

A Feasibility Metric for Trajectory Optimization of Legged Robots using Wrench Polytopes

Romeo Orsolino¹ Michele Focchi¹ Carlos Mastalli^{1,2}
Hongkai Dai³ Darwin G. Caldwell¹ Claudio Semini¹

Abstract—Motion planning in multi-contact scenarios has recently gathered interest within the legged robotics community, however actuator force/torque limits are rarely considered. We believe that these limits gain paramount importance when the complexity of the terrains to be traversed increases. For this reason we propose two new six-dimensional bounded polytopes named the Actuation Wrench Polytope (AWP) and the Feasible Wrench Polytope (FWP). We define the AWP as the set of all the wrenches that a robot can generate on its own Center of Mass (CoM) while considering its actuation limits. This considers the admissible contact forces that the robot can generate given its current configuration and actuation capabilities but does not include features of the environment such as the contact normal or the friction coefficient. These are considered by the Contact Wrench Cone (CWC); the AWP can therefore be seen as complementary with respect to the CWC. The intersection of the AWP and of the CWC results in a polytope, the FWP, which turns out to be more descriptive of the real robot capabilities, while maintaining the same compact representation. We explain how to efficiently compute the vertex-description of the FWP and we also introduce a new locomotion stability metric based on the FWP, that we call *feasibility margin*, which allows us to optimize for robustness to external disturbance wrenches. Based on this, we present an implementation of a motion planner for our quadruped robot HyQ that provides *online* CoM trajectories that are guaranteed to be stable and actuation-consistent.

I. INTRODUCTION

Legged locomotion in rough terrains requires the careful selection of a contact sequence along with a feasible motion of the CoM. In case of an unexpected event (e.g. changes in the terrain conditions, human operator commands, external force disturbance, inaccuracies in the state estimation and in the terrain mapping, etc.) replanning is an important feature to avoid accumulation of errors. As a consequence ideal motion planners for complex terrains should be fast but accurate. Approaches that use simplified dynamic models are especially fast but they only capture the main dynamics of the system [1], [2]. On the other hand other approaches use the whole-body model of the robot and provide particularly accurate joint torques and position trajectories but are not suitable for online applications in arbitrary terrains. A third option consists in offline learning primitives and behaviors generated with the more accurate whole-body models that can be later quickly realized in real-time [3].

The present paper tackles this issue using simplified dynamic models that still contain sufficient details of the system.

The use of the centroidal dynamics [4] coupled with the CWC-based planning represents a step in this direction, allowing to remove the limitation of having coplanar contacts (as for Zero Moment Point (ZMP) based approaches) and thus increasing the complexity of motions that can be generated [5], [6]. This has also led to the formulation of algorithms that can efficiently verify robots stability in multi-contact scenarios [7], [8]. Such approaches however still fail to capture some properties of the robot - such as the actuation limits, the joints kinematic limits and the possible self-collisions. These properties become more and more important with the increasing complexity of the environment and we believe that they should not be neglected in motion planning. To the best of the author's knowledge, while actuation constraints have been considered at the control level ([9], [10]), this is the first time that a framework for the formulation of actuation consistent online motion planners is provided. As later explained, the strategy consists in devising CoM trajectories that are guaranteed to respect the actuation and friction constraints, without explicitly optimizing neither the joint torques nor the contact forces.

A. Contribution

In this paper we address the problem of devising actuation-consistent motions for legged robots and, in particular, we propose the four following contributions: a) first, we introduce the concept of Actuation Wrench Polytope (AWP) which complements the CWC, adding the robot related constraints such as its configuration and actuation capabilities. The consideration of both the environment related constraints (the CWC) and the robot related constraints (the AWP) leads us to the definition of a second convex wrench polytope that we call Feasible Wrench Polytope (FWP). Discarding the constraints due to self-collision and kinematic joints limits, the FWP can then be used as a sufficient criterion for legged robots stability; b) second, we describe an alternative procedure for an efficient computation of the vertex-description (\mathcal{V} -description) of the FWP that drastically decreases the computation time with respect to standard half-space-description (\mathcal{H} -description) based methods; c) third, we propose a *feasibility metric* that can be computed from the FWP, by solving a Linear Program (LP), to evaluate the feasibility of a motion. d) Finally, we show how the efficient computation of the FWP \mathcal{V} -description can also be exploited for the online generation of CoM trajectories.

¹Department of Advanced Robotics, Istituto Italiano di Tecnologia, Genova, Italy. *email*: {romeo.orsolino, michele.focchi, carlos.mastalli, darwin.caldwell, claudio.semini}@iit.it.

²CNRS, LAAS, University of Toulouse, France. *email*: carlos.mastalli@laas.fr.

³Toyota Research Institute, Los Altos, USA. *email*: hongkai.dai@tri.global.

B. Outline

The rest of this paper is organized as follows: we first discuss the previous research in the field of motion planning with a special consideration for the robot stability and actuation-consistency (Section II). We then introduce the computation of the AWP [11] and an efficient strategy to calculate the \mathcal{V} -description of the FWP (Section III). Section IV explains how the FWP can be used as a criterion for robot stability by defining a FWP-based *feasibility margin*. Section V describes how the FWP can be used for online motion planning using an algorithm that uses the a vertices-based description of the considered sets. Section VI presents the simulations and experimental results we obtained by implementing our strategy on IIT's Hydraulically actuated Quadruped (HyQ) robot [12]. Finally, Section VII draws the conclusions with a brief discussion on the obtained results, mentioning possible future developments for wrench-based online motion planning.

II. RELATED WORK

Wrench-based feasibility analysis is not a novel idea in robotics. For instance, in the field of cable-driven robots under the name of Wrench-Feasible Workspace (WFW) it is intended the set of all the configurations that can be realized respecting the maximum tension in the ropes [13]. The WFW is used to analyze the robot's capability to carry loads, but it does not consider constraints that might arise from the interaction with the environment, such as unilaterality and frictional constraints. The idea of modeling the wrench admissible region is also present in the field of mechanical fixtures and tolerance analysis [14] where reciprocity of twists and screws is exploited to characterize the mobility conditions of any couple of faces in tolerance chains.

In legged locomotion the seminal work of Takao et al. has studied the problem of finding Feasible Solution of Wrench (FSW) in multi-contact configurations [15]. Wrench sets have appeared with the CWC [5], that is a stability criterion for locomotion that is suitable for *non-coplanar* contacts and finite friction coefficients. Dai et al. in [6], [16] have shown how to exploit a CWC-based stability margin to obtain a convex optimization formulation that can plan CoM and joints trajectories of legged robots on complex terrains. On a similar line, Caron et al. [17] have focused on improving the real-time performances of CWC-based motion planning either exploiting the double-description of the 6D polyhedra or by considering lower dimensional projections of the CWC defining full-support areas that, coupled with a linear pendulum model, led to the definition of the *pendular support area* [18].

Despite the excellent results shown in this field the lack of successful experimental implementations on the hardware is mainly due to the fact that often the desired complex movements require torques that are beyond the limits of the actuators. Indeed, the actuation capabilities become even more critical when the robot interacts with an environment of complex geometry. Therefore, an accurate evaluation of the robot actuation capabilities takes on paramount importance.

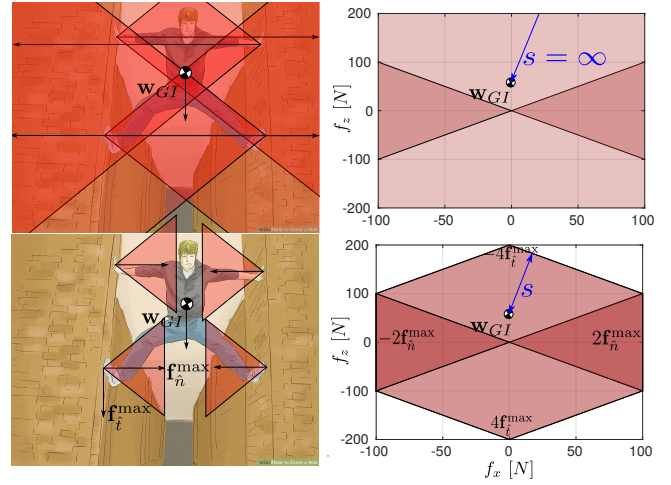


Fig. 1. Unbounded friction cones (*top-left*) give origin to unbounded feasible wrench sets (*top-right*). As an opposite bounded friction polytopes (*bottom-left*) give origin to bounded feasible wrench sets (*bottom-right*).

III. WRENCH-BASED ANALYSIS

Considering the actuation limits considerably affects the wrench margin of a legged robot. As an example let us consider a human(oid) trying to climb a vertical chimney shown in Fig. 1. Here, the CWC is obtained through the Minkowski sum of the friction cones, represented by the pink areas (Fig. 1 *top-left*). In the CWC-based approach the margin is quantified as the minimal distance between the gravito-inertial wrench w_{GI} and the boundary of the CWC. The CWC margin represents the maximum allowed wrench that can be applied (or rejected) in order to keep the system stable. Next, we show that the CWC margin s has an infinite value $s = \infty$, i.e. the *force closure* condition is achieved (Fig. 1 *top-right*). This happens because the friction cone representation assumes that a contact force with an infinite normal component can be realized at the contact. This misleading result is the consequence of not taking the actuation limits into account.

On the other hand, imposing the actuation limits (here shown as a simple saturation of the normal components of the contact forces $f_{\bar{n}}$) can be rephrased as constraining the magnitude of the admissible contact forces represented by the friction *cones* that become friction *polytopes* as shown in Fig. 1(*bottom-left*). This limits the set of applicable body wrenches (feasible wrenches) that the human(oid) can apply on its own CoM to keep himself stable. In Fig. 1(*bottom-right*) the hexagonal area represents the result of the sum of the four *bounded friction polytopes*. This new convex region is a subset of the CWC and we call it the Feasible Wrench Polytope (FWP). This is computed as the intersection of CWC and AWP, for more details see Section III-B. According to our new definition, the margin s is limited to a finite value, as in Fig. 1(*bottom-right*), showing that the human(oid) might fall if his limbs are not strong enough to support his body's weight.

A. The Actuation Wrench Polytope (AWP)

In this section we illustrate the procedure to compute the AWP, the wrench polytope devoted to take actuation limits into account. Let us consider the Equation of Motion (EoM) of a floating-base robot with n_l branches (e.g. legs) in contact with the environment, each of them with a number n_a of actuated Degree of Freedoms (DoFs), $n = \sum_{k=1}^{n_l} n_a^k$ being the total number of actuated joints:

$$\mathbf{M}(\mathbf{q})\ddot{\mathbf{q}} + \mathbf{c}(\mathbf{q}, \dot{\mathbf{q}}) + \mathbf{g}(\mathbf{q}) = \mathbf{B}\boldsymbol{\tau} + \mathbf{J}_s^T(\mathbf{q})\mathbf{f} \quad (1)$$

where $\mathbf{q} = [\mathbf{q}_b^T \quad \mathbf{q}_j^T]^T$ is the generalized coordinates vector, $\mathbf{q}_b \in \mathbb{R}^6$ represents the pose of the floating base and $\mathbf{q}_j \in \mathbb{R}^n$ contains the joint coordinates describing the angular positions of the n actuated joints. $\boldsymbol{\tau} \in \mathbb{R}^n$ is the vector of actuated joint torques while $\mathbf{c}(\mathbf{q})$ and $\mathbf{g}(\mathbf{q}) \in \mathbb{R}^{6+n}$ are the centrifugal/coriolis and gravity terms, respectively. $\mathbf{B} \in \mathbb{R}^{(6+n) \times n}$ is the matrix that selects the actuated joints of the system. $\mathbf{f} \in \mathbb{R}^{3n_l}$ is the vector of contact forces that are mapped into joint torques through the stack of Jacobians $\mathbf{J}_s(\mathbf{q}) \in \mathbb{R}^{3n_l \times (6+n)}$. Note that our robot has nearly point feet, thus we only consider pure contact forces at the contact point and no contact torque. If we split (1) into its underactuated and actuated parts, we get:

$$\underbrace{\begin{bmatrix} \mathbf{M}_b & \mathbf{M}_{bj} \\ \mathbf{M}_{bj}^T & \mathbf{M}_j \end{bmatrix}}_{\mathbf{M}(\mathbf{q})} \underbrace{\begin{bmatrix} \ddot{\mathbf{q}}_b \\ \ddot{\mathbf{q}}_j \end{bmatrix}}_{\ddot{\mathbf{q}}} + \underbrace{\begin{bmatrix} \mathbf{c}_b \\ \mathbf{c}_j \end{bmatrix}}_{\mathbf{c}(\mathbf{q}, \dot{\mathbf{q}})} + \underbrace{\begin{bmatrix} \mathbf{g}_b \\ \mathbf{g}_j \end{bmatrix}}_{\mathbf{g}(\mathbf{q})} = \underbrace{\begin{bmatrix} \mathbf{0}_{6 \times n} \\ \mathbf{I}_{n \times n} \end{bmatrix}}_{\mathbf{B}} \boldsymbol{\tau} + \underbrace{\begin{bmatrix} \mathbf{J}_{sb}^T \\ \mathbf{J}_{sq}^T \end{bmatrix}}_{\mathbf{J}_s(\mathbf{q})^T} \mathbf{f}. \quad (2)$$

By inspecting the actuated part (n bottom equations) we can see that it is possible to map joint torques into contact forces for each leg *separately* through the $\mathbf{J}_{sq} \in \mathbb{R}^{3n_l \times n}$ block diagonal. We will see in Section V-A that this is convenient because it allows us to neglect the coupling term \mathbf{J}_{sb} .

For motion planning we are interested in estimating the maximum $\mathbf{f}^{\max} \in \mathbb{R}^{3n_l}$ and minimum $\mathbf{f}^{\min} \in \mathbb{R}^{3n_l}$ contact forces that the end-effectors are able to apply on the environment. This quantity can be estimated by considering the maximal torques achievable by the actuation system $\boldsymbol{\tau}^{\max} \in \mathbb{R}^n$ in a given configuration of the actuated joints $\mathbf{q}_j \in \mathbb{R}^n$. As previously anticipated, we can estimate the maximum admissible contact force $\mathbf{f}_i^{\max} \in \mathbb{R}^3$ for each end-effector i separately, by considering a subset of the EoM (represented by the i sub-script) describing the dynamics of the actuated joints of that specific leg:

$$\mathbf{f}_i^{\max} = \mathbf{J}_i^{T\#} \left(\underbrace{\mathbf{M}_{bi}^T \ddot{\mathbf{q}}_b + \mathbf{M}_i \ddot{\mathbf{q}}_i + \mathbf{c}_i + \mathbf{g}_i}_{\delta} - \boldsymbol{\tau}_i^{\max} \right), \quad (3)$$

where $(.)^\#$ is the Moore-Penrose pseudoinverse, and $\mathbf{J}_i \in \mathbb{R}^{3 \times n_a}$ is the Jacobian matrix for the i -th foot. In our robot, each leg has three DoFs ($n_a = 3$), thus a simple inversion is sufficient since \mathbf{J}_i is square¹. $\boldsymbol{\tau}_i^{\max} \in \mathbb{R}^{n_a}$ is a vector that contains the upper and lower bounds of the joint torques

¹Note that (3) might not have a solution unless the torque belongs to the image of $\mathbf{J}_i^T(\mathbf{q}_i)$, i.e.: $\boldsymbol{\tau}_i \in \text{Im}(\mathbf{J}_i^T(\mathbf{q}_i))$ for redundant manipulators ($n_a > m$), where m is the task dimension. However, [19] describes an algorithm that solves (3) for these cases.

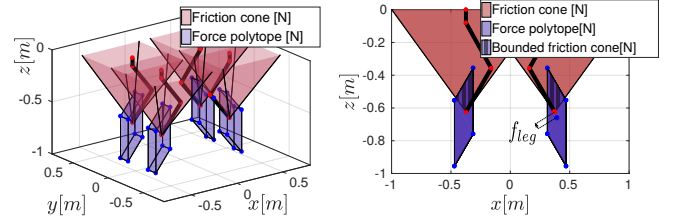


Fig. 2. Representation of the force polytopes (blue) and of the friction cones (pink) of each single leg in 3D (left) and projected on the (F_x, F_z) plane (right). The offset f_{leg} between the center of the force polytope and the foot is due to the bias term δ in (3).

of each single leg. Considering all their combinations, these limits result in 2^{n_a} values of \mathbf{f}_i^{\max} that form the vertices of the *force polytope* \mathcal{F}_k . In the case of our quadruped robot HyQ \mathcal{F}_k is a polytope with 8 vertices and its shape changes nonlinearly with the joint configuration because of the nonlinearities in \mathbf{J}_i^2 . As an example, we compute the force polytope for each leg in a quadruped robot. Fig. 2(left) shows the four force polytopes (together with the friction cones) obtained for a typical quadruped robot. Fig. 2(right) shows a lateral view that depicts the same force polytopes projected onto the (F_x, F_z) plane.

To compute the AWP, the next step is adding the coordinates relative to the torque that the maximum contact forces can apply on the CoM of the robot:

$$\mathbf{w}_{i,k} = \begin{bmatrix} \mathbf{f}_{i,k}^{\max} \\ \mathbf{p}_i \times \mathbf{f}_{i,k}^{\max} \end{bmatrix} \quad \text{with } k = 1, \dots, 2^{n_a}, \quad (4)$$

where $\mathbf{p}_i \in \mathbb{R}^3$ represents the position of the i -th foot and $\mathbf{w}_{i,k} \in \mathbb{R}^6$ represents the wrench that can be realized at that foot, both quantities are expressed in a fixed frame. Therefore, the set of admissible wrenches that can be applied at the CoM by the i -th foot/end-effector is:

$$\mathcal{W}_i = \text{ConvHull}(\mathbf{w}_{i,1}, \dots, \mathbf{w}_{i,2^{n_a}}) \quad (5)$$

with $i = 1, \dots, n_l$. We now have n_l *wrench polytopes* \mathcal{W}_i , one for each limb in contact with the environment. Finally, the AWP corresponds to the Minkowski sum of all the n_l wrench polytopes:

$$\text{AWP} = \mathcal{W}_1 \oplus \dots \oplus \mathcal{W}_{n_l}. \quad (6)$$

As defined above, the AWP is a bounded convex polytope in \mathbb{R}^6 (Fig. 3 left) that contains all the admissible wrenches that can be applied to the robot's CoM that do not violate the actuation limits of the limbs in contact with the environment.

B. The Feasible Wrench Polytope (FWP)

Note that the AWP does not include the constraints imposed by the environment such as the terrain normal, the friction coefficient and the unilateral contact condition

²Note that the torque limits $\boldsymbol{\tau}_i^{\max}$ can depend on the joint positions, making the dependency of the polytope from the joints configuration even more complex (e.g. a revolute joint connected to a linear actuator with nonlinear lever-arm).

(e.g. the legs can not pull on the ground). However, those constraints can be accounted by the CWC [5] (Fig. 3 *center*):

$$CWC = \text{ConvexCone}(\hat{\mathbf{e}}_i^k) \quad k = 1, \dots, n_e \quad (7)$$

and $i = 1, \dots, n_l$. Here n_e is the number of edges of the linearized friction cone and:

$$\hat{\mathbf{e}}_i^k = \begin{bmatrix} \mathbf{e}_i^k \\ \mathbf{p}_i \times \mathbf{e}_i^k \end{bmatrix} \in \mathbb{R}^6, \quad \text{with } k = 1, \dots, n_e \quad (8)$$

where $\mathbf{e}_i^k \in \mathbb{R}^3$ is the k -th edge of the contact point i .

We subsequently perform the intersection of the CWC with the AWP obtaining a convex polytope that we define as Feasible Wrench Polytope (FWP) (Fig. 3 *right*):

$$FWP = CWC \cap AWP. \quad (9)$$

However, performing the *intersection* of polytopes in 6D is an expensive operation that requires the *double-description* [20] of both operands. We propose a more efficient approach for the computation of the FWP that: 1) first computes the intersection between the friction cones \mathcal{C}_i and the force polytopes \mathcal{F}_i obtaining, for each i -th contact, a 3D bounded friction cone \mathcal{B}_i (Fig. 2 *right*) with μ vertices $\mathbf{b}_i^k \in \mathbb{R}^3$:

$$\mathcal{B}_i = \text{ConvHull}(\mathbf{b}_i^k), \quad \text{with } k = 1, \dots, \mu \quad (10)$$

2) then composes the wrench adding the torque, as in (4):

$$\hat{\mathbf{b}}_i^k = \begin{bmatrix} \mathbf{b}_i^k \\ \mathbf{p}_i \times \mathbf{b}_i^k \end{bmatrix} \in \mathbb{R}^6 \quad \text{with } k = 1, \dots, \mu \quad (11)$$

obtaining in this way the intermediate sets $\mathcal{G}_i \in \mathbb{R}^6$:

$$\mathcal{G}_i = \text{ConvHull}(\hat{\mathbf{b}}_i^k), \quad \text{with } k = 1, \dots, \mu \quad (12)$$

3) finally, the FWP is computed through the Minkowski sum of all the \mathcal{G}_i of all the n_l contacts:

$$FWP = \mathcal{G}_1 \oplus \dots \oplus \mathcal{G}_{n_l}. \quad (13)$$

The advantage of this proposed method is that the intersection is performed in 3D rather than in 6D, which is computationally faster. This is advantageous also for the final step in (13) because it avoids computing vertices that will be removed later (e.g. all the vertices from the AWP with negative contact forces are removed by intersecting with the CWC). Additionally, the Minkowski sum can be efficiently obtained using the \mathcal{V} -description only as in [20].

C. Polytope representation for a Planar model

To achieve a better understanding of the nature of these polytopes, it is useful to have a pictorial representation. Let us consider the simplified case of a planar dynamic model (as the one depicted in Fig. 2 *right*) where each point of the space is represented through the (x, z) coordinates. In this case the wrench space has three coordinates (F_x, F_z, τ_y) and can be represented in 3D. Fig. 3 depicts the AWP (*left*), the CWC (*center*) and the FWP (*right*) for this simplified model.

IV. FWP-BASED FEASIBILITY MARGIN

The CWC has been proven to be an universal criterion for dynamic legged stability [5]. However, the CWC still lacks knowledge of robot's feasibility constraints such as the joint limits and self-collision. Using the CWC naturally brings to the definition of the *margin* as a *stability* criterion. By considering complex motions in 3D environments there is the need to introduce a more restrictive metric, that we call the *feasibility margin*. This margin includes all the properties of the CWC criterion, and additionally the robot's actuation limits. To obtain the *feasibility margin* we first need to compute the robot's gravito-inertial wrench $\mathbf{w}_{GI} \in \mathbb{R}^6$ in the specific robot state that we want to evaluate:

$$\mathbf{w}_{GI} = \dot{\mathbf{h}} - \mathbf{w}_G, \quad (14)$$

with:

$$\dot{\mathbf{h}} = \begin{bmatrix} m\ddot{\mathbf{c}} \\ \dot{\mathbf{k}}_{\mathcal{W}} \end{bmatrix}, \quad \mathbf{w}_G = \begin{bmatrix} m\mathbf{g} \\ \mathbf{c} \times m\mathbf{g} \end{bmatrix}, \quad (15)$$

where $\mathbf{k}_{\mathcal{W}} \in \mathbb{R}^3$ is the robot's angular momentum and \mathbf{c} is its CoM position (both expressed in the fixed coordinate frame \mathcal{W}). The criterion of feasibility can then be defined as:

$$\mathbf{w}_{GI} \in FWP. \quad (16)$$

Since we are also interested in assessing the robustness of such feasibility we intend to define a feasibility margin s . However, this depends on the type of the representation chosen for the FWP set, i.e. half-plane description (\mathcal{H} -description) or a vertex description (\mathcal{V} -description).

A. Half-plane description

In the \mathcal{H} -description, the FWP set can be written in terms of half-spaces as:

$$FWP = \{\mathbf{w} | \hat{\mathbf{a}}_j^T \mathbf{w} \leq 0, j = 1, \dots, n_h\}, \quad (17)$$

where n_h is the number of half-spaces of the FWP and $\hat{\mathbf{a}}_j \in \mathbb{R}^6$ is the normal vector to the j -th facet. The feasibility criterion expressed in (16) can thus be written as:

$$\mathbf{H}^T \mathbf{w}_{GI} \leq \mathbf{0} \quad (18)$$

where $\mathbf{H} \in \mathbb{R}^{6 \times n_h}$ is the matrix whose columns are the normals to all the half-spaces of the FWP and \leq is a component-wise operator. The columns of \mathbf{H} can be divided into two blocks \mathbf{H}_c and \mathbf{H}_a in order to differentiate the CWC half-spaces from the AWP half-spaces, respectively: $\mathbf{H} = [\mathbf{H}_c | \mathbf{H}_a]$. If $\mathbf{H}_c^T \mathbf{w}_{GI} > \mathbf{0}$ but $\mathbf{H}_a^T \mathbf{w}_{GI} \leq \mathbf{0}$ then the robot's state is consistent with its actuation capabilities but its contact condition is unstable (e.g. friction limits are violated). Viceversa, if $\mathbf{H}_c^T \mathbf{w}_{GI} \leq \mathbf{0}$ but $\mathbf{H}_a^T \mathbf{w}_{GI} > \mathbf{0}$ then the system has stable contacts but it does not respect the actuation limits. In the latter case, the legged system might still not fall but it will not be able to realize the desired task.

If the \mathcal{H} -description of the FWP is given, we can provide a definition of robustness that extends the properties of the CWC [6]. In this case computing the margin s corresponds to finding the biggest disturbance wrench $\mathbf{w}_d \in \mathbb{R}^6$ that the

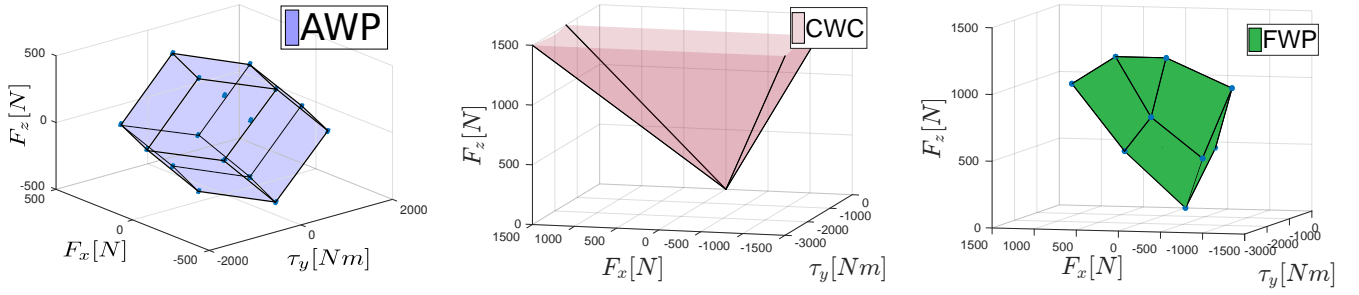


Fig. 3. The Actuation Wrench Polytope (AWP) (left), the Contact Wrench Cone (CWC) (center) and the Feasible Wrench Polytope (FWP) (right). These drawings refer to a planar dynamic model where the only non zero wrench components are (F_x, F_z, τ_y) and that can therefore be represented in 3D.

system can reject. This is equivalent to computing the largest residual radius s such that the s -ball \mathcal{B}_s (centered in \mathbf{w}_{gi}) still lies within the FWP:

$$\mathcal{B}_s \in FWP, \quad (19)$$

and it is defined as:

$$\mathcal{B}_s = \{\mathbf{w}_{gi} + \mathbf{T}(\mathbf{p}_{wd})\mathbf{w}_d \mid \mathbf{w}_d^T \mathbf{Q} \mathbf{w}_d \leq s\}. \quad (20)$$

where \mathbf{p}_{wd} is the disturbance application point and $\mathbf{T}(\mathbf{p}_{wd})$ is the adjoint spatial transform that expresses it in the frame \mathcal{W} [6]. \mathbf{Q} is a positive definite matrix that encodes the norm of the wrench space.

B. Vertex description

If only a \mathcal{V} -description of the FWP is available, we provide another definition of the feasibility margin s based on the vertices of the FWP. Let us consider a matrix $\mathbf{V} \in \mathbb{R}^{6 \times n_v}$ whose columns are the vertices \mathbf{v}_i of the FWP, and a vector $\boldsymbol{\lambda} \in \mathbb{R}_+^{n_v}$ of positive weights where n_v is the number of vertices of the FWP. Every point inside the FWP can be described with a combination of weights λ_i such that $\sum_{i=1}^{n_v} \lambda_i = 1$. We therefore define the robot to be in a feasible state if, for the corresponding wrench \mathbf{w}_{GI} , there exists a $\boldsymbol{\lambda}$ such that:

$$\begin{cases} \mathbf{V}\boldsymbol{\lambda} = \mathbf{w}_{GI} \\ \|\boldsymbol{\lambda}\|_1 = 1 \\ \lambda_i \geq 0 \quad i = 1, 2, \dots, n_v \end{cases}, \quad (21)$$

where we have written the term $\sum_{i=1}^{n_v} \lambda_i = 1$ in a more compact form exploiting the L_1 norm of $\boldsymbol{\lambda}$. At this point, we can connect the concept of feasibility margin with a quantity that somewhat encodes the idea of “distance” between the point \mathbf{w}_{GI} and the vertices of the FWP.

A preliminary step consists in subtracting the centroid \mathbf{v}_c from all the FWP vertices \mathbf{v}_i , obtaining new translated vertices $\hat{\mathbf{v}}_i$ ($\hat{\mathbf{v}}_i = \mathbf{v}_i - \mathbf{v}_c$). This has the effect of shifting the origin of the wrench space in the centroid, that, in a \mathcal{V} -description, is a good approximation of the most “robust” point (e.g. the Chebishev centre). We then define a new *shrunk* polytope \mathcal{P}^s centered in the origin (which is now also the centroid of the FWP). \mathcal{P}^s can be expressed in terms

of its own vertices \mathbf{v}_i^s and of a set of multipliers λ_i^s :

$$\mathcal{P}^s = \left\{ \mathbf{w} \in \mathbb{R}^6 \mid \mathbf{w} = \sum_{i=1}^{n_v} \lambda_i^s \hat{\mathbf{v}}_i^s, \lambda_i^s \geq 0, \|\boldsymbol{\lambda}^s\|_1 = 1 \right\} \quad (22)$$

For a better understanding Fig. 5 (left) illustrates the idea of the shrunk polytope for a 2D representation.

The FWP’s vertices are linked to the vertices of \mathcal{P}^s through a scalar $s \in [0, 1]$ which is the *feasibility margin*:

$$\hat{\mathbf{v}}_i^s = \hat{\mathbf{v}}_i(1 - s), \quad 0 \leq s \leq 1. \quad (23)$$

If, by instance, $s = 1$ then \mathcal{P}^s shrinks into the origin. If we impose $\lambda_i = (1 - s)\lambda_i^s$, then we can write the shrunk polytope \mathcal{P}^s in terms of the vertices \mathbf{v}_i of the FWP, i.e.:

$$\mathcal{P}^s = \left\{ \mathbf{w} \in \mathbb{R}^6 \mid \mathbf{w} = \sum_{i=1}^{n_v} \lambda_i \mathbf{v}_i, \lambda_i \geq 0, \|\boldsymbol{\lambda}\|_1 = 1 - s \right\} \quad (24)$$

We define the feasibility margin as the scalar s that corresponds to the smallest shrunk polytope that still contains the point \mathbf{w}_{GI} . This problem can be formulated as a LP that can be efficiently carried out by any general-purpose linear solver:

$$\begin{aligned} \max_{\boldsymbol{\lambda}, s} \quad & s \\ \text{s. t.} \quad & \mathbf{V}\boldsymbol{\lambda} = \mathbf{w}_{GI} \\ & \|\boldsymbol{\lambda}\|_1 = 1 - s \\ & \lambda_i \geq 0 \quad i = 1, 2, \dots, n_v. \end{aligned} \quad (25)$$

Note that the larger is s , the more robust is the system against disturbances. A negative s means that the point is out of the polytope and the wrench is unfeasible. When s becomes zero, it means the point is on the polytope boundary and that either the friction or actuation limits are violated. Table I shows the computation time of a Intel(R) Core(TM) i5-4440 CPU @ 3.10GHz with 4 cores for three and four contacts scenarios.

TABLE I
COMPUTATION TIME OF THE FEASIBILITY MARGIN s

	3 non-coplanar contacts	4 non-coplanar contacts
FWP vertices	436	1118
variables	437	1119
constraints	7	7
LP time [ms]	90	350

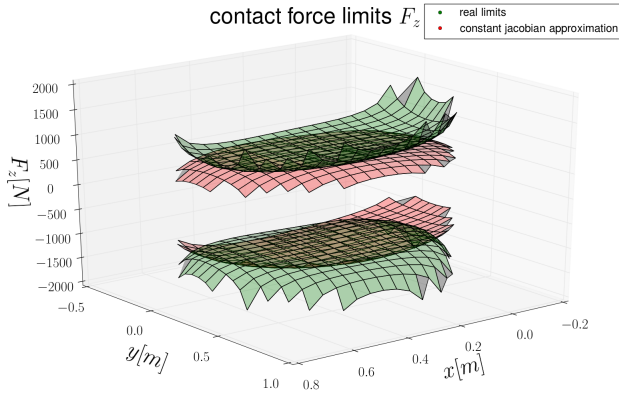


Fig. 4. Contact force limits (Z component) on the left-front (LF) leg of HyQ as a function of the foot position computed with real torque limits (green) and with jacobian approximation (red).

V. ONLINE TRAJECTORY OPTIMIZATION (TO)

The FWP margin can be used to devise a motion planner that provides robust CoM trajectories. Hereafter, we present a brief description of how we used the proposed criterion to plan online motions for our quadruped robot, that do not violate the actuation limits. We further extended the capabilities of the locomotion framework [21], [22] by replacing the original (*heuristic*) planner with an optimized planning that exploits the proposed feasibility criterion. This framework realizes a *crawling* gait where a *base motion* phase and a *swing motion* phase are alternated (therefore the CoM of the robot does not move when a leg is in swing).

To be able to compare with the heuristic planner, we opted for a decoupled planning approach where the footholds and the CoM trajectory are determined sequentially. Our online TO computes during every swing phase the CoM trajectory to be realized in the next base motion phase using a *one-step horizon*. The decision variables of the optimization problem are the X , Y components of the CoM positions, the velocities and the duration of the *base motion* phase Δt_{bm} : $\Gamma = \{\mathbf{c}_x[k], \mathbf{c}_y[k], \dot{\mathbf{c}}_x[k], \dot{\mathbf{c}}_y[k], \Delta t_{bm}\}$ with $k = 1, \dots, N$. The trajectory is discretized in N equally spaced knots (at time intervals $h = \Delta t_{mb}/N$). Note, that we here do not optimize the angular dynamics nor the coordinate z , parallel to gravity. This is because, for quasi static motions, the predominant acceleration term acting on the system is gravity itself, and therefore its influence on the stability or on the joint torques is limited compared to the role of the X and Y components. We aim to maximize the FWP margin s , as in 25, while we enforce back-ward Euler integration constraints along the trajectory and zero velocity at the trajectory extremes:

$$\begin{aligned} \min_{\Gamma} \quad & \sum_{k=1}^N \mathcal{L}(\mathbf{c}[k], \dot{\mathbf{h}}[k], \mathbf{V}) \\ \text{s. t.} \quad & \dot{\mathbf{c}}_x[k+1] = (\mathbf{c}_x[k+1] - \mathbf{c}_x[k])/h \\ & \dot{\mathbf{c}}_y[k+1] = (\mathbf{c}_y[k+1] - \mathbf{c}_y[k])/h \\ & \dot{\mathbf{c}}_x[0] = \dot{\mathbf{c}}_y[0] = \dot{\mathbf{c}}_x[N] = \dot{\mathbf{c}}_y[N] = \mathbf{0} \end{aligned} \quad (26)$$

TABLE II

FWP's \mathcal{V} - AND \mathcal{H} -DESCRIPTION COMPUTATION TIME WITH POLITOPIX [23].

	2 contact points	3 contact points	4 contact points
\mathcal{V} -description	0.030s	0.150s	0.500s
\mathcal{H} -description	0.040s	1.000s	30.000s

TABLE III

TO OF THE VARIABLES Γ USING THE FWP \mathcal{V} -DESCRIPTION

	3 non-coplanar contacts	4 non-coplanar contacts
timesteps	10	10
FWP vertices	436	1118
variables Γ	41	41
constraints	24	24
time [ms]	75	85

As a first step, we evaluate the FWP polytope considering the robot contact configuration. Exploiting the approximation in Section V-A we can do this just once at the beginning of the optimization. Then, to compute the running cost $\sum_{k=1}^N \mathcal{L}$, for each optimization loop, we evaluate the CoM acceleration along the trajectory ($h \cdot \ddot{\mathbf{c}}_{x,y}[k+1] = \dot{\mathbf{c}}_{x,y}[k+1] - \dot{\mathbf{c}}_{x,y}[k], \forall k$) and evaluate the gravito-inertial wrench at each knot through (14). In order to exploit the \mathcal{V} -description, for each node, we should add the λ vector as decision variable and the constraints in (25). However, the amount of decision variables would significantly increase due to the high number of vertices in the polytope (i.e. λ may have hundreds of elements for each optimization knot) leading to computation times that do not meet the requirements for online planning. Thus we tackled this problem by computing the set of λ through a simple Moore-Penrose pseudo-inversion: $\lambda[k] = \mathbf{V}^\# \mathbf{w}_{GI}[k]$. In this way the decision variables will be only the states Γ of the system and the number of vertices will influence the size of the TO problem only marginally (see Table III). We noticed that adding a *bias* term in the nullspace of \mathbf{V} to “drive” the solution $\lambda[k]$ toward $\lambda_0 \in \mathbb{R}^{n_v}$ was giving satisfactory results:

$$\lambda[k] = \mathbf{V}^\# \mathbf{w}_{GI}[k] + \mathbf{N}_V \lambda_0, \quad \lambda \in \mathbb{R}^{n_v} \quad (27)$$

where $\mathbf{N}_V \in \mathbb{R}^{n_v \times n_v}$ is the null-space projector associated to \mathbf{V} . Indeed, if we set $\lambda_0 = [1/n_v, \dots, 1/n_v]$ as the geometric center of the FWP, the constraints $\|\lambda_0\|_1 = 1, \lambda_{0i} > 0$ are satisfied by construction. Thanks to the one-to-one correspondence between the gravito-inertial wrench and the weights λ , penalizing the deviation of $\lambda \in \mathbb{R}^{n_v}$ from λ_0 is equivalent to maximizing the feasibility margin. Therefore, we formulate the running cost computation as:

$$\mathcal{L}[\mathbf{c}[k], \dot{\mathbf{h}}[k], \mathbf{V}] = \|\lambda[k] - \lambda_0\|_2. \quad (28)$$

A. Computational issues and approximations

Despite the remarkable computational speed-up obtained by the use of the \mathcal{V} -description (see Table II), the evaluation of the FWP still represents the most time-consuming (a) of this pipeline (about 150ms needed for a triple-stance configuration). The subsequent step (b), i.e. the solution of the TO problem for a given FWP, requires instead about 75–85ms for 10 nodes trajectory. In theory we should recompute step a and step b iteratively, concurrently optimizing the vertices

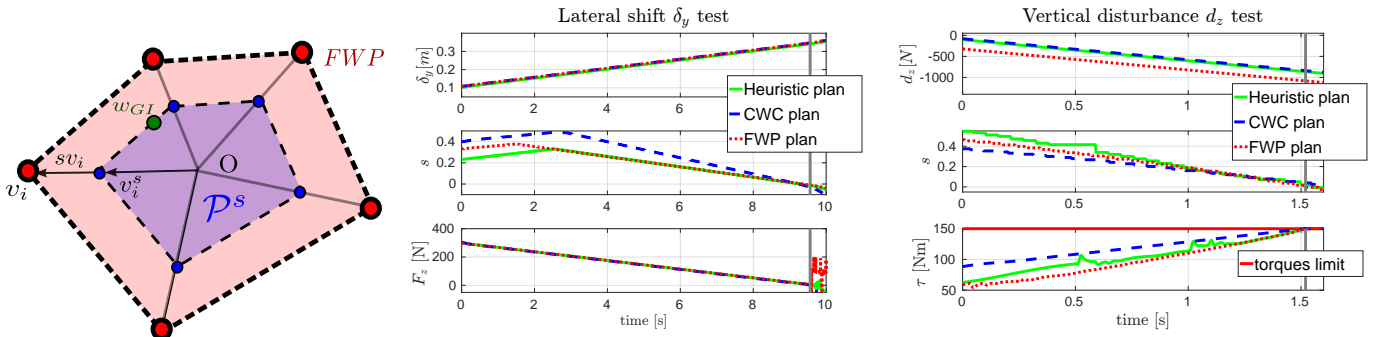


Fig. 5. A 2D pictorial representation of the shrunk polytope \mathcal{P}^s (left): we see that the margin s can be seen as the shrinkage rate of the reduced polytope with respect to the FWP; simulation data of the horizontal displacement test (center) and vertical disturbance test (right).

of the FWP and the trajectory inside it. We decided instead to compute the FWP only once per step, assuming that its vertices do not change during the execution of each phase, as explained in the following paragraph.

We analyze the influence on the estimated maximum and minimum contact force when the Jacobian matrix is approximated to be constant along a trajectory in the case of the HyQ robot. In Fig. 4 we can see the contact force boundaries (Z component only) when a foot spans its workspace (the foot covers all the X and Y positions on a plane located at $Z = -0.6m$). The green surfaces represent the real boundaries of the vertical contact force considering the correct leg Jacobian and correct piston lever-arm for each considered position. The red surfaces show instead the same force boundaries when the correct piston lever-arm and a constant Jacobian is evaluated. We can see that in a neighborhood of the default foot configuration ($[0.3, 0.2, -0.6]m$ with respect to the base frame of the robot) the approximation is accurate and becomes rough in proximity of the workspace boundaries. We chose to use a constant Jacobian matrix corresponding to a joint configuration \mathbf{q}_0 of the trajectory coming from the heuristic plan. In this way the Jacobian remains constant and we remove the AWP's dependency from the joints position. As a further simplification we assume a quasi-static motion ($\ddot{\mathbf{q}} = \dot{\mathbf{q}} = \mathbf{0}$), this allows to compute the FWP only once at each stance change and (3) can thus be written as:

$$\mathbf{f}_i^{\max} = \mathbf{J}_i(\mathbf{q}_{0_i})^\# \left(\mathbf{g}(\mathbf{q}_{0_i}) - \mathbf{B}\boldsymbol{\tau}^{\max}(\mathbf{q}_{0_i}) \right). \quad (29)$$

Note that all the wrenches are expressed in the inertial frame.

VI. EXPERIMENTAL RESULTS

In this section we present simulation results and real experiments with HyQ. The first two simulations validate the feasibility margin formulation based on the \mathcal{V} -description of the FWP. After that we highlight the differences between the state of the art stability measure and the feasibility metric we propose. Finally we present a few examples of the behaviors we can obtain with the TO presented in Section V.

A. Vertex-based FWP margin validation

In a first test, we consider three different motion planners: the heuristic planner, a CWC planner (that incorporates only frictional constraints) and our FWP planner. We have the robot crawling where the CoM trajectory is planned by

the three different methods and, in all cases, we stop the robot during a triple-stance phase (being more critical for robustness than four-stance phases). We then make the robot displace laterally with an increasing offset $\delta_y = \epsilon 0.5 m$ ($\epsilon \in \mathbb{R}$ increases linearly from 0 to 1). The objective is to obtain a gradual unloading of the lateral legs and therefore violate the unilaterality constraints. Figure 5 (center) shows the evolution of the displacement δ_y and of the normal component of the contact force F_z at the left-front (LF) leg. As expected the plot shows that for all the cases s drops to zero when the leg LF becomes unloaded ($F_z = 0$).

In the second test the robot is again stopped during a walk in a three-legs stance configuration. This time a vertical disturbance force was applied at the origin of the base link (i.e. the geometric center of the torso). The force is vertical and pointing downwards with increasing magnitude $d_z = -\epsilon 1000 N$ where $\epsilon \in \mathbb{R}$ is linearly increasing from 0 to 1. The joint torques will increase because of the action of this force, eventually making one (or more) of them hit the limits. Since the test is performed in a static configuration and the disturbance force is always vertical, the Center of Pressure (CoP) of the system will not change, being the robot always statically stable. Fig. 5 (right) shows a plot of the magnitude of the vertical pushing force d_z together with the knee joint torque of the LF leg and the feasibility margins s in the three cases. We can see that, in the case of the static configuration found with the FWP planner, the torque limit is reached for a higher amplitude of the disturbing force (about $-1100N$ compared to $-900N$), showing that this is more robust against external disturbance forces than the configurations selected by the heuristic and by the CWC planners. We can see that in all the cases the feasibility margin s goes to zero when a torque limit is violated.

B. CWC- vs. FWP-margin Comparison

The last test highlights the main differences between the feasibility metrics s and the traditional stability measures. As state-of-the-art stability metric we consider the CWC-margin, which is obtained by applying Eq. 25 on the \mathcal{V} -description of the CWC, rather than the FWP as explained in Section IV-B. Fig. 6 (above) shows the results when a crawl gait is evaluated using this method. The red line represents the value of the CWC-margin during the triple-stance phase of

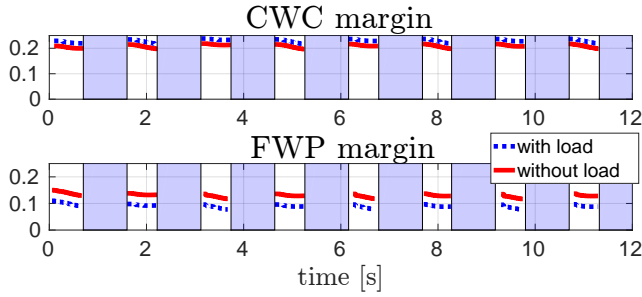


Fig. 6. Evaluation of the same heuristic crawl gait (with and without 20kg load) with the CWC margin (top) and with the FWP margin (bottom).

a crawling gait. The dashed blue line represents the same walk, evaluated again with the CWC-margin, in the case that an external load of 20kg is applied on the CoM of the robot during its walk. The margin referring to the four-legs stance is not directly comparable to the triple-stance phase because the vertices of the FWP have a different scaling. For this reason we only show the values referring to the triple stance. We can see that the same two trials, with and without external load, provide a completely different result if evaluated with the FWP-margin (Fig. 6 bottom): the blue-dashed line shows that the feasibility is lower in the case with external load. This implies that, when the load increases, even if the stability might improve, the risk of hitting the torque limits is higher.

C. Crawling Simulations

As shown in the accompanying video³, we report a few simulation and hardware experiments of HyQ performing a crawling gait. At first we see that the heuristic crawl easily hits the torque limits while crawling on a flat ground while carrying an external load of 20kg (about 25% of the robot total weight) placed on its CoM. We can then see that the FWP planner, as explained in Section V, finds a new duration Δt_{bm} of the *base motion* phase and a new CoM trajectory that avoids hitting the torque limits at all times, while maintaining the desired linear speed.

Final simulations show the capability of the planner to optimize feasible trajectories when the robot has a *hindered* joint (i.e. when a specific joint can only realize a significantly smaller torque than the other joints), or when we limit the normal force that a specific leg can realize on the ground. The video also shows hardware experiments of HyQ crawling on a rough terrain without hitting the torque limits.

VII. CONCLUSION

The complexity of a motion increases with the complexity of the terrain to be traversed. Moreover, there is a need for online motion replanning to avoid error accumulation. For this reason in this paper we presented the concepts of AWP and FWP and a method to efficiently compute their \mathcal{V} -description. Based on this we defined a *feasibility margin* that considers the stability and the actuation-consistency of a given motion plan. Finally we showed how this metrics can be used in a CoM trajectory optimization not only for

motion evaluation but also for motion planning.

Thanks to the efficiency of the vertex-based approach we are able to perform *online* TO where the robot plans each *swing phase* the trajectory for the next *base motion* phase. Our approach does not take any assumption on the environment and it is therefore suitable for complex terrain scenarios.

Future works will concentrate on the removal of the approximations mentioned in Section V-A and on the speed-up of the computations required to obtain the wrench polytopes.

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³<https://www.dropbox.com/s/mymoep3ffc0iup/orsolino17ral.m4?dl=0>