№1 $\Delta\vec{A} = -\frac{4\pi}{c}\vec{j}$

$$\Delta\vec{A} = \begin{cases} -\frac{4\pi}{c}\rho\omega r\vec{e}_\varphi, & 0\le r<a \\ 0, & a\le r \end{cases}$$

С помощью Wolfram Math

$$\Rightarrow \begin{cases} \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial A_r}{\partial r}\right) - \frac{A_r}{r^2} = 0 \\ \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial A_\varphi}{\partial r}\right) - \frac{A_\varphi}{r^2} = -\frac{4\pi}{c}\rho\omega r \\ \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial A_z}{\partial r}\right) = 0 \end{cases}$$

И тоже самое для $\Delta\vec{A} = 0$, только все $= 0$!

$\Rightarrow$ Для $\Delta\vec{A} = -\frac{4\pi}{c}\rho\omega r\vec{e}_\varphi$:

$A_r = C_1 r + \frac{C_2}{r}$, $A_\varphi = -\frac{\pi}{2c}\rho\omega r^3 + C_3 r + \frac{C_4}{r}$;

$A_z = C_5\ln r + C_6$

Аналогично для $\Delta\vec{A} = 0$:

$A_r = C_7 r + \frac{C_8}{r}$; $A_\varphi = C_9 r + \frac{C_{10}}{r}$; $A_z = C_{11}\ln r + C_{12}$

Т.к. при $r\to\infty$ поле огр.:

$A_r = \frac{C_8}{r}$; $A_\varphi = \frac{C_{10}}{r}$; $A_z = C_{12}$

$\cancel{A_\varphi(a) = -\frac{\pi}{2c}\rho\omega a^3 + C_3 a + \frac{C_4}{a}} =$

$\frac{dA_{<,z}}{dr} = \frac{dA_{>,z}}{dr} \Rightarrow C_5 = 0$ Аналогично для всех, получаем:

$0\le r<a$:
$$\begin{cases} A_r = \frac{C_1}{r}; \\ A_\varphi = -\frac{\pi}{2c}\rho\omega r^3 + \frac{\pi}{c}\rho\omega a^2 r + \frac{C_2}{r}; \\ A_z = C_3; \end{cases}$$

$a\le r$:
$$\begin{cases} A_r = \frac{C_1}{r} \\ A_\varphi = \frac{C_2}{r} + \frac{\pi}{2cr}\rho\omega a^4 \\ A_z = C_3 \end{cases}$$

$$\vec{H} = \nabla\times\vec{A} = \left(\frac{1}{\rho}\frac{\partial A_z}{\partial\varphi} - \frac{\partial A_\varphi}{\partial z}\right)\vec{e}_r + \left(\frac{\partial A_\rho}{\partial z} - \frac{\partial A_z}{\partial r}\right)\vec{e}_\varphi + \frac{1}{\rho}\left(\frac{\partial(\rho A_\varphi)}{\partial r} - \frac{\partial A_r}{\partial\varphi}\right)\vec{e}_z$$

$= 0 + 0 + \frac{1}{r}\frac{\partial}{\partial r}(rA_\varphi) = \frac{2\pi}{c}\rho\omega(a^2 - r^2)\vec{e}_z$ для $0\le r<a$

и $\vec{H} = \nabla\times\vec{A} = 0$ для $r\ge a$

№2.

$$\Delta\varphi = \begin{cases} -4\pi e\,\delta(\vec{r}), & 0\le r\le R_2 \\ 0, & R_2\le r< R_1 \\ 0, & R_1\le r \end{cases}$$

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\varphi}{\partial r}\right)+\frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\varphi}{\partial\theta}\right)=0$$

$$\varphi(r,\theta)=R(r)\cdot\Theta(\theta)$$

$$\Rightarrow \frac{1}{r^2}\left[2rR'+r^2R''\right]\Theta+\frac{1}{r^2}\left[\operatorname{ctg}\theta\cdot\Theta'+\Theta''\right]R=0$$

$$\Rightarrow 2r\frac{R'}{R}+r^2\frac{R''}{R}=-\operatorname{ctg}\theta\frac{\Theta'}{\Theta}+\frac{\Theta''}{\Theta}=\alpha$$

$$\begin{cases} r^2R''+2rR'-\alpha R=0 \\ \Theta''+\operatorname{ctg}\theta\,\Theta'+\alpha\Theta=0 \end{cases}$$

$$\Rightarrow \varphi=\sum_{n=0}^{+\infty}\left[A_nr^n+\frac{B_n}{r^{n+1}}\right]P_n(\cos\theta)$$

1. $\Delta\varphi=-4\pi e\,\delta(\vec{r})\Rightarrow \varphi=C_1+\frac{e}{r}$

2. $\begin{cases}\Delta\varphi=0\\ E=0\Rightarrow\varphi=0\end{cases} \Rightarrow \varphi=C_2$

3. $\begin{cases}\Delta\varphi=0\\ \lim\limits_{r\to\infty}\vec{E}=E\cos\theta\,\vec{e}_r-E\sin\theta\,\vec{e}_\theta\end{cases} \Rightarrow \varphi=\left(C_3+\frac{C_4}{r}\right)+\left(-Er+\frac{C_5}{r^2}\right)\cos\theta$

$$\begin{cases} C_1+\frac{e}{r_2}=C_2 \\ C_2=\left(C_3+\frac{C_4}{r_1}\right)+\left(-Er_1+\frac{C_5}{r_1^2}\right)\cos\theta \end{cases}$$

Т.к. при $r\to\infty$: $\varphi=-Er\cos\theta \Rightarrow C_3=0$,

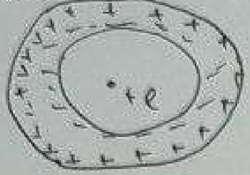
Т.к. тан. сост. E на R_1 равна нулю, то:

$$\frac{d\varphi}{d\theta}=0\Rightarrow -ER_1+\frac{C_5}{R_1^2}=0\Rightarrow$$

$$\Rightarrow C_5=ER_1^3$$

Т.к. $\cancel{\oint (\vec{E}\, d\vec{S}) = 4\pi e \to C_1}$

В случае $E = 0$:



Из теоремы Гаусса:

$C_4 = e$.

Тогда $C_2 = \dfrac{e}{R_1}$ и $C_1 = e\left(\dfrac{1}{R_1} - \dfrac{1}{R_2}\right)$

$$\Rightarrow \varphi = \begin{cases} \dfrac{e}{r} + e\left(\dfrac{1}{R_1} - \dfrac{1}{R_2}\right), & 0 \le r < R_2; \\ \dfrac{e}{R_1}, & R_2 \le r < R_1; \\ \dfrac{e}{r} - E\left(r - \dfrac{R_1^3}{r^2}\right)\cos\theta, & R_1 \le r; \end{cases}$$

$\vec{E} = -\operatorname{grad}\varphi \Rightarrow$

$$\vec{E} = \begin{cases} \dfrac{e}{r^2}\vec{e}_r, & 0 \le r < R_2; \\ 0, & R_2 \le r < R_1; \\ \left(\dfrac{e}{r^2} + E\left(1 + \dfrac{2R_1^3}{r^3}\right)\cos\theta\right)\vec{e}_r - E\left(1 - \dfrac{R_1^3}{r^3}\right)\sin\theta\, \vec{e}_\theta, & R_1 \le r \end{cases}$$

№3 Рассмотрим сл. интеграл:

$$\frac{1}{2\pi i}\int_{C_1}\frac{(t^2-1)^l}{t-x}dt = (x^2-1)^l$$

Возьмем пр. $\frac{d^l}{dx^l}\int_{C_1}\frac{(t^2-1)^l}{t-x}dt = l!\int_{C_1}\frac{(t^2-1)^l}{(t-x)^{l+1}}dt \Rightarrow$

$$\Rightarrow \frac{1}{2\pi i}\frac{d^n}{dx^n}(x^2-1)^l = \frac{l!}{2\pi i}\int_{C_1}\frac{(t^2-1)^l}{(t-x)^{l+1}}dt$$

$$\Rightarrow P_l(x) = \frac{1}{2^l l!}\frac{d^l}{dx^l}(x^2-1)^l =$$

$$= \frac{1}{2^{l+1}\pi i}\oint_{C_1}\frac{(t^2-1)^l}{(t-x)^{l+1}}dt$$, где C_1 – любой контур, огр. точку $t=x$!

$$(1-x^2)P_l''(x) - 2xP_l'(x) + l(l+1)P_l(x) =$$

$$= \frac{l+1}{2^{l+1}\pi i}\oint_{C_1}\frac{(t^2-1)^l}{(t-x)^{l+3}}\left[lt^2-2(l+1)tx+(l+2)\right]dt = 0$$

$$F(t,x) = lt^2 - 2xt(l+1) + (l+2)$$

Заметим, что

$$\frac{(t^2-1)^l}{(t-x)^{l+3}}\left[lt^2-2xt(l+1)+(l+2)\right] = \frac{d}{dt}\frac{(t^2-1)^{l+1}}{(t-x)^{l+2}}$$

$$\Rightarrow \oint_{C_1} = 0 \Rightarrow (1-x^2)P_l''(x) - 2xP_l'(x) + l(l+1)P_l(x) = 0$$

ч.т.д.