Assignment 9

Chirag Mehta - AI20BTECH11006

Download all python codes from

```
https://github.com/cmapsi/AI1103−Probability−and
    −random−variables/tree/main/Assignment−9/
    codes
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and latex-tikz codes from

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https://github.com/cmapsi/AI1103−Probability−and
    −random−variables/blob/main/Assignment−9/
    main.tex
```

1 Problem

(CSIR UGC NET EXAM (Dec 2015), Q.109)
Suppose $\begin{pmatrix} X \\ Y \end{pmatrix}$ is a random vector such that the marginal distribution of X and the marginal distribution of Y are the same and each is normally distributed with mean 0 and variance 1. Then, which of the following conditions imply independence of X and Y?

1) $\text{Cov}(X, Y) = 0$
2) $aX + bY$ is normally distributed with mean 0 and variance $a^2 + b^2$ for all real a and b
3) $\Pr(X \le 0, Y \le 0) = \frac{1}{4}$
4) $E\left[e^{itX+isY}\right] = E\left[e^{itX}\right]E\left[e^{isY}\right]$ for all real s and t

2 solution

An important property of dirac delta function that will be used at multiple ocassions in this solution is

$$\int_{-\infty}^{\infty} f(x)\delta(x-a)dx = f(a) \tag{2.0.1}$$

Given $X \sim N(0,1), Y \sim N(0,1)$

1)

$$Cov(X, Y) = 0 \tag{2.0.2}$$
$$E[XY] - E[X]E[Y] = 0 \tag{2.0.3}$$
$$E[XY] = 0 \tag{2.0.4}$$
$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} xy f_{XY}(x, y)dxdy = 0 \tag{2.0.5}$$

This doesn't imply independence. Counter example given below
Lets consider a case where X and Y are dependent based on the following relation

$$X = KY \tag{2.0.6}$$

PMF for K is given as

$$p_K(k) = \begin{cases} \frac{1}{2} & k = 1 \\ \frac{1}{2} & k = -1 \\ 0 & \text{otherwise} \end{cases} \tag{2.0.7}$$

A simulation is given below, Y is gaussian, then X also follows gaussian

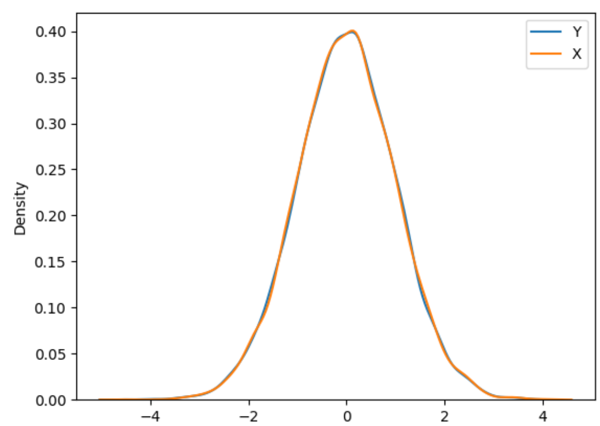

Fig. 1: X and Y, if Y is normal

The joint probability distribution is therefore

$$\begin{aligned} f_{XY}(x, y) &= f_{X|Y}(x|y)f_X(x) \\ &= f_X(x)\frac{1}{2}(\delta(x+y) + \delta(x-y)) \end{aligned} \tag{2.0.8}$$

The marginal probability distribution function for X is given as

$$\int_{-\infty}^{\infty} f_X(x)\frac{1}{2}(\delta(x+y) + \delta(x-y))dy \tag{2.0.9}$$

Using (2.0.1), we get

$$\int_{-\infty}^{\infty} f_X(x)\frac{1}{2}(\delta(x+y)+\delta(x-y))dy = f_X(x) \tag{2.0.10}$$

We know that $X \sim N(0,1)$, $f_X(x)$ represents gaussian probability distribution function.
Futher, using symmetry of (2.0.6), we can establish that marginal distribution of Y is gaussian

$$Cov(X,Y) = E[XY] - E[X]E[Y] = E[XY] \tag{2.0.11}$$

$$E[XY] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} xyf_{XY}(x,y)dydx \tag{2.0.12}$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} xyf_X(x)\frac{1}{2}(\delta(x+y)+\delta(x-y))dydx \tag{2.0.13}$$

$$= \int_{-\infty}^{\infty} xf_X(x)\int_{-\infty}^{\infty} y\frac{1}{2}(\delta(x+y)+\delta(x-y))dydx \tag{2.0.14}$$

Using (2.0.1)

$$E[XY] = \int_{-\infty}^{\infty} xf_X(x)\frac{1}{2}(x-x)dx = 0 \tag{2.0.15}$$

2) Defining the following matrices/vectors Given

vector/matrix	expression
$\mathbf{Z}$	$\begin{pmatrix} X & Y \end{pmatrix}^\top$
$\mathbf{C}$	$\begin{pmatrix} a & b \end{pmatrix}^\top$
$\boldsymbol{\mu}$	$\begin{pmatrix} 0 & 0 \end{pmatrix}^\top$
$\boldsymbol{\Sigma}$	$\begin{pmatrix} 1 & \rho \\ \rho & 1 \end{pmatrix}$

TABLE 2: vectors/matrices and their expressions

$$\mathbf{C}^\top\mathbf{Z} \sim N\left(0, a^2+b^2\right) \tag{2.0.16}$$

Since this is true for all a and b, it is equivalent to X and Y being jointly gaussian

$$\mathbf{Z} \sim N(\boldsymbol{\mu}, \boldsymbol{\Sigma}) \tag{2.0.17}$$

For correlated random variables X and Y in bivariate normal distribution, we have

$$\sigma_Z^2 = \sum_{i,j}\Sigma_{ij} \tag{2.0.18}$$

$$a^2+b^2 = a^2+b^2+2\rho ab \tag{2.0.19}$$

$$\therefore \rho = 0 \tag{2.0.20}$$

The joint distribution is given as

$$f_Z(x,y) = \frac{\exp\left(-\frac{1}{2}\left(\mathbf{z}-\boldsymbol{\mu}\right)^\top\boldsymbol{\Sigma}^{-1}\left(\mathbf{z}-\boldsymbol{\mu}\right)\right)}{\sqrt{(2\pi)^2|\boldsymbol{\Sigma}|}} \tag{2.0.21}$$

$$f_Z(x,y) = \frac{\exp\left(-\frac{1}{2}\begin{pmatrix} x & y\end{pmatrix}I_2\begin{pmatrix} x & y\end{pmatrix}^\top\right)}{\sqrt{(2\pi)^2}} \tag{2.0.22}$$

Where I_2 is the identity matrix of order 2

$$f_Z(x,y) = \frac{\exp\left(-\frac{1}{2}\begin{pmatrix} x & y\end{pmatrix}\begin{pmatrix} x & y\end{pmatrix}^\top\right)}{\sqrt{(2\pi)^2}} \tag{2.0.23}$$

$$f_Z(x,y) = \frac{\exp\left(-\frac{1}{2}\left(x^2+y^2\right)\right)}{\sqrt{(2\pi)^2}} = f_X(x)f_Y(y) \tag{2.0.24}$$

$\therefore$ **Option(2) is correct**, A simulation for bivariate gaussian is given below

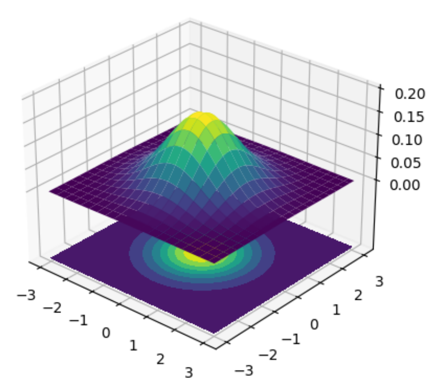

Fig. 2: bivariate gaussian while 0 mean vector and identity covariance matrix

3) The counter example is (2.0.8), the joint probability function is symmetric across all 4 quadrants

$$\therefore \Pr(X \le 0, Y \le 0) = \frac{1}{4} \tag{2.0.25}$$

Alternatively, here is proof

$$\Pr(X \le 0) = F_X(0) = \frac{1}{2} \tag{2.0.26}$$

Using (2.0.6)

$$\Pr(Y \le 0 | X \le 0) = \frac{1}{2} \tag{2.0.27}$$

Using (2.0.26) and (2.0.27)

$$\Pr(X \le 0, Y \le 0) = \frac{1}{4} \tag{2.0.28}$$

4)

$$E\left[e^{itX+isY}\right] = E\left[e^{itX}\right] E\left[e^{isY}\right] \tag{2.0.29}$$

$$E\left[e^{itX+isY}\right] = \varphi_X(t)\varphi_Y(s) \tag{2.0.30}$$

The inverse is given as

$$f_{XY}(x,y) = \frac{1}{4\pi^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-itX-isY} E\left[e^{itX+isY}\right] ds dt \tag{2.0.31}$$

Using (2.0.30)

$$f_{XY}(x,y) = \frac{1}{4\pi^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-itX-isY} \varphi_X(t)\varphi_Y(s) ds dt \tag{2.0.32}$$

$$f_{XY}(x,y) = f_X(x) f_Y(y) \tag{2.0.33}$$

$\therefore$ **Option(4) is correct**