zero matrix, while under indeterminacy it allows for sunspot shocks, or belief shocks, to affect endogenous forecast errors and expectations.[25] As in the full-information literature, Assumption 2 sees endogenous forecast errors as a linear combination of exogenous fundamental shocks and non-fundamental belief shocks. In addition, the belief shocks are assumed to be normally distributed, which is not entirely innocuous in that it restricts the stochastic nature of the equilibrium outcomes. While this assumption is immaterial for the solution under full information, it allows us to derive an exact Kalman-filter

[25] In a linear equilibrium, where (24) holds, the belief shocks affect outcomes only via the product $\boldsymbol{\Gamma}_b\, \boldsymbol{b}_t$. Thus, without loss of generality, $\boldsymbol{b}_t$ can be normalized to have a variance-covariance matrix equal to the identity matrix.

representation of expectation formation under imperfect information. Extension to non-Gaussian shocks are beyond the scope of the paper, but constitute important further research.

We now define and characterize a class of stationary, linear and time-invariant equilibria, where expectations are represented by a Kalman filter and where we treat the shock loadings $\boldsymbol{\Gamma}_\varepsilon$ and $\boldsymbol{\Gamma}_b$ as given. In a subsequent step, we turn to solution methods that determine values for $\boldsymbol{\Gamma}_\varepsilon$ and $\boldsymbol{\Gamma}_b$ consistent with these equilibria.

DEFINITION 1 (Stationary, linear, time-invariant equilibrium) *In a stationary, linear, and time-invariant equilibrium, backward- and forward-looking variables, $\boldsymbol{Y}_t$ and$\boldsymbol{X}_t$, and the instrument, $\boldsymbol{i}_t$, are stationary and their equilibrium dynamics satisfy the expectational difference system described by* (15) *and* (16).[26] *All expectations are rational, and the imperfectly informed agent's information set is described by* (19). *In addition, Assumption 2 holds, which means that the forecast errors of the forward-looking variables are a linear combination of fundamental shocks and belief shocks, with time-invariant loadings $\boldsymbol{\Gamma}_\varepsilon$ and $\boldsymbol{\Gamma}_b$, as in* (24) *and all shocks are joint normally distributed.*

We now show that in such an equilibrium the policymaker's conditional expectations, as defined in (18), can be represented by a Kalman filter. The policymaker observes the measurement equation (19), and the state equation of the filtering problem is given by:

$$\boldsymbol{S}_{t+1} + \hat{\boldsymbol{J}}\boldsymbol{S}_{t+1|t} = \boldsymbol{A}\boldsymbol{S}_t + \hat{\boldsymbol{A}}\boldsymbol{S}_{t|t} + \boldsymbol{A}_i\,\boldsymbol{i}_t + \boldsymbol{B}\boldsymbol{w}_{t+1}\,, \qquad \text{with} \quad \boldsymbol{B} = \begin{bmatrix} \boldsymbol{B}_{x\varepsilon} & \boldsymbol{0} \\ \boldsymbol{\Gamma}_\varepsilon & \boldsymbol{\Gamma}_b \end{bmatrix}\,, \tag{25}$$

which combines the expectational difference equation (15) with the implications of Assumption 2 for the endogenous forecast errors. The appearance of projections $\boldsymbol{S}_{t+1|t}$ and $\boldsymbol{S}_{t|t}$ in (25) lends this state equation a slightly non-standard format. However, when expressed in terms of innovations, the filtering problem can be cast in the canonical "ABCD" form, studied, among others, by Fernández-Villaverde, Rubio-Ramírez, Sargent & Watson (2007):[27]

$$\tilde{\boldsymbol{S}}_{t+1} = \boldsymbol{A}\left(\tilde{\boldsymbol{S}}_t - \tilde{\boldsymbol{S}}_{t|t}\right) + \boldsymbol{B}\boldsymbol{w}_{t+1}\,, \tag{26}$$

$$\tilde{\boldsymbol{Z}}_{t+1} = \boldsymbol{C}\left(\tilde{\boldsymbol{S}}_t - \tilde{\boldsymbol{S}}_{t|t}\right) + \boldsymbol{D}\boldsymbol{w}_{t+1}\,, \tag{27}$$

$$\text{with} \quad \boldsymbol{C} = \boldsymbol{H}\boldsymbol{A}\,, \quad \boldsymbol{D} = \boldsymbol{H}\boldsymbol{B}\,, \quad \text{since} \quad \tilde{\boldsymbol{Z}}_{t+1} = \boldsymbol{H}\tilde{\boldsymbol{S}}_{t+1}\,. \tag{28}$$

[26] Linear equilibria considered in this paper are driven by normally distributed shocks, leading to normally distributed outcomes, such that covariance stationarity also implies strict stationarity. Hence, we will not distinguish between both concepts and merely refer to stationarity.

[27] The innovations form is obtained by projecting both sides of (25) onto $\boldsymbol{Z}^t$ and subtracting these projections from (25). Note that the control $\boldsymbol{i}_t$ is always in the policymaker's information set. The innovations form given by (26) and (27) is identical to the innovations form of a state-space system with $\boldsymbol{S}_{t+1} = \boldsymbol{A}\boldsymbol{S}_t + \boldsymbol{B}\boldsymbol{w}_{t+1}$ in place of (25) while maintaining (19) as measurement equation, as noted also by Baxter, Graham & Wright (2011).

To ensure a well-behaved filtering problem, we impose the following assumption on the shocks to the policymaker's measurement vector $\boldsymbol{D}\boldsymbol{w}_t = \boldsymbol{Z}_t - E_{t-1}\boldsymbol{Z}_t$.

ASSUMPTION 3 (Non-degenerate shocks to the signal equation) *The variance-covariance matrix of the shocks to the measurement equation has full rank:* $|\boldsymbol{D}\boldsymbol{D}'| \neq 0$.

A necessary condition for Assumption 3 to hold is that the signal vector does not have more elements than the sum of endogenous and exogenous forecast errors: $N_z \leq N_\varepsilon + N_y \leq N_x + N_y$.

Existence of a steady-state Kalman filter requires certain conditions on $\boldsymbol{A}$, $\boldsymbol{B}$, and $\boldsymbol{H}$ to hold, known in the literature as "observability" and "unit-circle controllability," formally defined in the appendix. As shown in (25), $\boldsymbol{B}$ depends on the yet to be determined shock loadings $\boldsymbol{\Gamma}_\varepsilon$, $\boldsymbol{\Gamma}_b$, which are endogenous in our framework. To ensure the existence of a steady-state filter, the following assumption restricts the set of potential equilibria that we consider. But, as discussed further below, these conditions impose only weak restrictions on the shock loadings in $\boldsymbol{B}$.

ASSUMPTION 4 (Sufficient condition for existence of a steady-state Kalman filter) *The equilibrium shock loadings $\boldsymbol{\Gamma}_\varepsilon$, $\boldsymbol{\Gamma}_b$ of the endogenous forecast errors $\boldsymbol{\eta}_t$ are such that $\boldsymbol{A}$, $\boldsymbol{B}$ and $\boldsymbol{H}$ are detectable and unit-circle controllable as stated in Definition 5 of the appendix.*

A central component of our framework is that the less informed agent observes data imperfectly, but employs optimal signal extraction, which is provided by the Kalman filter in this linear Gaussian environment. The optimal projections $\boldsymbol{S}_{t|t}$ can be decomposed into the sum of previous period's forecasts $\boldsymbol{S}_{t|t-1}$ and an update that is linear in the innovations in measurement vector. When a steady-state filter exists, a constant Kalman gain relates the projected innovations in the state vector, $\tilde{\boldsymbol{S}}_{t|t}$, to innovations in the measurement vector, $\tilde{\boldsymbol{Z}}_t$:

$$\boldsymbol{S}_{t|t} = \boldsymbol{S}_{t|t-1} + \tilde{\boldsymbol{S}}_{t|t} \quad \text{with} \quad \tilde{\boldsymbol{S}}_{t|t} = \boldsymbol{K}\tilde{\boldsymbol{Z}}_t\,, \quad \text{and} \quad \boldsymbol{K} = \operatorname{Cov}(\tilde{\boldsymbol{S}}_t, \tilde{\boldsymbol{Z}}_t)\left(\operatorname{Var}(\tilde{\boldsymbol{Z}}_t)\right)^{-1}. \tag{29}$$

The Kalman gain matrix, $\boldsymbol{K}$, is given by the solution of a standard Riccati equation involving $\boldsymbol{A}$, $\boldsymbol{B}$ and $\boldsymbol{H}$. We detail the derivation in the appendix.

We focus on the case when a steady-state Kalman filter exists, which allows us to represent the policymaker's conditional expectations as a recursive linear system with time-invariant coefficients. We formally state these results in the following two propositions.

PROPOSITION 1 (Existence of steady-state Kalman filter) *When Assumptions 3 and 4 hold in a linear, stationary, and time-invariant equilibrium, a steady-state Kalman filter exists that describes the projection of innovations in the state vector, $\tilde{\boldsymbol{S}}_{t|t} = \boldsymbol{K}\tilde{\boldsymbol{Z}}_t$ with a constant Kalman gain $\boldsymbol{K}$ as in* (29). *The variance-covariance matrix of projection residuals is constant,* $\operatorname{Var}\left(\boldsymbol{S}_t|\boldsymbol{Z}^t\right) = \operatorname{Var}(\boldsymbol{S}_t^*) = \boldsymbol{\Sigma}^*$. *Existence of a steady-state Kalman filter ensures that innovations $\tilde{\boldsymbol{S}}_t = \boldsymbol{S}_t - \boldsymbol{S}_{t|t-1}$ and residuals $\boldsymbol{S}_t^* = \boldsymbol{S}_t - \boldsymbol{S}_{t|t}$ are stationary as are innovations to the measurement equation, $\tilde{\boldsymbol{Z}}_t$.*

Proof. See Theorem 3 of Appendix B. The stationarity of $\tilde{\boldsymbol{Z}}_t = \boldsymbol{H}\tilde{\boldsymbol{S}}_t$ follows then from the stationarity of $\tilde{\boldsymbol{S}}_t$. ■

Given the existence of a Kalman filter, equation (29) describes the optimal update from forecasts, $\boldsymbol{S}_{t|t-1}$, to current projections, $\boldsymbol{S}_{t|t}$. What remains to be characterized is the transition equation from projections $\boldsymbol{S}_{t|t}$ to forecasts, $\boldsymbol{S}_{t+1|t}$. This relationship is restricted by the linear difference equations (15) and the rule (16) for the policy instrument $\boldsymbol{i}_t$. Since the instrument can depend on its own lagged value via $\boldsymbol{\Phi}_i \neq \boldsymbol{0}$ in (16), we construct the transition from $\boldsymbol{S}_{t|t}$ to $\boldsymbol{S}_{t+1|t}$ based on the vector $\boldsymbol{\mathcal{S}}_t$, which includes $\boldsymbol{S}_t$ and $\boldsymbol{i}_{t-1}$, as defined in (20).

We derive the transition equation from projections to forecasts by noting that the structure of the dynamic system conditioned down onto the policymaker's information takes a familiar form. That is, conditioning down (15) and (16) onto $\boldsymbol{Z}^t$ yields a system of expectational linear difference equations in $\boldsymbol{\mathcal{S}}_t$ that is akin to the full-information system shown in (21), except for the use of policymaker projections in lieu of full-information expectations:

$$\boldsymbol{\mathcal{J}}\boldsymbol{\mathcal{S}}_{t+1|t} = \boldsymbol{\mathcal{A}}\boldsymbol{\mathcal{S}}_{t|t} \tag{30}$$

with $\boldsymbol{\mathcal{J}}$, and $\boldsymbol{\mathcal{A}}$ as defined in (21) above. In a stationary equilibrium consistent with Definition 1, $\boldsymbol{\mathcal{S}}_t$ is stationary, and so is its projection $\boldsymbol{\mathcal{S}}_{t|t}$. When the equilibrium is linear and time-invariant, the projections $\boldsymbol{\mathcal{S}}_{t|t}$ must follow

$$\boldsymbol{\mathcal{S}}_{t+1|t} = \boldsymbol{\mathcal{T}}\boldsymbol{\mathcal{S}}_{t|t}\ , \qquad \text{with} \quad (\boldsymbol{\mathcal{J}}\boldsymbol{\mathcal{T}} - \boldsymbol{\mathcal{A}})\,\boldsymbol{\mathcal{S}}_{t|t} = 0 \qquad \text{for some stable matrix } \boldsymbol{\mathcal{T}}. \tag{31}$$

$\boldsymbol{\mathcal{T}}$ spans a stable, invariant subspace of $|\boldsymbol{\mathcal{J}}\,z - \boldsymbol{\mathcal{A}}|$. In addition, $\boldsymbol{\mathcal{J}}\boldsymbol{\mathcal{T}} - \boldsymbol{\mathcal{A}}$ must be orthogonal to the space of projections $\boldsymbol{\mathcal{S}}_{t|t}$. In principle, there can be multiple solutions for $\boldsymbol{\mathcal{T}}$ in (31). A valid choice for $\boldsymbol{\mathcal{T}}$ is $\bar{\boldsymbol{\mathcal{T}}}$, known from the full-information solution given in (23).[28] However, there need not be a unique solution for $\boldsymbol{\mathcal{T}}$.[29] However, the multiplicity of equilibria highlighted in our paper does not stem from the implications of choosing different $\boldsymbol{\mathcal{T}}$. Instead, we restrict ourselves to a class of imperfect information equilibria such that $\boldsymbol{\mathcal{T}} = \bar{\boldsymbol{\mathcal{T}}}$.[30].

We can now state the second key result in this section. In a linear equilibrium with normally distributed shocks and with a given linear transformation from projections $\boldsymbol{\mathcal{S}}_{t|t}$ into forecasts $\boldsymbol{\mathcal{S}}_{t+1|t}$,

[28] To verify the validity of $\bar{\boldsymbol{\mathcal{T}}}$ as a solution to (30), note that $\bar{\boldsymbol{\mathcal{T}}}$ solves (21) and that (30) represents the same difference equation, when projected onto $\boldsymbol{Z}^t$. In the full information case, the matrices $\boldsymbol{\mathcal{G}}$ and $\boldsymbol{\mathcal{P}}$, used to construct $\bar{\boldsymbol{\mathcal{T}}}$ are unique, and so are the resulting outcomes for all variables. But, as discussed by Mertens (2016*a*), alternative choices of $\boldsymbol{\mathcal{T}}$ can describe the same outcomes, as all imply a unique product between transition matrix $\boldsymbol{\mathcal{T}}$ and the shock loadings $\bar{\boldsymbol{\mathcal{H}}}$.

[29] In principle, several choices of $\boldsymbol{\mathcal{T}}$ could satisfy the criterion in (31). To see this, think of any $\boldsymbol{\mathcal{T}}$ whose columns include a (sub)set of the stable eigenvectors of the matrix pencil $|\boldsymbol{\mathcal{J}}\,z - \boldsymbol{\mathcal{A}}|$ and columns of zeros otherwise.

[30] In a setup similar to ours, Svensson & Woodford (2004) also assume $\boldsymbol{\mathcal{T}} = \bar{\boldsymbol{\mathcal{T}}}$. In contrast to our paper, they restrict the number of state variables, in a way that disregards a role for belief shocks, and assume equilibrium uniqueness. Applications that build on Svensson & Woodford (2004) are, for example, Dotsey & Hornstein (2003), Aoki (2006),Nimark (2008*b*), Carboni & Ellison (2011).

the Kalman filter represents conditional expectations of $\boldsymbol{\mathcal{S}}_t$ (and thus also $\boldsymbol{S}_t$).

PROPOSITION 2 (Kalman filter represents conditional expectations) *In a linear, time-invariant stationary equilibrium, when the conditions for Proposition 1 hold, and for a given stable transition matrix $\boldsymbol{\mathcal{T}}$ between policymaker projections and forecasts as in* (31)*, the steady-state Kalman filter represents conditional expectations $\boldsymbol{\mathcal{S}}_{t|t} = E\left(\boldsymbol{\mathcal{S}}_t|\boldsymbol{Z}^t\right)$. For a given sequence of innovations in the measurement vector, $\tilde{\boldsymbol{Z}}_t$, the Kalman filter implies a stationary evolution of projections:*

$$\boldsymbol{\mathcal{S}}_{t+1|t+1} = \boldsymbol{\mathcal{T}}\boldsymbol{\mathcal{S}}_{t|t} + \boldsymbol{\mathcal{K}}\tilde{\boldsymbol{Z}}_{t+1} \ , \qquad \textit{with} \quad \boldsymbol{\mathcal{K}} = \operatorname{Cov}\left(\tilde{\boldsymbol{\mathcal{S}}}_t, \tilde{\boldsymbol{Z}}_t\right)\left(\operatorname{Var}\left(\tilde{\boldsymbol{Z}}_t\right)\right)^{-1} = \begin{bmatrix}\boldsymbol{0}' & \boldsymbol{K}_x' & \boldsymbol{K}_y' & \boldsymbol{K}_i'\end{bmatrix}' . \tag{32}$$

Proof. In a linear, time-invariant stationary equilibrium, shocks are jointly normal and propagate linearly so that the sequences of $\boldsymbol{\mathcal{S}}_t$ and $\boldsymbol{Z}_t$ are joint normally distributed, and conditional expectations are identical to mean-squared-error optimal linear projections. By the law of iterated projections, we can decompose $E\left(\boldsymbol{\mathcal{S}}_{t+1}|\boldsymbol{Z}^{t+1}\right) = E\left(\boldsymbol{\mathcal{S}}_{t+1}|\boldsymbol{Z}^t\right) + E\left(\tilde{\boldsymbol{\mathcal{S}}}_{t+1}|\tilde{\boldsymbol{Z}}_{t+1}\right)$ and, based on (31), we have $E\left(\boldsymbol{\mathcal{S}}_{t+1}|\boldsymbol{Z}^t\right) = \boldsymbol{\mathcal{T}}\boldsymbol{\mathcal{S}}_{t|t}$. From Proposition 1 follows $E\left(\tilde{\boldsymbol{\mathcal{S}}}_{t+1}|\tilde{\boldsymbol{Z}}_{t+1}\right) = \boldsymbol{\mathcal{K}}\tilde{\boldsymbol{Z}}_{t+1}$. $\boldsymbol{K}_x$ and $\boldsymbol{K}_y$ are appropriate partitions of $\boldsymbol{K}$ as defined in (29) and $\boldsymbol{K}_i = \boldsymbol{\mathcal{G}}_{ix}\boldsymbol{K}_x$. The upper block of $\boldsymbol{\mathcal{K}}$, corresponding to the gain coefficients for the lagged policy instrument, is zero since $\boldsymbol{i}_{t-1} = \boldsymbol{i}_{t-1|t-1} \Rightarrow \tilde{\boldsymbol{i}}_{t-1|t} = \boldsymbol{i}_{t-1|t} - \boldsymbol{i}_{t-1|t-1} = 0$. Projections are stationary since $\boldsymbol{\mathcal{T}}$ is stable. ■

To summarize, in this section we derive two key insights for our linear rational expectations framework with heterogeneous information sets. First, we show that under Gaussian shocks a Kalman filter exists, which is the optimal filter in this environment. The filter is available to the less-informed agent to gain information about realizations of variables that are available to the fully informed agent. Second, we show that the Kalman filter represents the conditional expectations of the less informed agent. This insight allows us to construct a solution approach for this framework which deviates from standard methods in that certainty equivalence no longer holds. Specifically, expectation formation of the less informed agents now depends on second moments embedded in the steady-state Kalman filter. Moreover, it is the intersection of the different expectation formation processes under rational expectations that lies at the heart of indeterminacy in this framework.

3.4 A Class of Imperfect Information Equilibria

We now put the results and insights from the previous sections to work to derive and solve a linear rational expectations difference system under imperfect information. The main result of this section, and key contribution of our paper, is that the equilibrium is generally indeterminate in this environment. Indeterminacy means that there is a multiplicity, possibly infinite, of initial conditions (or adjustment paths) to a long-run equilibrium (or steady state). An implication of indeterminacy is that along such adjustment paths the economy is subject to sunspot (or belief) shocks that are non-fundamental.

As noted before, we restrict ourselves to a class of imperfect information equilibria such that $\boldsymbol{\mathcal{T}} = \bar{\boldsymbol{\mathcal{T}}}$, which leads to the following condition.

DEFINITION 2 (Projection Condition) *The projection condition restricts the mapping between projected backward- and forward-looking variables to be identical to the full-information case:*

$$\boldsymbol{\mathcal{Y}}_{t|t} = \boldsymbol{\mathcal{G}} \boldsymbol{\mathcal{X}}_{t|t}\,, \qquad \textit{and} \quad \boldsymbol{\mathcal{X}}_{t+1|t} = \boldsymbol{\mathcal{P}} \boldsymbol{\mathcal{X}}_{t|t}\,. \tag{33}$$

where $\boldsymbol{\mathcal{G}}$ and $\boldsymbol{\mathcal{P}}$ are the unique solution coefficients in the corresponding full-information case.

The projection condition is an equilibrium condition that imposes a linear mapping between projections of backward- and forward-looking variables. In particular, the it imposes a *second-moment* restriction on the joint distribution of the innovations $\tilde{\boldsymbol{X}}_t$, $\tilde{\boldsymbol{Y}}_t$.[31] As a second-moment restriction, the projection condition only restricts co-movements of the innovations *on average*, but not for any particular realization of $\tilde{\boldsymbol{X}}_t$ and $\tilde{\boldsymbol{Y}}_t$.

The projection condition (33) implies the following restriction between the Kalman gains of forward- and backward-looking variables:

$$\boldsymbol{Y}_{t|t} = \boldsymbol{\mathcal{G}}_{yx}\, \boldsymbol{X}_{t|t} + \boldsymbol{\mathcal{G}}_{yi}\, \boldsymbol{i}_{t-1|t} \quad \Longrightarrow \tilde{\boldsymbol{Y}}_{t|t} = \boldsymbol{\mathcal{G}}_{yx}\, \tilde{\boldsymbol{X}}_{t|t} \quad \Longleftrightarrow \quad \left(\boldsymbol{K}_y - \boldsymbol{\mathcal{G}}_{yx}\, \boldsymbol{K}_x\right) \tilde{\boldsymbol{Z}}_t = 0\,, \tag{34}$$

where $\boldsymbol{K}_y$ and $\boldsymbol{K}_x$ denote the corresponding partitions of the Kalman gain, $\boldsymbol{K}$, defined in (29). Since equation (34) must hold for every $\tilde{\boldsymbol{Z}}_t$, the projection condition therefore implies a restriction on the Kalman gains, which we summarize in the following proposition.

PROPOSITION 3 (Projection Condition for Kalman gains) *The projection condition* (33) *holds only if the Kalman gains satisfy $\boldsymbol{K}_y = \boldsymbol{\mathcal{G}}_{yx}\, \boldsymbol{K}_x$.*

Proof. As noted in (34), a necessary condition for the projection to hold is $\left(\boldsymbol{K}_y - \boldsymbol{\mathcal{G}}_{yx}\, \boldsymbol{K}_x\right) \tilde{\boldsymbol{Z}}_t = 0$ for all realizations of $\tilde{\boldsymbol{Z}}_t$. $\tilde{\boldsymbol{Z}}_t$ has a joint normal distribution and Assumption 3 implies that $\mathrm{Var}\,(\tilde{\boldsymbol{Z}}_t) = \boldsymbol{C}\,\mathrm{Var}\,(\boldsymbol{S}_t^*)\boldsymbol{C}' + \boldsymbol{D}\boldsymbol{D}'$ is strictly positive definite, so that the distribution of $\tilde{\boldsymbol{Z}}_t$ is non-degenerate. Thus, for (34) to hold we must have $\boldsymbol{K}_y = \boldsymbol{\mathcal{G}}_{yx}\, \boldsymbol{K}_x$. Since

$$\boldsymbol{K}_y = \mathrm{Cov}\,(\tilde{\boldsymbol{Y}}_t, \tilde{\boldsymbol{Z}}_t)\, \mathrm{Var}\,(\tilde{\boldsymbol{Z}}_t)^{-1}, \text{ and } \boldsymbol{K}_x = \mathrm{Cov}\,(\tilde{\boldsymbol{X}}_t, \tilde{\boldsymbol{Z}}_t)\, \mathrm{Var}\,(\tilde{\boldsymbol{Z}}_t)^{-1},$$

the projection condition equivalently requires $\mathrm{Cov}\,(\tilde{\boldsymbol{Y}}_t, \tilde{\boldsymbol{Z}}_t) = \boldsymbol{\mathcal{G}}_{yx}\; \mathrm{Cov}\,(\tilde{\boldsymbol{X}}_t, \tilde{\boldsymbol{Z}}_t)$. ■

[31] In addition to $\boldsymbol{Y}_t$ and $\boldsymbol{X}_t$, $\boldsymbol{\mathcal{Y}}_t$ and $\boldsymbol{\mathcal{X}}_t$ also contain the current and lagged policy instrument, respectively. However, the projection condition does not impose a direct restriction on innovations in the policy instrument since $\boldsymbol{i}_t = \boldsymbol{i}_{t|t}$ and thus $\tilde{\boldsymbol{i}}_{t-1|t} = \boldsymbol{i}_{t-1|t} - \boldsymbol{i}_{t-1|t-1} = 0$.

Intuitively, the Kalman gains are multivariate regression slopes and, as shown in Proposition 3, the projection condition imposes a linear restriction on covariances between $\tilde{\boldsymbol{Y}}_t$ and $\tilde{\boldsymbol{X}}_t$. It is this feature of our framework that moves the solution of the underlying linear rational expectations model out of certainty equivalence and poses an intricate fixed-point problem with a highly non-linear solution: The Kalman gains depend on the second moments of the solution, which in turn depends on the Kalman gains via the projection condition.

We are now in a position to derive a representation of the system of expectational difference equations (15) - (17), where the expectations of the fully informed agent are captured by the concept of endogenous forecast errors known from Definition 2 and the expectation formation of the less-informed agent is represented by the Kalman filter. As summarized in the following theorem, equilibrium dynamics then follow a state vector that tracks both projections and actual values of backward- and forward-looking variables as well as the policy instrument.[32]

THEOREM 1 (Difference System Under Imperfect Information) *Consider the system of difference equations* (15) *and* (16) *with a measurement vector that is linear in $\boldsymbol{S}_t$ as defined in* (19)*. Let Assumptions 1, 2, 3, and 4 hold and consider stationary, linear, time-invariant equilibria that satisfy the projection condition, as stated in Definitions 1 and 2. Equilibrium dynamics are then characterized by the evolution of the following vector system:*

$$\overline{\boldsymbol{\mathcal{S}}}_{t+1} \equiv \begin{bmatrix} \boldsymbol{S}^*_{t+1} \\ \boldsymbol{\mathcal{X}}_{t+1|t+1} \end{bmatrix} = \underbrace{\begin{bmatrix} (\boldsymbol{A}-\boldsymbol{K}\boldsymbol{C}) & 0 \\ \boldsymbol{\mathcal{K}}_x \boldsymbol{C} & \boldsymbol{\mathcal{P}} \end{bmatrix}}_{\overline{\boldsymbol{\mathcal{A}}}} \overline{\boldsymbol{\mathcal{S}}}_t + \begin{bmatrix} (\boldsymbol{I}-\boldsymbol{K}\boldsymbol{H}) \\ \boldsymbol{\mathcal{K}}_x \boldsymbol{H} \end{bmatrix} \begin{bmatrix} \boldsymbol{B}_{x\varepsilon} & 0 \\ 0 & \boldsymbol{I} \end{bmatrix} \begin{bmatrix} \boldsymbol{\varepsilon}_{t+1} \\ \boldsymbol{\eta}_{t+1} \end{bmatrix}, \tag{35}$$

where $\boldsymbol{\mathcal{K}}_x' = \begin{bmatrix} 0 & \boldsymbol{K}_x' \end{bmatrix}$, $\boldsymbol{C}$ as defined in (28)*, and $\boldsymbol{\mathcal{P}}$ known from the unique full-information solution in* (22)*.*

Proof. $\boldsymbol{S}_t$ can be decomposed into $\boldsymbol{S}_t = \boldsymbol{S}^*_t + \boldsymbol{S}_{t|t}$. We need to show that $\boldsymbol{S}_{t|t}$ can be constructed from $\overline{\boldsymbol{\mathcal{S}}}_t$, and that the policy instrument $\boldsymbol{i}_t = \boldsymbol{i}_{t|t}$ can be constructed from the proposed state vector $\overline{\boldsymbol{\mathcal{S}}}_t$. Recalling the definitions of $\boldsymbol{S}_t$, $\boldsymbol{\mathcal{X}}_t$ and $\boldsymbol{\mathcal{Y}}_t$ in (17), and (20), the projection condition then implies that:

$$\boldsymbol{S}_{t|t} = \begin{bmatrix} \boldsymbol{X}_{t|t} \\ \boldsymbol{Y}_{t|t} \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & \boldsymbol{I} \\ \boldsymbol{\mathcal{G}}_{yi} & \boldsymbol{\mathcal{G}}_{yx} \end{bmatrix}}_{\boldsymbol{\mathcal{G}}_S} \boldsymbol{\mathcal{X}}_{t|t}, \qquad \boldsymbol{i}_t = \underbrace{\begin{bmatrix} \boldsymbol{\mathcal{G}}_{ii} & \boldsymbol{\mathcal{G}}_{ix} \end{bmatrix}}_{\boldsymbol{\mathcal{G}}_i} \boldsymbol{\mathcal{X}}_{t|t}, \qquad \text{with} \quad \boldsymbol{\mathcal{X}}_{t|t} = \begin{bmatrix} \boldsymbol{i}_{t-1} \\ \boldsymbol{X}_{t|t} \end{bmatrix},$$

[32] In fact, the joint vector of $\boldsymbol{\mathcal{S}}_t$ and $\boldsymbol{\mathcal{S}}_{t|t}$ does not need to tracked in its entirety. First, $\boldsymbol{\mathcal{S}}_t$ already includes the policy instrument $\boldsymbol{i}_t$, which lies in the space of central bank projections. The state of the economy can thus be described by $\boldsymbol{\mathcal{S}}_{t|t}$ and $\boldsymbol{S}_t$, which differs from $\boldsymbol{\mathcal{S}}_t$ in omitting $\boldsymbol{i}_t$. Second, the state of the economy is equivalently described by $\boldsymbol{S}^*_t = \boldsymbol{S}_t - \boldsymbol{S}_{t|t}$ and $\boldsymbol{\mathcal{S}}_{t|t}$. Third, when the projection condition (33) is satisfied, we need only track $\boldsymbol{\mathcal{X}}_{t|t}$ rather than $\boldsymbol{\mathcal{S}}'_{t|t} = \begin{bmatrix} \boldsymbol{\mathcal{X}}'_{t|t} & \boldsymbol{\mathcal{Y}}'_{t|t} \end{bmatrix}$, since $\boldsymbol{\mathcal{Y}}_{t|t} = \boldsymbol{\mathcal{G}}\boldsymbol{\mathcal{X}}_{t|t}$.

where block matrices are partitioned along the lines of $\boldsymbol{\mathcal{X}}_{t|t}$ above. The various coefficient matrices "$\boldsymbol{\mathcal{G}}_{..}$" are known from the full-information solution given in (22). The dynamics of $\boldsymbol{S}^*_{t+1}$, as captured by the top rows of $\overline{\boldsymbol{\mathcal{S}}}$, follow from the innovation state space (26) - (27) and the steady-state Kalman filter described in Appendix B. The dynamics of $\boldsymbol{\mathcal{X}}_{t+1|t+1}$, as captured by the bottom rows of $\overline{\boldsymbol{\mathcal{S}}}$, follow from (32) together with the projection condition (33) and the dynamics of $\tilde{\boldsymbol{Z}}_t$ given in (27). ■

Equation (35) forms the basis for solving a rational expectations model with heterogeneous information sets. Based on this representation we can derive key insights into the nature of the equilibria in this setting. The state vector $\overline{\boldsymbol{\mathcal{S}}}_t$ follows a first-order linear difference system given in (35). The stability of the system depends on the eigenvalues of its transition matrix $\overline{\boldsymbol{\mathcal{A}}}$. The transition matrix $\overline{\boldsymbol{\mathcal{A}}}$ depends on the Kalman gain $\boldsymbol{K}$, which depends on the yet to be determined shock loadings $\boldsymbol{\Gamma}_\varepsilon$ and $\boldsymbol{\Gamma}_b$ of the endogenous forecast errors $\boldsymbol{\eta}_t$. Nevertheless, existence of a steady-state Kalman filter allows us to conclude that $\overline{\boldsymbol{\mathcal{A}}}$, is stable.

COROLLARY 1 (Stable Transition Matrix) *Provided that a steady-state Kalman filter exists, the transition matrix $\overline{\boldsymbol{\mathcal{A}}}$ in* (35) *is stable. The eigenvalues of $\overline{\boldsymbol{\mathcal{A}}}$ are given by the eigenvalues of $\boldsymbol{\mathcal{P}}$, which is stable and known from the full-information solution* (22)*, and $\boldsymbol{A} - \boldsymbol{KC}$, whose stability is assured by the existence of the steady-state Kalman filter.*

Proof. The stability of $\boldsymbol{\mathcal{P}}$ follows from Assumption 1 and the resulting solution of the full-information case. The stability of $\boldsymbol{A} - \boldsymbol{KC}$ follows from Theorem 3 in Appendix B. ■

One key implication of Proposition 1 is that the usual root-counting arguments, as in Blanchard & Kahn (1980) or generalized in Sims (2002), do not pin down the shock loadings $\boldsymbol{\Gamma}_\varepsilon$ and $\boldsymbol{\Gamma}_b$ of the endogenous forecast errors $\boldsymbol{\eta}_{t+1}$ in (35). This follows from the fact that $\overline{\boldsymbol{\mathcal{A}}}$ is a stable matrix for any choice of $\boldsymbol{\Gamma}_\varepsilon$ and $\boldsymbol{\Gamma}_b$ consistent with the existence of a steady-state Kalman filter. Moreover, the projection condition does typically not place sufficiently many restrictions on $\boldsymbol{\Gamma}_\varepsilon$ and $\boldsymbol{\Gamma}_b$ to uniquely identify the shock loadings. This observation is the key result of our paper, namely that equilibria are generically indeterminate.

PROPOSITION 4 (Generic Indeterminacy) *With $\overline{\boldsymbol{\mathcal{A}}}$ stable, the endogenous forecast errors are only restricted by the projection condition given in Definition 2. The shocks loadings of the endogenous forecast errors, $\boldsymbol{\Gamma}_\varepsilon$ and $\boldsymbol{\Gamma}_b$, have $N_y \times (N_\varepsilon + N_y)$ unknown conditions. The projection condition stated in* (34) *imposes only $N_y \times N_z$ restrictions. However, a necessary condition for Assumption 3 to hold is $N_z \leq N_\varepsilon + N_y$. As a result, the projection condition cannot uniquely identify the shock loadings $\boldsymbol{\Gamma}_\varepsilon$ and $\boldsymbol{\Gamma}_b$.*

The standard macroeconomic literature on indeterminacy rests on the idea that forward-looking behavior is not restrained by a mechanism that pins down expectations. This mechanism is still present in our framework as it is embedded in the stability assumptions governing the solutions of the conditioned-down system and the existence conditions for the Kalman-filter. Intuitively, in the context of the introductory example, the Taylor principle enforces uniqueness in a simple monetary model via an aggressive response to movements in inflation expectations, whether driven by fundamental or belief shocks. However, optimal filtering introduces, loosely speaking, too much stability in that it validates sunspot shocks as drivers of endogenous variables through the projection equation, which in turn affects expectation formation of the fully informed agent. We provide more discussion and simulation results from a simple model in a following section.

More formally, Theorem 1 rests on the assumption of joint detectability and unit-circle controllability of $(\boldsymbol{A}, \boldsymbol{B}, \boldsymbol{H})$ where $\boldsymbol{B}$ depends on the endogenous shock loadings $\boldsymbol{\Gamma}_\varepsilon$ and $\boldsymbol{\Gamma}_b$, while $\boldsymbol{A}$ and $\boldsymbol{H}$ are primitives of the model. The detectability condition can be verified independently of solving for $\boldsymbol{\Gamma}_\varepsilon$ and $\boldsymbol{\Gamma}_b$. In addition, as described in the appendix (see Proposition 6), a full rank of the matrix $\boldsymbol{B}$, defined in (25), is sufficient to ensure unit-circle controllability. As part of the model setup, $\boldsymbol{B}_{x\varepsilon}$ is assumed to have full rank. Consequently, the criterion of a full rank of $\boldsymbol{B}$ is satisfied when the belief shock loadings $\boldsymbol{\Gamma}_b$ have full rank, and thus $\boldsymbol{\Gamma}_b \neq \boldsymbol{0}$. Non-zero belief shock loadings are a hallmark of equilibrium indeterminacy. While the projection condition places restrictions on $\boldsymbol{\Gamma}_b$, non-zero belief shock loadings are thus a sufficient condition for the existence of a steady state Kalman filter, which in turn assures the stability of $\overline{\boldsymbol{\mathcal{A}}}$.[33]

We summarize the construction of an equilibrium for a given solution of the endogenous forecast errors in the following Theorem.

THEOREM 2 (Equilibria under Imperfect Information) *Consider the difference system characterized in Theorem 1 and let $\boldsymbol{\eta}_t = \boldsymbol{\Gamma}_\varepsilon \boldsymbol{\varepsilon}_t + \boldsymbol{\Gamma}_b \boldsymbol{b}_t$ with shock loadings $\boldsymbol{\Gamma}_\varepsilon$ and $\boldsymbol{\Gamma}_b$ such that the projection condition (33) is satisfied. Equilibrium outcomes are then given by:*

$$\boldsymbol{S}_t = \begin{bmatrix} \boldsymbol{I} & \boldsymbol{\mathcal{G}}_s \end{bmatrix} \overline{\boldsymbol{\mathcal{S}}}_t \qquad \boldsymbol{i}_t = \begin{bmatrix} 0 & \boldsymbol{\mathcal{G}}_i \end{bmatrix} \overline{\boldsymbol{\mathcal{S}}}_t \qquad \overline{\boldsymbol{\mathcal{S}}}_{t+1} = \overline{\boldsymbol{\mathcal{A}}}\ \overline{\boldsymbol{\mathcal{S}}}_t + \overline{\boldsymbol{\mathcal{B}}}\ \boldsymbol{w}_{t+1}\,,$$

$$\textit{with} \quad \overline{\boldsymbol{\mathcal{S}}}_t \equiv \begin{bmatrix} \boldsymbol{S}_t^* \\ \boldsymbol{\mathcal{X}}_t \end{bmatrix}, \qquad \boldsymbol{w}_{t+1} \equiv \begin{bmatrix} \boldsymbol{\varepsilon}_{t+1} \\ \boldsymbol{b}_{t+1} \end{bmatrix}, \qquad \textit{and} \quad \overline{\boldsymbol{\mathcal{B}}} \equiv \begin{bmatrix} (\boldsymbol{B} - \boldsymbol{K}\boldsymbol{D}) \\ \boldsymbol{\mathcal{K}}_{\mathcal{X}} \boldsymbol{D} \end{bmatrix}.$$

where $\boldsymbol{B}$ and $\boldsymbol{D}$ encode the shock loading $\boldsymbol{\Gamma}_\varepsilon$ and $\boldsymbol{\Gamma}_b$ as stated in (28); $\boldsymbol{\mathcal{K}}_x$, $\boldsymbol{\mathcal{G}}_s$ and $\boldsymbol{\mathcal{G}}_i$ are defined in the proof of Theorem 1. Block matrices are partitioned along the lines of $\overline{\boldsymbol{\mathcal{S}}}_t$ as stated above.

Proof. The proof follows directly from Theorem 1. ■

Theorem 2 fully describes a solution to the linear rational expectations model with imperfect information and nested information sets. As a final step, we only need to provide a description of

[33] It is possible to construct an equilibrium under indeterminacy where the loadings on the belief shocks are zero. In a standard framework under full information Lubik & Schorfheide (2003) call this a sunspot equilibrium without sunspot. In the working paper version of this present paper, we show that such a degenerate equilibrium can also exist in a simple monetary model.

how the endogenous forecast errors, specifically the loadings on the shocks are determined. The Kalman gain $\boldsymbol{K}_x$, defined in Proposition 2, depends on the equilibrium distribution of endogenous forecast errors $\boldsymbol{\eta}_t$. According to Assumption 2, $\boldsymbol{\eta}_t$ is a linear combination of exogenous shocks $\boldsymbol{\varepsilon}_t$ and belief shocks $\boldsymbol{b}_t$ with endogenous shock loadings $\boldsymbol{\Gamma}_\varepsilon$ and $\boldsymbol{\Gamma}_b$ that are yet to be determined so as to satisfy the projection condition stated in Definition 2.

3.5 Determination of the Endogenous Forecast Errors

In the full-information literature, restrictions for the endogenous forecast errors $\boldsymbol{\eta}_t$ emanate from explosive roots in the dynamic system, as described by Sims (2002) and Lubik & Schorfheide (2003). In our imperfect information case, further restrictions result from the projection condition stated in Definition 2. As noted in Corollary 1 above, the transition matrix $\overline{\boldsymbol{\mathcal{A}}}$ is always stable in a time-invariant equilibrium with a steady-state Kalman filter. Restrictions on $\boldsymbol{\eta}_t$ can therefore only result from the projection condition. At the same time, the projection condition does generally not provide sufficiently many restrictions to pin down $\boldsymbol{\eta}_t$ uniquely, as discussed in Proposition 4.

Determination of shock loadings $\boldsymbol{\Gamma}_\varepsilon$, $\boldsymbol{\Gamma}_b$ for the endogenous forecast errors that are consistent with the projection condition poses an intricate fixed problem between shock loadings and Kalman gains. As noted already by Sargent (1991), the Kalman gains are endogenous equilibrium objects when the observable signals reflect information contained in endogenous variables. In contrast, as discussed in an earlier working paper version (Lubik et al. 2019), the Kalman filtering problem can be solved independently of the equilibrium dynamics of the system when the signal vector consists only of exogenous variables.[34]

Equation (19) describes the central bank's measurement vector generically as a linear combination of backward- and forward-looking variables, $\boldsymbol{Z}_t = \boldsymbol{H}_x\boldsymbol{X}_t + \boldsymbol{H}_y\boldsymbol{Y}_t$. To simplify some of the algebra, we limit ourselves to signal vectors that have the same length as the vector of forward-looking variables ($\boldsymbol{Y}_t$) and that have no rank-deficient loading on $\boldsymbol{Y}_t$, so that $\boldsymbol{H}_y$ is square and invertible, and can be normalized to the identity matrix:[35]

$$\boldsymbol{Z}_t = \boldsymbol{H}_x\boldsymbol{X}_t + \boldsymbol{Y}_t \qquad \text{and thus} \quad \boldsymbol{H} = \begin{bmatrix}\boldsymbol{H}_x & \boldsymbol{I}\end{bmatrix}. \tag{36}$$

[34] The case of a purely exogenous signal arises when $\boldsymbol{H}_y = \boldsymbol{0}$ in (19) and $\boldsymbol{X}_t$ exogenous (i.e. without dependence on lagged forward-looking variables). The working paper version of this article contains further discussion and a general analytical characterization of the solution for this case. In Lubik et al. (2019), we derive two general results for the exogenous-signal case: First, the projection condition does not restrict the belief shock loadings of the endogenous forecast errors, $\boldsymbol{\Gamma}_b$, when the signal is exogenous. Second, we derive an analytical expression for the restrictions on the loadings of the endogenous forecast errors on fundamental shocks (including the measurement errors) that result from the projection condition. However, the exogenous-signal case is arguably less realistic since variables relevant to the policymaker are typical endogenous.

[35] Consider the case of a signal $\hat{\boldsymbol{Z}}_t = \hat{\boldsymbol{H}}_x\boldsymbol{X}_t + \hat{\boldsymbol{H}}_y\boldsymbol{Y}_t$, where $\hat{\boldsymbol{H}}_y$ is square and invertible. The information content provided by $\hat{\boldsymbol{Z}}_t$ is equivalent to what is spanned by $\boldsymbol{Z}_t = \hat{\boldsymbol{H}}_y^{-1}\hat{\boldsymbol{Z}}_t$ with $\boldsymbol{H}_x = \hat{\boldsymbol{H}}_y^{-1}\hat{\boldsymbol{H}}_x$.

This setup also includes the case where each forward-looking variable is observed with error, as in $\boldsymbol{Z}_t = \boldsymbol{Y}_t + \boldsymbol{\nu}_t$, where $\boldsymbol{\nu}_t$ is an exogenous measurement error to be included among the set of backward-looking variables in $\boldsymbol{X}_t$.

We pursue a numerical the solution for the shock loadings $\boldsymbol{\Gamma}_\varepsilon$ and $\boldsymbol{\Gamma}_b$ that are consistent with the projection condition. Our procedure combines conventional techniques for solving linear RE models with standard algorithms for solving the Kalman filter's Riccati equation, while ensuring consistency with the projection condition. The algorithm searches for shock loadings $\boldsymbol{\Gamma}_\varepsilon$ and $\boldsymbol{\Gamma}_b$ that satisfy the projection condition for a Kalman filter that is consistent with equilibrium outcomes of all variables.

From the measurement equation (36), we can deduce that $\boldsymbol{Y}_t^* = -\boldsymbol{H}_x\boldsymbol{X}_t^*$, which allows to simplify the innovation system given by (26) and (27) as follows: $\tilde{\boldsymbol{X}}_{t+1} = \tilde{\boldsymbol{A}}\boldsymbol{X}_t^* + \tilde{\boldsymbol{B}}\boldsymbol{w}_{t+1}$, and $\tilde{\boldsymbol{Z}}_{t+1} = \tilde{\boldsymbol{C}}\boldsymbol{X}_t^* + \tilde{\boldsymbol{D}}\boldsymbol{w}_{t+1}$ with $\tilde{\boldsymbol{A}} = \boldsymbol{A}_{xx} - \boldsymbol{A}_{xy}\boldsymbol{H}_x$, $\tilde{\boldsymbol{B}} = \begin{bmatrix}\boldsymbol{B}_{x\varepsilon} & 0\end{bmatrix}$, $\tilde{\boldsymbol{C}} = \boldsymbol{H}_x\ (\boldsymbol{A}_{xx} - \boldsymbol{A}_{xy}\boldsymbol{H}_x) + \boldsymbol{A}_{yx} - \boldsymbol{A}_{yy}\boldsymbol{H}_x$,[36]

$$\text{and} \quad \tilde{\boldsymbol{D}} = \begin{bmatrix}(\boldsymbol{H}_x\boldsymbol{B}_{x\varepsilon} + \boldsymbol{\Gamma}_\varepsilon) & \boldsymbol{\Gamma}_b\end{bmatrix}. \tag{37}$$

For a given $\tilde{\boldsymbol{D}}$, which embodies a guess of $\boldsymbol{\Gamma}_\varepsilon$ and $\boldsymbol{\Gamma}_b$, the Kalman-filtering solution to this system generates a Kalman gain $\boldsymbol{K}_x$, which can be used to form projections $\tilde{\boldsymbol{X}}_{t|t} = \boldsymbol{K}_x\tilde{\boldsymbol{Z}}_t$. The algorithm checks whether this guess for $\tilde{\boldsymbol{D}}$ also satisfies the projection condition, which requires $\boldsymbol{K}_y = \boldsymbol{\mathcal{G}}_{yx}\boldsymbol{K}_x$. Together with the projection condition, the measurement equation (36) implies $\boldsymbol{I} = \boldsymbol{H}_x\boldsymbol{K}_x + \boldsymbol{K}_y = (\boldsymbol{H}_x + \boldsymbol{\mathcal{G}}_{yx})\,\boldsymbol{K}_x$. We employ a numerical solver that searches for a $\tilde{\boldsymbol{D}}$ that generates a Kalman gain $\boldsymbol{K}_x$ such that $\boldsymbol{L}\boldsymbol{K}_x = \boldsymbol{I}$, where $\boldsymbol{L} = \boldsymbol{H}_x + \boldsymbol{\mathcal{G}}_{yx}$. Given a solution for $\tilde{\boldsymbol{D}}$ that satisfies the projection condition $\boldsymbol{L}\boldsymbol{K}_x = \boldsymbol{I}$, we can then back out $\boldsymbol{\Gamma}_\varepsilon$ and $\boldsymbol{\Gamma}_b$ based on (37).

4 Quantitative Results for the Simple Example

We now provide further insight into the mechanics and implications of our framework by solving for the full set of resulting equilibria.[37] We apply the solution techniques described in section 3 and consider the case of the policy rule $i_t = \phi\pi_{t|t}$ where the endogenous information set $Z_t = \pi_t + \nu_t$. For purposes of illustration, we set the policy parameter $\phi = 1.5$ and assume that the real rate follows an AR(1) process with persistence $\rho = 0.9$ and a unit innovation variance $\sigma_\varepsilon^2 = 1$.[38] Some experimentation shows that in this simple example the measurement error on inflation has to be

[36] $\boldsymbol{A}_{xx}$, $\boldsymbol{A}_{xy}$, etc. denote suitable sub-matrices of $\boldsymbol{A}$.

[37] We limit attention to equilibria, where consistent with Assumption 4 a stabilizing solution for the policymaker's Kalman filter exists. As discussed in Appendix A, there is also a particular equilibrium for the simple example where the real rate is perfectly revealed. This equilibrium represents a knife-edge case where the Kalman filter has a non-stabilizing solution with $\mathrm{Var}\,(r_t|Z^t) = 0$. This particular exists, however, for any level of noise variance.

[38] As discussed in section 2, the variance of the belief shock σ_b^2 is normalized to unity without loss of generality.

large for an equilibrium to exist where the real is rate is not revealed without error.[39] We therefore set the variance of the $i.i.d.$ measurement error ν_t to $\sigma_\nu^2 = 2.5^2$. We generate 2000 starting points from which our algorithm is able to find valid equilibria in 99 percent of all cases. Each equilibrium is associated with a triple $(\gamma_\varepsilon, \gamma_b, \gamma_\nu)$ of loadings on the shocks in the forecast error decomposition. The fact that there are multiple such loadings for the parameter space simply reflects that the RE solution is indeterminate.

We plot the range of impulse responses to each shock obtained over the entire set of equilibria in Figure 1. We select an arbitrary example equilibrium, and show its impulse responses in Figure 2. Each figure displays the corresponding full-information impulse responses. Under full information, a unit innovation to the real rate raises inflation by $1/(\phi - \rho) = 5/3$ which then decays at the constant rate ρ. The interest response follows the same pattern, which reflects the Fisher effect in that a higher real rate requires a higher nominal rate and in turn a higher inflation rate. The unique full-information equilibrium has zero response to the measurement error since the model is not defined as having such error, and there is also a zero-response to the belief shock since, with $\phi = 1.5$, the solution is unique.

When we introduce a limited information set for the policymaker the set of equilibria is notably different: Belief shocks play a role and multiple values of shock loadings are possible. On impact, a unit innovation in the real rate can either lead to an increase or a decrease in inflation over a range of about $(-1.9, 1.7)$ depending on the equilibrium the economy is in. Similarly, the nominal rate response can be positive or negative. In effect, inflation and the nominal rate can comove positively or negatively under different equilibria.[40] Figure 2 displays impulse response for one of the possible limited information equilibria where, in contrast to the full-information solution, inflation and the nominal interest rate comove negatively. While the nominal rate follows the real rate increase, inflation can fall on account of a negative loading γ_ε on the forecast error. The figure also shows that a unit measurement error shock lowers inflation and the nominal rate which indicates a negative loading on ν_t in the solution, $\gamma_\nu < 0$. In addition, belief shocks play a non-trivial role. Critically, the range of possible responses to the belief shock across equilibria shown in Figure 1 is bounded.[41]

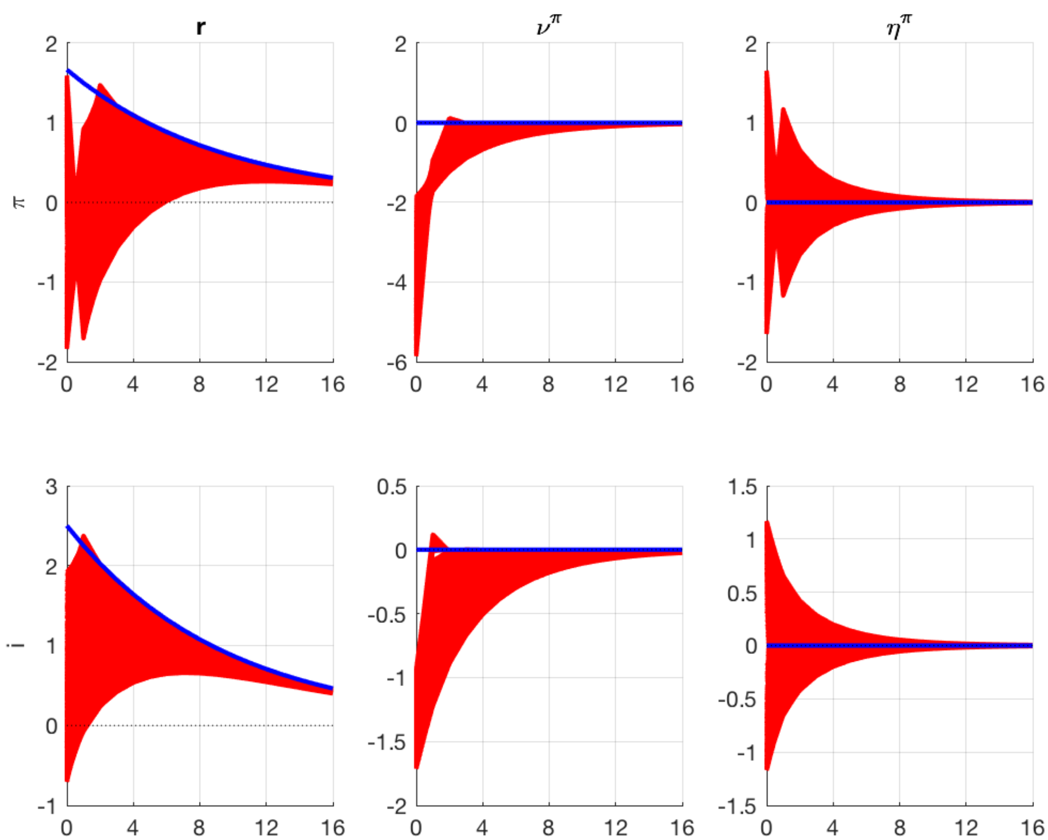
Figure 3 reports the autocorrelation function (ACF) and the standard deviation relative to the

[39] As noted before, in Lubik et al. (2019), we show that meaningful multiplicity of equilibria can easily arise in a standard New Keynesian model where the measurement error process is calibrated to match revisions in US macroeconomic data.

[40] The findings are reminiscent of the observation by Lubik and Schorfheide (2004) that changes in comovement patterns are a hallmark of equilibrium indeterminacy and thereby allow econometricians to identify different sets of equilibria. Moreover, their observation that indeterminate equilibria do impose some restrictions on the behavior of the economy in response to fundamental shocks thus carries over to our framework.

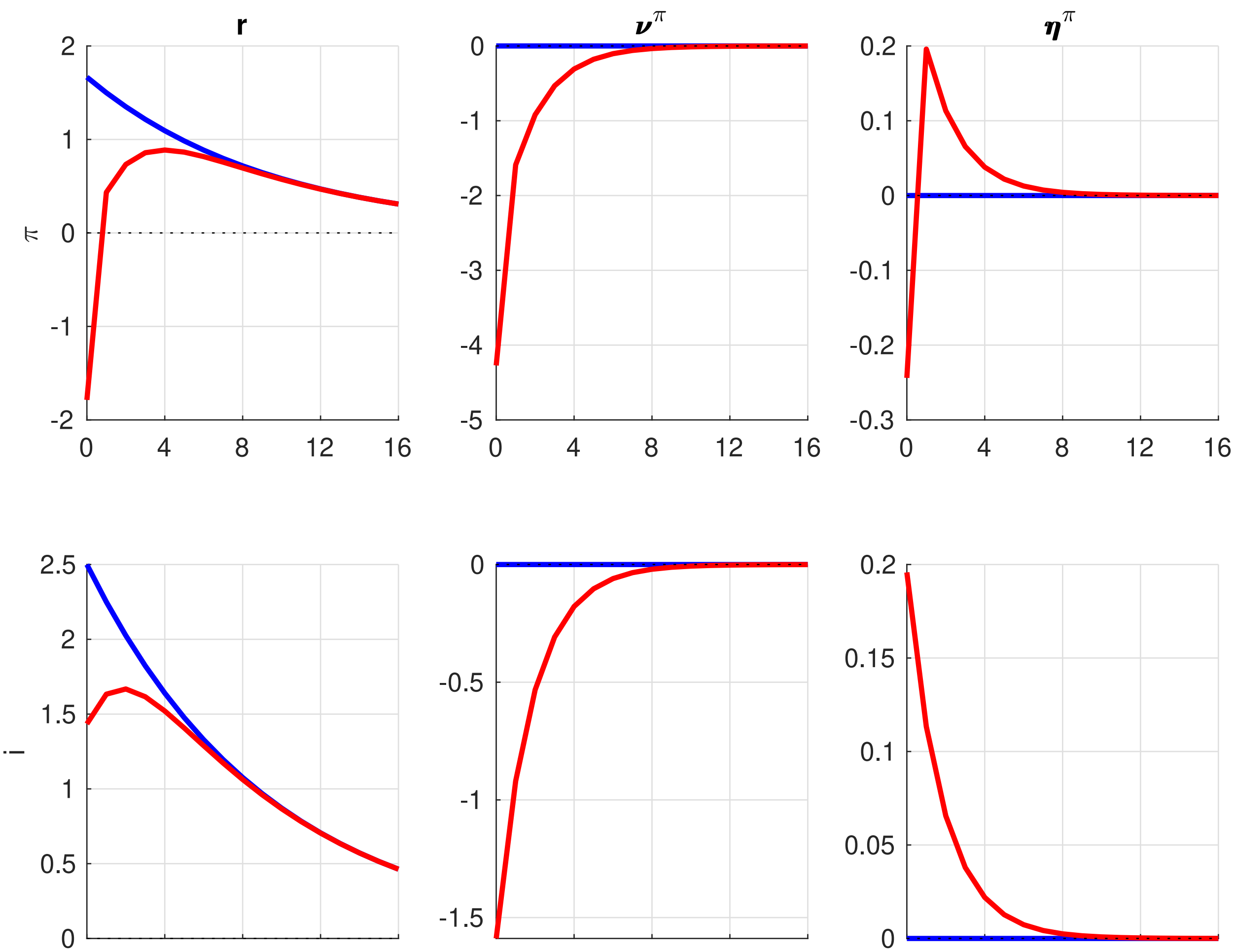
[41] The range of possible belief-shock responses is symmetric, since equilibrium conditions depend only on the squared loading, γ_b^2, as shown in section 2.

Figure 1: IRFs of Various Equilibria



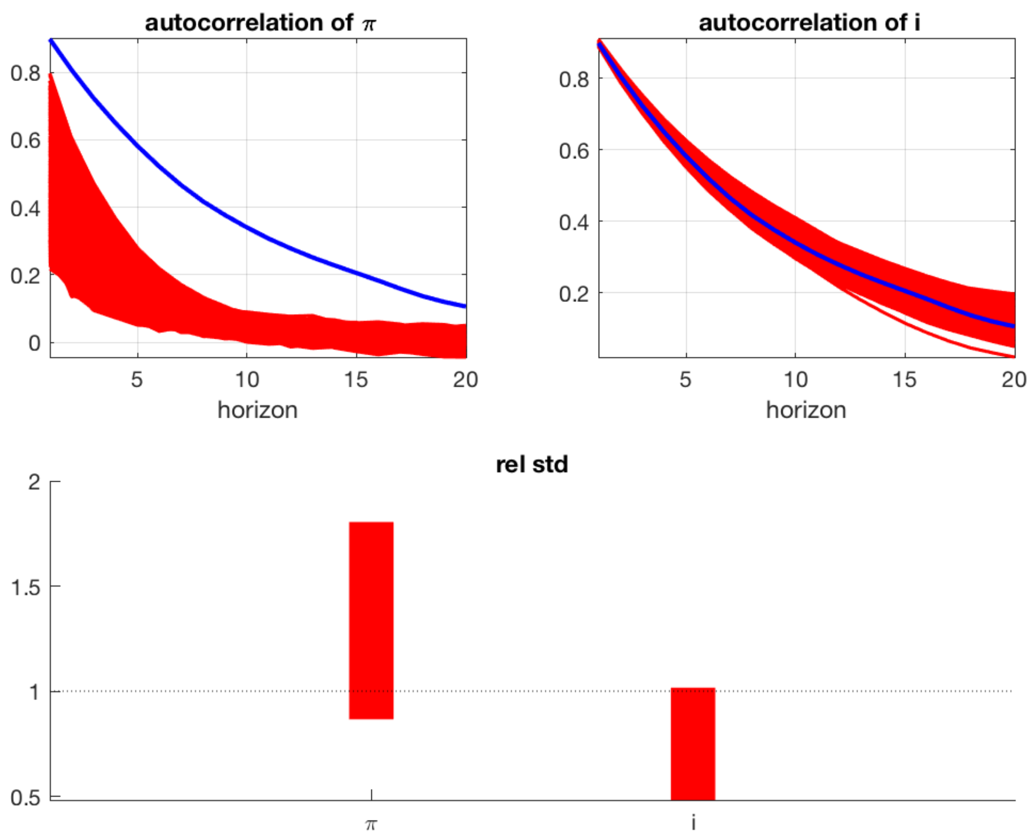
Note: Impulse response functions (IRF) under full information (blue) as well as various limited-information equilibria (red). Each row represents the response of a specific variable to the shocks in the model whereas each column represent the responses of the endogenous variables to a specific shock. The column labeled η^π reports responses to belief shocks.

Figure 2: IRF from an Arbitrarily Selected Equilibrium



Note: Impulse response functions (IRF) under full information (blue) as well as an example from the limited-information equilibria (red). Each row represents the response of a specific variable to the shocks in the model whereas each column represent the responses of the endogenous variables to a specific shock. The column labeled η^{π} reports responses to belief shocks.

Figure 3: Second Moments of Limited-Information Equilibria



Note: Top panels show moments of endogenous variables for the simple example economy under full information (blue) as well as various limited information equilibria (red). Bottom panel reports ranges of relative standard deviations of outcomes under limited information relative to the full-information outcomes.

full information scenario.[42] As shown in the upper row of panels, the full-information solution displays the typical autocorrelation pattern of a first-order autoregressive process. Comparing the set of outcomes under limited information against the unique equilibrium, the persistence of inflation is generally lower and its serial correlation decays much more rapidly, whereas for the nominal rate the ACF closely resembles that under full information. Since $i_t = \phi\pi_{t|t} = \frac{\phi}{\phi-\rho} r_{t|t}$ the nominal rate behaves like the real rate projection. The lower panel of the figure shows ranges of relative standard deviations of outcomes under limited information relative to the respective full-information outcomes. The interest rate under limited information is generally less volatile despite the presence of two additional shocks. The lower interest-rate volatility echoes our discussion in section 2 that the *effective* response to inflation under limited information after taking into account the filtering problem constitutes a violation of the Taylor principle. Imperfect information prevents the central bank from moving the policy rate aggressively in response to actual inflation. Instead, optimal filtering leads to attenuation of the policy response, the flipside of which is heightened inflation volatility.[43]

In standard models such as Clarida et al. (2000) or Lubik & Schorfheide (2004) indeterminacy arises because the central bank conducts a policy that does not satisfy the Taylor principle. The Taylor principle prescribes a sufficiently strong response of the nominal policy rate to actual inflation. In our limited information setting, the central bank applies the Taylor principle with respect to projections derived from an optimal filter. Since the optimal filter attenuates its response to the incoming inflation signal, the policy response to actual inflation ends up being too weak to ensure determinacy. Deviations from the Taylor principle lead to sunspot-driven movements in private sector expectations that the central bank cannot invalidate through its actions. Even though there is a unique mapping between central bank projections of outcomes and economic conditions in our framework, actual outcomes remain indeterminate . The source of the indeterminacy is the interaction of expectations formed under the two information sets.

When outcomes are not uniquely determined by economic fundamentals, there is a role for belief shocks to drive economic fluctuations. The term "belief shocks" refers to a set of economic disturbances that matter since people believe that they do. In general, these disturbances are otherwise unrelated to economic fundamentals.[44] We can think of the implications of belief shocks in terms of the following thought experiment. Suppose that the realization of a sunspot leads the private sector to believe that inflation is higher than warranted by economic fundamentals. This

[42]Moments are computed via simulation for 20,000 periods with the first 1,000 periods discarded as burn-in to avoid dependence on initial conditions.

[43]We obtain similar results in a richer a New Keynesian model, described in an earlier working paper version (Lubik et al. 2019).

[44]The use of the term "beliefs" is conceptually distinct from the "projections" described as part of our imperfect information setup, where projections are the result of the policymaker's optimal signal extraction efforts.

implies a reassessment of the nominal interest-rate path and a higher i_t in compensation for higher expected inflation. At this point, the behavior of the central bank is crucial. In the full-information model, if the Taylor principle holds, the central bank would raise the policy rate by proportionally more than the private-sector's sunspot-driven increase in inflation. This is only consistent with the Fisher equation if expected inflation rose by more than current inflation. However, this would imply an explosive trajectory for inflation, which is ruled out as an equilibrium path. If the Taylor principle does not hold, the policy rate rises less than the sunspot-driven increase in current inflation. In this case, the Fisher equation implies a less than proportionate and thereby stable increase in expected inflation, which validates the sunspot (or belief shock) to be consistent with a stationary equilibrium.[45] Consequently, the resulting equilibrium can be subject to belief shocks and is indeterminate.

In our imperfect information setup, the policy rule cannot apply the Taylor principle to actual inflation, which is unobserved. Instead, the policymaker applies the disproportionate response recommended by the Taylor principle to an estimate of current inflation. The estimate solves a signal extraction problem that attenuates the noisy signal of inflation so that policy ends up responding less than proportionally to actual inflation. As discussed in section 2, making the policy response to projected inflation more aggressive does not restore determinacy as it also attenuates the effect of actual inflation on the policymaker's optimal inflation projection. However, consistency of expectations held by the private sector and the imperfectly informed policymaker places a bound on the contribution of belief shocks to inflation.

5 Conclusion

This paper studies the implications of imperfect information for equilibrium determination in linear dynamic models when differently informed agents interact. We introduce a single deviation from full information rational expectations: one group of agents is strictly less informed than another. While we differentiate between types of agents that have different information sets, we continue to assume that each agent forms rational expectations conditional on available information. The implications of this model structure are stark. We show that indeterminacy of equilibrium is generic in this environment, even if the corresponding full information setting implies uniqueness. A given amount of noise, with optimal filtering of a less informed agent, can produce outcomes where there is a sunspot component to economic fluctuations. Our paper thereby contributes to a recent literature on informational frictions in rational expectations models.

Throughout our analysis, we maintain the assumption that the policymaker's limited informa-

[45] As can be seen from the equilibrium process for inflation in (12), next period's expected inflation is a fraction ϕ of this period's inflation rate, and thus stable for $|\phi| < 1$.

tion set is nested inside the public's information set. The indeterminacy issues identified by our paper should, however, extend to richer informational environments as long as the policymaker cannot perfectly observe forward-looking choice variables of the private sector. The key condition behind our indeterminacy results is that imperfect information impedes the policymaker from directly responding to belief-shock induced variations in private-sector decisions, which would also hinder policy in a setting of non-nested information.

The findings in this paper suggest various avenues for further investigation. For example, our framework has strong implications for empirical research: The general model under limited information has a state-space representation like any other linear dynamic framework so that a likelihood function can be constructed. The key difference and main complication with respect to standard frameworks is that the solution of the model is not certainty equivalent. Conditional on the Kalman gains the model implies a standard representation, but the gains are equilibrium objects and depend on second moment properties of the solution. This can be taken into account in solution and estimation, albeit at the cost of posing non-trivial computational challenges.

While we illustrate our framework by a monetary model with an imperfectly informed central bank, it is not limited to applications in monetary policy. In fact, section 3 shows that indeterminacy is a generic feature for a general class of economies where private-sector behavior is characterized by a set of expectational linear difference equations, exogenous driving processes are Gaussian, policy is described by a linear rule that responds to the policymaker's projections of economic conditions, and the projections are rational. A related issue is the choice of the information set. In our example, we endow the central bank with specific information sets. Alternatively, one could imagine a scenario where the policymaker chooses an optimal information set that minimizes the impact of sunspot shocks and possibly reduces the incidence of multiplicity. This direction has relevance for policy as central banks operate in a real-time environment fraught with measurement error and regularly face judgment calls on the importance of incoming data. An important extension should be to look beyond a given class of linear policy rules, as considered here, and model the optimal policy choice for a given set of preferences. Such an exercise could also consider how a desirable policy could be implemented with a suitable policy rule, which requires an analysis of equilibrium selection in the presence of indeterminacy.

Appendix

A Generic Indeterminacy in the Simple Example

Here we derive that $|\phi \cdot \kappa_\pi| < 1$ in the simple example of Section 2 with the policy rule (6), signal (5) and iid real rate shocks. The projection condition requires $\phi \cdot \kappa_\pi = \kappa_r$ where κ_r is the Kalman gain in the central bank's signal extraction about the real rate, $r_{t|t} = \kappa_r \left(\pi_t + \nu_t\right)$. As shown next, optimal signal extraction implies $|\kappa_r| < 1$ for any equilibrium where r_t cannot be perfectly inferred by the central bank.[46]

In the iid case we have $\rho = 0$ and $r_t = \varepsilon_t$, and the central bank's real-rate projections are given by $r_{t|t} = \kappa_r \left(\pi_t + \nu_t\right)$ with $\kappa_r = \operatorname{Cov}\left(r_t, \pi_t + \nu_t \,|\, Z^{t-1}\right) / \operatorname{Var}\left(\pi_t + \nu_t \,|\, Z^{t-1}\right)$. Furthermore, recall from (12) the equilibrium process for inflation:

$$\pi_{t+1} = -\left(r_t - r_{t|t}\right) + \gamma_\varepsilon \varepsilon_{t+1} + \gamma_\nu \nu_{t+1} + \gamma_b b_{t+1} \tag{12}$$

We now show that $|\kappa_r| < 1$ for any γ_ε, γ_ν, and γ_b. This carries over to $|\phi \cdot \kappa_\pi| < 1$ when the projection condition is satisfied so that $\phi \cdot \kappa_\pi = \kappa_r$. To begin, define the following variance ratio:

$$\begin{aligned}
R^2_\varepsilon &\equiv \frac{\operatorname{Var}\left(\varepsilon_{t|t}\right)}{\operatorname{Var}\left(\varepsilon_t\right)} \qquad\qquad \text{with} \quad 0 \leq R^2_\varepsilon \leq 1. && \text{(A.1)}\\
&= \frac{\operatorname{Cov}\left(\varepsilon_t, \pi_t + \nu_t\right)^2}{\operatorname{Var}\left(\pi_t + \nu_t\right) \cdot \operatorname{Var}\left(\varepsilon_t\right)} \\
&= \frac{\gamma^2_\varepsilon \cdot \sigma^2_\varepsilon}{\operatorname{Var}\left(\pi_t + \nu_t\right)} = \gamma_\varepsilon \cdot \kappa_r && \text{(A.2)}
\end{aligned}$$

where the second line follows from optimal signal extraction, and the last line from (12).[47] Based on (12) and (A.1) we also have

$$\operatorname{Var}\left(\pi_t + \nu_t\right) = \left(1 - R^2_\varepsilon\right) \cdot \sigma^2_\varepsilon + \gamma^2_\varepsilon \cdot \sigma^2_\varepsilon + \left(1 + \gamma_\nu\right)^2 \cdot \sigma^2_\nu + \gamma^2_b$$

and (A.2) can be rewritten as

$$R^2_\varepsilon \cdot \left(\left(1 - R^2_\varepsilon\right) \cdot \sigma^2_\varepsilon + \left(1 + \gamma_\nu\right)^2 \cdot \sigma^2_\nu + \gamma^2_b\right) = \left(1 - R^2_\varepsilon\right) \cdot \left(\gamma^2_\varepsilon \cdot \sigma^2_\varepsilon\right) \tag{A.3}$$

First, consider the case of $R^2_\varepsilon < 1$ so that the real rate is not perfectly revealed in equilibrium.

[46] When r_t cannot be perfectly inferred by the central bank we have $\operatorname{Var}\left(r_{t|t}\right) < \operatorname{Var}\left(r_t\right)$.

[47] Recall further that $r_t = \varepsilon_t$ in the iid case.

Equation (A.3) then implies the following:

$$\gamma_\varepsilon^2 = R_\varepsilon^2 \cdot \left(1 + \frac{(1+\gamma_\nu)^2 \cdot \sigma_\nu^2 + \gamma_b^2}{(1-R_\varepsilon^2)\cdot \sigma_\varepsilon^2}\right) \geq R_\varepsilon^2$$

and with (A.2) we have $\kappa_r^2 = R_\varepsilon^2 \cdot (R_\varepsilon^2/\gamma_\varepsilon^2) < 1$ and thus $|\kappa_r| < 1$.[48]

Second, for $R_\varepsilon^2 = 1$, we can also construct an alternative equilibrium with $|\kappa_r| = |\phi| > 1$. In this case, (A.3) requires $\gamma_\nu = -1$ and $\gamma_b = 0$, and it follows that $\pi_t = \bar{g} \cdot r_t - \nu_t$ with $\bar{g} = 1/\phi$ and thus $\kappa_r = \phi$. As discussed in the main text, this equilibrium generates an unstable root in the linear RE system characterizing the economy since we have $\kappa_\pi = 1$ and thus $|\phi \cdot \kappa_\pi| > 1$. But since the unstable root reflects the endogenous Kalman gain, it is not necessarily a unique equilibrium. Of note, this equilibrium is identical in outcomes to what is generated by the policy rule described in (10). The construction of this alternative equilibrium straightforwardly applies also in the case where the real rate is not iid but follows an AR(1), as in (2), with $\bar{g} = 1/(\phi - \rho)$.

B The Steady State Kalman Filter

This section describes details of the steady-state Kalman filter for the innovations state space (26) and (27) when Assumption 3 holds. Existence of a steady-state Kalman filter relies on finding an ergodic distribution for $\boldsymbol{S}_t^*$ (and thus $\tilde{\boldsymbol{S}}_t$) with constant second moments $\boldsymbol{\Sigma} \equiv \mathrm{Var}\left(\boldsymbol{S}_t^*\right)$. When a steady-state filter exists, a constant Kalman gain, $\boldsymbol{K}$ relates projected innovations of $\tilde{\boldsymbol{S}}_t$ to innovations in the signal, $\tilde{\boldsymbol{S}}_{t|t} = \boldsymbol{K}\tilde{\boldsymbol{Z}}_t$ with:[49]

$$\boldsymbol{K} = \mathrm{Cov}\left(\tilde{\boldsymbol{S}}_t, \tilde{\boldsymbol{Z}}_t\right)\left(\mathrm{Var}\left(\tilde{\boldsymbol{Z}}_t\right)\right)^{-1} = \left(\boldsymbol{A}\boldsymbol{\Sigma}\boldsymbol{C}' + \boldsymbol{B}\boldsymbol{D}'\right)\left(\boldsymbol{C}\boldsymbol{\Sigma}\boldsymbol{C}' + \boldsymbol{D}\boldsymbol{D}'\right)^{-1} .$$

The dynamics of $\boldsymbol{S}_t^*$ are then characterized by

$$\boldsymbol{S}_{t+1}^* = \left(\boldsymbol{A} - \boldsymbol{K}\boldsymbol{C}\right)\, \boldsymbol{S}_t^* + \left(\boldsymbol{B} - \boldsymbol{K}\boldsymbol{D}\right)\boldsymbol{w}_{t+1} \tag{B.1}$$

Existence of a steady-state filter depends on finding a symmetric, positive (semi) definite solution $\boldsymbol{\Sigma}$ to the following Riccati equation:

$$\begin{aligned}\boldsymbol{\Sigma} &= (\boldsymbol{A} - \boldsymbol{K}\boldsymbol{C})\boldsymbol{\Sigma}(\boldsymbol{A} - \boldsymbol{K}\boldsymbol{C})' + \boldsymbol{B}\boldsymbol{B}' \\ &= \boldsymbol{A}\boldsymbol{\Sigma}\boldsymbol{A}' + \boldsymbol{B}\boldsymbol{B}' - \left(\boldsymbol{A}\boldsymbol{\Sigma}\boldsymbol{C}' + \boldsymbol{B}\boldsymbol{D}'\right)\left(\boldsymbol{C}\boldsymbol{\Sigma}\boldsymbol{C}' + \boldsymbol{D}\boldsymbol{D}'\right)^{-1}\left(\boldsymbol{A}\boldsymbol{\Sigma}\boldsymbol{C}' + \boldsymbol{B}\boldsymbol{D}'\right)' .\end{aligned} \tag{B.2}$$

Intuitively, the Kalman filter seeks to construct mean-squared error optimal projections $\boldsymbol{S}_{t|t}$ that

[48] The inequality is strict, since $R_\varepsilon^2 < 1$ in this case.

[49] See also (29) in the paper.

minimize Σ. A necessary condition for the existence of a solution to this minimization problem is the ability to find at least some gain $\hat{\boldsymbol{K}}$ for which $\boldsymbol{A} - \hat{\boldsymbol{K}}\boldsymbol{C}$ is stable; otherwise, $\boldsymbol{S}^*$ would have unstable dynamics as can be seen from (B.1). Thus, existence of the second moment for the residuals, $\mathrm{Var}\left(\boldsymbol{S}_t^*\right) = \boldsymbol{\Sigma} \geq \boldsymbol{0}$, is synonymous with a stable transition matrix $\boldsymbol{A} - \boldsymbol{K}\boldsymbol{C}$.

Formal conditions for the existence of a time-invariant Kalman filter have been stated, among others, by Anderson & Moore (1979), Anderson, McGrattan, Hansen & Sargent (1996), Kailath, Sayed & Hassibi (2000), and Hansen & Sargent (2007). Necessary and sufficient conditions for the existence of a unique and stabilizing solution that is also positive semi-definite depend on the "detectability" and "unit-circle controllability" of certain matrices in our state space. We restate those concepts next.

DEFINITION 3 (Detectability) *A pair of matrices $(\boldsymbol{A}, \boldsymbol{C})$ is detectable when no right eigenvector of $\boldsymbol{A}$ that is associated with an unstable eigenvalue is orthogonal to the row space of $\boldsymbol{C}$. That is, there is no non-zero column vector $\boldsymbol{v}$ such that $\boldsymbol{A}\boldsymbol{v} = \boldsymbol{v}\lambda$ and $|\lambda| \geq 1$ with $\boldsymbol{C}\boldsymbol{v} = \boldsymbol{0}$.*

Detectability alone is already sufficient for the existence of *some* solution to the Riccati equation such that $\boldsymbol{A} - \boldsymbol{K}\boldsymbol{C}$ is stable; see (Kailath et al. 2000, Table E.1). Evidently, detectability is assured when $\boldsymbol{A}$ is a stable matrix, regardless of $\boldsymbol{C}$. To gain further intuition for the role of detectability, consider transforming $\boldsymbol{S}_t$ into "canonical variables" by premultiplying $\boldsymbol{S}_t$ with the matrix of eigenvectors of $\boldsymbol{A}$ — this transformation into canonical variables is at the heart of procedures for solving rational expectations models known from Blanchard & Kahn (1980), King & Watson (1998), Klein (2000), Sims (2002). Detectability then requires the signal equation (27) to provide some signal (i.e. to have non-zero loadings) for any unstable canonical variables.[50]

To establish existence of a solution to the Riccati equation that is unique and positive semi-definite, we follow Kailath et al. (2000) and require unit-circle controllability, defined as follows.

DEFINITION 4 (Unit-circle controllability) *The pair $(\boldsymbol{A}, \boldsymbol{B})$ is unit-circle controllable when no left-eigenvector of $\boldsymbol{A}$ associated with an eigenvalue on the unit circle is orthogonal to the column space of $\boldsymbol{B}$. That is, there is no non-zero row vector $\boldsymbol{v}$ such that $\boldsymbol{v}\boldsymbol{A} = \boldsymbol{v}\lambda$ with $|\lambda| = 1$ and $\boldsymbol{v}\boldsymbol{B} = \boldsymbol{0}$.*

In our state space, with $\boldsymbol{B}\boldsymbol{D}' \neq \boldsymbol{0}$, shocks to state and measurement equation are generally

[50] Specifically, let $\boldsymbol{A} = \boldsymbol{V}\boldsymbol{\Lambda}\boldsymbol{V}^{-1}$ with $\boldsymbol{\Lambda}$ diagonal be the eigenvalue-eigenvector factorization of $\boldsymbol{A}$ so that the columns of $\boldsymbol{V}$ correspond to the right eigenvectors of $\boldsymbol{A}$. Define canonical variables $\boldsymbol{S}_t^C \equiv \boldsymbol{V}^{-1}\boldsymbol{S}_t$. The signal equation can then be stated as $\boldsymbol{Z}_t = \boldsymbol{C}\boldsymbol{V}\boldsymbol{S}_t^C$ and detectability requires the signal equation to have non-zero loadings on at least every canonical variable associated with an unstable eigenvalue in $\boldsymbol{\Lambda}$.

correlated. Unit-circle controllability is thus applied to the following transformations of $\boldsymbol{A}$, $\boldsymbol{B}$:[51]

$$\boldsymbol{A}^C \equiv \boldsymbol{A} - \boldsymbol{B}\boldsymbol{D}' \left(\boldsymbol{D}\boldsymbol{D}'\right)^{-1} \boldsymbol{C} \qquad\qquad \boldsymbol{B}^C \equiv \boldsymbol{B}\left(\boldsymbol{I} - \boldsymbol{D}' \left(\boldsymbol{D}\boldsymbol{D}'\right)^{-1} \boldsymbol{D}\right)$$

Based on these definitions, the following theorem restates results from Kailath et al. (2000) in our notation:

THEOREM 3 (Stabilizing Solution to Riccati Equation) *Provided Assumption 3 holds, a stabilizing and positive semi-definite solution to the Riccati equation* (B.2) *exists when* $(\boldsymbol{A}^C, \boldsymbol{B}^C)$ *is unit-circle controllable and* $(\boldsymbol{A}, \boldsymbol{C})$ *is detectable. The steady-state Kalman gain is such that* $\boldsymbol{A} - \boldsymbol{K}\boldsymbol{C}$ *is a stable matrix; moreover, the stabilizing solution is unique.*[52]

Proof. See Theorem E.5.1 of Kailath et al. (2000); related results are also presented in Anderson et al. (1996), or Chapter 4 of Anderson & Moore (1979). ■

In our context, with $\boldsymbol{C} = \boldsymbol{H}\boldsymbol{A}$ and $\boldsymbol{D} = \boldsymbol{H}\boldsymbol{B}$, the conditions for detectability and unit-circle controllability can also be restated as follows.

PROPOSITION 5 (Detectability of $(\boldsymbol{A}, \boldsymbol{H})$) *With* $\boldsymbol{C} = \boldsymbol{H}\boldsymbol{A}$*, detectability of* $(\boldsymbol{A}, \boldsymbol{C})$ *is equivalent to detectability of* $(\boldsymbol{A}, \boldsymbol{H})$

Proof. When $(\boldsymbol{A}, \boldsymbol{C})$ are detectable, we have $\boldsymbol{C}\boldsymbol{v} \neq \boldsymbol{0}$ for any right-eigenvector of $\boldsymbol{A}$ associated with an eigenvalue λ on or outside the unit circle, $|\lambda| \geq 1$. With $\boldsymbol{C} = \boldsymbol{H}\boldsymbol{A}$ we then also have $\boldsymbol{C}\boldsymbol{v} = \boldsymbol{H}\boldsymbol{A}\boldsymbol{v} = \boldsymbol{H}\boldsymbol{v}\lambda \neq 0 \Leftrightarrow \boldsymbol{H}\boldsymbol{v} \neq 0$ ■

Furthermore, with $\boldsymbol{C} = \boldsymbol{H}\boldsymbol{A}$ and $\boldsymbol{D} = \boldsymbol{H}\boldsymbol{B}$, the above expressions for $\boldsymbol{A}^C$ and $\boldsymbol{B}^C$ can be transformed as follows:

$$\boldsymbol{A}^C = (\boldsymbol{I} - \boldsymbol{P}^C)\boldsymbol{A} \quad \text{and} \quad \boldsymbol{B}^C = (\boldsymbol{I} - \boldsymbol{P}^C)\boldsymbol{B} \quad \text{with} \quad \boldsymbol{P}^C \equiv \boldsymbol{B}\boldsymbol{H}' \left(\boldsymbol{H}\boldsymbol{B}\boldsymbol{B}'\boldsymbol{H}'\right)^{-1} \boldsymbol{H}. \tag{B.3}$$

$\boldsymbol{P}^C$ is a non-symmetric, idempotent projection matrix with $\boldsymbol{H}\boldsymbol{P}^C = \boldsymbol{H}$.[53]

PROPOSITION 6 (Unit-circle controllability of $(\boldsymbol{A}(\boldsymbol{I} - \boldsymbol{P}^C), \boldsymbol{B})$) *With* $\boldsymbol{C} = \boldsymbol{H}\boldsymbol{A}$ *and* $\boldsymbol{D} = \boldsymbol{H}\boldsymbol{B}$*, unit-circle controllability of* $(\boldsymbol{A}^C, \boldsymbol{B}^C)$ *is equivalent to unit-circle controllability of* $(\boldsymbol{A}(\boldsymbol{I} - \boldsymbol{P}^C), \boldsymbol{B})$ *with* $\boldsymbol{P}^C$ *defined in* (B.3).

[51]Notice that $\boldsymbol{B}^C = \boldsymbol{B}\boldsymbol{\mathcal{M}}^D$ where $\boldsymbol{\mathcal{M}}^D = \boldsymbol{I} - \boldsymbol{D}' \left(\boldsymbol{D}\boldsymbol{D}'\right)^{-1} \boldsymbol{D}$ is a projection matrix, which is symmetric and idempotent, $\boldsymbol{\mathcal{M}}^D = \boldsymbol{\mathcal{M}}^D\boldsymbol{\mathcal{M}}^D$, and orthogonal to the row space of $\boldsymbol{D}$.To appreciate the role of $\mathcal{M}^D$, consider the following thought experiment: $\boldsymbol{\mathcal{M}}^D$ construct the residual in projecting the shocks of the system *off* the shocks in the signal equation, $\boldsymbol{w}_t - E(\boldsymbol{w}_t|\boldsymbol{D}\boldsymbol{w}_t) = \boldsymbol{\mathcal{M}}^D\boldsymbol{w}_t$.

[52]There may be other, non-stabilizing positive semi-definite solutions.

[53]An idempotent matrix is equal to its own square, that is $\boldsymbol{P}^C = \boldsymbol{P}^C\boldsymbol{P}^C$, and the eigenvalues of an idempotent matrix are either zero or one and we have $|\boldsymbol{P}^C| = \boldsymbol{0}$.

Proof. Suppose $(\boldsymbol{A}^C, \boldsymbol{B}^C)$ are unit-circle controllable. Let $\tilde{\boldsymbol{v}} \equiv \boldsymbol{v}(\boldsymbol{I} - \boldsymbol{P}^C)$ and note that left-eigenvectors of $\boldsymbol{A}^C$ associated with eigenvalues on the unit circle cannot be orthogonal to $\boldsymbol{P}^C$ (otherwise we would have $\boldsymbol{v}\boldsymbol{A}^C = \boldsymbol{0}$). Accordingly, $\boldsymbol{v}\boldsymbol{A}^C = \boldsymbol{v}\lambda$ with $|\lambda| = 1$, $\boldsymbol{v}\boldsymbol{B}^C \neq \boldsymbol{0}$ and $\boldsymbol{v} \neq \boldsymbol{0}$ is equivalent to $\tilde{\boldsymbol{v}}\boldsymbol{A}(\boldsymbol{I} - \boldsymbol{P}^C) = \tilde{\boldsymbol{v}}\lambda$ with $|\lambda| = 1$, $\tilde{\boldsymbol{v}} \neq \boldsymbol{0}$ $\tilde{\boldsymbol{v}}\boldsymbol{B} \neq \boldsymbol{0}$. The converse reasoning applies as well. ■

As discussed in the main text, an upshot of Proposition 6 is that a sufficient condition for unit-circle controllability of $(\boldsymbol{A}^C, \boldsymbol{B}^C)$ is for $\boldsymbol{B}$ to have full rank.

Finally, for convenience, we define the the joint concept of detectability and unit-circle controllability for the triplet $(\boldsymbol{A}, \boldsymbol{B}, \boldsymbol{H})$.

DEFINITION 5 (Joint detectability and unit-circle controllability) *The triplet $\boldsymbol{A}, \boldsymbol{B}, \boldsymbol{H}$ is detectable and unit-circle controllable when $(\boldsymbol{A}, \boldsymbol{H})$ is detectable and $(\boldsymbol{A}(\boldsymbol{I} - \boldsymbol{P}^C), \boldsymbol{B})$ is unit-circle controllable, where $\boldsymbol{P}^C$ is defined in* (B.3).

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