19.07

№2

$\cancel{e^{z^2} = 1 - \frac{1}{z^2} + \frac{1}{2!\,z^4} - \ldots}$

$e^{-z^2} \Rightarrow e^{-\frac{1}{t^2}} \quad \left(z = \frac{1}{t}\right)$

$e^{-\frac{1}{t^2}} = 1 - \frac{1}{\cancel{z}\,t^2} + \frac{1}{2!\,t^4} - \ldots$

При $t = 0$; $z \to \infty$ ~~это~~ $\Rightarrow$ сущ. особая точка

№3 $\sin\frac{\pi}{z^2}$

$\lim\limits_{z \to 0}\left(\sin\frac{\pi}{z^2}\right)$ не сущ. $\to$ сущ. особая точка

19.15

№1

$\frac{(1+z^2)^2}{1-z^2}$

$((1+z^2)^2 = 0 \Rightarrow z = \pm i)$

$z = \pm i$ — полюс второго порядка

$z = \pm 1$ — полюса ~~второго~~ первого порядка.

$z = \pm\infty$ — полюса порядка

№2.

$\mathrm{ctg}\, z = 0 = \frac{\cos z}{\sin z} \Rightarrow z = \frac{\pi}{2} + \pi k,\ k \in \mathbb{Z}$ —
~~нули~~ нули первого порядка

$\frac{1}{\mathrm{ctg}\, z} = \frac{\sin z}{\cos z} = 0 \Rightarrow z = \pi n,\ n \in \mathbb{Z}$ —
полюса первого порядка

№3. $z\,\mathrm{tg}^2 z = f(z)$

$f'(z) = \frac{\cos z \sin^2 z + 2z \sin z}{\cos^3 z}$

$f''(z) = \frac{z \sin(2z) + 2z\cos^2 z + 6z \sin z}{\cos^4 z}$

$f'''(z) = \ldots$

$z = 0$ — нуль тр. порядка

$z = \pi n,\ n \in \mathbb{Z}/0$ — нули вт. порядка

$z = \frac{\pi}{2} + \pi n$ — полюса вт. порядка

20.01

№1. $\sum\limits_{n=-\infty}^{+\infty} 2^{-|n|} z^n$

$r < |z - z_0| < R$

где $r = \lim\limits_{n\to\infty} \sqrt[n]{|c_n|}$ или $r = \lim\limits_{n\to\infty} \left|\frac{c_{-(n+1)}}{c_{-n}}\right|$;

$R = \frac{1}{\lim\limits_{n\to+\infty} \sqrt[n]{|c_n|}}$ или $R = \lim\limits_{n\to+\infty} \left|\frac{c_n}{c_{n+1}}\right|$

$r = \lim\limits_{n\to\infty} 2^{-\frac{|n|}{n}} = \frac{1}{2}$

$R = \lim \frac{1}{\sqrt[n]{2^{-|n|}}} = 2$ $\Rightarrow \frac{1}{2} < |z| < 2$

№2. $\sum\limits_{n=-\infty}^{+\infty} \frac{z^n}{3^n+1}$

$r = \lim\limits_{n\to+\infty} \left|\frac{3^{-n}+1}{3^{-n-1}+1}\right| = 1$

$R = \lim\limits_{n\to+\infty} \left|\frac{1+3^n}{1+3^{n+1}}\right| = \lim\limits_{n\to+\infty} \left|\frac{3+\frac{1}{3^n}}{1+\frac{1}{3^n}}\right| = 3$

$\Rightarrow 1 < |z| < 3$

№4. $\sum\limits_{n=-\infty}^{+\infty} 2^{-n^2}(z+1)^n$

$r = \lim\limits_{n\to\infty} \left(2^{-n^2\cdot\frac{1}{n}}\right) = 0$

$R = \lim\limits_{n\to\infty} \left(2^{+n^2\frac{1}{n}}\right) = +\infty$

$\Rightarrow 0 < |z+1| < \infty$

20.09

№1. $\frac{1}{z(z-3)^2} = \frac{1}{9z} - \frac{1}{9(z-3)} + \frac{1}{3(z-3)^2}$

Д.к. $1 < |z-1| < 2$:

$$\frac{1}{9(1+(z-1))} = \frac{1}{9(z-1)\left(1+\frac{1}{z-1}\right)} = \frac{1}{9(z-1)}\sum_{n=0}^{+\infty}(-1)^n\left(\frac{1}{z-1}\right)^n =$$

$$= \cancel{\frac{1}{9z}} \sum_{-\infty}^{-1} \frac{(-1)^{n-1}}{9}(z-1)^n$$

$$-\frac{1}{9(z-1-2)} = \frac{1}{18\left(1-\frac{z-1}{2}\right)} = \frac{1}{18}\sum_{n=0}^{+\infty}\left(\frac{z-1}{2}\right)^n$$

$$\frac{1}{3(z-1-2)^2} = \frac{1}{3\cdot 2^2\left(1-\frac{z-1}{2}\right)^2} = \sum_{n=1}^{+\infty}\frac{1}{12}\, n\left(\frac{z-1}{2}\right)^{n-1} =$$

$$= \sum_{n=0}^{+\infty}\frac{1}{12}(n+1)\left(\frac{z-1}{2}\right)^n$$

$$\Rightarrow \frac{1}{z(z-3)^2}\cancel{z} = \sum_{-\infty}^{-1}\frac{(-1)^{n-1}}{9}(z-1)^n +$$

$$+ \sum_{n=0}^{+\infty}\frac{3n+5}{9\cdot 2^{n+2}}(z-1)^n$$

№5

$$\frac{1}{z(z-1)(z-2)} = \frac{1}{2z} + \frac{1}{2(z-2)} - \frac{1}{z-1}$$

$-\frac{3}{2} \in D$!

$$\frac{1}{z-1} = \frac{1}{z\left(1-\frac{1}{z}\right)} \quad \left(\left|\frac{1}{z}\right| < 1 \Rightarrow |z| > 1 \Leftarrow -\frac{3}{2} \in D\right) =$$

$$= \frac{1}{z}\sum_{n=0}^{\infty}\left(\frac{1}{z}\right)^n = \sum_{n=0}^{\infty} z^{-n-1}$$

$$\cancel{\frac{1}{2(z-2)} = \frac{1}{2z\left(1-\frac{2}{z}\right)}} \qquad \frac{1}{2(z-2)} = -\frac{1}{4\left(1-\frac{z}{2}\right)} =$$

$$= -\frac{1}{4}\sum_{n=0}^{+\infty}\left(\frac{z}{2}\right)^n = -\sum_{n=0}^{+\infty} 2^{-n-2} \cdot z^n$$

$$\frac{1}{z(z-1)(z-2)} = -\sum_{n=0}^{\infty} z^{-n-1} + \Sigma\ \frac{1}{2} z^{-1} - \sum_{n=0}^{+\infty} 2^{-n-2} z^n =$$

$$= -\sum_{n=1}^{\infty} z^{-n-1} - \frac{1}{z} + \frac{1}{2z} - \sum_{n=0}^{+\infty} 2^{-n-2} z^n =$$

$$= -\frac{1}{2}z - \sum_{-\infty}^{-2} z^n - \sum_{n=0}^{+\infty} 2^{-n-2} z^n$$

N7. $\frac{z^3}{(z+1)(z-2)} = z^3\left(\frac{1}{3(z-2)} + \frac{1}{3(z+1)}\right) =$

$$= \left| \begin{matrix} t = z+1 \\ 0<|t|<3 \end{matrix} \right| = \frac{(t-1)^3}{t(t-3)} = \frac{t^3-3t^2+3t-1}{t(t-3)} =$$

$$= \frac{t^2}{(t-3)} - \frac{3t}{t-3\cancel{t}} + \frac{3}{t-3} - \frac{1}{t(t-3)}$$

$$\frac{t^2}{t-3} = -\frac{t^2}{3(1-\frac{t}{3})} = -\frac{1}{3}\sum_{n=0}^{\infty}\frac{t^{n+2}}{3^n} = -\sum_{n=0}^{+\infty}\frac{t^{n+2}}{3^{n+1}}$$

$$-\frac{3t}{t-3} = \frac{t}{1-\frac{t}{3}} = \sum_{n=0}^{+\infty}\frac{t^{n+1}}{3^n}$$

$$\frac{3}{t-3} = -\frac{1}{(1-\frac{t}{3})} = -\sum_{n=0}^{+\infty}\frac{t^n}{3^n}$$

$$-\frac{1}{t(t-3)} = +\frac{1}{3t(1-\frac{t}{3})} = +\sum_{n=0}^{\infty}\frac{t^{n-1}}{3^{n+1}}$$

$$f(t) = \frac{1}{3}t^{-1} + \frac{1}{9} + \frac{t}{27} + \sum_{n=3}^{\infty}\frac{t^{n-1}}{3^{n+1}} +$$

$$-\cancel{\tfrac{8}{9}}\,1 - \frac{t}{3} - \sum_{n=2}^{+\infty}\frac{t^n}{3^n} + t + \sum_{n=1}^{+\infty}\frac{t^{n+1}}{3^n} + -$$

$$-\sum_{n=0}^{+\infty}\frac{t^{n+2}}{3^{n+1}} =$$

$$= \frac{1}{3}t^{-1} - \frac{8}{9} + \frac{19}{27}t + \sum_{n=2}^{\infty}\left[\frac{t^n}{3^{n+2}} - \frac{t^n}{3^n} + \frac{t^n}{3^{n-1}} - \frac{t^n}{3^{n-1}}\right] = \frac{1}{3}(z+1)^{-1} - \frac{8}{9} + \frac{19}{27}(z+1) -$$

$$-\sum_{n=2}^{+\infty}\frac{8}{3^{n+2}}(z+1)^n$$

N8

$$\frac{1}{(z^2-1)(z^2+4)} = \frac{1}{5}\left(\frac{1}{z^2-1} - \frac{1}{z^2+4}\right)$$

$$\frac{1}{z^2-1} = \frac{1}{z^2}\cdot\frac{1}{1-\frac{1}{z^2}} = \sum_{n=0}^{+\infty} z^{2n-2}$$

$$\frac{1}{z^2+4} = \frac{1}{z^2\left(1+\frac{4}{z^2}\right)} = \sum_{n=0}^{+\infty} 4^n(-1)^n z^{-2n-2}$$

$$\Rightarrow f(z) = \frac{1}{5}\sum_{n=0}^{+\infty}\left(1+4^n(-1)^n\right)z^{-2n-2} = \quad \Big| -2n-2 \to 2n$$

$$= \sum_{-\infty}^{-1} \frac{1+4^{-n-1}(-1)^{n}}{5} z^{2n}$$

20.16

N 1

$$z^3 e^{\frac{1}{z}} = z^3 + z^2 + \frac{1}{2} z + \frac{1}{6} + \sum_{n=1}^{+\infty} \frac{z^{-n}}{(n+3)!}$$

N 2

$$z^2 \sin\left(\pi \frac{z+1}{z}\right) = z^2 \sin\left(\pi + \frac{\pi}{z}\right) = -z^2 \sin\left(\frac{\pi}{z}\right) =$$

$$= -z^2 \frac{\pi}{z} - z^2 \sum_{n=1}^{\infty} \frac{(-1)^n \left(\frac{\pi}{z}\right)^{2n+1}}{(2n+1)!} =$$

$$= -\pi z + \sum_{n=1}^{\infty} \frac{(-1)^{n+1} \pi^{2n+1}}{(3n+1)!} z^{-2n+1}$$

20.21

№1

$$\frac{z}{(z+2)^2} = \frac{z+2-2}{(z+2)^2} = -\frac{2}{(z+2)^2} + \frac{1}{z+2}$$

№2

$$\frac{e^{z}+1}{e^{z}-1} = \frac{e^{\frac{z+2i\pi k}{2}}\left[e^{i\left(\frac{-iz-2\pi k}{2}\right)} + e^{i\left(\frac{iz+2\pi k}{2}\right)}\right]}{e^{\frac{z+2i\pi k}{2}}\left[e^{i\left(\frac{iz-2\pi k}{2}\right)} - e^{i\left(\frac{iz+2\pi k}{2}\right)}\right]} =$$

$$= \frac{2\cos\left(\frac{iz+2\pi k}{2}\right)}{-2i\sin\left(\frac{iz+2\pi k}{2}\right)} \approx \frac{2}{-2i\left(i\frac{z}{2}+\pi k\right)} =$$

$$= \frac{2}{z-2i\pi k}$$

№3. $$\frac{z-1}{\sin^2 z} \approx \frac{z-1}{z^2} = -\frac{1}{z^2} + \frac{1}{z}$$

№4. $$\frac{e^{iz}}{z^2+b^2} = \frac{e^{iz}}{(z-ib)(z+ib)} =$$

$$\approx \frac{e^{-b}}{2ib\,(z-ib)} = -\frac{ie^{-b}}{2b\,(z-ib)}$$

№5. $$\frac{(z^2+1)^2}{z^2+b^2} = \frac{z^4+2z^2+1}{z^2+b^2} =$$

$$= \frac{z^4}{z^2+e^2} + \frac{2}{1+\frac{b^2}{z^2}} + \frac{1}{z^2+b^2} \approx$$

$$\approx z^2$$

№6. $\dfrac{ze^{iz}}{(z^2+b^2)^2} = \dfrac{ze^{iz}}{(z-ib)^2(z+ib)^2} =$

$$= \frac{(z-ib)e^{iz}+ibe^{iz}}{(z-ib)^2(z+ib)^2} =$$

$$= \frac{ibe^{iz}}{(z-ib)^2(z+ib)^2} + \frac{e^{iz}}{(z-ib)^{\cancel{2}}(z+ib)^2} \simeq$$

$$\simeq -\frac{ie^{-b}}{4b(z-ib)^2} - \frac{e^{-b}}{4b^2}\,\frac{1}{z-ib} =$$

$$= -\frac{(ib+z+ib)e^{-b}}{4b^2(z-ib)^2} = -\frac{ie^{-b}}{4b}\,\frac{1}{(z-ib)^2}$$

№7. $\dfrac{z}{(z^2+b^2)^2} = \dfrac{1}{(z-ib)(z+ib)^2} + \dfrac{ib}{(z-ib)^2(z+ib)^2} \simeq$

$$\simeq -\frac{1}{4b^2(z-ib)} - \frac{i\cancel{b}}{4b(z-ib)^2} =$$

$$= -\frac{i}{4b}\cdot\frac{1}{(z-ib)^2}$$

№ 5.

$$\operatorname{ctg}\pi z = \frac{1}{\operatorname{tg}\pi z}$$

$$\xi = z - z_0 \to 0;\ z_0 = 0, \pm 1, \dots$$

$$\Rightarrow z = z_0 + \xi$$

$$\frac{1}{\operatorname{tg}\pi z} = \frac{1}{\operatorname{tg}(\pi z_0 + \pi\xi)} = \frac{1 - \overset{0}{\cancel{\operatorname{tg}\pi z_0}}\operatorname{tg}\pi\xi}{\underbrace{\operatorname{tg}\pi z_0}_{0} + \operatorname{tg}\pi\xi} =$$

$$\approx \frac{1}{\operatorname{tg}\pi\xi} \approx \frac{1}{\operatorname{tg}\pi(z - z_0)} \approx \frac{1}{\pi(z - z_0)}$$

№ 6

$$\frac{1}{\sin\pi z} = \left| \begin{matrix} z = z_0 + \xi \\ \xi \to 0 \end{matrix} \right| = \frac{1}{\sin\pi(z_0 + \xi)} =$$

$$= \frac{1}{\cos\pi z_0 \sin\pi\xi + \cos\pi\xi \underbrace{\sin\pi z_0}_{0}} \approx$$

$$\approx \frac{1}{\cos(\pi z_0)\,\pi\xi} = \frac{(-1)^{z_0}}{\pi(z - z_0)}$$