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# 2.3 QUICKSORT

- *quicksort*
- *selection*
- *duplicate keys*
- *system sorts*

## Two classic sorting algorithms

Critical components in the world's computational infrastructure.

- Full scientific understanding of their properties has enabled us to develop them into practical system sorts.
- Quicksort honored as one of top 10 algorithms of 20$^{th}$ century in science and engineering.

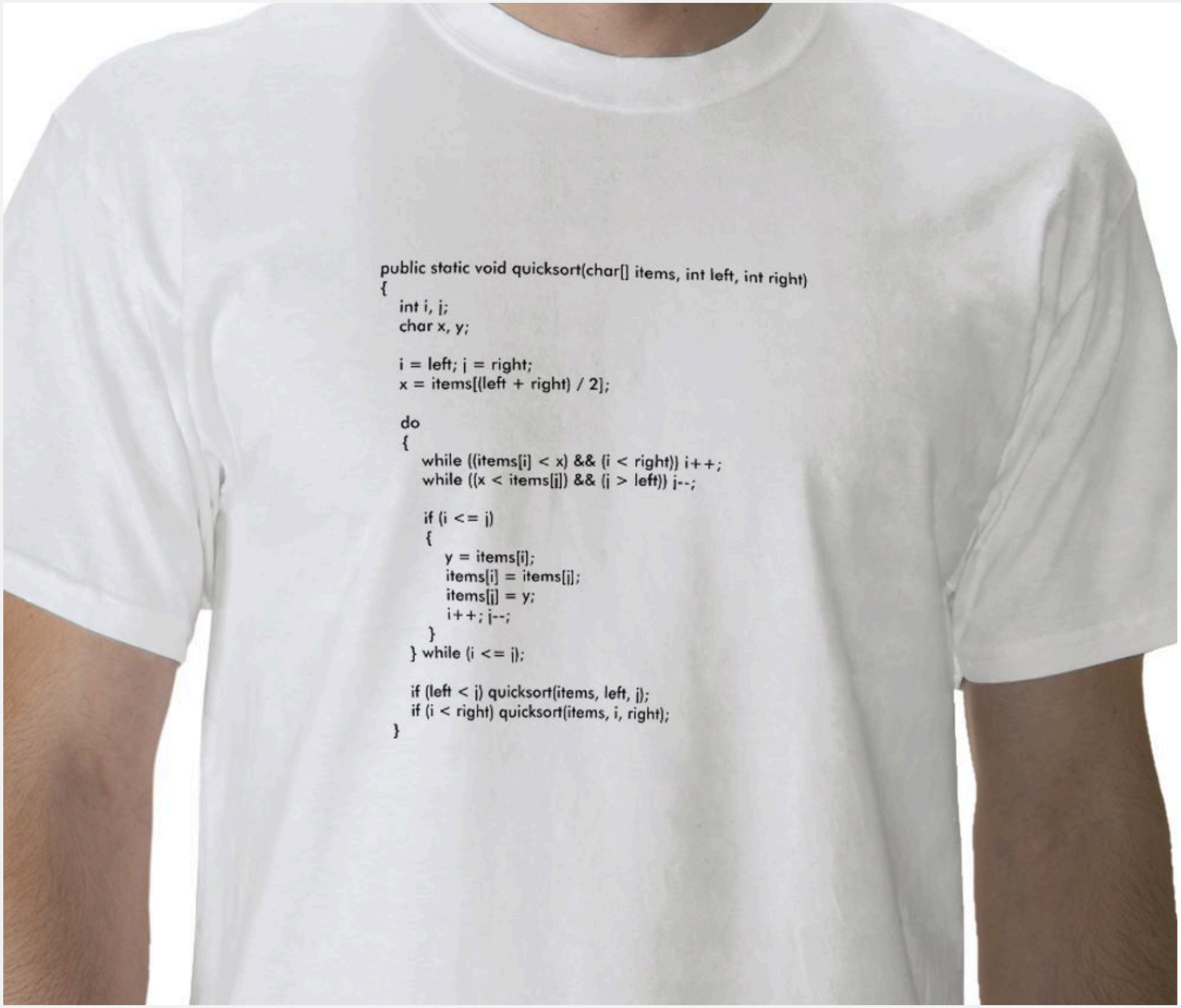
Mergesort. ← last lecture

- Java sort for objects.
- Perl, C++ stable sort, Python stable sort, Firefox JavaScript, ...

Quicksort. ← this lecture

- Java sort for primitive types.
- C qsort, Unix, Visual C++, Python, Matlab, Chrome JavaScript, ...

# Quicksort t-shirt



Algorithms

Robert Sedgewick | Kevin Wayne

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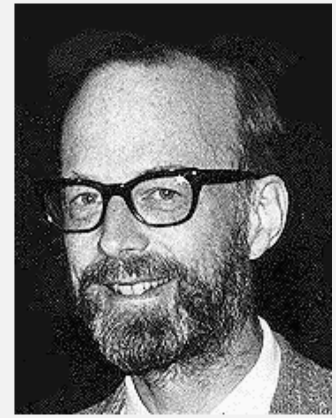
# 2.3 Quicksort

- *quicksort*
- *selection*
- *duplicate keys*
- *system sorts*

# Quicksort

Basic plan.

- Shuffle the array.
- Partition so that, for some `j`
  - entry `a[j]` is in place
  - no larger entry to the left of `j`
  - no smaller entry to the right of `j`
- Sort each piece recursively.



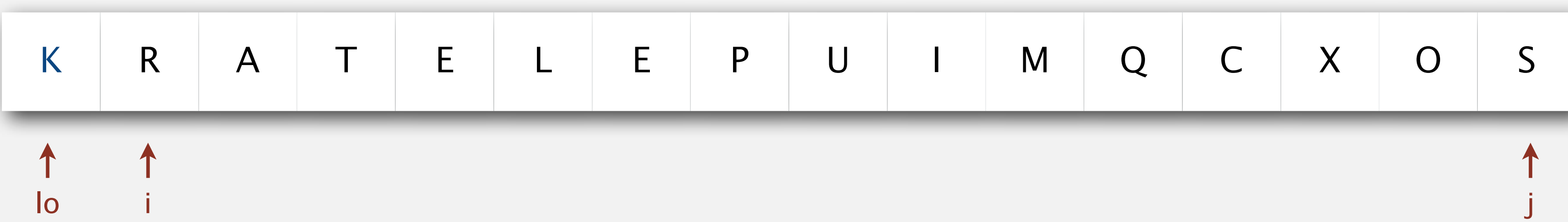
**Sir Charles Antony Richard Hoare**
**1980 Turing Award**

| | | | | | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| input | Q | U | I | C | K | S | O | R | T | E | X | A | M | P | L | E |
| shuffle | K | R | A | T | E | L | E | P | U | I | M | Q | C | X | O | S |
| | | | | | | | | | *partitioning item* | | | | | | | |
| partition | E | C | A | I | E | K | L | P | U | T | M | Q | R | X | O | S |
| | | | | *not greater* | | | | | *not less* | | | | | | | |
| sort left | A | C | E | E | I | K | L | P | U | T | M | Q | R | X | O | S |
| sort right | A | C | E | E | I | K | L | M | O | P | Q | R | S | T | U | X |
| result | A | C | E | E | I | K | L | M | O | P | Q | R | S | T | U | X |

# Quicksort partitioning demo

Repeat until `i` and `j` pointers cross.

- Scan `i` from left to right so long as `(a[i] < a[lo])`.
- Scan `j` from right to left so long as `(a[j] > a[lo])`.
- Exchange `a[i]` with `a[j]`.



# Quicksort partitioning demo

Repeat until `i` and `j` pointers cross.

- Scan `i` from left to right so long as `(a[i] < a[lo])`.
- Scan `j` from right to left so long as `(a[j] > a[lo])`.
- Exchange `a[i]` with `a[j]`.

When pointers cross.

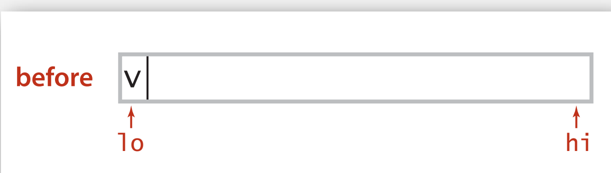
- Exchange `a[lo]` with `a[j]`.

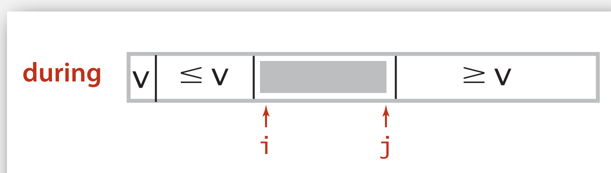
| | | | | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| E | C | A | I | E | K | L | P | U | T | M | Q | R | X | O | S |
| ↑ lo | | | | | ↑ j | | | | | | | | | | ↑ hi |

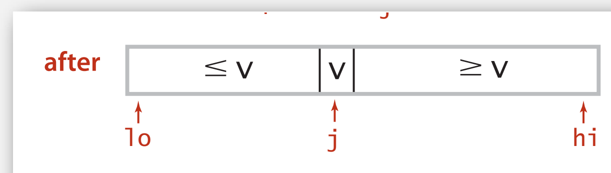
partitioned!

# Quicksort: Java code for partitioning

```
private static int partition(Comparable[] a, int lo, int hi)
{
   int i = lo, j = hi+1;
   while (true)
   {
      while (less(a[++i], a[lo]))                 find item on left to swap
         if (i == hi) break;

      while (less(a[lo], a[--j]))                 find item on right to swap
         if (j == lo) break;

      if (i >= j) break;                          check if pointers cross
      exch(a, i, j);                              swap
   }

   exch(a, lo, j);                                swap with partitioning item
   return j;                                      return index of item now known to be in place
}
```



# Quicksort: Java implementation

```
public class Quick
{
   private static int partition(Comparable[] a, int lo, int hi)
   {  /* see previous slide */  }

   public static void sort(Comparable[] a)
   {
      StdRandom.shuffle(a);
      sort(a, 0, a.length - 1);
   }

   private static void sort(Comparable[] a, int lo, int hi)
   {
      if (hi <= lo) return;
      int j = partition(a, lo, hi);
      sort(a, lo, j-1);
      sort(a, j+1, hi);
  }
}
```

shuffle needed for performance guarantee (stay tuned)

# Quicksort trace

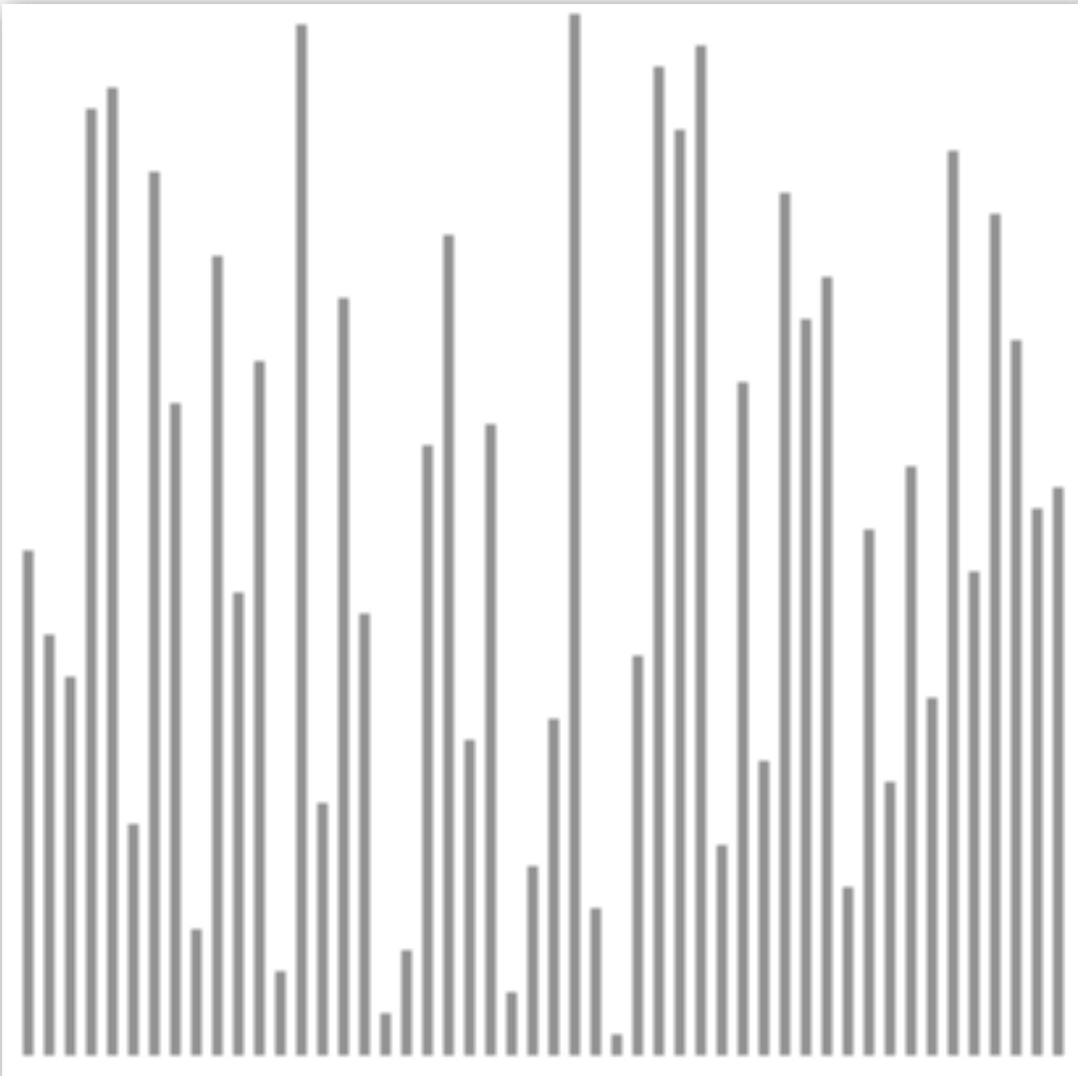
| | lo | j | hi | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| **initial values** | | | | Q | U | I | C | K | S | O | R | T | E | X | A | M | P | L | E |
| **random shuffle** | | | | K | R | A | T | E | L | E | P | U | I | M | Q | C | X | O | S |
| | 0 | 5 | 15 | E | C | A | I | E | K | L | P | U | T | M | Q | R | X | O | S |
| | 0 | 3 | 4 | E | C | A | E | I | K | L | P | U | T | M | Q | R | X | O | S |
| | 0 | 2 | 2 | A | C | E | E | I | K | L | P | U | T | M | Q | R | X | O | S |
| | 0 | 0 | 1 | A | C | E | E | I | K | L | P | U | T | M | Q | R | X | O | S |
| | 1 | | 1 | A | C | E | E | I | K | L | P | U | T | M | Q | R | X | O | S |
| | 4 | | 4 | A | C | E | E | I | K | L | P | U | T | M | Q | R | X | O | S |
| | 6 | 6 | 15 | A | C | E | E | I | K | L | P | U | T | M | Q | R | X | O | S |
| | 7 | 9 | 15 | A | C | E | E | I | K | L | M | O | P | T | Q | R | X | U | S |
| | 7 | 7 | 8 | A | C | E | E | I | K | L | M | O | P | T | Q | R | X | U | S |
| | 8 | | 8 | A | C | E | E | I | K | L | M | O | P | T | Q | R | X | U | S |
| | 10 | 13 | 15 | A | C | E | E | I | K | L | M | O | P | S | Q | R | T | U | X |
| | 10 | 12 | 12 | A | C | E | E | I | K | L | M | O | P | R | Q | S | T | U | X |
| | 10 | 11 | 11 | A | C | E | E | I | K | L | M | O | P | Q | R | S | T | U | X |
| | 10 | | 10 | A | C | E | E | I | K | L | M | O | P | Q | R | S | T | U | X |
| | 14 | 14 | 15 | A | C | E | E | I | K | L | M | O | P | Q | R | S | T | U | X |
| | 15 | | 15 | A | C | E | E | I | K | L | M | O | P | Q | R | S | T | U | X |
| **result** | | | | A | C | E | E | I | K | L | M | O | P | Q | R | S | T | U | X |

*no partition for subarrays of size 1*

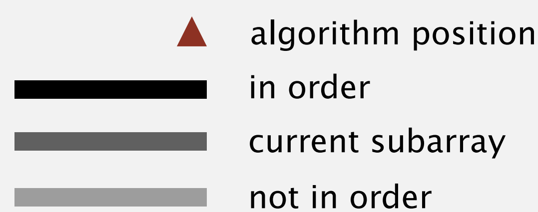
**Quicksort trace (array contents after each partition)**

# Quicksort animation

**50 random items**



http://www.sorting-algorithms.com/quick-sort



## Quicksort: implementation details

Partitioning in-place. Using an extra array makes partitioning easier (and stable), but is not worth the cost.

Terminating the loop. Testing whether the pointers cross is a bit trickier than it might seem.

Staying in bounds. The `(j == lo)` test is redundant (why?), but the `(i == hi)` test is not.

Preserving randomness. Shuffling is needed for performance guarantee.

Equal keys. When duplicates are present, it is (counter-intuitively) better to stop on keys equal to the partitioning item's key.

# Quicksort: empirical analysis

Running time estimates:

- Home PC executes $10^8$ compares/second.
- Supercomputer executes $10^{12}$ compares/second.

| | insertion sort ($N^2$) | | | mergesort (N log N) | | | quicksort (N log N) | | |
|---|---|---|---|---|---|---|---|---|---|
| computer | thousand | million | billion | thousand | million | billion | thousand | million | billion |
| home | instant | 2.8 hours | 317 years | instant | 1 second | 18 min | instant | 0.6 sec | 12 min |
| super | instant | 1 second | 1 week | instant | instant | instant | instant | instant | instant |

Lesson 1. Good algorithms are better than supercomputers.

Lesson 2. Great algorithms are better than good ones.

# Quicksort: best-case analysis

Best case. Number of compares is $\sim N \lg N$.

| lo | j | hi | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| initial values | | | H | A | C | B | F | E | G | D | L | I | K | J | N | M | O |
| random shuffle | | | H | A | C | B | F | E | G | D | L | I | K | J | N | M | O |
| 0 | 7 | 14 | D | A | C | B | F | E | G | H | L | I | K | J | N | M | O |
| 0 | 3 | 6 | B | A | C | D | F | E | G | H | L | I | K | J | N | M | O |
| 0 | 1 | 2 | A | B | C | D | F | E | G | H | L | I | K | J | N | M | O |
| 0 | | 0 | A | B | C | D | F | E | G | H | L | I | K | J | N | M | O |
| 2 | | 2 | A | B | C | D | F | E | G | H | L | I | K | J | N | M | O |
| 4 | 5 | 6 | A | B | C | D | E | F | G | H | L | I | K | J | N | M | O |
| 4 | | 4 | A | B | C | D | E | F | G | H | L | I | K | J | N | M | O |
| 6 | | 6 | A | B | C | D | E | F | G | H | L | I | K | J | N | M | O |
| 8 | 11 | 14 | A | B | C | D | E | F | G | H | J | I | K | L | N | M | O |
| 8 | 9 | 10 | A | B | C | D | E | F | G | H | I | J | K | L | N | M | O |
| 8 | | 8 | A | B | C | D | E | F | G | H | I | J | K | L | N | M | O |
| 10 | | 10 | A | B | C | D | E | F | G | H | I | J | K | L | N | M | O |
| 12 | 13 | 14 | A | B | C | D | E | F | G | H | I | J | K | L | M | N | O |
| 12 | | 12 | A | B | C | D | E | F | G | H | I | J | K | L | M | N | O |
| 14 | | 14 | A | B | C | D | E | F | G | H | I | J | K | L | M | N | O |
| | | | A | B | C | D | E | F | G | H | I | J | K | L | M | N | O |

# Quicksort: worst-case analysis

**Worst case.** Number of compares is $\sim \frac{1}{2} N^2$ .

| lo | j | hi | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| initial values | | | A | B | C | D | E | F | G | H | I | J | K | L | M | N | O |
| random shuffle | | | A | B | C | D | E | F | G | H | I | J | K | L | M | N | O |
| 0 | 0 | 14 | A | B | C | D | E | F | G | H | I | J | K | L | M | N | O |
| 1 | 1 | 14 | A | B | C | D | E | F | G | H | I | J | K | L | M | N | O |
| 2 | 2 | 14 | A | B | C | D | E | F | G | H | I | J | K | L | M | N | O |
| 3 | 3 | 14 | A | B | C | D | E | F | G | H | I | J | K | L | M | N | O |
| 4 | 4 | 14 | A | B | C | D | E | F | G | H | I | J | K | L | M | N | O |
| 5 | 5 | 14 | A | B | C | D | E | F | G | H | I | J | K | L | M | N | O |
| 6 | 6 | 14 | A | B | C | D | E | F | G | H | I | J | K | L | M | N | O |
| 7 | 7 | 14 | A | B | C | D | E | F | G | H | I | J | K | L | M | N | O |
| 8 | 8 | 14 | A | B | C | D | E | F | G | H | I | J | K | L | M | N | O |
| 9 | 9 | 14 | A | B | C | D | E | F | G | H | I | J | K | L | M | N | O |
| 10 | 10 | 14 | A | B | C | D | E | F | G | H | I | J | K | L | M | N | O |
| 11 | 11 | 14 | A | B | C | D | E | F | G | H | I | J | K | L | M | N | O |
| 12 | 12 | 14 | A | B | C | D | E | F | G | H | I | J | K | L | M | N | O |
| 13 | 13 | 14 | A | B | C | D | E | F | G | H | I | J | K | L | M | N | O |
| 14 | | 14 | A | B | C | D | E | F | G | H | I | J | K | L | M | N | O |
| | | | A | B | C | D | E | F | G | H | I | J | K | L | M | N | O |

## Quicksort: average-case analysis

**Proposition.** The average number of compares $C_N$ to quicksort an array of $N$ distinct keys is $\sim 2N \ln N$ (and the number of exchanges is $\sim \frac{1}{3} N \ln N$).

**Pf.** $C_N$ satisfies the recurrence $C_0 = C_1 = 0$ and for $N \geq 2$:

$$C_N = \underbrace{(N+1)}_{\text{partitioning}} + \left(\frac{C_0 + C_{N-1}}{N}\right) + \left(\frac{\overbrace{C_1}^{\text{left}} + \overbrace{C_{N-2}}^{\text{right}}}{\underbrace{N}_{\text{partitioning probability}}}\right) + \ldots + \left(\frac{C_{N-1} + C_0}{N}\right)$$

- Multiply both sides by $N$ and collect terms:

$$NC_N = N(N+1) + 2(C_0 + C_1 + \ldots + C_{N-1})$$

- Subtract this from the same equation for $N - 1$:

$$NC_N - (N-1)C_{N-1} = 2N + 2C_{N-1}$$

- Rearrange terms and divide by $N(N+1)$:

$$\frac{C_N}{N+1} = \frac{C_{N-1}}{N} + \frac{2}{N+1}$$

# Quicksort: average-case analysis

- Repeatedly apply above equation:

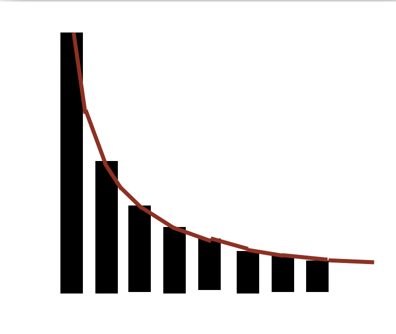
$$
\begin{aligned}
\frac{C_N}{N+1} &= \frac{C_{N-1}}{N} + \frac{2}{N+1} \\
&= \frac{C_{N-2}}{N-1} + \frac{2}{N} + \frac{2}{N+1} \\
&= \frac{C_{N-3}}{N-2} + \frac{2}{N-1} + \frac{2}{N} + \frac{2}{N+1} \\
&= \frac{2}{3} + \frac{2}{4} + \frac{2}{5} + \ldots + \frac{2}{N+1}
\end{aligned}
$$

previous equation

substitute previous equation

- Approximate sum by an integral:

$$
\begin{aligned}
C_N &= 2(N+1)\left(\frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \ldots \frac{1}{N+1}\right) \\
&\sim 2(N+1)\int_3^{N+1} \frac{1}{x}\,dx
\end{aligned}
$$



- Finally, the desired result:

$$C_N \sim 2(N+1)\ln N \approx 1.39 N \lg N$$

## Quicksort: summary of performance characteristics

Worst case. Number of compares is quadratic.

- $N + (N-1) + (N-2) + \ldots + 1 \sim \frac{1}{2} N^2$.
- More likely that your computer is struck by lightning bolt.

Average case. Number of compares is $\sim 1.39\, N \lg N$.

- 39% more compares than mergesort.
- But faster than mergesort in practice because of less data movement.

Random shuffle.

- Probabilistic guarantee against worst case.
- Basis for math model that can be validated with experiments.

Caveat emptor. Many textbook implementations go quadratic if array

- Is sorted or reverse sorted.
- Has many duplicates (even if randomized!)

## Quicksort properties

Proposition. Quicksort is an in-place sorting algorithm.

Pf.

- Partitioning: constant extra space.
- Depth of recursion: logarithmic extra space (with high probability).

can guarantee logarithmic depth by recurring on smaller subarray before larger subarray

Proposition. Quicksort is not stable.

Pf.

| i | j | 0 | 1 | 2 | 3 |
|---|---|---|---|---|---|
| | | $B_1$ | $C_1$ | $C_2$ | $A_1$ |
| 1 | 3 | $B_1$ | $C_1$ | $C_2$ | $A_1$ |
| 1 | 3 | $B_1$ | $A_1$ | $C_2$ | $C_1$ |
| 0 | 1 | $A_1$ | $B_1$ | $C_2$ | $C_1$ |

# Quicksort: practical improvements

Insertion sort small subarrays.

- Even quicksort has too much overhead for tiny subarrays.
- Cutoff to insertion sort for ≈ 10 items.
- Note: could delay insertion sort until one pass at end.

```
private static void sort(Comparable[] a, int lo, int hi)
{
   if (hi <= lo + CUTOFF - 1)
   {
      Insertion.sort(a, lo, hi);
      return;
   }
   int j = partition(a, lo, hi);
   sort(a, lo, j-1);
   sort(a, j+1, hi);
}
```

# Quicksort: practical improvements

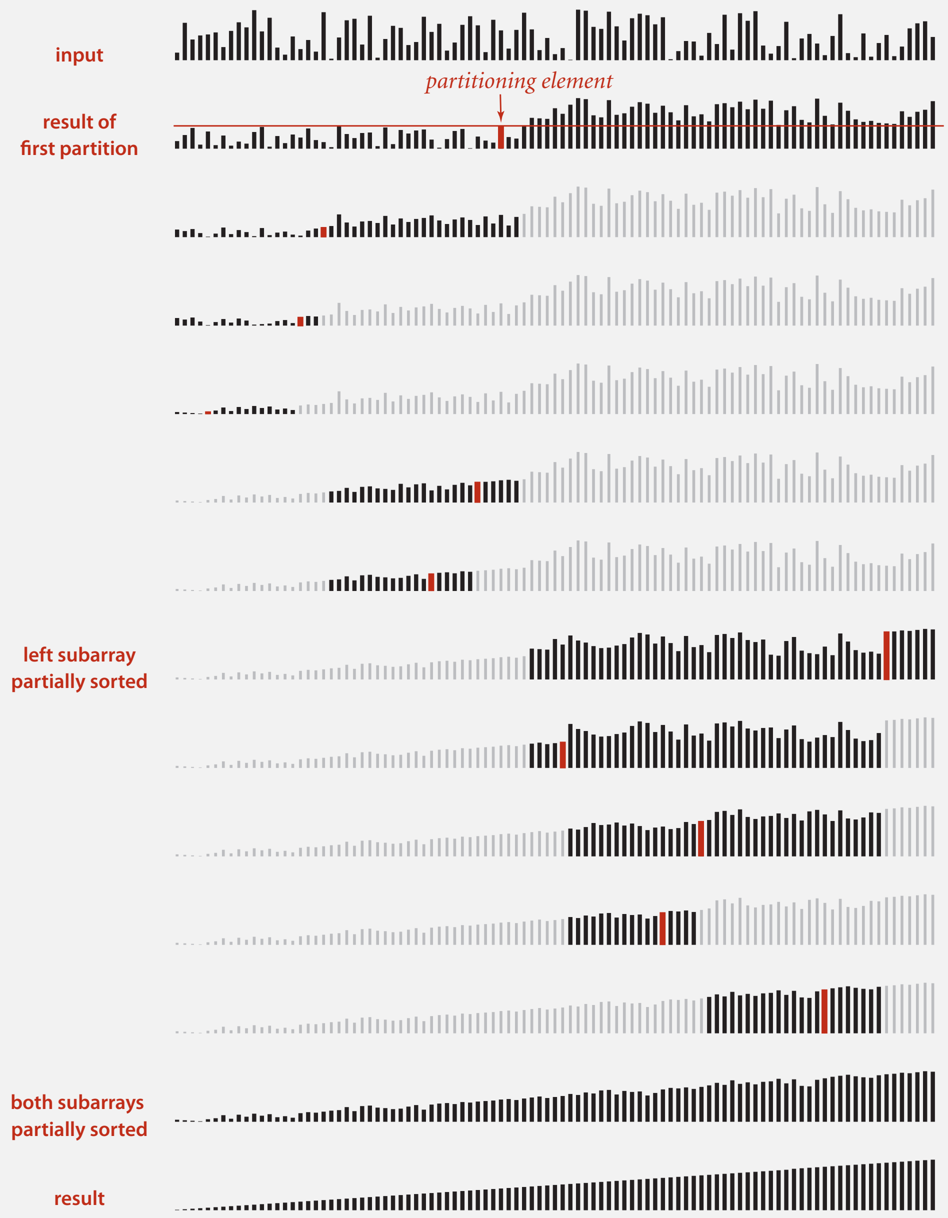
## Median of sample.

- Best choice of pivot item = median.
- Estimate true median by taking median of sample.
- Median-of-3 (random) items.

~ 12/7 N ln N compares (slightly fewer)
~ 12/35 N ln N exchanges (slightly more)

```
private static void sort(Comparable[] a, int lo, int hi)
{
   if (hi <= lo) return;

   int m = medianOf3(a, lo, lo + (hi - lo)/2, hi);
   swap(a, lo, m);

   int j = partition(a, lo, hi);
   sort(a, lo, j-1);
   sort(a, j+1, hi);
}
```

# Quicksort with median-of-3 and cutoff to insertion sort: visualization



Algorithms

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# 2.3 Quicksort

- quicksort
- **selection**
- duplicate keys
- system sorts

# Selection

Goal. Given an array of $N$ items, find a $k^{th}$ smallest item.

Ex. Min $(k = 0)$, max $(k = N - 1)$, median $(k = N/2)$.

Applications.

- Order statistics.
- Find the "top $k$."

Use theory as a guide.

- Easy $N \log N$ upper bound. How?
- Easy $N$ upper bound for $k = 1, 2, 3$. How?
- Easy $N$ lower bound. Why?

Which is true?

- $N \log N$ lower bound? ← is selection as hard as sorting?
- $N$ upper bound? ← is there a linear-time algorithm for each k?

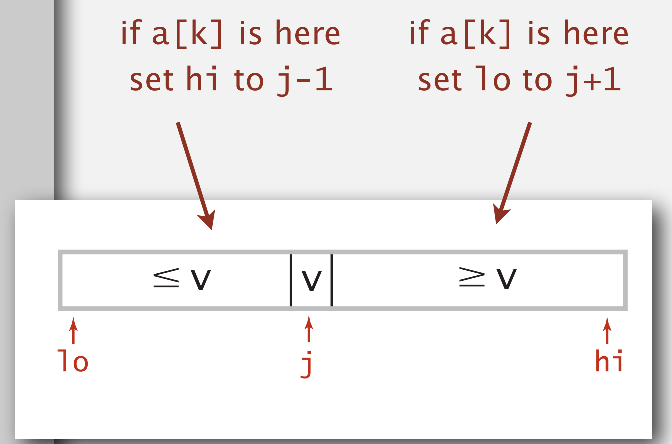
# Quick-select

Partition array so that:

- Entry `a[j]` is in place.
- No larger entry to the left of `j`.
- No smaller entry to the right of `j`.

Repeat in one subarray, depending on `j`; finished when `j` equals `k`.

```
public static Comparable select(Comparable[] a, int k)
{
    StdRandom.shuffle(a);
    int lo = 0, hi = a.length - 1;
    while (hi > lo)
    {
       int j = partition(a, lo, hi);
       if      (j < k) lo = j + 1;
       else if (j > k) hi = j - 1;
       else            return a[k];
    }
    return a[k];
}
```



## Quick-select: mathematical analysis

**Proposition.** Quick-select takes linear time on average.

**Pf sketch.**

- Intuitively, each partitioning step splits array approximately in half: $N + N/2 + N/4 + \ldots + 1 \sim 2N$ compares.
- Formal analysis similar to quicksort analysis yields:

$$C_N = 2\,N + 2\,k \ln(N/k) + 2\,(N-k) \ln(N/(N-k))$$

(2 + 2 ln 2) N to find the median

**Remark.** Quick-select uses $\sim \frac{1}{2} N^2$ compares in the worst case, but (as with quicksort) the random shuffle provides a probabilistic guarantee.

# Theoretical context for selection

Proposition. [Blum, Floyd, Pratt, Rivest, Tarjan, 1973] Compare-based selection algorithm whose worst-case running time is linear.

Time Bounds for Selection

by .

Manuel Blum, Robert W. Floyd, Vaughan Pratt,
Ronald L. Rivest, and Robert E. Tarjan

Abstract

The number of comparisons required to select the i-th smallest of n numbers is shown to be at most a linear function of n by analysis of a new selection algorithm -- PICK. Specifically, no more than 5.4305 n comparisons are ever required. This bound is improved for

Remark. But, constants are too high $\Rightarrow$ not used in practice.

Use theory as a guide.

- Still worthwhile to seek practical linear-time (worst-case) algorithm.
- Until one is discovered, use quick-select if you don't need a full sort.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# 2.3 Quicksort

- *quicksort*
- ***selection***
- *duplicate keys*
- *system sorts*


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# 2.3 QUICKSORT

- quicksort
- selection
- duplicate keys
- system sorts

# Duplicate keys

Often, purpose of sort is to bring items with equal keys together.

- Sort population by age.
- Remove duplicates from mailing list.
- Sort job applicants by college attended.

Typical characteristics of such applications.

- Huge array.
- Small number of key values.

```
Chicago 09:25:52
Chicago 09:03:13
Chicago 09:21:05
Chicago 09:19:46
Chicago 09:19:32
Chicago 09:00:00
Chicago 09:35:21
Chicago 09:00:59
Houston 09:01:10
Houston 09:00:13
Phoenix 09:37:44
Phoenix 09:00:03
Phoenix 09:14:25
Seattle 09:10:25
Seattle 09:36:14
Seattle 09:22:43
Seattle 09:10:11
Seattle 09:22:54
```

key

## Duplicate keys

Mergesort with duplicate keys. Between $\frac{1}{2} N \lg N$ and $N \lg N$ compares.

Quicksort with duplicate keys.

- Algorithm goes quadratic unless partitioning stops on equal keys!
- 1990s C user found this defect in `qsort()`.

several textbook and system implementation also have this defect

S T O P O N E Q U A L K E Y S

swap

if we don't stop on equal keys

if we stop on equal keys

## Duplicate keys: the problem

Mistake. Put all items equal to the partitioning item on one side.

Consequence. $\sim \frac{1}{2} N^2$ compares when all keys equal.

B A A B A B B B C C C          A A A A A A A A A A A

Recommended. Stop scans on items equal to the partitioning item.

Consequence. $\sim N \lg N$ compares when all keys equal.

B A A B A B C C B C B          A A A A A A A A A A A

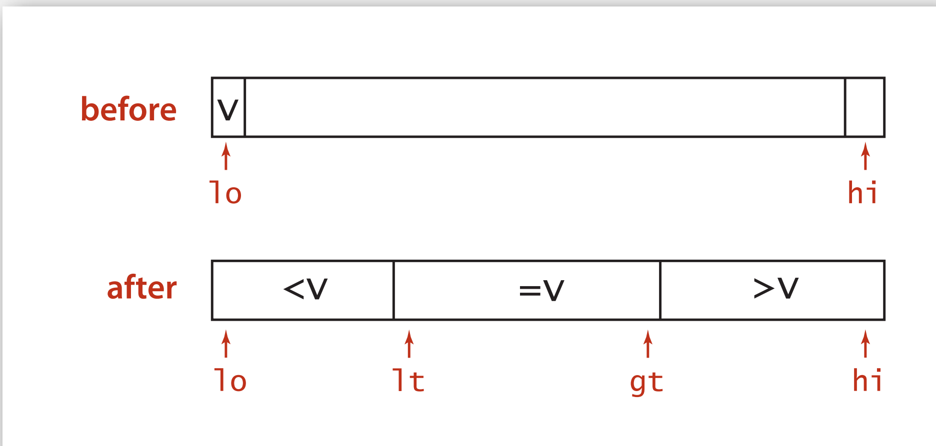
Desirable. Put all items equal to the partitioning item in place.

A A A B B B B B C C C          A A A A A A A A A A A

## 3-way partitioning

Goal. Partition array into 3 parts so that:

- Entries between `lt` and `gt` equal to partition item `v`.
- No larger entries to left of `lt`.
- No smaller entries to right of `gt`.



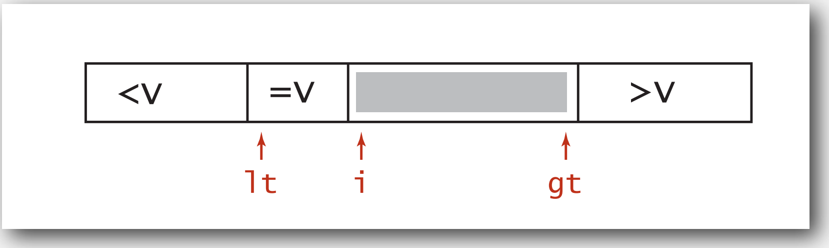

Dutch national flag problem. [Edsger Dijkstra]

- Conventional wisdom until mid 1990s: not worth doing.
- New approach discovered when fixing mistake in C library `qsort()`.
- Now incorporated into `qsort()` and Java system sort.

# Dijkstra 3-way partitioning demo

- Let `v` be partitioning item `a[lo]`.
- Scan `i` from left to right.
  - `(a[i] < v)`: exchange `a[lt]` with `a[i]`; increment both `lt` and `i`
  - `(a[i] > v)`: exchange `a[gt]` with `a[i]`; decrement `gt`
  - `(a[i] == v)`: increment `i`


**invariant**



# Dijkstra 3-way partitioning demo

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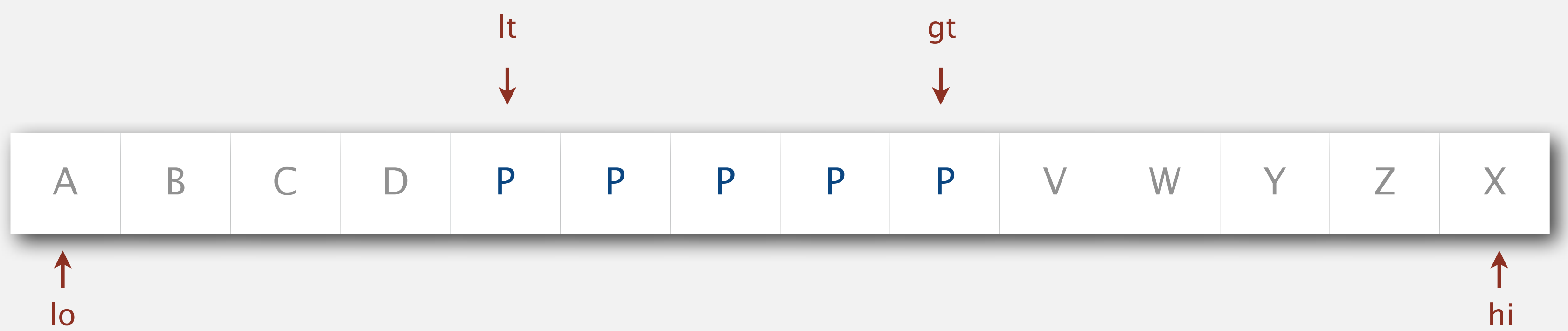
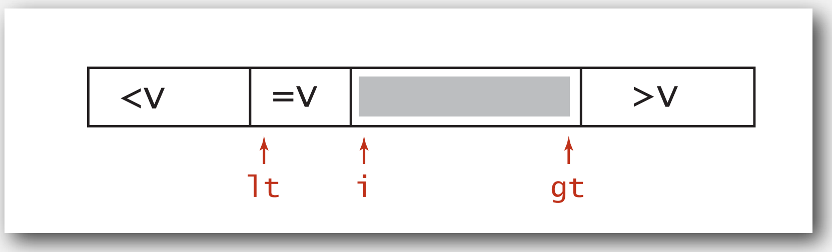



**invariant**



# Dijkstra's 3-way partitioning: trace

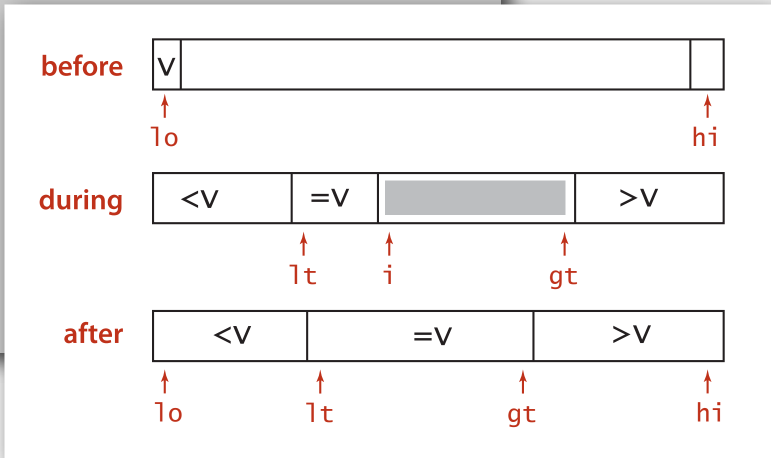
| | | | | | | | | a[] | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| lt | i | gt | v | | | | | | | | | | | |
| | | | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 |
| 0 | 0 | 11 | R | B | W | W | R | W | B | R | R | W | B | R |
| 0 | 1 | 11 | R | B | W | W | R | W | B | R | R | W | B | R |
| 1 | 2 | 11 | B | R | W | W | R | W | B | R | R | W | B | R |
| 1 | 2 | 10 | B | R | R | W | R | W | B | R | R | W | B | W |
| 1 | 3 | 10 | B | R | R | W | R | W | B | R | R | W | B | W |
| 1 | 3 | 9 | B | R | R | B | R | W | B | R | R | W | W | W |
| 2 | 4 | 9 | B | B | R | R | R | W | B | R | R | W | W | W |
| 2 | 5 | 9 | B | B | R | R | R | W | B | R | R | W | W | W |
| 2 | 5 | 8 | B | B | R | R | R | W | B | R | R | W | W | W |
| 2 | 5 | 7 | B | B | R | R | R | R | B | R | W | W | W | W |
| 2 | 6 | 7 | B | B | R | R | R | R | B | R | W | W | W | W |
| 3 | 7 | 7 | B | B | B | R | R | R | R | R | W | W | W | W |
| 3 | 8 | 7 | B | B | B | R | R | R | R | R | W | W | W | W |
| 3 | 8 | 7 | B | B | B | R | R | R | R | R | W | W | W | W |

**3-way partitioning trace (array contents after each loop iteration)**

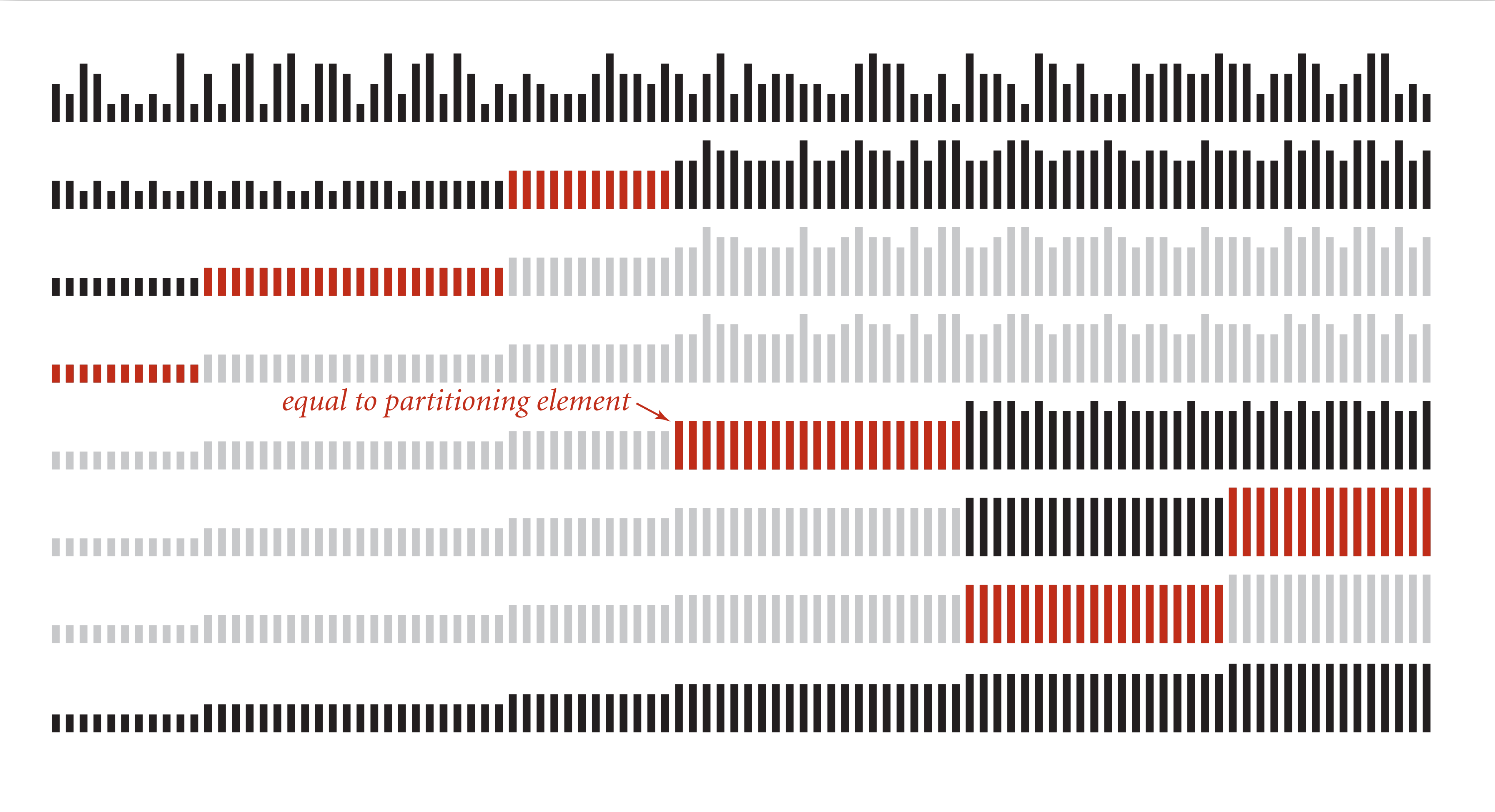
# 3-way quicksort: Java implementation

```
private static void sort(Comparable[] a, int lo, int hi)
{
   if (hi <= lo) return;
   int lt = lo, gt = hi;
   Comparable v = a[lo];
   int i = lo;
   while (i <= gt)
   {
      int cmp = a[i].compareTo(v);
      if      (cmp < 0) exch(a, lt++, i++);
      else if (cmp > 0) exch(a, i, gt--);
      else              i++;
   }

   sort(a, lo, lt - 1);
   sort(a, gt + 1, hi);
}
```



# 3-way quicksort: visual trace



## Duplicate keys: lower bound

Sorting lower bound. If there are $n$ distinct keys and the $i^{th}$ one occurs $x_i$ times, any compare-based sorting algorithm must use at least

$$\lg\left(\frac{N!}{x_1!\; x_2! \cdots x_n!}\right) \;\sim\; -\sum_{i=1}^{n} x_i \lg \frac{x_i}{N}$$

$N \lg N$ when all distinct;
linear when only a constant number of distinct keys

compares in the worst case.

Proposition. [Sedgewick-Bentley, 1997]
Quicksort with 3-way partitioning is entropy-optimal.
Pf. [beyond scope of course]

proportional to lower bound

Bottom line. Randomized quicksort with 3-way partitioning reduces running time from linearithmic to linear in broad class of applications.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# 2.3 Quicksort

- quicksort
- selection
- *duplicate keys*
- system sorts


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# 2.3 Quicksort

- *quicksort*
- *selection*
- *duplicate keys*
- *system sorts*

# Sorting applications

Sorting algorithms are essential in a broad variety of applications:

- Sort a list of names.
- Organize an MP3 library.
- Display Google PageRank results.
- List RSS feed in reverse chronological order.

obvious applications

- Find the median.
- Identify statistical outliers.
- Binary search in a database.
- Find duplicates in a mailing list.

problems become easy once items are in sorted order

- Data compression.
- Computer graphics.
- Computational biology.
- Load balancing on a parallel computer.

. . .

non-obvious applications

# Java system sorts

`Arrays.sort()`.

- Has different method for each primitive type.
- Has a method for data types that implement `Comparable`.
- Has a method that uses a `Comparator`.
- Uses tuned quicksort for primitive types; tuned mergesort for objects.

```
import java.util.Arrays;

public class StringSort
{
   public static void main(String[] args)
   {
      String[] a = StdIn.readStrings());
      Arrays.sort(a);
      for (int i = 0; i < N; i++)
         StdOut.println(a[i]);
   }
}
```

Q. Why use different algorithms for primitive and reference types?

## War story (C qsort function)

AT&T Bell Labs (1991). Allan Wilks and Rick Becker discovered that a `qsort()` call that should have taken seconds was taking minutes.


At the time, almost all `qsort()` implementations based on those in:

- Version 7 Unix (1979): quadratic time to sort organ-pipe arrays.
- BSD Unix (1983): quadratic time to sort random arrays of 0s and 1s.

# Engineering a system sort

Basic algorithm = quicksort.

- Cutoff to insertion sort for small subarrays.
- Partitioning scheme: Bentley-McIlroy 3-way partitioning.
- Partitioning item.
  - small arrays: middle entry
  - medium arrays: median of 3
  - large arrays: Tukey's ninther [next slide]

Engineering a Sort Function

JON L. BENTLEY
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**SUMMARY**

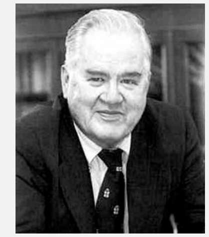
**We recount the history of a new `qsort` function for a C library. Our function is clearer, faster and more robust than existing sorts. It chooses partitioning elements by a new sampling scheme; it partitions by a novel solution to Dijkstra's Dutch National Flag problem; and it swaps efficiently. Its behavior was assessed with timing and debugging testbeds, and with a program to certify performance. The design techniques apply in domains beyond sorting.**

Now widely used. C, C++, Java 6, ….

# Tukey's ninther

Tukey's ninther. Median of the median of 3 samples, each of 3 entries.

- Approximates the median of 9.
- Uses at most 12 compares.



| | | | | | | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| nine evenly spaced entries | R | L | A | P | M | C | G | A | X | Z | K | R | B | R | J | J | E |
| groups of 3 | R | A | M | | G | X | K | | B | J | E | | | | | | |
| medians | M | K | E | | | | | | | | | | | | | | |
| ninther | K | | | | | | | | | | | | | | | | |

Q. Why use Tukey's ninther?

A. Better partitioning than random shuffle and less costly.

# Achilles heel in Bentley-McIlroy implementation (Java system sort)

Q. Based on all this research, Java's system sort is solid, right?

A. No: a killer input.

- Overflows function call stack in Java and crashes program.
- Would take quadratic time if it didn't crash first.

```
% more 250000.txt
0
218750
222662
11
166672
247070
83339
...
```

250,000 integers
between 0 and 250,000

```
% java IntegerSort 250000 < 250000.txt
Exception in thread "main"
java.lang.StackOverflowError
   at java.util.Arrays.sort1(Arrays.java:562)
   at java.util.Arrays.sort1(Arrays.java:606)
   at java.util.Arrays.sort1(Arrays.java:608)
   at java.util.Arrays.sort1(Arrays.java:608)
   at java.util.Arrays.sort1(Arrays.java:608)
   ...
```

Java's sorting library crashes, even if
you give it as much stack space as Windows allows

# System sort: Which algorithm to use?

Many sorting algorithms to choose from:

Internal sorts.
- Insertion sort, selection sort, bubblesort, shaker sort.
- Quicksort, mergesort, heapsort, samplesort, shellsort.
- Solitaire sort, red-black sort, splaysort, Yaroslavskiy sort, psort, ...

External sorts. Poly-phase mergesort, cascade-merge, oscillating sort.

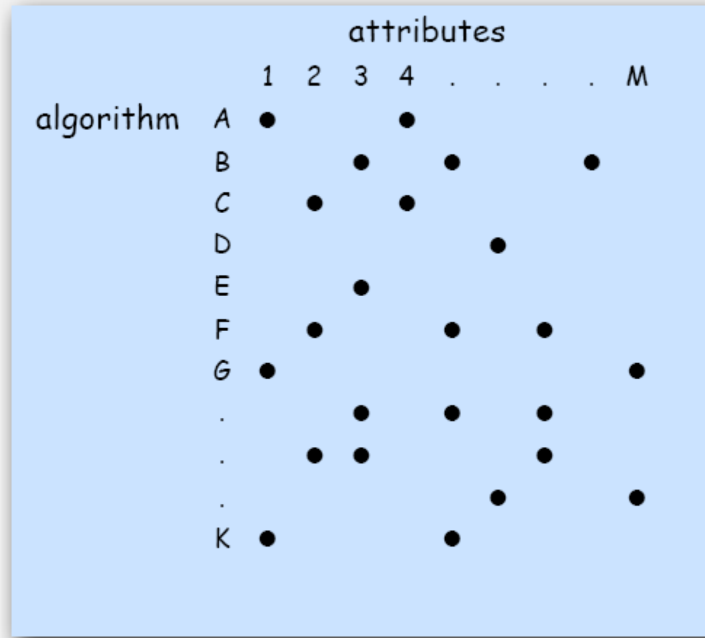
String/radix sorts. Distribution, MSD, LSD, 3-way string quicksort.

Parallel sorts.
- Bitonic sort, Batcher even-odd sort.
- Smooth sort, cube sort, column sort.
- GPUsort.

# System sort: Which algorithm to use?

Applications have diverse attributes.

- Stable?
- Parallel?
- Deterministic?
- Keys all distinct?
- Multiple key types?
- Linked list or arrays?
- Large or small items?
- Is your array randomly ordered?
- Need guaranteed performance?




many more combinations of attributes than algorithms

Elementary sort may be method of choice for some combination.

Cannot cover all combinations of attributes.

Q. Is the system sort good enough?

A. Usually.

# Sorting summary

| | inplace? | stable? | worst | average | best | remarks |
|---|---|---|---|---|---|---|
| selection | ✔ | | $N^2 / 2$ | $N^2 / 2$ | $N^2 / 2$ | *N* exchanges |
| insertion | ✔ | ✔ | $N^2 / 2$ | $N^2 / 4$ | N | use for small *N* or partially ordered |
| shell | ✔ | | ? | ? | N | tight code, subquadratic |
| merge | | ✔ | N lg N | N lg N | N lg N | *N* log *N* guarantee, stable |
| quick | ✔ | | $N^2 / 2$ | 2 N ln N | N lg N | *N* log *N* probabilistic guarantee fastest in practice |
| 3-way quick | ✔ | | $N^2 / 2$ | 2 N ln N | N | improves quicksort in presence of duplicate keys |
| ??? | ✔ | ✔ | N lg N | N lg N | N | holy sorting grail |

Algorithms

Robert Sedgewick | Kevin Wayne

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