Introduction to the FitzHugh-Nagumo model

Workshop on Dynamics of Excitable Systems

9 April 2022

FitzHugh-Nagumo Model

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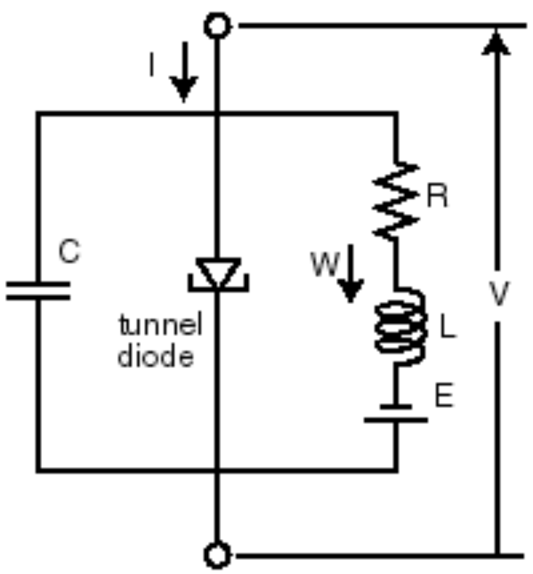

Scholarpedia;
Nagumo et al., (1962)

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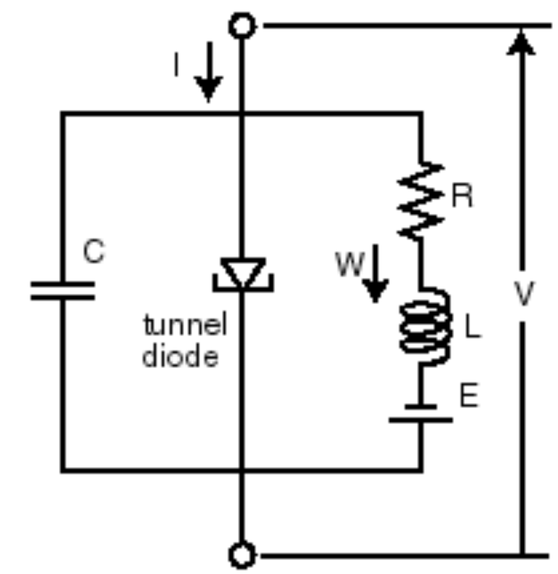
- Delineate the essential features of mathematical excitation from physiological details
    - The low dimensionality of the system makes it amenable to visual exploration
    - Everything about the system can be shown in the plane



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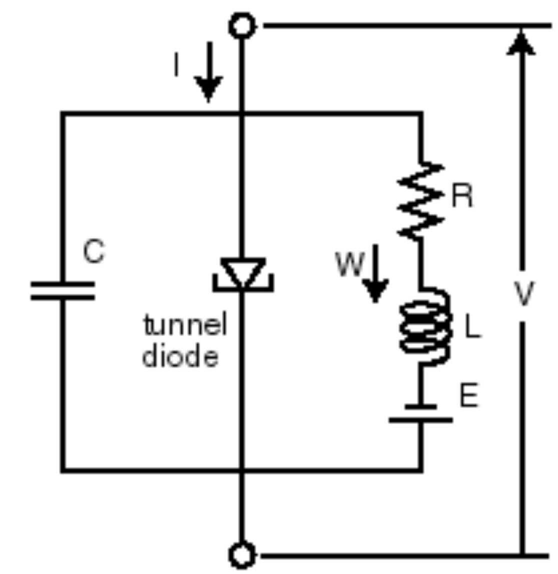
- $\dot{u} = u - u^3/3 - v + I,\ \ \dot{v} = 0.08(u - 0.8v + 0.7)$



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    - Near these curves, the dynamics are slower for smooth systems, so they organize the dynamics in the phase plane

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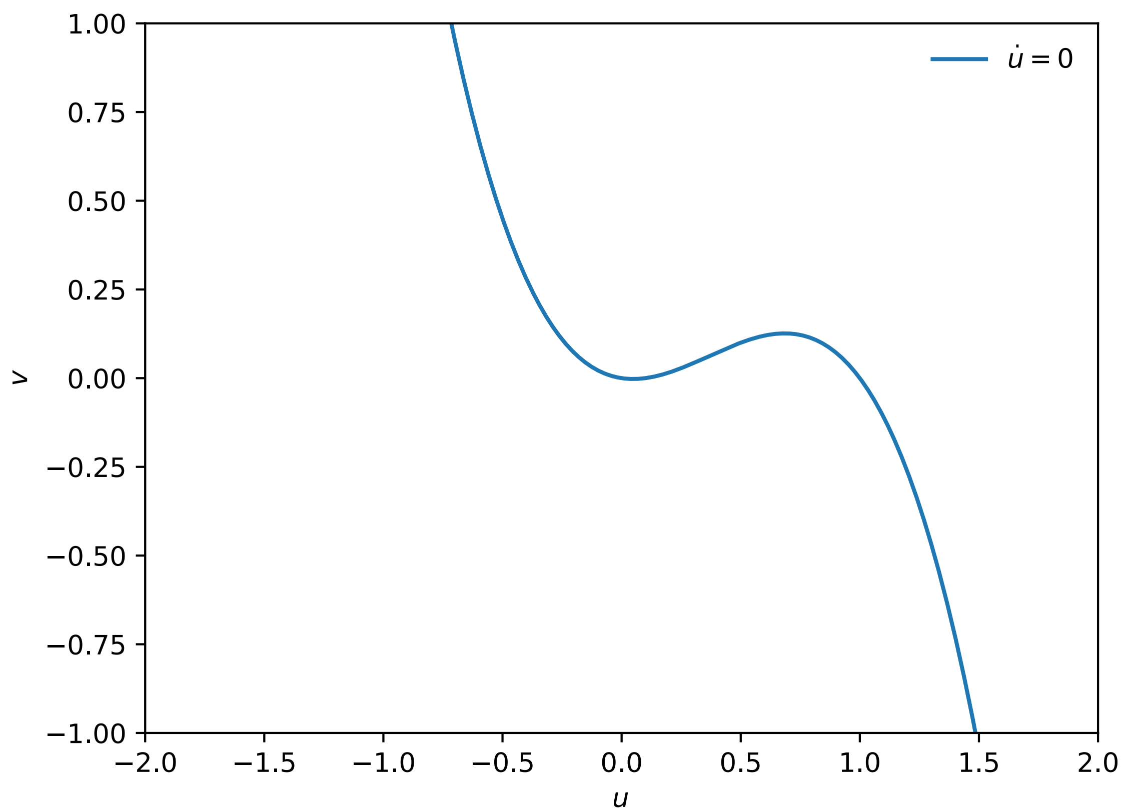

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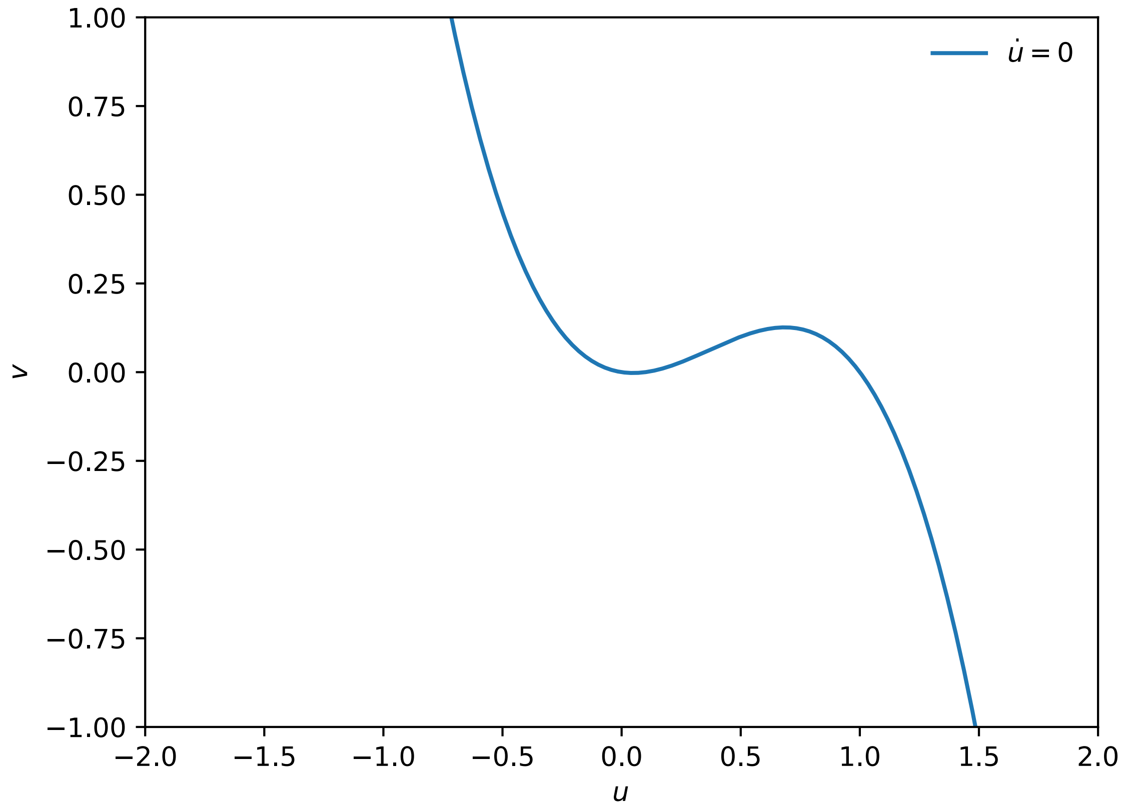

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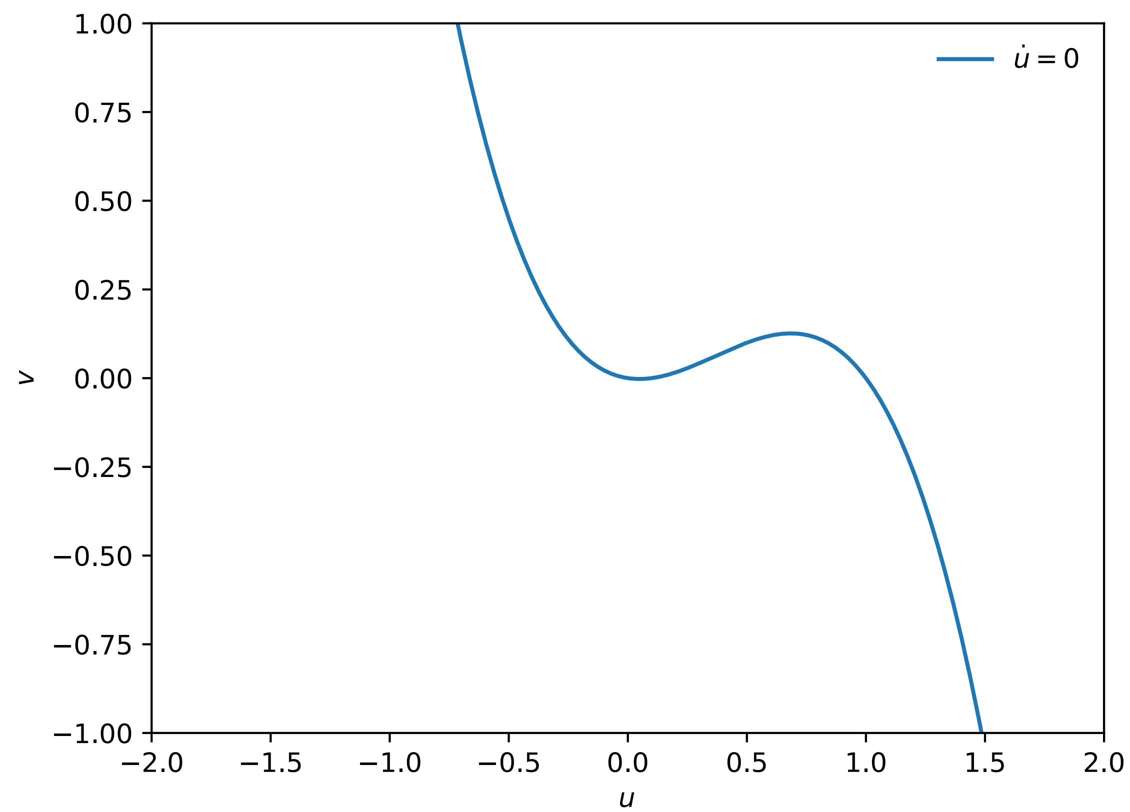

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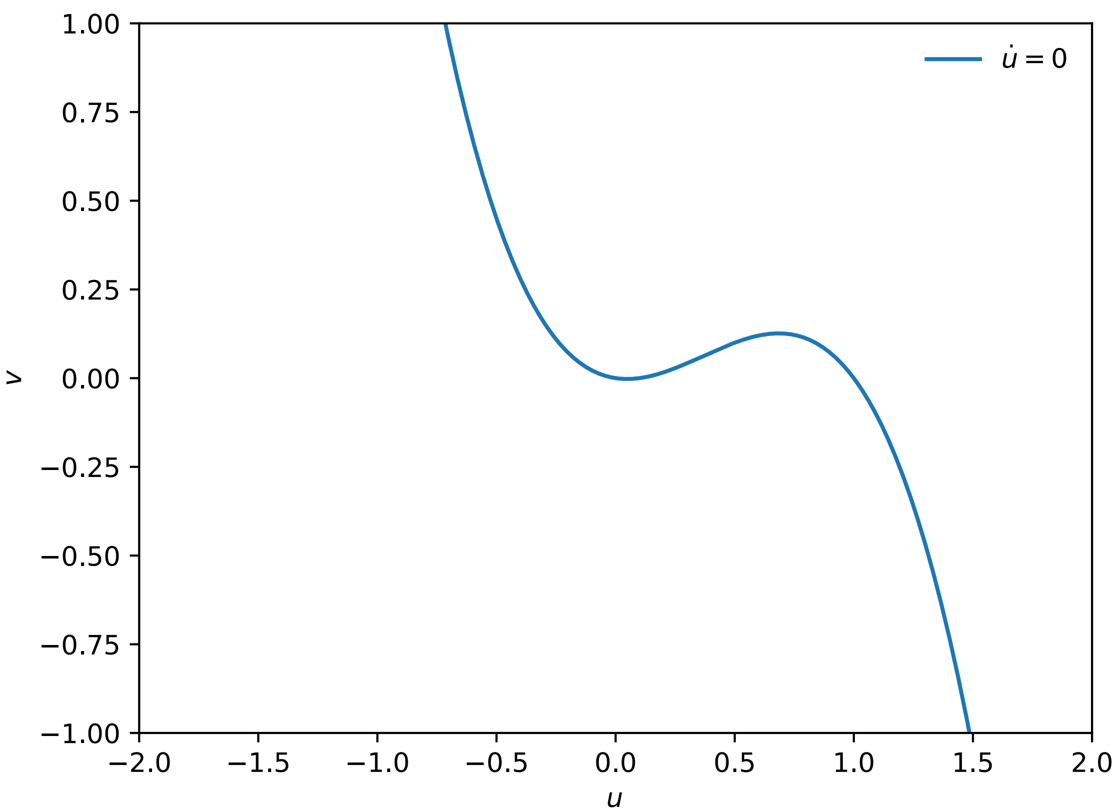

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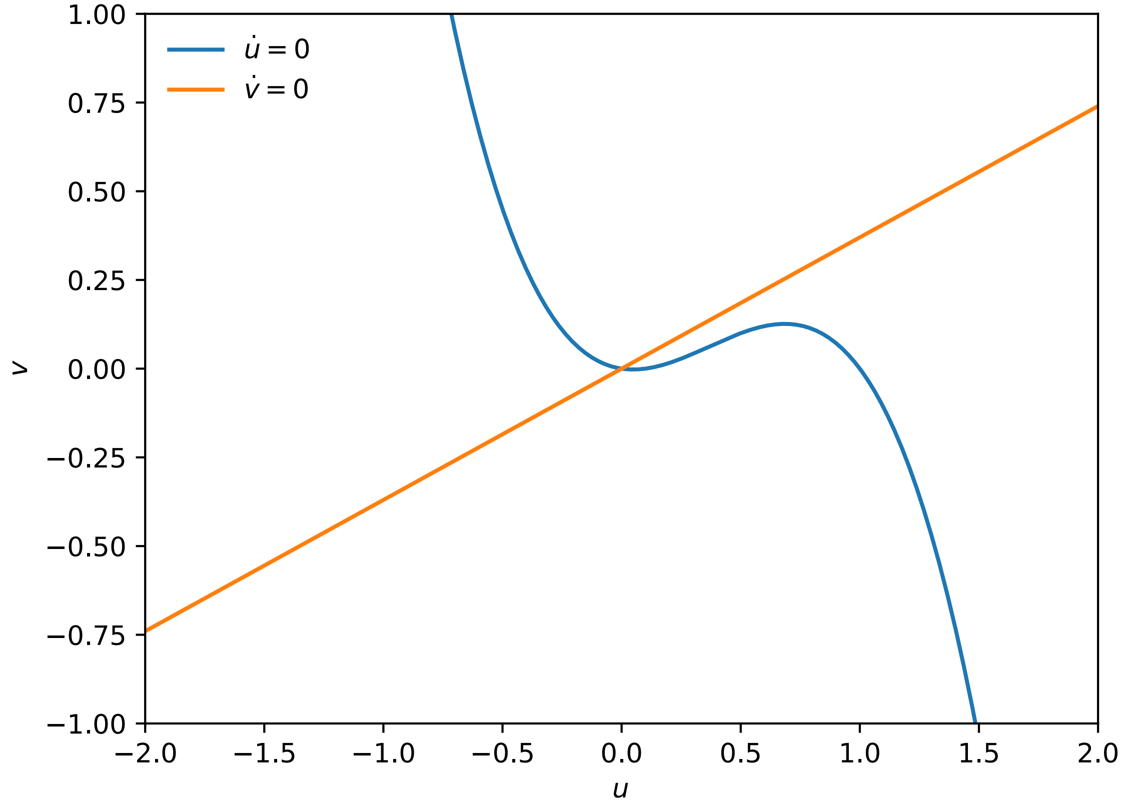

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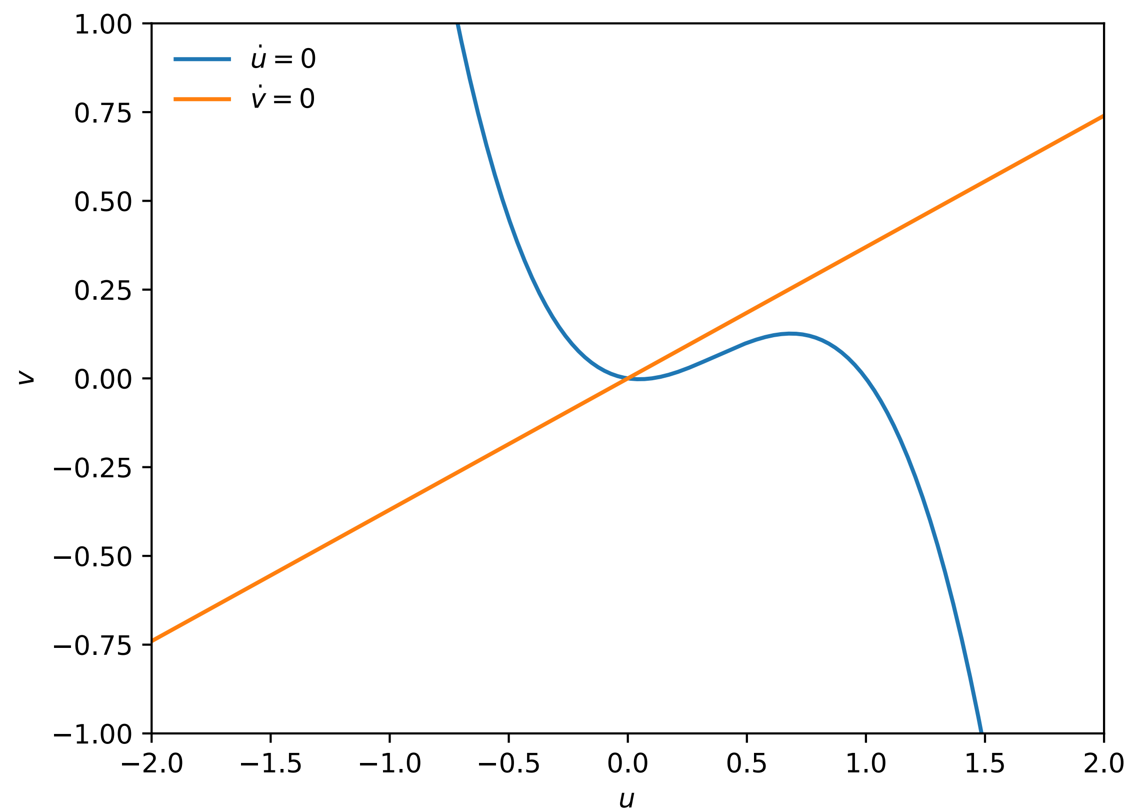

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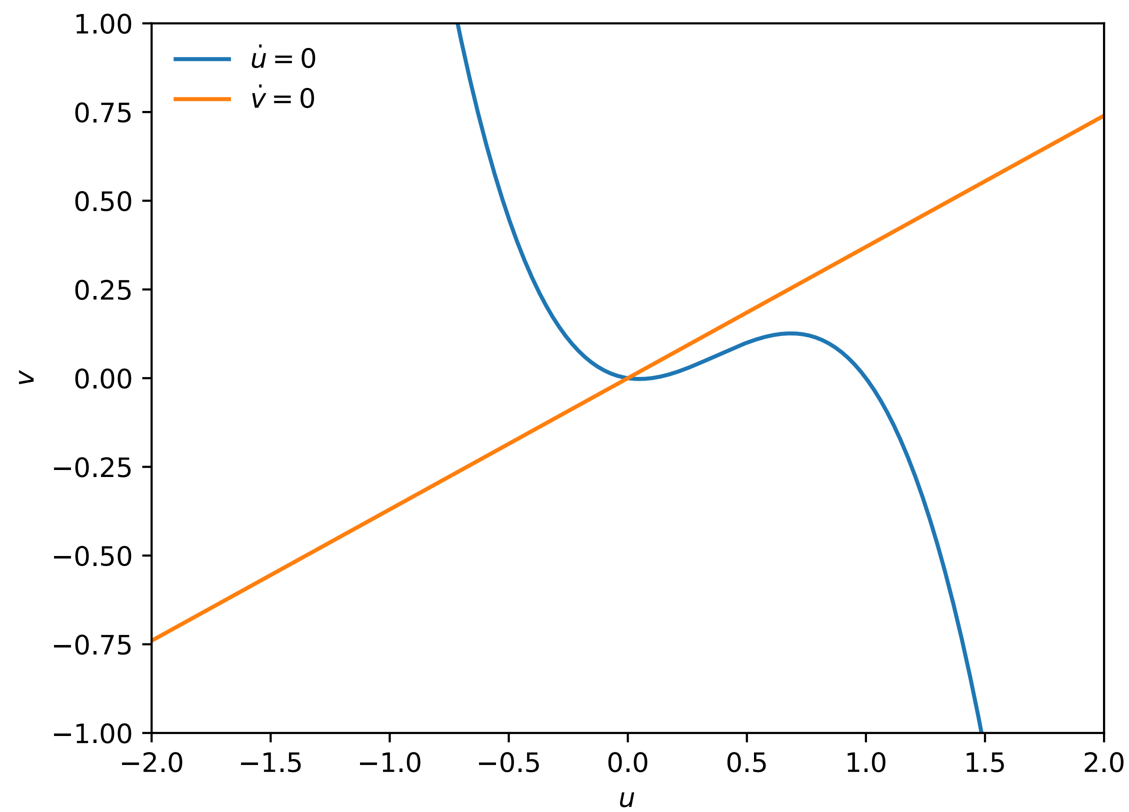

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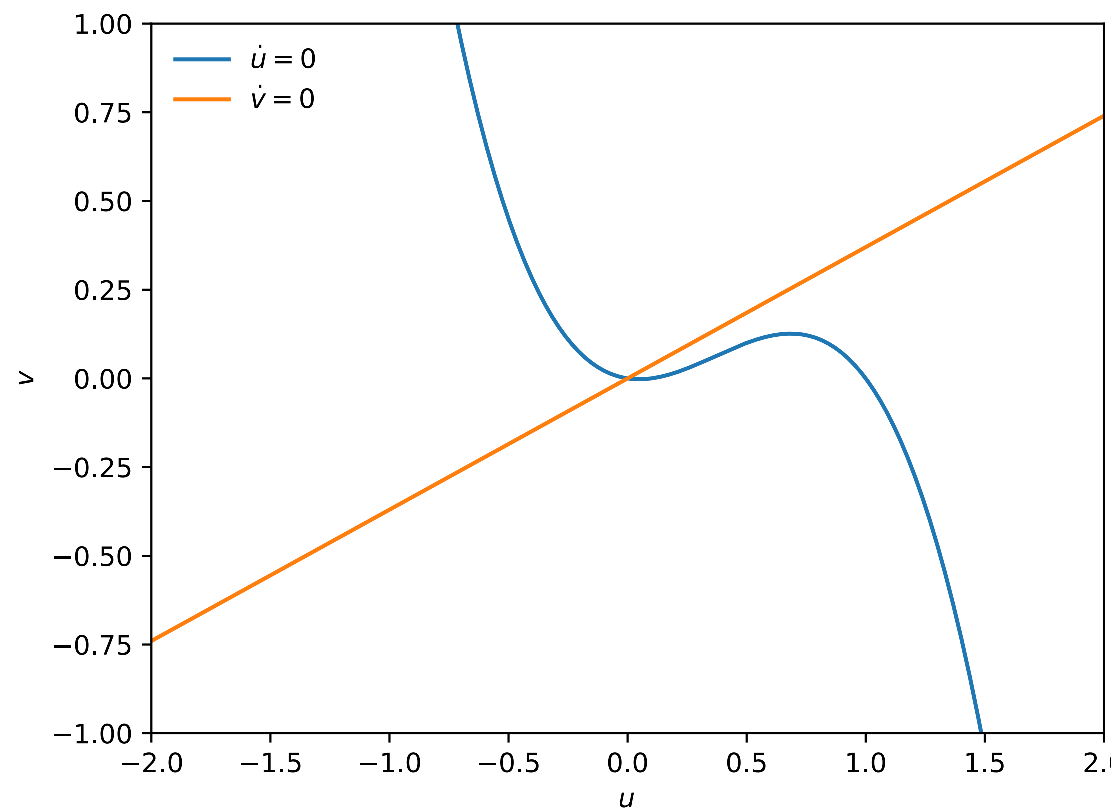

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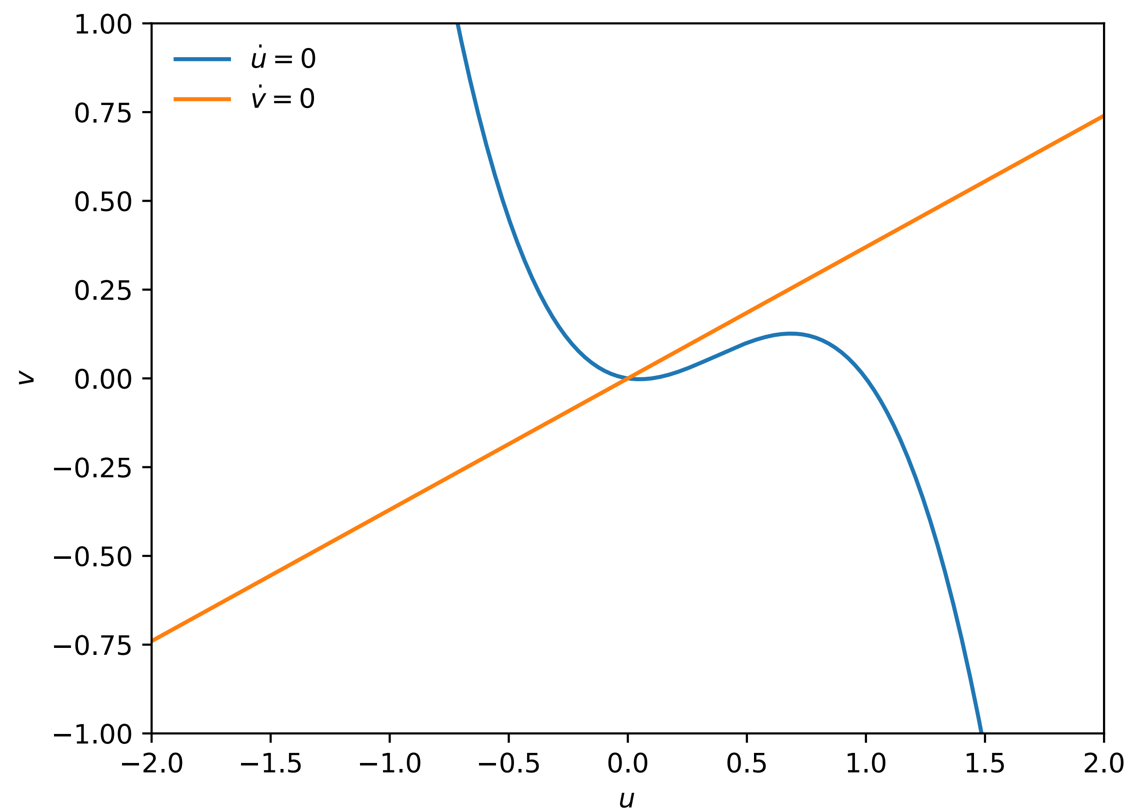

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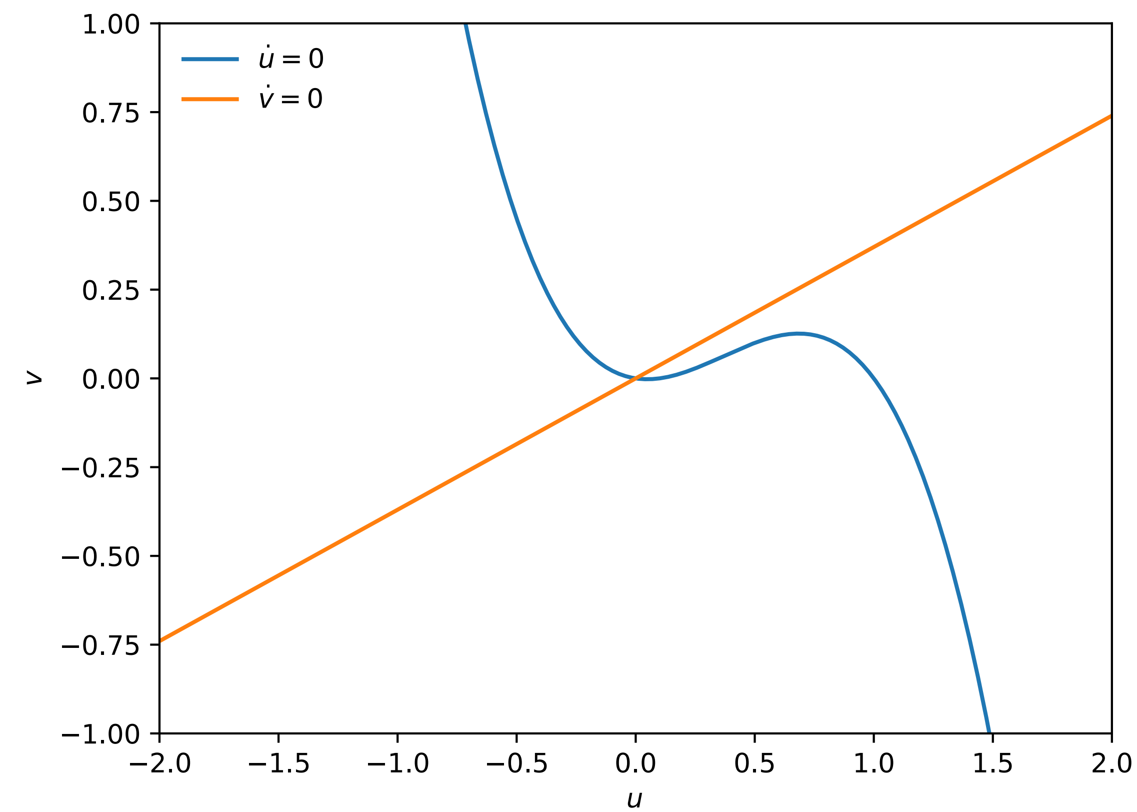

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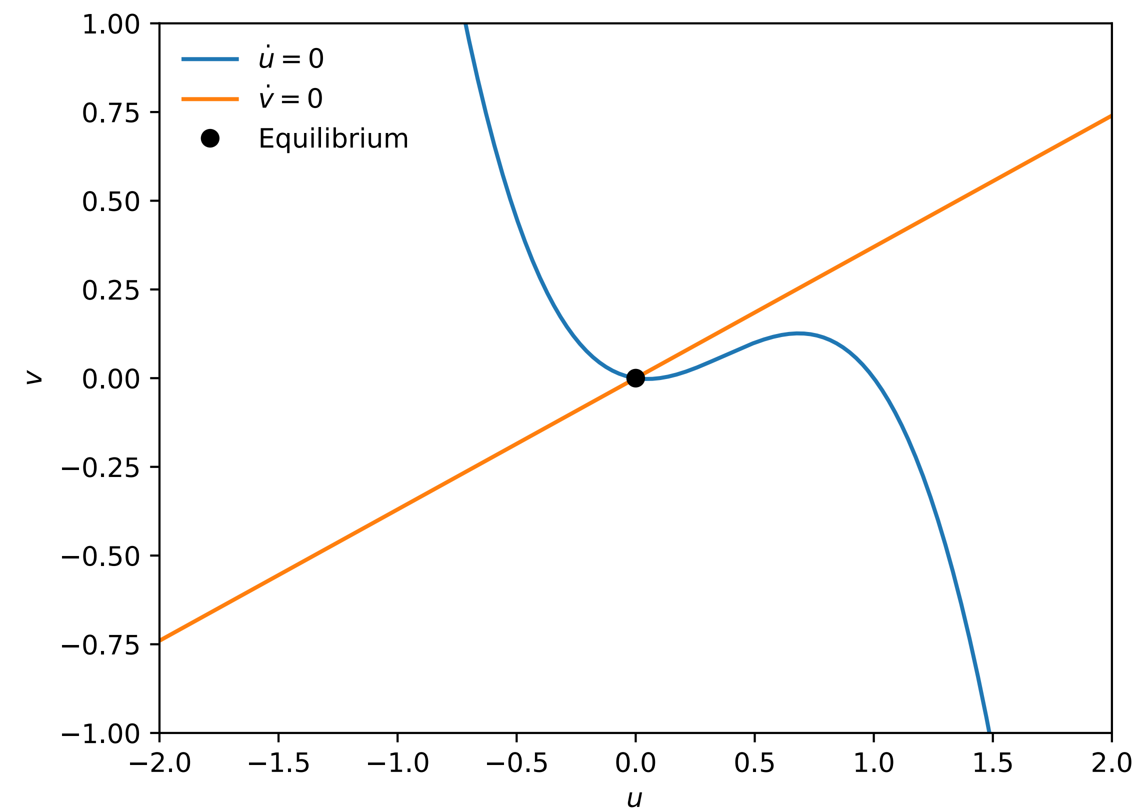

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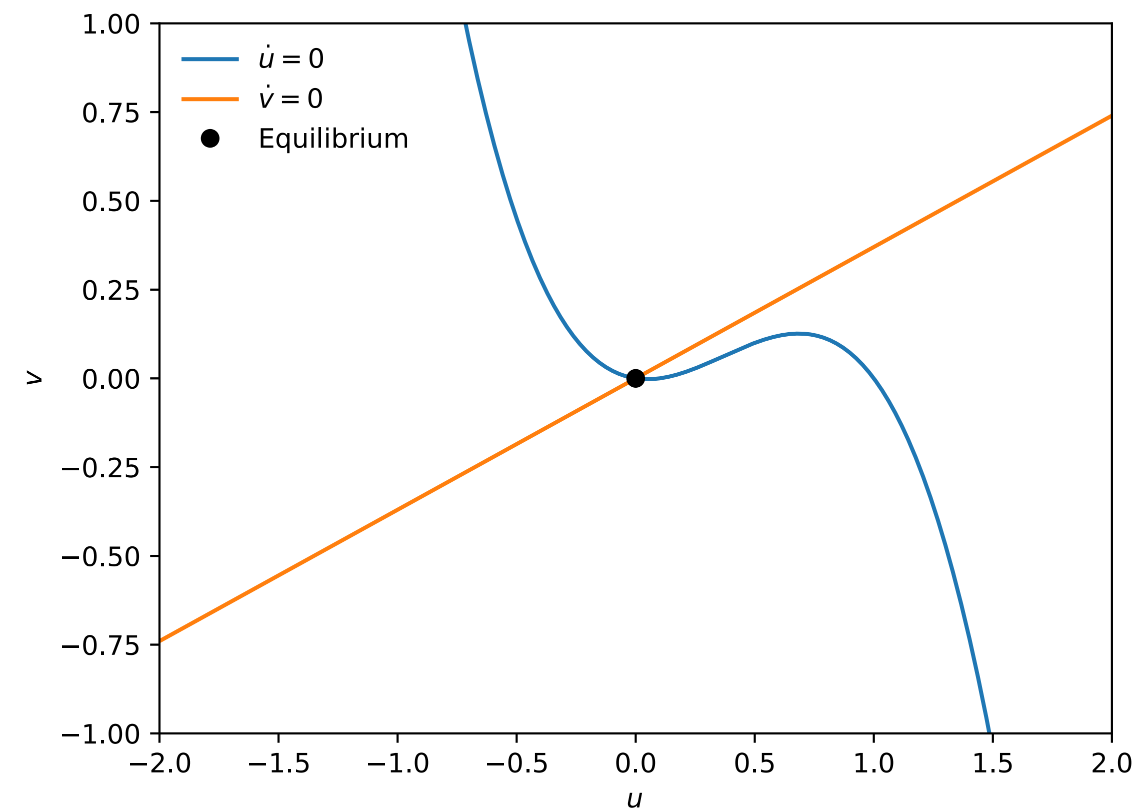

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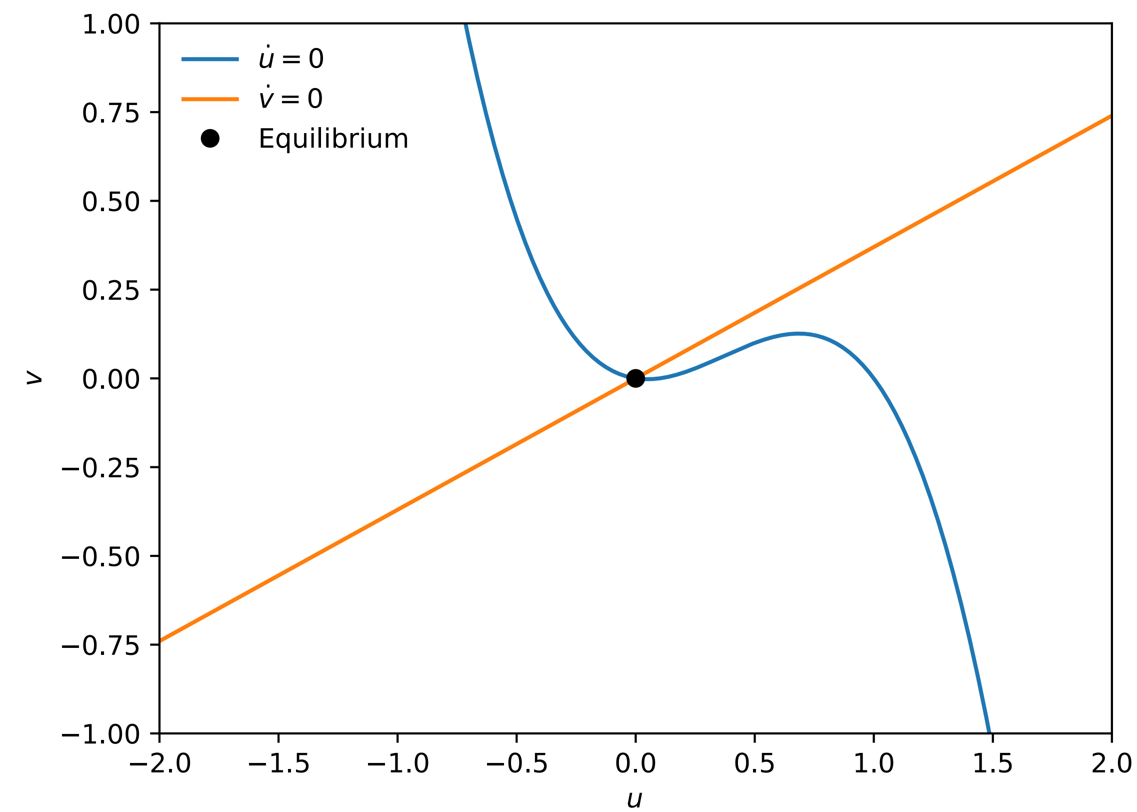
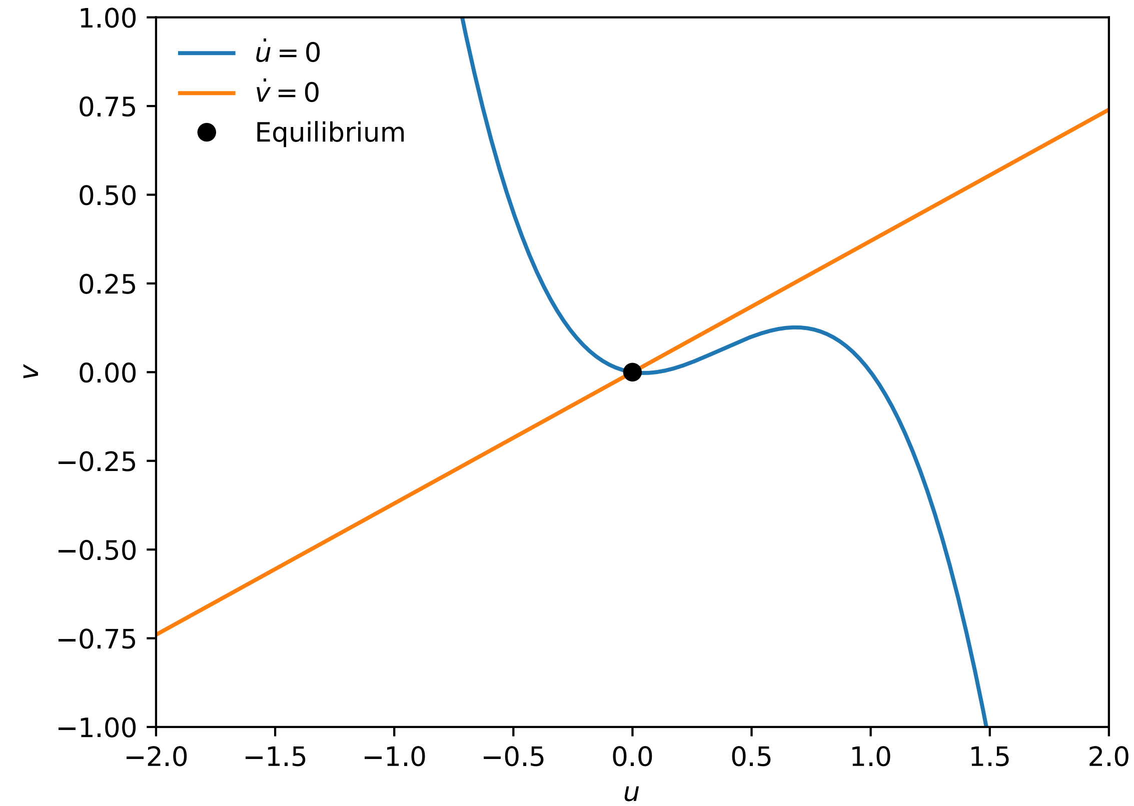


- Only one (apparent!) intersection, $(\bar{u}, \bar{v}) = (0{,}0)$
- How do we check this?
    - Plug it back into the equations and verify that $f(0{,}0) = 0$ and $\gamma(\alpha 0 - 0) = 0$

Phase Plane

Streamplots, Nullclines, and Equilibria



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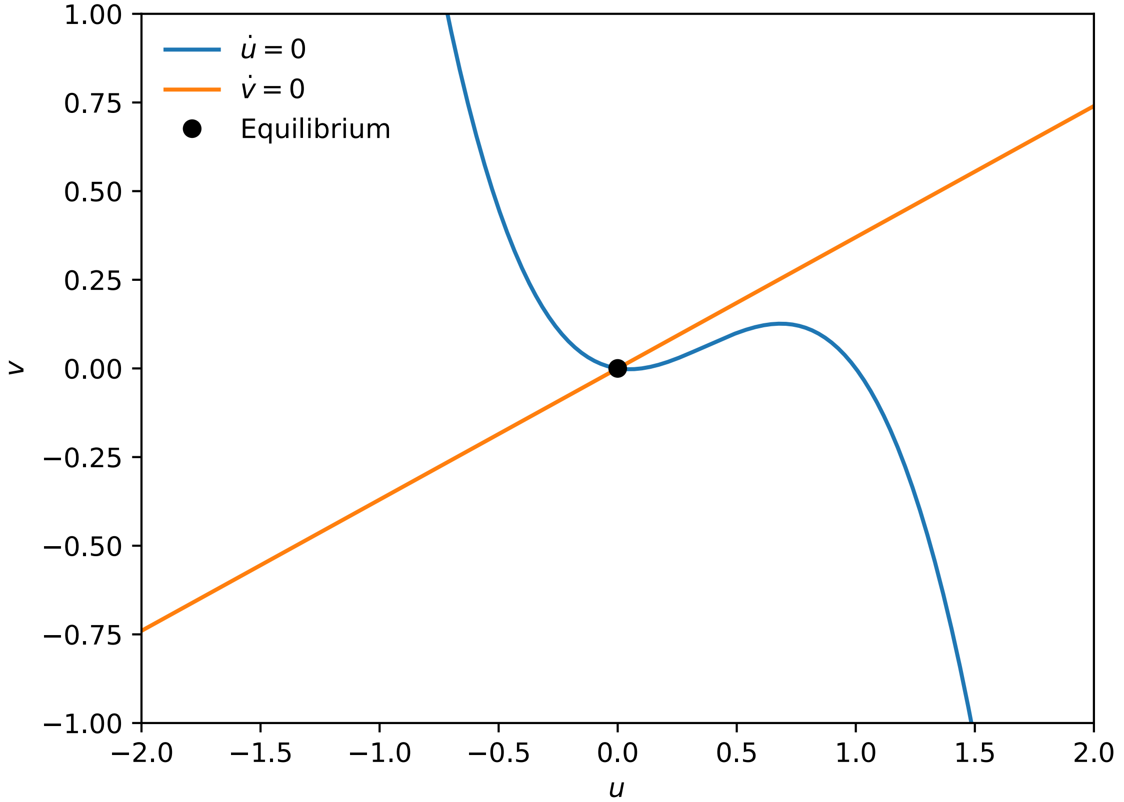

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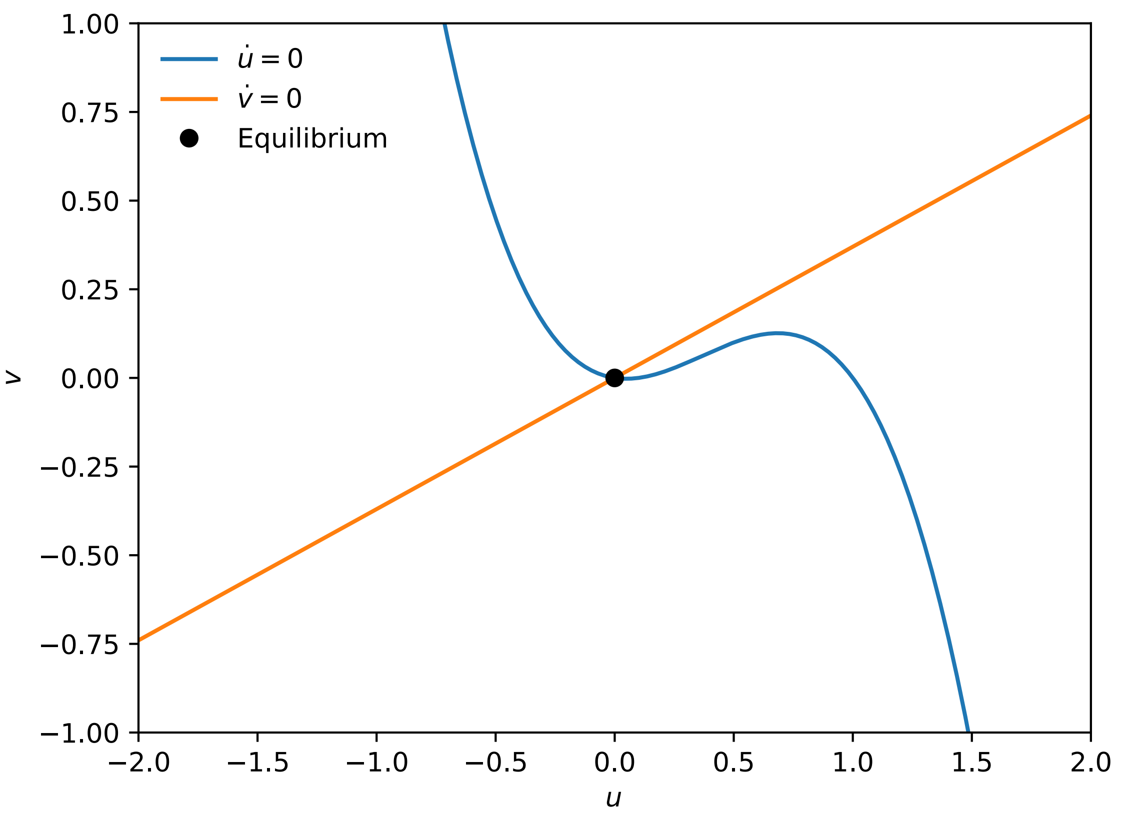

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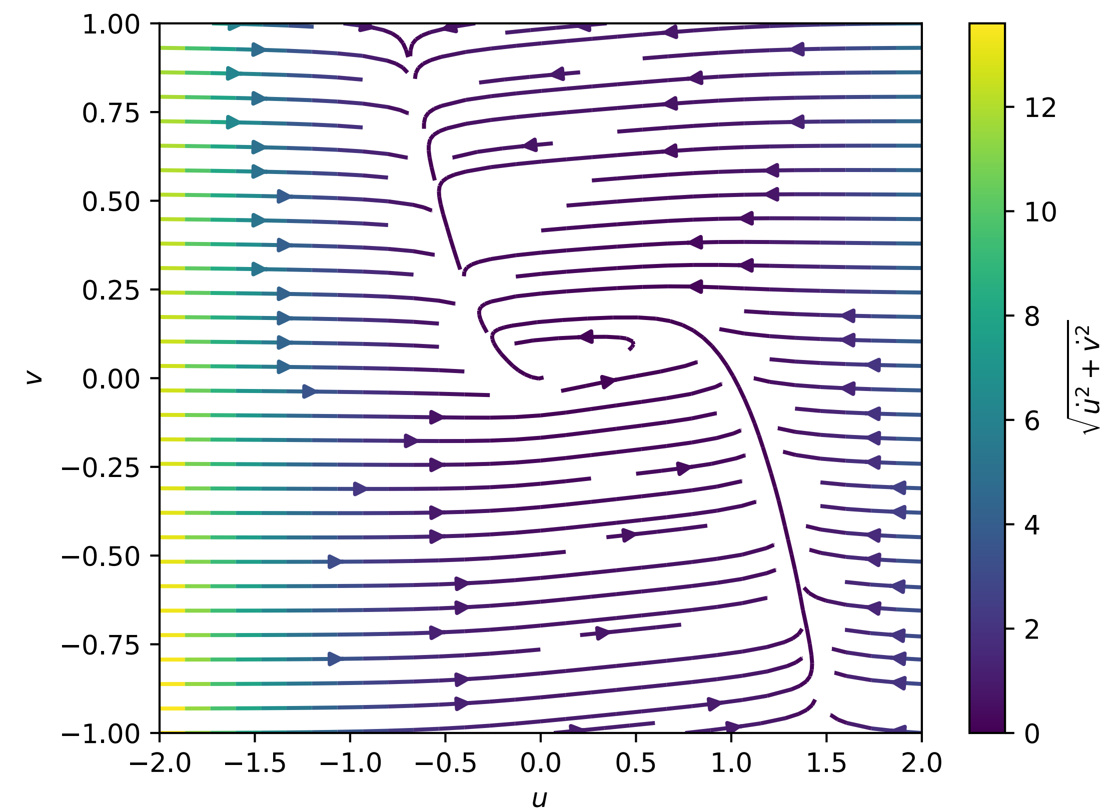

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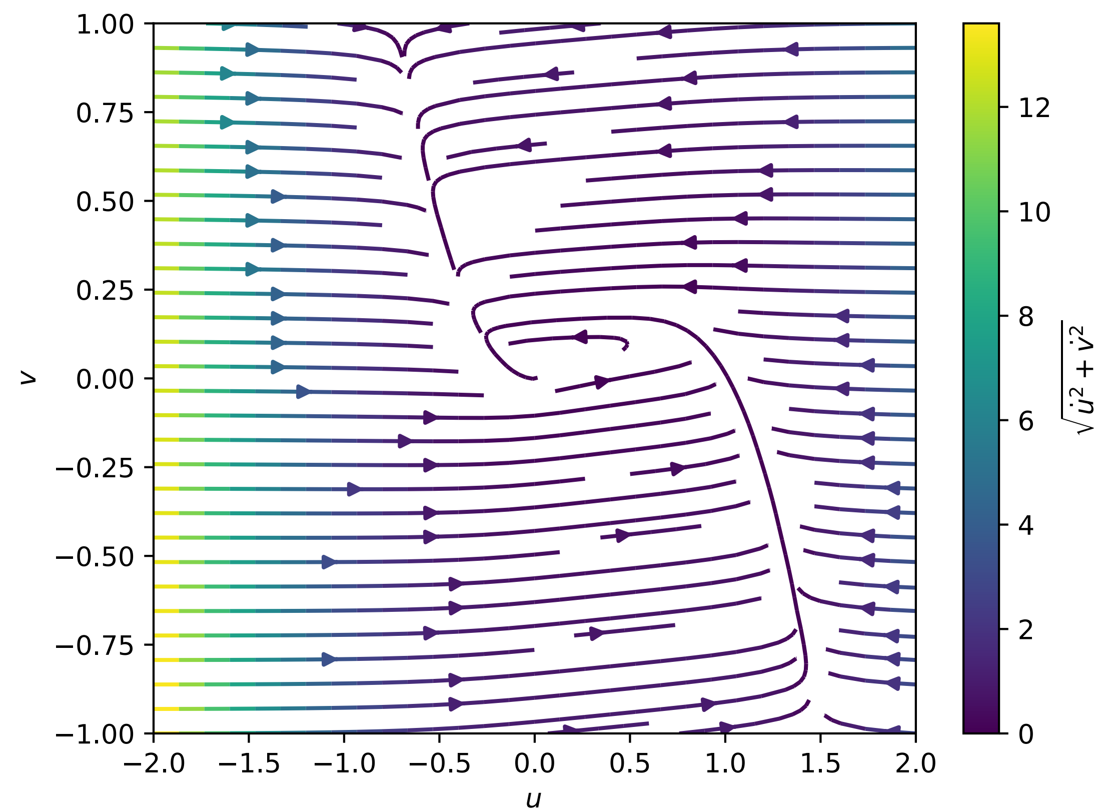

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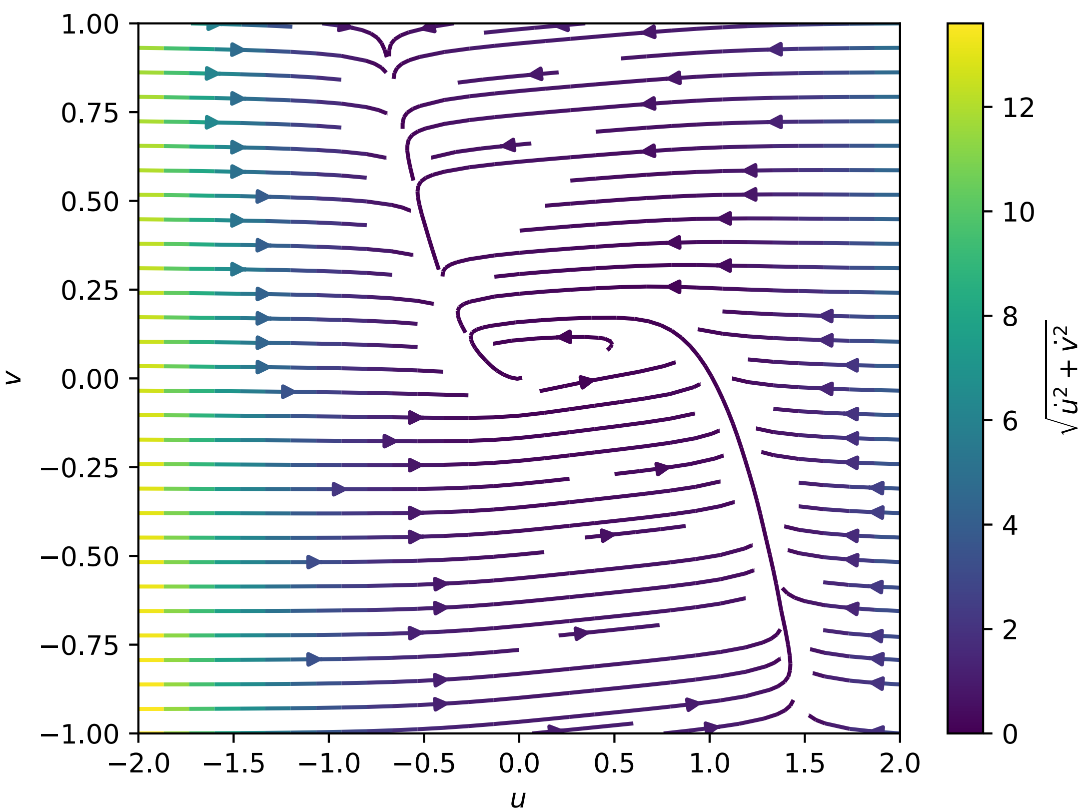

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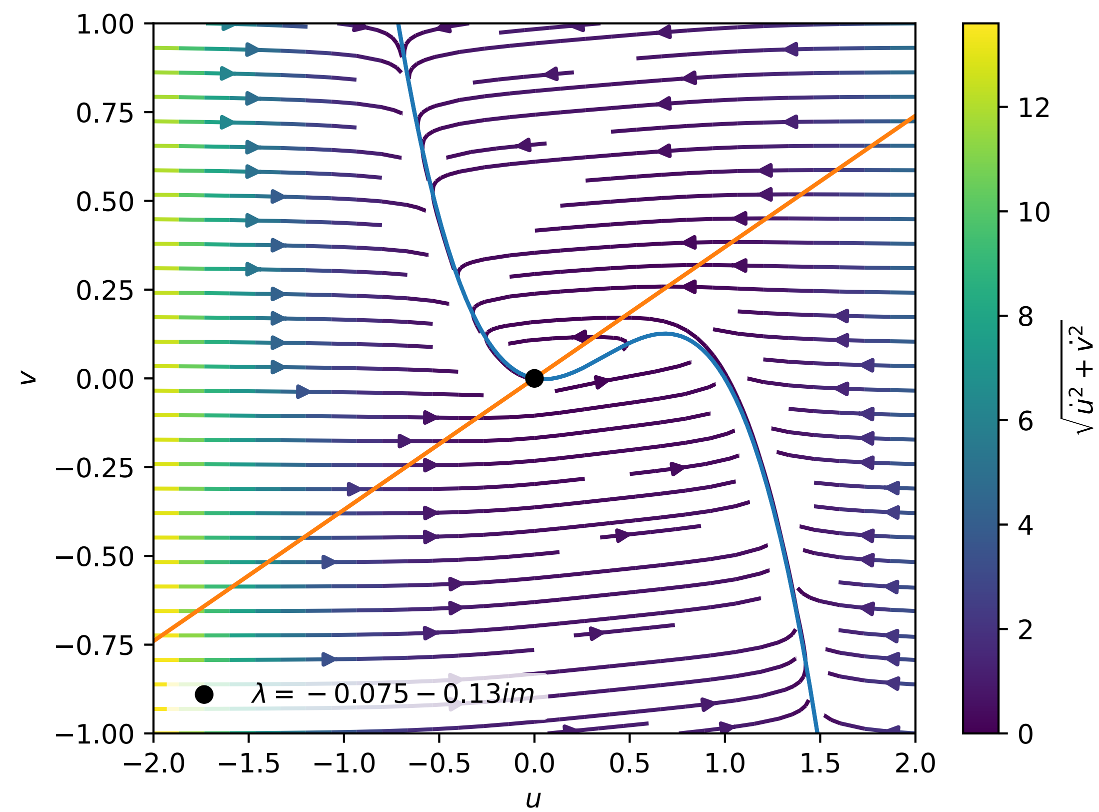

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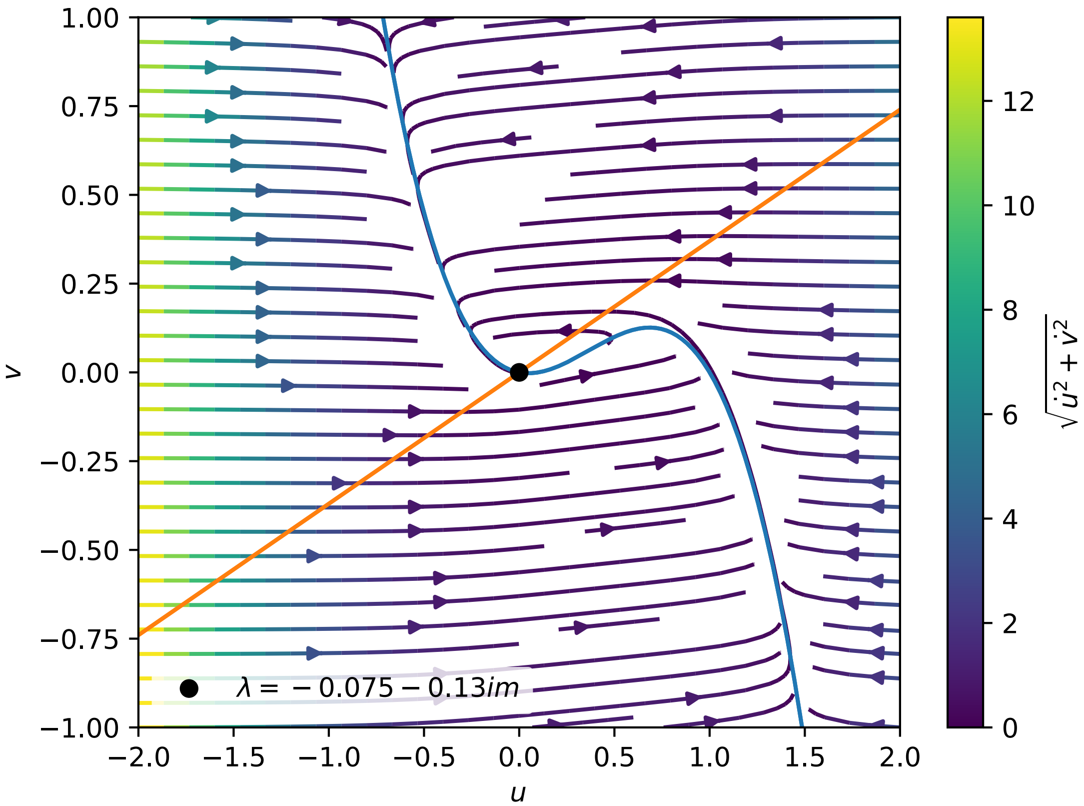

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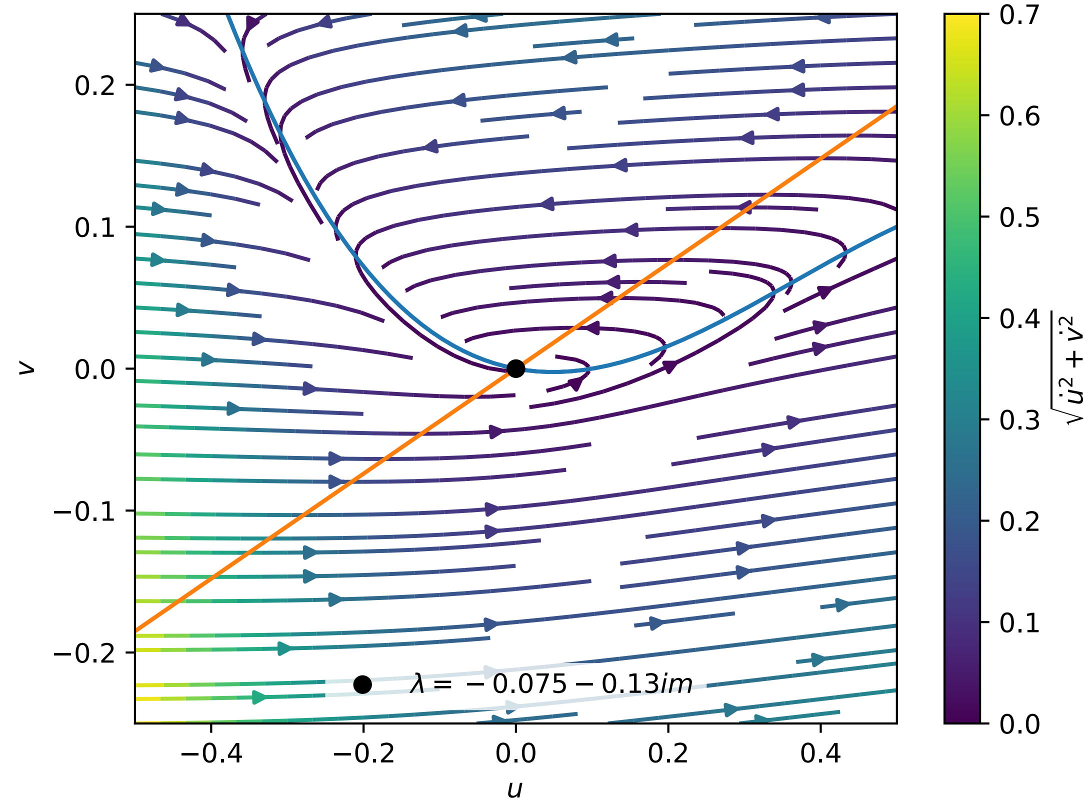

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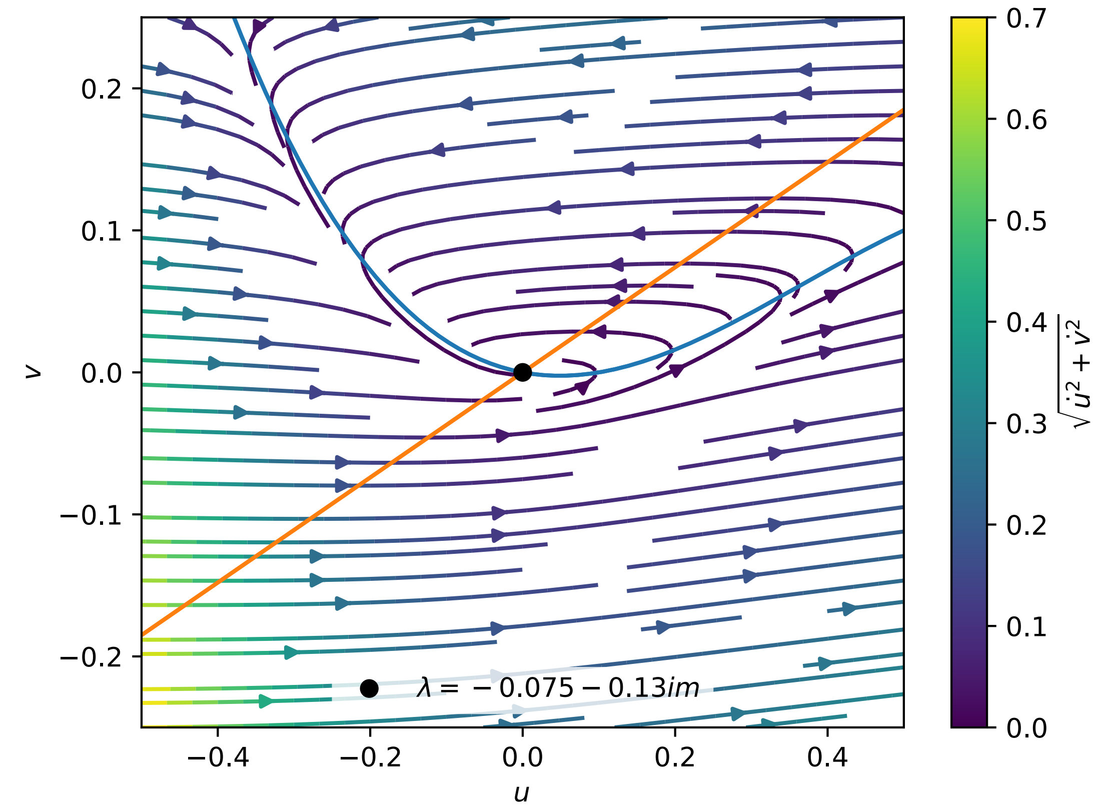

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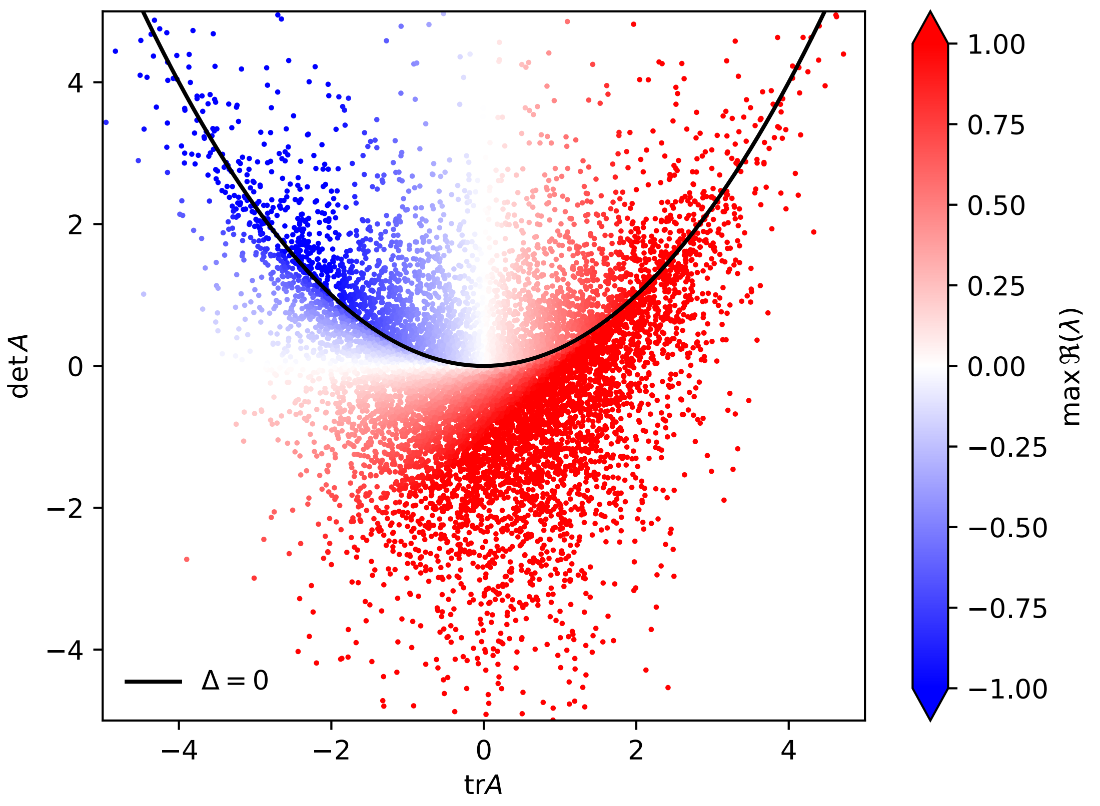

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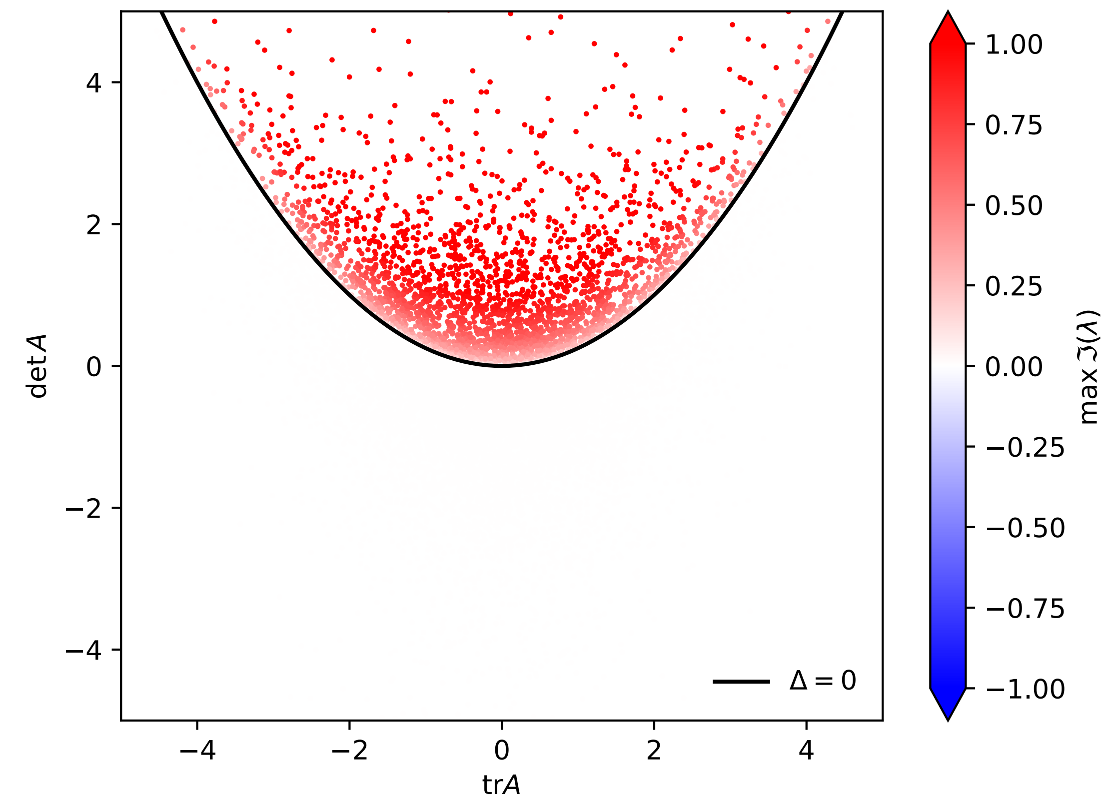
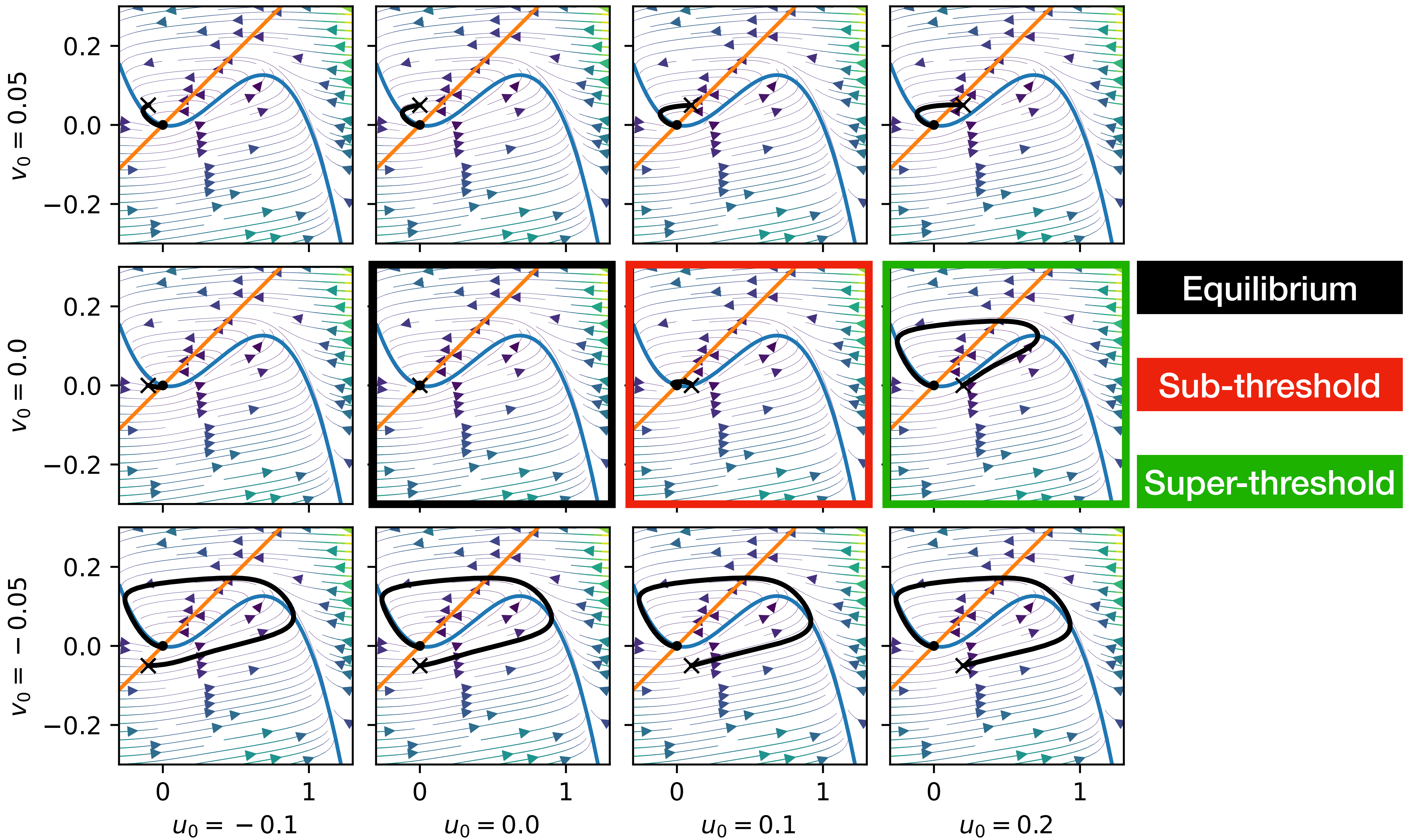


- $\Delta(A) = \mathrm{tr}(A)^2 - 4\det(A)$
  - Oscillations: $\Delta(A) < 0$
- For general systems, $A = \mathrm{d}\dot{\mathbf{x}}/\mathrm{d}\mathbf{x}$ evaluated at $\mathbf{x} : \dot{\mathbf{x}} = \mathbf{0}$

Where is the excitability?

What makes FHN an archetypal model



Bifurcations

Qualitative changes of the dynamics

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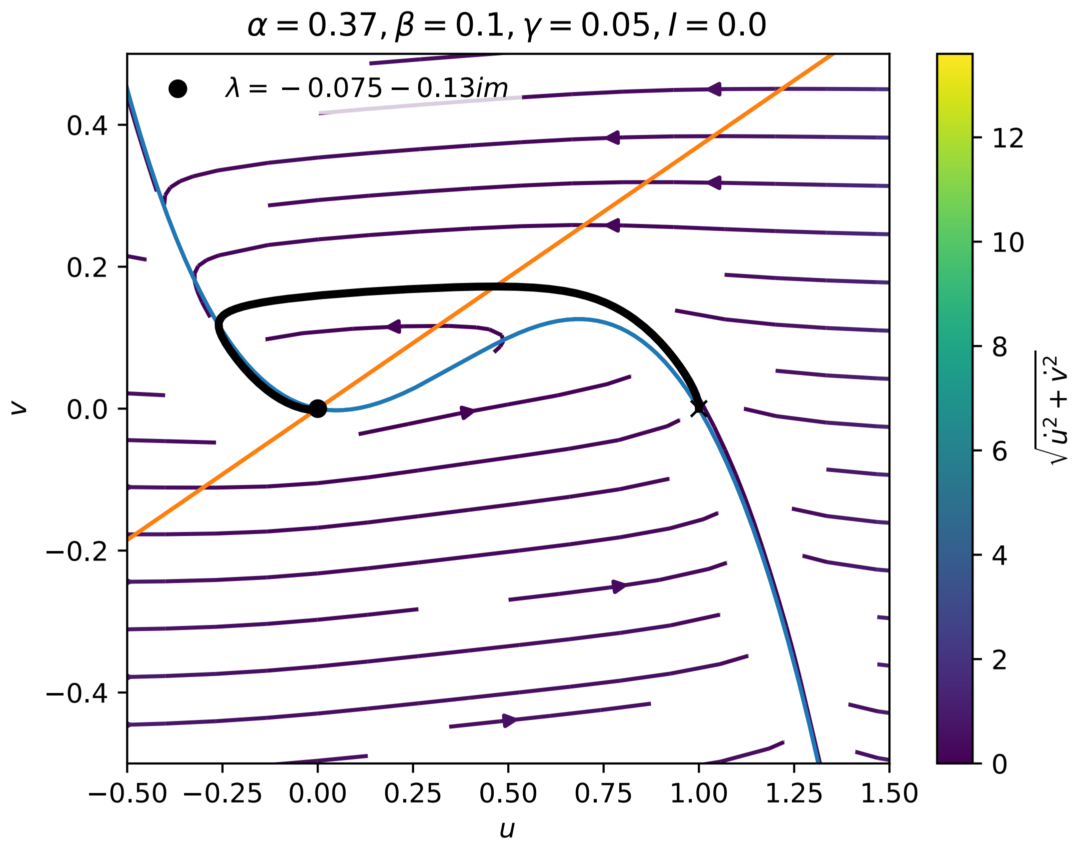

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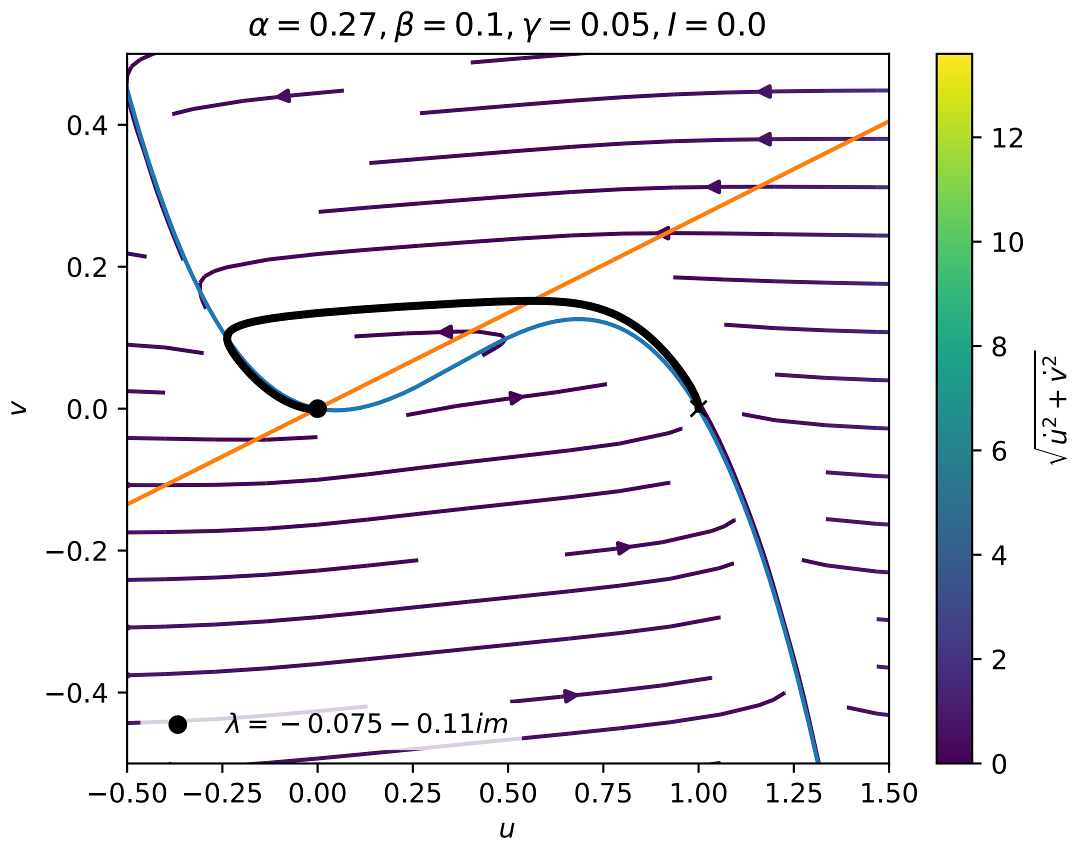

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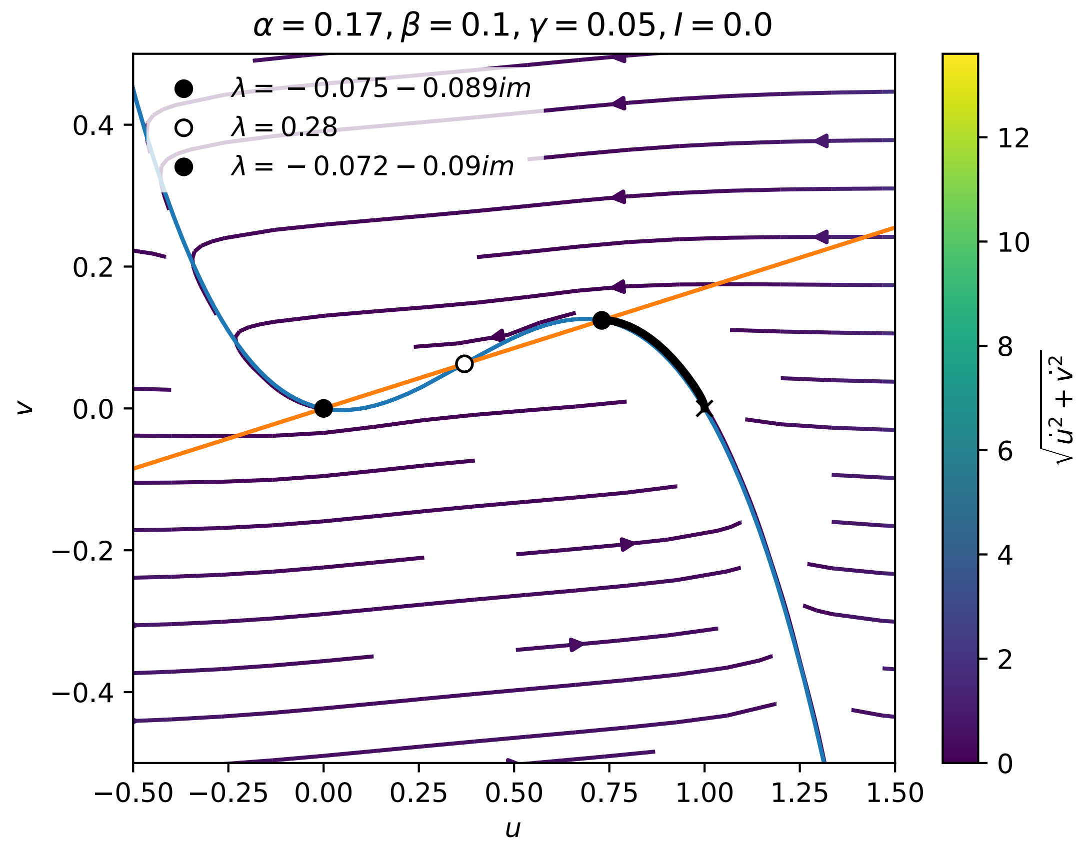

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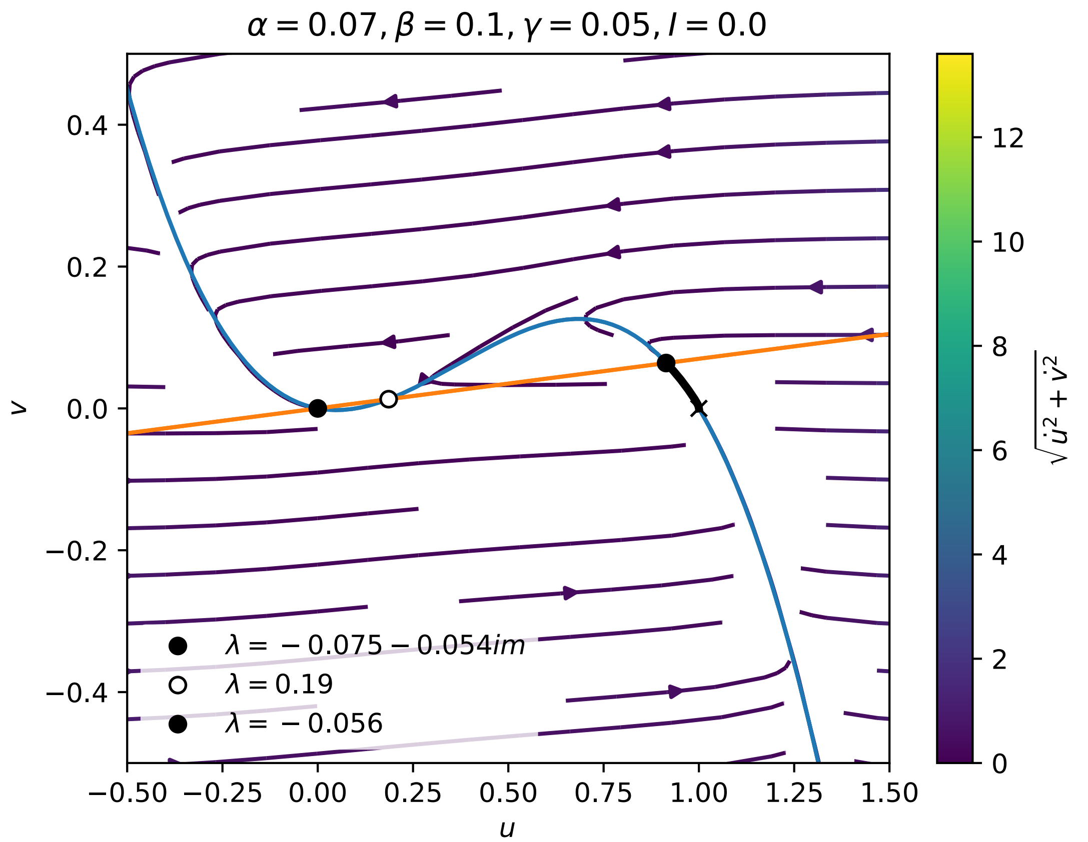

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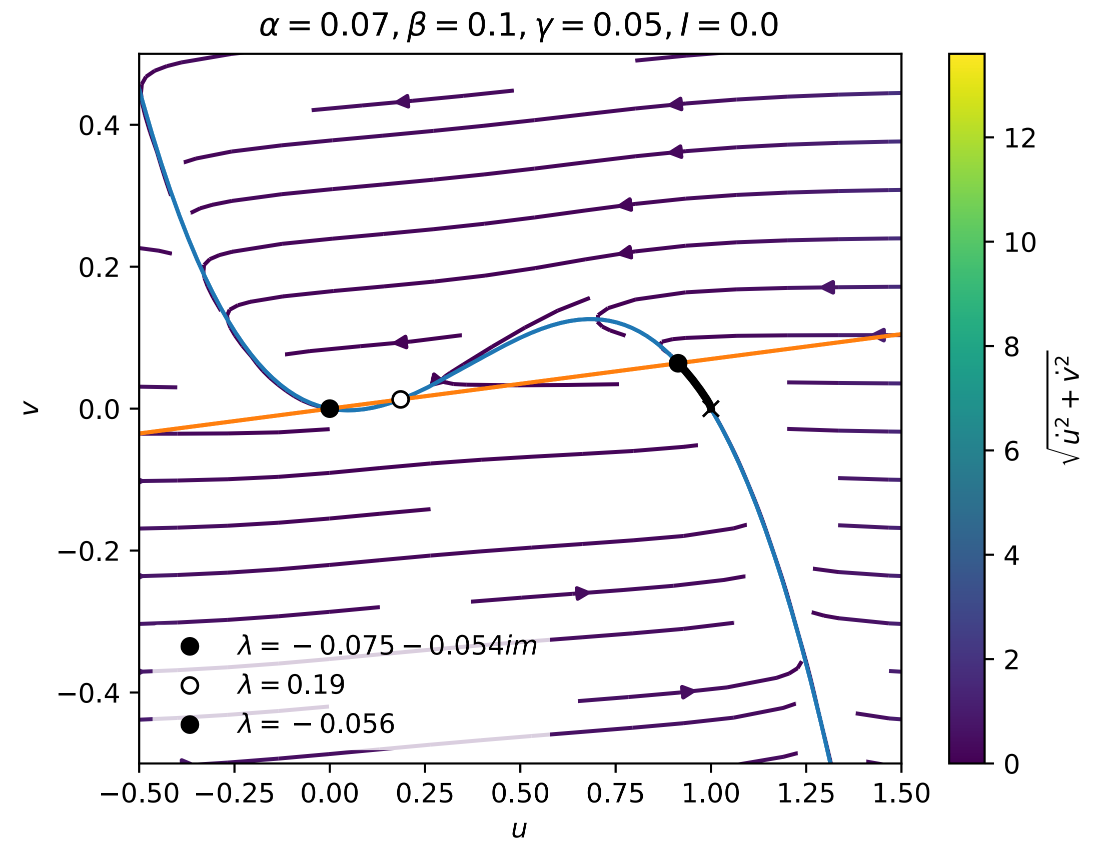

Bifurcations

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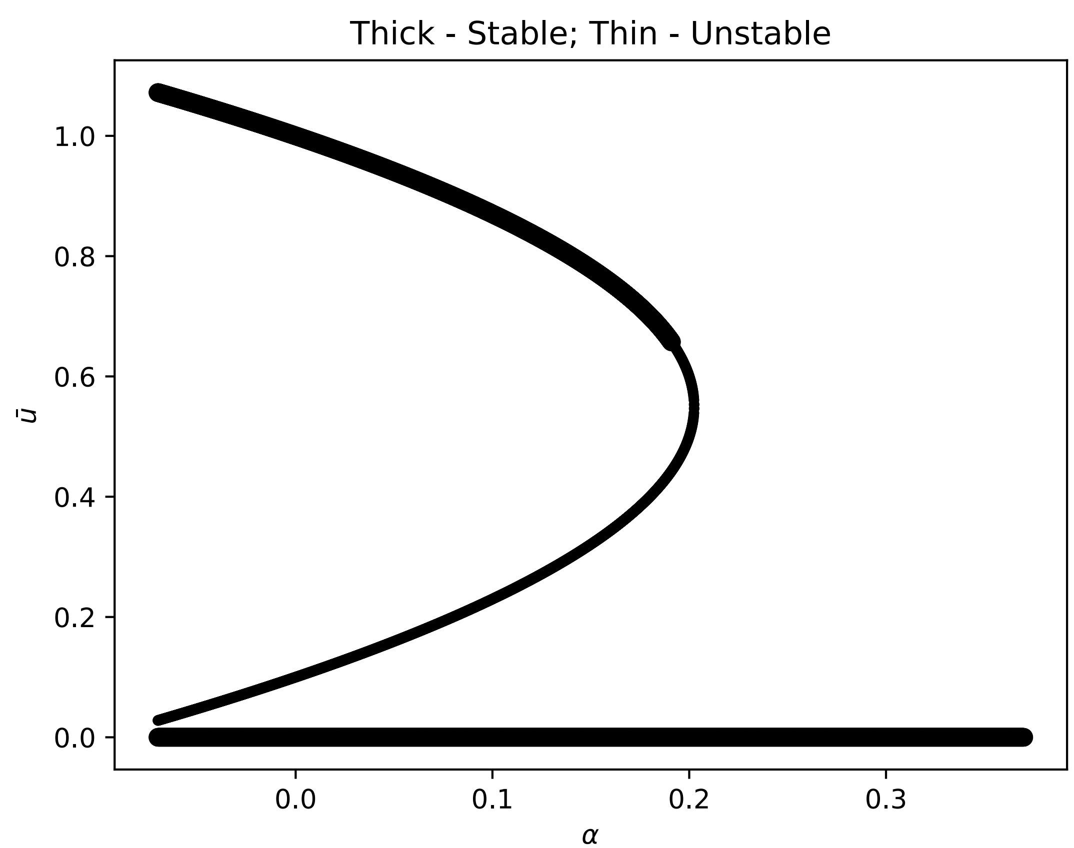
- Bifurcation diagrams are an important part of understanding dynamical systems with parameters



- "Saddle-node" bifurcation
    - New solutions appear out of collision of a saddle (unstable) and node (stable)
- Normal form: $\dot{x} = r + x^2$ in 1D, harder in 2D
- Verify using Taylor expansion around bif. pt.

Changing Parameter values

Bifurcations and qualitative changes to the dynamics

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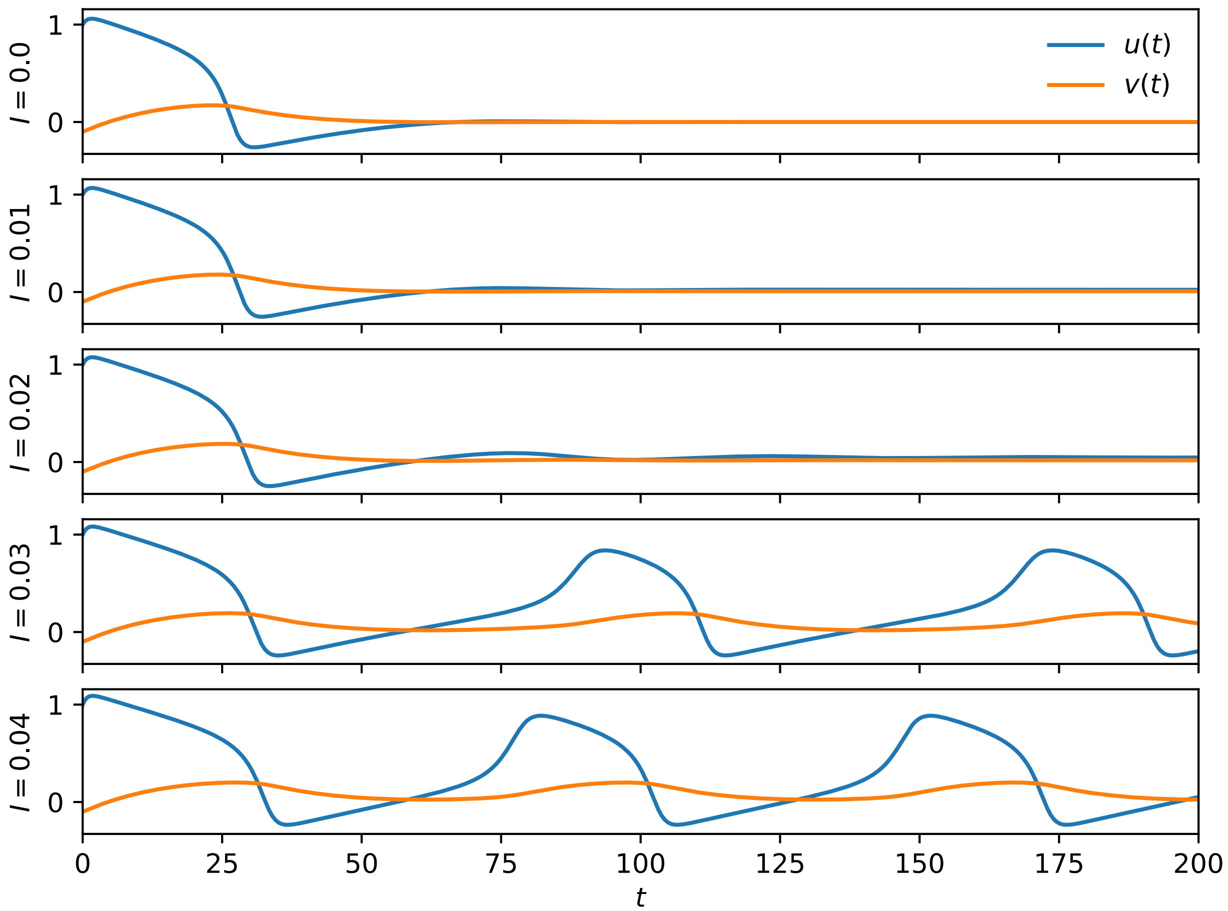
- For $I < 0$, constant outflowing current lowers equilibrium u-value
- For $I > 0$, constant inflowing current raises equilibrium u-value
- But for sufficiently large $I > 0$, something interesting happens

Excitable to Oscillatory

Increasing stimulation current



- For sufficiently high input current $I \gtrsim 0.03$, solution spontaneously oscillates
  - Why do the dynamics change?

- The equilibrium stability changes at $I \approx 0.0298$
  - This is called a Hopf bifurcation, and occurs with the creation of a periodic orbit or limit cycle

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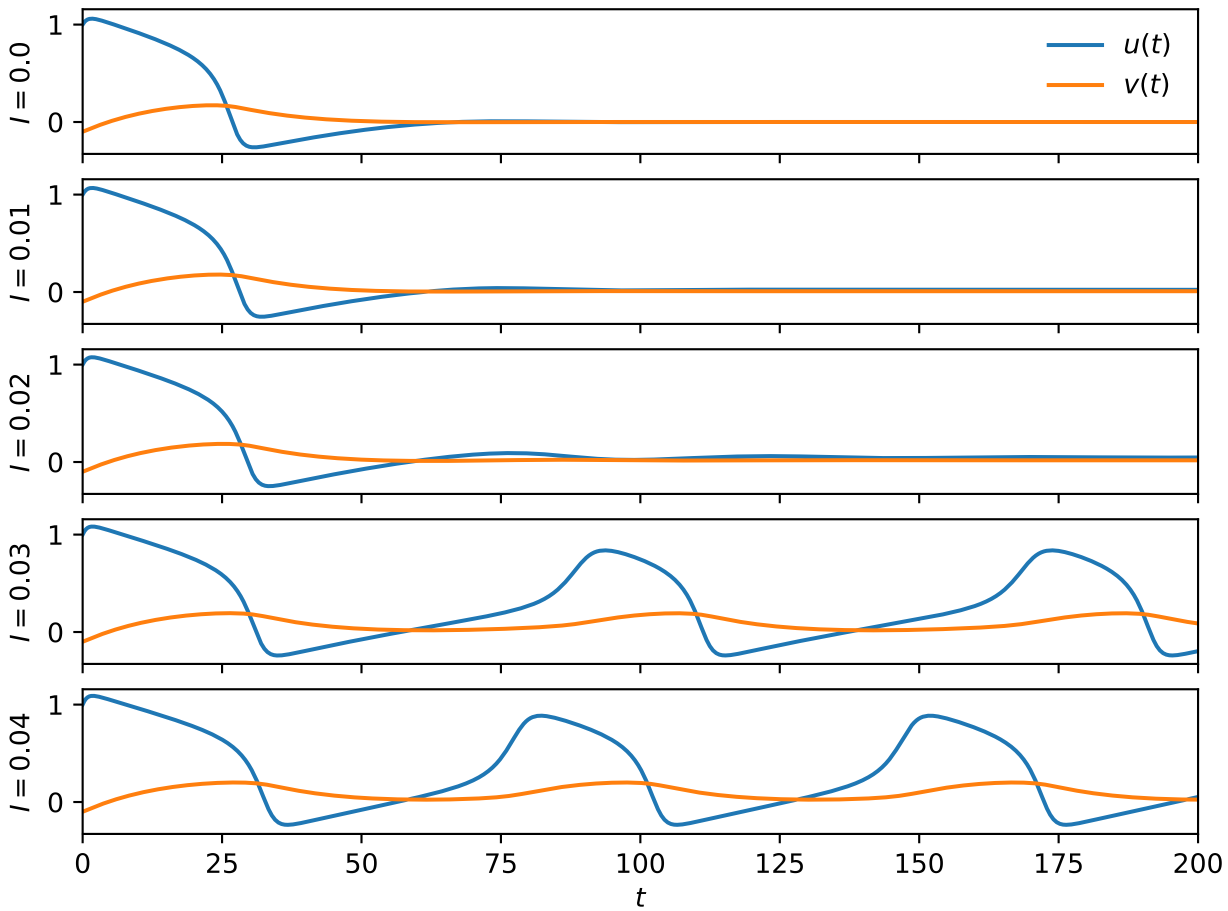

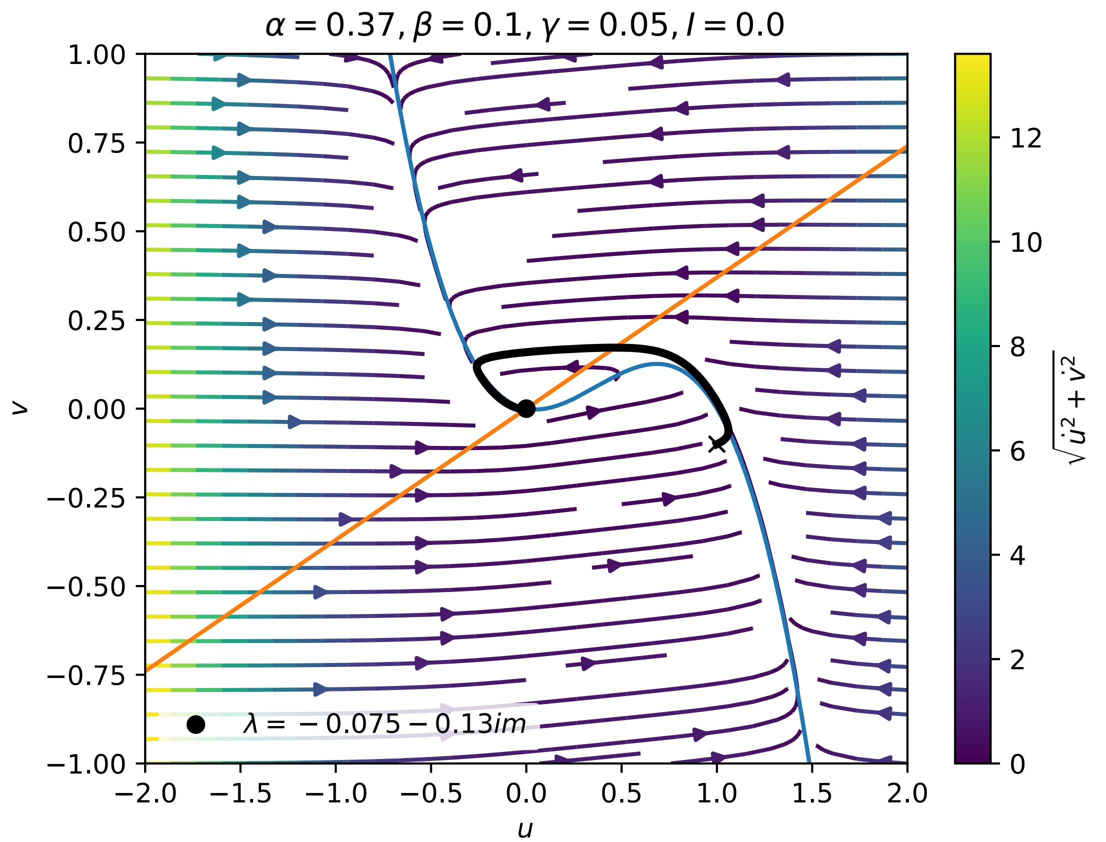

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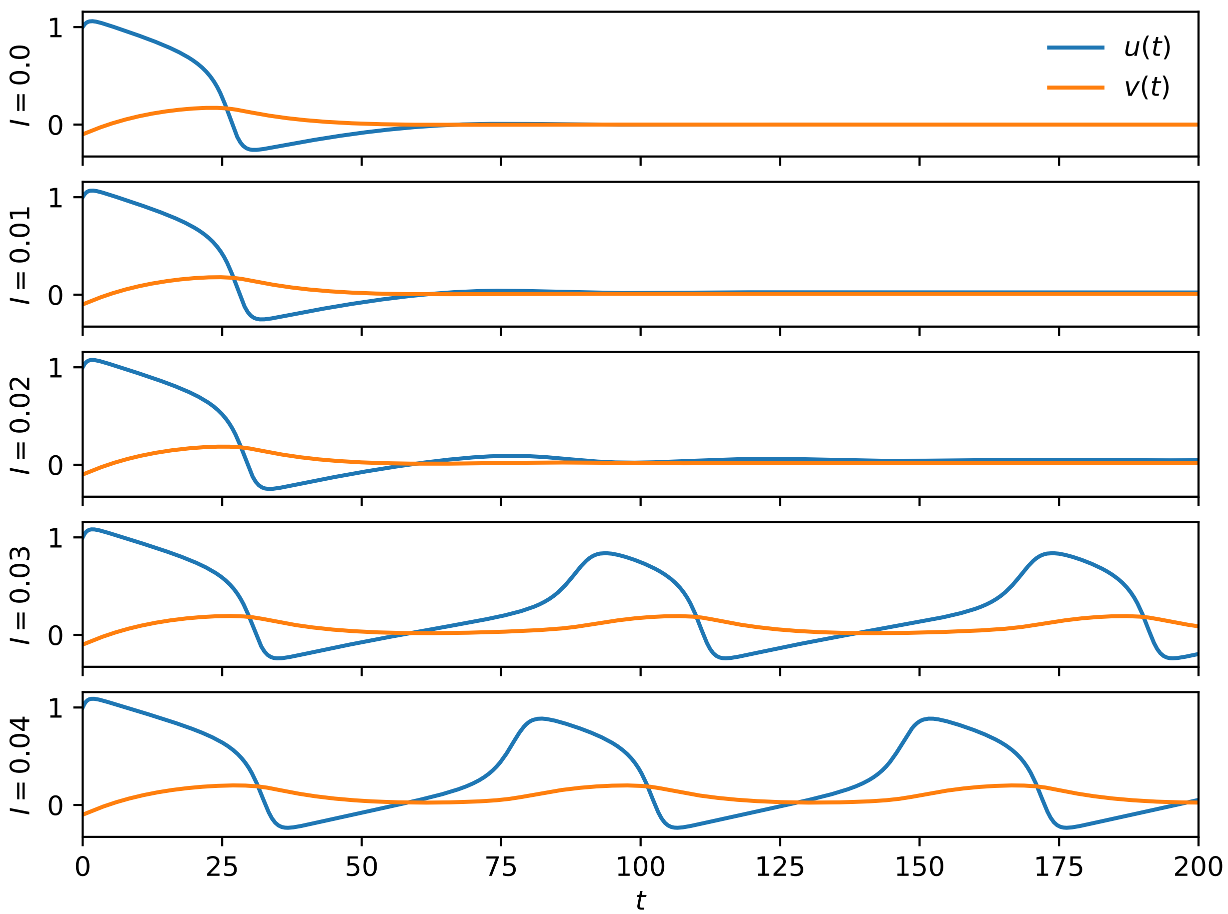

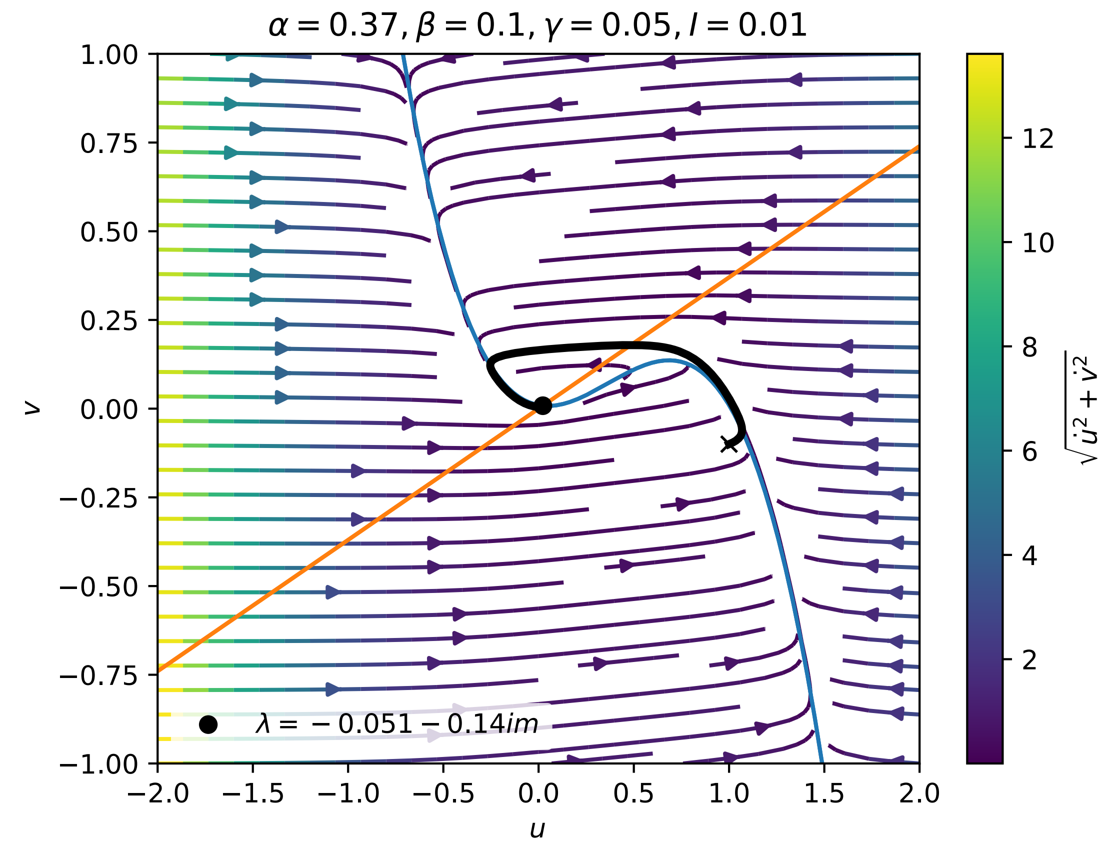

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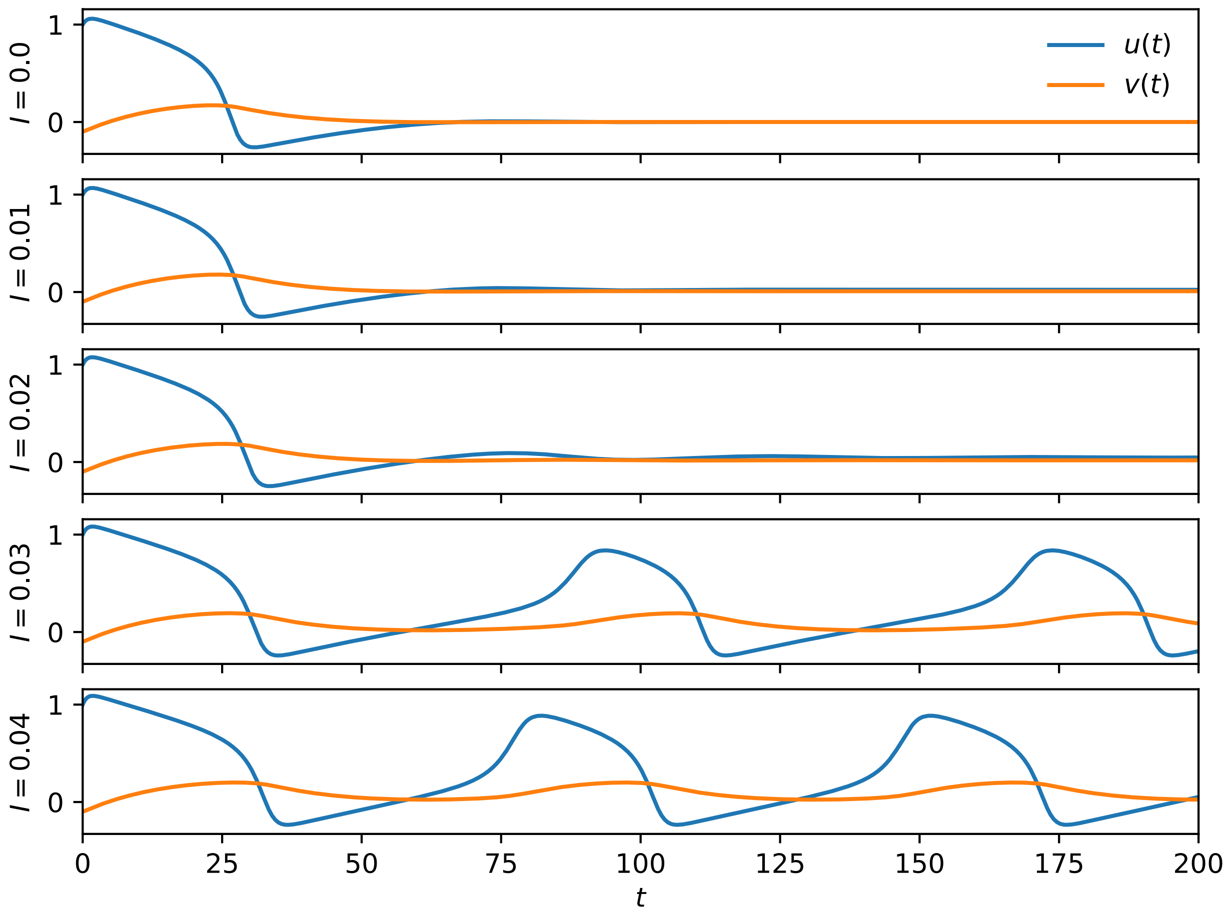

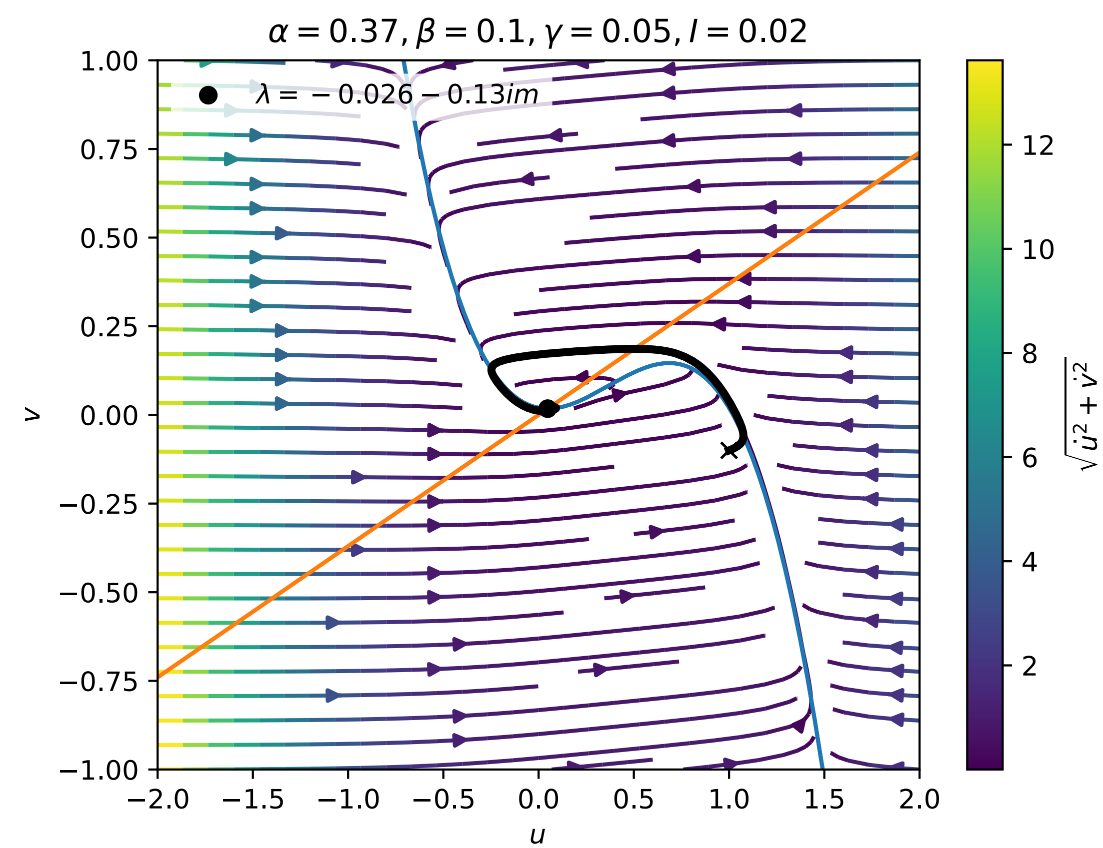

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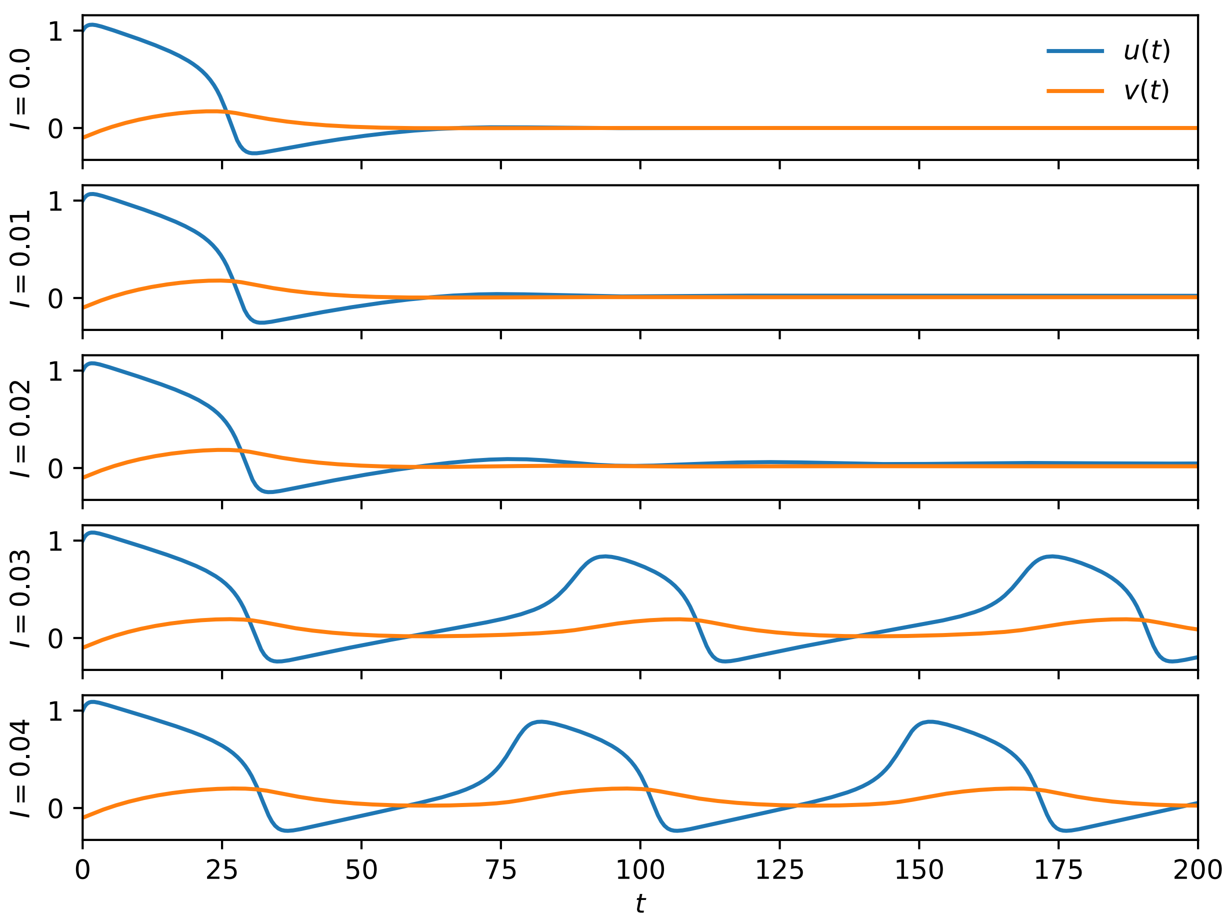

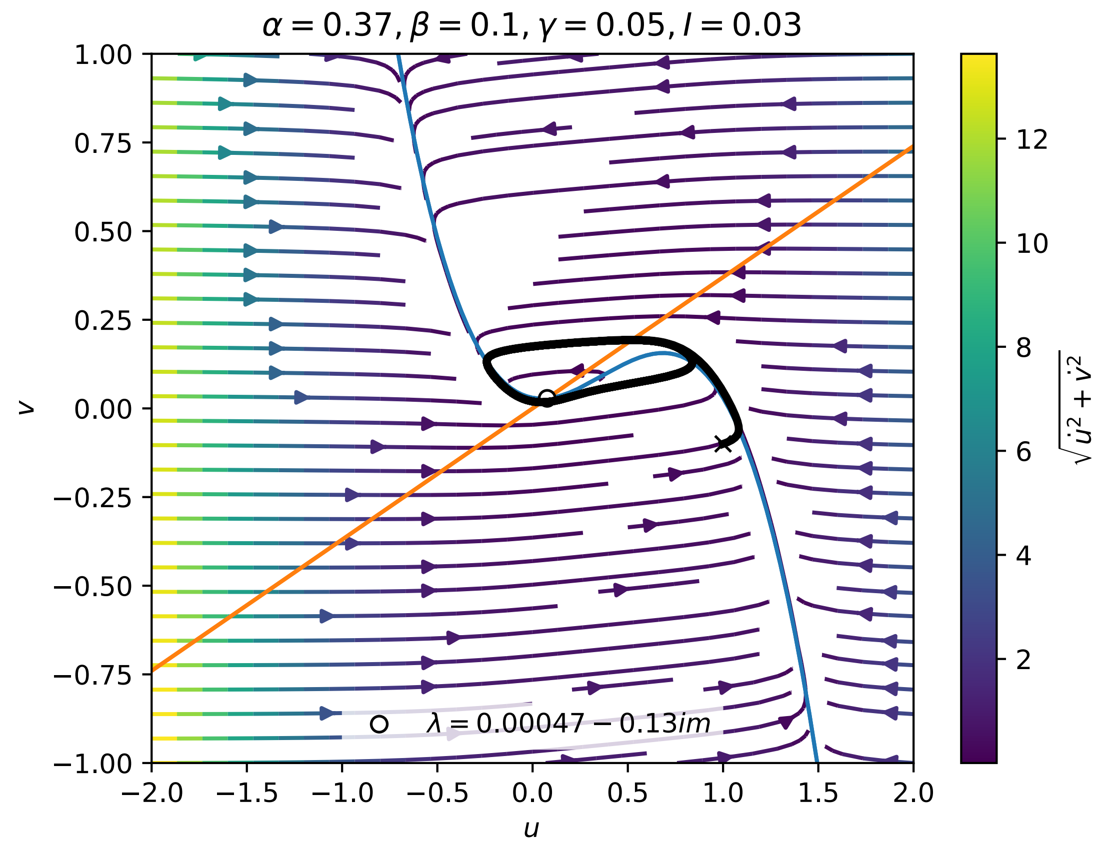

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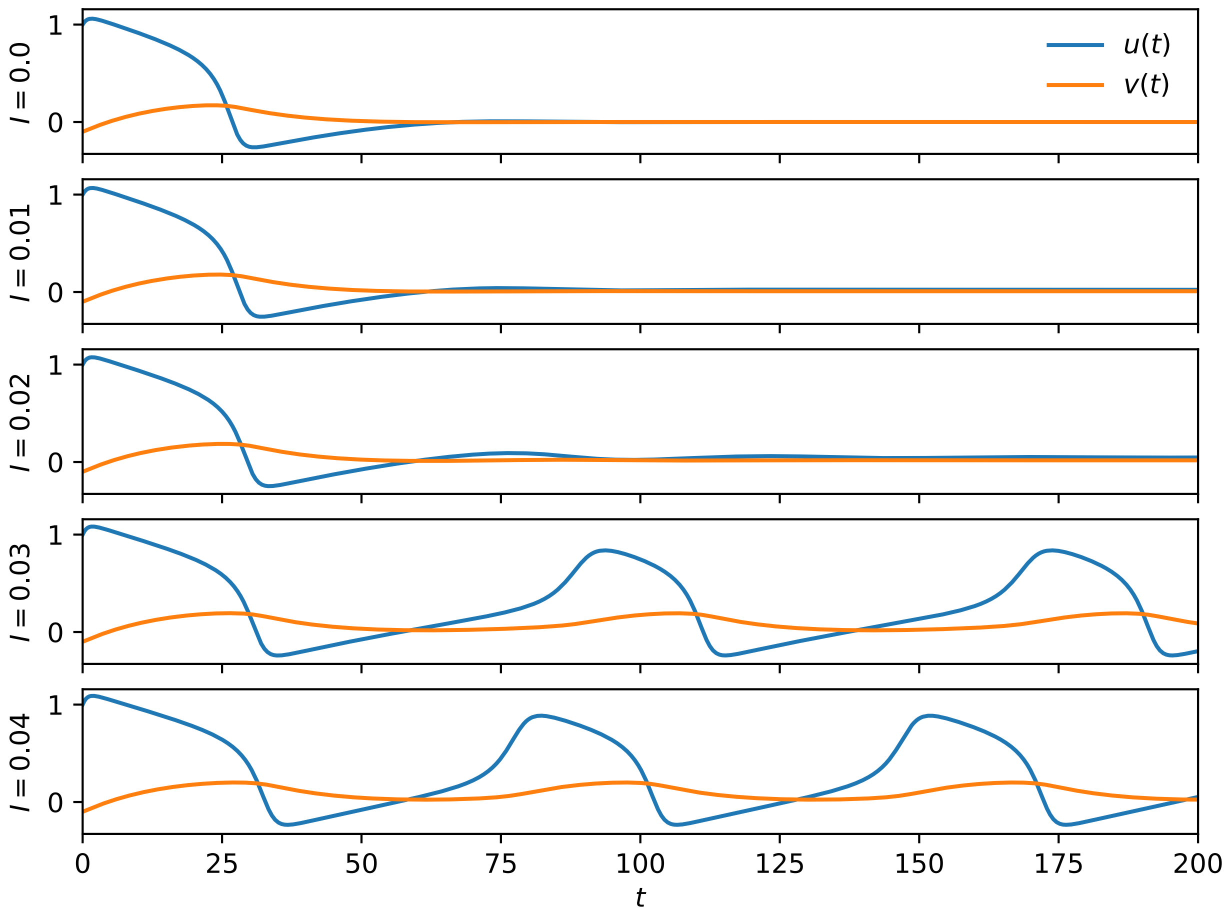

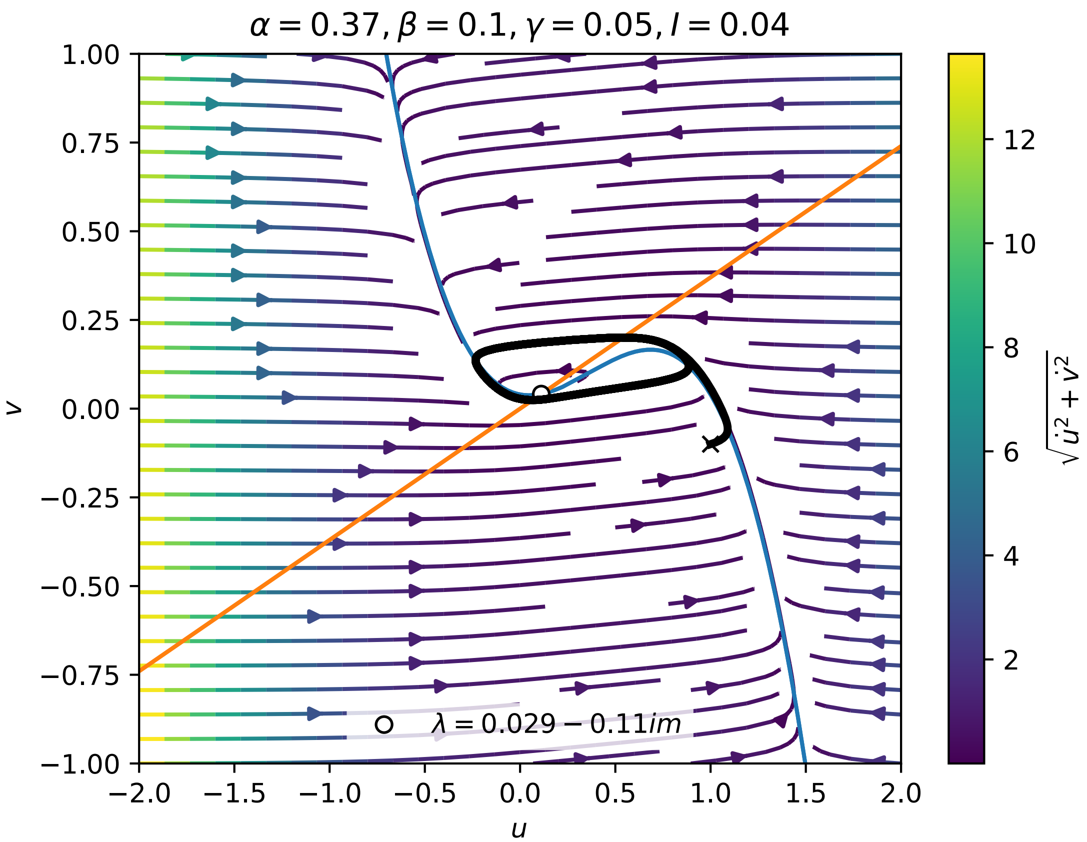

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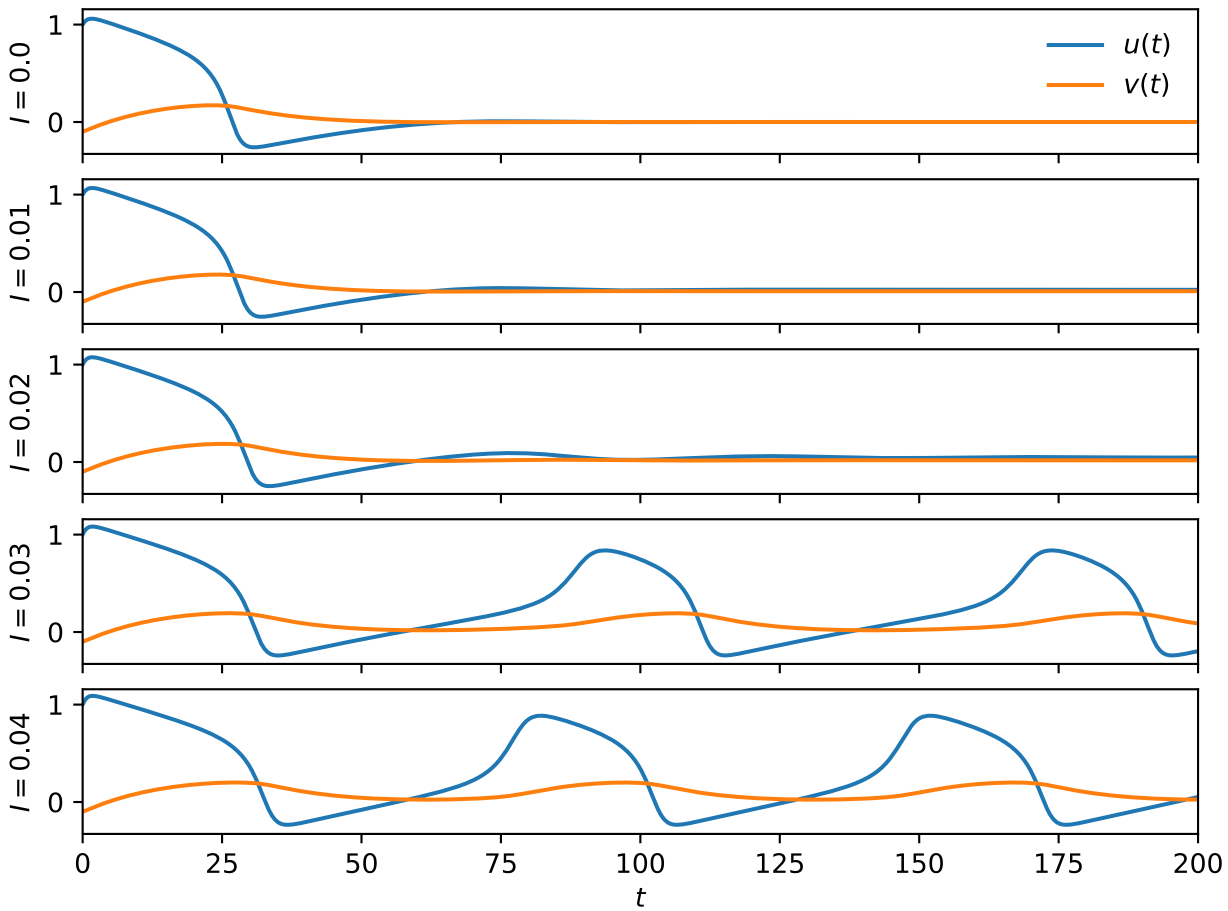

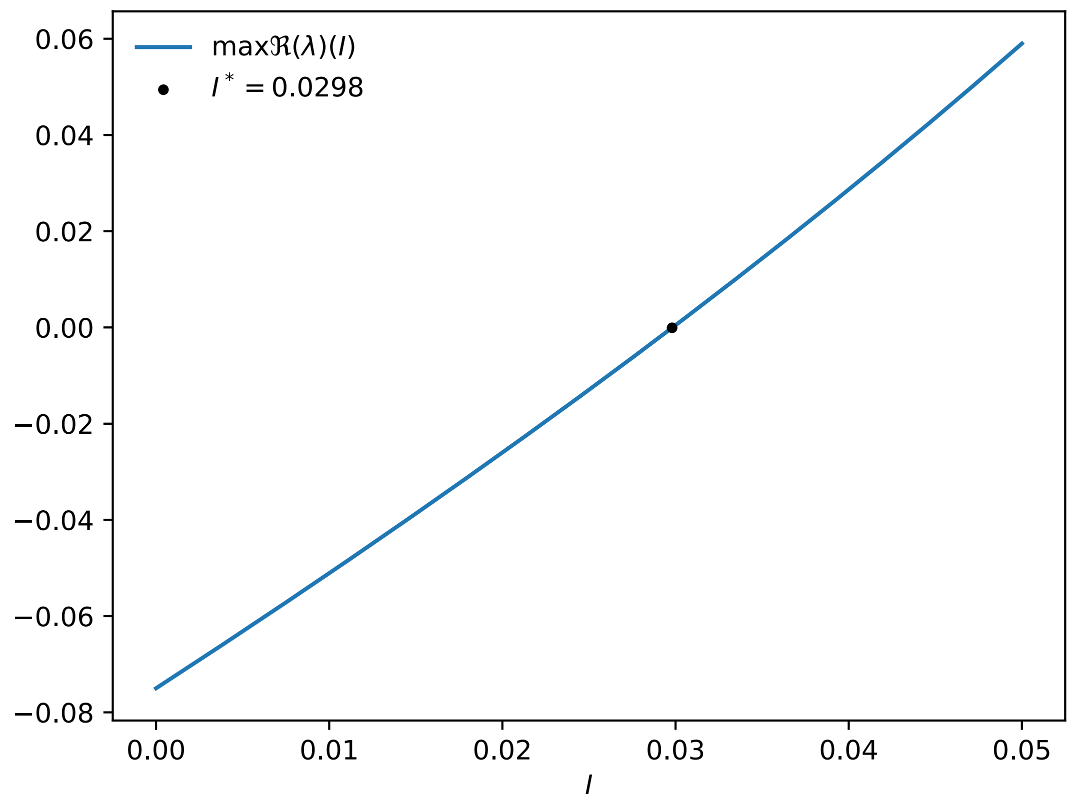

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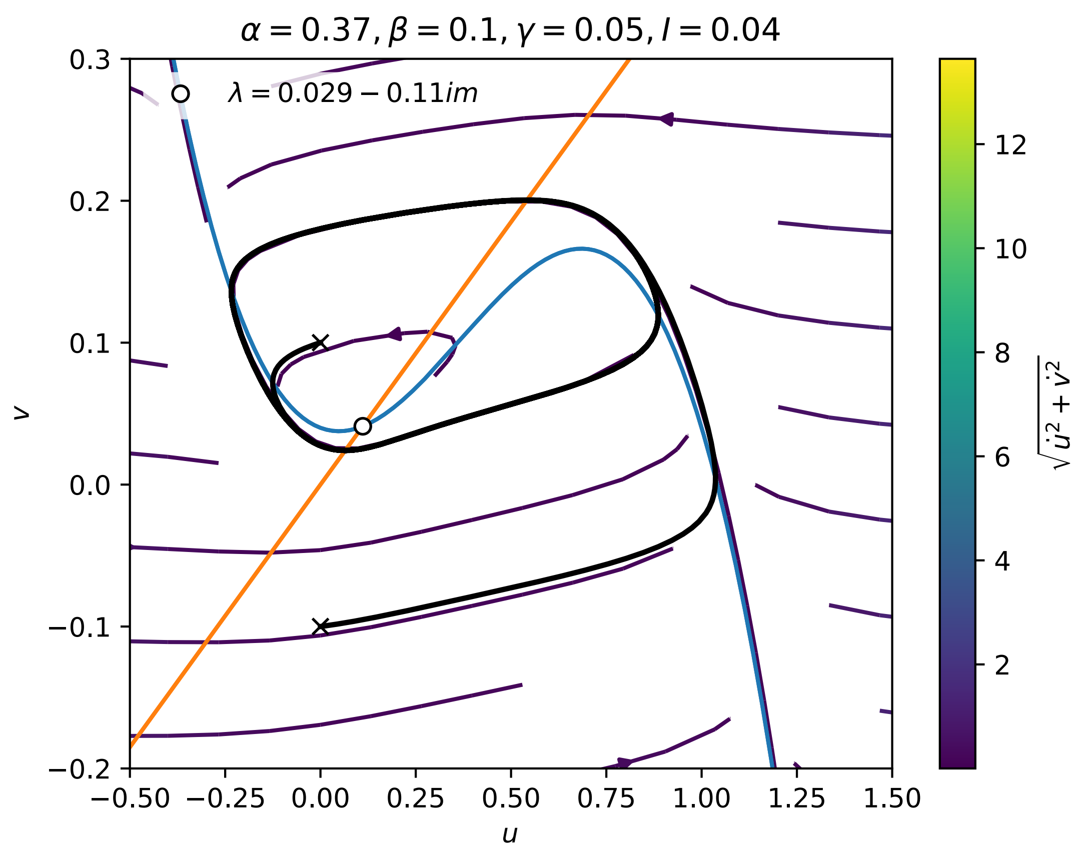
Periodic Orbits

Limit Cycles and Poincaré-Bendixson

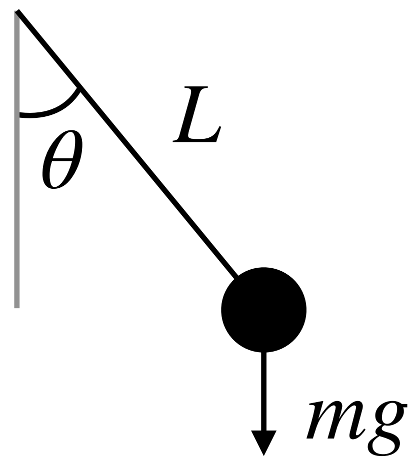
"The Poincaré-Bendixson theorem is one of the central results of nonlinear dynamics. It says that the dynamical possibilities in the phase plane are very limited: if a trajectory is confined to a closed, bounded region that contains no fixed points, then the trajectory must eventually approach a closed orbit. Nothing more complicated is possible." — *Nonlinear Dynamics and Chaos, with Applications to Physics, Biology, Chemistry, and Engineering*, S. H. Strogatz, §7.3



- A corollary to this is that we can not exhibit chaotic dynamics with two or less state variables related by ordinary differential equations
- An example of a chaotic 3D system will be looked at in detail later!

FHN Oscillations

What makes them different from the pendulum?



- $$\frac{d^2\theta}{dt^2} + \frac{g}{L}\sin\theta = 0$$
- Rescale time by natural frequency: $\ddot{\theta} + \sin\theta = 0$
  - $\dot{\theta}(\ddot{\theta} + \sin\theta) = 0$ implies energy $E(\theta, \dot{\theta}) = \dot{\theta}^2/2 - \cos\theta$ is constant!
- $\dot{\theta} = \omega, \dot{\omega} = -\sin\theta$

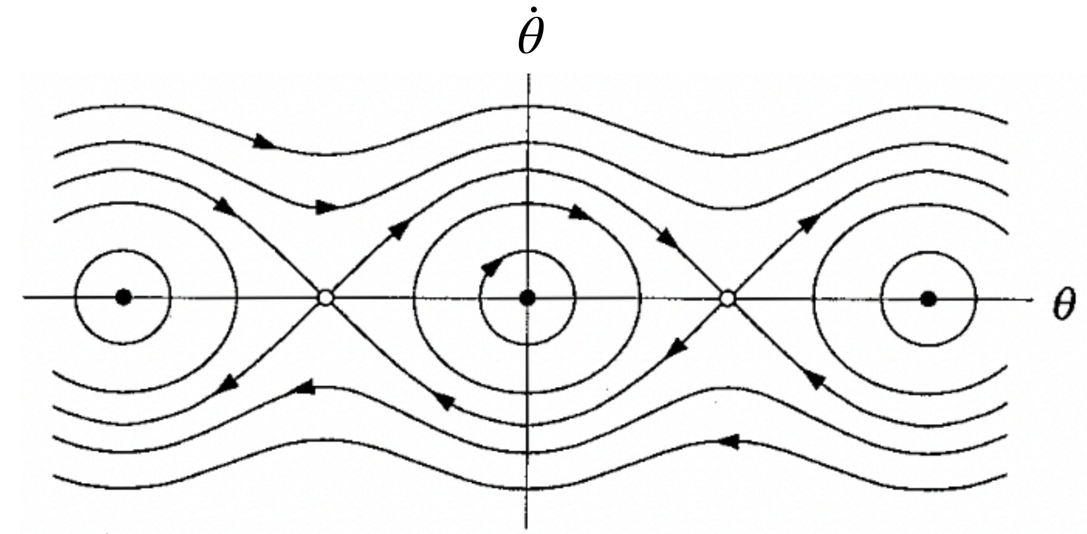


Figure 6.7.3 Strogatz

- The trajectories are just the isocontours of $E(\theta, \dot{\theta})$
  - No attracting or repelling orbits — all separated by energy

Other slow-fast systems

[Keener's listing of slow-fast systems]