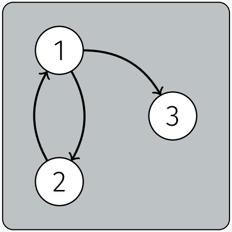
Algebraic Graphs with Class

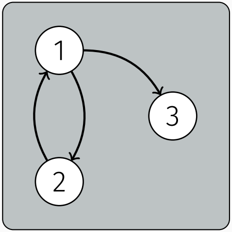
Christoph Madlener

01.06.2021

Introduction

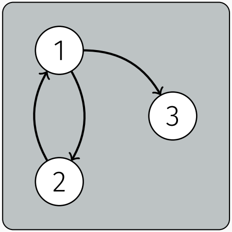


G_1



G_1

$G = (V, E)$ s.t. $E \subseteq V \times V$



$$G_1 = \Big(\{1,2,3\}, \{(1,2),(1,3),(2,1)\}\Big)$$

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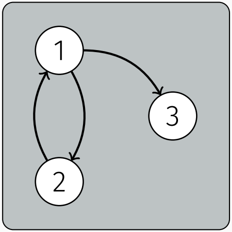

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In Haskell?

```haskell
data G a = G { vs :: Set a, es :: Set (a,a) }
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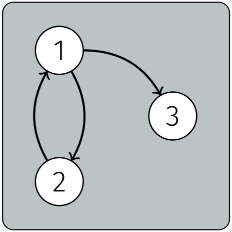

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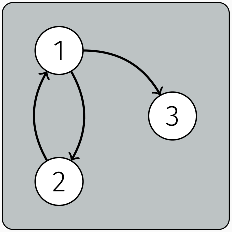

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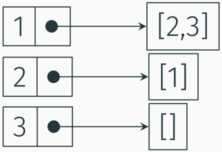
```
g1 = G [1,2,3] [(1,2), (1,3), (2,1)]
g2 = G [1,2] [(1,3)]
```

$E \not\subseteq V \times V$

containers
adjacency lists



containers & fgl

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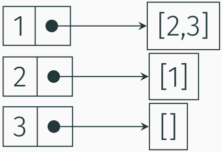

fgl
inductive graphs

- inductive datatype: Context of a vertex + Graph

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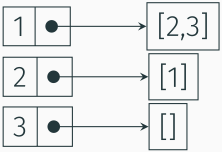

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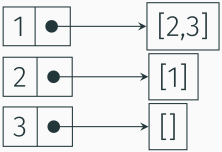

$E \subseteq V \times V$ **?**

- partial functions →runtime errors

`fgl`
inductive graphs

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Algebraic Graphs

- complete and consistent graph representation
- simple construction primitives (*“the core”*)

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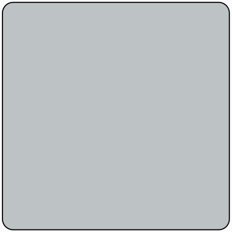
- simple construction primitives (*“the core”*)

Achieved by the datatype

```
data Graph a = Empty
             | Vertex a
             | Overlay (Graph a) (Graph a)
             | Connect (Graph a) (Graph a)
```

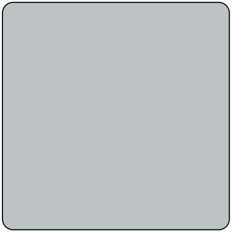
Empty - ε



$\texttt{Empty} = \varepsilon = (\emptyset, \emptyset)$

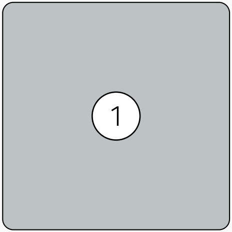
Empty (ε) & Vertex

Empty - ε



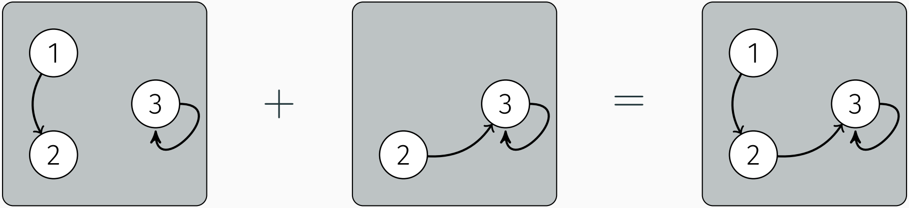
`Empty` $= \varepsilon = (\emptyset, \emptyset)$

Vertex



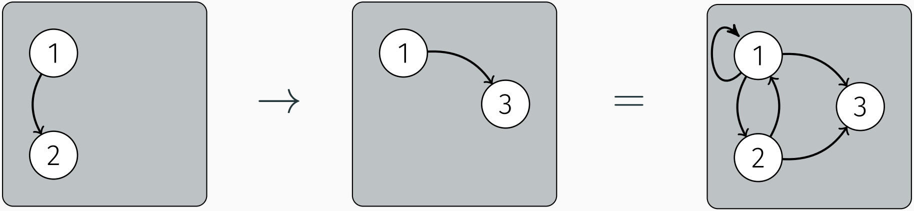
`Vertex 1` $= (\{1\}, \emptyset)$

Overlay ($+$)



$$(V_1, E_1) + (V_2, E_2) := (V_1 \cup V_2, E_1 \cup E_2)$$

Connect ($\rightarrow$ / $*$)



$$(V_1, E_1) \rightarrow (V_2, E_2) := (V_1 \cup V_2, E_1 \cup E_2 \cup V_1 \times V_2)$$

Graph Construction

```haskell
vertices :: [a] -> Graph a
vertices = foldr Overlay Empty . map Vertex
```

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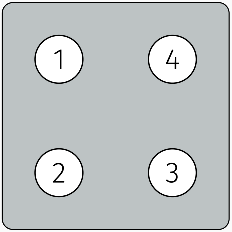
`vertices [1,2,3,4]` $=$ 

Graph Construction

```haskell
edge :: a -> a -> Graph a
edge u v = Connect (Vertex u) (Vertex v)
```

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edge :: a -> a -> Graph a
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edges :: [(a,a)] -> Graph a
edges = foldr Overlay Empty . map (uncurry edge)
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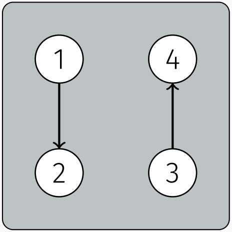
`edges [(1,2),(3,4)]` $=$



Graph Construction

```
graph :: [a] -> [(a,a)] -> Graph a
graph vs es = Overlay (vertices vs) (edges es)
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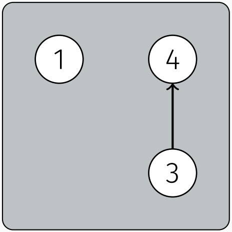
`graph [1] [(3,4)]` =



Graph Transformation - Functor

Functor

```
instance Functor Graph where
  ...
  fmap f (Vertex u) = Vertex (f u)
  ...
```

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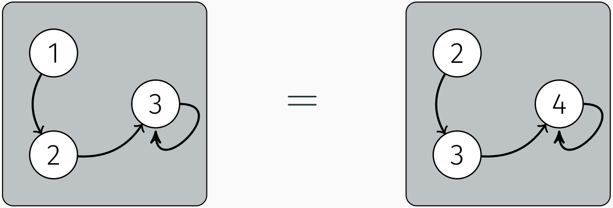
`fmap (+1)` 

Graph Transformation - Merging Vertices

```
mergeVs :: (a -> Bool) -> a -> Graph a -> Graph a
mergeVs p v = fmap $ \u -> if p u then v else u
```

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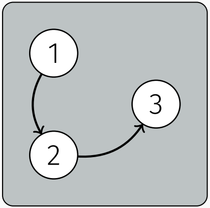
`mergeVs (>1) 4` 

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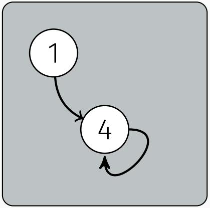

Graph Transformation - Monad

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  (Vertex u) >>= f = f u
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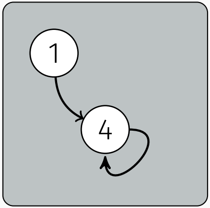
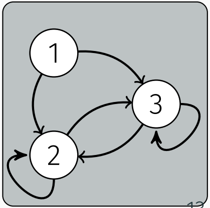
splitVertex 4 [2,3]  = 

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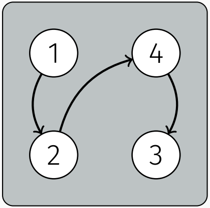
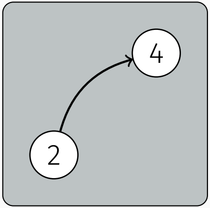
`induce even`  = 

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- "shared" identity ε
- no "zero" for $\rightarrow$ (i.e. $g \rightarrow 0 = 0$)

Decomposition

Additional axiom for algebra of graphs

$$x \to y \to z = x \to y + x \to z + y \to z$$

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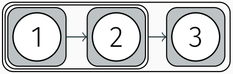

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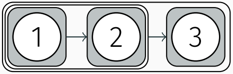

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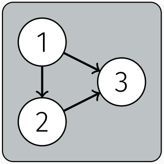

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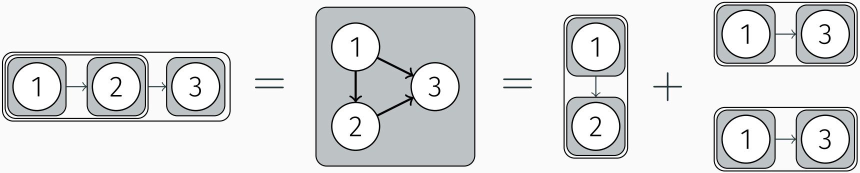

Subgraph Relation

Define subgraphs in terms of $(+)$ and equality:

$$x \sqsubseteq y \iff x + y = y$$

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Beyond Directed Graphs

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- Preorders (reflexive + transitive), equivalence relations (preorder + undirected)
- Hypergraphs

Graph Equality

Structural equality is not suitable:

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- Current implementation: build adjacency map
- Possible approach: *minimize* graph expressions →*Modular Graph Decomposition*

Compact Representation for Dense Graphs

Fully connected (undirected) graph

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Open question

- Can we exploit this compact representation, i.e. find algorithms that work directly on algebraic graphs?

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Future work

- Minimization
- Algorithms, applications beyond graph construction/transformation

```haskell
class Graph g where
  type Vertex g
  empty :: g
  vertex :: Vertex g -> g
  overlay :: g -> g -> g
  connect :: g -> g -> g
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→Polymorphic graph construction/transformation library

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- formal proof of completeness and consistency

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→ *Verified* polymorphic graph library

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- alternative approach to graph construction/transformation
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- alternative approach to graph construction/transformation
  - small core of primitives - safe and complete
- two main directions for future work
  1. exploit compact representation/other applications of data type
  2. exploit polymorphism of type class/locale