

Assignment 9

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Download all python codes from

<https://github.com/cmapsi/AI1103-Probability-and-random-variables/tree/main/Assignment-9/codes>

and latex-tikz codes from

<https://github.com/cmapsi/AI1103-Probability-and-random-variables/blob/main/Assignment-9/main.tex>

1 PROBLEM

(CSIR UGC NET EXAM (Dec 2015), Q.109)

Suppose $\begin{pmatrix} X \\ Y \end{pmatrix}$ is a random vector such that the marginal distribution of X and the marginal distribution of Y are the same and each is normally distributed with mean 0 and variance 1. Then, which of the following conditions imply independence of X and Y ?

- 1) $\text{Cov}(X, Y) = 0$
- 2) $aX + bY$ is normally distributed with mean 0 and variance $a^2 + b^2$ for all real a and b
- 3) $\Pr(X \leq 0, Y \leq 0) = \frac{1}{4}$
- 4) $E[e^{itX+isY}] = E[e^{itX}]E[e^{isY}]$ for all real s and t

2 SOLUTION

An important property of dirac delta function that will be used at multiple occasions in this solution is

$$\int_{-\infty}^{\infty} f(x)\delta(x-a)dx = f(a) \quad (2.0.1)$$

Given $X \sim N(0, 1)$, $Y \sim N(0, 1)$

1)

$$\text{Cov}(X, Y) = 0 \quad (2.0.2)$$

$$E[XY] - E[X]E[Y] = 0 \quad (2.0.3)$$

$$E[XY] = 0 \quad (2.0.4)$$

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} xy f_{XY}(x, y) dx dy = 0 \quad (2.0.5)$$

This doesn't imply independence. Counter example given below

Lets consider a case where X and Y are dependent based on the following relation, Y being independent of K

$$X = KY \quad (2.0.6)$$

PMF for K is given as

$$p_K(k) = \begin{cases} \frac{1}{2} & k = 1 \\ \frac{1}{2} & k = -1 \\ 0 & \text{otherwise} \end{cases} \quad (2.0.7)$$

A simulation is given below, Y is gaussian, then X also follows gaussian

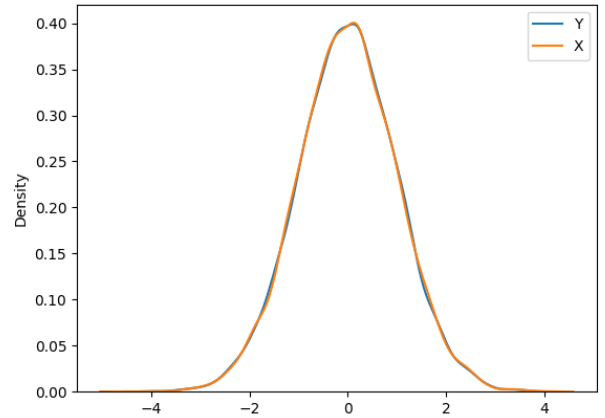


Fig. 1: X and Y , if Y is normal

Theoretically it can be proved in the following manner, Since K and Y are independent

$$f_X(x) = \Pr(K = 1) f_Y(x) + \Pr(K = -1) f_Y(-x) \quad (2.0.8)$$

$$= \frac{1}{2} (f_Y(x) + f_Y(-x)) \quad (2.0.9)$$

$$= f_Y(x) \quad (2.0.10)$$

Therefore, X follows identical but not independent distribution as Y , An alternative proof is

given below as a proof for marginal probability. Now consider that X is normally distributed, we will establish Y is also normally distributed. The joint probability distribution is therefore

$$\begin{aligned} f_{XY}(x, y) &= f_{X|Y}(x|y)f_X(x) \\ &= f_X(x)\frac{1}{2}(\delta(x+y) + \delta(x-y)) \end{aligned} \quad (2.0.11)$$

The marginal probability distribution function for X is given as

$$\int_{-\infty}^{\infty} f_X(x)\frac{1}{2}(\delta(x+y) + \delta(x-y))dy \quad (2.0.12)$$

Using (2.0.1), we get

$$\int_{-\infty}^{\infty} f_X(x)\frac{1}{2}(\delta(x+y) + \delta(x-y))dy = f_X(x) \quad (2.0.13)$$

We know that $X \sim N(0, 1)$, $f_X(x)$ represents gaussian probability distribution function. Further, using symmetry of (2.0.6), we can establish that marginal distribution of Y is gaussian. Here is a proof anyways

$$f_Y(y) = \int_{-\infty}^{\infty} f_X(x)\frac{1}{2}(\delta(x+y) + \delta(x-y))dx \quad (2.0.14)$$

Using (2.0.1), we get

$$f_Y(y) = \frac{1}{2}(f_X(y) + f_X(-y)) = f_X(y) \quad (2.0.15)$$

Since Y has identical probability distribution function, $Y \sim N(0, 1)$

The covariance is given as

$$\text{Cov}(X, Y) = E[XY] - E[X]E[Y] = E[XY] \quad (2.0.16)$$

$$E[XY] = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} xy f_{XY}(x, y) dy dx \quad (2.0.17)$$

$$= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} xy f_X(x) \frac{1}{2}(\delta(x+y) + \delta(x-y)) dy dx \quad (2.0.18)$$

$$= \int_{-\infty}^{\infty} x f_X(x) \int_{-\infty}^{\infty} y \frac{1}{2}(\delta(x+y) + \delta(x-y)) dy dx \quad (2.0.19)$$

Using (2.0.1)

$$E[XY] = \int_{-\infty}^{\infty} x f_X(x) \frac{1}{2}(x - x) dx = 0 \quad (2.0.20)$$

2) Defining the following matrices/vectors Given

vector/matrix	expression
\mathbf{Z}	$\begin{pmatrix} X & Y \end{pmatrix}^T$
\mathbf{C}	$\begin{pmatrix} a & b \end{pmatrix}^T$
$\boldsymbol{\mu}$	$\begin{pmatrix} 0 & 0 \end{pmatrix}^T$
$\boldsymbol{\Sigma}$	$\begin{pmatrix} 1 & \rho \\ \rho & 1 \end{pmatrix}$

TABLE 2: vectors/matrices and their expressions

$$\mathbf{C}^T \mathbf{Z} \sim N(0, a^2 + b^2) \quad (2.0.21)$$

Since this is true for all a and b , it is equivalent to X and Y being jointly gaussian

$$\mathbf{Z} \sim N(\boldsymbol{\mu}, \boldsymbol{\Sigma}) \quad (2.0.22)$$

For correlated random variables X and Y in bivariate normal distribution, we have

$$\sigma_Z^2 = \sum_{i,j} \Sigma_{ij} \quad (2.0.23)$$

$$a^2 + b^2 = a^2 + b^2 + 2\rho ab \quad (2.0.24)$$

$$\therefore \rho = 0 \quad (2.0.25)$$

The joint distribution is given as

$$f_{\mathbf{Z}}(x, y) = \frac{\exp\left(-\frac{1}{2}(\mathbf{z} - \boldsymbol{\mu})^\top \boldsymbol{\Sigma}^{-1}(\mathbf{z} - \boldsymbol{\mu})\right)}{\sqrt{(2\pi)^2 |\boldsymbol{\Sigma}|}} \quad (2.0.26)$$

$$f_{\mathbf{Z}}(x, y) = \frac{\exp\left(-\frac{1}{2}\begin{pmatrix} x & y \end{pmatrix} I_2 \begin{pmatrix} x & y \end{pmatrix}^\top\right)}{\sqrt{(2\pi)^2}} \quad (2.0.27)$$

Where I_2 is the identity matrix of order 2

$$f_{\mathbf{Z}}(x, y) = \frac{\exp\left(-\frac{1}{2}\begin{pmatrix} x & y \end{pmatrix} \begin{pmatrix} x & y \end{pmatrix}^\top\right)}{\sqrt{(2\pi)^2}} \quad (2.0.28)$$

$$f_{\mathbf{Z}}(x, y) = \frac{\exp\left(-\frac{1}{2}(x^2 + y^2)\right)}{\sqrt{(2\pi)^2}} = f_X(x)f_Y(y) \quad (2.0.29)$$

\therefore **Option(2) is correct**, A simulation for bi-variate gaussian is given below

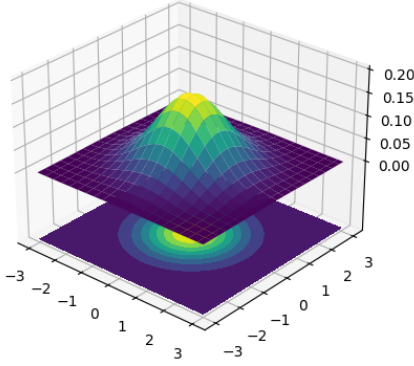


Fig. 2: bivariate gaussian while 0 mean vector and identity covariance matrix

3)

$$\Pr(X \leq 0, Y \leq 0) = \frac{1}{4} \quad (2.0.30)$$

This doesn't imply independence, it can be true even for dependent X and Y , the counter example is (2.0.11), the joint probability function is symmetric across all 4 quadrants

$$\therefore \Pr(X \leq 0, Y \leq 0) = \frac{1}{4} \quad (2.0.31)$$

Alternatively, here is proof

$$\Pr(X \leq 0) = F_X(0) = \frac{1}{2} \quad (2.0.32)$$

Using (2.0.6)

$$\Pr(Y \leq 0 | X \leq 0) = \frac{1}{2} \quad (2.0.33)$$

Using (2.0.32) and (2.0.33)

$$\Pr(X \leq 0, Y \leq 0) = \frac{1}{4} \quad (2.0.34)$$

4)

$$E[e^{itX+isY}] = E[e^{itX}] E[e^{isY}] \quad (2.0.35)$$

$$E[e^{itX+isY}] = \varphi_X(t)\varphi_Y(s) \quad (2.0.36)$$

The inverse is given as

$$f_{XY}(x, y) = \frac{1}{4\pi^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-itX-isY} E[e^{itX+isY}] ds dt \quad (2.0.37)$$

Using (2.0.36)

$$f_{XY}(x, y) = \frac{1}{4\pi^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-itX-isY} \varphi_X(t)\varphi_Y(s) ds dt \quad (2.0.38)$$

$$f_{XY}(x, y) = f_X(x)f_Y(y) \quad (2.0.39)$$

\therefore **Option(4) is correct**