

Assignment 9

Chirag Mehta - AI20BTECH11006

Download all latex-tikz codes from

<https://github.com/cmapsi/AI1103-Probability-and-random-variables/blob/main/Assignment-9/main.tex>

1 PROBLEM

(CSIR UGC NET EXAM (Dec 2015), Q.109)

Suppose $\begin{pmatrix} X \\ Y \end{pmatrix}$ is a random vector such that the marginal distribution of X and the marginal distribution of Y are the same and each is normally distributed with mean 0 and variance 1. Then, which of the following conditions imply independence of X and Y ?

- 1) $\text{Cov}(X, Y) = 0$
- 2) $aX + bY$ is normally distributed with mean 0 and variance $a^2 + b^2$ for all real a and b
- 3) $\Pr(X \leq 0, Y \leq 0) = \frac{1}{4}$
- 4) $E[e^{itX+isY}] = E[e^{itX}]E[e^{isY}]$ for all real s and t

2 SOLUTION

Given $X \sim N(0, 1)$, $Y \sim N(0, 1)$

1)

$$\text{Cov}(X, Y) = 0 \quad (2.0.1)$$

$$E[XY] - E[X]E[Y] = 0 \quad (2.0.2)$$

$$E[XY] = 0 \quad (2.0.3)$$

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} xy f_{XY}(x, y) dx dy = 0 \quad (2.0.4)$$

This doesn't imply independence. Refer (2.0.5) for counter example.

$$f_{XY}(x, y) = \frac{1}{\pi} e^{-x^2 - y^2} \quad (2.0.5)$$

2)

$$\begin{pmatrix} aX \\ bY \end{pmatrix} \sim N\left(\begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} a^2 & \rho ab \\ \rho ab & b^2 \end{bmatrix}\right) \quad (2.0.6)$$

For correlated random variables X and Y in normal distribution, we have

$$\sigma_{X+Y}^2 = \sigma_X^2 + \sigma_Y^2 + 2\rho\sigma_X\sigma_Y \quad (2.0.7)$$

Replacing X by aX , Y by bY , we get

$$a^2 + b^2 = a^2 + b^2 + 2\rho ab \quad (2.0.8)$$

$$\therefore \rho = 0 \quad (2.0.9)$$

For standard bivariate normal distribution, with correlation coefficient ρ , the joint pdf is given by

$$f_{XY}(x, y) = \frac{1}{2\pi\sqrt{1-\rho^2}} e^{\frac{-x^2-2\rho xy-y^2}{2(1-\rho^2)}} \quad (2.0.10)$$

Using (2.0.9) and (2.0.10)

$$f_{XY}(x, y) = \frac{1}{2\pi} e^{\frac{-x^2-y^2}{2}} = f_X(x)f_Y(y) \quad (2.0.11)$$

\therefore **Option(2) is correct**

- 3) The counter example is (2.0.5), using properties of even function it is easy to obtain

$$\Pr(X \leq 0, Y \leq 0) = \frac{1}{4} \quad (2.0.12)$$

4)

$$E[e^{itX+isY}] = E[e^{itX}]E[e^{isY}] \quad (2.0.13)$$

$$E[e^{itX+isY}] = \varphi_X(t)\varphi_Y(s) \quad (2.0.14)$$

The inverse is given as

$$f_{XY}(x, y) = \frac{1}{4\pi^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-itX-isY} E[e^{itX+isY}] ds dt \quad (2.0.15)$$

Using (2.0.14)

$$f_{XY}(x, y) = \frac{1}{4\pi^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-itX-isY} \varphi_X(t)\varphi_Y(s) ds dt \quad (2.0.16)$$

$$f_{XY}(x, y) = f_X(x)f_Y(y) \quad (2.0.17)$$

\therefore Option(4) is correct