

CS4495/6495

Introduction to Computer Vision

2C-L2 *Convolution in frequency domain*

Fourier Transform and Convolution

Let $g = f * h$

Fourier Transform and Convolution

Then $G(u) = \int_{-\infty}^{\infty} g(x) e^{-i2\pi ux} dx$

$$= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(\tau) h(x - \tau) e^{-i2\pi ux} d\tau dx$$

$$= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \left[f(\tau) e^{-i2\pi u\tau} d\tau \right] \left[h(x - \tau) e^{-i2\pi u(x - \tau)} dx \right]$$

$$= \int_{-\infty}^{\infty} \left[f(\tau) e^{-i2\pi u\tau} d\tau \right] \int_{-\infty}^{\infty} \left[h(x') e^{-i2\pi ux'} dx' \right]$$

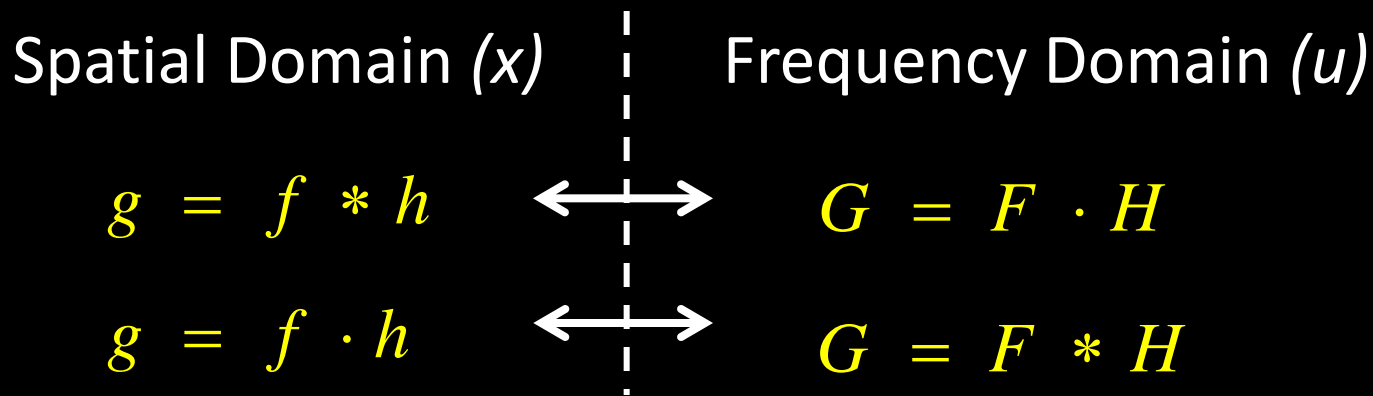
$$= F(u) H(u)$$

Convolution in spatial domain



Multiplication in frequency domain

Fourier Transform and Convolution



Fourier Transform and Convolution

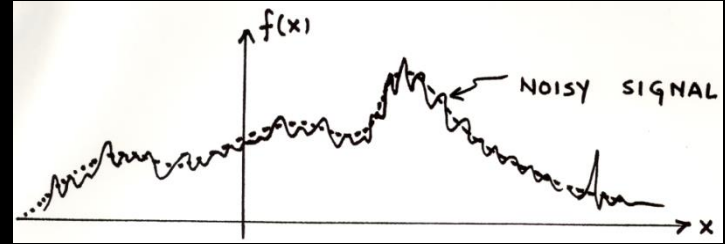
So, we can find $g(x)$ by Fourier transform

$$\begin{array}{ccccc} g & = & f & * & h \\ \uparrow & & \downarrow & & \downarrow \\ \boxed{\text{IFT}} & & \boxed{\text{FT}} & & \boxed{\text{FT}} \\ \downarrow & & \uparrow & & \uparrow \\ G & = & F & * & H \end{array}$$

Example use: Smoothing/Blurring

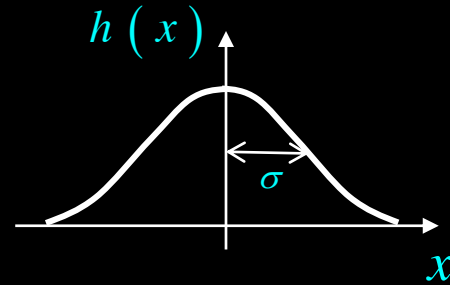
- We want a smoothed function of $f(x)$

$$g(x) = f(x) * h(x)$$



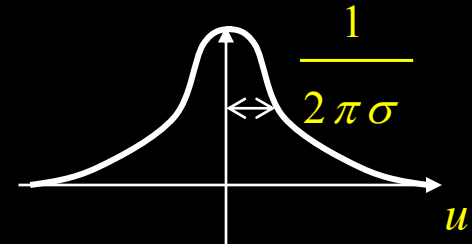
- Let us use a Gaussian kernel

$$h(x) = \frac{1}{\sqrt{2\pi}\sigma} \exp\left[-\frac{1}{2} \frac{x^2}{\sigma^2}\right]$$



- The Fourier transform of a Gaussian is a Gaussian

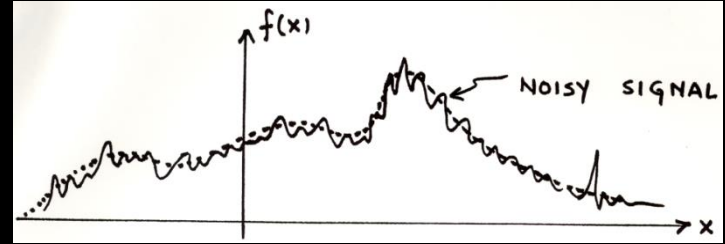
$$H(u) = \exp\left[-\frac{1}{2} (2\pi u)^2 \sigma^2\right]$$



Example use: Smoothing/Blurring

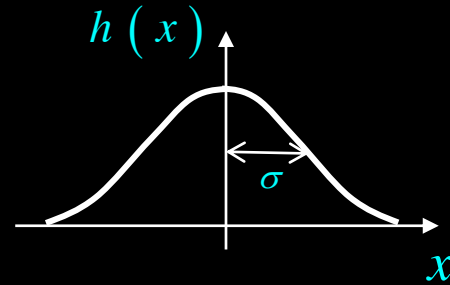
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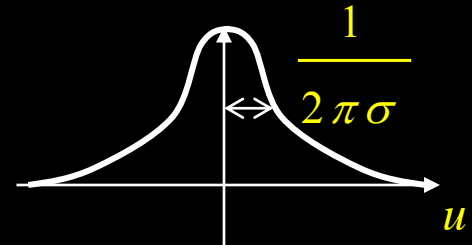
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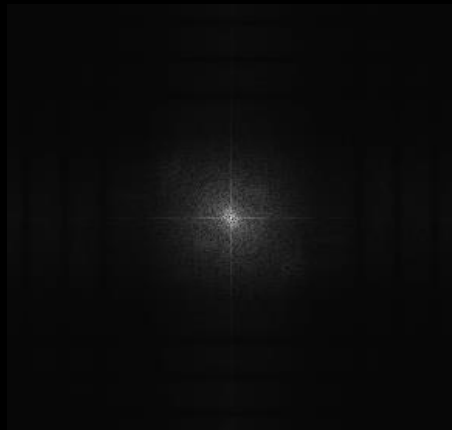
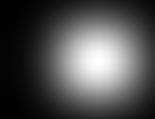
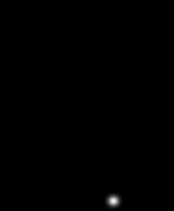


- Convolution in space is multiplication in freq: $H(u)$

$$G(u) = F(u) H(u)$$

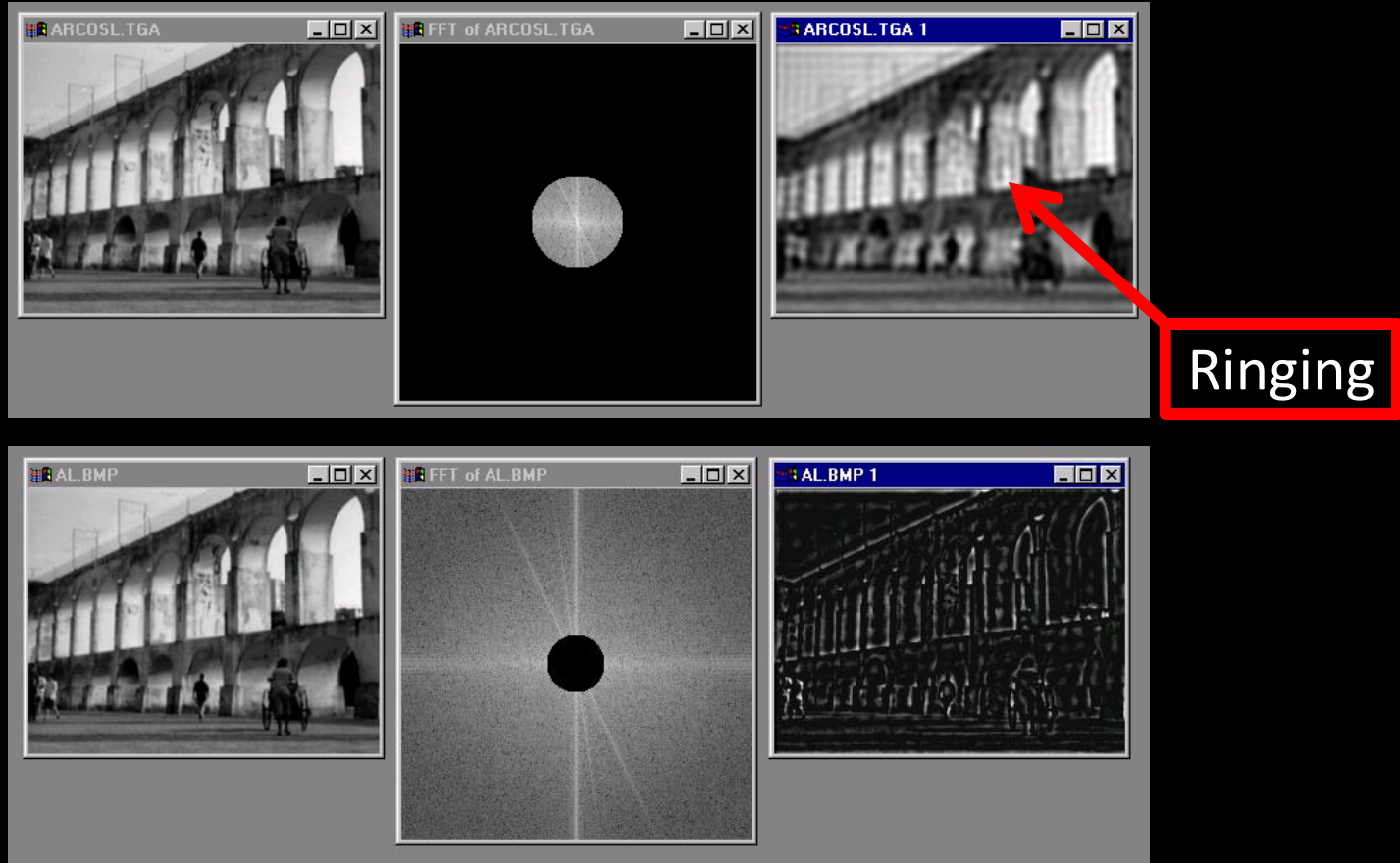


$$f(x,y) * h(x,y) = g(x,y)$$



$$|F(u,v)| \times |H(s_x, s_y)| \Rightarrow |G(s_x, s_y)|$$

Low and High Pass filtering



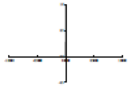
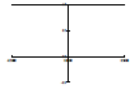
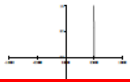
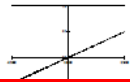

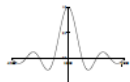
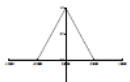

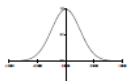


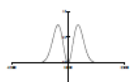
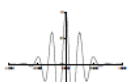
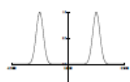

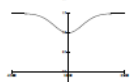


Properties of Fourier Transform

	Spatial Domain (x)	Frequency Domain (u)
Linearity	$c_1 f(x) + c_2 g(x)$	$c_1 F(u) + c_2 G(u)$
Convolution	$f(x) * g(x)$	$F(u) G(u)$
Scaling	$f(ax)$	$\frac{1}{ a } F\left(\frac{u}{a}\right)$
Differentiation	$\frac{d^n f(x)}{dx^n}$	$(i2\pi u)^n F(u)$

Diagram illustrating the mapping between Spatial Domain (x) and Frequency Domain (u) for various properties:

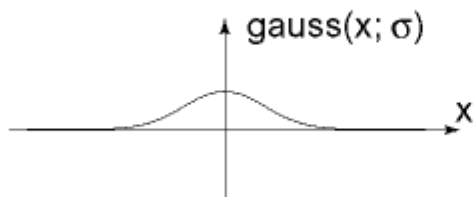
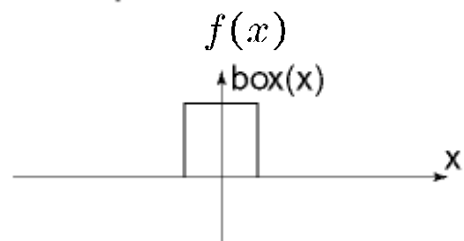
- Scaling:** A function $f(ax)$ in the Spatial Domain is transformed to $\frac{1}{|a|} F\left(\frac{u}{a}\right)$ in the Frequency Domain. This is labeled "Shrink" (Spatial) and "Stretch" (Frequency).
- Differentiation:** A function $\frac{d^n f(x)}{dx^n}$ in the Spatial Domain is transformed to $(i2\pi u)^n F(u)$ in the Frequency Domain. This is labeled "Differentiate" (Spatial) and "Multiply by u " (Frequency).

Fourier Pairs (from Szeliski)

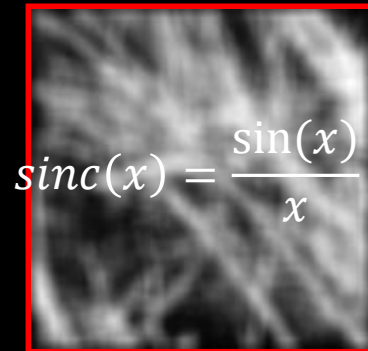
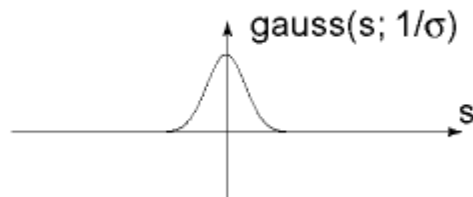
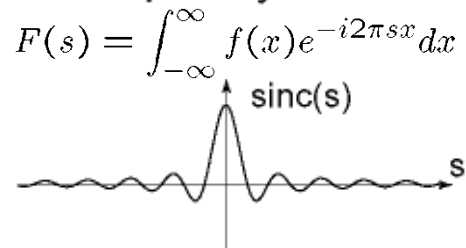
Name	Signal	Transform
impulse	 $\delta(x)$	\Leftrightarrow 1 
shifted impulse	 $\delta(x - u)$	\Leftrightarrow $e^{-j\omega u}$ 
box filter	 $\text{box}(x/a)$	\Leftrightarrow $a\text{sinc}(a\omega)$ 
tent	 $\text{tent}(x/a)$	\Leftrightarrow $a\text{sinc}^2(a\omega)$ 
Gaussian	 $G(x; \sigma)$	\Leftrightarrow $\frac{\sqrt{2\pi}}{\sigma} G(\omega; \sigma^{-1})$ 
Laplacian of Gaussian	 $(\frac{x^2}{\sigma^4} - \frac{1}{\sigma^2})G(x; \sigma)$	\Leftrightarrow $-\frac{\sqrt{2\pi}}{\sigma} \omega^2 G(\omega; \sigma^{-1})$ 
Gabor	 $\cos(\omega_0 x)G(x; \sigma)$	\Leftrightarrow $\frac{\sqrt{2\pi}}{\sigma} G(\omega \pm \omega_0; \sigma^{-1})$ 
unsharp mask	 $(1 + \gamma)\delta(x) - \gamma G(x; \sigma)$	\Leftrightarrow $(1 + \gamma) - \frac{\sqrt{2\pi}\gamma}{\sigma} G(\omega; \sigma^{-1})$ 
windowed sinc	 $\text{rcos}(x/(aW)) \text{sinc}(x/a)$	\Leftrightarrow (see Figure 3.29) 

Fourier Transform smoothing pairs

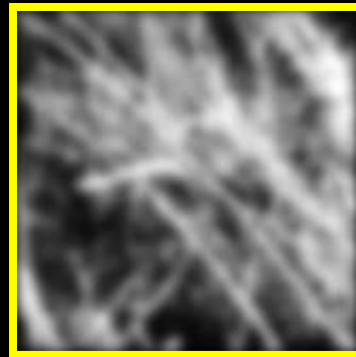
Spatial domain



Frequency domain

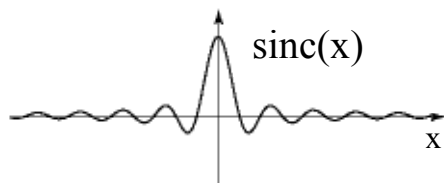
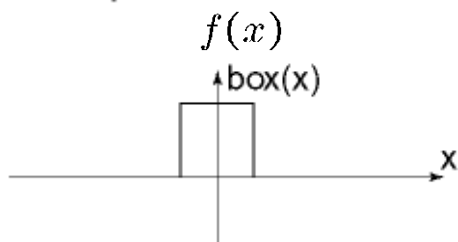


$$\text{sinc}(x) = \frac{\sin(x)}{x}$$



Fourier Transform smoothing pairs

Spatial domain



Frequency domain

