Problem-2

Part-a

For P

$$\|P(v) - P(u)\| \leq \gamma_1 \|v - u\|$$

and for Q

$$\|Q(v) - Q(u)\| \leq \gamma_2 \|v - u\|$$

In the first equation substitute v by $Q(v)$ and u by $Q(u)$. We get,

$$\|P(Q(v)) - P(Q(u))\| \leq \gamma_1 \|Q(v) - Q(u)\|$$

From the second equation

$$\|P(Q(v)) - P(Q(u))\| \leq \gamma_1 \gamma_2 \|v - u\|$$

$\therefore$ $P \circ Q$ is a contraction mapping

Now, there's nothing unique about P or Q, we could just swap them and we would still get the same result. Or we would follow the same procedure and get

$$\|Q(P(v)) - Q(P(u))\| \leq \gamma_1 \gamma_2 \|v - u\|$$

Part-b

From the solution to above subpart, we can see that the contraction coefficient for both the composite functions is $\gamma_1 \gamma_2$ where γ_1, γ_2 are the contraction coefficients for P and Q respectively.

Part-c

The value iteration scheme under the operator would converge to a unqiue solution if it follows

$$\|B(v) - B(u)\| \leq \|v - u\|$$

Writing B as a composite function, we can arrive at a general solution with a specific behaviour

$$\|F(L(v)) - F(L(u))\| \leq \|v - u\|$$

Assume $\|F(v) - F(u)\| \leq \eta \|v - u\|$ Now

$$\|F(L(v)) - F(L(u))\| \leq \eta \|L(v) - L(u)\| \leq \eta\gamma \|v - u\|$$

We get that $\eta < \frac{1}{\gamma}$