To motivate the optimization approach, consider the BTL model, where $w^\star \in \mathbb{R}^n$ is the ground-truth parameter vector. Let $P^\star \in \mathbb{R}^{n\times n}$ be the matrix with $P^\star_{ij} = \frac{w_i^\star}{w_i^\star + w_j^\star}$, so that $P^\star_{ij}$ is the probability that the i^{th} item beats the j^{th} item.

Given the results of comparisons, we construct the matrix $P \in \mathbb{R}^n$, where P_{ij} records the fraction of comparisons between i and j that resulted in a win for i (and if the items are not compared at all, we record a zero). Thus, for pairs (i, j) that are compared, we have $\mathbb{E}[P_{ij}] = P^\star_{ij}$.

In order to estimate the parameters $w^\star$, we use the hidden low-rank structure of $P^\star$, as noted by Rajkumar and Agarwal [1]. Namely, let $\Psi(x) = \log\left(\frac{x}{1-x}\right)$ be the logit function. Then $\Psi(P^\star_{ij}) = \log(w_i^\star) - \log(w_j^\star)$. Letting $s^\star \in \mathbb{R}^n$ be the vector with $s_i^\star = \log(w_i^\star)$ and $\mathbf{e} \in \mathbb{R}^n$ be the all-ones vector, we have $\Psi(P^\star) = s^\star \mathbf{e}^\top - \mathbf{e} s^{\star\top}$ (where Ψ is applied to $P^\star$ entrywise).

Letting $\Omega(\cdot)$ denote the restriction of a matrix to the set of ordered pairs (i, j) which are compared, we thus expect

$$\|\Omega\left(\Psi(P) - \Psi(P^\star)\right)\|$$

to be small (where the norm is not specified). It turns out that the 1-norm is convenient for optimization. We thus propose the following algorithm:

Algorithm 1 Estimate Ranking (BTL Model)

Input: Pairwise comparison data matrix P, score range $[w^\star_{\min}, w^\star_{\max}]$
Output: Estimated score vector w, estimated ranking π

1: Construct link-transformed matrix $\hat{P}$ where $\hat{P}_{ij} = \psi(P_{ij})$
2: Construct the optimization problem as follows:
Let x be a vector of size n, where $\log(w^\star_{\min}) \leq x_i \leq \log(w^\star_{\max})$ for all $i \in [n]$.
Let Y be a matrix defined by $Y_{ij} = x_i - x_j$.
Solve by finding the vector x which minimizes the matrix norm:

$$\min ||Y - \hat{P}||_1.$$

3: Set $w_i = e^{x_i}$, and construct π so that $\pi(i) \leq \pi(j)$ whenever $w_i \geq w_j$.

We note that the optimization problem can be formulated as a linear program, as follows:

$$\begin{aligned}
&\min \sum_{i=1}^{n}\sum_{j=1}^{n} Z_{ij} \\
\text{subject to } &\log(w^\star_{\min}) \leq x_i \leq \log(w^\star_{\max}) && \forall i \in [n] \\
&Y_{ij} = x_i - x_j && \forall i, j \in [n] \\
&-Z_{ij} \leq Y_{ij} - \hat{P}_{ij} \leq Z_{ij} && \forall i, j \in [n].
\end{aligned}$$

The Thurstone model can be handled similarly, as $\overline{\Psi}(P^{\star})$ is low-rank for $\overline{\Psi}(\cdot)$ which is the probit function (inverse of the normal cumulative distribution function) [1].

References

[1] Arun Rajkumar and Shivani Agarwal. When can we rank well from comparisons of O(n log(n)) non-actively chosen pairs? In *Conference on Learning Theory*, pages 1376–1401. PMLR, 2016.