CS4495/6495
Introduction to Computer Vision

2C-L1 *Fourier transform*

What do you see?

Salvador Dali
"Gala Contemplating the Mediterranean Sea, which at 30 meters becomes the portrait of Abraham Lincoln", 1976

What do you see?

Salvador Dali
"Gala Contemplating the Mediterranean Sea, which at 30 meters becomes the portrait of Abraham Lincoln", 1976

Decomposing an image: *basis sets*

A basis set is (edit from Wikipedia):

- A basis B of a vector space V is a linearly independent subset of V that spans V.

Decomposing an image: *basis sets*

Suppose that $B = \{ \boldsymbol{v_1}, \ldots, \boldsymbol{v_n} \}$ is a finite subset of a vector space V over a field F (such as the real or complex numbers R or C). Then B is a basis if it satisfies the following conditions:

- Linear independence:

  For all $a_1, \ldots, a_n \in F$, if $a_1\boldsymbol{v}_1 + \ldots + a_n\boldsymbol{v}_n = \mathbf{0}$, then necessarily

  $\mathrm{a}_1 = \cdots = a_n = 0$

Decomposing an image: *basis sets*

Suppose that $B = \{ \boldsymbol{v_1}, \dots, \boldsymbol{v_n} \}$ is a finite subset of a vector space V over a field F (such as the real or complex numbers R or C). Then B is a basis if it satisfies the following conditions:

- Spanning property,
  - For every $\mathbf{x}$ in V it is possible to choose $a_1, \dots, a_n \in F$ such that $\mathbf{x} = a_1\boldsymbol{v}_1 + \dots + a_n\boldsymbol{v}_n = \mathbf{0}$.

Decomposing an image: *basis sets*

Suppose that $B = \{ \boldsymbol{v_1}, \dots, \boldsymbol{v_n} \}$ is a finite subset of a vector space V over a field F (such as the real or complex numbers R or C). Then B is a basis if it satisfies the following conditions:

- Not necessarily orthogonal… but helpful if they are. (Why?)

Images as points in a vector space

- Consider an image as a point in a NxN size space – can rasterize into a single vector

$$[x_{00} x_{10} x_{20} \ldots x_{(n-1)0} x_{10} \ldots x_{(n-1)(n-1)}]^T$$

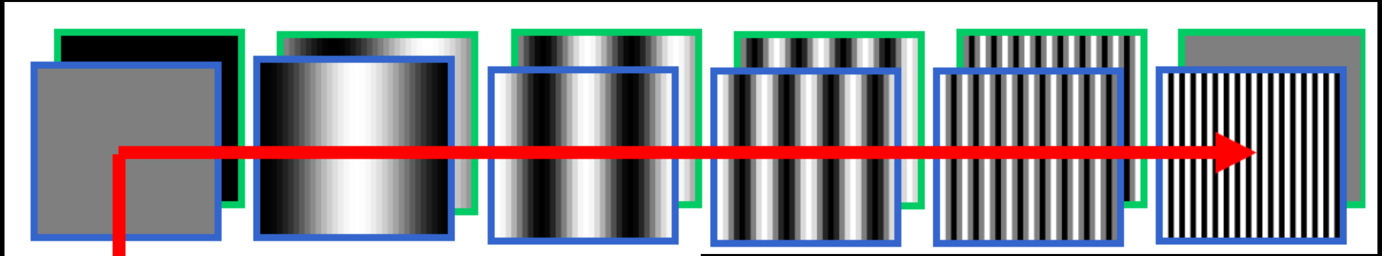
- The "normal" basis is just the vectors:

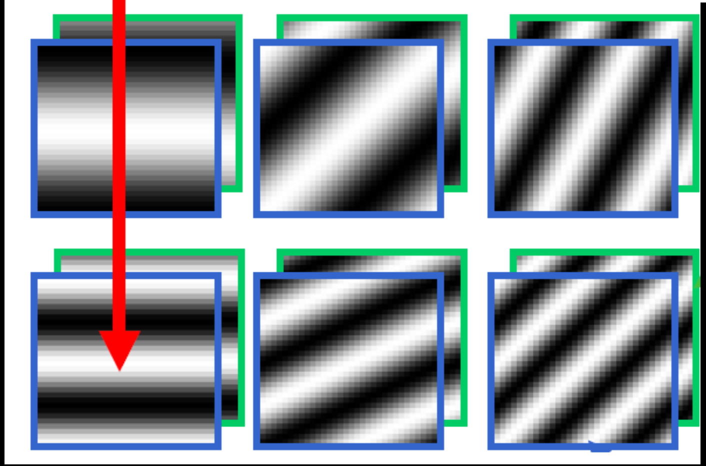
$$[0\ 0\ 0\ 0 \ldots 0 1\ 0\ 0\ 0\ 0\ 0\ 0\ 0 \ldots\ 0]^T$$

  - Independent
  - Can create any image
  - Not very helpful…

A nice set of basis

Teases away fast vs. slow changes in the image.





Jean Baptiste Joseph Fourier (1768-1830)

Had crazy idea (1807):

- Any periodic function can be rewritten as a weighted sum of sines and cosines of different frequencies.

Jean Baptiste Joseph Fourier (1768-1830)

Don’t believe it?

- Neither did Lagrange, Laplace, Poisson and other big wigs
- Not translated into English until 1878!

Jean Baptiste Joseph Fourier (1768-1830)

But it’s true!

- Called Fourier Series

A sum of sines

Our building block:

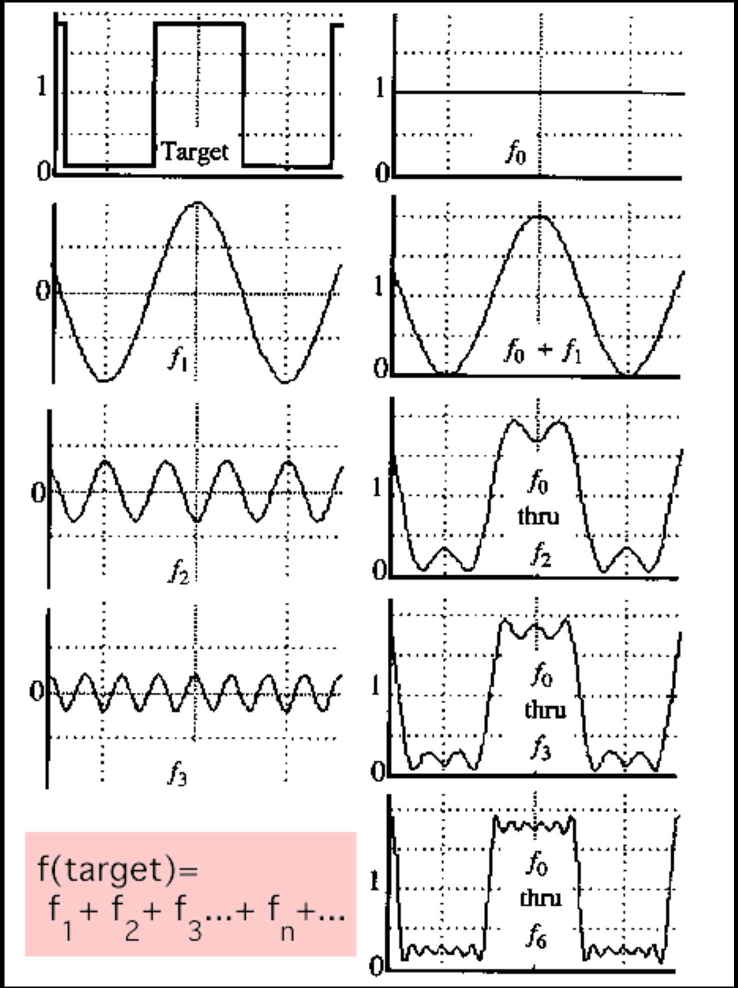
$$A \sin(\omega x + \varphi)$$

Add enough of them to get any signal *f(x)* you want!

How many degrees of freedom?

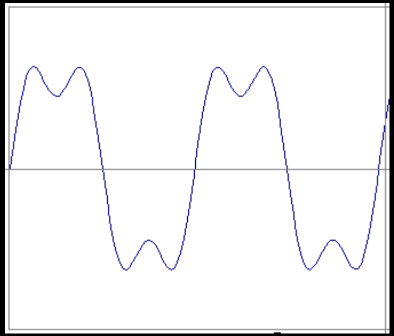
What does each control?

Which one encodes the coarse vs. fine structure of the signal?



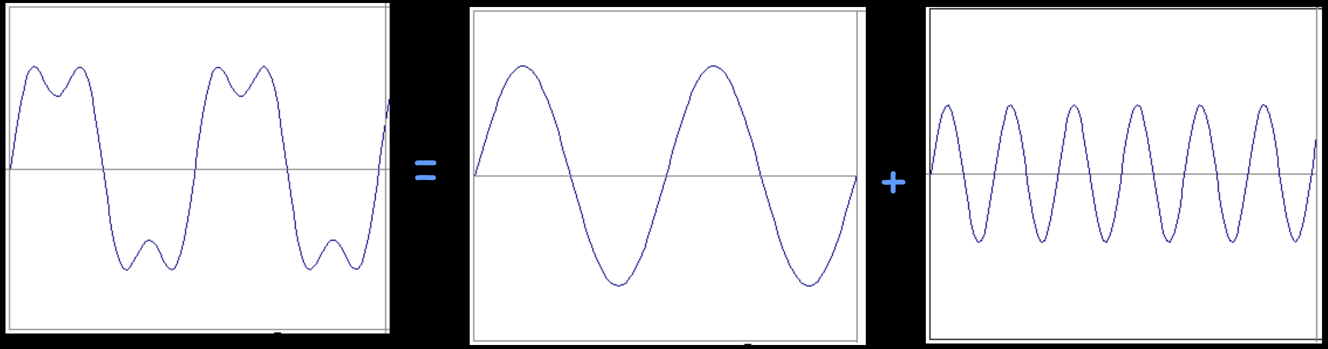
Time and Frequency

- Example: $g(t) = \sin(2\pi f t) + \frac{1}{3}\sin(2\pi(3f)t)$



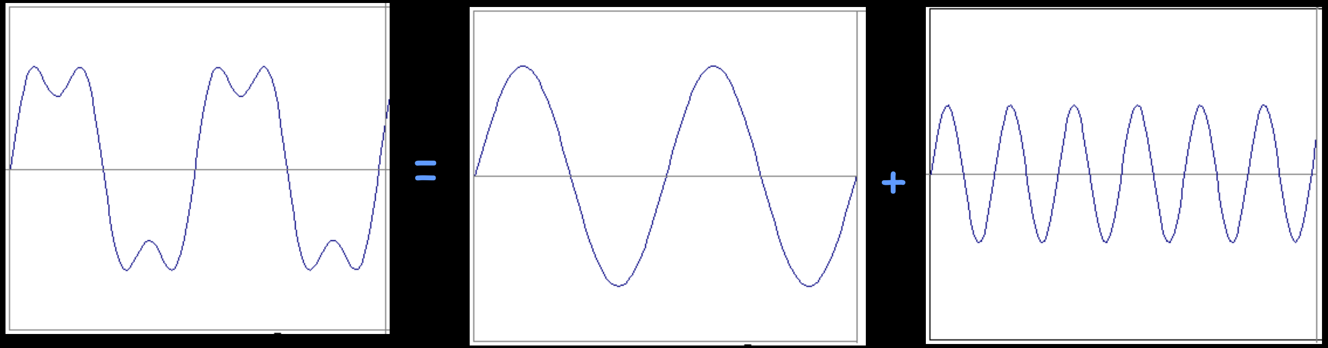
Time and Frequency

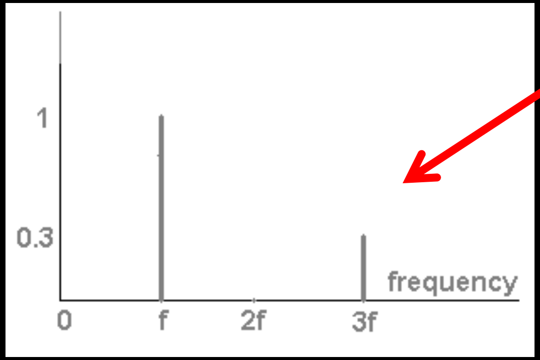
- Example: $g(t) = \sin(2\pi f t) + \frac{1}{3}\sin(2\pi(3f)t)$



Frequency Spectra - Series

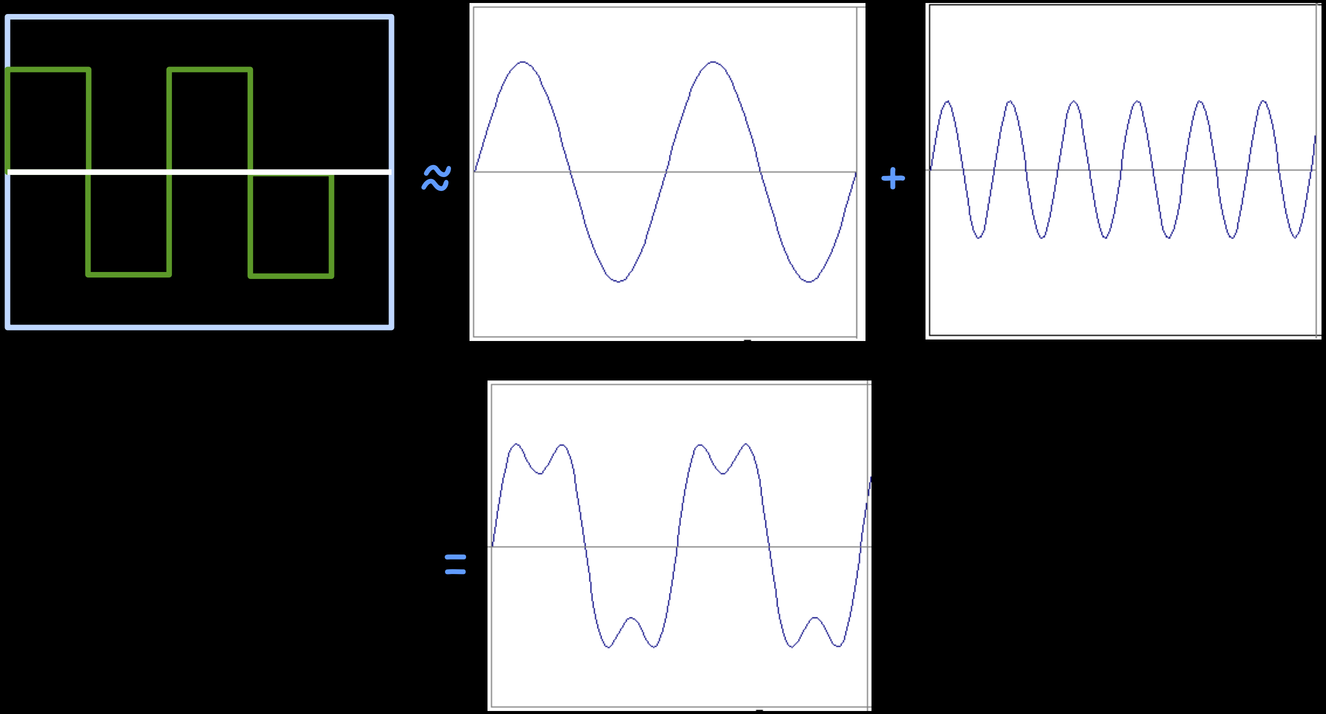
- Example: $g(t) = \sin(2\pi f t) + \frac{1}{3}\sin(2\pi(3f)t)$



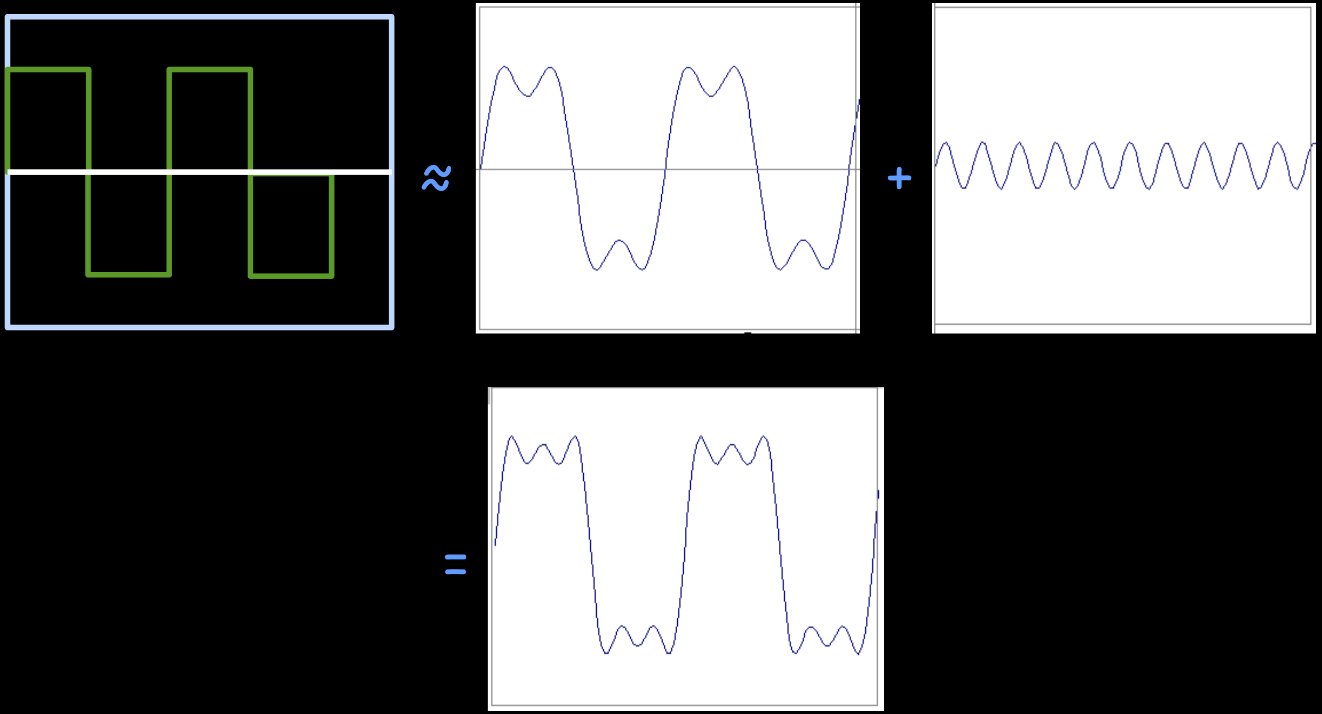


One form of *spectrum* – more in a bit

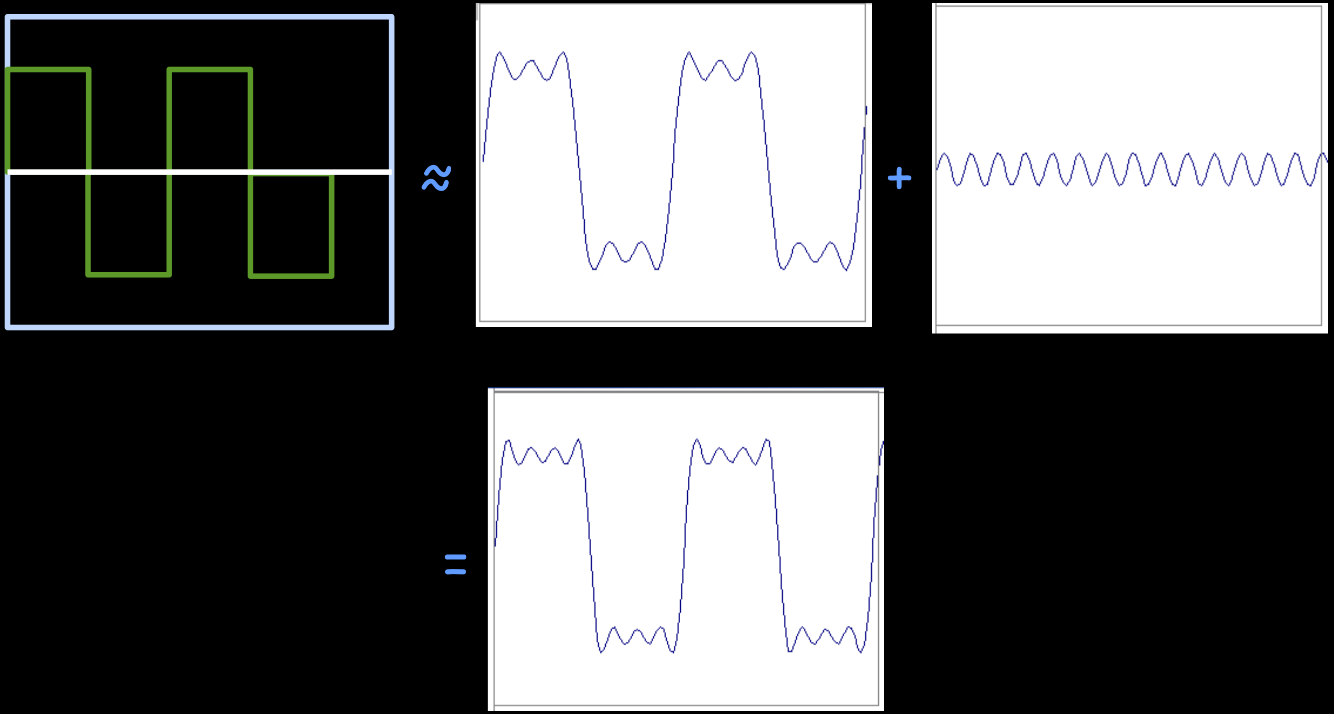
Frequency Spectra - Series



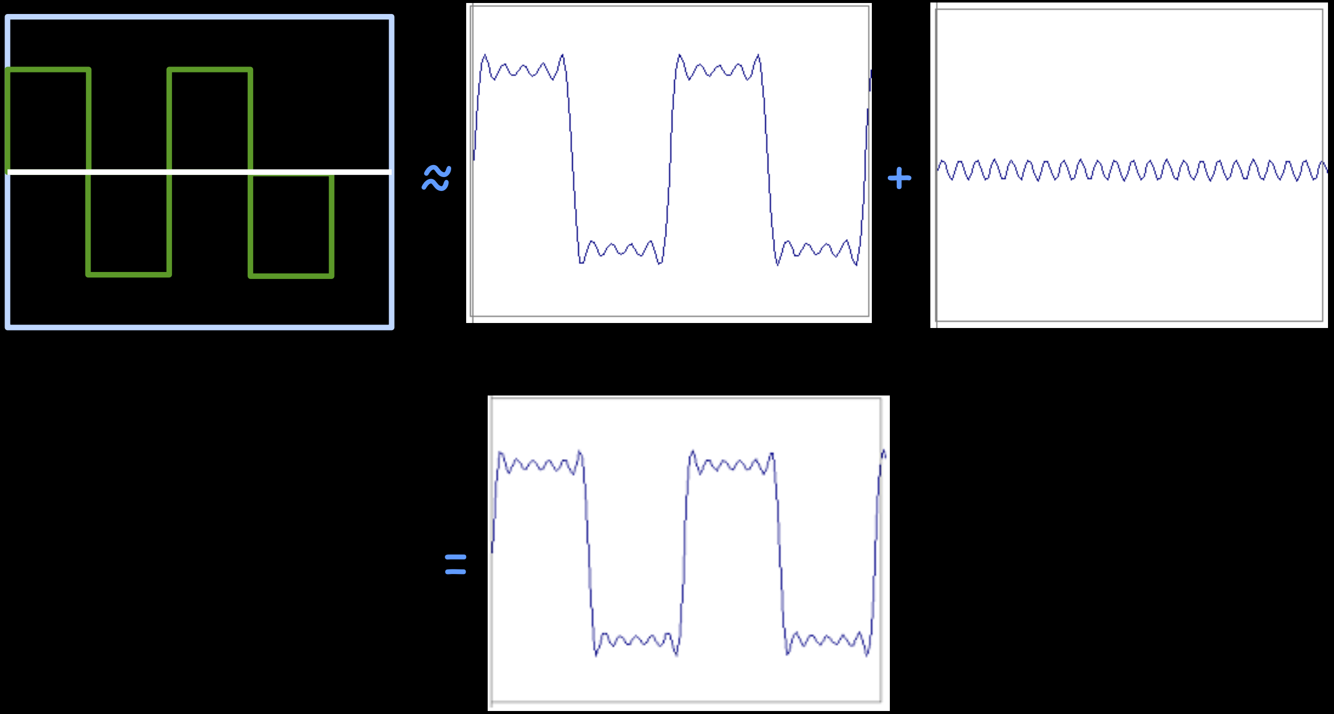
Frequency Spectra - Series



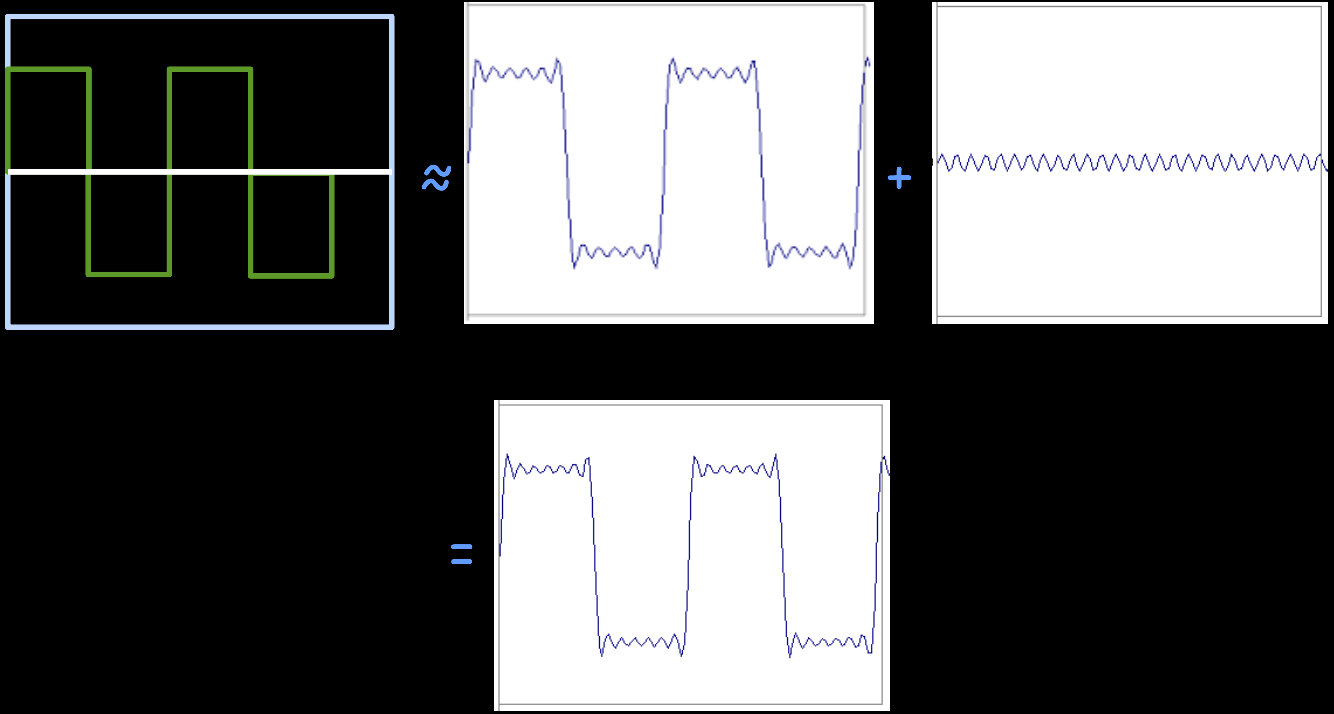
Frequency Spectra - Series



Frequency Spectra - Series

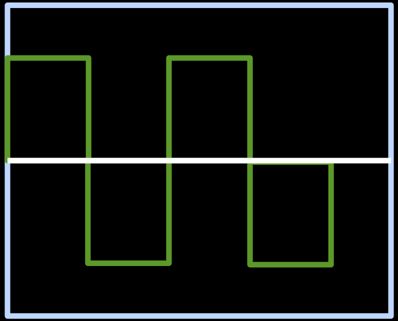


Frequency Spectra - Series

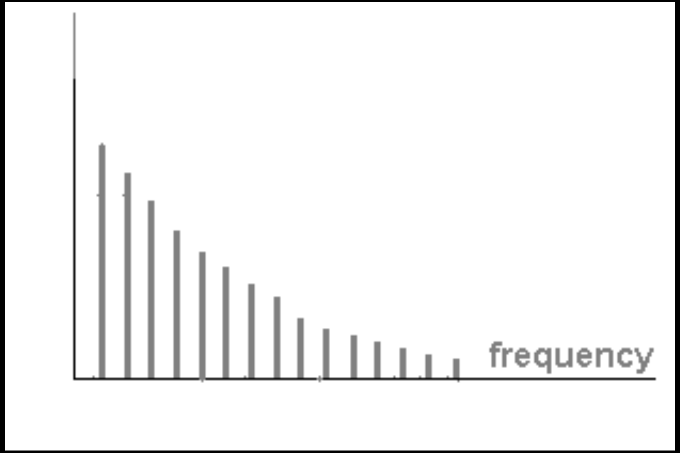


Frequency Spectra - Series

$$A\sum_{k=1}^{\infty}\frac{1}{k}\sin(2\pi kt)$$



=



Usually, frequency magnitude is more interesting than the phase for CV because we're not reconstructing the image

Fourier *Transform*

We want to understand the frequency ω of our signal.
So, let's reparametrize the signal by ω instead of x:

$$A\sin(\omega x + \varphi)$$

$$\boldsymbol{f}(\boldsymbol{x}) \longrightarrow \boxed{\text{Fourier Transform}} \longrightarrow \boldsymbol{F}(\boldsymbol{\omega})$$

For every ω from 0 to ∞ (actually $-\infty$ to ∞), $F(\omega)$ holds the amplitude A and phase f of the corresponding sinusoid

Fourier *Transform*

We want to understand the frequency ω of our signal.

So, let's reparametrize the signal by ω instead of *x*:

$$A \sin(\omega x + \varphi)$$

$$\boldsymbol{f(x)} \longrightarrow \boxed{\text{Fourier Transform}} \longrightarrow \boldsymbol{F(\omega)}$$

How can *F* hold both amplitude and phase?

Complex number trick!

Fourier *Transform*

We want to understand the frequency ω of our signal.

So, let's reparametrize the signal by ω instead of *x*:

$$A \sin(\omega x + \varphi)$$

$$\boldsymbol{f}(\boldsymbol{x}) \longrightarrow \boxed{\text{Fourier Transform}} \longrightarrow \boldsymbol{F}(\boldsymbol{\omega})$$

$$F(\omega) = \underbrace{R(\omega)}_{\textit{Even}} + i\underbrace{I(\omega)}_{\textit{Odd}}$$

$$A = \pm\sqrt{R(\omega)^2 + I(\omega)^2}$$

$$\varphi = \tan^{-1}\frac{I(\omega)}{R(\omega)}$$

Computing FT: Just a basis

The infinite integral of the product of two sinusoids of *different* frequency is zero. (Why?)

$$\int_{-\infty}^{\infty} \sin(ax+\phi)\sin(bx+\varphi)\,dx = 0, \text{ if } a \neq b$$

And if same frequency the integral is infinite:

$$\int_{-\infty}^{\infty} \sin(ax+\phi)\sin(ax+\varphi)\,dx = \pm\infty$$

.... if ϕ and φ not exactly pi/2 out of phase (sin and cos).

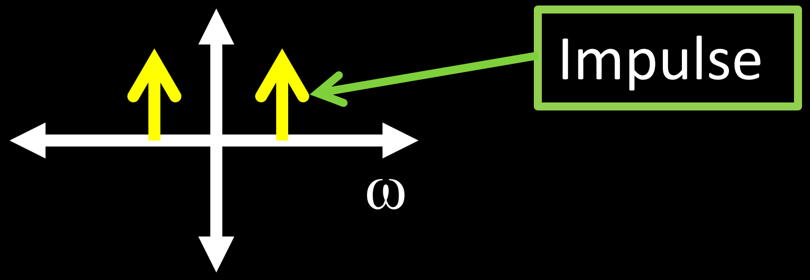
Computing FT: Just a basis

So, suppose f(x) is a cosine wave of freq ω:

$$f(x) = \cos(2\pi\omega x)$$

Then: $$C(u) = \int_{-\infty}^{\infty} f(x)\cos(2\pi u x)\,dx$$

… which is infinite if *u* is equal to ω (or - ω) and zero otherwise:



Computing FT: Just a basis

- We can do that for all frequencies *u.*
- But we'd have to do that for all *phases,* don't we???
- No! Any phase can be created by a weighted sum of cosine and sine. Only need each piece:

$$C(u) = \int_{-\infty}^{\infty} f(x)\cos(2\pi ux)\,dx$$

$$S(u) = \int_{-\infty}^{\infty} f(x)\sin(2\pi ux)\,dx$$

- Sinusoid demo?

Fourier Transform – more formally

Represent the signal as an infinite weighted sum of an infinite number of sinusoids:

$$F(u) = \int_{-\infty}^{\infty} f(x) e^{-i 2 \pi u x} dx$$

Again: $e^{ik} = \cos k + i \sin k \qquad i = \sqrt{-1}$

Spatial Domain (x) $\longrightarrow$ Frequency Domain (ω or u or *even* s)
(Frequency Spectrum $F(u)$ or $F(\omega)$)

Fourier Transform – more formally

Inverse Fourier Transform (IFT) – add up all the sinusoids at x:

$$f(x) = \int_{-\infty}^{\infty} F(u)\, e^{i 2 \pi u x}\, du$$

Fourier Transform - limitations

- The integral $\int_{-\infty}^{\infty} f(x)e^{-i2\pi ux}dx$ exists if the function f is integrable:

$$\int_{-\infty}^{\infty} |f(x)|dx < \infty$$

- If there is a bound of width T outside of which f is zero then obviously could integrate from just $-T/2$ to $T/2$

Fourier Transform ⇔ Fourier Series

- The ***Discrete FT***:

$$F(k) = \frac{1}{N} \sum_{x=0}^{x=N-1} f(x) e^{-i\frac{2\pi kx}{N}}$$

… where x is discrete and goes from the start of the signal to the end *(N-1)*

… and *k* is the number “cycles per period of the signal” or “cycles per image.

  - Only makes sense k = $-N/2$ to $N/2$. Why? What’s the highest frequency you can unambiguously have in a discrete image?

2D Fourier Transforms

- The two dimensional version: .

$$F(u,v) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\, e^{-i2\pi(ux+vy)}\, dx\, dy \ \frac{1}{2}$$

- And the 2D ***Discrete FT***:

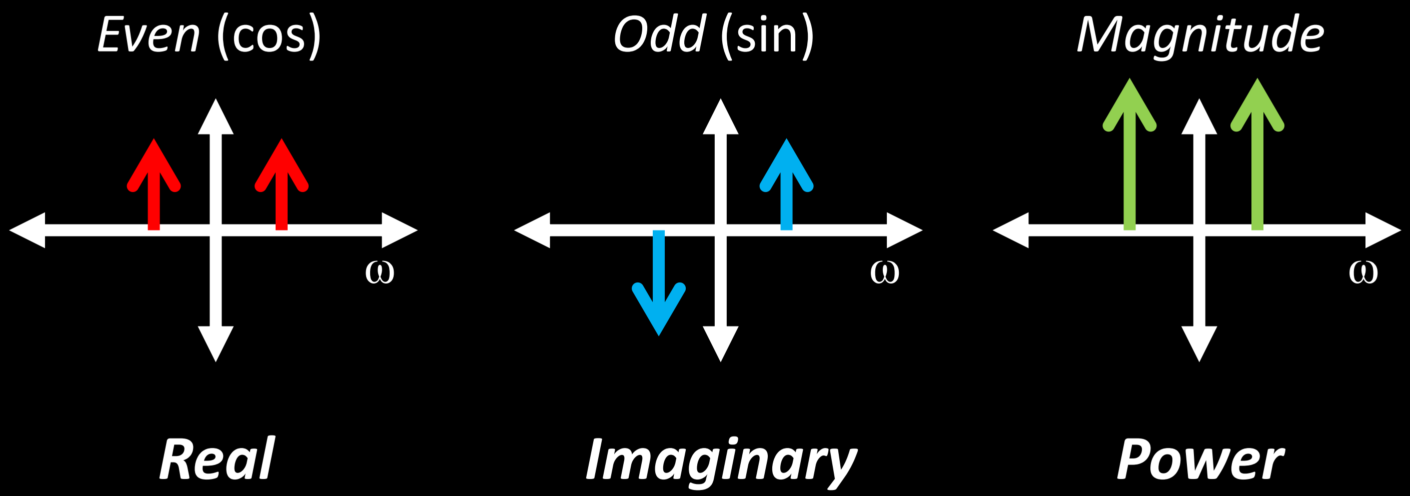
$$F(k_x, k_y) = \frac{1}{N} \sum_{x=0}^{x=N-1} \sum_{y=0}^{y=N-1} f(x,y)\, e^{-i\frac{2\pi(k_x x + k_y y)}{N}}$$

- Works best when you put the origin of *k* in the middle….

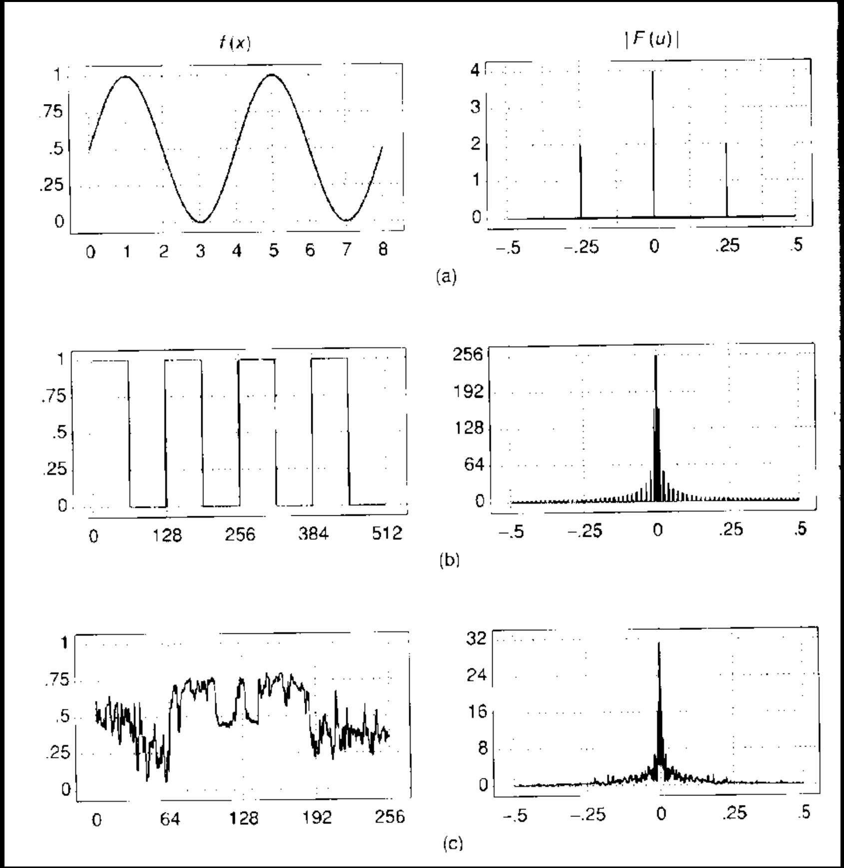
Frequency Spectra – Even/Odd

Frequency actually goes from $-\infty$ to $+\infty$

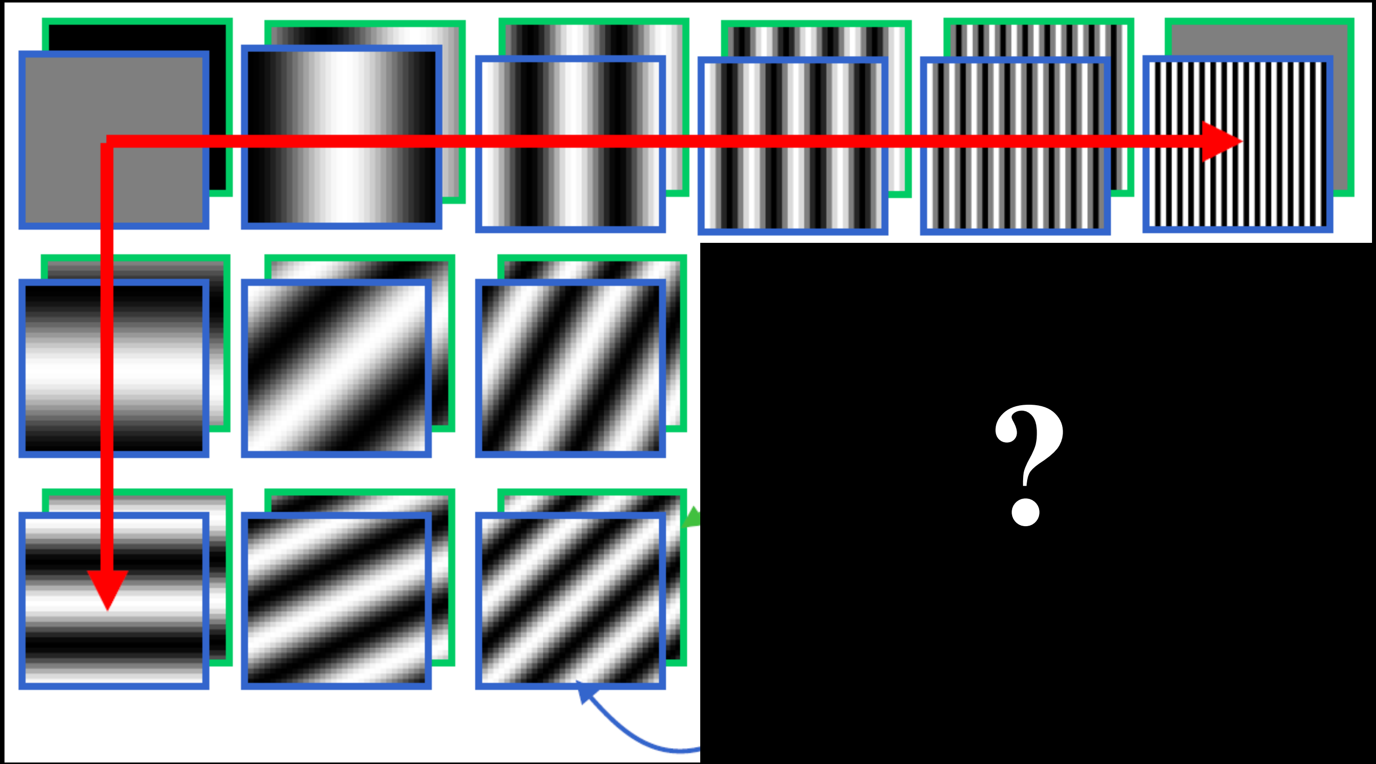
Sinusoid example:



Frequency Spectra

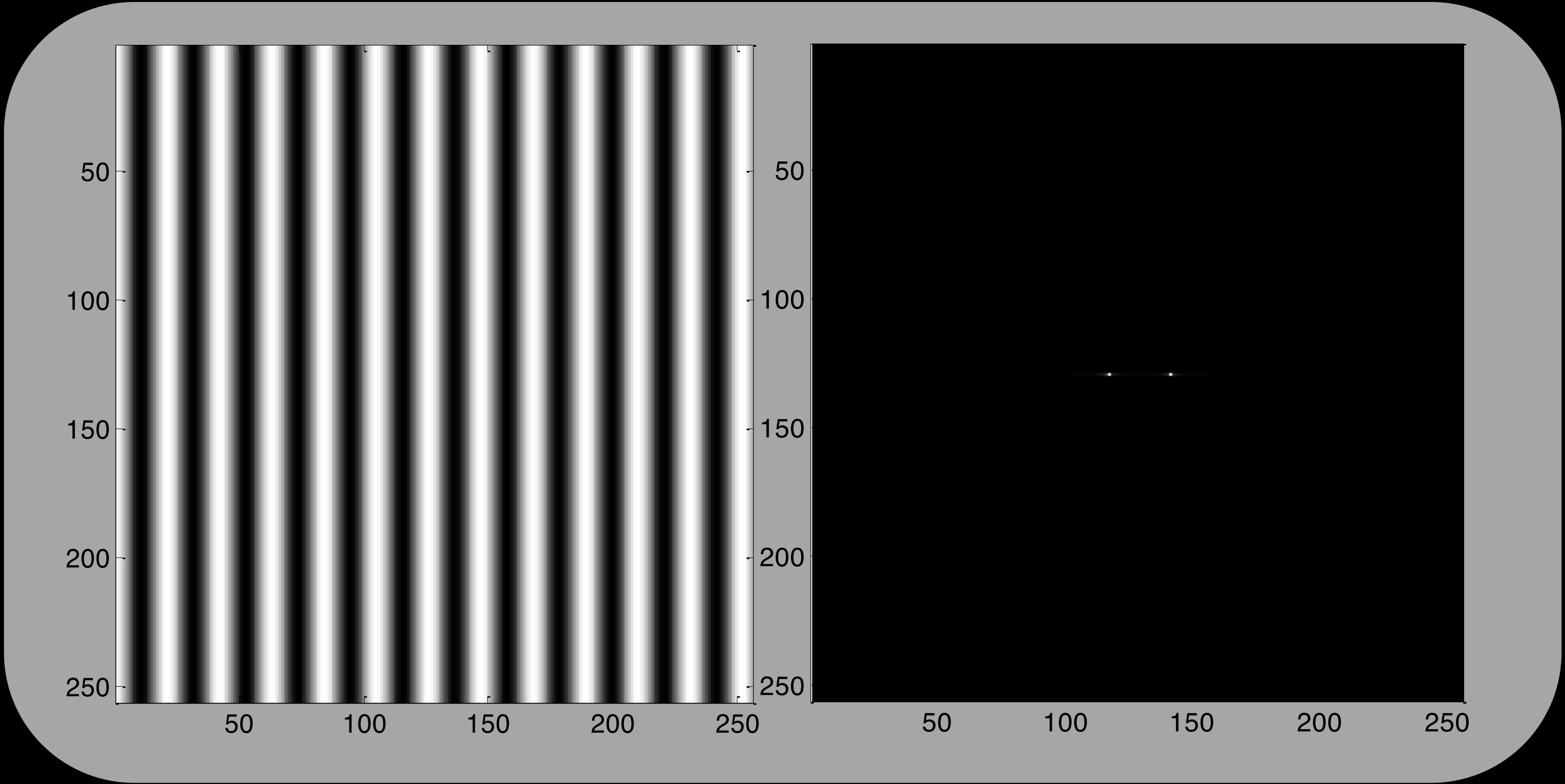


Extension to 2D

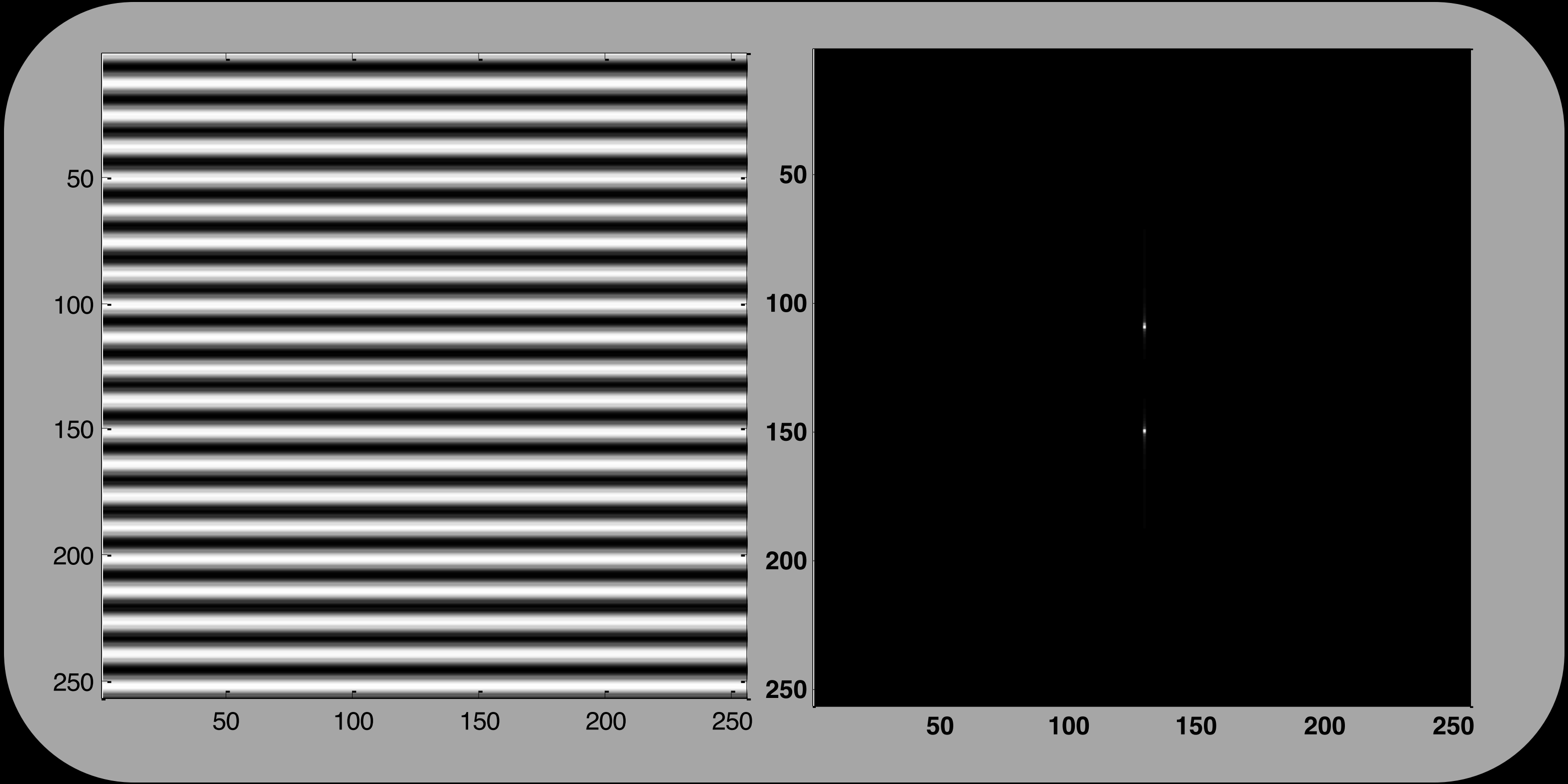


?

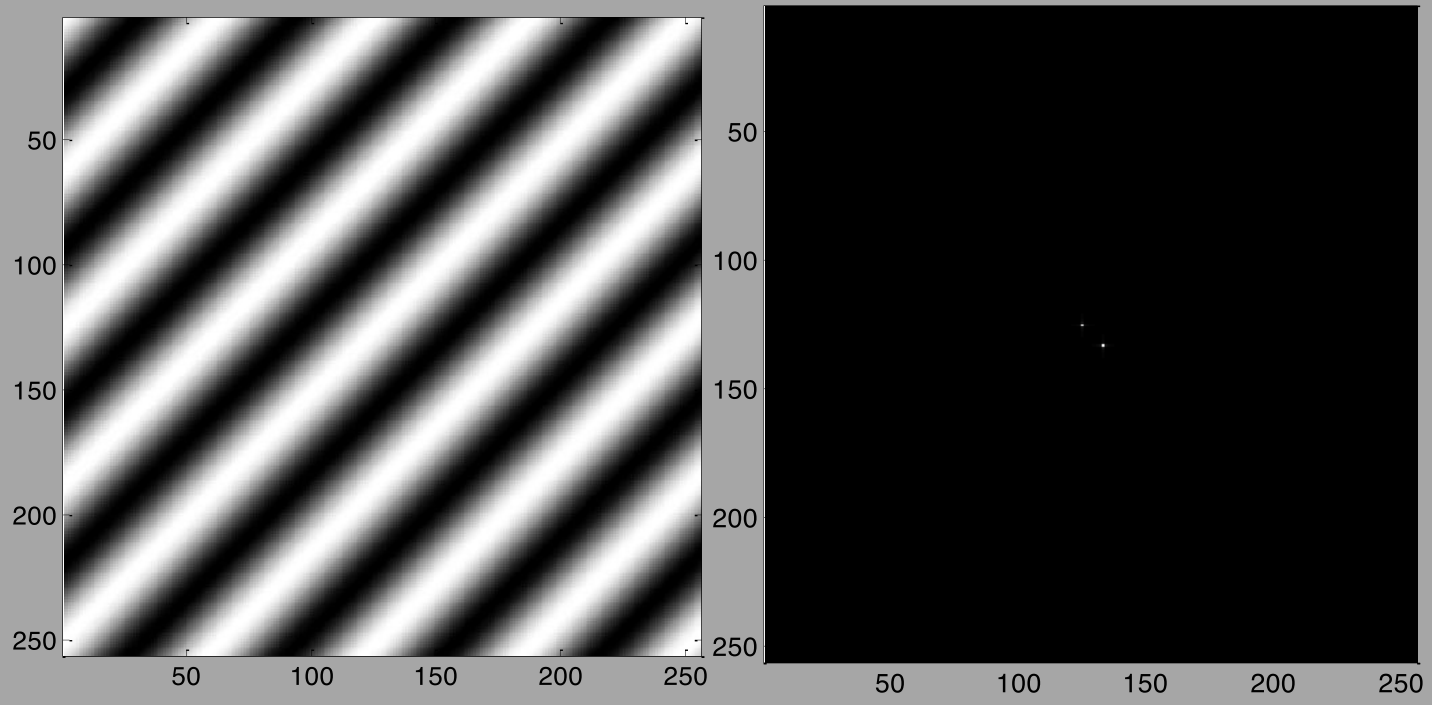
2D Examples – sinusoid magnitudes



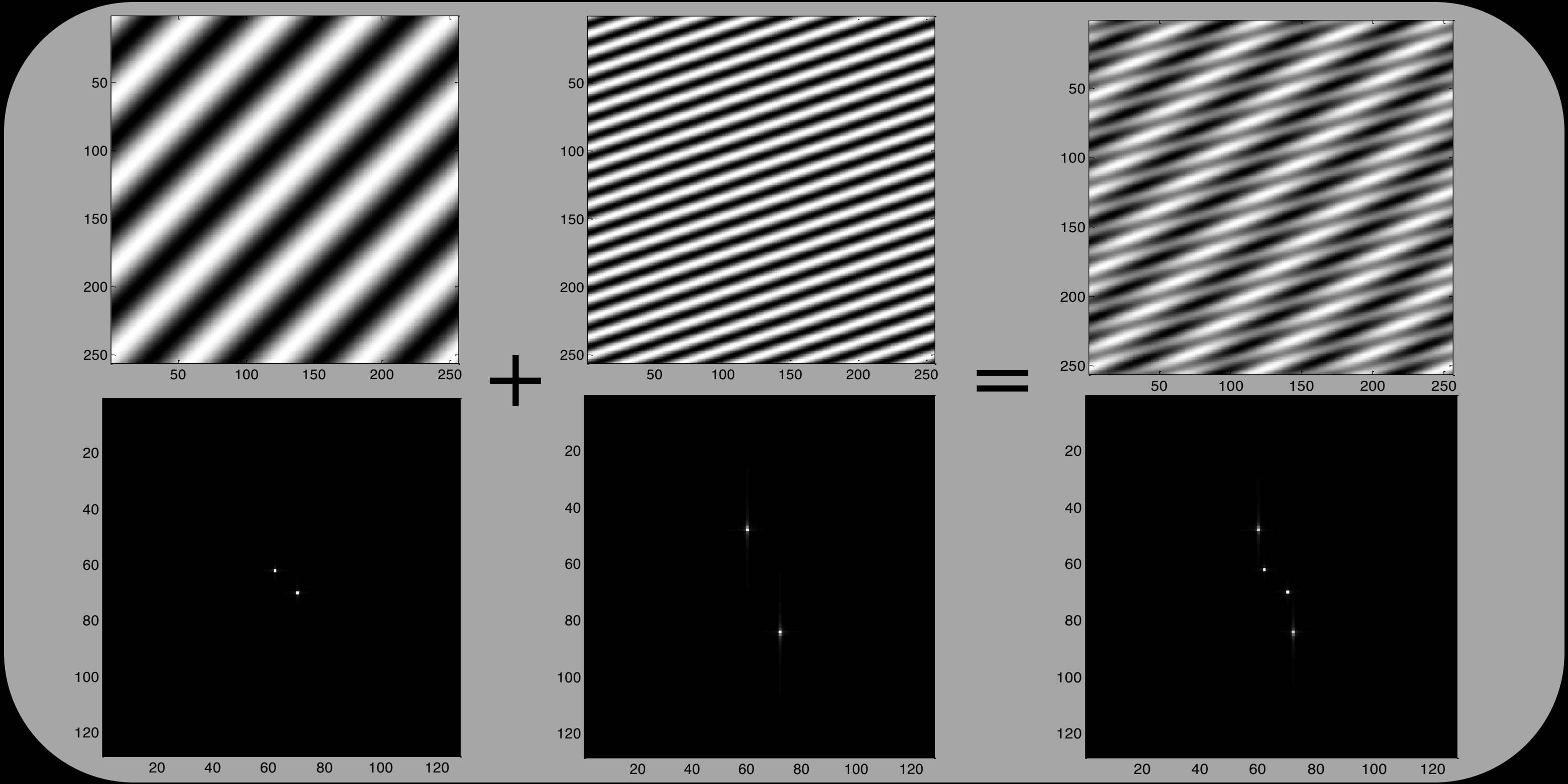
2D Examples – sinusoid magnitudes



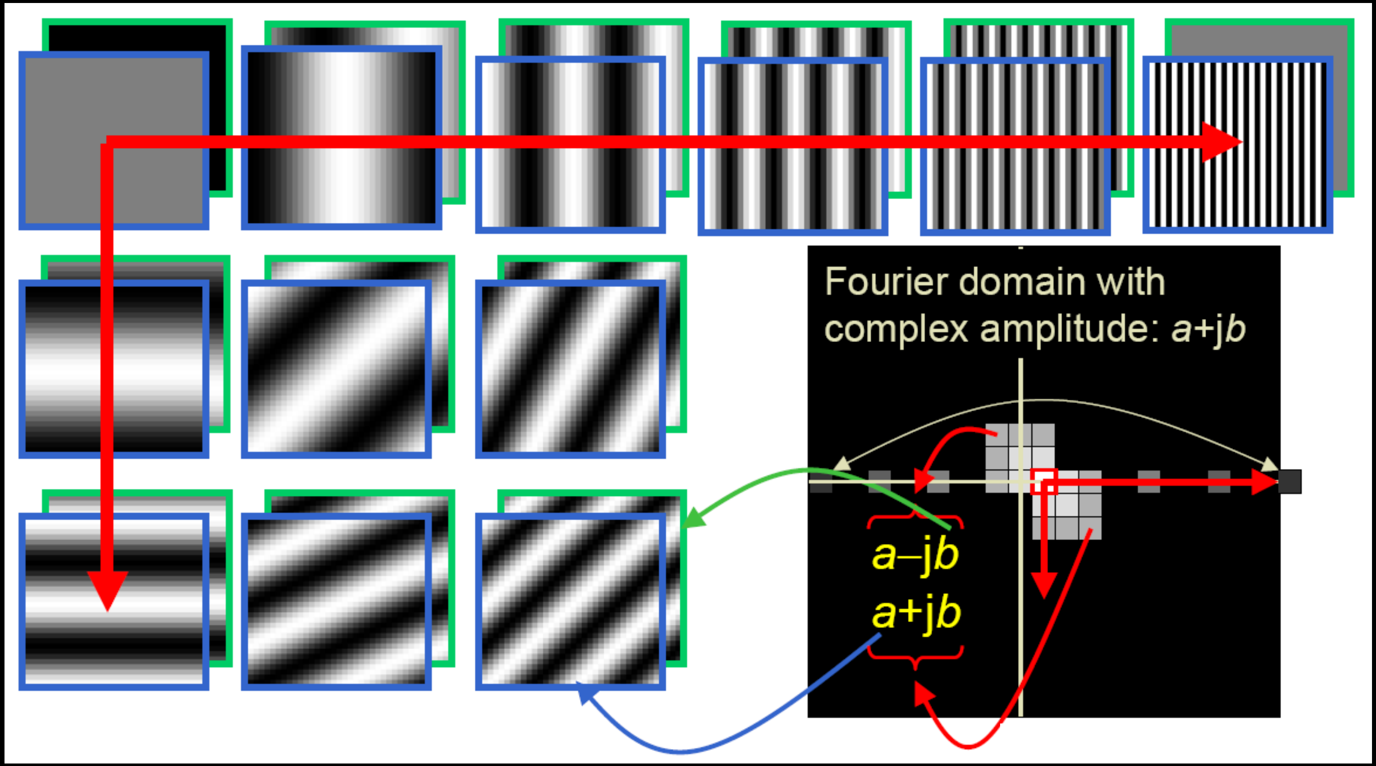
2D Examples – sinusoid magnitudes



Linearity of Sum

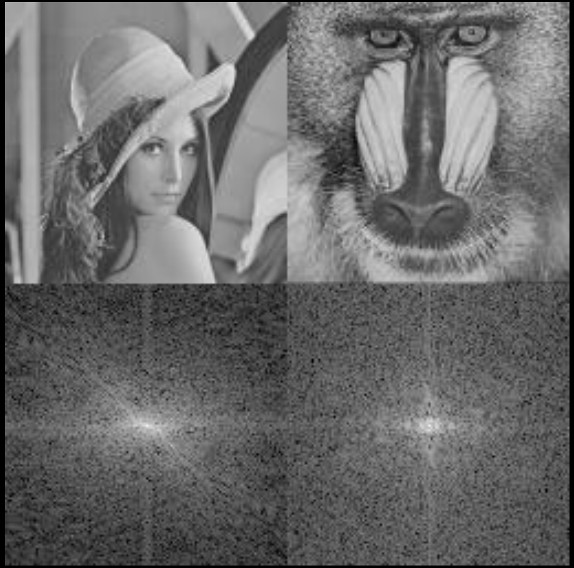


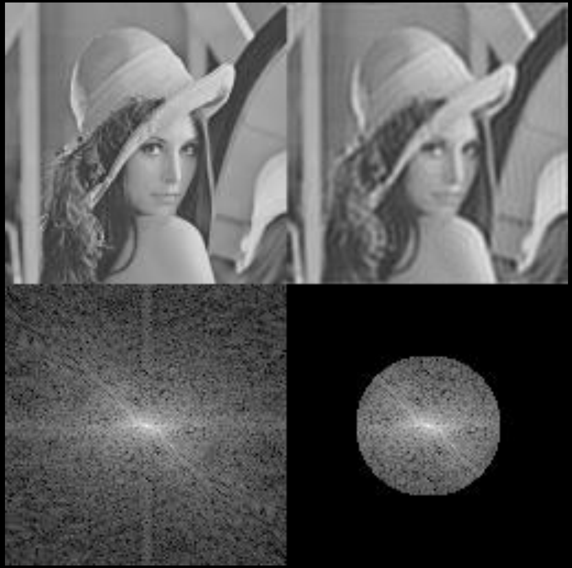
Extension to 2D – Complex plane



Both a Real and Im version

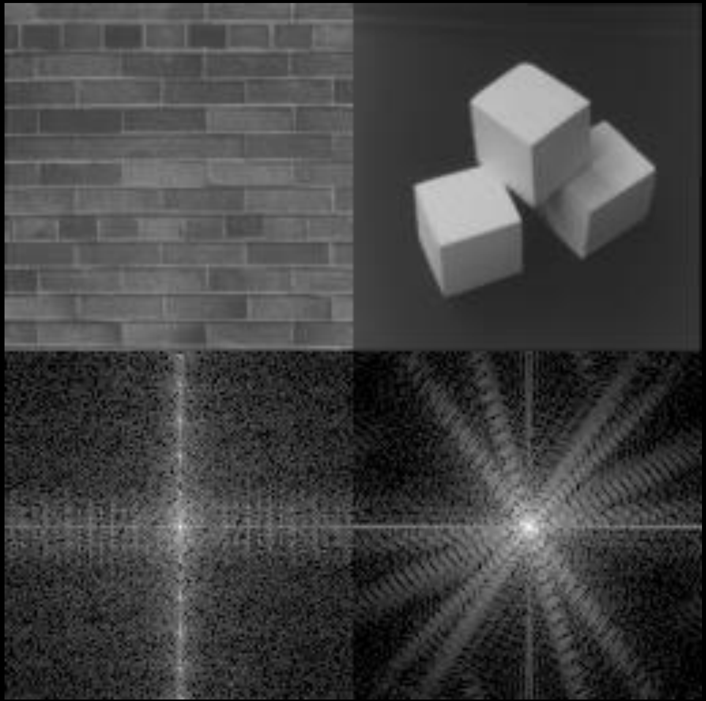
Examples





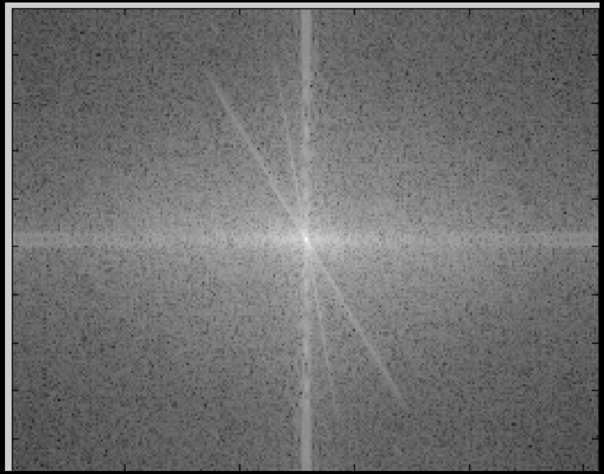
B.K. Gunturk

Examples



B.K. Gunturk

Man-made Scene



Where is this strong horizontal suggested by vertical center line?