Problem 1.

$S_{ij} = S_{ji}$

$S'_{ij} = d_{ik} d_{jl} S_{kl} = d_{ik} d_{jl} S_{lk} = S'_{ji}$ Q.E.D.

Problem 2.

$\Pi'_{ij} = d_{ik} d_{jl} \Pi_{kl} = d_{ik} \Pi_{kl} d^T_{lj} = (d \Pi d^T)_{ij}$

$\Rightarrow \Pi' = d \Pi d^T$

Problem 3.

$d = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$, $\varepsilon_{ij} = -\varepsilon_{ji}$, $\varepsilon_{12} = 1$

$\Rightarrow \varepsilon = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, $d^T = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$

$$\varepsilon'_{ij} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} =$$

$$= \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} \sin\theta & \cos\theta \\ -\cos\theta & \sin\theta \end{pmatrix} =$$

$$= \begin{pmatrix} \cos\theta\sin\theta - \cos\theta\sin\theta & \cos^2\theta + \sin^2\theta \\ -\sin^2\theta - \cos^2\theta & \cos\theta\sin\theta - \cos\theta\sin\theta \end{pmatrix} =$$

$$= \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = \varepsilon_{ij} \quad \text{Q.E.D.}$$

Problem 4.

1. $C_{ijkl} = A_{ij} B_{kl}$ has n^4 components.

$C'_{ijkl} = A'_{ij} B'_{kl} = d_{is} d_{jt} A_{st} d_{km} d_{ln} B_{mn} =$

$= d_{is} d_{jt} d_{km} d_{ln} C_{stmn} \Rightarrow C$ - Tensor of 4 rank.

2. $D'_{il} = \sum_j C'_{ijjl} = \sum_j A_{ij} B_{jl} = \sum_j d_{im} d_{jn} d_{jr} d_{ls} C_{mnrs} =$

$= \sum_j d_{im} d_{ls} d^T_{nj} d_{jr} C_{mnrs} = d_{im} d_{ls} \delta_{nr} C_{mnrs} =$

$= d_{im} d_{ls} C_{mjjs} = d_{im} d_{ls} D_{ms}$ Q.E.D.

Problem 5.

$$\varphi = \varphi(x_1, x_2, x_3),\ D_{ij} = \frac{\partial^2 \varphi}{\partial x_i \partial x_j};$$

$$\frac{\partial \varphi}{\partial x'_j} = \frac{\partial f}{\partial x_l}\frac{\partial x_l}{\partial x'_j};\quad \left[\begin{array}{l} x_i = (d^{-1})_{ij} x_j \\ r^2 = x^T x = (x')^T \cdot x' \Rightarrow \\ \Rightarrow x^T d^T d x = x^T \cdot x \Rightarrow \\ \Rightarrow d^T \cdot d = E \Rightarrow d^T = d^{-1} \end{array}\right]$$

$$\frac{\partial}{\partial x'_i}\left(\frac{\partial \varphi}{\partial x'_j}\right) = \frac{\partial}{\partial x'_i}\left(\frac{\partial \varphi}{\partial x_l}\frac{\partial x_l}{\partial x'_j}\right) =$$

$$= \frac{\partial}{\partial x_k}\left(\frac{\partial \varphi}{\partial x_l}\frac{\partial x_l}{\partial x'_j}\right)\frac{\partial x_k}{\partial x'_i} = \left[\frac{\partial^2 \varphi}{\partial x_k \partial x_l}\frac{\partial x_l}{\partial x'_j}\frac{\partial x_k}{\partial x'_i} + \frac{\partial \varphi}{\partial x_l}\frac{\partial x_k}{\partial x'_i}\frac{\partial^2 x_l}{\partial x'_j \partial x_k}\right]$$

Because $\frac{\partial x_l}{\partial x_k} = \delta_{lk} \Rightarrow \frac{\partial}{\partial x'_j}\left(\frac{\partial x_l}{\partial x_k}\right) = 0$

$$\Rightarrow \frac{\partial^2 \varphi}{\partial x'_i \partial x'_j} = \frac{\partial x_l}{\partial x'_j}\frac{\partial x_k}{\partial x'_i}\frac{\partial^2 \varphi}{\partial x_k \partial x_l} =$$

$$= (d^{-1})_{lj}(d^{-1})_{ki}\frac{\partial^2 \varphi}{\partial x_k \partial x_l} = d_{ik} d_{jl}\frac{\partial^2 \varphi}{\partial x_k \partial x_l} \Rightarrow$$

$\Rightarrow D'_{ij} = d_{ik} d_{jl} D_{ij} \Rightarrow D$ - Tensor of rank 2.

Q.E.D.

Problem 6.

$$A = \begin{pmatrix} 0 & \omega_3 & -\omega_2 \\ -\omega_3 & 0 & \omega_1 \\ \omega_2 & -\omega_1 & 0 \end{pmatrix},\ \omega_k = \frac{1}{2} A_{ij}\varepsilon_{ijk}$$

$$\omega'_k = \frac{1}{2} A'_{ij}\varepsilon_{ijk} = \frac{1}{2} d_{im} d_{jn} d_{ir} d_{js} d_{kt} A_{mn}\varepsilon_{rst} =$$

$$= \frac{1}{2} d^T_{mi} d_{ir}\, d^T_{nj} d_{js}\, d_{kt} A_{mn}\varepsilon_{rst} =$$

$$= \frac{1}{2}\delta_{mr}\delta_{ns} d_{kt} A_{mn}\varepsilon_{rst} = \frac{1}{2} d_{kt} A_{ij}\varepsilon_{ijt}$$

$$= \frac{1}{2} d_{kt}\omega_t \quad \text{Q.E.D.}$$

Problem 4.31

$$D' = D'_{ii} = d_{ik} d_{il} D_{kl} = d^T_{ki} \cdot d_{il} \cdot D_{kl} =$$

$$= \delta_{kl} D_{kl} = D_{ii} = D \quad \text{Q.E.D}$$

Problem 7.

1. $\sum_j A_{ij}\delta_{jk} = A_{ik}$, $\sum_i A_{ij}\delta_{ik} = A_{kj}$

$\sum_{ij} A_{ij}\delta_{ij} = A_{ii} = A_{11}+A_{22}+A_{33}$

2. $\sum_k \delta_{ik}\delta_{kj} = \delta_{ij}$, $\sum_{ik}\delta_{ik}\delta_{ik} = 3$, $\sum_{ik}\delta_{ik}\delta_{ki} = 3$

Problem 8.

$$\varepsilon_{ij} = \begin{vmatrix}\delta_{1i} & \delta_{1j}\\ \delta_{2i} & \delta_{2j}\end{vmatrix} , \quad \varepsilon_{lm} = \begin{vmatrix}\delta_{1l} & \delta_{1m}\\ \delta_{2l} & \delta_{2m}\end{vmatrix}$$

$$\varepsilon_{ij}\varepsilon_{lm} = \det\left[\begin{pmatrix}\delta_{1i} & \delta_{1j}\\ \delta_{2i} & \delta_{2j}\end{pmatrix}\begin{pmatrix}\delta_{1l} & \delta_{1m}\\ \delta_{2l} & \delta_{2m}\end{pmatrix}\right] = \left|\begin{matrix}\det A^T = \det A\\ \det AB = \det A\cdot\det B\end{matrix}\right|$$

$$= \begin{vmatrix}\delta_{1i}\delta_{1l}+\delta_{2i}\delta_{2l} & \delta_{1i}\delta_{1m}+\delta_{2i}\delta_{2m}\\ \delta_{1j}\delta_{1l}+\delta_{2j}\delta_{2l} & \delta_{1j}\delta_{1m}+\delta_{2j}\delta_{2m}\end{vmatrix} = \begin{vmatrix}\delta_{di}\delta_{dl} & \delta_{di}\delta_{dm}\\ \delta_{dj}\delta_{dl} & \delta_{dj}\delta_{dm}\end{vmatrix} =$$

$$= \begin{vmatrix}\delta_{il} & \delta_{im}\\ \delta_{jl} & \delta_{jm}\end{vmatrix} = \delta_{il}\delta_{jm} - \delta_{im}\delta_{jl}\text{. Q.E.D.}$$

Problem 9.

1. $$\varepsilon_{ijk}\varepsilon_{lmn} = \det\begin{pmatrix}\delta_{il} & \delta_{im} & \delta_{in}\\ \delta_{jl} & \delta_{jm} & \delta_{jn}\\ \delta_{kl} & \delta_{km} & \delta_{kn}\end{pmatrix} = \delta_{il}(\delta_{jm}\delta_{kn} - \delta_{km}\delta_{jn}) + \delta_{im}(\delta_{jn}\delta_{kl} - \delta_{jl}\delta_{kn}) + \delta_{in}(\delta_{jl}\delta_{km} - \delta_{kl}\delta_{jm})$$

2. $$\varepsilon_{ijk}\varepsilon_{lmk} = \det\begin{pmatrix}\delta_{il} & \delta_{jl} & \delta_{kl}\\ \delta_{im} & \delta_{jm} & \delta_{km}\\ \delta_{ik} & \delta_{jk} & \delta_{kk}\end{pmatrix} =$$

$$= \delta_{ik}\begin{vmatrix}\delta_{jl} & \delta_{kl}\\ \delta_{jm} & \delta_{km}\end{vmatrix} - \delta_{jk}\begin{vmatrix}\delta_{il} & \delta_{kl}\\ \delta_{im} & \delta_{km}\end{vmatrix} + \delta_{kk}\begin{vmatrix}\delta_{il} & \delta_{jl}\\ \delta_{im} & \delta_{jm}\end{vmatrix} =$$

$$= \begin{vmatrix}\delta_{jl} & \delta_{ik}\delta_{kl}\\ \delta_{jm} & \delta_{ik}\delta_{km}\end{vmatrix} - \begin{vmatrix}\delta_{il} & \delta_{jk}\delta_{kl}\\ \delta_{im} & \delta_{jk}\delta_{km}\end{vmatrix} + 3\begin{vmatrix}\delta_{il} & \delta_{jl}\\ \delta_{im} & \delta_{jm}\end{vmatrix} =$$

$$= \begin{vmatrix}\delta_{jl} & \delta_{il}\\ \delta_{jm} & \delta_{im}\end{vmatrix} - \begin{vmatrix}\delta_{il} & \delta_{jl}\\ \delta_{im} & \delta_{jm}\end{vmatrix} + 3\begin{vmatrix}\delta_{il} & \delta_{jl}\\ \delta_{im} & \delta_{jm}\end{vmatrix} = \begin{vmatrix}\delta_{il} & \delta_{jl}\\ \delta_{im} & \delta_{jm}\end{vmatrix} =$$

$$= \delta_{jm}\delta_{il} - \delta_{im}\delta_{jl}$$

3. $\varepsilon_{ijk}\varepsilon_{ljk} = \delta_{jj}\delta_{il} - \delta_{ij}\delta_{jl} = 3\delta_{il} - \delta_{il} = 2\delta_{il}$

4. $\varepsilon_{ijk}\varepsilon_{ijk} = 2\delta_{ii} = 2\cdot 3 = 6$

Problem 10.

$A_{ij} = -A_{ji}, \quad S_{ij} = S_{ji}$

$\sum_{ij} A_{ij} S_{ij} = -\sum_{ij} A_{ji} S_{ji} = -\sum_{ji} A_{ij} S_{ij} \Rightarrow$

$\Rightarrow 2 \sum_{ij} A_{ij} S_{ij} = 0 \Rightarrow \sum_{ij} A_{ij} S_{ij} = 0$ Q.E.D

Problem 11

1) $b_i = \Pi_{ij} a_j \Rightarrow b'_i = \Pi'_{ij} a'_j = d_{ik} d_{jl} \Pi_{kl} d_{jm} a_m =$

$= d_{ik} d^T_{mj} d_{jl} \Pi_{kl} a_m = d_{ik} \delta_{ml} \Pi_{kl} a_m =$

$= d_{ik} \Pi_{kj} a_j = d_{ik} b_k$ Q.E.D

2) $c_j = \sum_i \Pi_{ij} a_i \Rightarrow c'_j = d_{ik} d_{jl} \Pi_{kl} d_{im} a_m =$

$= d^T_{mi} d_{ik} d_{ik} \Pi_{kl} a_m = \delta_{mk} d_{jl} \Pi_{kl} a_m =$

$= \cancel{d_{ij} \Pi_{jk} a_j}\ d_{jl} \Pi_{sl} a_s = d_{jl} c_l$ Q.E.D