Assignment 9

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Download all latex-tikz codes from

https://github.com/cmapsi/AI1103−Probability−and−random−variables/blob/main/Assignment−9/main.tex

1 Problem

(CSIR UGC NET EXAM (Dec 2015), Q.109)

Suppose $\begin{pmatrix} X \\ Y \end{pmatrix}$ is a random vector such that the marginal distribution of X and the marginal distribution of Y are the same and each is normally distributed with mean 0 and variance 1. Then, which of the following conditions imply independence of X and Y?

1) $\text{Cov}(X, Y) = 0$
2) $aX + bY$ is normally distributed with mean 0 and variance $a^2 + b^2$ for all real a and b
3) $\Pr(X \le 0, Y \le 0) = \frac{1}{4}$
4) $E\left[e^{itX+isY}\right] = E\left[e^{itX}\right] E\left[e^{isY}\right]$ for all real s and t

2 solution

Given $X \sim N(0, 1), Y \sim N(0, 1)$

1)

$$Cov(X, Y) = 0 \tag{2.0.1}$$
$$E[XY] - E[X]E[Y] = 0 \tag{2.0.2}$$
$$E[XY] = 0 \tag{2.0.3}$$
$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} xy f_{XY}(x, y) dx dy = 0 \tag{2.0.4}$$

This doesn't imply independence. Refer (2.0.5) for counter example.

$$f_{XY}(x, y) = \frac{1}{\pi} e^{-x^2-y^2} \tag{2.0.5}$$

2)

$$\begin{bmatrix} a & b \end{bmatrix} \cdot \begin{bmatrix} X \\ Y \end{bmatrix} \sim N\left(\begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} a^2 & \rho ab \\ \rho ab & b^2 \end{bmatrix}\right) \tag{2.0.6}$$

For correlated random variables X and Y in normal distribution, we have

$$\sigma^2_{X+Y} = \sigma^2_X + \sigma^2_Y + 2\rho\sigma_X\sigma_Y \tag{2.0.7}$$

Replacing X by aX, Y by bY, we get

$$a^2 + b^2 = a^2 + b^2 + 2\rho ab \tag{2.0.8}$$
$$\therefore \rho = 0 \tag{2.0.9}$$

For standard bivariate normal distribution, with correlation coefficient ρ, the joint pdf is given by

$$f_{XY}(x, y) = \frac{1}{2\pi \sqrt[2]{1 - \rho^2}} e^{\frac{-x^2-2\rho xy-y^2}{2(1-\rho^2)}} \tag{2.0.10}$$

Using (2.0.9) and (2.0.10)

$$f_{XY}(x, y) = \frac{1}{2\pi} e^{\frac{-x^2-y^2}{2}} = f_X(x) f_Y(y) \tag{2.0.11}$$

$\therefore$ **Option(2) is correct**

3) The counter example is (2.0.5), using properties of even function it is easy to obtain

$$\Pr(X \le 0, Y \le 0) = \frac{1}{4} \tag{2.0.12}$$

4)

$$E\left[e^{itX+isY}\right] = E\left[e^{itX}\right] E\left[e^{isY}\right] \tag{2.0.13}$$
$$E\left[e^{itX+isY}\right] = \varphi_X(t)\varphi_Y(s) \tag{2.0.14}$$

The inverse is given as

$$f_{XY}(x, y) = \frac{1}{4\pi^2} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-itX-isY} E\left[e^{itX+isY}\right] ds dt \tag{2.0.15}$$

Using (2.0.14)

$$f_{XY}(x, y) = \frac{1}{4\pi^2} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-itX-isY} \varphi_X(t)\varphi_Y(s) ds dt \tag{2.0.16}$$

$$f_{XY}(x, y) = f_X(x) f_Y(y) \tag{2.0.17}$$

$\therefore$ **Option(4) is correct**