Assignment 5 Presentation

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Question

Quadratic forms Q2.66

Find the point at which the line $\begin{pmatrix} -1 & 1 \end{pmatrix} \mathbf{x} = 1$ is a tangent to the curve $y^2 = 4x$

Theorem

Theorem

The solution to a under-determined system of equations $\mathrm{Ax} = \mathrm{b}$ *is given by*

$$\mathrm{x} = \mathrm{A}^{+}\mathrm{b} + (\mathrm{I} - \mathrm{A}^{+}\mathrm{A})\mathrm{w} \tag{1}$$

where A^{+} *is the pseudoinverse of the matrix* A

Proof.

Let $Ax = b$ have at least one solution

$$Ax = b \tag{2}$$

$$AA^+(Ax) = AA^+(b) \tag{3}$$

Using property of pseudoinverse

$$Ax = AA^+b = b \tag{4}$$

Therefore, A^+b is a specific solution.

Proof.

The entire set of solution is given by $A^{+}b + k$, where k is a vector in kernel space or null space of A

$$A\left(A^{+}b + k\right) = b \tag{2}$$

$$b + Ak = b \tag{3}$$

$$Ak = 0 \tag{4}$$

Any vector in the null space of A can be written as

$$k = (I - A^{+}A)w \tag{5}$$

where w is any vector with appropriate dimension.

Proof.

Now, we will prove that the above equation holds true

$$k = (I - A^{+}A)k \tag{2}$$

$$k = k + A^{+}(Ak) \tag{3}$$

$$\therefore k = k + A^{+}(0) \tag{4}$$

Therefore any vector in null space of A is also in the image space of $I - A^{+}A$

$$A(I - A^{+}A)w = (A - A)w = 0 \tag{5}$$

Therefore, the null space of A and image space of $I - A^{+}A$ are the same

$$x = A^{+}b + (I - A^{+}A)w \tag{6}$$

$\square$

Lemma

For an idempotent matrix the pseudoinverse is the matrix itself

Proof.

For an symmetric idempotent matrix

$$A^n = A \tag{7}$$

$$AAA = A \tag{8}$$

Therefore, $A = A^+$ satisfies all the conditions of pseudoinverse which are listed as follows

1. $AA^+A = A$
2. $A^+AA^+ = A^+$
3. AA^+ is symmetric
4. A^+A is symmetric

□

Solution

The general form of a conic is given by

$$\mathrm{x}^T\mathrm{V}\mathrm{x} + 2\mathrm{u}^T\mathrm{x} + f = 0 \tag{9}$$

for $y^2 = 4x$

$$\mathrm{V} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix},\ \mathrm{u} = \begin{pmatrix} -2 \\ 0 \end{pmatrix},\ f = 0 \tag{10}$$

Solution Contd.

For any given conic such that V is non-invertible, the point of tangency is given by

$$\mathrm{V}\mathbf{q} = k\mathbf{n} - u \tag{11}$$

$$\text{where, } k = \frac{\mathbf{p}_1^T \mathbf{u}}{\mathbf{p}_1^T \mathbf{n}}, \ \mathrm{V}\mathbf{p}_1 = 0 \tag{12}$$

Clearly

$$\mathbf{p}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \tag{13}$$

$$\implies k = \frac{-2}{-1} = 2 \tag{14}$$

Solution Contd.

Using the obtained values

$$\mathrm{Vq} = 2\mathrm{n} - \mathrm{u} \tag{15}$$

$$\mathrm{Vq} = \begin{pmatrix} -2 \\ 2 \end{pmatrix} - \begin{pmatrix} -2 \\ 0 \end{pmatrix} = 2\mathrm{e}_2 \tag{16}$$

Using (1) and (2)

$$\mathrm{q} = 2\mathrm{Ve}_2 + (\mathrm{I} - \mathrm{V}^2)\mathrm{w} \tag{17}$$

$$\mathrm{q} = 2\mathrm{e}_2 + \lambda\mathrm{e}_1 \tag{18}$$

Solution Contd.

Using (18) and (9), we get

$$2\mathbf{q}^T\mathbf{e}_2 - 4\mathbf{e}_1^T\mathbf{q} = 0 \tag{19}$$

Evaluating this, we get

$$4 - \lambda = 0 \tag{20}$$

$$\implies \mathbf{q} = \begin{pmatrix} 1 \\ 2 \end{pmatrix} \tag{21}$$

Solution Contd.

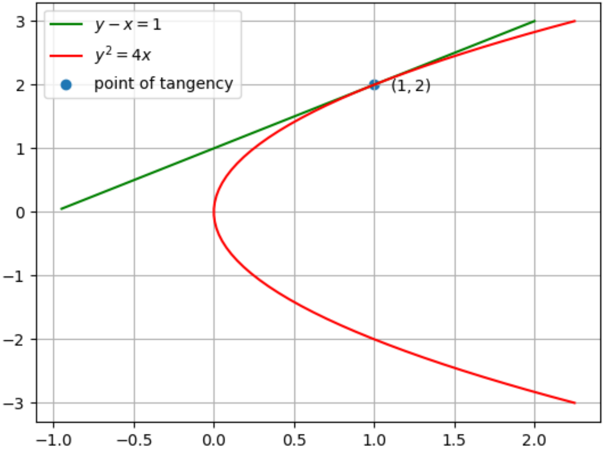

Figure: Plot of the line and parabola