Online Matching and Contention Resolution for Edge Arrivals with Vanishing Probabilities

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Abstract

We study the performance of sequential contention resolution and matching algorithms on random graphs with vanishing edge probabilities. When the edges of the graph are processed in an adversarially-chosen order, we derive a new OCRS that is 0.382-selectable, attaining the "independence benchmark" from the literature under the vanishing edge probabilities assumption. Complementary to this positive result, we show that no OCRS can be more than 0.390-selectable, significantly improving upon the upper bound of 0.428 from the literature. We also derive negative results that are specialized to bipartite graphs or subfamilies of OCRS's. Meanwhile, when the edges of the graph are processed in a uniformly random order, we show that the simple greedy contention resolution scheme which accepts all active and feasible edges is 1/2-selectable. This result is tight due to a known upper bound. Finally, when the algorithm can choose the processing order, we show that a slight tweak to the random order—give each vertex a random priority and process edges in lexicographic order—results in a strictly better contention resolution scheme that is $1 - \ln(2 - 1/e) \approx 0.510$-selectable. Our positive results also apply to online matching on 1-uniform random graphs with vanishing (non-identical) edge probabilities, extending and unifying some results from the random graphs literature.

1 Introduction

We consider the online Bayesian selection of edges in a graph $G = (V, E)$. In particular, each edge $e \in E$ has a random state $X_e \in \{0, 1\}$ that is unknown a priori, but known to be *active* (i.e., $X_e = 1$) independently with probability x_e. The edges arrive one-by-one, at which point their random state is revealed. If the edge is not active, then it is discarded; if the edge is active, then an immediate decision must be made about whether to select the edge. The edge is only feasible to select if it does not share a vertex with any previously-selected edge. That is, the set of edges selected must form a *matching* in the graph.

In the *contention resolution* version of this problem, the probabilities $(x_e)_{e \in E}$ form a fractional matching in the graph. That is,

$$\sum_{e \in \partial(v)} x_e \leq 1 \qquad\qquad \forall v \in V, \qquad\qquad (1.1)$$

which says that the expected number of active edges incident to any vertex v is at most 1. The selection problem can be interpreted as "rounding" a fractional matching into an integer one, where the rounding occurs online, and only active edges can be rounded up. The goal is to provide a

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uniform guarantee that every edge $e \in E$ is selected with probability at least cx_e, for a constant c as large as possible. We note that to provide this type of guarantee, the algorithm will generally need to be randomized, and is called a contention resolution scheme.

The contention resolution problem arises naturally in applications, such as posted-price mechanism design. Here, edges represent agents, and x_e is the probability that an optimal mechanism would accept agent e. The optimal mechanism is also bound by selecting a set of agents that form a matching in the graph, and hence (1.1) is satisfied. The above rounding procedure with a uniform guarantee of c would translate into a posted-price mechanism that obtains at least c times the optimal social welfare, demonstrating the applicability of contention resolution. Similar reductions exist for other problems such as prophet inequalities and stochastic probing; we defer to Feldman et al. (2021) for further details. The feasibility constraint of selecting matchings in graphs is also well-motivated, e.g. with edges representing the offering of jobs to gig workers (Pollner et al., 2022).

Aside from contention resolution, this online Bayesian edge selection problem is also related to matching in random graphs. There, x_e can be interpreted as the probability that edge e exists, and of particular interest is the *greedy* algorithm, which processes the edges in some order and adds any feasible existent edge to the matching. Questions of interest include the expected number of edges matched in Erdős–Rényi random graphs as the number of vertices approaches ∞. If the graph is sufficiently sparse such that (1.1) is satisfied, then contention resolution can be applied and its uniform guarantee will imply a bound on the expected number of edges matched.

In both contention resolution and greedy matching, the order in which edges are processed is important. This leads to the following problem variants.

- **Adversarial Order**: the edges are processed in a worst-case[1] order. An online selection algorithm in this setting is called an Online Contention Resolution Scheme (OCRS).

- **Random Order**: the edges are processed in a uniformly random order. An online selection algorithm in this setting is called a Random-order Contention Resolution Scheme (RCRS).

- **Free Order**: the edges are processed in an order chosen by the algorithm. An online selection algorithm in this setting is called a Free-order Contention Resolution Scheme (FO-CRS).

We note that contention resolution was originally studied in the offline setting (Chekuri et al., 2014), where all random states are revealed before making any selections.

In this paper we consider OCRS, RCRS, and FO-CRS in the *vanishing regime*, which was previously resolved by Nuti and Vondrák (2024) in the offline setting. The idea of this regime is to assume every edge is active with a probability less than some small ε, and provide limiting guarantees as $\varepsilon \to 0$. This is an important regime for the following reasons.

- **Precedent**: x_e being small has also been a special case of interest in other models of online matching:

  - Literature on the online matching with stochastic rewards model (Mehta and Panigrahi, 2012; Mehta et al., 2014; Huang and Zhang, 2020; Goyal and Udwani, 2023; Huang et al., 2023) generally focuses on this regime.

  - Motivated by online edge-coloring, rounding fractional matchings online with small x_e has also been studied extensively. Cohen et al. (2019) showed that for bipartite one-sided vertex arrivals, a $1 - o(1)$ guarantee is attainable, where $o(1)$ is a function that tends to 0

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[1]We assume that this order is chosen by an adversary who knows the description of the algorithm. They cannot however change the order after the algorithm begins its execution.

as $\max_{e \in E} x_e \to 0$. For edge arrivals, Kulkarni et al. (2022) proved a rounding guarantee of $1 - 1/e - o(1)$, and this was recently improved to $1 - o(1)$ by Blikstad et al. (2024).

– The random graphs literature analyzes what happens when $|V| \to \infty$, with $x_e = p \to 0$ for all e if the graph is sparse, which is a special case of the vanishing regime with equal probabilities. We relate our results to this literature at the end of Subsection 1.1.

- **Tightness**: The vanishing or "Poisson" regime is often where worst-case examples occur in online Bayesian selection, especially in problems like contention resolution where one is competing against a fractional benchmark. The simplest example arises in RCRS for rank-1 matroids (Lee and Singla, 2018), where the guarantee c cannot be larger than $1 - 1/e$ due to the vanishing regime. More specifically for matchings in graphs, three settings where optimal contention resolution schemes are known have tight examples that lie in the vanishing regime: offline monotone bipartite contention resolution (Bruggmann and Zenklusen, 2022), OCRS for two-sided vertex arrivals (Ezra et al., 2022), and, RCRS for two-sided bipartite vertex arrivals (MacRury and Ma, 2024).

  Our results suggest that a plausible strategy for optimal contention resolution is to start with an algorithm for the vanishing regime, and then add adjustments as necessary to "force" the worst case to lie here (see Brubach et al. (2021)), even though we acknowledge the second step may not be easy. Bruggmann and Zenklusen (2022) already show how to reduce to the vanishing regime if the reduced graph can have parallel edges; however, our results are for graphs *without parallel edges*.

We are therefore interested in the following question:

What is the optimal OCRS, RCRS, and FO-CRS in the vanishing regime?

1.1 Outline of Results

We say that an OCRS (resp. RCRS, FO-CRS) is *c-selectable* if it selects every edge $e \in E$ with probability at least cx_e, under an adversarial (resp. random, free) arrival order, for all graphs $G = (V, E)$ and vectors $(x_e)_{e \in E}$ satisfying (1.1). We say that it is *c-selectable in the vanishing regime* if it is $(c - f(\max_{e \in E} x_e))$-selectable for all G and $(x_e)_{e \in E}$ satisfying (1.1), where f is a function satisfying $\lim_{\varepsilon \to 0} f(\varepsilon) = 0$. We call constant c the *selection guarantee* or just guarantee, which lies in [0,1]. We use *selectability* to refer to largest c for which an OCRS, RCRS, or FO-CRS can be *c*-selectable, possibly in the vanishing regime. By *optimal* contention resolution scheme, we refer to one that achieves the selectability for a particular setting (it does not have to be instance-optimal).

Theorem 1.1. *There is a polytime OCRS that is $\frac{3 - \sqrt{5}}{2} \approx 0.382$-selectable in the vanishing regime.*

We let α denote the constant $\frac{3 - \sqrt{5}}{2}$, which is the smaller real number satisfying $\alpha = (1 - \alpha)^2$. This is the guarantee that *would* be achieved, under a (false) assumption about the independence of edges being selected (Ezra et al., 2022, §4.2). We show that this guarantee is achieved in the vanishing regime. Formalizing the way in which small activeness probabilities x_e imply approximate independence is the main technical challenge in this work, as we explain in Subsection 1.2. We also note that we are using a different OCRS for selecting edges than Ezra et al. (2022), because their algorithm has a sophisticated behavior whose independence properties are difficult to analyze.

Theorem 1.2.

1. *No OCRS can be better than $\beta \approx 0.390$-selectable, even in the vanishing regime.*

2. *No OCRS can be better than $\alpha = \frac{3-\sqrt{5}}{2}$-selectable, even in the vanishing regime over tree graphs, if it satisfies a concentration property (see Subsection 3.2).*

3. *No OCRS can be better than $\gamma \approx 0.393$-selectable, even in the vanishing regime over tree graphs.*

In parts (1) and (3) of Theorem 1.2, we establish upper bounds of 0.390 and 0.393 over vanishing general graphs and vanishing tree graphs (which are bipartite graphs), respectively. Part (2) of Theorem 1.2 says that the 'independence benchmark' of α is tight over OCRS's that satisfy a certain concentration property.[2] We believe it is intuitive to posit that an optimal OCRS should satisfy this property, which is also satisfied by all OCRS's in the literature (we discuss this in greater detail in Subsection 1.2). As we explain in Subsection 1.2, these upper bounds use different arguments than existing ones in the literature, and also offer a significant improvement. Indeed, the previous upper bounds were $4/9 \approx 0.444$ (Gravin and Wang, 2019) followed by $3/7 \approx 0.428$ (Correa et al., 2022) for bipartite graphs, and 0.4 (MacRury et al., 2023) for general graphs (this final result did not use vanishing edge values).

Turning to RCRS and FO-CRS, we will assume that G is *1-regular*, which makes it easier to describe optimal algorithms. This is the assumption that $\sum_{e \in \partial(v)} x_e = 1$ for all v, i.e. (1.1) is tight, which has been shown to be the worst case via reductions (see Section 4). In the random order setting, the optimal algorithm will then be the greedy[3] CRS, which accepts every arriving edge that is active and feasible.

Theorem 1.3. *The greedy CRS is a $\frac{1}{2}$-selectable RCRS in the vanishing regime.*

Theorem 1.3 is tight because no RCRS can be better than 1/2-selectable, even in the vanishing regime (MacRury et al., 2023). It is worth noting that the 0.474-selectable RCRS of MacRury et al. (2023), which is state-of-the-art for non-vanishing edges, behaves identically to the greedy algorithm in the vanishing regime, and hence is 1/2-selectable. This contrasts with the OCRS variant, where the state-of-the-art algorithm originating from Ezra et al. (2022) cannot attain the selection guarantee of α from the vanishing regime (its guarantee is upper-bounded by 0.361, as shown by MacRury et al. (2023)).

Theorem 1.4. *There exist random orderings of edges under which the greedy CRS is a $1 - \ln\left(2 - \frac{1}{e}\right) \approx 0.510$-selectable FO-CRS in the vanishing regime.*

Theorem 1.4 establishes a strict separation between free and random order, for edge arrivals in graphs in the vanishing regime. To our knowledge, results that show how to improve upon random order are generally rare. In stochastic probing and price of information problems on graphs in which the algorithm can choose the order, many results (Gamlath et al., 2019; Fu et al., 2021) still defer to analyzing the random order setting at some point.

Relation to oblivious matching and query-commit. In the edge-weighted *oblivious matching problem*, the edges have known weights, however whether they exist or not is determined by an unknown (arbitrary) distribution. The algorithm probes edges in an order of its choosing, subject to the constraint that if an edge is found to exist, it must be added to the matching. By randomly

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[2] Optimal strategies for various other kinds of matching problems do satisfy this concentration property, so it seems intuitive that an optimal OCRS would also want to have this concentration property, even though we cannot formally prove it.

[3] This is not to be confused with the notion of a greedy OCRS from Feldman et al. (2021).

perturbing the edge weights and executing greedily, Tang et al. (2023) prove a 0.501-approximation ratio against the matching of largest expected weight. The oblivious matching setting generalizes the edge-weighted *query-commit matching problem*, the latter of which has been studied extensively, beginning with the work of Chen et al. (2009); Bansal et al. (2012) (see Borodin and MacRury (2023) for a detailed overview of this problem and its variants). In this problem, the edges are realized independently according to known probabilities. Fu et al. (2021) design a 0.533-selectable random-order *vertex arrival* OCRS which they use with the linear program of Costello et al. (2012); Gamlath et al. (2019) to get an approximation ratio of 0.533 for this problem. (This selection guarantee, and thus approximation ratio, was recently improved to 0.535 by MacRury and Ma (2024)). Assuming the edge probabilities are vanishing, we can apply our free-order OCRS to get a 0.510 approximation ratio for this problem, and thus beat $1/2$ in a different way.

We summarize our results and how they position in the literature in Table 1.

Selectability Bounds	General Edge Values x_e	Vanishing Edge Values $x_e \to 0$
OCRS	≥ 0.344 [3]	$\to \geq \mathbf{0.382}$ [§2]
	≤ 0.4 [3] $\to \leq \mathbf{0.390}$ [§3]	≤ 0.428 [2] $\to \leq \mathbf{0.390}, \leq \mathbf{0.393}$ (bipartite) [§3]
RCRS	≥ 0.474 [3]	$\to \geq \mathbf{0.5}$ [§4]
	≤ 0.5 [3]	≤ 0.5 [3]
FO-CRS	≥ 0.474 [3]	$\to \geq \mathbf{0.510}$ [§4]
	≤ 0.544 [1]	≤ 0.544 [1]

Table 1: Summary of known results, with new results **bolded**. "$\geq$" refers to lower bounds on c (algorithmic results), "$\leq$" refers to upper bounds (impossibility results), and arrows indicate improvement from state of the art. [1] is Karp and Sipser (1981), [2] is Correa et al. (2022), and [3] is MacRury et al. (2023).

Relation to matching in random graphs. Dyer et al. (1993) show, among other results, that the greedy algorithm leaves 50% of the vertices matched in expectation when it is executed on an Erdős–Rényi random graph with n vertices and $n/2$ edges (i.e. average degree 1), as $n \to \infty$, if it processes the edges in a uniformly random order. They show that a modified order, in which a vertex is first chosen uniformly at random and then an edge incident to it (if any) is chosen uniformly at random, leaves $1 - \ln\left(2 - \frac{1}{e}\right) \approx 51\%$ of the vertices matched. Their arguments are combinatorial. Mastin and Jaillet (2013) show that these answers (under either order) are unchanged if one considers a complete bipartite Erdős–Rényi graph instead, using the differential equation method (Wormald et al., 1999). Our Theorems 1.3 and 1.4 can be viewed as a stronger version of these results (in the special case with average degree 1), because:

- We allow for an *arbitrary* graph with vanishing probabilities and average degree 1, whereas both the results for complete and complete bipartite Erdős–Rényi graphs assume that edge probabilities are equal;

- Despite the asymmetry in our probabilities, we provide *uniform* guarantees to every edge for being selected, and show that the guarantee does not get worse. Put another way, our guarantee holds for max-weight matching instead of just max-cardinality matching.

As explained in Subsection 1.2, we also use a differential equation method based on the Galton-Watson trees from Nuti and Vondrák (2024). Interestingly, the random edge orderings we consider in Theorem 1.4 are different[4] from the modified order described above.

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[4] We draw a random priority for each vertex. The property we require in our analysis for the ordering of the edges is that an edge (u, w) is processed before (u, v) exactly when w's priority is less than v's priority.

We note here that some proofs in the following sections are omitted, but can be found in the appendices.

1.2 Techniques

Before explaining the techniques used in proving our results, we formalize what we mean for an input $(G, \boldsymbol{x})$ to have vanishing edges values. Our definition includes the Erdős–Rényi random graph with vanishing edge probabilities case, but allows us to work with a sequence of inputs $(G(n), \boldsymbol{x}(n))_{n \geq 1}$ dependent on n in an arbitrary way.

Definition 1. Given an input $(G, \boldsymbol{x}) = (G(n), \boldsymbol{x}(n))$ indexed by a parameter $n \in \mathbb{N}$, we say it has *vanishing edge values*, provided there exists $(\varepsilon(n))_{n \geq 1}$, such that $x_e(n) \leq \varepsilon(n)$ for each edge e of $G(n)$, where $\varepsilon(n) \to 0$ as $n \to \infty$. In this case, we say that $(G(n), \boldsymbol{x}(n))$ has *vanishing edge values with respect to* $\varepsilon(n)$. When the context is clear, we drop the dependence on n in our notation.

In addition, we say a contention resolution scheme is c-selectable on an input $(G, \boldsymbol{x})$ with vanishing edge values, provided it is c_n-selectable on $(G(n), \boldsymbol{x}(n))$, where $c_n \to c$ as $n \to \infty$.

Finally, we say that a sequence of events $(\mathcal{E}(n))_{n \geq 1}$ occurs *with high probability* (w.h.p.), provided $\Pr[\mathcal{E}(n)] \to 1$ as $n \to \infty$.

In the rest of this section, we will summarize the techniques used in proving our results. We begin with our positive results for RCRS and FO-CRS, as the algorithms are very simple to describe, and the analysis is based on the Galton-Watson approach developed by Nuti and Vondrák (2024) for an offline CRS with vanishing edge values. We then discuss our positive result for OCRS, where we first explain why the Galton-Watson approach does not seem applicable in this setting. Finally, we discuss the constructions used in proving our negative results for OCRS.

RCRS and FO-CRS positive results. Let us assume that $(G, \boldsymbol{x})$, with $G = (V, E)$ is a 1-regular input, i.e., $\sum_{e \in \partial(v)} x_e = 1$ for each $v \in V$. (This is without loss, as we explain in Section 4). Both positive results then analyze the *greedy CRS* on $G = (V, E)$, which accepts any active edge whenever possible, where the only difference between the settings is the order in which the edges are processed. For RCRS, we assume each edge e has an *arrival time* drawn independently and uniformly at random (u.a.r.) from $[0, 1]$. For FO-CRS, we first draw a *seed* from $[0, 1]$ independently and u.a.r for each vertex. We then define $e = (u, v)$ to arrive at time $\min\{t_u, t_v\} + \varepsilon \max\{t_u, t_v\}$, where t_u and t_v are the seeds of u and v, and ε is any number with $\varepsilon < \min_{u \neq v} |t_u - t_v|$. This ensures that the ordering is primarily decided by $\min\{t_u, t_v\}$, and in case there is a tie, the ordering is then decided by $\max\{t_u, t_v\}$. We refer to this order as a *lexicographical* random ordering.

For general edge values, the state-of-the-art is the RCRS of MacRury et al. (2023). This algorithm is very similar to the greedy CRS, except that edges with large x_e are necessarily *dropped* or *attenuated* with probability $a(x_e)$ for a carefully engineered function a. Since $a(0) = 1$, this algorithm is asymptotically identical to the greedy CRS for vanishing edge values. Due to this connection, a natural approach to prove Theorem 1.3 would be to improve the analysis of MacRury et al. (2023) for vanishing edge values. Unfortunately, their analysis is inherently lossy—when lower bounding the probability e is matched, it only considers vertices at distance *at most* 3 from u or v.

We follow instead the approach of Nuti and Vondrák (2024) and *reduce* the problem to analyzing a greedy matching algorithm on a certain randomly generated tree. To explain this approach, fix $e = (u, v) \in E$ which is active, and define S_e to be the connected component of e induced by the *active* edges of G. There are three essential observations underlying our analysis:

1. As $n \to \infty$, S_e looks like a (finite) random tree $\mathcal{T}$, constructed by drawing independent Galton-Watson trees $\mathcal{T}_u$ (at u), $\mathcal{T}_v$ (at v) with Poisson parameter 1, and setting $\mathcal{T} = \mathcal{T}_u \cup \mathcal{T}_v \cup \{e\}$.

2. Whether the greedy CRS selects a particular active edge e depends entirely upon on S_e, and the order in which the edges of S_e are presented to the algorithm.

3. The edges of S_e are themselves presented to the greedy CRS in a random order.

Using these observations, we equate the probability the greedy CRS selects e to the probability the greedy matching algorithm on $\mathcal{T}$ selects e, when presented the edges of $\mathcal{T}$ in a random order (see Theorem 4.3). Assuming this reduction, let us now sketch how the analysis on $\mathcal{T}$ proceeds for u.a.r. edge arrivals.

We define $q(t)$ to be the probability the vertex u in edge e is not matched when $e = (u, v)$ arrives, given it arrives at time t. Note that due to the symmetry between u and v, this is also the conditional probability v is not matched. Then, since $\mathcal{T}_u$ and $\mathcal{T}_v$ are drawn independently, and e is matched if only if u and v are *not* matched when e arrives, e is matched w.p. $\int_0^1 q(t)^2 dt$ after integrating over the arrival time of e. By considering the children of u in $\mathcal{T}_u$, and using the recursive nature of the Galton-Watson tree process, we then prove that $q(t)$ satisfies a certain integral equation. Using the fact that $q(0) = 1$, we show that the integral equation has the unique solution $q(t) = 1/(t+1)$, and so $\int_0^1 q(t)^2 dt = 1/2$ as desired.

For a lexicographical random ordering, while the analysis is similar in broad strokes, it is more involved, as we have to consider the function $q(t_1, t_2)$, which is the probability vertex u of e is unmatched, given the seed of u is t_1 and the seed of v is t_2.

OCRS positive result. The techniques used in the proof of Theorem 1.1 differ substantially from the RCRS/FO-CRS setting. This is because the decisions of our OCRS depend on the active *and* inactive edges of G. As such, (2) does *not* hold, and so we cannot reduce the problem to analyzing a matching algorithm on $\mathcal{T}$. The OCRS of Ezra et al. (2022), which provides the state-of-the-art lower bound on OCRS selectability (MacRury et al., 2023), also has this limitation.

Before we describe our OCRS, we first review the OCRS of Ezra et al. (2022), and explain why it is challenging to prove it is $\alpha \approx 0.382$-selectable on inputs with vanishing edge values (even if we abandon the Galton-Watson approach). This OCRS is given a graph $G = (V, E)$ with an arbitrary fractional matching $\boldsymbol{x} = (x_e)_{e \in E}$. It then selects arriving edge e with probability $c/\Pr[u \text{ and } v \text{ unmatched before } e]$, when e is active and its endpoints are unmatched. This ensures it selects each edge $e \in E$ with probability *exactly* cx_e, and the goal is to make c as large as possible.

In order to prove this is well-defined, one must show $c \leq \Pr[u \text{ and } v \text{ unmatched before } e]$, while assuming the induction hypothesis that every previous edge f before e was selected w.p. cx_f. Now, if G is a tree, then the events "u is matched before e" and "v is matched before e" are independent, and so the induction hypothesis yields $\Pr[u \text{ unmatched before } e] \cdot \Pr[v \text{ unmatched before } e] \geq (1 - c)^2$, so one can set $c = \alpha$, as $\alpha = (1 - \alpha)^2$.

Suppose now that $(G, \boldsymbol{x}) = (G(n), \boldsymbol{x}(n))$ has vanishing edges values. Nuti and Vondrák (2024) proved that S_e is cycle-free w.h.p (here S_e is defined as before). Thus, restricted to the active edges of G, the neighborhood of e looks *tree-like*. Suppose that conditional on S_e being cycle-free, the events "u is unmatched before e" and "v is unmatched before e" are independent. Since S_e occurs w.h.p., this would be enough to prove $\Pr[u \text{ unmatched before } e] \cdot \Pr[v \text{ unmatched before } e] \geq (1 - o(1))(1 - c)^2$, and thus prove Theorem 1.1. However, this conditional independence does *not* hold for arbitrary inputs[5], and so we do not analyze this OCRS.

We instead prove Theorem 1.1 in two self-contained parts: Theorem 2.1 and Theorem 2.2. In Theorem 2.1, we allow $\boldsymbol{x}$ of $G = (V, E)$ to be an arbitrary fractional matching. We say that an

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[5]Consider a graph on 3 vertices u, v, w with uniform edge values, where (u, v) arrives last. Then $\Pr[u \text{ and } v \text{ matched before } (u, v)] = 0$, yet $\Pr[S_{u,v} \text{ is cycle-free}] > 0$, and $\Pr[u \text{ matched before } e | S_e \text{ cycle-free}]$, $\Pr[v \text{ matched before } e | S_e \text{ cycle-free}] > 0$.

edge e is *ℓ-locally-tree-like*, provided there are no cycles in the ℓ-neighborhood of e. We design an OCRS (Algorithm 1) for G in which each active edge which is ℓ-locally-tree-like is matched with probability α_ℓ, where α_ℓ converges to $\alpha \approx 0.382$ as $\ell \to \infty$ (see (2.2) for the explicit definition of α_ℓ). Note that if G has girth $g \geq 3$, then any edge is $\lfloor (g-1)/2 \rfloor$ locally-tree-like, and so Algorithm 1 is $\alpha_{\lfloor (g-1)/2 \rfloor}$-selectable on such inputs. The convergence $\alpha_\ell \to \alpha$ occurs rapidly; in particular, $\alpha_7 \geq 0.3658$, and so Algorithm 1 beats the state-of-the-art of 0.344 for inputs with girth $g \geq 15$.

Algorithm 1 is related to the OCRS of Ezra et al. (2022), however it is *not* defined recursively, and so it is easier to analyze. Specifically, when an active edge $e = (u, v)$ arrives and u, v are unmatched, e is added to the matching with probability $\frac{\alpha}{(1-\alpha x_u(e))(1-\alpha x_v(e))}$, where $x_v(e)$ (respectively, $x_u(e)$) is the fractional degree of u (respectively, v) prior to the arrival of e. Note that if G is a tree, then Algorithm 1 is identical to the OCRS of Ezra et al. (2022), and so it is exactly α-selectable.

The core of the analysis is a *correlation decay* argument in terms of ℓ. We argue that since e is ℓ-locally-tree-like, there is no *short* path between u in v in $G \setminus e$, and so the events "u is matched before e" and "v is matched before e" are *asymptotically* independent as $\ell \to \infty$. The analysis here is similar to the online fractional rounding algorithm of Kulkarni et al. (2022), with the main difference being that our analysis works for a general matching $\boldsymbol{x}$, whereas theirs requires $\boldsymbol{x}$ to have vanishing edge values (and in fact be uniform).

The second step of the proof is Theorem 2.2, which provides a *black-box reduction* from the *vanishing edge value regime* to the *locally-tree-like regime*. That is, given a CRS ψ_ℓ which is c_ℓ-selectable for ℓ-locally-tree-like edges, we show how to use it to design a new CRS which is c-selectable for inputs with edge values which are vanishing with respect to ε, where $c = \lim_{\ell \to \infty} c_\ell$. The reduction proceeds via a "two-round exposure" argument, where we split the randomness of each edge state X_e into two random variables, and analyze each separately. We first set $y_e := \log(1/\varepsilon)x_e$ and $z_e := 1/\log(1/\varepsilon)$ for each $e \in E$. Since $\varepsilon = o(1)$, $y_e, z_e \in [0, 1]$, and so we can sample $Y_e \sim \mathrm{Ber}(y_e)$ and $Z_e \sim \mathrm{Ber}(z_e)$ for each $e \in E$, such that $X_e = Y_e Z_e$, and for which $((Y_e)_{e \in E}, (Z_e)_{e \in E})$ are independent (see Proposition 2.8).

In the first round, we expose the $(Y_e)_{e \in E}$ random variables, and consider the sub-graph G' of G with edge-set $E' = \{e \in E : Y_e = 1\}$. We argue that for a specific choice of $\ell \to \infty$, any fixed edge of G' is ℓ-locally-tree-like w.h.p. (see Lemma 2.12). At this point, we'd like to expose the $(Z_e)_{e \in E'}$ random variables and execute ψ_ℓ on $(G', (z_e)_{e \in E'})$. However, $(z_e)_{e \in E'}$ may not be a fractional matching of G'. Thus, we first greedily process the edges of G', and create a fractional matching $\boldsymbol{z}' = (z'_e)_{e \in E'}$, where $z'_e \in \{0, z_e\}$ for each $e \in E'$. In Lemma 2.11, we argue that for any fixed edge $e \in E'$, w.h.p. $z'_e = 1/\log(1/\varepsilon)$ (i.e., z'_e will *not* be rounded down from z_e). Let us say an edge $e \in E'$ is *well-behaved*, provided e is ℓ-locally-tree-like, and $z'_e = 1/\log(1/\varepsilon)$. In the second round, we expose the $(Z_e)_{e \in E'}$ random variables, and execute ψ_ℓ on $(G', (z'_e)_{e \in E'})$ to ensure that each well-behaved edge $e \in E'$ is selected with probability $c_\ell z_e$. Since $e \in E$ is an edge of G' with probability y_e, and w.h.p. $e \in E'$ is well-behaved, this ensures that e is matched with an overall probability of $(1 - o(1))c_\ell z_e y_e = (1 - o(1))cx_e$ after averaging over G'.

In order to execute our reduction online, we need to be able to compute $(z'_e)_{e \in E'}$ online. While this can't in general be done for an arbitrary OCRS ψ_ℓ, it can be done if ψ_ℓ is *strongly online*. Roughly speaking, the OCRS should work even if the fractional values of the matching are adversarially revealed. Algorithm 1 of Theorem 2.1 is strongly online, and so Theorem 2.1 and Theorem 2.2 together imply Theorem 1.1.

OCRS negative result. We prove the various results of Theorem 1.2 by considering explicit graphs on which any OCRS cannot perform well.

For part (1) of the theorem, consider a single vertex u and suppose it has edges $\{(u, v_i)\}_{i=1}^n$

around it, with $x_{u,v_i} \approx 1/n$ for all i. For each $i \in [n]$, add a set W_i of $n-1$ children to v_i, with $x_{v_i,w} \approx 1/n$ for each $w \in W_i$. Finally, we add a complete graph of edges amongst the v_i with tiny edge values.

All the edges (v_i, w), $w \in \cup W_i$ are presented to the OCRS first. Let us say that $A_{v_i} = 1$ provided that v_i is unmatched after this first phase of edges is presented to the OCRS. The tiny edges are presented to the OCRS *after* all the edges of the first phase but *before* the edges (u, v_i). The edges (u, v_i) are presented to the OCRS last.

We consider two cases, depending on the size of the minimum pairwise covariance amongst the A_{v_i} (see Lemmas 3.2 and 3.3). Suppose first that after the first phase there is a pair A_{v_i} and A_{v_j} which are not very positively correlated with each other. Then, examining the newly added tiny edge between v_i and v_j tells us that no OCRS can perform particularly well. Indeed, if the scheme was c-selectable,

$$c \leq \Pr[(v_i, v_j) \in \mathcal{M} \mid X_{v_i,v_j} = 1] \leq \Pr[A_{v_i} = 1, A_{v_j} = 1] = \mathbb{E}[A_{v_i}]\mathbb{E}[A_{v_j}] + \mathrm{Cov}[A_{v_i}, A_{v_j}]$$
$$\approx (1-c)^2 + \text{something small.}$$

Otherwise, let us suppose all the pairs A_{v_i} and A_{v_j} are positively correlated with each other, and we consider the last edge (u, v_n) that is presented to the algorithm. Note first that it must be the case that if $A_{v_n} = 0$, then a higher than average number of the other A_{v_i} are also zero. Conversely, a fewer than average number of the A_{v_i} are one, since all the random variables are positively correlated.

Suppose that the OCRS does not impose any strong positive correlation between A_{v_n} and the event $\mathcal{E}$ that u is unmatched prior to observing (u, v_n). In this case, exactly like the argument we just made, if the scheme was c-selectable, then

$$c \leq \Pr[\mathcal{E}, A_{v_n} = 1 \mid X_{u,v_n} = 1] = \Pr[\mathcal{E}]\mathbb{E}[A_{v_n}] \approx (1-c)^2 + \text{something small.} \qquad (1.2)$$

So the OCRS must try to correlate A_{v_n} with $\mathcal{E}$. But there is a limit to how positively correlated $A_{v_n} = 0$ can be with the event $\neg\mathcal{E}$, because $\neg\mathcal{E}$ can only occur if one of the edges (u, v_i) with $A_{v_i} = 1$ is actually active, and there are a fewer than average number of v_i with this property in the case we are considering.

Next, consider part (2) of the theorem. First, we would like to work with a tree, so we eliminate all the tiny edges we previous had between the v_i. But we also want to obtain an impossibility bound of exactly α. So we need a new mechanism to ensure that A_{v_n} (or perhaps some variation on A_{v_n}) has essentially no positive correlation with $\mathcal{E}$ (or perhaps some variation on $\mathcal{E}$).

We thus consider a *concentrated* OCRS. An OCRS is said to be *concentrated* on an input $(G, \boldsymbol{x})$, provided the matching $\mathcal{M}$ on $(G, \boldsymbol{x})$ it returns satisfies $||\mathcal{M}| - \mathbb{E}[|\mathcal{M}|]| = o(\mathbb{E}[|\mathcal{M}|])$ w.h.p. If the OCRS is concentrated, it must be the case that $\sum A_{v_i}$, while random, still exhibits concentration.

It follows then that then we can find a subset S of size $m = o(n)$ such that $\sum_{v_i \in S} A_{v_i}$ is also concentrated. We prove this by taking a uniformly at random subset of size m, and arguing that there's a non-zero chance it has small variance. Now, suppose that all these edges (u, v_i) with $v_i \in S$ are presented to the OCRS last, and we imagine treating all of the edges as if they were a single 'mega-edge'. It is impossible for the OCRS to strongly correlate u being matched prior to this mega-edge being presented to the algorithm with $\sum_{v_i \in S} A_{v_i}$, since the latter sum is approximately deterministic. We use this fact to argue that any concentrated OCRS is, in fact, at most α-selectable.

Finally, to establish part (3) of the theorem, we start by arguing that there is a rearrangement of the v_i so that A_{v_n} is positively correlated with $\sum A_{v_i}$. Then, it must be the case that given $A_{v_n} = 0$, a higher than average number of the other A_{v_i} are also zero. The rest of the argument is then very similar to the argument we used to prove (1.2) in part (1) of the theorem.

2 OCRS Positive Result

As discussed in Subsection 1.2, there are two separate self-contained parts to the proof of Theorem 1.1. Let $\alpha \in (0,1)$ be the unique solution to the equation

$$\alpha = (1-\alpha)^2, \tag{2.1}$$

where $\alpha \approx 0.382$. For each $\ell \geq 1$, we define

$$\alpha_\ell := \left(1 - \left(\frac{\alpha}{1-\alpha}\right)^{2\lceil \frac{\ell}{2}\rceil}\right)^2 \alpha, \tag{2.2}$$

which we observe satisfies $\alpha_\ell \to \alpha$ as $\ell \to \infty$. Recall that for an edge $e = (u,v)$, we define the ℓ-neighborhood of e to be those vertices of G at graph distance $\leq \ell$ from u or v, which we denote by $N^\ell(e)$. We prove that if $N^\ell(e)$ is cycle-free, then e is matched by Algorithm 1 with probability at least $\alpha_\ell x_e$. This motivates the following definition:

Definition 2. Suppose that $\ell \in \mathbb{N}$, $c_\ell \in [0,1]$, and ψ is a CRS for the matching polytope, whose matching on $(G, \boldsymbol{x})$ we denote by $\psi(G, \boldsymbol{x})$. We say that ψ is c_ℓ-*selectable for ℓ-locally-tree-like edges*, provided the following holds for any input $(G, \boldsymbol{x})$ where $G = (V, E)$:

- For each $e \in E$, if $N^\ell(e)$ is cycle-free, then $\Pr[e \in \psi(G, \boldsymbol{x}) \mid X_e = 1] \geq c_\ell$.

Theorem 2.1. *For each $\ell \geq 1$, Algorithm 1 is α_ℓ-selectable for ℓ-locally-tree-like edges, where α_ℓ is defined in (2.2).*

In order to state our reduction for the second part of the proof of Theorem 1.1, we require the OCRS to satisfy an additional property: When processing an edge e, the decision to match e depends on only (x_e, X_e), as well as the previous edges' fractional values and states (as well as any potential internal randomization used). Thus, the OCRS initially only knows V of G, and learns both X_e *and* x_e of each edge e in an adversarial order. This is in contrast to the usual definition of an OCRS, where it is assumed that the edges of G and their edges values are known upfront. Note that we still work with an oblivious adversary who decides on the arrival order of E prior to drawing $(X_e)_{e \in E}$. We refer to an OCRS with this property as *strongly online*, and provided a formal definition in Definition 3 of Appendix A. We now state our reduction.

Theorem 2.2. *Suppose $(\psi_\ell)_{\ell \geq 1}$ is a sequence of CRSs, where each ψ_ℓ is c_ℓ-selectable on ℓ-locally-tree-like edges, and $c := \lim_{\ell \to \infty} c_\ell$. The following then hold:*

1. *There exists an offline CRS ψ which is c-selectable on graphs with vanishing edge values.*

2. *If each ψ_ℓ is strongly online, then ψ can be implemented as an OCRS.*

Since Algorithm 1 is a strongly online CRS, and $\alpha = \lim_\ell \alpha_\ell$, Theorems 2.1 and 2.2 together imply Theorem 1.1. The rest of the section is devoted to proving each individual theorem.

2.1 OCRS for Locally-Tree-Like Edges: Proving Theorem 2.1

Given $G = (V, E)$ with fractional matching $\boldsymbol{x} = (x_e)_{e \in E}$, assume the edges E of G arrive in some adversarial order, where $e' < e$ indicates e' arrives before e. Let $x_v(e) := \sum_{\substack{f < e: \\ f \in \partial(v)}} x_f$ for each $v \in V$ and $e \in \partial(v)$. Observe that $x_v(e)$ is the fractional degree of v, prior to the arrival of e.

Algorithm 1 Tree-OCRS

Input: $G = (V, E)$, $\boldsymbol{x} = (x_e)_{e \in E}$, and α which satisfies (2.1).
Output: subset of active edges forming a matching $\mathcal{M}$
 1: $\mathcal{M} \leftarrow \emptyset$
 2: **for** arriving $e = (u, v) \in E$ with edge state X_e **do**
 3:      Draw $A_e \sim \text{Ber} \left(\frac{\alpha}{(1-\alpha x_u(e))(1-\alpha x_v(e))} \right)$ independently.
 4:      **if** u and v are unmatched, $X_e = 1$ and $A_e = 1$ **then**
 5:         $\mathcal{M} \leftarrow \mathcal{M} \cup \{(u, v)\}$
 6: **return** $\mathcal{M}$

Remark 2.3. Algorithm 1 is well-defined, as $\max\{x_v(e), x_u(e)\} \leq 1$, and $\alpha/(1-\alpha)^2 = 1$ by (2.1). This is the only place we use the specific value of α. Algorithm 1 is well-defined if it was executed with $c \leq \alpha$ instead of α.

In order to prove Theorem 2.1, we must show that if $\mathcal{M}$ is returned by Algorithm 1, then for each $e \in E$ with $N^\ell(e)$ cycle-free,

$$\Pr[e \in \mathcal{M} \mid X_e = 1] \geq \alpha_\ell = \left(1 - \left(\frac{\alpha}{1-\alpha} \right)^{2 \lceil \frac{\ell}{2} \rceil} \right)^2 \alpha. \tag{2.3}$$

Fix $e = (u, v)$, and let T be the graph induced by the vertices in the ℓ-neighborhood of e. Note that by assumption, T is cycle-free and connected, and so a tree. Denote by ∂T the edges between $N^\ell(e)$ and $N^{\ell+1}(e)$, which we call the *boundary* of T. Let us assume it is non-empty, since otherwise, the neighborhood of e is a tree, and there is nothing to prove.

One difficulty with establishing guarantees on the selectability of an OCRS is the potentially arbitrary order in which edges might be presented to the algorithm. However, when studying *one particular edge e*, we may assume that the edges appear in a much more specifc order. We begin with a preliminary lemma (closely related to the 'tree-matching game' discussed in Kulkarni et al. (2022)) that demonstrates that this is the case:

Lemma 2.4. *When lower bounding* $\Pr[e \in \mathcal{M} \mid X_e = 1]$, *we may assume that the edges of $T \cup \partial T$ are processed "bottom up": The edges between $N^i(e)$ and $N^{i+1}(e)$ are processed before the edges between $N^{i-1}(e)$ and $N^i(e)$ for $i = 1, 2, \ldots, \ell$.*

For the remainder of the proof, we assume that the order of the edges in T and $\partial(T)$ is as described in Lemma 2.4. This lemma is immensely useful, because it lets us come up with a recurrence letting us calculate the value of $\Pr[e \in \mathcal{M} \mid X_e = 1]$. In order to establish this recurrence, first, consider vertices $s \in N^\ell(e)$, and define the random variables Q_s, with $Q_s = 1$ if the OCRS (after making decision on the edges in $\partial(T)$) chooses to leave s unmatched, and 0 otherwise.

Suppose we fix $r \in T$, $r \notin N^\ell(e)$ whose children (in the order they will be processed) are $r_1, \ldots, r_k$ for $k \geq 0$ (if $k = 0$, then r has no children). We can explicitly write out the probability that a vertex r is unmatched after $e_1 = (r, r_1), \ldots, e_k = (r, r_k)$ are processed, conditional on the values of Q_s. Denote this probability by q_r. The key point is that since T is a tree, and edges are processed in a bottom up fashion, once we condition on any particular values of Q_s, the children $r_1, \ldots, r_k$ of r are matched with their children independently. Thus,

$$q_r = \prod_{i=1}^{k} \Pr[e_i \notin \mathcal{M} \mid e_j \notin \mathcal{M}, \forall j < i]$$

$$= \prod_{i=1}^{k} (1 - \Pr[e_i \in \mathcal{M} \mid e_j \notin \mathcal{M}, \forall j < i])$$

$$= \prod_{i=1}^{k} (1 - \Pr[r_i \text{ is not matched with any of its children}, X_{e_i} = 1, A_{e_i} = 1 \mid e_j \notin \mathcal{M}, \forall j < i])$$

$$= \prod_{i=1}^{k} (1 - \Pr[r_i \text{ is not matched with any of its children}, X_{e_i} = 1, A_{e_i} = 1])$$

$$= \prod_{i=1}^{k} (1 - \Pr[r_i \text{ is not matched with any of its children}] \Pr[X_{e_i} = 1] \Pr[A_{e_i} = 1])$$

$$= \prod_{i=1}^{k} \left(1 - \frac{q_{r_i} x_{e_i} \alpha}{(1 - \alpha x_r(e_i))(1 - \alpha x_{r_i}(e_i))}\right).$$

So,

$$q_r = \prod_{i=1}^{k} \left(1 - \frac{q_{r_i} x_{e_i} \alpha}{(1 - \alpha x_r(e_i))(1 - \alpha x_{r_i}(e_i))}\right), \tag{2.4}$$

where $x_r(e_i) = \sum_{j=1}^{i-1} x_{r,r_j}$. If r has no children, then $q_r = 1$. We implicitly understand in the above expression that when we write q_s for $s \in N^{\ell}(e)$, we actually mean Q_s. In our next lemma, also similar to the work of Kulkarni et al. (2022), we provide a bound on q_r that does not depend on the specific values of the Q_s. We can then use the law of total probability to obtain a bound on the unconditional probability that that a vertex r is unmatched after $e_1 = (r, r_1), \ldots, e_k = (r, r_k)$ are processed. This simplifies the study of our recurrence:

Lemma 2.5. *When lower bounding* $\Pr[e \in \mathcal{M} \mid X_e = 1]$ *using the recurrence, we may assume that all the* Q_s *($s \in N^{\ell}(e)$) are deterministically 0 or 1, and furthermore, that* $Q_s = 1$ *for all s when ℓ is odd, and* $Q_s = 0$ *for all s when ℓ is even.*

Of course, the OCRS itself does not choose Q_s deterministically; the lemma merely tells us that this is a convenient way to analyze the recurrence. As a consequence of the lemma, we will stop thinking about the Q_s as random variables, and instead say $Q_s = q_s \in \{0, 1\}$ for $s \in N^l(e)$.

Let $x_r = \sum_{j=1}^{k} x_{r,r_j}$ for convenience, where $x_r = 0$ if $k = 0$ (as x_r never includes the parent of r). We wish to show that if r is "high-enough" up the tree, then $q_r \approx (1 - x_r \alpha)$. To this end, define $\varepsilon_r = 1 - \frac{q_r}{1 - \alpha x_r}$. Our goal is to convert the recurrence (2.4) on the q_r into a recurrence on the ε_r. This proceeds similarly to Kulkarni et al. (2022), though we include the derivation for completeness. We first use the equation $q_{r_i} = (1 - \varepsilon_{r_i})(1 - \alpha x_{r_i})$ for $i = 1, \ldots, k$. By applying this

to the right-hand of (2.4),

$$q_r = \prod_{i=1}^{k}\left(1 - \frac{q_{r_i}\alpha x_{e_i}}{(1-\alpha x_r(e_i))(1-\alpha x_{r_i}(e_i))}\right) = \prod_{i=1}^{k}\left(1 - \frac{(1-\varepsilon_{r_i})(1-\alpha x_{r_i})\alpha x_{e_i}}{(1-\alpha x_r(e_i))(1-\alpha x_{r_i}(e_i))}\right)$$

$$= \prod_{i=1}^{k}\left(1 - (1-\varepsilon_{r_i})\frac{\alpha x_{e_i}}{(1-\alpha x_r(e_i))}\right)$$

$$= \prod_{i=1}^{k}\left(\frac{1-\alpha x_r(e_i) - \alpha x_{e_i} + \varepsilon_{r_i}\alpha x_{e_i}}{1 - \alpha x_r(e_i)}\right)$$

$$= \prod_{i=1}^{k}\left(\frac{1 - \alpha\sum_{j\le i}x_{r,r_j} + \varepsilon_{r_i}\alpha x_{e_i}}{1 - \alpha\sum_{j<i}x_{r,r_j}}\right)$$

$$= \prod_{i=1}^{k}\frac{1-\alpha\sum_{j\le i}x_{r,r_j}}{1-\alpha\sum_{j<i}x_{r,r_j}}\left(1 + \frac{\varepsilon_{r_i}\alpha x_{e_i}}{1-\alpha\sum_{j\le i}x_{r,r_j}}\right)$$

$$= (1 - \alpha x_r)\prod_{i=1}^{k}\left(1 + \frac{\varepsilon_{r_i}\alpha x_{e_i}}{1-\alpha\sum_{j\le i}x_{r,r_j}}\right).$$

Thus, applied to (2.4), since $\varepsilon_r = 1 - \frac{q_r}{1-\alpha x_r}$, we get that

$$\varepsilon_r = 1 - \prod_{i=1}^{k}\left(1 + \frac{\varepsilon_{r_i}\alpha x_{e_i}}{1-\alpha\sum_{j\le i}x_{r,r_j}}\right). \tag{2.5}$$

Note that for $r \in N^\ell(e)$, since $q_r \in \{0,1\}$, $\varepsilon_r = 1$ if $q_r = 0$, and $\varepsilon_r = -\alpha x_r/(1-\alpha x_r) \ge -\alpha/(1-\alpha)$ if $q_r = 1$. Finally, $\varepsilon_r = 0$ if r has no children.

Given $0 \le j \le \ell$, define the *maximum error* at distance $\ell - j$ from e by $\varepsilon_j^{\max} := \max_{r\in N^{\ell-j}(e)}|\varepsilon_r|$. We argue that the maximum error *decreases* the larger j becomes (and so higher up the tree we go).

Lemma 2.6. *For each* $0 \le j \le \ell - 1$, $\varepsilon_{j+1}^{\max} \le \varepsilon_j^{\max}\left(\frac{\alpha}{1-\alpha}\right)$. *Thus*, $\varepsilon_\ell^{\max} \le \left(\frac{\alpha}{1-\alpha}\right)^{2\lceil\frac{\ell}{2}\rceil}$, *since* $\varepsilon_0^{\max} \le \alpha/(1-\alpha)$ *if* ℓ *is odd, and* $\varepsilon_0^{\max} \le 1$ *otherwise.*

Proof of Lemma 2.6. Fix $r \in N^{\ell-j-1}(e)$, and assume that r has $k \ge 1$ children, as otherwise $\varepsilon_r = 0$. Then, since $|\varepsilon_{r_i}| \le \varepsilon_\ell^{\max}$ for $i \in [k]$,

$$\varepsilon_r = 1 - \prod_{i=1}^{k}\left(1 + \frac{\varepsilon_{r_i}\alpha x_{e_i}}{1-\alpha\sum_{i'\le i}x_{r,r_{i'}}}\right)$$

$$\le 1 - \prod_{i=1}^{k}\left(1 - \frac{\varepsilon_\ell^{\max}\alpha x_{e_i}}{1-\alpha\sum_{i'\le i}x_{r,r_{i'}}}\right) \le 1 - \left(1 - \frac{\varepsilon_\ell^{\max}\alpha x_r}{1-\alpha x_r}\right) \le \varepsilon_\ell^{\max}\left(\frac{\alpha}{1-\alpha}\right),$$

where the second inequality uses Lemma 2.7, and the last uses $x_r \le 1$. Similarly, $\varepsilon_r \ge -\varepsilon_\ell^{\max}\left(\frac{\alpha}{1-\alpha}\right)$, and so $|\varepsilon_r| \le \varepsilon_\ell^{\max}\cdot\left(\frac{\alpha}{1-\alpha}\right)$ for all $r \in N^{\ell-j-1}(e)$. $\qquad\square$

The following optimization problem is critical to the proof of Lemma 2.6:

Lemma 2.7. *Fix $k \in \mathbb{N}$, $\alpha \in (0, 1)$, and $\varepsilon, \bar{x} \in [0, 1]$. Suppose $\boldsymbol{x} = (x_i)_{i=1}^k$ has non-negative entries, and $\sum_{i=1}^k x_i = \bar{x}$. Then, by maximizing (respectively, minimizing) over all such $\boldsymbol{x}$:*

$$\max_{\boldsymbol{x}} \prod_{i=1}^k \left(1 + \frac{\varepsilon \alpha x_i}{1 - \alpha \sum_{j \leq i} x_j} \right) = 1 + \frac{\varepsilon \alpha \bar{x}}{1 - \alpha \bar{x}}, \quad \min_{\boldsymbol{x}} \prod_{i=1}^k \left(1 - \frac{\varepsilon \alpha x_i}{1 - \alpha \sum_{j \leq i} x_j} \right) = 1 - \frac{\varepsilon \alpha \bar{x}}{1 - \alpha \bar{x}}.$$

Proof of Theorem 2.1. It suffices to prove (2.3). First observe that

$$\Pr[e \in \mathcal{M} \mid X_e = 1] = \Pr[u \text{ and } v \text{ unmatched when } e \text{ arrives}] \frac{\alpha}{(1 - \alpha x_u(e))(1 - \alpha x_v(e))}.$$

Now, by Lemma 2.4 and Lemma 2.5, $\Pr[u \text{ and } v \text{ unmatched when } e \text{ arrives}] \geq q_u q_v$. But, $q_u = (1 - \alpha x_u(e))(1 - \varepsilon_u)$ and $q_v = (1 - \alpha x_v(e))(1 - \varepsilon_v)$. Thus, since $\max\{\varepsilon_u, \varepsilon_v\} \leq \left(\frac{\alpha}{1-\alpha} \right)^{2\lceil \frac{\ell}{2} \rceil}$ by Lemma 2.6, it follows that $\Pr[e \in \mathcal{M} \mid X_e = 1] \geq (1 - \left(\frac{\alpha}{1-\alpha} \right)^{2\lceil \frac{\ell}{2} \rceil})^2 \alpha$ after cancellation, and so (2.3) is proven. The proof of Theorem 2.1 is thus complete. $\qquad\square$

2.2 Reduction to Locally-Tree-Like Edges: Proving Theorem 2.2

We shall make use of the following elementary coupling.

Proposition 2.8. *Suppose $x, y, z \in (0, 1)$ satisfy $x \geq yz$. There exists a coupling of Bernoulli random variables $X \sim \mathrm{Ber}(x)$, $Z \sim \mathrm{Ber}(z)$, and $Y \sim \mathrm{Ber}(y)$, where Y, Z are independent, and $X \geq Y \cdot Z$.*

Suppose that $(G, \boldsymbol{x}) = (G(n), \boldsymbol{x}(n))$ satisfies $x_e \leq \varepsilon$ for all $e \in E$, where $\varepsilon = \varepsilon(n)$ satisfies $\varepsilon = o(1)$. Fix $\ell = \ell(n)$ where $\ell(n) = \log\log(1/\varepsilon)$. Observe that $\ell(n) \to \infty$ as $n \to \infty$, since $\varepsilon = o(1)$. We describe a CRS ψ for $(G, \boldsymbol{x})$ which makes use of the CRS ψ_ℓ assumed in Theorem 2.2. The idea is to first write $y_e = (1 - 1/\log^{1/4}(\varepsilon^{-1})) x_e \log(1/\varepsilon)$ and $z_e = 1/\log(1/\varepsilon)$, where we observe that $x_e \geq y_e z_e$ for each e. There are then 3 steps to ψ, which we describe below:

1. Sample $G' \subseteq G$: Apply Proposition 2.8 to each $e \in E$, to get $(Y_e)_{e \in E}$ and $(Z_e)_{e \in E}$ such that $X_e \geq Y_e Z_e$, and for which $((Y_e)_{e \in E}, (Z_e)_{e \in E})$ are independent. Let $G' = (V, E')$ be the subgraph of G where $E' := \{e \in E : Y_e = 1\}$.

2. Compute a fractional matching $(z'_e)_{e \in E}$ of G supported on E': Initialize $z'_e := 0$ for each $e \in E$. Then, process the edges $e = (u, v)$ of E in an arbitrary order (such as the one presented by an oblivious adversary if we wish ψ to be an OCRS), and update $z'_e \leftarrow 1/\log(1/\varepsilon)$, if the (current) fractional matching constraints on u and v are not violated. (See Algorithm 2 below for a formal description of this step).

3. Run ψ_g on $(G', \boldsymbol{z}')$ where $\boldsymbol{z}' = (z'_e)_{e \in E'}$: For each $e \in E$, define $Z'_e = Z_e \cdot \mathbf{1}_{[z'_e > 0]}$. Execute ψ_g on $(G', \boldsymbol{z}')$ using $(Z'_e)_{e \in E'}$, and return the matching $\psi_g(G', \boldsymbol{z}')$.

Let us denote $\mathcal{M} = \psi_\ell(G', \boldsymbol{z}')$ to be the matching output by ψ. First observe that $\mathcal{M}$ only contains active edges of G, and so ψ is indeed a valid CRS. To see this, observe that a necessary condition for $e \in \mathcal{M}$ is that $Y_e Z'_e = 1$. But since $X_e \geq Y_e Z_e$ by Proposition 2.8, and $Z'_e := Z_e \cdot \mathbf{1}_{[z'_e > 0]}$, it follows that $X_e = 1$.

Algorithm 2 Fractional-Matching

Input: $G = (V, E)$ and $\varepsilon \in [0, 1]$.

Output: a fractional matching $\mathbf{z}' = (z'_e)_{e \in E}$ of G supported on E'.

1: Set $z'_e = 0$ for all $e \in E$.

2: **for** arriving $e = (u, v) \in E$ in an arbitrary order $<$ **do**

3:      **if** $Y_e = 1$, and $\max\{\sum_{\substack{f \in \partial_G(v): \\ f < e}} z'_f, \sum_{\substack{f \in \partial_G(u): \\ f < e}} z'_f\} \leq 1 - \frac{1}{\log(1/\varepsilon)}$ **then**

4:          $z'_e \leftarrow \frac{1}{\log(1/\varepsilon)}$.

5: **return** $\mathbf{z}' = (z'_e)_{e \in E}$.

Remark 2.9. Algorithm 2 is stated and analyzed for an arbitrary edge ordering. When proving the OCRS part of Theorem 2.2, we will take this to be the one chosen by an oblivious adversary.

To analyze ψ, we prove three lemmas. The first is a technical point that ensures that once we condition on G', the random variables $(Z'_e)_{e \in E}$ are independent. This ensures that we are indeed passing a valid CRS input to ψ_ℓ in step 3. This follows from the fact that $(z'_e)_{e \in E}$ is a function of G', and $((Y_e)_{e \in E}, (Z_e)_{e \in E})$ are independent by construction.

Lemma 2.10. *Conditional on G' (equivalently $(Y_e)_{e \in E}$), the random variables $(Z'_e)_{e \in E}$ are independently distributed with marginals described by $(z'_e)_{e \in E}$.*

Proof of Lemma 2.10. Observe that the fractional matching $(z'_e)_{e \in E}$ is a function of $(Y_e)_{e \in E}$ (i.e., it is $(Y_e)_{e \in E}$-measurable). Thus, if we define the random subset $\mathcal{S} := \{e \in E : z'_e > 0\}$, then $\mathcal{S}$ is also a function of $(Y_e)_{e \in E}$. On the other hand, observe that once $\mathcal{S}$ is fixed, $(Z'_e)_{e \in \mathcal{S}}$ and $(Z_e)_{e \in \mathcal{S}}$ are equal, and so distributed identically. Since the random variables $((Y_e)_{e \in E}, (Z_e)_{e \in E})$ are independent by construction, and $Z'_e = 0$ for all $e \in E \setminus \mathcal{S}$, the claim follows. $\square$

We next show that if $Y_e = 1$, then w.h.p. $z'_e = 1/\log(1/\varepsilon)$. In other words, z'_e will typically be rounded up to the desired value, assuming e is an edge of G'.

Lemma 2.11. *For each $e \in E$, $\Pr[z'_e = 1/\log(1/\varepsilon) \mid Y_e = 1] \geq 1 - 2\exp\left(-\Omega(\log^{1/2}(1/\varepsilon))\right)$.*

Proof of Lemma 2.11. Consider the randomly generated fractional matching $(z'_f)_{f \in E}$ output by Algorithm 2. Given $e = (u, v) \in E$, define $z'_v(e) := \sum_{\substack{f \in \partial(v): \\ f < e}} z'_f$ and $z'_u(e) := \sum_{\substack{f \in \partial(u): \\ f < e}} z'_f$. Observe that $z'_e = 1/\log(1/\varepsilon)$ if and only if $Y_e = 1$ and $\max\{z'_v(e), z'_u(e)\} \leq 1 - \frac{1}{\log(1/\varepsilon)}$. On the other hand, since $z'_v(e)$ and $z'_u(e)$ are independent of Y_e, to prove the lemma it suffices to show that

$$\Pr\left[\max\{z'_v(e), z'_u(e)\} \leq 1 - \frac{1}{\log(1/\varepsilon)}\right] \geq 1 - 2\exp\left(-\Omega(\log^{1/2}(1/\varepsilon))\right) \tag{2.6}$$

We begin by considering $z'_v(e)$. Observe that

$$z'_v(e) = \sum_{\substack{f \in \partial(v): \\ f < e}} z'_f \leq \sum_{f \in \partial(v) \setminus e} \frac{Y_f}{\log(1/\varepsilon)}, \tag{2.7}$$

where $(Y_f)_{f \in \partial(v) \setminus e}$ are independently drawn Bernoulli random-variables. Moreover,

$$\sum_{f \in \partial(v) \setminus e} \mathbb{E}[Y_f] = \sum_{f \in \partial(v) \setminus e} (1 - \log^{-1/4}(\varepsilon^{-1})) x_f \log(1/\varepsilon) \leq (1 - \log^{-1/4}(\varepsilon^{-1})) \log(1/\varepsilon),$$

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since $\sum_{f \in \partial(v)} x_f \le 1$. Setting $\delta = \log^{-1/4}(\varepsilon^{-1})$ and $\mu = (1 - \log^{-1/4}(\varepsilon^{-1})) \log(1/\varepsilon)$, we can thus apply Chernoff bound to get that for n sufficiently large,

$$\Pr\left[\sum_{f \in \partial(v)\backslash e} Y_f \ge (1+\delta)\left(1 - \log^{-1/4}(\varepsilon^{-1})\right) \log(1/\varepsilon)\right] \le \exp\left(-\frac{\delta^2 \mu}{2+\delta}\right) \le \exp\left(-\frac{\log^{1/2}(\varepsilon^{-1})}{3}\right).$$

On the other hand, $(1+\delta)\left(1 - \log^{-1/4}(\varepsilon^{-1})\right) \le 1 - \log^{-1}(1/\varepsilon)$, so

$$\Pr\left[\sum_{f \in \partial(v)\backslash e} Y_f \ge \log(1/\varepsilon)\left(1 - \frac{1}{\log(1/\varepsilon)}\right)\right] \le \exp\left(-\frac{\log^{1/2}(\varepsilon^{-1})}{3}\right).$$

Thus, $\Pr[z_v(e) > 1 - 1/\log(1/\varepsilon)] \le \exp\left(-\Omega(\log^{1/2}(1/\varepsilon))\right)$ after applying this bound to (2.7). An analogous argument applies to $z'_u(e)$, and so (2.6) follows after applying a union bound. The proof of the lemma is thus complete. $\qquad\square$

Define $C_\ell(e)$ to be the event that there *exists* a cycle in the ℓ-neighborhood of e in G'. We next upper bound the probability that $C_\ell(e)$ occurs, conditional on e being in G' (i.e., $Y_e = 1$).

Lemma 2.12. *For each $e \in E$, $\Pr[C_\ell(e) \mid Y_e = 1] = O(\varepsilon(\log(1/\varepsilon))^{2\ell+1})$.*

Proof of Lemma 2.12. In order to prove the lemma, we first upper bound the probability there exists a cycle in $N_{G'}^\ell(e)$ which contains e. Afterwards, we upper bound the probability there exists a cycle in $N_{G'}^\ell(e)$ which does *not* contain e. Both upper bounds are of the form $O(\varepsilon(\log(1/\varepsilon))^{2\ell+1})$, so a simple union bound will then complete the proof.

Let N_k be a random variable counting the number of k length cycles in G' which contain e. Since we only care about cycles of this form in the ℓ-neighborhood of e, it suffices to prove that

$$\Pr\left[\sum_{k=3}^{2\ell+2} N_k > 0 \,\middle|\, Y_e = 1\right] = O\left(\varepsilon(\log(1/\varepsilon))^{2\ell+1}\right). \tag{2.8}$$

Assume for now that

$$\mathbb{E}[N_k \mid Y_e = 1] \le \varepsilon(\log(1/\varepsilon))^{k-1}. \tag{2.9}$$

In this case, by Markov's inequality and linearity of expectation,

$$\Pr\left[\sum_{k=3}^{2\ell+2} N_k > 0 \,\middle|\, Y_e = 1\right] \le \sum_{k=3}^{2\ell+2} \varepsilon(\log(1/\varepsilon))^{k-1} = O\left(\varepsilon(\log(1/\varepsilon))^{2\ell+1}\right),$$

where the final inequality holds from the geometric series formula.

In order to establish (2.9), let $w_1, w_2, \ldots, w_{k-2}$ denote $k-2$ distinct vertices in G, all of them different from u and v. For convenience, let us write $u = w_0$, and $v = w_{k-1}$. Then, we have that:

$$\mathbb{E}[N_k \mid Y_e = 1] = \sum_{w_1, w_2, \ldots, w_{k-2}} \prod_{i=0}^{k-2} y_{w_i w_{i+1}} \le (\log(1/\varepsilon))^{k-1} \sum_{w_1, w_2, \ldots, w_{k-2}} \prod_{i=0}^{k-2} x_{w_i w_{i+1}},$$

since $y_e \le x_e \log(1/\varepsilon)$. Now, we know $x_{w_{k-2}v} \le \epsilon$, for all choices of w_{k-2}. Therefore, it follows that:

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$$\mathbb{E}[N_k \mid Y_e = 1] \leq \varepsilon(\log(1/\varepsilon))^{k-1} \sum_{w_1,w_2,\ldots,w_{k-2}} \prod_{i=0}^{k-3} x_{w_i w_{i+1}}.$$

Finally, we will show using a simple induction on k that

$$\sum_{w_1,w_2,\ldots,w_{k-2}} \prod_{i=0}^{k-3} x_{w_i w_{i+1}} \leq 1. \tag{2.10}$$

The statement is true for $k = 3$, since $\sum_{w_1} x_{uw_1} \leq 1$ as $\boldsymbol{x}$ is fractional matching. Suppose we have already established the statement for $k = r \geq 3$. Then, for $k = r + 1$,

$$\sum_{w_1,w_2,\ldots,w_{r-1}} \prod_{i=0}^{r-2} x_{w_i w_{i+1}} = \sum_{w_1,w_2,\ldots,w_{r-2}} \prod_{i=0}^{r-3} x_{w_i w_{i+1}} \cdot \left(\sum_{w_{r-1}} x_{w_{r-2} w_{r-1}} \right)$$

$$\leq \sum_{w_1,w_2,\ldots,w_{r-2}} \prod_{i=0}^{r-3} x_{w_i w_{i+1}} \leq 1,$$

where the first inequality follows from the fact that $\boldsymbol{x}$ is a fractional matching, and the second from the induction hypothesis. We conclude that (2.10) as desired, and so (2.9) holds.

Next, let us consider cycles in $N_{G'}^{\ell}(e)$ which do *not* contain the edge e. The existence of such a cycle guarantees the existence of a path of length $k \leq \ell - 1$ from either u of v to a distinguished vertex w (possibly equal to u or v), and a cycle of length at most $2(\ell - k) + 1$ that contains w. Note that we can upper bound the probability that such a structure by:

$$\sum_{w \in V, k \leq \ell - 1} \Pr[\exists \text{ a path of length } k \text{ from } u \text{ to } w] \Pr[\exists \text{ a cycle of length } \leq 2(g - k) + 1 \text{ containing } w].$$

Now, first of all,

$$\Pr[\text{there is a cycle of length} \leq 2(g - k) + 1 \text{ containing } w] = O(\varepsilon(\log(1/\varepsilon))^{2(g-k)+1}),$$

by an argument analogous to the one used to establish (2.8) (we let $w_0 = w_{g-k-1} = w$). Furthermore, note that if we write $u = w_0$, and $w = w_k$, then:

$$\sum_{w \in G} \Pr[\text{there is path of length } k \text{ from } u \text{ or } v \text{ to } w] \leq 2 \sum_{w \in G} \Pr[\text{there is path of length } k \text{ from } u \text{ to } w]$$

$$\leq 2 \sum_{w_1,w_2,\ldots,w_{k-1},w_k} \prod_{i=0}^{k-1} y_{w_i w_{i+1}}$$

$$\leq 2(\log(1/\varepsilon))^k \sum_{w_1,w_2,\ldots,w_{k-1},w_k} \prod_{i=0}^{k-1} x_{w_i w_{i+1}}$$

$$\leq 2(\log(1/\varepsilon))^k.$$

where the final inequality follows from (2.10). Finally it follows that the probability with which such a structure exists is at most:

$$\sum_{k=0}^{\ell-1} O(\varepsilon(\log(1/\varepsilon))^{2(\ell-k)+1}) \cdot 2(\log(1/\varepsilon))^k = O\left(\varepsilon(\log(1/\varepsilon))^{2\ell+1} \right).$$

Together with the previous upper bound of (2.8), this completes the proof. $\qquad\square$

We are now ready to prove Theorem 2.2.

Proof of Theorem 2.2. In order to analyze ψ on G, fix $e = (u, v) \in E$. Now, $e \in \mathcal{M}$ if and only if $e \in \psi_g(G', \boldsymbol{z}')$. We shall first show that

$$\Pr[e \in \psi_g(G', \boldsymbol{z}')] \geq (1 - o(1))cx_e, \tag{2.11}$$

Recall that $C_\ell(e)$ is the event that there *exists* a cycle in the ℓ-neighborhood of e in G'. Observe that $C_\ell(e)$ is a function of $(Y_f)_{f \in E}$ (i.e., it is $(Y_f)_{f \in E}$-measurable). Moreover, since ψ_ℓ is c_ℓ-selectable on ℓ-locally-tree-like edges by assumption, we can apply Lemma 2.10 to get that

$$\Pr[e \in \psi_\ell(G', \boldsymbol{z}') \mid (Y_f)_{f \in E \setminus e}, Y_e = 1] \cdot \mathbf{1}_{\neg C_\ell(e)} \geq c_\ell z_e' \cdot \mathbf{1}_{\neg C_\ell(e)}.$$

Thus, after taking expectations over $(Y_f)_{f \in E \setminus e}$,

$$\Pr[\{e \in \psi_\ell(G', \boldsymbol{z}')\} \cap \neg C_\ell(e) \mid Y_e = 1] \geq c_\ell \mathbb{E}[z_e' \cdot \mathbf{1}_{\neg C_\ell(e)} \mid Y_e = 1]. \tag{2.12}$$

We now must lower bound $\mathbb{E}[z_e' \cdot \mathbf{1}_{\neg C_\ell(e)} \mid Y_e = 1]$. By applying Lemmas 2.11 and 2.12, we get that

$$\mathbb{E}[z_e' \cdot \mathbf{1}_{\neg C_\ell(e)} \mid Y_e = 1] \geq \frac{1}{\log(1/\varepsilon)} \left(1 - O(\varepsilon(\log(1/\varepsilon))^{2\ell+1}) - 2\exp\left(-\Omega(\log^{1/2}(1/\varepsilon))\right)\right)$$

$$= (1 - o(1)) \frac{1}{\log(1/\varepsilon)},$$

where the final line uses that $\varepsilon(\log(1/\varepsilon))^{2\ell+1} = o(1)$ for $\ell(n) = \log \log(1/\varepsilon)$. By combining everything together,

$$\Pr[e \in \psi_\ell(G', \boldsymbol{z}') \mid Y_e = 1] \geq \Pr[\{e \in \psi_\ell(G', \boldsymbol{z}')\} \cap \neg C_\ell(e) \mid Y_e = 1] \geq (1 - o(1)) \frac{c_\ell}{\log(1/\varepsilon)}.$$

Finally, recalling that $y_e = (1 - 1/\log^{1/4}(\varepsilon^{-1}))x_e, \log(1/\varepsilon)$, we get that

$$\Pr[e \in \psi_\ell(G', \boldsymbol{z}')] \geq (1 - o(1))c_\ell = (1 - o(1))c$$

after cancellation, and using the fact that $c_\ell \to c$ as $n \to \infty$, since $\ell \to \infty$ as $n \to \infty$. Thus, ψ is c-selectable for graphs with vanishing edges values.

To complete the proof, we must argue that if each ψ_ℓ is strongly-online, then ψ can be implemented as an OCRS as the edge states $(X_e)_{e \in E}$ of $(G, \boldsymbol{x})$ are revealed. First observe that ψ is given $(G, \boldsymbol{x})$ in advance, and thus knows ε. This means that steps 1. and 2. (i.e., Algorithm 2) of ψ involving edge $e \in E$ can be implemented when e arrives. In order to execute step 3. online, we pass on matching e if $Y_e = 0$ (i.e., e is not an edge of G'). Otherwise, we make use of the strongly online property of ψ_ℓ: Observe that the values $(z_f', Z_f')_{f \leq e, Y_f = 1}$ are known when e arrives, and determine whether or not $e \in \psi_\ell(\boldsymbol{z}', G')$. Thus, when $Y_e = 1$, we can present ψ_ℓ the values $(z_f', Z_f')_{f \leq e, Y_f = 1}$ to determine whether to match e. The resulting matching will be identical to the offline setting in which ψ_ℓ is given the entire input $(G', \boldsymbol{z}')$ in advance. $\square$

3 OCRS Negative Results

3.1 General Graphs

We define a sequence of inputs dependent on $n \in \mathbb{N}$ to prove the hardness result. Start with a vertex u whose neighbors $N(u) = \{v_1, \ldots, v_n\}$ have edge values $x_{u,v_i} = 1/n$ for $i \in [n]$. Next, add

a clique amongst the vertices of $N(u)$ where $x_{v_i,v_j} = 1/n(n-1)$ for each $i, j \in [n]$, $i \neq j$. Finally, each v_i has its own $n-2$ additional neighbors, denoted W_i, where $x_{v_i,w} = 1/n$ for each $w \in W_i$ and $i = 1, \ldots, n$. We define $W = \cup_{i=1}^{n} W_i$ for convenience, and refer to the input by $(G(n), \boldsymbol{x}(n))$, which we denote by $(G, \boldsymbol{x})$ when clear (see Figure 1 for an illustration). We next specify the order in which the edges are presented to the OCRS. This is done in 3 *phases*:

1. Present the edges of $N(u) \times W$ in an arbitrary order.

2. Present the edges between the vertices of $N(u)$ in an arbitrary order.

3. Present the edges of $\partial(u)$ in an arbitrary order.

We refer to an order specified in this way as a *phase-based order*. Our negative result holds for any such order, even if it is presented to the OCRS ahead of time. Let us define $\beta \in (0, 1)$ to be the unique solution to the below equation:

$$0 = 1 - 2\beta - \beta \exp(2 - 1/\beta), \tag{3.1}$$

where $\beta \approx 0.3895$.

Theorem 3.1. *Any OCRS which processes the edges of $(G, \boldsymbol{x})$ in a phase-based order is at most $(\beta + o(1))$-selectable on $(G, \boldsymbol{x})$.*
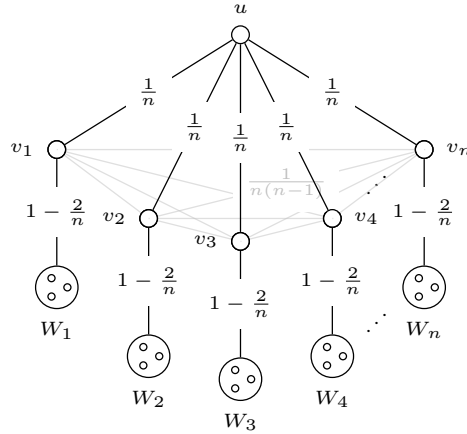

Figure 1: Graph to prove negative result. Each W_i consists of $n-2$ vertices.

In order to prove Theorem 3.1, let us fix an OCRS for $(G, \boldsymbol{x})$ which returns the matching $\mathcal{M}$ when presented the edges in a phase-based order, and which is c-selectable for $c \in (0, 1)$. Our goal is to upper bound c. It is not hard to show that without loss of generality, we may assume that the OCRS is *exactly*[6] c-selectable. That is, $\Pr[e \in \mathcal{M} \mid X_e = 1] = c$ for each edge e of G.

Define M_{v_i} (respectively, $A_{v_i} = 1 - M_{v_i}$) to be an indicator random variable for the event vertex v_i is matched (respectively, not matched) after the first phase. Observe first that for each $i \in [n]$,

$$\mathbb{E}[M_{v_i}] = \Pr[\cup_{w \in W_i}(v_i, w) \in \mathcal{M}] = \sum_{w \in W_i} c x_{v,w} = c \left(1 - \frac{2}{n}\right), \tag{3.2}$$

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[6] Given a c-selectable OCRS for $(G, \boldsymbol{x})$ which returns the matching $\mathcal{M}$, we can turn it into an exactly c-selectable OCRS by independently *dropping* any selected edge with probability $\Pr[e \in \mathcal{M} \mid X_e = 1] - c$. This reduction can be executed as the edges arrive online, and it preserves concentration properties on $(G, \boldsymbol{x})$, provided c is bounded away from 0.

where the last equality uses that $\sum_{w\in W_i} x_{v_i,w} = 1-2/n$. At this point, let us consider the minimum covariance amongst the $(A_{v_i})_{i=1}^n$ random variables. Specifically, define $\theta := \min_{i\neq j} \operatorname{Cov}[A_{v_i}, A_{v_j}]$, where $\operatorname{Cov}[A_{v_i}, A_{v_j}] := \mathbb{E}[A_{v_i} A_{v_j}] - \mathbb{E}[A_{v_i}]\mathbb{E}[A_{v_j}]$. We first lower bound θ as a function of c for $n \to \infty$. This gives a tighter upper bound on c the *smaller* θ is.

Lemma 3.2 (Small covariance regime). $\theta \geq (1+o(1))(c-(1-c)^2)$.

Proof of Lemma 3.2. Let suppose that $v_i \neq v_j$ satisfy $\theta = \operatorname{Cov}[A_{v_i}, A_{v_j}]$. Consider when (v_i, v_j) is presented to the OCRS in the second phase, and (v_i, v_j) is active. Observe that in order for (v_i, v_j) to be matched, both v_i and v_j must be *available* (i.e., $A_{v_i} = A_{v_j} = 1$). Thus, since A_{v_i} and A_{v_j} are independent of X_{v_i,v_j},

$$c = \Pr[(v_i, v_j) \in \mathcal{M} \mid X_{v_i,v_j} = 1] \leq \Pr[A_{v_i} = 1, A_{v_j} = 1] = \mathbb{E}[A_{v_i}]\mathbb{E}[A_{v_j}] + \operatorname{Cov}[A_{v_i}, A_{v_j}]$$
$$\leq \left(1 - c\left(1 - \frac{2}{n}\right)\right)^2 + \theta,$$

where the final inequality uses (3.2) together with $A_{v_i} = 1 - M_{v_i}$, as well as the definition of θ. As such, after rearranging for θ and taking $n \to \infty$,

$$\theta \geq c - \left(1 - c\left(1 - \frac{2}{n}\right)\right)^2 = (1+o(1))(c-(1-c)^2).$$

$\square$

We next upper bound a function of c and θ by 1. This leads to a tighter upper bound on c the *larger* θ is.

Lemma 3.3 (Large covariance regime). $2c + c\exp(c + \theta/c - 1) \leq 1 + o(1)$.

Proof of Lemma 3.3. In order to prove the lemma, we analyze the third and final phase. Let us assume that the edges of $\partial(u)$ are presented in order $(u, v_1), \ldots, (u, v_n)$ to the OCRS, and consider the arrival of the last edge $e_n = (u, v_n)$. Define $M_u(e_n)$ to be the indicator random variable for the event u is matched when e_n arrives. Observe then that conditional on $X_{e_n} = 1$, e_n is matched only if $M_{v_n} = 0$ *and* $M_u(e_n) = 0$. Thus, since X_{e_n} is independent of the latter two events,

$$c = \Pr[e_n \in \mathcal{M} \mid X_{e_n} = 1] \leq \Pr[M_{v_n} = 0, M_u(e_n) = 0]$$
$$= 1 - \Pr[M_{v_n} = 1] - \Pr[M_u(e_n) = 1] + \Pr[M_{v_n} = 1, M_u(e_n) = 1]$$
$$= 1 - \Pr[M_{v_n} = 1](1 - \Pr[M_u(e_n) = 1 \mid M_{v_n} = 1]) - \Pr[M_u(e_n) = 1]$$

Now, $\Pr[M_u(e_n) = 1] = \sum_{i=1}^{n-1} \Pr[(u, e_i) \in \mathcal{M}] = c\left(1 - \frac{1}{n}\right)$, so after simplifying and taking $n \to \infty$, we get that

$$2c + \Pr[M_{v_n} = 1](1 - \Pr[M_u(e_n) = 1 \mid M_{v_n} = 1]) \leq 1 + o(1). \tag{3.3}$$

We shall now prove that

$$\Pr[M_u(e_n) = 1 \mid M_{v_n} = 1] \leq (1+o(1))(1 - \exp(c + \theta/\Pr[M_{v_n} = 1] - 1)). \tag{3.4}$$

This will complete the proof. To see this, first observe that by applying (3.4) to (3.3),

$$2c + \Pr[M_{v_n} = 1](\exp(c + \theta/\Pr[M_{v_n} = 1] - 1)) \leq 1 + o(1).$$

Thus, since $\Pr[M_{v_n} = 1] = c(1 - 2/n)$ by (3.2), $2c + c(\exp(c + \theta/c - 1)) \leq 1 + o(1)$ as required.

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In order to prove (3.4), recall that $A_{v_n} = 1 - M_{v_n}$. We shall instead condition on $A_{v_n} = 0$, and upper bound $\Pr[M_u(e_n) = 1 \mid A_{v_n} = 0]$. Consider the vertices amongst $v_1, \ldots, v_{n-1}$ which were available after the first phase. Observe that if we condition on $A_{v_n} = 0$, then in order for $M_u(e_n) = 1$, there must be an active edge from u to one of these vertices. Thus,

$$\Pr[M_u(e_n) = 1 \mid A_{v_n} = 0] \le \Pr[\cup_{j=1}^{n-1}\{X_{u,v_j} = 1\} \cap \{A_{v_j} = 1\} \mid A_{v_n} = 0]. \tag{3.5}$$

On the other hand, the edge states $(X_{u,v_j})_{j=1}^n$ are independent from $(A_{v_i})_{i=1}^n$, and so

$$\Pr[\cup_{j=1}^{n-1}\{X_{u,v_j} = 1\} \cap \{A_{v_j} = 1\} \mid (A_{v_i})_{i=1}^{n-1}, A_{v_n} = 0] = 1 - \prod_{j=1}^{n-1}\left(1 - \frac{A_{v_j}}{n}\right)$$

$$= (1 + o(1))\left(1 - \exp\left(-\frac{\sum_{j=1}^{n-1} A_{v_j}}{n}\right)\right).$$

Now, the function $z \to 1 - \exp(-z)$ is concave for $z \in [0, 1]$. Thus, if we take expectations over the $(A_{v_i})_{i=1}^{n-1}$ random variables (while conditioning on $A_{v_n} = 0$) then Jensen's inequality implies that

$$\Pr[\cup_{j=1}^{n-1}\{X_{u,v_j} = 1\} \cap \{A_{v_j} = 1\} \mid A_{v_n} = 0] \le (1 + o(1))\left(1 - \exp\left(-\frac{\sum_{j=1}^{n-1}\mathbb{E}[A_{v_j} \mid A_{v_n} = 0]}{n}\right)\right),$$

and so combined with (3.5), we get that

$$\Pr[M_u(e_n) = 1 \mid A_{v_n} = 0] \le (1 + o(1))\left(1 - \exp\left(-\frac{\sum_{i=1}^{n-1}\mathbb{E}[A_{v_i} \mid A_{v_n} = 0]}{n}\right)\right). \tag{3.6}$$

As such, we focus on upper bounding $\sum_{i=1}^{n-1}\mathbb{E}[A_{v_i} \mid A_{v_n} = 0]$. We argue that for each $1 \le i \le n-1$, the events $A_{v_n} = 0$ and $A_{v_i} = 1$ are *negatively* correlated, which we quantify in terms of θ. Specifically, by the definition of θ, $\mathbb{E}[A_{v_i} \mid A_{v_n} = 1]\Pr[A_{v_n} = 1] = \mathbb{E}[A_{v_i}]\mathbb{E}[A_{v_n}] + \mathrm{Cov}[A_{v_i}, A_{v_n}] \ge \mathbb{E}[A_{v_i}]\mathbb{E}[A_{v_n}] + \theta$. Thus,

$$\mathbb{E}[A_{v_i} \mid A_{v_n} = 0]\Pr[A_{v_n} = 0] = \mathbb{E}[A_{v_i}] - \mathbb{E}[A_{v_i} \mid A_{v_n} = 1]\Pr[A_{v_n} = 1]$$

$$\le \mathbb{E}[A_{v_i}] - \mathbb{E}[A_{v_i}]\mathbb{E}[A_{v_n}] - \theta$$

$$= \mathbb{E}[A_{v_i}]\Pr[A_{v_n} = 0] - \theta.$$

Since $\Pr[A_{v_n} = 0] > 0$ for $n > 2$, and $\mathbb{E}[A_{v_i}] = 1 - c\left(1 - \frac{2}{n}\right)$, it follows that

$$\sum_{i=1}^{n-1}\mathbb{E}[A_{v_i} \mid A_{v_n} = 0] \le (1 + o(1))n\left((1 - c) - \frac{\theta}{\Pr[A_{v_n} = 0]}\right). \tag{3.7}$$

By applying (3.7) to (3.6), $\Pr[M_u(e_n) = 1 \mid A_{v_n} = 0] \le (1 + o(1))(1 - \exp(-1 + c + \theta/\Pr[A_{v_n} = 0]))$, and so (3.4) is proven, thereby completing the proof. □

We are now ready to prove Theorem 3.1. We combine the inequalities from Lemmas 3.2 and 3.3 to determine the optimal choice of θ for the OCRS.

Proof of Theorem 3.1. Recall that β satisfies (3.1). Now, for any OCRS which is exactly c-selectable on $(G, \boldsymbol{x})$, Lemmas 3.2 and 3.3 together imply that

$$2c + c\exp(c + (c - (1 - c)^2)/c - 1) \le 1 + o(1).$$

Thus, $1 - 2c - c\exp(2 - 1/c) \ge o(1)$ after rearranging and simplifying. Now, the function $z \to 1 - 2z - z\exp(2 - 1/z)$ is continuous, and non-negative for $z \le \beta$, so we may conclude that $c \le \beta + o(1)$, as required. □

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3.2 Trees

In this section, we first establish a negative result for *concentrated* OCRS's on trees. We refer to an OCRS as *concentrated* on $(G, \boldsymbol{x})$, provided the matching $\mathcal{M}$ on $(G, \boldsymbol{x})$ it returns satisfies $||\mathcal{M}| - \mathbb{E}[|\mathcal{M}|]| = o(\mathbb{E}[|\mathcal{M}|])$ w.h.p. Throughout this section, let us fix a particular concentrated OCRS for $(G, \boldsymbol{x})$ which returns a matching $\mathcal{M}$, and is c-selectable for $c \in (0, 1)$, i.e., $\Pr[e \in \mathcal{M} \mid X_e = 1] = c$ for each edge e of G.

As in the hardness result for general graphs, we begin by defining a sequence of inputs dependent on $n \in \mathbb{N}$ to prove the hardness result. Start with a vertex u whose neighbors $N(u) = \{v_1, \ldots, v_n\}$ have edge values $x_{u,v_i} = 1/n$ for $i \in [n]$. Each v_i has its own $n - 1$ additional neighbors, denoted W_i, where $x_{v_i, w} = 1/n$ for each $w \in W_i$ and $i = 1, \ldots, n$. We refer to this input by $(G(n), \boldsymbol{x}(n))$ (see Figure 2 for an illustration). The edges are presented to the OCRS in 2 phases. In the first
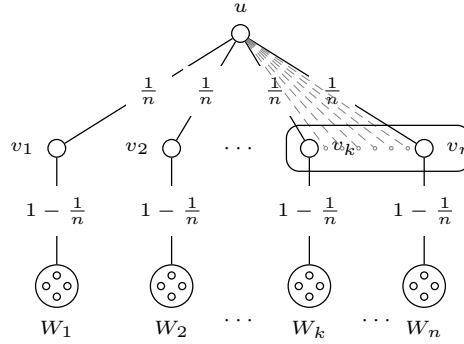

Figure 2: Tree to prove negative result. Each W_i consists of $n - 1$ vertices.

phase, the edges of $N(u) \times \cup_{i=1}^n W_i$ are presented in an arbitrary order. At this point, we define M_{v_i} (respectively, $A_{v_i} = 1 - M_{v_i}$) as an indicator random variable for the event that vertex v_i is matched (respectively, not matched) after the first phase. Before we can discuss the order in which the edges of the second phase are presented to the OCRS, we must briefly digress to discuss an important property of a concentrated OCRS.

Since $\sum_{i=1}^n M_{v_i}$ and $|\mathcal{M}|$ differ by at most 1 and we've assumed that $|\mathcal{M}|$ is concentrated about $\mathbb{E}[|\mathcal{M}|]$, we know that $\sum_{i=1}^n M_{v_i}$ is also concentrated. Thus, by applying elementary bounds, we get that $\mathrm{Var}[\sum_{i=1}^n M_{v_i}] = o(n^2)$, and so we must have that $\mathrm{Var}[\sum_{i=1}^n A_{v_i}] = o(n^2)$. For concreteness, let us say that $\mathrm{Var}[\sum_{i=1}^n A_{v_i}] \leq \theta_n n^2$, with $\theta_n \to 0$, and $\theta_n \geq \frac{1}{\sqrt{n}}$. Note that since we are only demanding an upper bound on $\mathrm{Var}[\sum_{i=1}^n A_{v_i}]$, we can assume w.l.o.g. $\theta_n \geq \frac{1}{\sqrt{n}}$. This will just be a convenient upper bound to use on the 'true' θ_n in the following calculations.

We have the following probabilistic lemma:

Lemma 3.4. *Suppose* $n \to \infty$. *Let* $A_1, A_2, \ldots, A_n$ *be random variables, with* $\mathrm{Var}(A_i) < \delta$. *Define* $A = \sum_{i=1}^n A_i$, *and* $A_S = \sum_{i \in S} A_i$. *Suppose* $\mathrm{Var}(A) \leq \theta_n n^2$ *and* $\theta_n \geq \frac{1}{\sqrt{n}}$. *Then, we can find* $S \subseteq \{1, 2, \ldots, n\}$ *with* $|S| = m = \theta_n^{1/4} n$ *such that* $\mathrm{Var}(A_S) \leq 3\theta_n m^2$ *for sufficiently large* n.

Proof of Lemma 3.4. Since

$$\mathrm{Var}(A) = \sum_{i \neq j} \mathrm{Cov}(A_i, A_j) + \sum_{i=1}^n \mathrm{Var}(A_i),$$

it follows that $\sum_{i \neq j} \mathrm{Cov}(A_i, A_j) \leq \theta_n n^2$. On the other hand,

$$\sum_{|S|=m} \mathrm{Var}(A_S) \leq \sum_{|S|=m} \left(\sum_{i \neq j, i, j \in S} \mathrm{Cov}(A_i, A_j) + \delta m \right)$$

$$= \left(\sum_{|S|=m} \sum_{i \neq j, i, j \in S} \mathrm{Cov}(A_i, A_j) \right) + \binom{n}{m} \cdot \delta m$$

$$= \sum_{i \neq j} \binom{n-2}{m-2} \mathrm{Cov}(A_i, A_j) + \binom{n}{m} \cdot \delta m.$$

It follows that

$$\frac{\sum_{|S|=m} \mathrm{Var}(A_S)}{\binom{n}{m}} \leq \frac{\binom{n-2}{m-2}}{\binom{n}{m}} \left(\sum_{i \neq j} \mathrm{Cov}(A_i, A_j) \right) + \delta m$$

$$\leq \frac{m(m-1)}{n(n-1)} \theta_n n^2 + \delta m$$

$$\leq 2\theta_n m^2 + \delta m.$$

Next, notice that,

$$\delta m \leq \theta_n m^2 \iff \delta \leq \theta_n m$$

$$\iff \delta \leq \theta_n^{5/4} n$$

$$\iff \delta \leq \theta_n^{5/4} n$$

$$\iff \frac{\delta^{4/5}}{n^{4/5}} \leq \theta_n.$$

Since we assumed that $\theta_n \geq \frac{1}{\sqrt{n}}$, the last inequality is true for sufficiently large n, so we conclude that:

$$\frac{\sum_{|S|=m} \mathrm{Var}(A_S)}{\binom{n}{m}} \leq 3\theta_n m^2.$$

Since the average value of $\mathrm{Var}(A_S)$ is $\leq 3\theta_n m^2$, it follows that there exists S such that $\mathrm{Var}(A_S) \leq 3\theta_n m^2$, as required. $\qquad\square$

The lemma implies that for $m = \theta_n^{1/4} n$, we can find a set $S \subseteq \{1, 2, \ldots, n\}$ of size m such that $\mathrm{Var}(\sum_{i \in S} A_{v_i}) \leq 3\theta_n m^2$. Let us assume that in phase 2, the edges of $\partial(u)$ are presented in an order that ensures that the edges $\{u\} \times S = \{(u, v_i) : v_i \in S\}$ come last. We emphasize that this is a valid ordering, as the adversary has access to the OCRS's description ahead of time.

Theorem 3.5. *Any concentrated OCRS which processes the edges of $(G, \boldsymbol{x})$ in two phases as described above is at most $(\alpha + o(1))$-selectable, where α is defined in (2.1).*

For convenience, let us immediately relabel the vertices so that the edges of $\partial(u)$ are presented in the order $e_1 = (u, v_1), \ldots, e_n = (u, v_n)$ to the OCRS. We will be group the last m vertices from

v_k to v_n together ($k = n - m + 1$), and think about them as if they were one vertex. It will be useful to define:

$$A_{end} = \sum_{i=k}^{n} A_{v_i}, \mathcal{B} = \{\cup_{j=k}^{n}\{X_{e_j} A_{v_j} = 1\}\}, \text{ and } \mathcal{C} = \{M_u(e_k) = 0\}.$$

Here $\mathcal{B}$ represents the event that there is something for the OCRS to select amongst the last m edges, and $\mathcal{C}$ represents the event that the OCRS even reaches the last m vertices, i.e., it has not already made a selection when it reaches e_k. Note that we have ensured that $\text{Var}(A_{end}) = o(m^2)$.

Since the OCRS must select each of the last m edges $e_k, \ldots, e_n$ with probability $\frac{c}{n}$, and it can only do this if both the events $\mathcal{B}$ and $\mathcal{C}$ happen, we conclude that:

$$\frac{cm}{n} \le \Pr[\mathcal{B} \cap \mathcal{C}] = \Pr[\mathcal{B} \mid \mathcal{C}]\Pr[\mathcal{C}] = \Pr[\mathcal{B} \mid \mathcal{C}] \cdot \left(1 - \frac{c(n-m)}{n}\right),$$

where the last equality follows from the fact that the algorithm selects each of first $k-1$ edges with probability exactly $\frac{c}{n}$. We have the following lemma:

Lemma 3.6. *Suppose* $\Pr[A_{end} > T] \le \varepsilon$. *Then,*

$$\Pr[\mathcal{B} \mid \mathcal{C}] \cdot \left(1 - \frac{c(n-m)}{n}\right) \le \left(\frac{T(1-c)(n-m)}{n^2} + \varepsilon\right) \cdot \frac{(1-c)n + cm}{(1-c)(n-m)}.$$

Proof of Lemma 3.6. Note that:

$$\Pr[\mathcal{B} \mid \mathcal{C}] = \sum_{\ell=0}^{m} \Pr[\mathcal{B} \mid \mathcal{C}, A_{end} = \ell] \cdot \Pr[A_{end} = \ell \mid \mathcal{C}]$$

$$= \sum_{\ell=0}^{m} \left(1 - \left(1 - \frac{1}{n}\right)^{\ell}\right) \cdot \Pr[A_{end} = \ell \mid \mathcal{C}],$$

since if we know that if $A_{end} = \ell$, then $\mathcal{B}$ occurs exactly when $X_{u,v} = 1$ for some v with $X_{u,v} = 1$. Fixing a threshold T, with $\Pr[A_{end} > T] \le \varepsilon$, it follows that:

$$\Pr[\mathcal{B} \mid \mathcal{C}] = \sum_{\ell=0}^{m} \left(1 - \left(1 - \frac{1}{n}\right)^{\ell}\right) \cdot \Pr[A_{end} = \ell \mid \mathcal{C}]$$

$$\le \left(1 - \left(1 - \frac{1}{n}\right)^{T}\right) \cdot \Pr[A_{end} \le T \mid \mathcal{C}] + \Pr[A_{end} > T \mid \mathcal{C}]$$

$$\le \frac{T}{n} \cdot \Pr[A_{end} \le T \mid \mathcal{C}] + \frac{\varepsilon}{\Pr[\mathcal{C}]}$$

$$\le \frac{T}{n} + \frac{\varepsilon n}{(1-c)(n-m)}.$$

We conclude that

$$\Pr[\mathcal{B} \mid \mathcal{C}] \cdot \left(1 - \frac{c(n-m)}{n}\right)$$

$$\le \left(\frac{T}{n} + \frac{\varepsilon n}{(1-c)(n-m)}\right) \cdot \left(1 - \frac{c(n-m)}{n}\right) \cdot \frac{(1-c)(n-m)}{(1-c)(n-m)}$$

$$= \left(\frac{T(1-c)(n-m)}{n^2} + \varepsilon\right) \cdot \frac{(1-c)n + cm}{(1-c)(n-m)}$$

□

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We are now ready to prove Theorem 3.5.

Proof of Theorem 3.5. Note that we assumed $m = \theta_n^{1/4} n$. Plugging this into the inequality, and using Lemma 3.6 we get that

$$c\theta_n^{1/4} \leq \left(\frac{T(1-c)(1-\theta_n^{1/4})}{n} + \varepsilon \right) \cdot \frac{1 - c + c\theta_n^{1/4}}{(1-c)(1-\theta_n^{1/4})}.$$

If we let $T = (1-c)m + \theta_n^{1/4} m$, Chebyshev's inequality tells us that $\Pr[A_{end} > T] \leq \frac{3\theta_n m^2}{\theta_n^{1/2} m^2} = 3\theta_n^{1/2}$. Therefore,

$$c \leq \left(((1-c) + \theta_n^{1/4})(1-c)(1-\theta_n^{1/4}) + 3\theta_n^{1/4} \right) \cdot \frac{1 - c + c\theta_n^{1/4}}{(1-c)(1-\theta_n^{1/4})}.$$

Taking $n \to \infty$ yields the desired result. $\qquad\square$

We next consider when the OCRS is not necessarily concentrated on $(G, \boldsymbol{x})$. The proof here follows closely to the proof of Theorem 3.1, with the main difference being that we focus on the average covariance of $(A_{v_i})_{i=1}^n$, as opposed to the minimum covariance.

Theorem 3.7. *Any OCRS is at most $(\gamma + o(1))$-selectable on $(G, \boldsymbol{x})$ where γ is the unique solution to the equation: $0 = 1 - 2\gamma - \gamma \exp(\gamma - 1)$, and $\gamma \approx 0.3929$.*

Proof of Theorem 3.7. The proof in this setting will, broadly speaking, resemble the proof of Lemma 3.3, with some key differences. We will work with the same sequence of input graphs useful in the proof of Theorem 3.5, and in the first phase, the edges are presented in the same order. However, we will choose a different order for the edges in $\partial(u)$.

Once again, let us fix an OCRS for $(G, \boldsymbol{x})$ which returns a matching $\mathcal{M}$, and is c-selectable for $c \in (0, 1)$, i.e., $\Pr[e \in \mathcal{M} \mid X_e = 1] = c$ for each edge e of G. We will assume that $\operatorname{Var}[\sum_{i=1}^n A_{v_i}] \geq \theta n^2$, for some $\theta \geq 0$. Before we explain the order in which the edges of $\partial(u)$ are presented, we note that since

$$\sum_{j=1}^n \left(\mathbb{E}[A_{v_j} \sum_{i=1}^n A_{v_i}] - \mathbb{E}[A_{v_j}]\mathbb{E}[\sum_{i=1}^n A_{v_i}] \right) = \operatorname{Var}[\sum_{i=1}^n A_{v_i}] \geq \theta n^2,$$

we can find j such that

$$\mathbb{E}[A_{v_j} \sum_{i=1}^n A_{v_i}] - \mathbb{E}[A_{v_j}]\mathbb{E}[\sum_{i=1}^n A_{v_i}] \geq \theta n.$$

But note that

$$\mathbb{E}[A_{v_j} \sum_{i=1}^{n} A_{v_i}] - \mathbb{E}[A_{v_j}]\mathbb{E}[\sum_{i=1}^{n} A_{v_i}] \geq \theta n$$

$$\iff \quad \mathbb{E}[\sum_{i=1}^{n} A_{v_i} \mid A_{v_j} = 1] \Pr[A_{v_j} = 1] - (1-c)^2 n \geq \theta n$$

$$\iff \quad \mathbb{E}[\sum_{i=1}^{n} A_{v_i}] - \mathbb{E}[\sum_{i=1}^{n} A_{v_i} A_{v_j} = 0] \Pr[A_{v_j} = 0] - (1-c)^2 n \geq \theta n$$

$$\iff \quad (1-c)n - \mathbb{E}[\sum_{i=1}^{n} A_{v_i} \mid A_{v_j} = 0]c - (1-c)^2 n \geq \theta n$$

$$\iff \quad \mathbb{E}[\sum_{i=1}^{n} A_{v_i} \mid A_{v_j} = 0]c \leq c(1-c)n - \theta n$$

$$\iff \quad (1-c)n - \mathbb{E}[\sum_{i=1}^{n} A_{v_i} \mid A_{v_j} = 0]c - (1-c)^2 n \geq \theta n$$

$$\iff \quad \sum_{i \neq j} \mathbb{E}[A_{v_i} \mid A_{v_j} = 0] \leq n\left((1-c) - \frac{\theta}{c}\right).$$

We conclude that we can find j such that $\sum_{i=1}^{n-1} \mathbb{E}[A_{v_i} \mid A_{v_j} = 0] \leq n\left((1-c) - \frac{\theta}{c}\right)$. Let us assume that in phase 2, the edges of $\partial(u)$ are presented in an order that ensures that v_j comes last.

For convenience, let us immediately relabel the vertices so that the edges of $\partial(u)$ are presented in order $e_1 = (u, v_1), \dots, e_n = (u, v_n)$ to the OCRS. Notice that by choosing the order of the edges in $\partial(u)$ correctly, we have already established equation (3.7) from the proof of Lemma 3.3. But this equation was the only requirement in the proof of Lemma 3.3 to establish its conclusion that:

$$2c + c\exp(c + \theta/c - 1) \leq 1 + o(1).$$

Note that the θ here has a different meaning than in the proof of Lemma 3.3—here, θ is just a number that satisfies the inequality $\mathrm{Var}[\sum_{i=1}^{n} A_{v_i}] \geq \theta n^2$. Clearly this inequality is always true for $\theta = 0$, so we conclude that:

$$2c + c\exp(c - 1) \leq 1 + o(1),$$

as was needed. $\qquad\square$

4 RO-CRS and FO-CRS Positive Results

We now investigate contention resolution schemes that assume that the edges of the input $(G, \boldsymbol{x}) = (G(n), \boldsymbol{x}(n))$ arrive in a random order drawn u.a.r., or in an order chosen by the CRS. In the latter case, the edge-ordering is a *non-uniform* random ordering. For both arrival orders, we study the GREEDY-CRS which matches an arriving edge $e = (u, v)$ provided e is active, and u and v were previously unmatched.

For general inputs with vanishing edge values, Brubach et al. (2021) showed that GREEDY-CRS is $(1 - e^{-2})/2 \approx 0.432$-selectable, and it is not hard to see that their analysis is tight[7]. In order to

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[7]Consider $G = (V, E)$ with $V = \{u_i, v_i\}_{i=0}^{n}$ and $E = \{(u_0, v_0)\} \cup \{(u_0, u_i), (v_0, v_i)\}_{i=1}^{n}$ The probability x_e of every $e \in E$ is $1/(n+1)$. When (u_0, v_0) is active, GREEDY-CRS selects (u_0, v_0) with probability *exactly* $(1 - e^{-2})/2$ as $n \to \infty$.

surpass this bound, one approach is to ensure that $(G, \boldsymbol{x})$ is 1-*regular*; that is, $\sum_{e \in \partial(v)} x_e = 1$ for each $v \in V$. Fu et al. (2021) were the first to prove a reduction to this setting in the related vertex arrival model. MacRury et al. (2023) applied this reduction for random-order edge arrivals, and Nuti and Vondrák (2024) applied it in the offline setting. Nuti and Vondrák (2024) also observed that this reduction preserves vanishing edge values, no matter the arrival order of the edges. We therefore make this assumption throughout the section.

Theorem 4.1. *For 1-regular inputs with vanishing edge values,* GREEDY-CRS *is $\frac{1}{2}$-selectable when the edges arrive u.a.r.*

Recall that it is impossible for an RCRS to be better than $\frac{1}{2}$-selectable even for vanishing edge values (MacRury et al., 2023). We next show that it is possible to surpass this impossibility result in the free-order setting, where we now analyze GREEDY-CRS in a *non-uniform* random order.

Theorem 4.2. *For 1-regular inputs with vanishing edge values,* GREEDY-CRS *is $1 - \ln\left(2 - \frac{1}{e}\right) \approx 0.510$-selectable when the edges arrive in a certain non-uniform random order.*

It will be convenient to assume for the proof of Theorem 4.1 that each edge e arrives at a time that is chosen independently and u.a.r. from $[0, 1]$. Furthermore, for the proof of Theorem 4.2, we will assume that we generate for each vertex an independent seed u.a.r. from $[0, 1]$, and that the edges then arrive in lexicographic order, i.e., each edge is associated with the seeds of its vertices $\{x, y\}$, and it arrives at time $\min\{x, y\} + \epsilon \max\{x, y\}$, where ϵ is any number less than the distance between any two vertex seeds. The primary property we will need for our ordering is that an edge (u, w) is processed before (u, v) exactly when w's seed is less than v's seed, so our proof will also work for orderings other than the lexicographic one. The proofs of Theorems 4.1 and 4.2 both first reduce to the random tree case, and so we treat them in a unified manner.

4.1 Reduction to the Random Tree Case

This section follows Nuti and Vondrák (2024) very closely; however for completeness we present all the proofs in Appendix B. Consider a simple method to construct a random tree $\mathcal{T}$ that will estimate S_e as $n \to \infty$. Let u and v be two special vertices connected by an edge, and generate the random tree $\mathcal{T}$ by generating two independent Galton-Watson processes at u and v. Explicitly, construct $\mathcal{T}$ as follows: Start with the two special vertices u and v unmarked and connected by an edge, and add them into $\mathcal{T}$. Then, for each unmarked vertex w added into $\mathcal{T}$, obtain an independent sample L_w of a Poisson random variable with mean 1, add L_w new children to w, and then mark w. End the construction when all vertices of $\mathcal{T}$ are marked. Standard results about Galton-Watson processes immediately tell us that $\mathcal{T}$ is finite with probability 1 (e.g., see Theorem 6.1 in Harris (1964)).

After we generate $\mathcal{T}$, clearly it is possible to run a greedy matching algorithm on $\mathcal{T}$ (either after generating random times of arrival for each edge $e \in \mathcal{T}$, or random seeds for each vertex $w \in \mathcal{T}$ as the case may be). This produces a matching $\mathcal{M}_{\mathcal{T}}$. We claim the following relation between the execution of the greedy CRS on $G(n)$ and the greedy matching algorithm on $\mathcal{T}$:

Theorem 4.3. *There exists a function f satisfying $\lim_{\varepsilon \to 0} f(\varepsilon) = 0$ such that for any 1-regular input $(G(n), \boldsymbol{x}(n))$ with vanishing edge values, and for all $e \in G(n)$, we have:*

$$|\Pr[e \in \mathcal{M} \mid X_e = 1] - \Pr[(u, v) \in \mathcal{M}_{\mathcal{T}}]| \le f(\varepsilon(n)).$$

Clearly, establishing Theorem 4.3 reduces the proofs of Theorems 4.1 and 4.2 to the calculation of $\Pr[(u, v) \in \mathcal{M}_{\mathcal{T}}]$ in two different random orderings of the edges in $\mathcal{T}$.

4.2 Proving Theorems 4.1 and 4.2

Before we begin the proofs of these theorems, we will establish some common notions between the two proofs. First, we say that a vertex $w \in \mathcal{T}$ has been matched if an edge adjacent to it has been matched by the greedy matching algorithm. Second, in order for u to be unmatched when (u, v) arrives, we need every edge (u, w) from u to one of its children w to satisfy:

1. It is processed after edge (u, v), *or*

2. w is already matched when (u, w) is processed.

Let us say that a child w of u is *bad* if it satisfies neither of these criteria, i.e., (u, w) is processed before (u, v) and w is unmatched at this point. Observe that (u, v) is matched by the greedy algorithm precisely when no child of u or v is bad.

Proof of Theorem 4.1. We aim to calculate $\Pr[(u, v) \in \mathcal{M}_{\mathcal{T}}]$ when each edge e of the random tree $\mathcal{T}$ arrives at a time that is chosen independently and uniformly at random from $[0, 1]$.

Let us define for any $x \in [0, 1]$, the function $q(x)$ which denotes the probability that the special vertex u is *unmatched* at the time x when (u, v) arrives. The probability is taken over all possible realizations of $\mathcal{T}$, and all possible arrival times of the edges in $\mathcal{T}$ (other than (u, v)). Note briefly that q is continuous—indeed,

$$|q(x) - q(x + h)| \leq \Pr[\text{some edge arrives in between times } x \text{ and } x + h].$$

The probability that (u, v) is selected by the greedy algorithm is $\int_0^1 q(x)^2 dx$. This is because:

1. The symmetry between u and v ensures $q(x)$ is also the conditional probability that v is unmatched.

2. Whether u or v are matched before the edge (u, v) arrives are independent events.

3. (u, v) is added into the matching exactly if neither u nor v are matched when it arrives.

It turns out that we can actually determine what $q(x)$ is using a recursion. Critically, for an edge (u, w) arriving at time z, the probability (taken over all possible realizations of how the process looks below w) that w is unmatched when (u, w) arrives is exactly $q(z)$, and so, the probability that w is bad is exactly $\int_0^x q(z) dz$. It follows that

$$q(x) = \sum_{k=0}^{\infty} \Pr[u \text{ has } k \text{ children and none of them are bad}] \quad = \quad \sum_{k=0}^{\infty} \frac{e^{-1}}{k!} \left(1 - \int_0^x q(z) dz \right)^k$$

$$= \quad \exp\left(-\int_0^x q(z) dz \right).$$

To solve this equation, note that q is differentiable in x (since $\int_0^x q(z) dz$ is differentiable in x) and $\frac{dq}{dx} = -q(x)^2$. We know that $q(0) = 1$, so this is a differential equation we can solve using separation of variables. We conclude that $q(x) = \frac{1}{x+1}$. Finally, note that

$$\int_0^1 q(x)^2 dx \quad = \quad -\int_0^1 \frac{dq}{dx} dx = q(0) - q(1) = \frac{1}{2}.$$

This concludes the proof of Theorem 4.1. □

The argument to prove Theorem 4.2 is similar to the previous proof, but we need to be slightly more careful in how we define q.

Proof of Theorem 4.2. Once more, we aim to calculate $\Pr[(u,v) \in \mathcal{M}_{\mathcal{T}}]$, but this time each edge e of the random tree $\mathcal{T}$ arrives at a time determined by the random seeds of the edge's associated vertices.

Let us define for any $x, y \in [0,1]$, the function $q(x,y)$ which denotes the probability that the special vertex u is unmatched when (u,v) arrives given u has the seed x and v has the seed y. The probability that (u,v) is selected by the greedy algorithm is now:

$$\int_0^1 \int_0^1 q(x,y)q(y,x)dxdy.$$

Once again, it turns out that we can establish a recurrence for q. Indeed, for an edge (u,w) in which w has seed z, the probability (taken over all possible realizations of how the process looks below w) that w is unmatched when (u,w) arrives is exactly $q(z,x)$, and so, the probability that (u,w) is bad is exactly $\int_0^y q(z,x)dz$ (since the edge (u,w) is processed before (u,v) exactly if $z \leq y$). It follows that

$$
\begin{aligned}
q(x,y) &= \sum_{k=0}^{\infty} \Pr[u \text{ has } k \text{ children and none of them are bad}] \\
&= \sum_{k=0}^{\infty} \frac{e^{-1}}{k!} \left(1 - \int_0^y q(z,x)dz\right)^k \\
&= \exp\left(-\int_0^y q(z,x)dz\right).
\end{aligned}
$$

To solve this equation, note that q is differentiable in y (since $\int_0^y q(z,x)dz$ is differentiable in y) and

$$\frac{\partial q}{\partial y} = -q(x,y)q(y,x).$$

Since $q(x,y)$ is differentiable in y, $\int_0^y q(z,x)dz$ is differentiable in x, and so q is differentiable in x as well. Then,

$$
\begin{aligned}
\frac{\partial q}{\partial x} &= q(x,y)\left(-\int_0^y \frac{\partial q(z,x)}{\partial x}dz\right) \\
&= q(x,y)\left(\int_0^y q(z,x)q(x,z)dz\right) \\
&= q(x,y)\left(-\int_0^y \frac{\partial q(x,z)}{\partial z}dz\right) \\
&= q(x,y)\left(1 - q(x,y)\right),
\end{aligned}
$$

since $q(x,0) = 1$. We know that $q(0,y) = e^{-y}$, so this is a differential equation we can solve for each value of y separately. It follows using variable separation that

$$q(x,y) = \frac{e^x}{e^x + e^y - 1}.$$

Finally, note that

$$
\begin{aligned}
\int_0^1 \int_0^1 q(x,y)q(y,x)dxdy &= -\int_0^1 \int_0^1 \frac{\partial q}{\partial y}dydx \\
&= \int_0^1 q(x,0) - q(x,1)dx \\
&= 1 - \int_0^1 \frac{e^x}{e^x + e - 1}dx \\
&= 1 - \ln\left(2 - \frac{1}{e}\right),
\end{aligned}
$$

where we computed the last integral by substituting $u = e^x$. $\qquad\square$

Acknowledgements

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A   Additions to Section 2

Definition 3. Given a deterministic OCRS ψ, we say that it is *strongly online*, provided the following holds for every input $(G, \boldsymbol{x})$ and ordering $e_1, \ldots, e_m$ of the edges E of G:

- For each $1 \le i \le m$, the event "e_i is matched by ψ" is a function of $(X_{e_j}, x_{e_j})_{j=1}^i$.

We say a randomized CRS is strongly online, provided it is supported on deterministic CRS's which are each strongly online.

Proof of Lemma 2.4. We roughly follow the argument of Kulkarni et al. (2022). For the tree T, define the witness tree $W(T)$. An edge f of $T \cup \partial T$ is in $W(T)$ if it is observed before every edge

f' that lies above it in T.[8] By definition, the edges of $W(T)$ are observed in a bottom-up fashion. Note that whether the Algorithm 1 matches e is a function of the edges in $W(T)$, and the decision it makes on those edges. Thus, so as far as bounding $\Pr[e \in \mathcal{M} \mid X_e = 1]$ is concerned, we can assume that $T \cup \partial T = W(T)$.

A few notes about this identification: identifying $T \cup \partial T$ with $W(T)$ might replace some of the x_e values with 0. For instance, if a vertex in $W(T)$ is a leaf, we think about it as a leaf of $T \cup \partial T$ after the identification. This might reduce the values of some of the x_e, but this is safe, since all we ever assume about the x_e in our arguments is that $\sum_{e \in \partial(u)} x_e \leq 1$, and this constraint remains satisfied even after reducing some x_e.

Furthermore, if $W(T)$ does not have an edge that goes from $N^\ell(e)$ to $N^{\ell+1}(e)$, there is nothing to prove, since in this case, the neighborhood of e is effectively a tree, so $\Pr[e \in \mathcal{M} \mid X_e = 1] \geq \alpha$. We will therefore assume that T still has depth ℓ even after $T \cup \partial T$'s identification with $W(T)$. This establishes the lemma. $\square$

Proof of Lemma 2.5. Once again, we roughly follow the argument of Kulkarni et al. (2022). The recurrence (2.4) implies q_u and q_v are linear functions in Q_s. As we go up a level of the tree calculating various q_r using (2.4), the sign in front of Q_s flips. Thus, q_u and q_v are decreasing functions of Q_s when ℓ is odd, and increasing functions of Q_s when ℓ is even.

Since $\Pr[e \in \mathcal{M} \mid X_e = 1]$ is proportional to $q_u q_v$, and in the worst case, we seek to minimize $\Pr[e \in \mathcal{M} \mid X_e = 1]$, we should make $Q_s = 1$ when ℓ is odd, and $Q_s = 0$ when ℓ is even. This is true for any vertex s, so this establishes the lemma. $\square$

Proof of Lemma 2.7. Let us concentrate first on establishing the maximizing version of the lemma. We prove the statement by induction on k. First, let us show that the statement is true for $k = 2$. Note that:

$$\left(1 + \frac{\varepsilon \alpha x_1}{1 - \alpha x_1}\right) \cdot \left(1 + \frac{\varepsilon \alpha x_2}{1 - \alpha(x_1 + x_2)}\right) \leq 1 + \frac{\varepsilon \alpha(x_1 + x_2)}{1 - \alpha(x_1 + x_2)}$$

$$\iff \quad 1 + \frac{\varepsilon \alpha x_1}{1 - \alpha x_1} + \frac{\varepsilon \alpha x_2}{1 - \alpha(x_1 + x_2)} + \frac{\varepsilon^2 \alpha^2 x_1 x_2}{(1 - \alpha x_1)(1 - \alpha(x_1 + x_2))} \leq 1 + \frac{\varepsilon \alpha(x_1 + x_2)}{1 - \alpha(x_1 + x_2)}$$

$$\iff \quad \frac{\varepsilon \alpha x_1}{1 - \alpha x_1} + \frac{\varepsilon^2 \alpha^2 x_1 x_2}{(1 - \alpha x_1)(1 - \alpha(x_1 + x_2))} \leq \frac{\varepsilon \alpha x_1}{1 - \alpha(x_1 + x_2)}$$

$$\iff \quad \varepsilon \alpha x_1 (1 - \alpha(x_1 + x_2)) + \varepsilon^2 \alpha^2 x_1 x_2 \leq \varepsilon \alpha x_1 (1 - \alpha x_1)$$

$$\iff \quad -\varepsilon \alpha^2 x_1 x_2 + \varepsilon^2 \alpha^2 x_1 x_2 \leq 0$$

$$\iff \quad \varepsilon \alpha^2 x_1 x_2 (\varepsilon - 1) \leq 0.$$

It follows immediately that the inequality we started with is true, and furthermore, that the inequality is in fact an equality when x_1 or x_2 is equal to 0. Hence the statement is true for $k = 2$, and the base case is established.

Suppose we have established the statement for $k = r$, and we are attempting to establish the statement for $k = r + 1$. Then, notice that:

---

[8]$W(T)$ is not literally a tree because it might have edges between $N^\ell(e)$ and $N^{\ell+1}(e)$ which create cycles. Here, we follow the notation in Kulkarni et al. (2022).

$$\prod_{i=1}^{r+1}\left(1+\frac{\varepsilon\alpha x_i}{1-\alpha\sum_{j\le i}x_j}\right)\le\left(1+\frac{\varepsilon\alpha(x_1+x_2)}{1-\alpha(x_1+x_2)}\right)\prod_{i=3}^{r+1}\left(1+\frac{\varepsilon\alpha x_i}{1-\alpha\sum_{j\le i}x_j}\right)$$

$$\le 1+\frac{\varepsilon\alpha\bar{x}}{1-\alpha\bar{x}}.$$

The first inequality follows from an application of the case $k=2$, and the second follows from the inductive hypothesis. Furthermore, the inequality is tight when $x_2=x_3=\ldots=x_n=0$. The desired result follows.

The proof of the minimizing version of the lemma is almost entirely identical, but we include a proof for completeness. We prove the statement by induction on k. First, let us show that the statement is true for $k=2$. Note that:

$$\left(1-\frac{\varepsilon\alpha x_1}{1-\alpha x_1}\right)\cdot\left(1-\frac{\varepsilon\alpha x_2}{1-\alpha(x_1+x_2)}\right)\ge 1-\frac{\varepsilon\alpha(x_1+x_2)}{1-\alpha(x_1+x_2)}$$

$$\iff \quad 1-\frac{\varepsilon\alpha x_1}{1-\alpha x_1}-\frac{\varepsilon\alpha x_2}{1-\alpha(x_1+x_2)}+\frac{\varepsilon^2\alpha^2 x_1 x_2}{(1-\alpha x_1)(1-\alpha(x_1+x_2))}\ge 1-\frac{\varepsilon\alpha(x_1+x_2)}{1-\alpha(x_1+x_2)}$$

$$\iff \quad -\frac{\varepsilon\alpha x_1}{1-\alpha x_1}+\frac{\varepsilon^2\alpha^2 x_1 x_2}{(1-\alpha x_1)(1-\alpha(x_1+x_2))}\ge -\frac{\varepsilon\alpha x_1}{1-\alpha(x_1+x_2)}$$

$$\iff \quad -\varepsilon\alpha x_1(1-\alpha(x_1+x_2))+\varepsilon^2\alpha^2 x_1 x_2\ge -\varepsilon\alpha x_1(1-\alpha x_1)$$

$$\iff \quad \varepsilon\alpha^2 x_1 x_2+\varepsilon^2\alpha^2 x_1 x_2\ge 0.$$

It follows immediately that the inequality we started with is true, and furthermore, that the inequality is in fact an equality when x_1 or x_2 is equal to 0. Hence the statement is true for $k=2$, and the base case is established.

Suppose we have established the statement for $k=r$, and we are attempting to establish the statement for $k=r+1$. Then, notice that:

$$\prod_{i=1}^{r+1}\left(1-\frac{\varepsilon\alpha x_i}{1-\alpha\sum_{j\le i}x_j}\right)\ge\left(1-\frac{\varepsilon\alpha(x_1+x_2)}{1-\alpha(x_1+x_2)}\right)\prod_{i=3}^{r+1}\left(1-\frac{\varepsilon\alpha x_i}{1-\alpha\sum_{j\le i}x_j}\right)$$

$$\ge 1-\frac{\varepsilon\alpha\bar{x}}{1-\alpha\bar{x}}.$$

The first inequality follows from an application of the case $k=2$, and the second follows from the inductive hypothesis. Furthermore, the inequality is tight when $x_2=x_3=\ldots=x_n=0$. The desired result follows. $\square$

Proof of Proposition 2.8. We first handle the case when $x=yz$. In this case, clearly $y,z\ge x$.

If $X=1$, then set $Y=Z=1$. Otherwise, if $X=0$, there are three outcomes to assign probability mass to: $Y=1$ and $Z=0$, $Y=0$ and $Z=1$, and $Y=0$ and $Z=0$. Denote the amount assigned to each by $a_{1,0},a_{0,1}$ and $a_{0,0}$, respectively. We wish to determine $a_{i,j}\in[0,1]$ for which the following hold:

1. $a_{0,0}+a_{0,1}+a_{1,0}=1$

2. $(1-x)a_{1,0}+x=y$

3. $(1-x)a_{0,1} + x = z$

Observe that $a_{1,0} = (y-x)/(1-x)$ and $a_{0,1} = (z-x)/(1-x)$. Clearly each is within $[0,1]$, as $y-x, z-x \geq 0$. Moreover, $a_{0,0} + a_{0,1} + a_{1,0} = (y+z-2x)/(1-x) + a_{0,0}$. So it suffices to show that $(y+z-2x)/(1-x) \leq 1$ to conclude $a_{0,0} \in [0,1]$. But this is equivalent to $y+z \leq 1+x = 1+yz$. Since $y+z \leq 1+yz$ for any $y, z \in [0,1]$, $a_{0,0} = 1 - a_{0,1} - a_{1,0} \in [0,1]$.

Finally, observe that Y and Z are independent. This is because $\Pr[Y=1, Z=1] = \Pr[X=1] = x = yz$.

Let us now suppose that $x \geq yz$. We first couple $X' \sim \text{Ber}(yz)$ and such that $X \geq X'$. By applying the previous construction to X', we get Y and Z which satisfy $X \geq X' = YZ$. □

B Additions to Section 4

In order to prove Theorem 4.3, we first restate the following result proven by Nuti and Vondrák (2024) (see Lemma 1 in this work).

Lemma B.1 (Nuti and Vondrák, 2024). *Let T_0 be any finite realization of $\mathcal{T}$. For any edge e of G:*

$$|\Pr[S_e = T_0 \mid X_e = 1] - \Pr[\mathcal{T} = T_0]| \leq 3\varepsilon(n)|T_0|^2.$$

Note that $S_e = T_0$ means that there is an isomorphism from the graphs S_e to T_0 which, if $e = (u_0, v_0)$, sends u_0 to the special vertex $u \in T_0$ and v_0 to the special vertex $v \in T_0$.

We next establish the key proposition needed to prove Theorem 4.3:

Proposition B.2. *Fix any set of finite trees $F = \{T_1, T_2, \ldots, T_m\}$, the following hold:*

1. $|\Pr[S_e \notin F \mid X_e = 1] - \Pr[\mathcal{T} \notin F]| \leq 3\varepsilon \sum |T_i|^2$.

2. $|\Pr[e \in \mathcal{M}, S_e \in F \mid X_e = 1] - \Pr[(u,v) \in \mathcal{M}_{\mathcal{T}}, \mathcal{T} \in F]| \leq 3\varepsilon \sum |T_i|^2$.

Proof of Proposition B.2. For the first result, note that Lemma B.1 implies

$$|\Pr[S_e \in F \mid X_e = 1] - \Pr[\mathcal{T} \in F]| \leq 3\varepsilon \sum |T_i|^2$$

and hence it follows that

$$|\Pr[S_e \notin F \mid X_e = 1] - \Pr[\mathcal{T} \notin F]| \leq 3\varepsilon \sum |T_i|^2.$$

For the second result, notice that since GREEDY-CRS only depends on the active edges of G,

$$\Pr[e \in \mathcal{M} \mid X_e = 1, S_e = T_i] = \Pr[(u,v) \in \mathcal{M}_{\mathcal{T}} \mid \mathcal{T} = T_i].$$

Hence, it follows that

$$
\begin{aligned}
&|\Pr[e \in \mathcal{M}, S_e \in F \mid X_e = 1] - \Pr[(u,v) \in \mathcal{M}_{\mathcal{T}}, \mathcal{T} \in F]| \\
&= \left| \sum_{i=1}^{m} \Pr[e \in \mathcal{M} \mid S_e = T_i, X_e = 1] \Pr[S_e = T_i \mid X_e = 1] - \sum_{i=1}^{m} \Pr[(u,v) \in \mathcal{M}_{\mathcal{T}} \mid \mathcal{T} = T_i] \Pr[\mathcal{T} = T_i] \right| \\
&= \left| \sum_{i=1}^{m} \Pr[(u,v) \in \mathcal{M}_{\mathcal{T}} \mid \mathcal{T} = T_i] (\Pr[S_e = T_i \mid X_e = 1] - \Pr[\mathcal{T} = T_i]) \right| \\
&\leq \sum_{i=1}^{m} |\Pr[S_e = T_i \mid X_e = 1] - \Pr[\mathcal{T} = T_i]| \\
&\leq 3\varepsilon \sum |T_i|^2.
\end{aligned}
$$

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□

Proof of Theorem 4.3. First of all, note that the triangle inequality implies

$$
\begin{aligned}
&|\Pr[e \in \mathcal{M} \mid X_e = 1] - \Pr[(u,v) \in \mathcal{M}_\mathcal{T}]| \\
\leq\ & |\Pr[e \in \mathcal{M}, S_e \notin F \mid X_e = 1] - \Pr[(u,v) \in \mathcal{M}_\mathcal{T}, \mathcal{T} \notin F]| \\
& + |\Pr[e \in \mathcal{M}, S_e \in F \mid X_e = 1] - \Pr[(u,v) \in \mathcal{M}_\mathcal{T}, \mathcal{T} \in F]| \\
\leq\ & \Pr[S_e \notin F \mid X_e = 1] + \Pr[\mathcal{T} \notin F] \\
& + |\Pr[e \in \mathcal{M}, S_e \in F \mid X_e = 1] - \Pr[(u,v) \in \mathcal{M}_\mathcal{T}, \mathcal{T} \in F]|,
\end{aligned}
$$

It follows from Proposition B.2 that:

$$
|\Pr[e \in \mathcal{M} \mid X_e = 1] - \Pr[(u,v) \in \mathcal{M}_\mathcal{T}]| \leq 2\Pr[\mathcal{T} \notin F] + 6\varepsilon \sum |T_i|^2,
$$

and the inequality is true for every F. Now fix a $\delta > 0$, and consider an F with the property that $\Pr[\mathcal{T} \notin F] < \delta/3$. Such an F exists since $\mathcal{T}$ is finite with probability 1. Then, find an ε so small that $6\varepsilon \sum |T_i|^2 < \delta/3$. For such an ε, we must have

$$
|\Pr[e \in \mathcal{M} \mid X_e = 1] - \Pr[(u,v) \in \mathcal{M}_\mathcal{T}]| \leq 2\Pr[\mathcal{T} \notin F] + 6\varepsilon \sum |T_i|^2 < \delta,
$$

and Theorem 4.3 immediately follows. □