Gate Assignment - 2

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Download all the python codes from

https://github.com/cmaspi/EE3900/tree/main/GateAssignment−2/code

latex-tikz codes from

https://github.com/cmaspi/EE3900/blob/main/GateAssignment−2/main.tex

1 Problem

(GATE EC 2009 Q42) The 4-point discrete fourier transform (DFT) of a discrete time sequence $[1, 0, 2, 3]$ is given by

1) $[0, -2 + 2j, 2, -2 - 2j]$
2) $[2, 2 + j, 6, 2 - 2j]$
3) $[6, 1 - 3j, 2, 1 + 3j]$
4) $[6, -1 + 3j, 0, -1, -3j]$

2 Solution

The discrete input signal is

$$x(n) = [1, 0, 2, 3] \tag{2.0.1}$$

Let $\mathbf{F}_N$ be the N-point DFT matrix.
Using the property of complex exponentials, we can express $\mathbf{F}_N$ in terms of $\mathbf{F}_{N/2}$

$$\mathbf{F}_N = \begin{pmatrix} \mathbf{I}_{N/2} & \mathbf{D}_{N/2} \\ \mathbf{I}_{N/2} & -\mathbf{D}_{N/2} \end{pmatrix} \begin{pmatrix} \mathbf{F}_{N/2} & \mathbf{0} \\ \mathbf{0} & \mathbf{F}_{N/2} \end{pmatrix} \mathbf{P}_N \tag{2.0.2}$$

Where

$$\mathbf{F}_N = \begin{pmatrix} 1 & 1 & \cdots & 1 \\ 1 & w & \cdots & w^{N-1} \\ \vdots & \vdots & \ddots & \vdots \\ 1 & w^{N-1} & \cdots & w^{(N-1)(N-1)} \end{pmatrix} \tag{2.0.3}$$

where $w = e^{\frac{-2\pi j}{N}}$

$$\mathbf{D}_N = \begin{pmatrix} w_{2N}^0 & 0 & \cdots \\ 0 & w_{2N}^1 & \cdots \\ \vdots & \vdots & \ddots \end{pmatrix} \tag{2.0.4}$$

$$\mathbf{D}_N = \text{diag}(w_{2N}^0, w_{2N}^1, \ldots, w_{2N}^{N-1}) \tag{2.0.5}$$

P_N is the permutation matrix defined as

$$\mathbf{P}_N = \left[a_{ij}\right]_{N \times N}, \; i, j \in \{0, 1, \ldots, N-1\} \tag{2.0.6}$$

$$a_{ij} = \begin{cases} 1 & j = 2i, \; i < \frac{N}{2} \\ 1 & j = 2\left(i - \frac{N}{2}\right) + 1, \; i \geq \frac{N}{2} \\ 0 & otherwise \end{cases} \tag{2.0.7}$$

For $N = 4$

$$\mathbf{F}_4 = \begin{pmatrix} \mathbf{I}_2 & \mathbf{D}_2 \\ \mathbf{I}_2 & -\mathbf{D}_2 \end{pmatrix} \begin{pmatrix} \mathbf{F}_2 & \mathbf{0} \\ \mathbf{0} & \mathbf{F}_2 \end{pmatrix} \mathbf{P}_4 \tag{2.0.8}$$

I_2 is 2×2 matrix

$$\mathbf{D}_2 = \begin{pmatrix} w_4^0 & 0 \\ 0 & w_4^1 \end{pmatrix} \tag{2.0.9}$$

$$\mathbf{P}_4 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{2.0.10}$$

Now,

$$\begin{pmatrix} X(0) \\ X(1) \end{pmatrix} = \begin{pmatrix} X_e(0) \\ X_e(1) \end{pmatrix} + \mathbf{D}_2 \begin{pmatrix} X_o(0) \\ X_o(1) \end{pmatrix} \tag{2.0.11}$$

$$\begin{pmatrix} X(2) \\ X(3) \end{pmatrix} = \begin{pmatrix} X_e(0) \\ X_e(1) \end{pmatrix} - \mathbf{D}_2 \begin{pmatrix} X_o(0) \\ X_o(1) \end{pmatrix} \tag{2.0.12}$$

This results in

$$\begin{pmatrix} X(0) \\ X(1) \end{pmatrix} = \mathbf{F}_2 \begin{pmatrix} 1 \\ 2 \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ 0 & -j \end{pmatrix} \mathbf{F}_2 \begin{pmatrix} 0 \\ 3 \end{pmatrix} \tag{2.0.13}$$

$$\begin{pmatrix} X(0) \\ X(1) \end{pmatrix} = \begin{pmatrix} 3 \\ -1 \end{pmatrix} + \begin{pmatrix} 3 \\ 3j \end{pmatrix} \tag{2.0.14}$$

$$\begin{pmatrix} X(0) \\ X(1) \end{pmatrix} = \begin{pmatrix} 6 \\ -1 + 3j \end{pmatrix} \tag{2.0.15}$$

and

$$\begin{pmatrix} X(2) \\ X(3) \end{pmatrix} = \mathbf{F}_2 \begin{pmatrix} 1 \\ 2 \end{pmatrix} - \begin{pmatrix} 1 & 0 \\ 0 & -j \end{pmatrix} \mathbf{F}_2 \begin{pmatrix} 0 \\ 3 \end{pmatrix} \tag{2.0.16}$$

$$\begin{pmatrix} X(2) \\ X(3) \end{pmatrix} = \begin{pmatrix} 3 \\ -1 \end{pmatrix} - \begin{pmatrix} 3 \\ 3j \end{pmatrix} \tag{2.0.17}$$

$$\begin{pmatrix} X(2) \\ X(3) \end{pmatrix} = \begin{pmatrix} 0 \\ -1 - 3j \end{pmatrix} \tag{2.0.18}$$

The correct answer is **option 4**

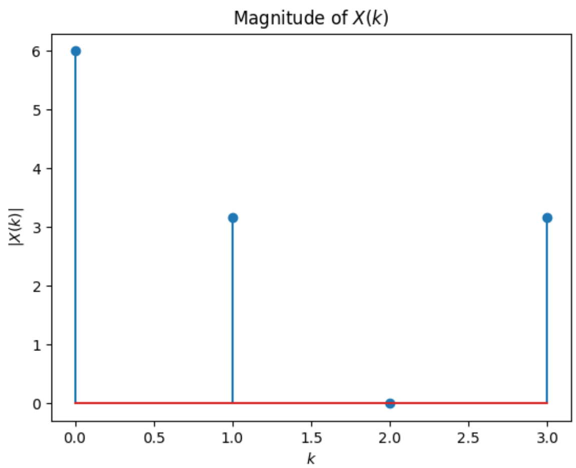

Fig. 4: Magnitude of $X(k)$

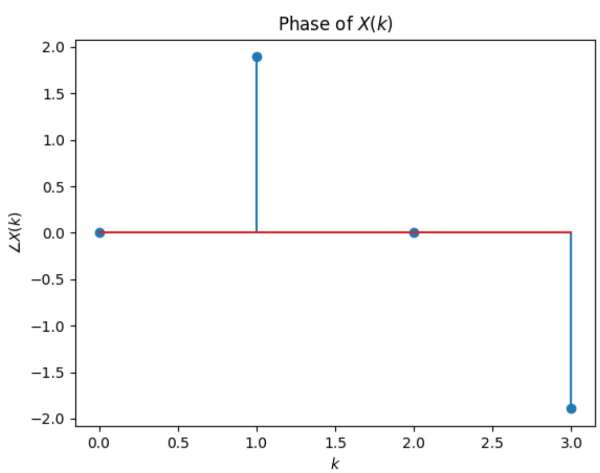

Fig. 4: Phase of $X(k)$

The following is a plot for time complexity of fast fourier transform obtained for multiple random singals. The value of k obtained from my computation is $3.25 \times 10^{-6} \pm 2\%$

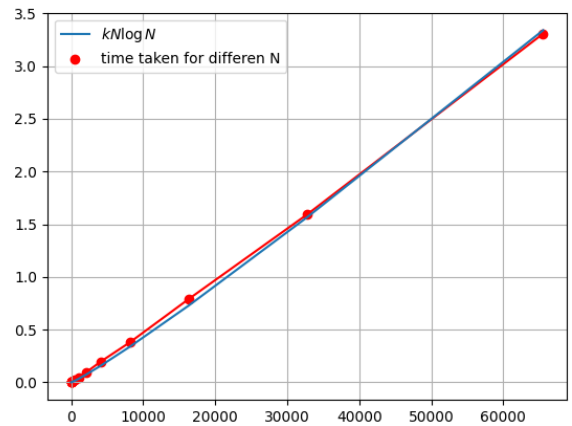

Fig. 4: Time Complexity of fast Fourier transform