

CSIR-UGC NET-Dec 2015-Problem(109)

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Multivariate gaussian

Multivariate Gaussian expression, definition

The multivariate normal distribution of a n dimensional vector $\mathbf{X} = (X_1, X_2, \dots, X_n)^\top$ can be written as

$$f_{\mathbf{X}}(x_1, x_2, \dots, x_n) = \frac{\exp\left(-\frac{1}{2}(\mathbf{x} - \boldsymbol{\mu})^\top \boldsymbol{\Sigma}^{-1}(\mathbf{x} - \boldsymbol{\mu})\right)}{\sqrt{(2\pi)^2 |\boldsymbol{\Sigma}|}} \quad (1)$$

Mean vector $\boldsymbol{\mu}$ is defined as

$$\boldsymbol{\mu} = E[\mathbf{X}] = (E[X_1], E[X_2], \dots, E[X_n])^\top \quad (2)$$

Covariance matrix $\boldsymbol{\Sigma}$ is defined as

$$\Sigma_{i,j} = \text{Cov}(X_i, X_j) \quad (3)$$

multivariate gaussian contd.

equivalence

A random vector $\mathbf{X} = (X_1, X_2, \dots, X_n)^\top$ has a multivariate gaussian distribution if it satisfies

- For every linear combination $Y = a_1X_1 + a_2X_2 + \dots + a_nX_n$ of its components is normally distributed. That is for any vector $\mathbf{a} \in \mathbb{R}^n$, the random variable $Y = \mathbf{a}^\top \mathbf{X}$ has a univariate normal distribution

marginal probability

$$f_{\mathbf{X}}(\mathbf{x}) = \int_{-\infty}^{\infty} f_{\mathbf{X}\mathbf{Y}}(\mathbf{x}\mathbf{y}) d\mathbf{y} \quad (4)$$

dirac delta function

dirac delta function

An important property of dirac delta function that will be used at multiple occasions in this solution is

$$\int_{-\infty}^{\infty} f(x)\delta(x-a)dx = f(a) \quad (5)$$

Question

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Suppose $\begin{pmatrix} X \\ Y \end{pmatrix}$ is a random vector such that the marginal distribution of X and the marginal distribution of Y are the same and each is normally distributed with mean 0 and variance 1. Then, which of the following conditions imply independence of X and Y ?

- ① $\text{Cov}(X, Y) = 0$
- ② $aX + bY$ is normally distributed with mean 0 and variance $a^2 + b^2$ for all real a and b
- ③ $\Pr(X \leq 0, Y \leq 0) = \frac{1}{4}$
- ④ $E[e^{itX+isY}] = E[e^{itX}] E[e^{isY}]$ for all real s and t

Solution

Given $X \sim N(0, 1)$, $Y \sim N(0, 1)$

1

$$\text{Cov}(X, Y) = 0 \quad (6)$$

$$E[XY] - E[X]E[Y] = 0 \quad (7)$$

$$E[XY] = 0 \quad (8)$$

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} xy f_{XY}(x, y) dx dy = 0 \quad (9)$$

This doesn't imply independence. Counter example given below

Solution Contd.

- ① Lets consider a case where X and Y are dependent based on the following relation, Y being independent of K

$$X = KY \quad (10)$$

PMF for K is given as

$$p_K(k) = \begin{cases} \frac{1}{2} & k = 1 \\ \frac{1}{2} & k = -1 \\ 0 & \text{otherwise} \end{cases} \quad (11)$$

Solution Contd.

- 1 A simulation is given below, Y is gaussian, then X also follows gaussian

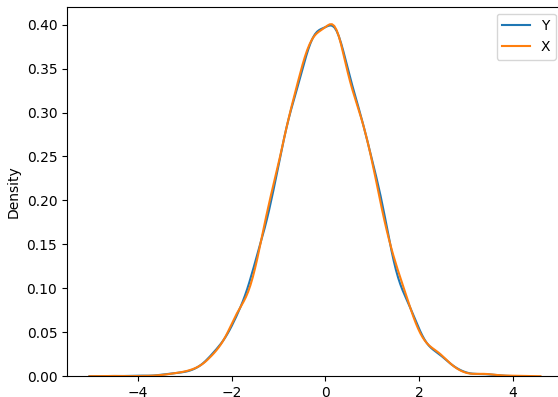


Figure: X and Y, if Y is normal

- ① Theoretically it can be proved in the following manner, Since K and Y are independent

$$f_X(x) = \Pr(K = 1) f_Y(x) + \Pr(K = -1) f_Y(-x) \quad (12)$$

$$= \frac{1}{2} (f_Y(x) + f_Y(-x)) \quad (13)$$

$$= f_Y(x) \quad (14)$$

Therefore, X follows identical but not independent distribution as Y ,
An alternative proof is given below as a proof for marginal probability

Solution Contd.

- ① Now consider that X is normally distributed, we will establish Y is also normally distributed. The joint probability distribution is therefore

$$\begin{aligned} f_{XY}(x, y) &= f_{X|Y}(x|y)f_X(x) \\ &= f_X(x)\frac{1}{2}(\delta(x+y) + \delta(x-y)) \end{aligned} \quad (15)$$

The marginal probability distribution function for X is given as

$$\int_{-\infty}^{\infty} f_X(x)\frac{1}{2}(\delta(x+y) + \delta(x-y))dy \quad (16)$$

Using (5), we get

$$\int_{-\infty}^{\infty} f_X(x)\frac{1}{2}(\delta(x+y) + \delta(x-y))dy = f_X(x) \quad (17)$$

We know that $X \sim N(0, 1)$, $f_X(x)$ represents gaussian probability distribution function.

Solution Contd.

- ① Further, using symmetry of (10), we can establish that marginal distribution of Y is gaussian. Here is a proof anyways

$$f_Y(y) = \int_{-\infty}^{\infty} f_X(x) \frac{1}{2} (\delta(x+y) + \delta(x-y)) dx \quad (18)$$

Using (5), we get

$$f_Y(y) = \frac{1}{2} (f_X(y) + f_X(-y)) = f_X(y) \quad (19)$$

Since Y has identical probability distribution function, $Y \sim N(0, 1)$

Solution Contd.

① The covariance is given as

$$\text{Cov}(X, Y) = E[XY] - E[X]E[Y] = E[XY] \quad (20)$$

$$E[XY] = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} xy f_{XY}(x, y) dy dx \quad (21)$$

$$= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} xy f_X(x) \frac{1}{2} (\delta(x+y) + \delta(x-y)) dy dx \quad (22)$$

$$= \int_{-\infty}^{\infty} x f_X(x) \int_{-\infty}^{\infty} y \frac{1}{2} (\delta(x+y) + \delta(x-y)) dy dx \quad (23)$$

Using (5)

$$E[XY] = \int_{-\infty}^{\infty} x f_X(x) \frac{1}{2} (x - x) dx = 0 \quad (24)$$

Solution Contd.

- 2 Defining the following matrices/vectors

vector/matrix	expression
Z	$(X \ Y)^T$
C	$(a \ b)^T$
μ	$(0 \ 0)^T$
Σ	$\begin{pmatrix} 1 & \rho \\ \rho & 1 \end{pmatrix}$

Table: vectors/matrices and their expressions

Solution Contd.

② Given

$$\mathbf{C}^T \mathbf{Z} \sim N(0, a^2 + b^2) \quad (25)$$

Since this is true for all a and b , it is equivalent to X and Y being jointly gaussian

$$\mathbf{Z} \sim N(\boldsymbol{\mu}, \boldsymbol{\Sigma}) \quad (26)$$

For correlated random variables X and Y in bivariate normal distribution, we have

$$\sigma_Z^2 = \sum_{i,j} \Sigma_{ij} \quad (27)$$

$$a^2 + b^2 = a^2 + b^2 + 2\rho ab \quad (28)$$

$$\therefore \rho = 0 \quad (29)$$

Solution Contd.

- ② The joint distribution is given as

$$f_{\mathbf{Z}}(x, y) = \frac{\exp\left(-\frac{1}{2}(\mathbf{z} - \boldsymbol{\mu})^T \boldsymbol{\Sigma}^{-1}(\mathbf{z} - \boldsymbol{\mu})\right)}{\sqrt{(2\pi)^2 |\boldsymbol{\Sigma}|}} \quad (30)$$

$$f_{\mathbf{Z}}(x, y) = \frac{\exp\left(-\frac{1}{2}(x \ y) I_2 (x \ y)^T\right)}{\sqrt{(2\pi)^2}} \quad (31)$$

Where I_2 is the identity matrix of order 2

$$f_{\mathbf{Z}}(x, y) = \frac{\exp\left(-\frac{1}{2}(x \ y)(x \ y)^T\right)}{\sqrt{(2\pi)^2}} \quad (32)$$

$$f_{\mathbf{Z}}(x, y) = \frac{\exp\left(-\frac{1}{2}(x^2 + y^2)\right)}{\sqrt{(2\pi)^2}} = f_X(x)f_Y(y) \quad (33)$$

∴ Option(2) is correct

Solution Contd.

- ② A simulation for bivariate gaussian is given below

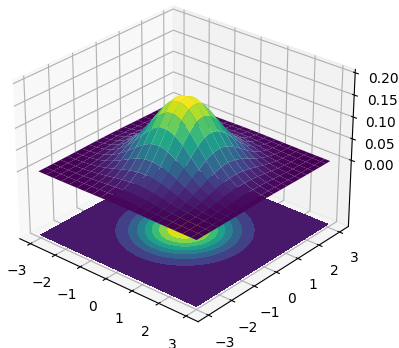


Figure: bivariate gaussian with 0 mean vector and identity covariance matrix

Solution Contd.

3

$$\Pr(X \leq 0, Y \leq 0) = \frac{1}{4} \quad (34)$$

This doesn't imply independence, it can be true even for dependent X and Y , the counter example is (15), the joint probability function is symmetric across all 4 quadrants

$$\therefore \Pr(X \leq 0, Y \leq 0) = \frac{1}{4} \quad (35)$$

Alternatively, here is proof

$$\Pr(X \leq 0) = F_X(0) = \frac{1}{2} \quad (36)$$

Using (10)

$$\Pr(Y \leq 0 | X \leq 0) = \frac{1}{2} \quad (37)$$

Solution Contd.

3 Using (36) and (37)

$$\Pr(X \leq 0, Y \leq 0) = \frac{1}{4} \quad (38)$$

4

$$E \left[e^{itX+isY} \right] = E \left[e^{itX} \right] E \left[e^{isY} \right] \quad (39)$$

$$E \left[e^{itX+isY} \right] = \varphi_X(t) \varphi_Y(s) \quad (40)$$

Solution Contd.

4 The inverse is given as

$$f_{XY}(x, y) = \frac{1}{4\pi^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-itX - isY} E \left[e^{itX + isY} \right] ds dt \quad (41)$$

Using (40)

$$f_{XY}(x, y) = \frac{1}{4\pi^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-itX - isY} \varphi_X(t) \varphi_Y(s) ds dt \quad (42)$$

$$f_{XY}(x, y) = f_X(x) f_Y(y) \quad (43)$$

\therefore Option(4) is correct

Reading material

- ① Sum of 2 gaussian random variables need not be gaussian

[https:](https://github.com/planetmath/62_Statistics/blob/master/pdf/62E15-SumsOfNormalRandomVariablesNeedNotBeNormal.pdf)

[//github.com/planetmath/62_Statistics/blob/master/pdf/62E15-SumsOfNormalRandomVariablesNeedNotBeNormal.pdf](https://github.com/planetmath/62_Statistics/blob/master/pdf/62E15-SumsOfNormalRandomVariablesNeedNotBeNormal.pdf)

- ② Multivariate gaussian

- ▶ <https://bit.ly/2RComEB>

- ▶ [https:](https://en.wikipedia.org/wiki/Multivariate_normal_distribution)

[//en.wikipedia.org/wiki/Multivariate_normal_distribution](https://en.wikipedia.org/wiki/Multivariate_normal_distribution)

- ③ Characteristic function of multivariate gaussian

<https://bit.ly/3gbWZeF>