Assignment 5

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Download all the python codes from

```
https://github.com/cmaspi/EE3900/tree/main/
    Assignment−5/code
```

latex-tikz codes from

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https://github.com/cmaspi/EE3900/blob/main/
    Assignment−5/main.tex
```

1 Problem

(Quadratic forms Q2.66) Find the point at which the line $\begin{pmatrix} -1 & 1 \end{pmatrix} \mathbf{x} = 1$ is a tangent to the curve $y^2 = 4x$

2 Solution

Theorem 2.1. *The solution to a under-determined system of equations* $\mathbf{Ax} = \mathbf{b}$ *is given by*

$$\mathbf{x} = \mathbf{A}^{+}\mathbf{b} + (\mathbf{I} - \mathbf{A}^{+}\mathbf{A})\mathbf{w} \tag{2.0.1}$$

where $\mathbf{A}^{+}$ *is the pseudoinverse of the matrix* $\mathbf{A}$

Proof. Let $\mathbf{Ax} = \mathbf{b}$ have at least one solution

$$\mathbf{Ax} = \mathbf{b} \tag{2.0.2}$$
$$\mathbf{AA}^{+}(\mathbf{Ax}) = \mathbf{AA}^{+}(\mathbf{b}) \tag{2.0.3}$$

Using property of pseudoinverse

$$\mathbf{Ax} = \mathbf{AA}^{+}\mathbf{b} = \mathbf{b} \tag{2.0.4}$$

Therefore, $\mathbf{A}^{+}\mathbf{b}$ is a specific solution.
The entire set of solution is given by $\mathbf{A}^{+}\mathbf{b}+\mathbf{k}$, where $\mathbf{k}$ is a vector in kernel space or null space of $\mathbf{A}$

$$\mathbf{A}\left(\mathbf{A}^{+}\mathbf{b} + \mathbf{k}\right) = \mathbf{b} \tag{2.0.5}$$
$$\mathbf{b} + \mathbf{Ak} = \mathbf{b} \tag{2.0.6}$$
$$\mathbf{Ak} = \mathbf{0} \tag{2.0.7}$$

Any vector in the null space of $\mathbf{A}$ can be written as

$$\mathbf{k} = (\mathbf{I} - \mathbf{A}^{+}\mathbf{A})\mathbf{w} \tag{2.0.8}$$

where $\mathbf{w}$ is any vector with appropriate dimension. Now, we will prove that (2.0.8) holds true

$$\mathbf{k} = (\mathbf{I} - \mathbf{A}^{+}\mathbf{A})\mathbf{k} \tag{2.0.9}$$
$$\mathbf{k} = \mathbf{k} + \mathbf{A}^{+}(\mathbf{Ak}) \tag{2.0.10}$$
$$\therefore \mathbf{k} = \mathbf{k} + \mathbf{A}^{+}(\mathbf{0}) \tag{2.0.11}$$

Therefore any vector in null space of $\mathbf{A}$ is also in the image space of $\mathbf{I} - \mathbf{A}^{+}\mathbf{A}$

$$\mathbf{A}(\mathbf{I} - \mathbf{A}^{+}\mathbf{A})\mathbf{w} = (\mathbf{A} - \mathbf{A})\mathbf{w} = \mathbf{0} \tag{2.0.12}$$

Therefore, the null space of $\mathbf{A}$ and image space of $\mathbf{I} - \mathbf{A}^{+}\mathbf{A}$ are the same

$$\mathbf{x} = \mathbf{A}^{+}\mathbf{b} + (\mathbf{I} - \mathbf{A}^{+}\mathbf{A})\mathbf{w} \tag{2.0.13}$$

□

Lemma 2.1. *For an idempotent matrix the pseudoinverse is the matrix itself*

Proof. For an symmetric idempotent matrix

$$\mathbf{A}^{n} = \mathbf{A} \tag{2.0.14}$$
$$\mathbf{AAA} = \mathbf{A} \tag{2.0.15}$$

Therefore, $\mathbf{A} = \mathbf{A}^{+}$ satisfies all the conditions of pseudoinverse which are listed as follows

1) $\mathbf{AA}^{+}\mathbf{A} = \mathbf{A}$
2) $\mathbf{A}^{+}\mathbf{AA}^{+} = \mathbf{A}^{+}$
3) $\mathbf{AA}^{+}$ is symmetric
4) $\mathbf{A}^{+}\mathbf{A}$ is symmetric

□

The general form of a conic is given by

$$\mathbf{x}^{T}\mathbf{Vx} + 2\mathbf{u}^{T}\mathbf{x} + f = 0 \tag{2.0.16}$$

for $y^2 = 4x$

$$\mathbf{V} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \ \mathbf{u} = \begin{pmatrix} -2 \\ 0 \end{pmatrix}, \ f = 0 \tag{2.0.17}$$

For any given conic such that $\mathbf{V}$ is non-invertible, the points of tangency are given by

$$\mathbf{Vq} = k\mathbf{n} - u \tag{2.0.18}$$

$$\text{where, } k = \frac{\mathbf{p_1}^T\mathbf{u}}{\mathbf{p_1}^T\mathbf{n}}, \mathbf{Vp_1} = 0 \tag{2.0.19}$$

Clearly

$$\mathbf{p_1} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \tag{2.0.20}$$

$$\implies k = \frac{-2}{-1} = 2 \tag{2.0.21}$$

Using the obtained values

$$\mathbf{Vq} = 2\mathbf{n} - \mathbf{u} \tag{2.0.22}$$

$$\mathbf{Vq} = \begin{pmatrix} -2 \\ 2 \end{pmatrix} - \begin{pmatrix} -2 \\ 0 \end{pmatrix} = 2\mathbf{e_2} \tag{2.0.23}$$

Using (2.0.1) and (2.1)

$$\mathbf{q} = 2\mathbf{Ve_2} + (\mathbf{I} - \mathbf{V}^2)\mathbf{w} \tag{2.0.24}$$

$$\mathbf{q} = 2\mathbf{e_2} + \lambda\mathbf{e_1} \tag{2.0.25}$$

Using (2.0.25) and (2.0.16), we get

$$2\mathbf{q}^T\mathbf{e_2} - 4\mathbf{e_1}^T\mathbf{q} = 0 \tag{2.0.26}$$

Evaluating this, we get

$$4 - \lambda = 0 \tag{2.0.27}$$

$$\implies \mathbf{q} = \begin{pmatrix} 1 \\ 2 \end{pmatrix} \tag{2.0.28}$$

A plot for the line and parabola is given below

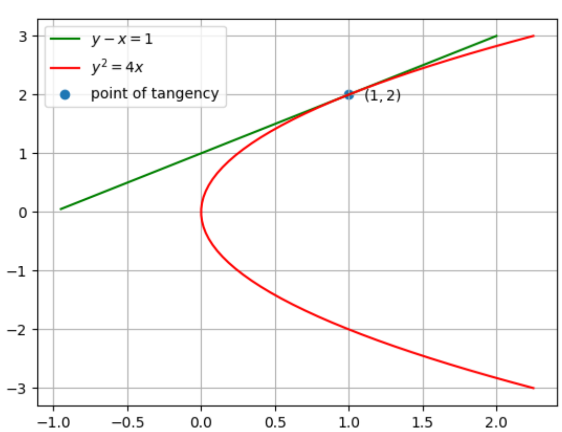

Fig. 4: Plot of the line and parabola