

Assignment 9

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Download all latex-tikz codes from

<https://github.com/cmapi/AI1103-Probability-and-random-variables/blob/main/Assignment-9/main.tex>

1 PROBLEM

(CSIR UGC NET EXAM (Dec 2015), Q.109)

Suppose $\begin{pmatrix} X \\ Y \end{pmatrix}$ is a random vector such that the marginal distribution of X and the marginal distribution of Y are the same and each is normally distributed with mean 0 and variance 1. Then, which of the following conditions imply independence of X and Y ?

- 1) $\text{Cov}(X, Y) = 0$
- 2) $aX + bY$ is normally distributed with mean 0 and variance $a^2 + b^2$ for all real a and b
- 3) $\Pr(X \leq 0, Y \leq 0) = \frac{1}{4}$
- 4) $E[e^{itX+isY}] = E[e^{itX}]E[e^{isY}]$ for all real s and t

2 SOLUTION

An important property of dirac delta function that will be used at multiple occasions in this solution is

$$\int_{-\infty}^{\infty} f(x)\delta(x-a)dx = f(a) \quad (2.0.1)$$

Given $X \sim N(0, 1)$, $Y \sim N(0, 1)$

1)

$$\text{Cov}(X, Y) = 0 \quad (2.0.2)$$

$$E[XY] - E[X]E[Y] = 0 \quad (2.0.3)$$

$$E[XY] = 0 \quad (2.0.4)$$

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} xy f_{XY}(x, y) dx dy = 0 \quad (2.0.5)$$

This doesn't imply independence. Counter example given below

Lets consider a case where X and Y are dependent based on the following relation

$$X = KY \quad (2.0.6)$$

PMF for K is given as

$$p_K(k) = \begin{cases} \frac{1}{2} & k = 1 \\ \frac{1}{2} & k = -1 \\ 0 & \text{otherwise} \end{cases} \quad (2.0.7)$$

The joint probability distribution is therefore

$$\begin{aligned} f_{XY}(x, y) &= f_{X|Y}(x|y)f_Y(y) \\ &= f_X(x) \frac{1}{2}(\delta(x+y) + \delta(x-y)) \end{aligned} \quad (2.0.8)$$

The marginal probability distribution function for X is given as

$$\int_{-\infty}^{\infty} f_X(x) \frac{1}{2}(\delta(x+y) + \delta(x-y)) dy \quad (2.0.9)$$

Using (2.0.1), we get

$$\int_{-\infty}^{\infty} f_X(x) \frac{1}{2}(\delta(x+y) + \delta(x-y)) dy = f_X(x) \quad (2.0.10)$$

We know that $X \sim N(0, 1)$, $f_X(x)$ represents gaussian probability distribution function.

Futher, using symmetry of (2.0.6), we can establish that marginal distribution of Y is gaussian

$$\text{Cov}(X, Y) = E[XY] - E[X]E[Y] = E[XY] \quad (2.0.11)$$

$$E[XY] = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} xy f_{XY}(x, y) dy dx \quad (2.0.12)$$

$$= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} xy f_X(x) \frac{1}{2} (\delta(x+y) + \delta(x-y)) dy dx \quad (2.0.13)$$

$$= \int_{-\infty}^{\infty} x f_X(x) \int_{-\infty}^{\infty} y \frac{1}{2} (\delta(x+y) + \delta(x-y)) dy dx \quad (2.0.14)$$

Using (2.0.1)

$$E[XY] = \int_{-\infty}^{\infty} x f_X(x) \frac{1}{2} (x - x) dx = 0 \quad (2.0.15)$$

2) Defining the following matrices/vectors

vector/matrix	expression
Z	$\begin{bmatrix} X & Y \end{bmatrix}^T$
C	$\begin{bmatrix} a & b \end{bmatrix}^T$
μ	$\begin{bmatrix} 0 & 0 \end{bmatrix}^T$
Σ	$\begin{bmatrix} a^2 & \rho ab \\ \rho ab & b^2 \end{bmatrix}$

TABLE 2: vectors/matrices and their expressions

$$C^T Z \sim N(\mu, \Sigma) \quad (2.0.16)$$

For correlated random variables X and Y in normal distribution, we have

$$\sigma_Z^2 = \sum_{i,j} \Sigma_{ij} \quad (2.0.17)$$

$$a^2 + b^2 = a^2 + b^2 + 2\rho ab \quad (2.0.18)$$

$$\therefore \rho = 0 \quad (2.0.19)$$

For standard bivariate normal distribution, with correlation coefficient ρ , the joint pdf is given by

$$f_{XY}(x, y) = \frac{1}{2\pi \sqrt{1-\rho^2}} e^{\frac{-x^2 - 2\rho xy - y^2}{2(1-\rho^2)}} \quad (2.0.20)$$

Using (2.0.19) and (2.0.20)

$$f_{XY}(x, y) = \frac{1}{2\pi} e^{\frac{-x^2 - y^2}{2}} = f_X(x) f_Y(y) \quad (2.0.21)$$

\therefore **Option(2) is correct**

3) The counter example is (2.0.8), the joint prob-

ability function is symmetric across all 4 quadrants

$$\therefore \Pr(X \leq 0, Y \leq 0) = \frac{1}{4} \quad (2.0.22)$$

Alternatively, here is proof

$$\Pr(X \leq 0) = F_X(0) = \frac{1}{2} \quad (2.0.23)$$

Using (2.0.6)

$$\Pr(Y \leq 0 | X \leq 0) = \frac{1}{2} \quad (2.0.24)$$

Using (2.0.23) and (2.0.24)

$$\Pr(X \leq 0, Y \leq 0) = \frac{1}{4} \quad (2.0.25)$$

4)

$$E[e^{itX+isY}] = E[e^{itX}] E[e^{isY}] \quad (2.0.26)$$

$$E[e^{itX+isY}] = \varphi_X(t) \varphi_Y(s) \quad (2.0.27)$$

The inverse is given as

$$f_{XY}(x, y) = \frac{1}{4\pi^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-itX-isY} E[e^{itX+isY}] ds dt \quad (2.0.28)$$

Using (2.0.27)

$$f_{XY}(x, y) = \frac{1}{4\pi^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-itX-isY} \varphi_X(t) \varphi_Y(s) ds dt \quad (2.0.29)$$

$$f_{XY}(x, y) = f_X(x) f_Y(y) \quad (2.0.30)$$

\therefore **Option(4) is correct**