

# Assignment 5

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Download all the python codes from

<https://github.com/cmaspi/EE3900/tree/main/Assignment-5/code>

latex-tikz codes from

<https://github.com/cmaspi/EE3900/blob/main/Assignment-5/main.tex>

where  $\mathbf{w}$  is any vector with appropriate dimension. Now, we will prove that (2.0.8) holds true

$$\mathbf{k} = (\mathbf{I} - \mathbf{A}^+\mathbf{A})\mathbf{k} \quad (2.0.9)$$

$$\mathbf{k} = \mathbf{k} + \mathbf{A}^+(\mathbf{A}\mathbf{k}) \quad (2.0.10)$$

$$\therefore \mathbf{k} = \mathbf{k} + \mathbf{A}^+(\mathbf{0}) \quad (2.0.11)$$

Therefore any vector in null space of  $\mathbf{A}$  is also in the null space of  $\mathbf{I} - \mathbf{A}^+\mathbf{A}$

$$\mathbf{A}(\mathbf{I} - \mathbf{A}^+\mathbf{A})\mathbf{w} = (\mathbf{A} - \mathbf{A})\mathbf{w} = \mathbf{0} \quad (2.0.12)$$

Therefore, the null space of  $\mathbf{A}$  and  $\mathbf{I} - \mathbf{A}^+\mathbf{A}$  are the same

$$\mathbf{x} = \mathbf{A}^+\mathbf{b} + (\mathbf{I} - \mathbf{A}^+\mathbf{A})\mathbf{w} \quad (2.0.13)$$

□

## 1 PROBLEM

(Quadratic forms Q2.66) Find the point at which the line  $(-1 \ 1)\mathbf{x} = 1$  is a tangent to the curve  $y^2 = 4x$

## 2 SOLUTION

**Theorem 2.1.** The solution to a under-determined system of equations  $\mathbf{Ax} = \mathbf{b}$  is given by

$$\mathbf{x} = \mathbf{A}^+\mathbf{b} + (\mathbf{I} - \mathbf{A}^+\mathbf{A})\mathbf{w} \quad (2.0.1)$$

where  $\mathbf{A}^+$  is the pseudoinverse of the matrix  $\mathbf{A}$

*Proof.* Let  $\mathbf{Ax} = \mathbf{b}$  have at least one solution

$$\mathbf{Ax} = \mathbf{b} \quad (2.0.2)$$

$$\mathbf{AA}^+(\mathbf{Ax}) = \mathbf{AA}^+(\mathbf{b}) \quad (2.0.3)$$

Using property of pseudoinverse

$$\mathbf{Ax} = \mathbf{A}^+\mathbf{Ab} = \mathbf{b} \quad (2.0.4)$$

Therefore,  $\mathbf{A}^+\mathbf{b}$  is a specific solution.

The entire set of solution is given by  $\mathbf{A}^+\mathbf{b} + \mathbf{k}$ , where  $\mathbf{k}$  is a vector in kernel space or null space of  $\mathbf{A}$

$$\mathbf{A}(\mathbf{A}^+\mathbf{b} + \mathbf{k}) = \mathbf{b} \quad (2.0.5)$$

$$\mathbf{b} + \mathbf{Ak} = \mathbf{b} \quad (2.0.6)$$

$$\mathbf{Ak} = \mathbf{0} \quad (2.0.7)$$

Any vector in the null space of  $\mathbf{A}$  can be written as

$$\mathbf{k} = (\mathbf{I} - \mathbf{A}^+\mathbf{A})\mathbf{w} \quad (2.0.8)$$

**Lemma 2.1.** For an idempotent matrix the pseudoinverse is the matrix itself

*Proof.* For an symmetric idempotent matrix

$$\mathbf{A}^n = \mathbf{A} \quad (2.0.14)$$

$$\mathbf{AAA} = \mathbf{A} \quad (2.0.15)$$

Therefore,  $\mathbf{A} = \mathbf{A}^+$  satisfies all the conditions of pseudoinverse which are listed as follows

- 1)  $\mathbf{AA}^+\mathbf{A} = \mathbf{A}$
- 2)  $\mathbf{A}^+\mathbf{AA}^+ = \mathbf{A}^+$
- 3)  $\mathbf{AA}^+$  is symmetric
- 4)  $\mathbf{A}^+\mathbf{A}$  is symmetric

□

The general form of a conic is given by

$$\mathbf{x}^T\mathbf{V}\mathbf{x} + 2\mathbf{u}^T\mathbf{x} + f = 0 \quad (2.0.16)$$

for  $y^2 = 4x$

$$\mathbf{V} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \mathbf{u} = \begin{pmatrix} -2 \\ 0 \end{pmatrix}, f = 0 \quad (2.0.17)$$

For any given conic such that  $\mathbf{V}$  is non-invertible,

the points of tangency are given by

$$\mathbf{V}\mathbf{q} = k\mathbf{n} - \mathbf{u} \quad (2.0.18)$$

$$\text{where, } k = \frac{\mathbf{p}_1^T \mathbf{u}}{\mathbf{p}_1^T \mathbf{n}}, \mathbf{V}\mathbf{p}_1 = 0 \quad (2.0.19)$$

Clearly

$$\mathbf{p}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad (2.0.20)$$

$$\Rightarrow k = \frac{-2}{-1} = 2 \quad (2.0.21)$$

Using the obtained values

$$\mathbf{V}\mathbf{q} = 2\mathbf{n} - \mathbf{u} \quad (2.0.22)$$

$$\mathbf{V}\mathbf{q} = \begin{pmatrix} -2 \\ 2 \end{pmatrix} - \begin{pmatrix} -2 \\ 0 \end{pmatrix} = 2\mathbf{e}_2 \quad (2.0.23)$$

Using (2.0.1) and (2.1)

$$\mathbf{q} = 2\mathbf{V}\mathbf{e}_2 + (\mathbf{I} - \mathbf{V}^2)\mathbf{w} \quad (2.0.24)$$

$$\mathbf{q} = 2\mathbf{e}_2 + \lambda\mathbf{e}_1 \quad (2.0.25)$$

Using (2.0.25) and (2.0.16), we get

$$2\mathbf{q}^T \mathbf{e}_2 - 4\mathbf{e}_1^T \mathbf{q} = 0 \quad (2.0.26)$$

Evaluating this, we get

$$4 - \lambda = 0 \quad (2.0.27)$$

$$\Rightarrow \mathbf{q} = \begin{pmatrix} 1 \\ 2 \end{pmatrix} \quad (2.0.28)$$

A plot for the line and parabola is given below

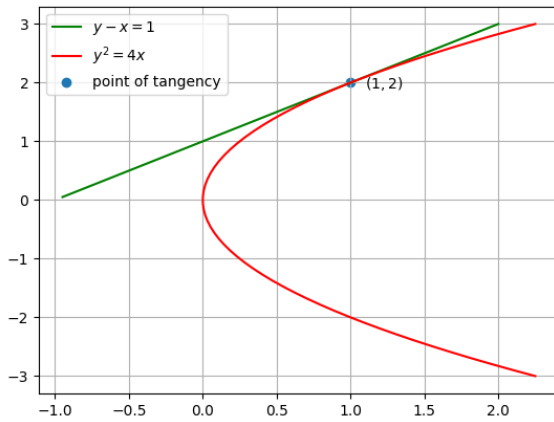


Fig. 4: Plot of the line and parabola