Assignment 9

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Download all latex-tikz codes from

https://github.com/cmapsi/AI1103−Probability−and−random−variables/blob/main/Assignment−9/main.tex

1 Problem

(CSIR UGC NET EXAM (Dec 2015), Q.109)
Suppose $\begin{pmatrix} X \\ Y \end{pmatrix}$ is a random vector such that the marginal distribution of X and the marginal distribution of Y are the same and each is normally distributed with mean 0 and variance 1. Then, which of the following conditions imply independence of X and Y?

1) $\text{Cov}(X, Y) = 0$
2) $aX + bY$ is normally distributed with mean 0 and variance $a^2 + b^2$ for all real a and b
3) $\Pr(X \le 0, Y \le 0) = \frac{1}{4}$
4) $E\left[e^{itX+isY}\right] = E\left[e^{itX}\right] E\left[e^{isY}\right]$ for all real s and t

2 solution

An important property of dirac delta function that will be used at multiple ocassions in this solution is

$$\int_{-\infty}^{\infty} f(x)\delta(x-a)dx = f(a) \tag{2.0.1}$$

Given $X \sim N(0, 1), Y \sim N(0, 1)$

1)

$$Cov(X, Y) = 0 \tag{2.0.2}$$
$$E[XY] - E[X]E[Y] = 0 \tag{2.0.3}$$
$$E[XY] = 0 \tag{2.0.4}$$
$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} xyf_{XY}(x, y)dxdy = 0 \tag{2.0.5}$$

This doesn't imply independence. Counter example given below

Lets consider a case where X and Y are dependent based on the following relation

$$X = KY \tag{2.0.6}$$

PMF for K is given as

$$p_K(k) = \begin{cases} \frac{1}{2} & k = 1 \\ \frac{1}{2} & k = -1 \\ 0 & \text{otherwise} \end{cases} \tag{2.0.7}$$

The joint probability distribution is therefore

$$\begin{aligned} f_{XY}(x, y) &= f_{X|Y}(x|y)f_X(x) \\ &= f_X(x)\frac{1}{2}(\delta(x+y) + \delta(x-y)) \end{aligned} \tag{2.0.8}$$

The marginal probability distribution function for X is given as

$$\int_{-\infty}^{\infty} f_X(x)\frac{1}{2}(\delta(x+y) + \delta(x-y))dy \tag{2.0.9}$$

Using (2.0.1), we get

$$\int_{-\infty}^{\infty} f_X(x)\frac{1}{2}(\delta(x+y) + \delta(x-y))dy = f_X(x) \tag{2.0.10}$$

We know that $X \sim N(0, 1)$, $f_X(x)$ represents gaussian probability distribution function.
Futher, using symmetry of (2.0.6), we can establish that marginal distribution of Y is gaussian

$$Cov(X, Y) = E[XY] - E[X]E[Y] = E[XY] \tag{2.0.11}$$

$$E[XY] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} xyf_{XY}(x, y)dydx \tag{2.0.12}$$

$$= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} xy f_X(x) \frac{1}{2} (\delta(x+y) + \delta(x-y)) dy dx \tag{2.0.13}$$

$$= \int_{-\infty}^{\infty} x f_X(x) \int_{-\infty}^{\infty} y \frac{1}{2} (\delta(x+y) + \delta(x-y)) dy dx \tag{2.0.14}$$

Using (2.0.1)

$$E[XY] = \int_{-\infty}^{\infty} x f_X(x) \frac{1}{2} (x - x) dx = 0 \tag{2.0.15}$$

2) Defining the following matrices/vectors

vector/matrix	expression
Z	$\begin{bmatrix} X & Y \end{bmatrix}^\top$
C	$\begin{bmatrix} a & b \end{bmatrix}^\top$
μ	$\begin{bmatrix} 0 & 0 \end{bmatrix}^\top$
Σ	$\begin{bmatrix} a^2 & \rho ab \\ \rho ab & b^2 \end{bmatrix}$

TABLE 2: vectors/matrices and their expressions

$$C^\top Z \sim N(\mu, \Sigma) \tag{2.0.16}$$

For correlated random variables X and Y in normal distribution, we have

$$\sigma_Z^2 = \sum_{i,j} \Sigma_{ij} \tag{2.0.17}$$

$$a^2 + b^2 = a^2 + b^2 + 2\rho ab \tag{2.0.18}$$

$$\therefore \rho = 0 \tag{2.0.19}$$

considering the case with $a = 1, b = 1$, the joint probability distribution is given as

$$f_Z(x,y) = \frac{\exp\left(-\frac{1}{2} (z-\mu)^\top \Sigma^{-1} (z-\mu)\right)}{\sqrt{(2\pi)^2 |\Sigma|}} \tag{2.0.20}$$

$$f_Z(x,y) = \frac{\exp\left(-\frac{1}{2} \left(\begin{bmatrix} x & y \end{bmatrix}\right) I_2 \left(\begin{bmatrix} x & y \end{bmatrix}\right)^\top\right)}{\sqrt{(2\pi)^2}} \tag{2.0.21}$$

Where I_2 is the identity matrix of order 2

$$f_Z(x,y) = \frac{\exp\left(-\frac{1}{2} \left(\begin{bmatrix} x & y \end{bmatrix}\right) \left(\begin{bmatrix} x & y \end{bmatrix}\right)^\top\right)}{\sqrt{(2\pi)^2}} \tag{2.0.22}$$

$$f_Z(x,y) = \frac{\exp\left(-\frac{1}{2} \left(x^2 + y^2\right)\right)}{\sqrt{(2\pi)^2}} = f_X(x) f_Y(y) \tag{2.0.23}$$

$\therefore$ **Option(2) is correct**

3) The counter example is (2.0.8), the joint probability function is symmetric across all 4 quadrants

$$\therefore \Pr(X \le 0, Y \le 0) = \frac{1}{4} \tag{2.0.24}$$

Alternatively, here is proof

$$\Pr(X \le 0) = F_X(0) = \frac{1}{2} \tag{2.0.25}$$

Using (2.0.6)

$$\Pr(Y \le 0 | X \le 0) = \frac{1}{2} \tag{2.0.26}$$

Using (2.0.25) and (2.0.26)

$$\Pr(X \le 0, Y \le 0) = \frac{1}{4} \tag{2.0.27}$$

4)

$$E\left[e^{itX+isY}\right] = E\left[e^{itX}\right] E\left[e^{isY}\right] \tag{2.0.28}$$

$$E\left[e^{itX+isY}\right] = \varphi_X(t) \varphi_Y(s) \tag{2.0.29}$$

The inverse is given as

$$f_{XY}(x,y) = \frac{1}{4\pi^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-itX-isY} E\left[e^{itX+isY}\right] ds dt \tag{2.0.30}$$

Using (2.0.29)

$$f_{XY}(x,y) = \frac{1}{4\pi^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-itX-isY} \varphi_X(t) \varphi_Y(s) ds dt \tag{2.0.31}$$

$$f_{XY}(x,y) = f_X(x) f_Y(y) \tag{2.0.32}$$

$\therefore$ **Option(4) is correct**