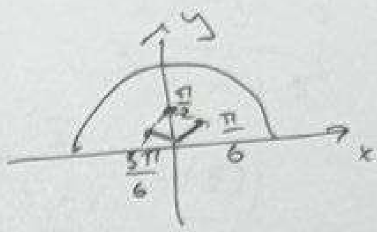
Problem 3.1

$$\frac{1}{R} \to 0$$

1. $$\int_{-\infty}^{+\infty} \frac{x^4}{1+x^6}\,dx = \oint + \cancel{\int_{C_R}} = 2\pi i \sum \operatorname*{res}_{z=z_0} f(z)$$



$$f(z) = \frac{z^4}{1+z^6}$$

$$z = e^{i\left(\frac{\pi}{6} + \frac{2\pi k}{3}\right)}$$

$$\operatorname*{res}_{z=e^{i\frac{\pi}{6}}} \frac{1}{6z} = \frac{1}{6} e^{-i\frac{\pi}{6}}\ ;\quad \operatorname*{res}_{z=\frac{\pi}{2}} \frac{1}{6z} = \frac{1}{6} e^{-i\frac{\pi}{2}}\ ;$$

$$\operatorname*{res}_{z=\frac{5\pi}{6}} \frac{1}{6z} = \frac{1}{6} e^{-i\frac{5\pi}{6}}$$

$$\Rightarrow I = \frac{2\pi i}{6}\left(\frac{\sqrt{3}}{2} + \frac{1}{2}i - \frac{\sqrt{3}}{2} + \frac{1}{2}i + i\right) = \frac{2}{3}\pi$$

N2

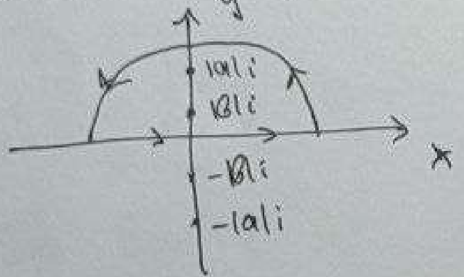
$$\int_{-\infty}^{+\infty}\frac{dx}{(x^2+a^2)(x^2+b^2)^2} =$$

$$= 2\pi i\left[\frac{1}{(z+|a|i)(z^2+b^2)^2}\right]\Bigg|_{z=|a|i} + 2\pi i\,\frac{d}{dz}\,\frac{1}{(z^2+a^2)(z+|b|i)^2}\Bigg|_{z=|b|i} =$$

$$= 2\pi i\left[\frac{1}{2|a|i\,(|b|^2-|a|^2)^2} + \frac{4z^3+6z^2|b|\cdot i-2z+2z\,a^2+2a^2|b|i}{(z^2+|a|^2)^2\cdot(z+i|b|)}\right.\Bigg|_{z=i|b|}$$

$$= \frac{\pi}{2}\,\frac{|a|+2|b|}{|a||b|^3(|a|+|b|)^2}\,,\qquad a\cdot b\neq 0$$

if $ab=0 \Rightarrow$ I расходится. $\left(I\sim\int_{-\infty}^{+\infty}\frac{dx}{x^4}\right)$



$$\int_0^{2\pi}\frac{\cos 2\theta}{2+\cos\theta}\,d\theta = \left|\cos\theta=\frac12\left(z+\frac1z\right)\right| = \left|\begin{array}{l}\cos 2\theta = 2\cos^2\theta-1 =\\ =\frac12\left(z^2+\frac{1}{z^2}\right)\end{array}\right| =$$

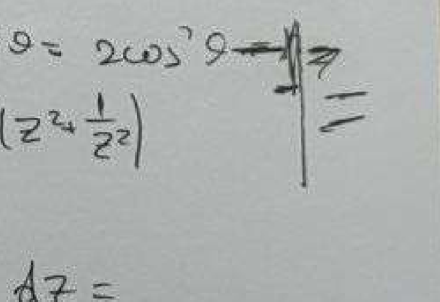
$$= \oint_{|z|=1}\frac{\frac12\left(z^2+\frac{1}{z^2}\right)}{\left[\frac12\left(z+\frac1z\right)+2\right]}\,\frac{dz}{iz} = \oint_{|z|=1}\frac{z^4+1}{iz^2(z^2+4z+1)}\,dz =$$

$$= 2\pi i\left[\operatorname*{res}_{z=0}(\dots)+\operatorname*{res}_{z=\sqrt3-2}(\dots)\right];\quad \operatorname*{res}_{z=0}(\dots)=c_{-1}$$

$$f(z)\approx\frac{1}{iz^2(1+4z)}\approx-\frac{i}{z^2}(1-4z+\dots)\Rightarrow$$
$$\Rightarrow c_{-1}=4i$$

$$\operatorname*{res}_{z=\sqrt3-2}f(z) = \frac{z^4+1}{iz^2(z+2+\sqrt3)}\Bigg|_{z=\sqrt3-2} = -\frac{7}{\sqrt3}\,i\Rightarrow$$

$$\Rightarrow I = 2\pi i\left(4i-\frac{7}{\sqrt3}\,i\right) = \underline{\underline{2\pi\left(\frac{7}{\sqrt3}-4\right)}}$$



№ 3.2

$z=\infty \quad f(z)=z^3\cos\left(\frac{1}{z-2}\right)$

$z-2=\varepsilon,\ \varepsilon\to\infty$ if $z\to\infty$

$$f(\varepsilon)=(\varepsilon+2)^3\left(1-\frac{1}{2!}\left(\frac{1}{\cancel{z-2}\ \varepsilon}\right)^2+\frac{1}{24}\left(\frac{1}{\cancel{z-2}\ \varepsilon}\right)^4+\dots\right)$$

$$=\left(\varepsilon^3+6\varepsilon^2+12\varepsilon+8\right)\left(1-\frac{1}{2}\left(\frac{1}{\cancel{z-2}\ \varepsilon}\right)^2+\frac{1}{24}\left(\frac{1}{\cancel{z-2}\ \varepsilon}\right)^4+\dots\right)=$$

$$=\dots-6\frac{1}{\varepsilon}+\frac{1}{24}\cdot\frac{1}{\varepsilon}+\dots$$

$$\operatorname*{res}_{z=\infty}f(z)=-C_{-1}=-\left(-6+\frac{1}{24}\right)=\frac{143}{24}$$

~~№ 3.5~~

$$\int_{-\infty}^{+\infty}\frac{\sin^2 x}{x^2(x^2+1)}$$

$$\cancel{\frac{1}{(n-1)!}\frac{d^{n-1}}{dz^{n-1}}\left[(z-a)^n f(z)\right]}$$

Problem 3.10

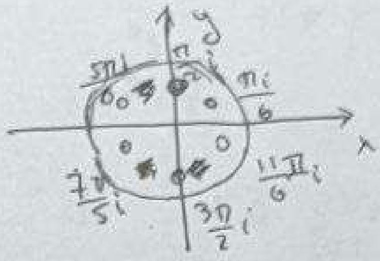
$$\text{P.V.}\ \cancel{\int_{0}^{+\infty}\frac{x\,dx}{(x^2+a^2)(\sin bx)}}=\cancel{\frac{1}{2}\int\frac{x\,dx}{(x^2+a^2)\sin bx}}=$$

$$\cancel{=\frac{1}{2}\cdot 2\pi i\operatorname*{Res}_{z=|a|i}}$$

Problem 3.4

$$\int_C \frac{z^5 dz}{1+z^6}$$

$$f(z) = \frac{z^5}{1+z^6} = \frac{1}{z}\left(1 - \frac{1}{z^6} + \dots\right) \approx \frac{1}{z}$$



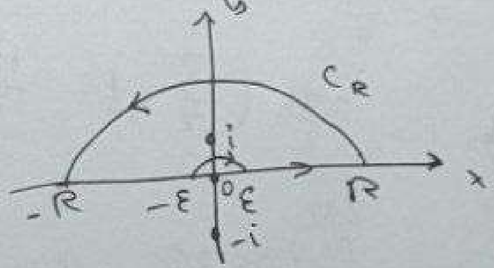
$$\operatorname{res} f(z) = c_{-1} = 1$$

$$\oint_C f(z)\, dz = 2\pi i$$

Problem 3.5

$$\sin^2 x = \frac{1}{2}(1 - \cos(2x))$$

$$I = \int_{-\infty}^{+\infty} \frac{\sin^2 x\, dx}{x^2(x^2+1)} =$$



$$= \frac{1}{2} \operatorname{Re}\left[\int_{-\infty}^{+\infty} \frac{1-e^{2ix}}{x^2(x^2+1)} dx\right]$$

$$\oint_C = \int_{-\infty}^{-\varepsilon} + \int_{\varepsilon}^{+\infty} + \int_{\varepsilon} + \int_{C_R}$$

$$f(z) \xrightarrow{z\to\infty} \approx \frac{1}{z^6} \Rightarrow \int_{C_R} \to 0 \sim \frac{1}{R^5}$$

$$\oint f(z) dz = 2\pi i \operatorname*{res}_{z=i} f(z) = 2\pi i \frac{1-e^{2iz}}{z^2(z+i)} = 2\pi i \frac{1-e^{-2}}{(-1)\cdot 2i} =$$

$$= \pi\left(\frac{1}{e^2} - 1\right)$$

$$\int_{\varepsilon} = \left| \begin{aligned} z &= \varepsilon e^{i\varphi} \\ dz &= \varepsilon i e^{i\varphi} d\varphi \end{aligned} \right. = \int_{\pi}^{0} \frac{1 - e^{2i\varepsilon e^{i\varphi}}}{(\varepsilon e^{i\varphi})^2\left((\varepsilon e^{i\varphi})^2 + 1\right)} i\varepsilon e^{i\varphi} d\varphi =$$

$$= \int_{\pi}^{0} \frac{1 - 1 - 2i\varepsilon e^{i\varphi}}{\varepsilon^2(\varepsilon^2+1)} i\varepsilon e^{i\varphi} d\varphi = 2\int_{\pi}^{0} d\varphi = -2\pi$$

$$\Rightarrow \int_{-\infty}^{+\infty} = \pi\left(\frac{1}{e^2} - 1\right) + 2\pi = \pi\left(1 + \frac{1}{e^2}\right) \Rightarrow I = \frac{1}{2}\operatorname{Re}\int_{-\infty}^{+\infty} = \frac{\pi}{2}\left(1 + \frac{1}{e^2}\right)$$

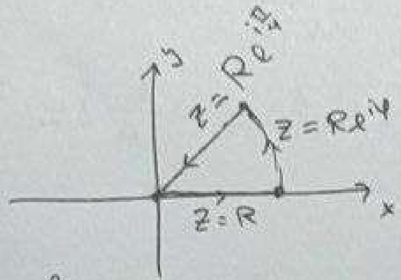
Problem 3.6

1. $$\lim_{R\to\infty}\int_{C_R} e^{iz}\,dz = \oint^{0} - \lim_{R\to\infty}\int_{-R}^{R} e^{iz}\,dz = -\frac{1}{i}\lim_{R\to\infty} e^{iz}\Big|_{-R}^{R} =$$

$$= -\lim_{R\to\infty}\frac{e^{+iR}-e^{-iR}}{i} = -\lim_{R\to\infty}\left(2\sin R\right).$$ не сущ.

$\Rightarrow$ Предела нет.

2. $$\lim_{R\to\infty}\int_{C_R} e^{iz^2}\,dz$$



$$\lim_{R\to\infty}\left(\int_{I} + \int_{C_R} + \int_{II}\right) = \oint = 0$$

$$\int_{I} = \cancel{\lim_{R\to\infty}}\ \lim_{R\to\infty}\int_0^R e^{ix^2}\,dx$$

$$\int_{II} = \lim\int_R^{0} e^{i\left(Re^{i\frac{\pi}{4}}\right)^2}\,d\left(Re^{i\frac{\pi}{4}}\right) = \lim\int_R^{0} e^{-R^2}e^{i\frac{\pi}{4}}\,dR$$

$$\int_{I} = \lim_{R\to\infty}\int_0^R e^{ix^2}\,dx = \left|\begin{array}{l} iy = e^{i\frac{\pi}{4}}x \\ -y^2 = ix^2 \\ i\,dy = e^{i\frac{\pi}{4}}dx \\ dy\cdot e^{i\frac{\pi}{4}} = dx \end{array}\right| = \lim_{R\to\infty}\int_0^R e^{-y^2}e^{i\frac{\pi}{4}}\,dy$$

$$\Rightarrow \int_{I} + \int_{II} = -\int_0^{\infty} e^{-R^2}e^{i\frac{\pi}{4}}\,dR + \int_0^{+\infty} e^{-y^2}e^{i\frac{\pi}{4}}\,dy = 0$$

$$\Rightarrow \lim_{R\to\infty}\int_{C_R} = 0$$ $\Rightarrow$ Предел существует

Problem 3.6

1. $\lim\limits_{R\to\infty}\int\limits_{C_R} e^{iz}dz = \lim\limits_{R\to\infty}\int R^2\, \text{func}(\varphi) \cdot \Rightarrow \infty$

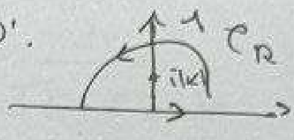
$\Rightarrow$ лимит не сущ.

Problem 3.8

$$\int_0^{+\infty}\frac{x\sin ax}{x^2+k^2}dx = \int_0^{+\infty}\frac{xe^{iax}}{2i(x^2+k^2)}dx + \int_0^{+\infty}\frac{(-x)e^{-iax}}{2i(x^2+k^2)}dx =$$

$$=\frac{1}{2i}\int_{-\infty}^{+\infty}\frac{xe^{iax}}{x^2+k^2}dx$$

Для $a>0$:



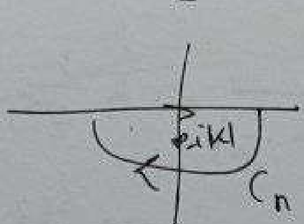
Лемма Жорд. выполняется

$$\Rightarrow \operatorname*{res}_{z\to ik} f(z) = \frac{ik\, e^{-ka}}{2ik} = \frac{e^{-ak}}{2}$$

$$I = 2\pi i\frac{e^{-ka}}{2} = \pi i e^{-ka}$$

$$\Rightarrow I = \frac{1}{2i}\cdot\pi i e^{-ka} = \frac{\pi}{2}e^{-ka}$$

Для $a<0$:



Аналогично:

$$I = \frac{1}{2i}\left(-2\pi i \operatorname*{res}_{z=-ik} f(z)\right) = -\frac{\pi}{2}e^{-ka}$$

$\Rightarrow$ Ответ: $I = \frac{\pi}{2}e^{-ak}\operatorname{sign}(a)$

Problem 3.3.

$$f(z) = \frac{1}{z^3 - z^5} = \frac{1}{z^3(1-z^2)}$$

$$\cancel{\underset{z=-1}{\operatorname{res}} f(z) = \frac{1}{(\varepsilon-1)^3(2+2\varepsilon+6\varepsilon}}$$

$$\underset{z=-1}{\operatorname{res}} f(z) = \frac{1}{(\varepsilon-1)^3(1-\varepsilon^2+2\varepsilon-1)} \approx \cancel{\frac{1}{(\varepsilon-1)^3(1+\varepsilon)}} =$$

$$= \cancel{\frac{1}{(\varepsilon^3-3\varepsilon^2+3\varepsilon-1)(1+\varepsilon)}} \approx \frac{1}{(\varepsilon^3-3\varepsilon^2+3\varepsilon-1)(2\varepsilon-\varepsilon^2)} \approx$$

$$\approx \cancel{\frac{1}{(2+8\varepsilon)}} \approx \cancel{\frac{1}{2(1+4\varepsilon)}}$$

$$\approx -\frac{1}{2\varepsilon} \quad ? \quad \underset{z=-1}{\operatorname{res}} f(z) = -\frac{1}{2}$$

Для $\underset{z=1}{\operatorname{res}} f(z)$ ~~для~~ аналогично. $= -\frac{1}{2}$

$$\underset{z=0}{\operatorname{res}} f(z) = \frac{1}{z^3(1-z^2)} \approx \frac{1}{z^3}(1+z^2+\dots) = \frac{1}{z^3} + \frac{1}{z}$$

$$\Rightarrow \underset{z=0}{\operatorname{res}} f(z) = 1$$

$$\underset{z=\infty}{\operatorname{res}} f(z) = -\frac{1}{z^5(1-\frac{1}{z^2})} = -\frac{1}{z^5}\left(1+\frac{1}{z^2}+\frac{1}{z^4}+\dots\right) =$$

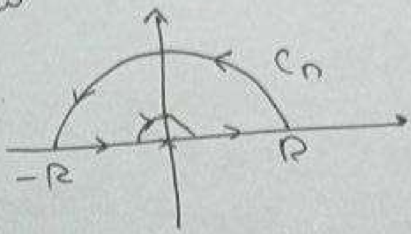
$$\Rightarrow \underset{z=\infty}{\operatorname{res}} f(z) = 0$$

Problem 3.7.

1. $\int_0^{+\infty} \frac{x-\sin x}{x^3}\,dx$

$$I = \frac{1}{2}\int_{-\infty}^{+\infty} \frac{x-\sin x}{x^3}\,dx = \frac{1}{2}\int_{-\infty}^{+\infty} \mathrm{Im}\left[\frac{1+ix-e^{ix}}{x^3}\right]dx =$$

$$= \frac{1}{2}\mathrm{Im}\int_{-\infty}^{+\infty}\frac{1+ix-e^{ix}}{x^3}$$



$$f(z) = \frac{1+iz-e^{iz}}{z^3}$$

$$\oint_C f(z)\,dz = 0 = \int_{-\infty}^{+\infty} + \int_{\varepsilon} + \int_{C_R} \to 0$$

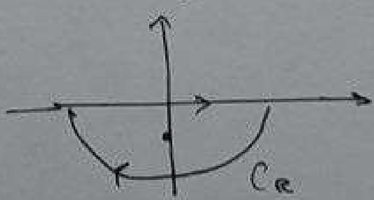
$$\int_{\varepsilon} = \left| \begin{matrix} z = \varepsilon e^{i\varphi} \\ dz = \varepsilon e^{i\varphi} i\,d\varphi \end{matrix} \right. = i\int_{\pi}^{0} \frac{1+i\varepsilon e^{i\varphi} - 1 - i\varepsilon e^{i\varphi} - \frac{(i\varepsilon e^{i\varphi})^2}{2!}}{\varepsilon^2 e^{2i\varphi}}\,d\varphi$$

$$= i\int_{\pi}^{0}\frac{\varepsilon^2 e^{2i\varphi}}{2!\,\varepsilon^2 e^{2i\varphi}} = -\frac{\pi}{2}i$$

$$\Rightarrow 0 = \int_{-\infty}^{+\infty} - \frac{\pi}{2}i \Rightarrow \int_{-\infty}^{+\infty} = \frac{\pi}{2}$$

$$\Rightarrow I = \frac{1}{2}\mathrm{Im}\int_{-\infty}^{+\infty} = \frac{1}{2}\cdot\frac{\pi}{2} = \frac{\pi}{4}$$

2. $\int_{-\infty}^{+\infty}\frac{e^{-iz}\,dz}{z^2+9}$



$$\oint = \int_{-\infty}^{+\infty} + \int_{C_R} \to 0$$

$$\oint_{-\infty}^{+\infty} \cancel{\frac{e^{-iz}}{z-3i}} = \mathrm{res}_{z=-3i} f(z) = \frac{e^{-i(-3i)}}{-6i} =$$

$$= \frac{1}{e^3\cdot 6i}$$

$$\int_{-\infty}^{+\infty}\frac{e^{-iz}\,dz}{z^2+9} = 2\pi i\cdot\frac{1}{3e^3} = \frac{\pi}{3e^3}$$

Problem 3.9

$$\int_{-\infty}^{+\infty} \frac{\cos\left(x-\frac{1}{x}\right)}{1+x^2}\,dx = \left| \begin{array}{l} u = x - \frac{1}{x} \\ du = \frac{(1+x^2)}{x^2}\,dx \end{array} \right. =$$

$$= 2\int_{-\infty}^{+\infty} \frac{\cos u}{u^2+4}\,du = 2\,\mathrm{Re}\int_{-\infty}^{+\infty} \frac{e^{iu}}{u^2+4} =$$

$$= 2\,\mathrm{Re}\cdot 2\pi i \operatorname*{res}_{u=2i} \frac{e^{iu}}{(u+i2)(u-i2)} =$$

$$= 2\,\mathrm{Re}\; 2\pi i\,\frac{e^{-2}}{4i} = 2\,\mathrm{Re}\,\frac{\pi}{2e^2} = \frac{\pi}{e^2}$$