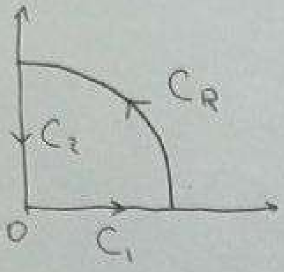
1 Задача.

1.1. $C = \int_0^{+\infty} \frac{\ln t}{t}\sin t\,dt = \frac{d}{da}\left[\int_0^{+\infty} t^{a-1}\sin t\,dt\right]\Big|_{a=0} = \frac{d}{da} f(a)\Big|_{a=0}$

$f'(0) = C$, $f(a) = \operatorname{Im} F(a)$, где $F(a) = \int_0^\infty t^{a-1} e^{it}\,dt$



$\oint = \int_{C_1} + \int_{C_2} + \cancel{\int_{C_R}} \to 0$, $\int_{C_1} = F(a)$

$\Rightarrow C_2 = \int_\infty^0 i(i\tau)^{a-1} e^{-\tau}\,d\tau = -e^{\frac{i\pi a}{2}} \int_0^\infty \tau^{a-1} e^{-\tau}\,d\tau$

$\Rightarrow F(a) - e^{\frac{i\pi a}{2}}\int_0^\infty \tau^{a-1}e^{-\tau}\,d\tau = \oint_C = 0 \Rightarrow$

$\Rightarrow F(a) = e^{\frac{i\pi a}{2}}\Gamma(a)$

$f(a) = \operatorname{Im}F(a) = \sin\frac{\pi a}{2}\cdot\Gamma(a)$

$C = f'(0) = \left(\frac{\pi}{2}\cos\frac{\pi a}{2}\cdot\Gamma(a) + \sin\frac{\pi a}{2}\,\Gamma'(a)\right)\Big|_{a=0} =$

$= \left(\frac{\pi}{2}\cdot\frac{\pi}{2\sin\left(\frac{\pi a}{2}\right)}\cdot\frac{1}{\Gamma(1-a)} + \frac{\pi}{2}\psi(1-a)\cdot\sec\left(\frac{\pi a}{2}\right)\right)\Big|_{a=0} = -\frac{\pi\gamma}{2}$

1.2 $I(\nu) = \int_0^{\pi/2}\sin^\nu x\,dx$, при $x\to+0$ $\sin x \simeq x$

$\Rightarrow I(\nu)$ сходится при $\nu+1>0 \Rightarrow \operatorname{Re}\nu > -1$

$I(\nu) = \int_0^{\pi/2}\sin^\nu x\,dx = \left[\begin{matrix} t=\sin^2 x,\ [0;\frac{\pi}{2}]\Rightarrow[0,1] \\ dt = 2\sin x\cos x\,dx = \sin 2x\,dx\end{matrix}\right] =$

$= \int_0^1 t^{\frac{\nu}{2}}\cdot\frac12 t^{-\frac12}(1-t)^{-\frac12}\,dt = \frac12\int_0^1 t^{\frac{\nu}{2}-\frac12}(1-t)^{-\frac12}\,dt =$

$= \frac12 B\left(\frac{\nu+1}{2};\frac12\right) = \frac{\sqrt{\pi}}{2}\,\frac{\Gamma\left(\frac{\nu+1}{2}\right)}{\Gamma\left(\frac{\nu}{2}+1\right)}$

1.3. $$h(a)=\int_0^1 \frac{t^{a-1}}{1+t}\,dt = \sum_{n=0}^{\infty}(-1)^n \int_0^1 t^{a+n-1}\,dt =$$

$$= \sum_{n=0}^{\infty}(-1)^n \left.\frac{t^{a+n}}{a+n}\right|_0^1 = \sum_{n=0}^{\infty}(-1)^n \frac{1}{a+n} = \sum_{n=0}^{\infty}\left(\frac{1}{a+2n} - \frac{1}{a+2n+1}\right) =$$

$$= \frac{1}{2}\sum_{n=0}^{\infty}\left(\frac{1}{n+\frac{a}{2}} - \frac{1}{n+\frac{a+1}{2}}\right) = \frac{1}{2}\Big[-\gamma + \sum_{k=0}^{\infty}\frac{1}{1+k} - \psi\left(\frac{a}{2}\right) + \gamma -$$

$$- \sum_{k=0}^{\infty}\frac{1}{1+k} + \psi\left(\frac{1+a}{2}\right)\Big] \Rightarrow h(a) = \frac{1}{2}\left[\psi\left(\frac{1+a}{2}\right) - \psi\left(\frac{a}{2}\right)\right]$$

1.4. $$H(a,b) = \int_0^1 \frac{t^{a-1} - t^{b-1}}{(1+t)\ln t}\,dt$$

Исследуем на сходимость.

$$\int_0^1 \frac{t^{a-1}}{1+t}\frac{dt}{\ln t} = [t = \tau + 1] = \int_{-1}^0 \frac{(1+\tau)^{a-1}}{2+\tau}\frac{d\tau}{\ln(1+\tau)} \approx (\text{при } \tau \to 0)$$

$$\approx \int_{-1}^0 \frac{1+\tau a - \tau}{2+\tau}\frac{d\tau}{\tau} \sim \int \frac{d\tau}{\tau} \Rightarrow \text{Расходится при } t \to 1.$$

$$\int_0^1 \frac{t^{a-1} - t^{b-1}}{1+t}\frac{dt}{\ln t} \sim \int_{-1}^0 \frac{a-b}{2+\tau}\,d\tau \sim \int_{-1}^0 (a-b)\,d\tau \text{ — сходится.}$$

$$\frac{d}{da}H(a,b) = \int_0^1 \frac{t^{a-1}}{1+t}\,dt = h(a), \quad \frac{d}{db}H(a,b) = -h(b)$$

$$H(1,1) = \int_0^1 0\,dt = 0$$

$$H(a,b) = \int h(a)\,da = -\int h(b)\,db$$

$$\int h(a)\,da = \int \frac{1}{2}\left(\psi\left(\frac{1+a}{2}\right) - \psi\left(\frac{a}{2}\right)\right)da = \ln\Gamma\left(\frac{1+a}{2}\right) - \ln\Gamma\left(\frac{a}{2}\right) + C(b)$$

$$\Rightarrow H(a,b) = \ln\frac{\Gamma\left(\frac{1+a}{2}\right)\Gamma\left(\frac{b}{2}\right)}{\Gamma\left(\frac{a}{2}\right)\Gamma\left(\frac{1+b}{2}\right)} + C$$

$$H(1,1) = \ln 1 + C \Rightarrow C = 0$$

$$\Rightarrow H(a,b) = \ln\frac{\Gamma\left(\frac{1+a}{2}\right)\Gamma\left(\frac{b}{2}\right)}{\Gamma\left(\frac{a}{2}\right)\Gamma\left(\frac{1+b}{2}\right)}$$

2. Задача

$$I(\alpha) = \frac{1}{\Gamma(\frac{\alpha+1}{2})} \int_0^\infty x^\alpha (1+x)^{2\alpha} (2+x)^{3\alpha} e^{-x} dx$$

$$x \to 0: \quad x^\alpha (1+x)^{2\alpha} (2+x)^{3\alpha} \sim x^\alpha \cancel{\ } 2^{3\alpha}$$

$$\Rightarrow I(\alpha) = \frac{1}{\Gamma(\frac{\alpha+1}{2})} \int_0^\infty x^\alpha \left[(1+x)^{2\alpha}(2+x)^{3\alpha} - \cancel{\ } 2^{3\alpha}\right] e^{-x} dx + \frac{2^{3\alpha} \alpha \Gamma(\alpha)}{\Gamma(\frac{\alpha+1}{2})}$$

$$2^{3\alpha} \alpha \frac{\Gamma(\alpha)}{\Gamma(\frac{\alpha+1}{2})} = 2^{3\alpha} \cancel{\ } 2^\alpha \frac{1}{\sqrt{\pi}} \Gamma\left(\frac{\alpha}{2}+1\right) = \alpha \frac{2^{4\alpha-1}}{\sqrt{\pi}} \Gamma\left(\frac{\alpha}{2}\right)$$

$$\Rightarrow I_{con}(-1) = -\frac{2^{-5}}{\sqrt{\pi}} \Gamma\left(-\frac{1}{2}\right) = \frac{1}{16}$$

3. Задача

$$G(in) = \sum_{n=1}^\infty \left(\frac{1}{-a+ik+in} - \frac{1}{-a-ik+in} + \frac{2\cancel{\ } i}{k} \right) =$$

$$= i\left[\sum_{k=1}^\infty \left(-\frac{1}{n-ia+k} - \frac{1}{-n+ia+k} + \frac{2}{k}\right)\right] =$$

$$= i\left[\sum_{k=0}^\infty \left(-\frac{1}{n-ia+k+1} - \frac{1}{-n+ia+k+1} + \frac{2}{1+k}\right)\right] =$$

$$= i\Big[\psi(1+n-ia) + \gamma - \sum_{k=0}^\infty \frac{1}{1+k} -$$

$$-\psi(1-n+ia) + \gamma - \sum_{k=0}^\infty \frac{1}{1+k} + \sum_{k=0}^\infty \frac{2}{k+1}\Big] =$$

$$= i\left[2\gamma + \psi(1+n+ia) + \psi(1-n-ia)\right]$$

$$in \to z \Rightarrow G(z) = i\left(\psi(1+i(z-a)) + \psi(1-i(z-a)) + 2\gamma\right)$$

$\psi(1+i(z-a))$ имеет плохие полюса

$$\psi(1-(n+ia)) = \psi(n+ia) + \pi\,\mathrm{ctg}\,\pi(n+ia) = \psi(n+ia) - i\cancel{\ }\pi\coth\pi a$$

$$in \to z: \quad G(z) = i\left[2\gamma + \psi(1+i(a-z)) + \cancel{\ }\psi(ia-iz) - \pi i\coth\pi a\right]$$

4 Задача

$$L(z) = \frac{1}{\Gamma(z)} \int_0^\infty \frac{x^{z-1} e^{-x}}{1+e^{-2x}}\, dx$$

$$L(1) = \frac{1}{\Gamma(1)} \int_0^\infty \frac{e^{-x}}{1+e^{-2x}} = \left[e^x = t\right] = \int_1^\infty \frac{1}{1+t^2}\, dt = \frac{\pi}{2} - \frac{\pi}{4} = \frac{\pi}{4}$$

$$L(0) = \lim_{z\to 0} L(z) = \lim_{z\to 0} \frac{\frac{1}{z}\frac{x^{+z} e^{-x}}{1+e^{-2x}}\Big|_0^\infty - \int_0^\infty \frac{x^z}{z}\left(\frac{e^{-x}}{1+e^{-2x}}\right)' dx}{\Gamma(z)} =$$

$$= \lim_{z\to 0} \frac{1}{\Gamma(z+1)} \int_0^\infty x^z \left(\frac{e^{-x}}{1+e^{-2x}}\right)' dx = -\frac{e^{-x}}{1+e^{-x2}}\Big|_0^\infty = +\frac{1}{2}$$

$$L(z) = \frac{1}{\Gamma(z)} \int_0^\infty x^{z-1} e^{-x} \sum_{n=0}^\infty (-1)^n e^{-2xn}\, dx =$$

$$= \sum_{n=0}^\infty \frac{(-1)^n}{\Gamma(z)} \int_0^\infty x^{z-1} e^{-x(1+2n)}\, dx = \sum_{n=0}^\infty \frac{(-1)^n}{\Gamma(z)} \int_0^\infty [x(1+2n)]^{z-1} \frac{e^{-x(1+2n)}}{(1+2n)^z}\, d(x(1+2n)) =$$

$$= \sum_{n=0}^\infty \frac{(-1)^n}{\Gamma(z)} \Gamma(z)(1+2n)^{-z} = \sum_{n=0}^\infty (-1)^n (1+2n)^{-z}$$

$$S = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \ldots = \sum_{n=0}^\infty (-1)^n (1+2n)^{-1} = L(+1) = \frac{\pi}{4}$$

$$L'(z) = \sum_{n=0}^\infty (-1)^{n+1} (1+2n)^{-z} \ln(1+2n) = \ln \prod_{n=0}^{+\infty} e^{(-1)^{n+1}(1+2n)^{-z}\ln(1+2n)}$$

$$\Rightarrow c_n = \frac{\ln(4n+3)}{(4n+3)^z} - \frac{\ln(4n+5)}{(4n+5)^z}$$

$$F_1 = \ln \prod_{n=0}^{N-1} \frac{4n+3}{4n+5} = \ln \left[N^{-\frac{1}{2}} \frac{(N-1)!\, N^{5/4}}{\prod_{n=0}^{N-1}(n+\frac{5}{4})} \frac{\prod_{n=0}^{N-1}(n+\frac{3}{4})}{(N-1)!\, N^{3/4}}\right] -$$

$$\sim \ln\left[\frac{\Gamma(\frac{5}{4})}{\Gamma(\frac{3}{4})} N^{-\frac{1}{2}}\right],\ N \gg 1!$$

$$F_2 = \int_N^\infty \left[\frac{\ln(4n+3)}{(4n+3)^z} - \frac{\ln(4n+5)}{(4n+5)^z}\right] dn = \int_{4N+3}^{4N+5} \frac{\ln(x)}{x^z}\, dx = \frac{1}{4}\left[\frac{\ln t}{1-z} t^{1-z} - \frac{t^{1-z}}{(1-z)^2}\right]\Bigg|_{4N+3}^{4N+5} =$$

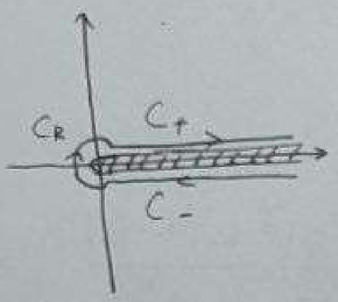
$$= \frac{1}{4}\Big[\frac{1}{1-z}(4N+5)^{1-z}\ln(4N+5) - \frac{1}{1-z}(N4+3)^{1-z}\ln(4N+3) -$$

$$- \frac{1}{(1-z)^2}\left[(4N+5)^{1-z} - (4N+3)^{1-z}\right]\Big] = \ln 2 - \frac{1}{2}\ln\frac{1}{N},\ N \gg 1$$

$$L'(0) = \ln \frac{\Gamma(\frac{5}{4})}{\Gamma(\frac{3}{4})} + \ln 2 = \ln \frac{2\Gamma(\frac{5}{4})}{\Gamma(\frac{3}{4})}$$

5 Задача

$$\oint_\gamma = \int_{C_+} + \int_{C_-} + \cancel{\int_{C_R}}^{\to 0}, \quad \int_{C_+} = \frac{1}{\Gamma(z)} \int_0^\infty \frac{x^{z-1} e^{-x}}{1+e^{-2x}} dx = I_+(z)$$



$$\Rightarrow \oint_\gamma = (1 - e^{2\pi i z}) \cdot L(z)$$

$$\Rightarrow L(z) = \frac{1}{1-e^{2\pi i z}} \frac{1}{\Gamma(z)} \oint_\gamma \frac{e^{-t} t^{z-1}}{1+e^{-2t}} dt$$

$$L(z) = \frac{2\pi i}{1-e^{2\pi i z}} \frac{1}{\Gamma(z)} \sum_{n=-\infty}^{+\infty} \operatorname{Res} \frac{e^{-t} t^{z-1}}{1+e^{-2t}} dt, \quad 1+e^{-2t} = 0 \Rightarrow t = \pm i\frac{\pi}{2}(2n+1)$$

$$\operatorname*{Res}_{t = i\frac{\pi}{2}(1+2n)} \frac{e^{-t} t^{z-1}}{1+e^{-2t}} = \frac{e^{-i\frac{\pi}{2}}}{2e^{i\frac{\pi}{2}}} \left(\frac{\pi}{2}\right)^{z-1} (1+2n)^{z-1} (-1)^n e^{i\frac{\pi z}{2}}$$

$$\operatorname*{Res}_{t = -i\frac{\pi}{2}(1+2n)} \frac{e^{-t} t^{z-1}}{1+e^{-2t}} = \frac{e^{-i\frac{\pi}{2}}}{2e^{i\frac{\pi}{2}}} \left(\frac{\pi}{2}\right)^{z-1} (1+2n)^{z-1} (-1)^n e^{i\frac{3\pi}{2}z}$$

$$L(z) = \frac{1}{1-e^{2\pi i z}} \frac{1}{\Gamma(z)} \cdot 2\pi i \cdot \frac{e^{-i\frac{\pi}{2}}}{2i} \left(e^{i\frac{\pi z}{2}} + e^{i\frac{3\pi}{2}z}\right) \sum (-1)^n (2n+1)^{z-1} =$$

$$= \frac{1}{\sin\frac{\pi z}{2}} \frac{1}{\Gamma(z)} \frac{L(1-z)}{\Gamma(z)} \left(\frac{\pi}{2}\right)^z$$

$$\Rightarrow L(1-z) = \left(\frac{\pi}{2}\right)^{-z} \sin\frac{\pi z}{2} \, \Gamma(z) \cdot L(z)$$

$$L(-2k-1) = \left(\frac{\pi}{2}\right)^{-2(k+1)} \overbrace{\sin(\cancel{\pi}(k+1))}^{\to 0} \cdot \Gamma(\ldots) \cdot L(\ldots) = 0$$

6 Задача

$$C = \int_0^\infty \frac{\ln x\, dx}{\cosh x}$$

$$2\frac{d}{dz}(L(z)\Gamma(z)) = 2\frac{d}{dz}\int_0^\infty \frac{x^{z-1}e^{-x}}{1+e^{-2x}}\,dx = 2\int_0^\infty \frac{x^{z-1}\ln x}{\cosh x}\,dx$$

$$\Rightarrow C = 2\frac{d}{dz}(L(z)\Gamma(z))\Big|_{z=1}$$

$$C = 2\frac{d}{dz}\left[\left(\frac{\pi}{2}\right)^z \frac{1}{\sin\frac{\pi z}{2}} L(1-z)\right]\Big|_{z=1} =$$

$$2\left[\left(\frac{\pi}{2}\right)^z \ln\frac{\pi}{2}\frac{L(1-z)^2}{\sin\frac{\pi z}{2}} - \left(\frac{\pi}{2}\right)^{z+1}\frac{\cos\frac{\pi z}{2}}{\sin^2\frac{\pi z}{2}} L(1-z) - \left(\frac{\pi}{2}\right)^z \frac{L'(1-z)}{\sin\frac{\pi z}{2}}\right]\Big|_{z=1} =$$

$$= \pi\ln\frac{\pi}{2} \cdot L(0) - \pi L'(0) =$$

$$= \pi\cdot\ln\frac{\pi}{2}\cdot\frac{1}{2} - \pi\ln\frac{2\Gamma(\frac{5}{4})}{\Gamma(\frac{3}{4})} = \pi\left(\ln\sqrt{\frac{\pi}{2}} - \ln\frac{2\Gamma(\frac{5}{4})}{\Gamma(\frac{3}{4})}\right) =$$

$$= \pi\left(\ln\sqrt{\frac{\pi}{2}}\cdot\frac{\Gamma^2(\frac{3}{4})}{2\Gamma(\frac{5}{4})\Gamma(\frac{3}{4})}\right) = \pi\ln\frac{\sqrt{\pi}\cdot\Gamma^2(\frac{3}{4})}{2^{3/2}\cdot 2^{-\frac{1}{2}}\sqrt{\pi}\cdot\Gamma(\frac{3}{2})} =$$

$$= \pi\ln\frac{\Gamma^2(\frac{3}{4})}{2\frac{1}{2}\sqrt{\pi}} = \pi\ln\frac{\Gamma^2(\frac{3}{4})}{\sqrt{\pi}}$$