Problem 3.4

$$\int_c \frac{z^5 dz}{1+z^6}$$

$$f(z) = \frac{z^5}{1+z^6} = \frac{1}{z}\left(1 - \frac{1}{z^6} + \dots\right) = \frac{1}{z}$$

$$\operatorname{res} f(z) = c_{-1} = 1$$

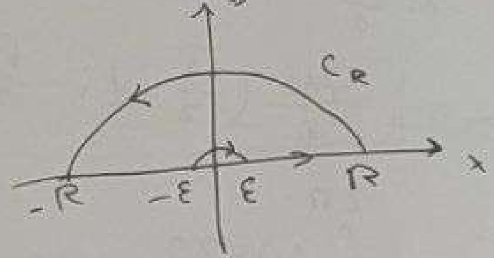
$$\oint_C f(z)\,dz = 2\pi i$$

Problem 3.5

$$\sin^2 x = \frac{1}{2}(1-\cos(2x))$$

$$I = \int_{-\infty}^{+\infty} \frac{\sin^2 x\,dx}{x^2(x^2+1)}{}^2$$

$$= \frac{1}{2}\operatorname{Re}\left[\int_{-\infty}^{+\infty} \frac{1-e^{2ix}}{x^2(x^2+1)}\,dx\right]$$



$$\oint_c = \int_{-\infty}^{-\varepsilon} + \int_{\varepsilon}^{+\infty} + \int_{\varepsilon} + \int_{C_R}$$

$$f(z) \xrightarrow{z\to\infty} \approx \frac{1}{z^6} \Rightarrow \int_{C_R} \to 0 \sim \frac{1}{R^5}$$

$$\oint f(z)\,dz = 2\pi i \operatorname*{res}_{z=i} f(z) = 2\pi i\, \frac{1-e^{2iz}}{z^2(z+i)} = 2\pi i\,\frac{1-e^{-2}}{(-1)\cdot 2i} =$$

$$= \pi\left(\frac{1}{e^2}-1\right)$$

$$\int_{\varepsilon} = \left| \begin{matrix} z = \varepsilon e^{i\varphi} \\ dz = \varepsilon i e^{i\varphi}\,d\varphi \end{matrix}\right. = \int_{\pi}^{0} \frac{1-e^{2i\varepsilon e^{i\varphi}}}{(\varepsilon e^{i\varphi})^2\left((\varepsilon e^{i\varphi})^2+1\right)}\, i\varepsilon e^{i\varphi}\,d\varphi =$$

$$= \int_{\pi}^{0} \frac{1-1-2i\varepsilon e^{i\varphi}}{\varepsilon^2(\varepsilon^2+1)}\, i\varepsilon e^{i\varphi}\,d\varphi = 2\int_{\pi}^{0} d\varphi = -2\pi$$

$$\Rightarrow \cancel{\oint}_{-\infty} \int_{-\infty}^{+\infty} = \pi\left(\frac{1}{e^2}-1\right) + 2\pi = \pi\left(1+\frac{1}{e^2}\right) \Rightarrow I = \frac{1}{2}\operatorname{Re}\int_{-\infty}^{+\infty} = \underline{\frac{\pi}{2}\left(1+\frac{1}{e^2}\right)}$$

Problem 3.3.

$$f(z) = \frac{1}{z^3 - z^5} = \frac{1}{z^3(1-z^2)}$$

$$\cancel{\operatorname*{res}_{z=-1} f(z) = \frac{1}{(\varepsilon-1)^3(2+2\varepsilon+6\varepsilon}}$$

$$\operatorname*{res}_{z=-1} f(z) = \frac{1}{(\varepsilon-1)^3(1-\varepsilon^2+2\varepsilon-1)} \approx \cancel{\frac{1}{\varepsilon(\varepsilon-1)^3(1+\varepsilon)}} =$$

$$= \cancel{\frac{1}{\varepsilon(\varepsilon^3-3\varepsilon^2+3\varepsilon-1)(1+\varepsilon)}} \approx \frac{1}{(\varepsilon^3-3\varepsilon^2+3\varepsilon-1)(2\varepsilon-\varepsilon^2)} =$$

$$\approx \cancel{\frac{1}{\varepsilon(2+8\varepsilon)}} \approx \cancel{\frac{1}{2(1+4\varepsilon)}}$$

$$\approx -\frac{1}{2\varepsilon}$$

Для $\operatorname*{res}_{z=1} f(z)$ ~~дел~~ аналогично.

$$\operatorname*{res}_{z=0} f(z) = \frac{1}{z^3(1-z^2)} \approx \frac{1}{z^3}(1+z^2+\dots) = \frac{1}{z^3} + \frac{1}{z}$$

$$\Rightarrow \operatorname*{res}_{z=0} f(z) = 1$$

$$\operatorname*{res}_{z=\infty} f(z) = -\frac{1}{z^5\left(1-\frac{1}{z^2}\right)} = -\frac{1}{z^5}\left(1+\frac{1}{z^2}+\frac{1}{z^4}+\dots\right) =$$

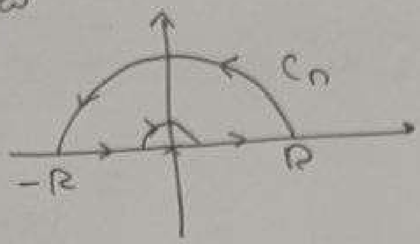
$$\Rightarrow \operatorname*{res}_{z=\infty} f(z) = 0$$

Problem 3.7.

1. $\int_0^{+\infty} \frac{x-\sin x}{x^3}\, dx$

$$I = \frac{1}{2}\int_{-\infty}^{+\infty} \frac{x-\sin x}{x^3}\, dx = \frac{1}{2}\int_{-\infty}^{+\infty} \mathrm{Im}\left[\frac{1+ix-e^{ix}}{x^3}\right] dx =$$

$$= \frac{1}{2}\mathrm{Im}\int_{-\infty}^{+\infty} \frac{1+ix-e^{ix}}{x^3}$$



$$f(z) = \frac{1+iz-e^{iz}}{z^3}$$

$$\oint_C f(z)\,dz = 0 = \int_{-\infty}^{+\infty} + \int_\varepsilon + \int_{C_R} \to 0$$

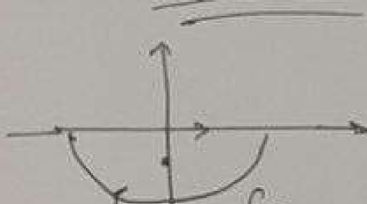
$$\int_\varepsilon = \left| \begin{matrix} z = \varepsilon e^{i\varphi} \\ dz = \varepsilon e^{i\varphi} i\, d\varphi \end{matrix} \right. = i\int_\pi^0 \frac{1+i\varepsilon e^{i\varphi} - 1 - i\varepsilon e^{i\varphi} - \frac{(i\varepsilon e^{i\varphi})^2}{2!}}{\varepsilon^2 e^{2i\varphi}}\, d\varphi$$

$$= i\int_\pi^0 \frac{\varepsilon^2 e^{2i\varphi}}{2!\,\varepsilon^2 e^{2i\varphi}} = -\frac{\pi}{2} i$$

$$\Rightarrow 0 = \int_{-\infty}^{+\infty} - \frac{\pi}{2} i \Rightarrow \int_{-\infty}^{+\infty} = \frac{\pi}{2}$$

$$\Rightarrow I = \frac{1}{2}\mathrm{Im}\int_{-\infty}^{+\infty} = \frac{1}{2}\cdot\frac{\pi}{2} = \frac{\pi}{4}$$

2. $\int_{-\infty}^{+\infty} \frac{e^{-iz}\, dz}{z^2+9}$



$$\oint = \int_{-\infty}^{+\infty} + \int_{C_R} \to 0$$

$$\mathrm{res}_{z=-3i} f(z) = \frac{e^{-i(-3i)}}{-6i} =$$

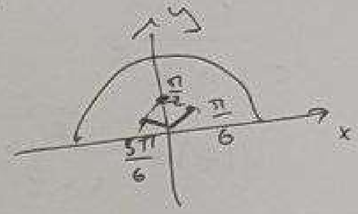
$$= \frac{1}{e^3 \cdot 6i}$$

$$\int_{-\infty}^{+\infty} \frac{e^{-iz}\, dz}{z^2+9} = 2\pi i \cdot \frac{1}{3e^3} = \frac{\pi}{3e^3}$$

Problem 3.1

$\frac{1}{R} \to 0$

1. $\int_{-\infty}^{+\infty} \frac{x^4}{1+x^6}\,dx = \oint + \cancel{\int_{C_R}} = 2\pi i \sum \underset{z=z_0}{\mathrm{res}} f(z)$



$f(z) = \frac{z^4}{1+z^6}$

$z = e^{i\left(\frac{\pi}{6} + \frac{2\pi k}{3}\right)}$

$\underset{z=e^{i\frac{\pi}{6}}}{\mathrm{res}} \frac{1}{6z} = \frac{1}{6} e^{-i\frac{\pi}{6}}$, $\underset{z=\frac{\pi}{2}}{\mathrm{res}} \frac{1}{6z} = \frac{1}{6} e^{-i\frac{\pi}{2}}$;

$\underset{z=\frac{5\pi}{6}}{\mathrm{res}} \frac{1}{6z} = \frac{1}{6} e^{-i\frac{5\pi}{6}}$

$\Rightarrow I = \frac{2\pi i}{6}\left(\frac{\sqrt{3}}{2} + \frac{1}{2}i - \frac{\sqrt{3}}{2} + \frac{1}{2}i + i\right) = \frac{2}{3}\pi$

2. $\int_0^{2\pi} \frac{\cos 2\theta}{2+\cos\theta}\,d\theta = \left|\cos 2\theta = \frac{1}{2}\left(z^2 + \frac{1}{z^2}\right)\right. =$

$\Leftrightarrow \oint \frac{z^4+1}{iz^2(z+2+\sqrt{3})(z+2-\sqrt{3})}$

$\underset{z=0}{\mathrm{res}} f(z) \simeq -i\frac{1}{z^2}(1-4z-z^2) \Rightarrow C_{-1} = 4i$

$\underset{z=\sqrt{3}-2}{\mathrm{Res}} f(z) = -7\frac{1}{\sqrt{3}}i$

$\Rightarrow \int = 2\pi\left(\frac{7}{\sqrt{3}} - 4\right)$

Problem 3.6

1. $\lim\limits_{R\to\infty}\int\limits_{C_R} e^{iz}\,dz = \lim\limits_{R\to\infty}\int R^2\,\text{func}(\varphi) \Rightarrow \infty$

$\Rightarrow$ лимит не нуль.

Problem 3.8

$$\int_0^{+\infty}\frac{x\sin ax}{x^2+k^2}\,dx = \int_0^{+\infty}\frac{xe^{iax}}{2i(x^2+k^2)}\,dx + \int_0^{+\infty}\frac{(-x)e^{-iax}}{2i(x^2+k^2)}\,dx =$$

$$= \frac{1}{2i}\int_{-\infty}^{+\infty}\frac{xe^{iax}}{x^2+k^2}\,dx$$

Для $a>0$:

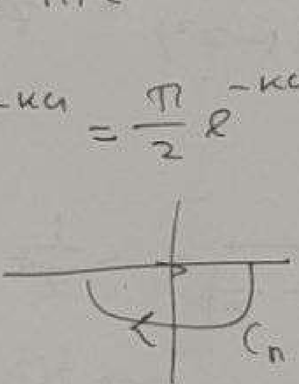
Лемма Жорд. выполняется

$$\Rightarrow \underset{z=ik}{\text{res}}\, f(z) = \frac{ike^{-ka}}{2ik} = \frac{e^{-ak}}{2}$$

$$I = 2\pi i\frac{e^{-ka}}{2} = \pi i e^{-ka}$$

$$\Rightarrow I = \frac{1}{2i}\,\pi i e^{-ka} = \frac{\pi}{2}e^{-ka}$$

Для $a<0$:



Аналогично:

$$I = \frac{1}{2i}\left(-2\pi i\,\underset{z=-ik}{\text{res}}\, f(z)\right) = -\frac{\pi}{2}e^{-ka}$$

$\Rightarrow$ Ответ: $I = \frac{\pi}{2}e^{-ak}\,\text{sign}(a)$