1. Задача

$$\mathrm{Im}(it - t^2) = \mathrm{const} \Rightarrow \mathrm{Im}\,t = \frac{\mathrm{const}}{\mathrm{Re}\,t} + \frac{1}{2}$$

$$\mathrm{Re}\; e^{\lambda(it - t^2)} = e^{\lambda(-\mathrm{Im}\,t - (\mathrm{Re}\,t)^2 - (\mathrm{Im}\,t)^2)} \cos[\lambda(\mathrm{Re}\,t - 2\,\mathrm{Re}\,t\,\mathrm{Im}\,t)]$$

$\mathrm{Im}\,t = y$, $\mathrm{Re}\,t = x$

$\Rightarrow$ Красный ~~линия~~ контур:

$$y = 0,\quad \mathrm{Re}\,t \in [-3, 3] \Rightarrow \mathrm{Re}\; e^{\lambda(it - t^2)} =$$

$$= e^{-\lambda x^2} \cos(\lambda x) \Rightarrow 1 \text{ рисунок}$$

Зеленый ~~линия~~ контур:

$$\begin{cases} y = 0, & x \in [-3, -2] \\ y = \frac{1}{2}, & x \in [-2, +2] \\ y = 0, & x \in [2, 3] \end{cases} \Rightarrow \mathrm{Re}\, e^{\lambda(it - t^2)} = \begin{cases} e^{-\lambda x^2} \cos(\lambda x), & x \in [-3, -2] \\ e^{-\lambda(x^2 + \frac{1}{2}^2)}, & x \in [-2; 2] \\ e^{-\lambda x^2} \cos \lambda x, & x \in [2; 3] \end{cases}$$

$\Rightarrow$ 3 рисунок

Черный ~~линия~~ контур:

$$\begin{cases} y = 0, & x \in [-3, -2] \\ y = \frac{1}{x} + \frac{1}{2}, & x \in [-2, -1] \\ y = -\frac{1}{2} & x \in [-1, 1] \\ y = -\frac{1}{x} + \frac{1}{2}, & x \in [1, 2] \\ y = 0, & x \in [2, 3] \end{cases} \Rightarrow \mathrm{Re}\, e^{\lambda(it - t^2)} = \begin{cases} e^{-\lambda x^2} \cos \lambda x, & x \in [-3, -2] \\ e^{-\lambda(x^2 + \frac{1}{x^2} + \frac{1}{4})} \cos 2\lambda, & x \in [-2, -1] \\ e^{-\lambda(x^2 - \frac{3}{4})} \cos(2\lambda x), & x \in [-1, 1] \\ e^{-\lambda(x^2 + \frac{1}{x^2} + \frac{1}{4})} \cos 2\lambda, & x \in [1, 2] \\ e^{-\lambda x^2} \cos \lambda x, & x \in [2, 3] \end{cases}$$

$\Rightarrow$ 2 рисунок

! Удобнее всего исп. зеленый контур.

$$I(\lambda) = \int_{-3}^{3} \frac{e^{\lambda(it - t^2)}}{1 + t^2}\, dt, \quad f(t) = it - t^2 \Rightarrow \begin{cases} f'(t) = i - 2t = 0 \Rightarrow t = i/2 \\ f''(t) = -2 \end{cases}$$

Метод Лапласа для $I(\lambda)$, $\lambda \to +\infty$ дает:

$$I(\lambda) \sim \frac{4}{3} e^{-\lambda/4} \int_{-\infty}^{+\infty} e^{-\lambda(t - i/2)^2}\, dt = \frac{4}{3} e^{-\frac{\lambda}{4}} \sqrt{\frac{\pi}{\lambda}}$$

2 Задача

$$Ai(x)=\frac{1}{2\pi}\int_{-\infty}^{+\infty} e^{i(xt+\frac{t^3}{3})}dt$$

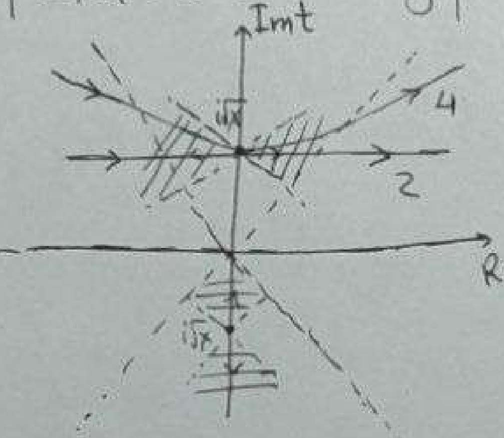
$$Re f(x,t) = -x|t|\sin\varphi - \frac{|t|^3}{3}\sin 3\varphi$$

$\Rightarrow \sin 3\varphi \geq 0 \Rightarrow 2\pi n < 3\varphi < \pi + 2\pi n \Rightarrow \frac{2\pi}{3}n < \varphi < \frac{\pi}{3}+\frac{2\pi}{3}n$

$\Rightarrow \varphi\in[0,\frac{\pi}{3}]\cup[\frac{2\pi}{3},\pi]\cup[\frac{4\pi}{3},\frac{5\pi}{3}]$

$f(x,t)=i(xt+\frac{t^3}{3}),\ f'(x,t)=i(x+t^2)=0\Rightarrow t_0=\pm i\sqrt{x}$

$f''(x,t)=2it \Rightarrow \arg f''(x,t_0)=\arg(\mp 2\sqrt{x})=\pi$ или 0



$\begin{cases} 2\varphi+\pi=\pi+2\pi n \\ 2\varphi+0=\pi+2\pi n \end{cases} \Leftrightarrow \begin{cases} \varphi=\pi n \\ \varphi=\frac{\pi}{2}+\pi n \end{cases} \Rightarrow$

$\Rightarrow$ нам подходит только контур 2 и 4

$$Ai(x)\sim \frac{1}{2\pi}e^{-\frac{2}{3}x^{3/2}}\int_{\sqrt{x}i-\infty}^{\sqrt{x}i+\infty} e^{-\sqrt{x}(t-i\sqrt{x})^2}dt =$$

$$= \frac{1}{2\sqrt{\pi}}x^{-\frac14}e^{-\frac23 x^{3/2}},\ x\to+\infty$$

$$\Rightarrow Ai(x)\sim\frac{1}{2\sqrt{\pi}\,x^{1/4}}e^{-\frac23 x^{3/2}}.$$

3 Задача

$$\frac{1}{2\pi}\int_{-\infty}^{+\infty} e^{i(xt+\frac{t^3}{3})}dt,\ x\to-\infty \Leftrightarrow \frac{1}{2\pi}\int_{-\infty}^{+\infty} e^{i(-|x|t+t^3/3)}dt,\ x\to+\infty$$

$$Re f(x,t) = |x||t|\sin\varphi - \frac{|t|^3}{3}\sin 3\varphi$$

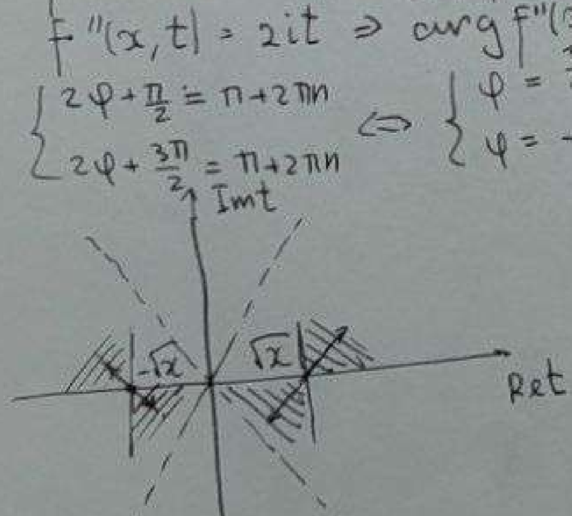
$\Rightarrow \varphi\in[0,\frac{\pi}{3}]\cup[\frac{2\pi}{3},\pi]\cup[\frac{4\pi}{3},\frac{5\pi}{3}]$

$f(x,t)=i(-|x|t+\frac{t^3}{3}),\ f'(x,t)=i(-|x|+t^2)=0\Rightarrow t_0=\pm\sqrt{|x|}$

$f''(x,t)=2it\Rightarrow \arg f''(x,t)=\arg(\pm 2i\sqrt{|x|})\Rightarrow \frac{\pi}{2}$ или $\frac{3\pi}{2}$

$\begin{cases} 2\varphi+\frac{\pi}{2}=\pi+2\pi n \\ 2\varphi+\frac{3\pi}{2}=\pi+2\pi n \end{cases} \Leftrightarrow \begin{cases} \varphi=\frac{\pi}{4}+\pi n \\ \varphi=-\frac{\pi}{4}+\pi n \end{cases} \Rightarrow$ а) контур D эквивалентен исходному

б) контур A подходит для оценки интеграла методом Лапласа



$$Ai(x)\sim\frac{1}{2\pi}\left(e^{-i\frac23|x|^{3/2}}e^{i\frac{\pi}{4}}\sqrt{\pi}|x|^{-\frac14} + e^{i\frac23|x|^{3/2}}e^{-i\frac{\pi}{4}}\sqrt{\pi}|x|^{-\frac14}\right)\Rightarrow$$

$$\Rightarrow Ai(x)\sim\frac{1}{\sqrt{\pi}\,x^{1/4}}\cos\left(\frac23|x|^{3/2}-\frac{\pi}{4}\right),$$

$x\to-\infty$

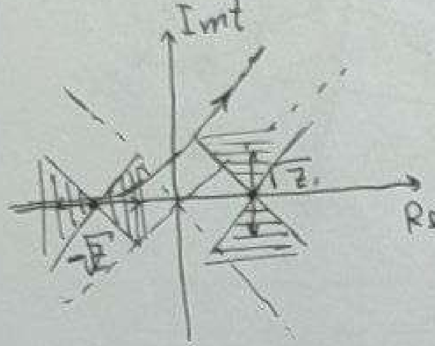
4 Задача

$$f(z)=\int\limits_{-\infty}^{e^{\pi i/3}\infty} e^{t^3/3-zt}\,dt,\quad z\to\infty \Leftrightarrow f(z)=\int\limits_{-\infty}^{e^{\frac{\pi i}{3}}\infty} e^{t^3/3-|z|t}\,dt,\ |z|\to+\infty$$

$$f(x,t)=\frac{t^3}{3}-|z|t \Rightarrow t_0=\pm\sqrt{|z|}$$

$$f''(x,t)=2t \Rightarrow \arg 2t = \arg \pm\sqrt{|z|} = 0 \text{ или } \pi$$

$$\begin{cases} 2\varphi+0=\pi+2\pi n \\ 2\varphi+\pi=\pi+2\pi n \end{cases} \Rightarrow \begin{cases} \varphi=\frac{\pi}{2}+\pi n \\ \varphi=\pi n \end{cases}$$



$$f(z)\sim e^{\frac{2}{3}|z|^{3/2}}\int\limits_{-\sqrt{z}-\infty}^{\sqrt{z}+\infty} e^{-\sqrt{|z|}(t+\sqrt{|z|})^2}\,dt=$$

$$=\sqrt{\pi}\,\frac{1}{z^{1/4}}\cdot e^{\frac{2}{3}|z|^{3/2}},\quad z\to+\infty$$

Контур не может быть деформирован так как

$$\cos 3\varphi<0 \Rightarrow -\frac{\pi}{2}+2\pi k<\varphi\cdot 3<\frac{\pi}{2}+2\pi k$$

$$\Rightarrow -\frac{\pi}{6}+\frac{2\pi k}{3}<\varphi<\frac{\pi}{6}+\frac{2\pi k}{3}$$

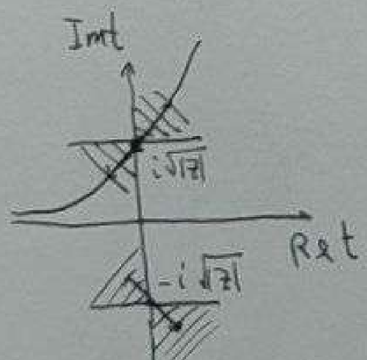
$\Rightarrow$ при $t\to\infty$ контур нельзя деформировать к реальной ~~оси~~ оси.

$$f(z)=\int\limits_{-\infty}^{e^{\pi i/3}\infty} e^{t^3/3+|z|t}\,dt,\quad |z|\to+\infty,\ z\to-\infty$$

$$f(x,t)=t^3/3+|z|t;\quad f'(x,t_0)=0 \Rightarrow t_0=\pm i\sqrt{|z|}$$

$$f''(x,t)=2t \Rightarrow \arg f''(x,t_0)=\frac{\pi}{2} \text{ или } -\frac{\pi}{2}$$

$$\Rightarrow \begin{cases} \varphi=\frac{\pi}{4}+\pi n \\ \varphi=-\frac{\pi}{4}+\pi n \end{cases}$$



$$f(z)\sim e^{i\frac{2}{3}|z|^{3/2}}\int\limits_{i\sqrt{|z|}+e^{-\frac{3\pi i}{4}}\infty}^{i\sqrt{|z|}+e^{\frac{i\pi}{4}}\infty} e^{i\sqrt{|z|}(t-i\sqrt{|z|})^2}\,dt=$$

$$=e^{\frac{i\pi}{4}}\sqrt{\pi}\,|z|^{-1/4}e^{i\frac{2}{3}|z|^{3/2}},\quad z\to-\infty$$

5 Задача

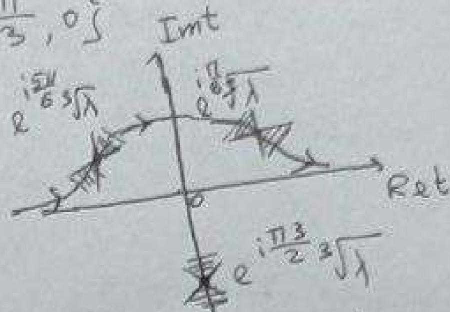
$$I(\lambda)=\int_{-\infty}^{+\infty} e^{-\frac{x^4}{4}+i\lambda x}\,dx=\int_{-\infty}^{+\infty} e^{-\frac{x^4}{4}}\cancel{\sin}\cos\lambda x\,dx+i\underbrace{\int_{-\infty}^{+\infty} e^{-\frac{x^4}{4}}\sin\lambda x\,dx}_{\to 0}$$

$$\int_{-\infty}^{+\infty} e^{-x^4/4+i\lambda x}\,dx,\ \lambda\to\infty$$

$$f(x,t)=-\frac{x^4}{4}+i\lambda x,\quad f'(\lambda,x_0)=0\Rightarrow x_0=e^{\frac{i\pi}{6}}|\lambda|^{\frac13};\ e^{\frac{i\pi 5}{6}}|\lambda|^{\frac13};\ e^{\frac{3\pi i}{2}}|\lambda|^{\frac13}$$

$$f''(x_0,\lambda)=-3x^2\Rightarrow \arg f''(x_0,\lambda)\in\left\{\frac{4\pi}{3},\frac{2\pi}{3},0\right\}$$

$$\begin{cases}2\varphi+0=\pi+2\pi n;\\ 2\varphi+\frac{2\pi}{3}=\pi+2\pi n;\\ 2\varphi+\frac{4\pi}{3}=\pi+2\pi n;\end{cases}\Rightarrow\begin{cases}\varphi=\frac{\pi}{2}+\pi n;\\ \varphi=\frac{\pi}{6}+\pi n;\\ \varphi=-\frac{\pi}{6}+\pi n\end{cases}$$



real λ = A + 4

$$\Rightarrow I(\lambda)=\sqrt{\frac{2\pi}{3}}\frac{1}{\sqrt[3]{\lambda}}\exp\left\{\frac34 e^{i\frac{4\pi}{3}}\lambda^{4/3}+i\frac{\pi}{6}\right\}-$$

$$-\sqrt{\frac{2\pi}{3}}\frac{1}{\sqrt[3]{\lambda}}\exp\left\{\frac34 e^{i\frac{2\pi}{3}}\cdot\lambda^{4/3}+i\frac{\pi}{6}\right\}=$$

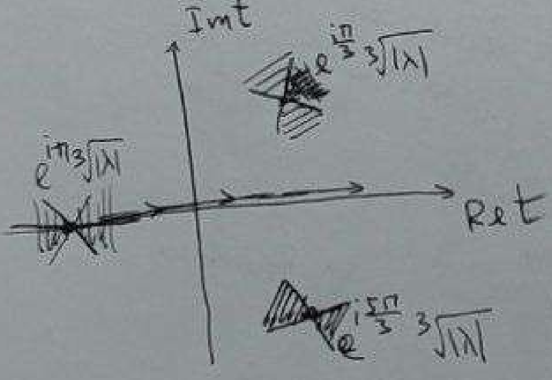
$$=2\sqrt{\frac{2\pi}{3}}\frac{1}{\lambda^{1/3}}e^{-\frac38\lambda^{4/3}}\cos\left(\frac{3\sqrt3}{8}\lambda^{4/3}-\frac{\pi}{6}\right),\ \lambda\to+\infty$$

$$\int_{-\infty}^{+\infty} e^{-x^4/4-|\lambda|x}\,dx,\ |\lambda|\to\infty$$

$$f(x,\lambda)=-x^4/4-|\lambda|x;\ f'(x_0,\lambda)=0\Rightarrow x_0=e^{\frac{i\pi}{3}}\sqrt[3]{|\lambda|},\ e^{i\pi}\sqrt[3]{|\lambda|},\ e^{\frac{i5\pi}{3}}\sqrt[3]{|\lambda|}$$

$$f''(x,\lambda)=-3x^2\Rightarrow\arg f''(x_0,\lambda)\in\left\{\frac{\pi}{3},\pi,\frac{5\pi}{3}\right\}$$

$$\begin{cases}2\varphi+\frac{\pi}{3}=\pi+2\pi n;\\ 2\varphi+\pi=\pi+2\pi n;\\ 2\varphi+\frac{5\pi}{3}=\pi+2\pi n;\end{cases}\Rightarrow\begin{cases}\frac{2\pi}{3}+\pi n;\\ \pi n;\\ \frac{\pi}{6}+\pi n\end{cases}$$



$$I(\lambda)\sim\sqrt{\frac{2\pi}{3}}e^{\frac34|\lambda|^{4/3}}\frac{1}{|\lambda|^{1/3}},\ \lambda\to+\infty$$

⇒ Imaginary λ = B + 7

т.к. $\cos 4\alpha>0$

$$\Rightarrow -\frac{\pi}{8}+\frac{\pi n}{2}<\alpha<\frac{\pi}{8}+\frac{\pi n}{2}$$

6 Задача

$$I(\lambda)=\int_{-\infty}^{+\infty} e^{-\lambda(x^2-3ix)} F(x)\,dx$$

$$F(x)=\int_0^{+\infty} \frac{(1+ix)\,y^{ix}}{(1+y)^{2ix+2}} e^{-y}dy$$

$f(x)=x^2-3ix$, $f'(x_0)=0 \Rightarrow x_0=i\frac{3}{2}$

$$F(\tfrac{3}{2}i)=-\frac{1}{2}\int_0^{+\infty}\frac{y^{-3/2}}{(1+y)^{-1}} e^{-y}dy=-\frac{1}{2}\left(\int_0^{\infty} y^{-3/2}e^{-y}dy+\int_0^{\infty} y^{-1/2}e^{-y}dy\right)=$$

$$=-\frac{\Gamma(\frac{1}{2})+\Gamma(-\frac{1}{2})}{2}=\frac{\sqrt{\pi}}{2}$$

$$\Rightarrow I(\lambda)\sim\frac{\pi}{2}\lambda^{-\frac{1}{2}}e^{-\frac{9}{4}\lambda},\ \lambda\to+\infty$$

$$F(x)=\int_0^{\infty}(1+ix)\left[(1+y)^{-2-2ix}-1\right]y^{ix}e^{-y}dy+\int_0^{\infty}(1+ix)\,y^{ix}e^{-y}dy=$$

$$=\underbrace{\int_0^{+\infty}(1+ix)\left[(1+y)^{-2-2ix}-1\right]y^{ix}e^{-y}dy}_{J}+\underbrace{(1+ix)\Gamma(1+ix)}_{\Gamma(2+ix)}$$

J сходится при $\operatorname{Im} x<2$!

$\Gamma(2+ix)$ имеет полюса $i(2+n)$!

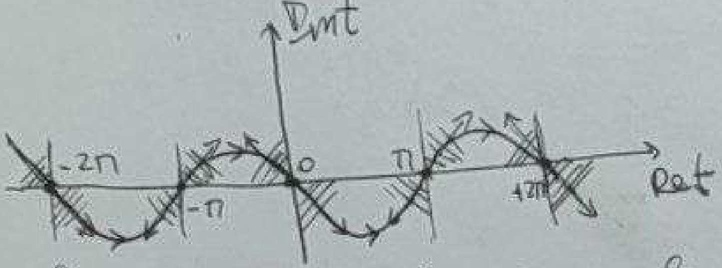
7 Задача

$$I(\lambda) = \int_{-\infty}^{+\infty} \cos(\lambda \cos x) \frac{\sin x}{x} dx = \mathrm{Re} \int_{-\infty}^{+\infty} e^{i\lambda \cos x} \frac{\sin x}{x} dx, \ \lambda \to +\infty$$

$$f(x) = i\cos x, \quad f'(x_0) = 0 \Rightarrow x_0 = \pi n, \ n \in \mathbb{Z}$$

$$f''(x) = -i\cos x \Rightarrow \arg f''(x_0) = \begin{cases} \pi/2, & n = 2k+1, \ k \in \mathbb{Z} \\ \frac{3\pi}{2}, & n = 2k, \ k \in \mathbb{Z} \end{cases}$$

$$\begin{cases} 2\varphi + \frac{\pi}{2} = \pi + 2\pi n \\ 2\varphi + \frac{3\pi}{2} = \pi + 2\pi n \end{cases} \Rightarrow \begin{cases} \varphi = \frac{\pi}{4} + \pi n \\ \varphi = -\frac{\pi}{4} + \pi n \end{cases}$$



$$I(\lambda) \sim \mathrm{Re}\left[e^{i\lambda} e^{-i\frac{\pi}{4}} \sqrt{\frac{2\pi}{\lambda}}\right] =$$

$$= \sqrt{\frac{2\pi}{\lambda}} \cos\left(\lambda - \frac{\pi}{4}\right), \ \lambda \to +\infty$$

Для вычисления второй поправки нам необходимо учесть бесконечное количество седел.

$$\Rightarrow I(\lambda) \sim \mathrm{Re} \sum_{n=-\infty}^{+\infty} \int_{\pi n + e^{-i(-1)^n \frac{\pi}{4}}(-\infty)}^{\pi n + e^{-i(-1)^n \frac{\pi}{4}} \infty} e^{\lambda i \cos x} \cdot \frac{\sin x}{x} dx = \left[e^{-i(-1)^n \frac{\pi}{4}} p = x - \pi n\right] =$$

$$= \mathrm{Re} \sum_{n=-\infty}^{+\infty} e^{-i(-1)^n \frac{\pi}{4}} e^{i(-1)^n \lambda} \int_{-\infty}^{+\infty} e^{\lambda\left(i(-1)^n \cos\left(e^{-i(-1)^n \frac{\pi}{4}} p\right) - i(-1)^n\right)} \frac{(-1)^n \sin\left(e^{-i(-1)^n \frac{\pi}{4}} p\right)}{e^{-i(-1)^n \frac{\pi}{4}} p + \pi n} dp =$$

$$= \left[-t^2 = (-1)^n \cos\left(e^{-i(-1)^n \frac{\pi}{4}} p\right) - i(-1)^n\right] = \ldots =$$

$$\sim \frac{1}{8} \sqrt{\frac{2\pi}{\lambda^3}} \sin\left(\lambda - \frac{\pi}{4}\right), \ \lambda \to +\infty$$