Задача 1

$$\frac{E_z^2}{E_2^2\sin^2\alpha} - \frac{2E_yE_z\cos\alpha}{E_1E_2\sin^2\alpha} + \frac{E_y^2}{E_1^2\sin^2\alpha} = 1$$

Заменой

$$E_y = E_y'\cos\theta + E_z'\sin\theta$$

$$E_z = -E_y'\sin\theta + E_z'\cos\theta$$

получим

$$\frac{E_y'^2\cos^2\theta - E_y'E_z'\sin2\theta + E_y'^2\sin^2\theta}{E_2^2\sin^2\alpha} - 2\frac{-\frac{1}{2}E_y'^2\sin2\theta - E_y'E_z'\sin^2\theta + E_y'E_z\cos^2\theta}{\frac{E_1E_2\sin^2\alpha}{\cos\alpha}} -$$

$$- 2\frac{E_z'^2\frac{1}{2}\sin2\theta}{\frac{E_1E_2\sin^2\alpha}{\cos\alpha}} + \frac{E_y'^2\cos^2\theta + E_y'E_z'\sin2\theta + E_z'^2\sin^2\theta}{E_1^2\sin^2\alpha} = 1$$

$$\Rightarrow \frac{E_y'^2}{\sin^2\alpha}\left(\frac{\sin^2\theta}{E_2^2} + 2\frac{\cos\theta\sin\theta}{E_1E_2}\cos\alpha + \frac{\cos^2\theta}{E_1^2}\right) +$$

$$+ \frac{2E_y'E_z'}{\sin^2\alpha}\left(-\frac{\sin2\theta}{2E_2^2} - \frac{1}{E_1E_2}\cos\alpha\cos2\theta + \frac{\sin2\theta}{2E_1^2}\right) +$$

$$+ \frac{E_z'^2}{\sin^2\alpha}\left(\frac{\cos^2\theta}{E_2^2} - 2\frac{\sin\theta\cos\theta}{E_1E_2}\cos\alpha + \frac{\sin^2\theta}{E_1^2}\right) = 1$$

1. $$a = \frac{\sin\alpha}{\sqrt{\frac{\sin^2\theta}{E_1^2} - \frac{\sin2\theta}{E_1E_2}\cos\alpha + \frac{\cos^2\theta}{E_2^2}}}$$

$$b = \frac{\sin\alpha}{\sqrt{\frac{\cos^2\theta}{E_1^2} + \frac{\sin2\theta}{E_1E_2}\cos\alpha + \frac{\sin^2\theta}{E_2^2}}}$$

2. $\frac{1}{2}\sin 2\vartheta\left(\frac{1}{E_1^2}-\frac{1}{E_2^2}\right)-\frac{1}{E_1E_2}\cos\alpha\cos 2\vartheta=0$

$$\frac{1}{2}\left(\frac{1}{E_1^2}-\frac{1}{E_2^2}\right)\cdot \operatorname{tg}2\vartheta=\frac{1}{E_1E_2}\cos\alpha$$

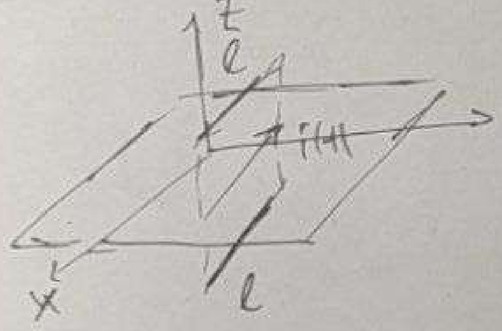
$$\operatorname{tg}2\vartheta=\frac{2E_1E_2}{E_2^2-E_1^2}\cos\alpha$$

3. Круговая пол. при $E_1=E_2$ и

$$\cos\alpha=0\Rightarrow\alpha=\frac{\pi}{2}+2\pi n;\ n\in\mathbb{Z}$$

Задача 10.

1.



$$\oint \vec{H}d\vec{l}=\frac{4\pi}{c}\vec{j}$$

$$2Hl=\frac{4\pi}{c}i(t)l$$

$$\vec{H}=\frac{2\pi}{c}i(t)$$

Мгновенное магнитное поле совпадает с ~~[illegible]~~ ~~с постоянным~~ полем с постоянным током $i(t)$

Задача 2

1. $t \in [-\pi; \pi]$

$f(t) = t^2$

$f(t) = \sum_{-\infty}^{+\infty} \hat{f}_k e^{ikt}$

$$\hat{f}_k = \frac{1}{2\pi}\int_{-\pi}^{+\pi} t^2 e^{-ikt} dt = \left| \begin{matrix} u = t^2 & dv = e^{-ikt} \\ du = 2t & v = -\frac{e^{-ikt}}{ik} \end{matrix} \right. =$$

$$= \frac{1}{2\pi}\left[- \frac{t^2 e^{-ikt}}{ik}\Big|_{-\pi}^{+\pi} \overset{=0}{} + \int_{-\pi}^{+\pi} \frac{2t}{ik} e^{-ikt} dt \right] = \left| \begin{matrix} u = t & dv = e^{-ikt} dt \\ du = dt & v = -\frac{e^{-ikt}}{ik} \end{matrix} \right.$$

$$= \frac{1}{2\pi}\left[-2 \frac{t e^{-ikt}}{i^2 k^2}\Big|_{-\pi}^{+\pi} + \int_{-\pi}^{+\pi} \frac{2 e^{-ikt}}{i^2 k^2} dt \overset{=0}{} \right] =$$

$$= \frac{2\pi (e^{-i\pi k} + e^{i\pi k})}{2\pi k^2} = \frac{2 \cdot 2\pi \cos \pi k}{2\pi k^2} = \frac{2}{k^2}(-1)^k$$

$$\Rightarrow f(t) = \sum_{-\infty}^{+\infty} \frac{2}{k^2}(-1)^k e^{ikt}$$

$$\hat{f}_0 = \frac{1}{2\pi}\int_{-\pi}^{+\pi} t^2 dt = \frac{\pi^2}{3}$$

$$\Rightarrow f(t) = \frac{\pi^2}{3} + \sum_{-\infty}^{-1} \frac{2}{k^2}(-1)^k e^{ikt} + \sum_{+1}^{+\infty} \frac{2}{k^2}(-1)^k e^{ikt} =$$

$$= \frac{\pi^2}{3} + \sum_{+1}^{+\infty} \frac{2}{k^2}(-1)^k \cdot 2\cos kt =$$

$$= \frac{\pi^2}{3} + \sum_{+1}^{+\infty} \frac{4}{k^2}(-1)^k \cos kt$$

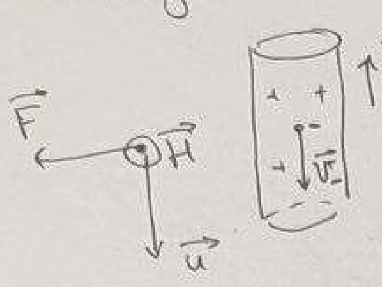
2. $$f(\pi) = \pi^2 = \frac{\pi^2}{3} + \sum_{k=1}^{+\infty} \frac{4}{k^2}(-1)^k \cos \pi k = \frac{\pi^2}{3} + \sum_{k=1}^{+\infty} \frac{4}{k^2}(-1)^k \cdot (-1)^k$$

$$= \frac{\pi^2}{3} + \sum_{k=1}^{+\infty} \frac{4}{k^2} \Rightarrow \sum_{k=1}^{+\infty} \frac{4}{k^2} = \frac{2\pi^2}{3} \Rightarrow$$

$$\Rightarrow \sum_{k=1}^{+\infty} \frac{1}{k^2} = \frac{\pi^2}{6}$$

Задача 3

Неподвижная СО: λ - лин. плотность.



$|\lambda_+| = |\lambda_-| = \lambda$ (т. к. стержень эл. нейтрален)

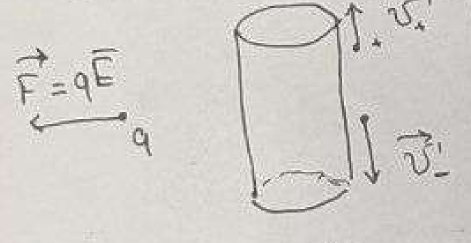
$H = \frac{2I}{cR}$, $I = \frac{dq}{dt} = \frac{d(\lambda v t)}{dt} = \lambda v_-$,

где v_- – ск. электронов

$\Rightarrow H = \frac{2\lambda v_-}{cR}$. $F = \frac{q}{c} \cdot u \cdot H =$

$= 2q \frac{u v_-}{c^2} \frac{\lambda}{R}$

Подвижная СО: (Магн. поле преобр. в электрическое)



Т.к. пол. заряды покоились в неподв. СО, то в этой СО у них будет скорость $|v'_+| = u$ (u – скорость системы отсчета)

$v'_- = \frac{u - v_-}{1 - \frac{u v_-}{c^2}}$; $v_- = \frac{v'_- + u}{1 + \frac{u v'_-}{c^2}}$.

Т.к. $Q = \text{const}$, то $\lambda \cdot l = \text{const} \Rightarrow \lambda \gamma^{-1} = \text{const}$. Тогда:

$$\lambda'_+ = \lambda_+ \frac{1}{\sqrt{1-\left(\frac{u}{c}\right)^2}}; \quad \lambda'_- = \lambda_- \frac{\sqrt{1-\left(\frac{v_-}{c}\right)^2}}{\sqrt{1-\left(\frac{v'_-}{c}\right)^2}}$$

$$\Rightarrow \lambda'_+ - \lambda'_- = \lambda\left[\frac{1}{\sqrt{1-\left(\frac{u}{c}\right)^2}} - \frac{\sqrt{1-\left(\frac{v_-}{c}\right)^2}}{\sqrt{1-\left(\frac{v'_-}{c}\right)^2}}\right] =$$

$$= \lambda\left[\frac{1}{\sqrt{1-\left(\frac{u}{c}\right)^2}} - \frac{\sqrt{1-\left(\frac{v_-}{c}\right)^2}}{\sqrt{1-\frac{u^2-2uv_-+v_-^2}{c^2-2uv_-+\frac{u^2v_-^2}{c^2}}}}\right] =$$

$$= \lambda\left[\frac{1}{\sqrt{1-\left(\frac{u}{c}\right)^2}} - \frac{\sqrt{1-\left(\frac{v_-}{c}\right)^2}}{\sqrt{\left(1-\left(\frac{v_-}{c}\right)^2\right)\left(1-\left(\frac{u}{c}\right)^2\right)}}\left(1-\frac{uv_-}{c^2}\right)\right] =$$

$$= \frac{\lambda}{\sqrt{1-\left(\frac{u}{c}\right)^2}}\left(1 - 1 + \frac{uv_-}{c^2}\right) = \lambda \frac{uv_-}{c^2}\gamma,$$

где $\gamma = \frac{1}{\sqrt{1-\left(\frac{u}{c}\right)^2}}$

Тогда: $E \cdot 2\pi R \cdot l = 4\pi \lambda' l \Rightarrow E = \frac{2\lambda'}{R}$

$$\lambda' = \lambda \frac{uv_-}{c^2}\gamma \Rightarrow E = 2\frac{uv_-}{c^2}\frac{\lambda}{R}\gamma$$

$$\Rightarrow F = 2q\frac{uv_-}{c^2}\frac{\lambda}{R}\gamma \ !$$

Заметим, что при $u = v$ формула сх. с полученной нами ~~в~~ на лекцией $F = 2q\frac{v^2}{c^2}\frac{\lambda}{R}\gamma$

Задача 4

1. $(\vec{a}\nabla)\vec{r} = \left(a_i \frac{\partial}{\partial x_i}\right) x_k \vec{e}_k = a_i \vec{e}_i = \vec{a}$

ч.т.д

$$(\vec{a}\nabla)\vec{r} = \left(a_x \frac{\partial}{\partial x} + a_y \frac{\partial}{\partial y} + a_z \frac{\partial}{\partial z}\right)(x\vec{i} + y\vec{j} + z\vec{k}) =$$

$$= a_x \vec{i} + a_y \vec{j} + a_z \vec{k} = \vec{a} \quad \text{ч.т.д}$$

2. ~~$\nabla \times (\vec{a} \times r$~~ $\nabla \times (\vec{a}(\vec{r}) \times \vec{b}) = -?$

$$\vec{a}(\vec{r}) \times \vec{b} = \vec{i}\,(a_y(\vec{r})\, b_z - a_z(\vec{r})\, b_y) + \dots$$

$$\nabla \times (\vec{a}(\vec{r}) \times \vec{b}) = \left[\frac{\partial(a_x b_y - a_y b_x)}{\partial y} - \frac{\partial(a_z b_x - a_x b_z)}{\partial z}\right]\vec{i} + \dots =$$

$$= \vec{i}\left[b_y \frac{\partial a_x}{\partial y} - b_x \frac{\partial a_y}{\partial y} - b_x \frac{\partial a_z}{\partial z} + b_z \frac{\partial a_x}{\partial z}\right] + \dots$$

$$= \vec{i}\left[b_x \frac{\partial a_x}{\partial x} + b_y \frac{\partial a_x}{\partial y} + b_z \frac{\partial a_x}{\partial z} - b_x \frac{\partial a_x}{\partial x} - b_x \frac{\partial a_y}{\partial y} - b_x \frac{\partial a_z}{\partial z}\right]$$

$$= \vec{i}\left[(\vec{b}\nabla)a_x - b_x(\nabla \vec{a})\right] + \dots$$

Аналогично находим и другие сост. Тогда:

$$\nabla \times (\vec{a}(\vec{r}) \times \vec{b}) = \vec{i}\left[(\vec{b}\nabla)a_x - b_x(\nabla\vec{a})\right] + \vec{j}\left[(\vec{b}\nabla)a_y - b_y(\nabla\vec{a})\right] + \vec{k}\left[(\vec{b}\nabla)a_z - b_z(\nabla\vec{a})\right] =$$

$$= (\vec{b}\nabla)\vec{a}(\vec{r}) - \vec{b}(\nabla\vec{a}(\vec{r})) \quad \text{ч.т.д.}$$

Задача 5.

1. $\nabla \times f(\vec{r})\cdot\vec{r} = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ \partial_x & \partial_y & \partial_z \\ f(r)x & f(r)y & f(r)z \end{vmatrix} =$

$= \bar{i}\left(z\frac{\partial f}{\partial y} - y\frac{\partial f}{\partial z}\right) +$
$+ \bar{j}\left(x\frac{\partial f}{\partial z} - z\frac{\partial f}{\partial x}\right) +$
$+ \bar{k}\left(y\frac{\partial f}{\partial x} - x\frac{\partial f}{\partial y}\right)$

$\boxed{\frac{\partial r}{\partial x_i} = \frac{2x_i}{2\cdot r} = \frac{x_i}{r}}$

$= \bar{i}\left(z\frac{\partial f}{\partial r}\frac{\partial r}{\partial y} - y\frac{\partial f}{\partial r}\frac{\partial r}{\partial z}\right) + \ldots =$

$= \bar{i}\left(\frac{zy}{r}\frac{\partial f}{\partial r} - \frac{zy}{r}\frac{\partial f}{\partial r}\right) + \ldots =$

$= \bar{i}\,0 + \bar{j}\,0 + \bar{k}\cdot 0 = 0$

2. $\nabla \times (\bar{a}\times\vec{r}) = -\nabla\times(\vec{r}\times\vec{a}) =$

$= -\left((\bar{a}\nabla)\vec{r} - \bar{a}(\nabla\vec{r})\right) =$

$= -(\vec{a} - 3\vec{a}) = 2\vec{a}$

Задача 6.

$$\frac{1}{c^2(x)}\frac{\partial^2 f}{\partial t^2}-\frac{\partial^2 f}{\partial x^2}=0$$

1. $\frac{\partial^2 f}{\partial t^2}=-\omega^2 f$ (т.к. волна монохроматическая)

$$\Rightarrow -\frac{\omega^2}{c^2(x)}f-\frac{\partial^2 f}{\partial x^2}=0 \Leftrightarrow$$

$$\Leftrightarrow \frac{\partial^2 f}{\partial x^2}+k^2(x)f=0$$

2. $c(x+\lambda)=c(x)+\frac{dc(x)}{dx}\lambda$

$$\Rightarrow c(x+\lambda)-c(x)=\frac{dc(x)}{dx}\frac{2\pi c(x)}{\omega}\Rightarrow$$

$$\Rightarrow \frac{c(x+\lambda)-c(x)}{c(x)}=\frac{2\pi}{\omega}\frac{dc(x)}{dx}$$

$$\left|\frac{c(x+\lambda)-c(x)}{c(x)}\right|\ll 1\Rightarrow$$

$$\Rightarrow \left|-2\pi\frac{dk(x)}{dx}\frac{1}{k^2}\right|=\left|2\pi\frac{dk(x)}{dx}\frac{1}{k^2}\right|\ll 1$$

$$\Rightarrow \frac{1}{k^2}\frac{dk(x)}{dx}\ll 1$$

3. $f=e^{S(x)}\Rightarrow \frac{d^2f}{dx^2}+k^2(x)f=\frac{d^2S}{dx^2}+\left(\frac{dS}{dx}\right)^2+k^2(x)=0$

4. $\frac{d^2S}{dx^2}\ll k^2(x)\Rightarrow$

$$\Rightarrow \left(\frac{dS}{dx}\right)^2+k^2(x)=0\Leftrightarrow\left(\frac{dS}{dx}-ik(x)\right)\left(\frac{dS}{dx}+ik(x)\right)=0\Rightarrow$$

$$\Rightarrow S=\pm i\int k(x)dx$$

$\frac{dS}{dx}=\pm ik(x)$; $\frac{d^2S}{dx^2}=\pm i\frac{dk(x)}{dx}\Rightarrow$ По усл.

$\frac{1}{k^2}\frac{dk}{dx}\ll 1$ имеем $\frac{d^2S}{dx^2}\ll k^2(x)$

Задача 7.

$$\nabla \times \vec{H} = \frac{4\pi}{c}\vec{j} + \frac{1}{c}\frac{\partial \vec{E}}{\partial t}$$

$$\left(\nabla \cdot [\nabla \times \vec{H}]\right) = \frac{4\pi}{c}(\nabla \cdot \vec{j}) + \frac{1}{c}\frac{\partial}{\partial t}(\nabla \cdot \vec{E})$$

Т.к. div rot = 0 и $\nabla \cdot \vec{E} = 4\pi\rho$:

$$0 = \frac{4\pi}{c}\nabla \cdot \vec{j} + \frac{4\pi}{c}\frac{\partial \rho}{\partial t} \Rightarrow$$

$$\Rightarrow \frac{\partial \rho}{\partial t} + \nabla \cdot \vec{j} = 0$$

Задача 8.

Калибровочная преобразование:

$$\vec{A}' = \vec{A} + \mathrm{grad} f, \quad \varphi' = \varphi - \frac{1}{c}\frac{\partial f}{\partial t}$$

f - произв. функция.

Подставим в $\nabla \vec{A}' + \frac{1}{c}\frac{\partial \varphi'}{\partial t} = 0$:

$$\nabla \vec{A} + \nabla \mathrm{grad} f + \frac{1}{c}\frac{\partial \varphi}{\partial t} - \frac{1}{c^2}\frac{\partial^2 f}{\partial t^2} =$$

$$= \nabla \vec{A} + \Delta f + \frac{1}{c}\frac{\partial \varphi}{\partial t} - \frac{1}{c^2}\frac{\partial^2 f}{\partial t^2} =$$

$$= \nabla \vec{A} + \frac{1}{c}\frac{\partial \varphi}{\partial t} + \Box f \overset{=0}{=} \left| \begin{array}{l} \text{Если } f \text{ уд.} \\ \text{волн. уравнению} \end{array} \right| =$$

$$= \nabla \vec{A} + \frac{1}{c}\frac{\partial \varphi}{\partial t} = -\Box f = 0$$

$$\Rightarrow \nabla \vec{A} + \frac{1}{c}\frac{\partial \varphi}{\partial t} = 0 \quad \text{ч.т.д}$$

Задача 9.

1. $B_z = B_0 - \alpha z$

$\cancel{\text{div}\, B_z}$

$$\text{div}\,\vec{B} = \partial_x B_x + \partial_z B_z = \partial_x B_x - \alpha = 0$$

$$\Rightarrow B_x = \alpha x + C$$

$$B_z(B_x) - 0 = \text{tg}\frac{\pi}{4}(B_x - 0)$$

$\frac{B_z}{B_x} = 1$ в начале коорд. $\Rightarrow$

$$\frac{B_z}{B_x} = \frac{B_0}{C} = 1 \Rightarrow C = B_0 \Rightarrow \begin{cases} B_z = B_0 - \alpha z \\ B_x = B_0 + \alpha x \end{cases}$$

2. $$\frac{dx}{B_x} = \frac{dz}{B_z} \Leftrightarrow \frac{dx}{B_0 + \alpha x} = \frac{dz}{B_0 - \alpha z} \Rightarrow$$

$$\Rightarrow \ln(\alpha x + B_0) = -\ln(B_0 - \alpha z) + C$$

$$\Rightarrow \alpha x + B_0 = \frac{C'}{B_0 - \alpha z},$$ C' – отл. от C константа

Семейство силовых линий – гиперболы с центром $(x;z) = \left(-\frac{B_0}{\alpha}; \frac{B_0}{\alpha}\right)$ и две прямые $x = \cancel{B_0} - \frac{B_0}{\alpha}$ и $z = \frac{B_0}{\alpha}$