

## Lecture 9

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April 12, 2022

### Quadratic Problems

A convex quadratic program is just a least squares problem with linear constraints. Consider the following formulation

$$\min_{\underline{x}} \underline{x}^T Q \underline{x} + \underline{c}^T \underline{x} \quad (1)$$

$$\text{s.t: } C \underline{x} \leq \underline{d} \quad (2)$$

RECALL: Every positive definite matrix is a gram matrix

We can write

$$\exists R : Q = R^T R \quad (3)$$

We can rewrite our optimization problem as

$$\min_{\underline{x}} \underline{x}^T R^T R \underline{x} + \underline{c}^T \underline{x} \quad (4)$$

$$\text{s.t: } C \underline{x} \leq \underline{d} \quad (5)$$

Further, we can write

$$\exists \underline{b} : \underline{c} = 2R^T \underline{b} \quad (6)$$

<sup>1</sup> Further, we can rewrite the optimization problem as a least square problems using (4) and (6)

<sup>1</sup> since  $R^T$  is a full rank matrix, a unique solution must exist for the above system of equations.

$$\min_{\underline{x}} \|\underline{R}\underline{x} - \underline{b}\|^2 - \underline{b}^T \underline{b} \quad (7)$$

$$\text{s.t: } C \underline{x} \leq \underline{d} \quad (8)$$

$$\underline{c} = 2R^T \underline{b}$$

**Problem:** Given two polyhedra, determine the distance between them.

$$P_1 : A \underline{x} \leq \underline{b} \quad (9)$$

$$P_2 : C \underline{x} \leq \underline{d} \quad (10)$$

The optimization problem can be formulated as follows

$$\min_{\underline{x}_1, \underline{x}_2} \|\underline{x}_1 - \underline{x}_2\| \quad (11)$$

$$\text{s.t: } \underline{x}_1 \in P_1 \quad (12)$$

$$\underline{x}_2 \in P_2$$

The above is a quadratic problem with linear constraints, next we will look at a Quadratically Constrained Quadratic Programs (QCQP)

$$\min_{\underline{x}} \underline{x}^T Q \underline{x} + \underline{b}^T \underline{x} \quad (13)$$

$$\text{s.t: } \underline{x}^T Q_i \underline{x} + \underline{b}_i^T \underline{x} \leq c_i \quad (14)$$

$$C \underline{x} \leq \underline{d}$$

### Example: Portfolio optimization

$$\min_{\underline{x}} \underline{x}^T \Sigma \underline{x} - \lambda \underline{\mu}^T \underline{x} \quad (15)$$

$$\text{s.t: } \underline{x}^T \underline{1} = 1 \quad (16)$$

$$\underline{x} \geq 0$$

where  $\Sigma$  is the covariance matrix that accounts for the risk.<sup>2</sup> The above problem is a quadratic program since the objective is quadratic, while the constraints are linear.

<sup>2</sup> if  $\lambda$  is small then we are taking low risk,  $\lambda$  controls the risk against expected returns.

**Example: Linear Discriminator** Lets say we have a dataset of two classes

$$C1 : \{x_1, x_2, \dots, x_m\}$$

$$C2 : \{y_1, y_2, \dots, y_m\}$$

Check whether the given datapoints are linearly separable.

*Initial Thought:* Check if the convex hulls of the given datapoints intersect, if yes, then the classes are not linearly separable.<sup>3</sup>

<sup>3</sup> Note that this problem is a feasibility problem.

*Another method:* Consider the following two inequalities for respective classes

$$C1 : \underline{a}^T \underline{x}_i - b \geq 1 \quad (17)$$

$$C2 : \underline{a}^T \underline{y}_i - b \leq -1$$

We can frame our feasibility problem as

$$\min_{\underline{a}, b} 0 \quad (18)$$

$$\text{s.t: } \underline{a}^T \underline{x}_i - b \geq 1, i = 1, 2, \dots, m \quad (19)$$

$$\underline{a}^T \underline{y}_i - b \leq -1, i = 1, 2, \dots, m$$

<sup>4</sup> Both the listed methods are equivalent.

Further, we can extend the problem from a feasibility problem to hard margin support vector machine.

<sup>4</sup> The reason why we choose 1 instead of 0 in the inequality is because of inequalities are treated the same way as equalities in cvxpy solver, the problem will then always be feasible by choose  $\underline{a}$  as zero vector, and  $b$  to be 0

$$\min_{\underline{a}, b} \|\underline{a}\|^2 \quad (20)$$

$$\text{s.t: } \underline{a}^T \underline{x}_i - b \geq 1, i = 1, 2, \dots, m \quad (21)$$

$$\underline{a}^T \underline{y}_i - b \leq -1, i = 1, 2, \dots, m$$

This is a quadratic program (QP), we can however, reformulate this problem as follows

$$\min_{\underline{a}, b, t} t \quad (22)$$

$$\text{s.t: } \underline{a}^T \underline{x}_i - b \geq t, i = 1, 2, \dots, m \quad (23)$$

$$\underline{a}^T \underline{y}_i - b \leq -t, i = 1, 2, \dots, m$$

$$\|\underline{a}\| \leq 1$$

The above formulation in Quadratically Constrained Quadratic Program (QCQP).<sup>5</sup>

**Assertion:** The above two problems are equivalent. Solving one would also solve the other, we can arrive at solution for latter from the prior formulation

Lets say after solving the prior, we get  $\underline{a}', b'$  as the optimal solution, we can divide both  $\underline{a}', b'$  by  $\|\underline{a}'\|$ , to get the solution to our latter formulation.

<sup>5</sup> Since the margin in our latter problem is  $\frac{2t}{\|\underline{a}\|}$ , but we are maximizing  $t$ , then we would get  $\|\underline{a}\| = 1$