**Preprints of the 18th IFAC World Congress**
**Milano (Italy) August 28 - September 2, 2011**

# Fully Parameterized Tube MPC ⋆

**Saša V. Raković** *,** **Basil Kouvaritakis** ** **Mark Cannon** **
**Christos Panos** *** **Rolf Findeisen** *

* *Otto–von–Guericke–Universität Magdeburg, Germany*
** *University of Oxford, UK*
*** *Imperial College London, UK*

**Abstract:** Reaching a sensible compromise between computational tractability and degree of optimality still remains a significant challenge in robust model predictive control (MPC). Tube MPC offers an efficient approach which is based on pseudo–closed loop optimization but can thus be conservative. The degree of conservatism is reduced through the so–called disturbance affine control policy and it is the aim of this paper to propose a new methodology that deploys a more general policy thereby improving on system theoretic properties yet is no more computationally intensive. The work is based on a suitable parameterization of state and control tubes and is underpinned by guarantees of strong system theoretic properties.

*Keywords:* Tube Model Predictive Control, Parameterized Tubes, Set–Dynamics.

## 1. INTRODUCTION

Robust model predictive control (MPC) may be thought as having reached a certain state of maturity, but despite the plethora of results (e.g. Rawlings and Mayne (2009)) there still remains a significant challenge, namely that of reaching a good compromise between computational complexity and degree of optimality. Optimal results can be achieved through the use of dynamic programming (Bertsekas, 2007) and min–max feedback MPC (Scokaert and Mayne, 1998) but are computationally impracticable. To avoid this, one can use pseudo closed–loop strategies and among these tube MPC (TMPC) provides some effective answers (Mayne et al., 2005; Raković et al., 2010d,c). The so–caled disturbance affine strategy (Löfberg, 2003; Goulart et al., 2006) provides an alternative with improved system theoretic properties at an increased but manageable computational cost. However, use of affine strategies constitutes a serious limitation and this was circumvented in (Raković et al., 2010a,b) through the use of an appropriate state/control decomposition. More precisely, the decomposition led to the replacement of the single pair of state and control tubes used in (Blanchini, 1990; Lee et al., 2002; Mayne et al., 2005; Raković et al., 2010d,c) with a number of pairs of partial state and control tubes, and to the use of a prediction scheme that had an upper triangular structure. The current paper generalizes the prediction structure of (Raković et al., 2010a,b) by introducing a fully parameterized tube (FPT) prediction structure which still guarantees invariance and robust stability. The required online optimization reduces to a linear program, in the case when linear partial stage and terminal cost functions are used, whose number of decision variables grows quadratically with the prediction horizon. This it shares in common with the method of (Raković et al., 2010a,b) and the so–called disturbance affine control strategy (Löfberg, 2003; Goulart et al., 2006), however the dependence of the predicted control moves on the future disturbances is nonlinear and is more general than the disturbance affine control strategy, and hence constitutes a significant advantage of the method reported here. A more detailed version of this manuscript, including the proofs and an illustrative example, can be found in (Raković et al., 2011).

*Notation and definitions:* The sets of non–negative, positive integers and non–negative reals are denoted by $\mathbb{N}$, $\mathbb{N}_+$, and $\mathbb{R}_+$, respectively. For $a \in \mathbb{N}$ and $b \in \mathbb{N}$ such that $a < b$ we denote $\mathbb{N}_{[a:b]} := \{a, a+1, \ldots, b-1, b\}$ and write $\mathbb{N}_b$ for $\mathbb{N}_{[0:b]}$. The largest absolute value of the eigenvalues of a matrix $M \in \mathbb{R}^{n\times n}$ is denoted $\rho(M)$. Given two sets $X \subset \mathbb{R}^n$ and $Y \subset \mathbb{R}^n$ and a vector $x \in \mathbb{R}^n$, the Minkowski set addition is defined by $X \oplus Y := \{x + y \ : \ x \in X,\ y \in Y\}$, and we write $x \oplus X$ instead of $\{x\} \oplus X$. Given a set $X$ and a real matrix $M$ of compatible dimensions the image of $X$ under $M$ is denoted by $MX := \{Mx \ : \ x \in X\}$. Given a set $Z \subset \mathbb{R}^{n+m}$ its projection onto $\mathbb{R}^n$ is denoted by $\mathrm{Proj}_{\mathbb{R}^n}(Z) = \{x \in \mathbb{R}^n \ : \ \exists y \in \mathbb{R}^m$ such that $(x, y) \in Z\}$. A set $X \subset \mathbb{R}^n$ is a $C$–set if it is compact, convex, and contains the origin. A set $X \subset \mathbb{R}^n$ is a $PC$–set if it is a $C$–set and contains the origin in its interior. A polyhedron is the (convex) intersection of a finite number of open and/or closed half–spaces and a polytope is a closed and bounded polyhedron. The convex hull of a set $X \subset \mathbb{R}^n$ is denoted $\mathrm{convh}(X)$. Given a $PC$–set $L$ in $\mathbb{R}^n$, the function $\mathrm{gauge}(L, \cdot)$ defined by $\mathrm{gauge}(L, x) := \min_\mu\{\mu \ : \ x \in \mu L,\ \mu \in \mathbb{R}_+\}$ for $x \in \mathbb{R}^n$ is called the gauge function of $L$. If $L$ is a symmetric $PC$–set in $\mathbb{R}^n$, then the gauge function of $L$ induces the vector norm $|x|_L := \mathrm{gauge}(L, x)$. For a symmetric $PC$–set $L$ in $\mathbb{R}^n$ and a non–empty closed set $X \subset \mathbb{R}^n$, the function $\mathrm{dist}(L, \cdot, X)$ given by $\mathrm{dist}(L, y, X) := \inf_x\{|x - y|_L \ : \ x \in X\}$ for $y \in \mathbb{R}^n$ is called the distance function with respect to the set $X$.

⋆ Saša V. Raković is a Visiting Academic Fellow at the University of Oxford, and is the corresponding author (svr@sasavrakovic.com).

**Copyright by the**
**International Federation of Automatic Control (IFAC)**

## 2. PRELIMINARIES

We consider linear, time–invariant, discrete time systems:

$$x^+ = Ax + Bu + w, \tag{1}$$

where $x \in \mathbb{R}^n$ is the current state, $u \in \mathbb{R}^m$ is the current control, $x^+$ is the successor state, $w \in \mathbb{R}^n$ is the disturbance taking values in the set $\mathbb{W} \subset \mathbb{R}^n$, $A \in \mathbb{R}^{n\times n}$ and $B \in \mathbb{R}^{n\times m}$. The system constraints are:

$$x \in \mathbb{X},\ u \in \mathbb{U} \text{ and } w \in \mathbb{W}. \tag{2}$$

*Assumption 2.1.* $(A, B) \in \mathbb{R}^{n\times n} \times \mathbb{R}^{n\times m}$ is stabilizable.

*Assumption 2.2.* The sets $\mathbb{X}$ and $\mathbb{U}$ are $PC$–polytopic sets in $\mathbb{R}^n$ and $\mathbb{R}^m$ given by irreducible representations:

$$\mathbb{X} := \{x \in \mathbb{R}^n\ :\ \forall i \in \mathbb{N}_{[1:p]},\ F_i^T x \leq 1\}, \text{ and,} \tag{3a}$$

$$\mathbb{U} := \{u \in \mathbb{R}^m\ :\ \forall i \in \mathbb{N}_{[1:r]},\ G_i^T u \leq 1\}. \tag{3b}$$

The set $\mathbb{W}$ is a non–trivial $C$–polytopic set in $\mathbb{R}^n$ given by:

$$\mathbb{W} := \operatorname{convh}(\{\tilde{w}_i \in \mathbb{R}^n\ :\ i \in \mathbb{N}_{[1:q]}\}), \tag{4}$$

whose extreme points $\tilde{w}_i \in \mathbb{R}^n,\ i \in \mathbb{N}_{[1:q]}$ are known.

*Interpretation 2.1.* At any time $k \in \mathbb{N}$, the state $x_k$ is known when the current control action $u_k$ is determined, while the current and future disturbances $w_{k+i},\ i \in \mathbb{N}$ are not known and can take any values $w_{k+i} \in \mathbb{W},\ i \in \mathbb{N}$.

For a given control function $\kappa(\cdot)\ :\ \mathbb{R}^n \to \mathbb{R}^m$, the controlled uncertain dynamics takes the form:

$$x^+ = Ax + B\kappa(x) + w, \tag{5}$$

whose variables are, in view of (2), subject to constraints:

$$x \in \mathbb{X}_\kappa := \{x \in \mathbb{X}\ :\ \kappa(x) \in \mathbb{U}\} \text{ and } w \in \mathbb{W}. \tag{6}$$

*Definition 2.1.* A set $\Omega \subseteq \mathbb{R}^n$ is said to be a robust positively invariant (RPI) set for the system $x^+ = Ax + B\kappa(x) + w$ given by (5) and the constraint set $(\mathbb{X}_\kappa, \mathbb{W})$ given by (6) if and only if $\Omega \subseteq \mathbb{X}_\kappa$ and for all $x \in \Omega$ and all $w \in \mathbb{W}$ it holds that $Ax + B\kappa(x) + w \in \Omega$.

*Definition 2.2.* A set $\underline{\Omega} \subseteq \mathbb{R}^n$ is robustly exponentially stable for the system $x^+ = Ax + B\kappa(x) + w$ given by (5) and the constraint set $(\mathbb{X}_\kappa, \mathbb{W})$ given by (6) with the basin of attraction being equal to the set $\bar{\Omega} \subseteq \mathbb{R}^n$ if and only if $\underline{\Omega} \subseteq \bar{\Omega} \subseteq \mathbb{X}_\kappa$, and, for any sequence $\{x_k\}_{k\in\mathbb{N}}$, generated by (5) with any $x_0 \in \bar{\Omega}$ and any disturbance sequence $\{w_k\}_{k\in\mathbb{N}}$, it holds that, for all $k \in \mathbb{N}$, $\operatorname{dist}(L, x_k, \bar{\Omega}) = 0$ and $\operatorname{dist}(L, x_k, \underline{\Omega}) \leq a^k b \operatorname{dist}(L, x_0, \underline{\Omega})$ for some $(a, b) \in [0, 1) \times [0, \infty)$ and where $L$ is a symmetric $PC$–set in $\mathbb{R}^n$.

Under a mild assumption (Kolmanovsky and Gilbert, 1998; Raković, 2007), the local dynamics under $u = Kx$:

$$x^+ = (A + BK)x + w, \tag{7}$$

guarantee that the minimal robust positively invariant set $X_\infty$ is the minimal robustly exponentially stable attractor for the system (7) and constraint set $(\mathbb{X}_K, \mathbb{W})$ with the basin of attraction being the maximal robust positively invariant set $\mathbb{X}_f$. The corresponding assumption is:

*Assumption 2.3.* (i) The matrix $K \in \mathbb{R}^{m\times n}$ is such that $A + BK$ is strictly stable, i.e. $\rho(A + BK) < 1$. (ii) The terminal constraint set $\mathbb{X}_f$ is the maximal robust positively invariant set for the system (7) and the constraint set $(\mathbb{X}_K, \mathbb{W})$ where $\mathbb{X}_K := \{x \in \mathbb{X}\ :\ Kx \in \mathbb{U}\}$, i.e. $\mathbb{X}_f$ is the maximal set (with respect to set inclusion) such that

$$(A + BK)\mathbb{X}_f \oplus \mathbb{W} \subseteq \mathbb{X}_f,\ \mathbb{X}_f \subseteq \mathbb{X}, \text{ and, } K\mathbb{X}_f \subseteq \mathbb{U}, \tag{8}$$

and, in addition, $\mathbb{X}_f$ is a $PC$–polytopic set in $\mathbb{R}^n$ with its irreducible representation given by:

$$\mathbb{X}_f := \{x \in \mathbb{R}^n\ :\ \forall i \in \mathbb{N}_{[1:t]},\ H_i^T x \leq 1\}. \tag{9}$$

## 3. FULLY PARAMETERIZED TUBES

### 3.1 FPT : Main Idea

At the time 0, the available knowledge of the disturbances $w_{j-1},\ j \in \mathbb{N}_{[1:N]}$ is that they will belong to the set $\mathbb{W}$ which is a convex hull of a finite number of extreme points $\tilde{w}_i,\ i \in \mathbb{N}_{[1:q]}$. Hence, as in (Raković et al., 2010a,b), linearity and convexity motivate the consideration of the 0–th partial state sequence $\mathbf{x}_{(0,N)} := \{x_{(0,k)}\}_{k\in\mathbb{N}_N}$ and the set of extreme partial state sequences $\{\mathbf{x}_{(i,j,N)}\ :\ i \in \mathbb{N}_{[1:q]},\ j \in \mathbb{N}_{[1:N]}\}$ induced, respectively, from the corresponding 0–th partial control sequence $\mathbf{u}_{(0,N-1)} := \{u_{(0,k)}\}_{k\in\mathbb{N}_{N-1}}$ and an initial condition $x_{(0,0)}$, and the set of extreme partial control sequences $\{\mathbf{u}_{(i,j,N-1)}\ :\ i \in \mathbb{N}_{[1:q]},\ j \in \mathbb{N}_{[1:N]}\}$ and initial conditions $\{x_{(i,j,0)}\ :\ i \in \mathbb{N}_{[1:q]},\ j \in \mathbb{N}_{[1:N]}\}$ according to, for all $k \in \mathbb{N}_{N-1}$,:

$$x_{(0,k+1)} = Ax_{(0,k)} + Bu_{(0,k)}, \tag{10}$$

and, for all $i \in \mathbb{N}_{[1:q]}$, all $j \in \mathbb{N}_{[1:N]}$ and all $k \in \mathbb{N}_{N-1}$,

$$x_{(i,j,k+1)} = \begin{cases} Ax_{(i,j,k)} + Bu_{(i,j,k)} + \tilde{w}_i, & k = j - 1, \\ Ax_{(i,j,k)} + Bu_{(i,j,k)}, & k \neq j - 1, \end{cases} \tag{11}$$

with the conditions that, for all $j \in \mathbb{N}_{[1:N]}$ and all $k \in \mathbb{N}_{j-1}$,:

$$x_{(1,j,k)} = x_{(2,j,k)} = \ldots = x_{(q,j,k)} =: x_{(j,k)}, \text{ and,} \tag{12a}$$

$$u_{(1,j,k)} = u_{(2,j,k)} = \ldots = u_{(q,j,k)} =: u_{(j,k)}. \tag{12b}$$

The 0–th partial initial condition $x_{(0,0)}$ and the set of extreme partial initial conditions $\{x_{(i,j,0)}\ :\ i \in \mathbb{N}_{[1:q]},\ j \in \mathbb{N}_{[1:N]}\}$ which, in view of (12), reduces to the set of partial initial conditions $\{x_{(j,0)}\ :\ j \in \mathbb{N}_{[1:N]}\}$ are required, for a given $x$, to satisfy:

$$x = \sum_{j=0}^{N} x_{(j,0)}, \tag{13}$$

but otherwise can be selected arbitrarily.

*Remark 3.1.* Note that, since the effect of the disturbance $w_{j-1}$ can be acted upon from the $j$–th prediction time instant, there is no loss of generality in imposing the requirements in (12). Note also that the sets of extreme partial state sequences $\{\mathbf{x}_{(i,j,N)}\ :\ i \in \mathbb{N}_{[1:q]},\ j \in \mathbb{N}_{[1:N]}\}$ and extreme partial control sequences $\{\mathbf{u}_{(i,j,N-1)}\ :\ i \in \mathbb{N}_{[1:q]},\ j \in \mathbb{N}_{[1:N]}\}$ are deterministic since the extreme disturbance points $\tilde{w}_i,\ i \in \mathbb{N}_{[1:q]}$ are known.

The 0–th partial state and control tubes $\mathbf{X}_{(0,N)} := \{X_{(0,k)}\}_{k\in\mathbb{N}_N}$ and $\mathbf{U}_{(0,N)} := \{U_{(0,k)}\}_{k\in\mathbb{N}_{N-1}}$ are singletons:

$$X_{(0,k)} := \{x_{(0,k)}\}, \text{ and, } U_{(0,k)} := \{u_{(0,k)}\}, \tag{14}$$

while, the partial state and control tubes $\mathbf{X}_{(j,N)} := \{X_{(j,k)}\}_{k\in\mathbb{N}_N}$ and $\mathbf{U}_{(j,N)} := \{U_{(j,k)}\}_{k\in\mathbb{N}_{N-1}}$ are given, for all $j \in \mathbb{N}_{[1:N]}$ and each $k$, by:

$$X_{(j,k)} := \operatorname{convh}(\{x_{(i,j,k)}\ :\ i \in \mathbb{N}_{[1:q]}\}), \text{ and,} \tag{15a}$$

$$U_{(j,k)} := \operatorname{convh}(\{u_{(i,j,k)}\ :\ i \in \mathbb{N}_{[1:q]}\}). \tag{15b}$$

The partial state and control tubes $\mathbf{X}_{(j,N)} = \{X_{(j,k)}\}_{k\in\mathbb{N}_N}$ and $\mathbf{U}_{(j,N)} = \{U_{(j,k)}\}_{k\in\mathbb{N}_{N-1}}$, with $j \in \mathbb{N}_N$, induce the overall, fully parameterized state and control tubes $\mathbf{X}_N := \{X_k\}_{k\in\mathbb{N}_N}$ and $\mathbf{U}_N := \{U_k\}_{k\in\mathbb{N}_{N-1}}$ where, for each $k$,:

$$X_k := \bigoplus_{j=0}^{N} X_{(j,k)}, \text{ and, } U_k := \bigoplus_{j=0}^{N} U_{(j,k)}. \tag{16}$$

The main idea behind the fully parameterized state and control tubes is to allow for all the partial state and control tubes $\mathbf{X}_{(j,N)} = \{X_{(j,k)}\}_{k\in\mathbb{N}_N}$, $j \in \mathbb{N}_N$ and $\mathbf{U}_{(j,N-1)} = \{U_{(j,k)}\}_{k\in\mathbb{N}_{N-1}}$, $j \in \mathbb{N}_{N-1}$ to commence at the time instant 0 rather than at the time instant $j$ within the prediction horizon $\mathbb{N}_N$ as done in (Raković et al., 2010a,b).

*Remark 3.2.* As in (Raković et al., 2010a,b), the Minkowski sum and convex hull operations are only used for the purpose of analysis. In fact, the state tubes are induced implicitly from the set of $(N+1)n + qN(N+1)n$ real variables (namely, variables $x_{(0,k)}$ and $x_{(i,j,k)}$) and, similarly, the control tubes from the set of $Nm + qN^2m$ real variables (namely, variables $u_{(0,k)}$ and $u_{(i,j,k)}$).

The FPT can be used to obtain controlled predictions which are robust to the uncertainty. We show this in two steps as follows. Let $\lambda := (\lambda_1, \lambda_2, \ldots, \lambda_q)$, $\Lambda := \{\lambda \in \mathbb{R}^q : \forall i \in \mathbb{N}_{[1:q]},\ \lambda_i \geq 0 \text{ and } \sum_{i=1}^{q} \lambda_i = 1\}$ and, let also, for all $w \in \mathbb{W}$,:

$$\lambda^0(w) := \arg\min_{\lambda}\{\lambda^T\lambda \ : \ \lambda \in \Lambda^*(w)\}, \text{ where,} \tag{17a}$$

$$\Lambda^*(w) := \{\lambda \in \Lambda : \ w = \sum_{i=1}^{q} \lambda_i \tilde{w}_i\}. \tag{17b}$$

Note that, by standard results in parametric mathematical programming (Bank et al., 1983), the function $\lambda^0(\cdot) : \mathbb{W} \to \Lambda$ is, in general, a continuous and piecewise affine function. We proceed and consider any partial state and control tubes $\mathbf{X}_{(j,N)}$ and $\mathbf{U}_{(j,N-1)}$, with $j \in \mathbb{N}_{[1:N]}$, satisfying (11), (12) and (15) and with arbitrary but fixed $x_{(j,0)}$. With any such pair of partial state and control tubes $\mathbf{X}_{(j,N)}$ and $\mathbf{U}_{(j,N-1)}$, with $j \in \mathbb{N}_{[1:N]}$, and any arbitrary disturbance $w_{j-1} \in \mathbb{W}$ we associate a partial state sequence $\mathbf{x}_{(j,N)}(x_{(j,0)}, w_{j-1}) := \{x_{(j,k)}(x_{(j,0)}, w_{j-1})\}_{k\in\mathbb{N}_N}$ as well as a partial control sequence $\mathbf{u}_{(j,N-1)}(x_{(j,0)}, w_{j-1}) := \{u_{(j,k)}(x_{(j,0)}, w_{j-1})\}_{k\in\mathbb{N}_{N-1}}$ whose terms are given by:

$$x_{(j,k)}(x_{(j,0)}, w_{j-1}) := \sum_{i=1}^{q} \lambda_i^0(w_{j-1}) x_{(i,j,k)}, \text{ and,} \tag{18a}$$

$$u_{(j,k)}(x_{(j,0)}, w_{j-1}) := \sum_{i=1}^{q} \lambda_i^0(w_{j-1}) u_{(i,j,k)}. \tag{18b}$$

Due to convexity, relations (15), (17) and (18) imply that, for all $k \in \mathbb{N}_N$, it holds that $x_{(j,k)}(x_{(j,0)}, w_{j-1}) \in X_{(j,k)}$ and, for all $k \in \mathbb{N}_{N-1}$, $u_{(j,k)}(x_{(j,0)}, w_{j-1}) \in U_{(j,k)}$. In addition, due to linearity and convexity, relations (11) and (17) imply that, for all $k \in \mathbb{N}_{N-1}$ for $k \neq j-1$, $x_{(j,k+1)}(x_{(j,0)}, w_{j-1}) = Ax_{(j,k)}(x_{(j,0)}, w_{j-1}) + Bu_{(j,k)}(x_{(j,0)}, w_{j-1})$ and for $k = j-1$, $x_{(j,j)}(x_{(j,0)}, w_{j-1}) = Ax_{(j,j-1)}(x_{(j,0)}, w_{j-1}) + Bu_{(j,j-1)}(x_{(j,0)}, w_{j-1}) + w_{j-1}$. In fact, these partial state and control sequences are generated by the functions $\mathbf{x}_{(j,N)}(x_{(j,0)}, \cdot) : \mathbb{W} \to X_{(j,0)} \times X_{(j,1)} \times \ldots \times X_{(j,N)}$ and $\mathbf{u}_{(j,N-1)}(x_{(j,0)}, \cdot) : \mathbb{W} \to U_{(j,0)} \times U_{(j,1)} \times \ldots \times U_{(j,N-1)}$. In particular, for all $k \in \mathbb{N}_N$, $x_{(j,k)}(x_{(j,0)}, \cdot) : \mathbb{W} \to X_{(j,k)}$ and, for all $k \in \mathbb{N}_{N-1}$, $u_{(j,k)}(x_{(j,0)}, \cdot) : \mathbb{W} \to U_{(j,k)}$ are, in view of (17)–(18), continuous piecewise affine functions (in the general case).

*Proposition 3.1.* Pick any $N \in \mathbb{N}_+$, any $j \in \mathbb{N}_{[1:N]}$, any $x_{(j,0)} \in \mathbb{R}^n$ and fix an arbitrary pair of partial state and control tubes $\mathbf{X}_{(j,N)}$ and $\mathbf{U}_{(j,N-1)}$ satisfying (11), (12) and (15). Then the functions $x_{(j,k)}(x_{(j,0)}, \cdot) : \mathbb{W} \to X_{(j,k)}$ and $u_{(j,k)}(x_{(j,0)}, \cdot) : \mathbb{W} \to U_{(j,k)}$ specified via (18) are, in general, continuous piecewise affine, and for all $w_{j-1} \in \mathbb{W}$ it holds that: $(i)$ for all $k \in \mathbb{N}_N$, $x_{(j,k)}(x_{(j,0)}, w_{j-1}) \in X_{(j,k)}$, $(ii)$ for all $k \in \mathbb{N}_{N-1}$, $u_{(j,k)}(x_{(j,0)}, w_{j-1}) \in U_{(j,k)}$, $(iii)$ for all $k \in \mathbb{N}_{N-1}$, $x_{(j,k+1)}(x_{(j,0)}, w_{j-1}) = Ax_{(j,k)}(x_{(j,0)}, w_{j-1}) + Bu_{(j,k)}(x_{(j,0)}, w_{j-1})$ if $k \neq j-1$, and, for $k = j-1$, $x_{(j,j)}(x_{(j,0)}, w_{j-1}) = Ax_{(j,j-1)}(x_{(j,0)}, w_{j-1}) + Bu_{(j,j-1)}(x_{(j,0)}, w_{j-1}) + w_{j-1}$.

We also note that if, for some $k \in \mathbb{N}_{N-1}$, we have that $x_k = \sum_{j=0}^{N} x_{(j,k)}$ and we set $u_k = \sum_{j=0}^{N} u_{(j,k)}$ then $x_{k+1} = Ax_k + Bu_k + w_k = A\sum_{j=0}^{N} x_{(j,k)} + B\sum_{j=0}^{N} u_{(j,k)} + w_k = \sum_{j=0,\ j\neq k+1}^{N} \left(Ax_{(j,k)} + Bu_{(j,k)}\right) + Ax_{(k+1,k)} + Bx_{(k+1,k)} + w_k = \sum_{j=0}^{N} x_{(j,k+1)}$ with $x_{(j,k+1)} := Ax_{(j,k)} + Bu_{(j,k)}$ when $j-1 \neq k$ and $x_{(k+1,k+1)} := Ax_{(k+1,k)} + Bu_{(k+1,k)} + w_k$ when $j-1 = k$. With this in mind, we proceed and consider any FPT $\mathbf{X}_N$ and $\mathbf{U}_{N-1}$, with $N \in \mathbb{N}_+$, satisfying (10), (11), (12), (13), (14), (15) and (16) and with arbitrary but fixed $x = x_0$. With any such FPT $\mathbf{X}_N$ and $\mathbf{U}_{N-1}$ and any arbitrary disturbance sequence $\mathbf{w}_{N-1} := \{w_k \in \mathbb{W}\}_{k\in\mathbb{N}_{N-1}}$ we associate state and control sequences $\mathbf{x}_N(x, \mathbf{w}_{N-1}) := \{x_k(x, \mathbf{w}_{N-1})\}_{k\in\mathbb{N}_N}$ and $\mathbf{u}_{N-1}(x, \mathbf{w}_{N-1}) := \{u_k(x, \mathbf{w}_{N-1})\}_{k\in\mathbb{N}_{N-1}}$ for which:

$$\begin{aligned}
x_k(x, \mathbf{w}_{N-1}) &:= \sum_{j=0}^{N} x_{(j,k)}(x, \mathbf{w}_{N-1}), \text{ where,}\\
x_{(0,k)}(x, \mathbf{w}_{N-1}) &:= x_{(0,k)}(x), \text{ and, for all } j \in \mathbb{N}_{[1:N]},\\
x_{(j,k)}(x, \mathbf{w}_{N-1}) &:= x_{(j,k)}(x_{(j,0)}(x), w_{j-1}), \text{ and,}
\end{aligned} \tag{19a}$$

$$\begin{aligned}
u_k(x, \mathbf{w}_{N-1}) &:= \sum_{j=0}^{N} u_{(j,k)}(x, \mathbf{w}_{N-1}) \text{ where,}\\
u_{(0,k)}(x, \mathbf{w}_{N-1}) &:= u_{(0,k)}(x), \text{ and, for all } j \in \mathbb{N}_{[1:N]},\\
u_{(j,k)}(x, \mathbf{w}_{N-1}) &:= u_{(j,k)}(x_{(j,0)}(x), w_{j-1}),
\end{aligned} \tag{19b}$$

where $x_{(0,k)}(x) = x_{(0,k)}$, $u_{(0,k)}(x) = u_{(0,k)}$, $x_{(j,0)}(x) = x_{(j,0)}$, $u_{(j,0)}(x) = u_{(j,0)}$ and the functions $x_{(j,k)}(x_{(j,0)}(x), \cdot)$ and $u_{(j,k)}(x_{(j,0)}(x), \cdot)$ are given as in (18). By construction and Proposition 3.1 it follows that, for all $k \in \mathbb{N}_N$ and any arbitrary disturbance sequence $\mathbf{w}_{N-1} = \{w_k \in \mathbb{W}\}_{k\in\mathbb{N}_{N-1}}$, $x_k(x, \mathbf{w}_{N-1}) := \sum_{j=0}^{N} x_{(j,k)}(x, \mathbf{w}_{N-1}) \in \bigoplus_{j=0}^{N} X_{(j,k)} = X_k$ and, similarly for all $k \in \mathbb{N}_{N-1}$, $u_k(x, \mathbf{w}_{N-1}) := \sum_{j=0}^{N} u_{(j,k)}(x, \mathbf{w}_{N-1}) \in \bigoplus_{j=0}^{N} U_{(j,k)} = U_k$. Furthermore, our construction and Proposition 3.1 also imply that, for all $k \in \mathbb{N}_{N-1}$ and any arbitrary disturbance sequence $\mathbf{w}_{N-1} = \{w_k \in \mathbb{W}\}_{k\in\mathbb{N}_{N-1}}$, we have $x_{k+1}(x, \mathbf{w}_{N-1}) = Ax_k(x, \mathbf{w}_{N-1}) + Bu_k(x, \mathbf{w}_{N-1}) + w_k$, and, in addition, that the functions $x_k(x, \cdot) : \mathbb{W}^N \to X_k$ and $u_k(x, \cdot) : \mathbb{W}^N \to U_k$ are, in general, continuous piecewise affine functions.

*Theorem 3.1.* Pick any $N \in \mathbb{N}_+$, any $x = x_0 \in \mathbb{R}^n$ and fix a FPT $\mathbf{X}_N$ and $\mathbf{U}_{N-1}$ satisfying (10)–(16). Then the functions $x_k(x, \cdot) : \mathbb{W}^N \to X_k$ and $u_k(x, \cdot) : \mathbb{W}^N \to U_k$ given by (19) are, in general, continuous piecewise affine, and for all $\mathbf{w}_{N-1} \in \mathbb{W}^N$ it holds that: $(i)$ for all $k \in \mathbb{N}_N$, $x_k(x, \mathbf{w}_{N-1}) \in X_k$, $(ii)$ for all $k \in \mathbb{N}_{N-1}$, $u_k(x, \mathbf{w}_{N-1}) \in U_k$, $(iii)$ for all $k \in \mathbb{N}_{N-1}$, $x_{k+1}(x, \mathbf{w}_{N-1}) = Ax_k(x, \mathbf{w}_{N-1}) + Bu_k(x, \mathbf{w}_{N-1}) + w_k$.

### *3.2 FPT : Equivalent Reparameterization*

We recall that the 0–th partial state and control sequences $\{x_{(0,k)}\}_{k\in\mathbb{N}_N}$ and $\{u_{(0,k)}\}_{k\in\mathbb{N}_{N-1}}$ as well as the $j$–th extreme partial state and control sequences $\{x_{(i,j,k)}\}_{k\in\mathbb{N}_N}$ and $\{u_{(i,j,k)}\}_{k\in\mathbb{N}_{N-1}}$ are deterministic. Hence, the 0–th partial state and control sequences $\{x_{(0,k)}\}_{k\in\mathbb{N}_N}$ and $\{u_{(0,k)}\}_{k\in\mathbb{N}_{N-1}}$ can be equivalently reparameterized as:

$$\forall k \in \mathbb{N}_N,\ x_{(0,k)} = z_{(0,k)} + e_{(0,k)},\ \text{and}, \tag{20a}$$
$$\forall k \in \mathbb{N}_{N-1},\ u_{(0,k)} = v_{(0,k)} + Ke_{(0,k)}, \tag{20b}$$

where, in order to guarantee (10), the sequences $\mathbf{z}_{(0,N)} := \{z_{(0,k)}\}_{k\in\mathbb{N}_N}$, $\mathbf{v}_{(0,N-1)} := \{v_{(0,k)}\}_{k\in\mathbb{N}_{N-1}}$ and $\mathbf{e}_{(0,N)} := \{e_{(0,k)}\}_{k\in\mathbb{N}_N}$ are required to satisfy:

$$\forall k \in \mathbb{N}_{N-1},\ z_{(0,k+1)} = Az_{(0,k)} + Bv_{(0,k)},\ \text{and}, \tag{21a}$$
$$\forall k \in \mathbb{N}_{N-1},\ e_{(0,k+1)} = (A+BK)e_{(0,k)}, \tag{21b}$$

with $x_{(0,0)} = z_{(0,0)} + e_{(0,0)}$. Likewise, for all $i \in \mathbb{N}_{[1:q]}$ and all $j \in \mathbb{N}_{[1:N]}$, the $j$–th extreme partial state and control sequences $\{x_{(i,j,k)}\}_{k\in\mathbb{N}_N}$ and $\{u_{(i,j,k)}\}_{k\in\mathbb{N}_{N-1}}$ are equivalently expressed as:

$$\forall k \in \mathbb{N}_N,\ x_{(i,j,k)} = z_{(i,j,k)} + e_{(i,j,k)},\ \text{and}, \tag{22a}$$
$$\forall k \in \mathbb{N}_{N-1},\ u_{(i,j,k)} = v_{(i,j,k)} + Ke_{(i,j,k)}, \tag{22b}$$

where, in order to guarantee (11), the sequences $\mathbf{z}_{(i,j,N)} := \{z_{(i,j,k)}\}_{k\in\mathbb{N}_N}$, $\mathbf{v}_{(i,j,N-1)} := \{v_{(i,j,k)}\}_{k\in\mathbb{N}_{N-1}}$ and $\mathbf{e}_{(i,j,N)} := \{e_{(i,j,k)}\}_{k\in\mathbb{N}_N}$ are required to satisfy, for all $i \in \mathbb{N}_{[1:q]}$, all $j \in \mathbb{N}_{[1:N]}$ and all $k \in \mathbb{N}_{N-1}$,:

$$z_{(i,j,k+1)} = Az_{(i,j,k)} + Bv_{(i,j,k)},\ \text{and}, \tag{23a}$$
$$e_{(i,j,k+1)} = \begin{cases} (A+BK)e_{(i,j,k)} + \tilde{w}_i, & k = j-1, \\ (A+BK)e_{(i,j,k)}, & k \neq j-1, \end{cases} \tag{23b}$$

with $x_{(i,j,0)} = z_{(i,j,0)} + e_{(i,j,0)}$. To ensure the satisfaction of relations (12) we impose an equivalent set of conditions given, for all $j \in \mathbb{N}_{[1:N]}$ and all $k \in \mathbb{N}_{j-1}$, by:

$$z_{(1,j,k)} = z_{(2,j,k)} = \ldots = z_{(q,j,k)} =: z_{(j,k)}, \tag{24a}$$
$$v_{(1,j,k)} = v_{(2,j,k)} = \ldots = v_{(q,j,k)} =: v_{(j,k)},\ \text{and}, \tag{24b}$$
$$e_{(1,j,k)} = e_{(2,j,k)} = \ldots = e_{(q,j,k)} =: e_{(j,k)},\ e_{(j,k)} := 0. \tag{24c}$$

Finally, relation (13) is equivalently expressed as:

$$x = \sum_{j=0}^{N}(z_{(j,0)} + e_{(j,0)}) = e_{(0,0)} + \sum_{j=0}^{N} z_{(j,0)}. \tag{25}$$

The equivalent reparameterization of the tubes is motivated by (Raković et al., 2010a,b, Proposition 4):

*Proposition 3.2.* Suppose Assumptions 2.1–2.3 hold and consider the local state and control tubes $\mathbf{E}_N := \{E_k\}_{k\in\mathbb{N}_N}$ and $\boldsymbol{\Delta}\mathbf{U}_{N-1} := \{\Delta U_k\}_{k\in\mathbb{N}_{N-1}}$ specified, for any $e \in \mathbb{X}_f$, by $E_0 := e$ and, for all $k \in \mathbb{N}_{[1:N]}$, $E_k := (A+BK)^k e \oplus \bigoplus_{j=1}^{k}(A+BK)^{k-j}\mathbb{W}$, and, $\Delta U_0 := Ke$ and, for all $k \in \mathbb{N}_{[1:N-1]}$, $\Delta U_k := K(A+BK)^k e \oplus \bigoplus_{j=1}^{k} K(A+BK)^{k-j}\mathbb{W}$. Then, for all $e \in \mathbb{X}_f$, it holds that:

$$\forall k \in \mathbb{N}_N,\ E_k \subseteq \mathbb{X}_f \subseteq \mathbb{X}, \tag{26a}$$
$$\forall k \in \mathbb{N}_{N-1},\ \Delta U_k \subseteq K\mathbb{X}_f \subseteq \mathbb{U},\ \text{and}, \tag{26b}$$
$$\forall k \in \mathbb{N}_{N-1},\ E_{k+1} = (A+BK)E_k \oplus \mathbb{W}. \tag{26c}$$

The local FPT specified in the Proposition form a feasible FPT for any $x \in \mathbb{X}_f$. This fact is helpful in selecting a cost function leading to the stabilizing properties of FPTMPC.

## 4. FPT OPTIMAL CONTROL

The main decision variable $\mathbf{d}_{(M,N)}$, for a given $x$, is obtained by vectorizing the sequences $\mathbf{z}_{(0,N)} = \{z_{(0,k)}\}_{k\in\mathbb{N}_N}$, $\mathbf{v}_{(0,N-1)} = \{v_{(0,k)}\}_{k\in\mathbb{N}_{N-1}}$, $\mathbf{z}_{(i,j,N)} = \{z_{(i,j,k)}\}_{k\in\mathbb{N}_N}$ and $\mathbf{v}_{(i,j,N-1)} = \{v_{(i,j,k)}\}_{k\in\mathbb{N}_{N-1}}$, with $i \in \mathbb{N}_{[1:q]}$ and $j \in \mathbb{N}_{[1:N]}$, so that $\mathbf{d}_{(M,N)} \in \mathbb{R}^{N_M}$ where $N_M := ((N+1)n + Nm)(qN+1)$. Let, for any $x \in \mathbb{R}^n$ and any $\mathbf{d}_{(M,N)} \in \mathbb{R}^{N_M}$,

$$e(x, \mathbf{d}_{(M,N)}) := x - \sum_{j=0}^{N} z_{(1,j,0)}. \tag{27}$$

Note that, in view of (21b), (23b), (24c) and (25) with $e_{(0,0)} = e(x, \mathbf{d}_{(M,N)})$ we have, for all $k \in \mathbb{N}_N$,:

$$e_{(0,k)} = (A+BK)^k(x - \sum_{j=0}^{N} z_{(1,j,0)}), \tag{28}$$

and, for all $i \in \mathbb{N}_{[1:q]}$ and all $j \in \mathbb{N}_{[1:N]}$,:

$$e_{(i,j,k)} = \begin{cases} 0, & k \in \mathbb{N}_{j-1}, \\ (A+BK)^{(k-j)}\tilde{w}_i, & k \in \mathbb{N}_{[j:N]}. \end{cases} \tag{29}$$

Hence, the sequence $\mathbf{e}_{(0,N)} = \{e_{(0,k)}\}_{k\in\mathbb{N}_N}$ depends in a linear/affine fashion on $x$ and $\mathbf{d}_{(M,N)}$ while the sequences $\mathbf{e}_{(i,j,N)} = \{e_{(i,j,k)}\}_{k\in\mathbb{N}_N}$ with $i \in \mathbb{N}_{[1:q]}$ and $j \in \mathbb{N}_{[1:N]}$ are deterministic since the points $\tilde{w}_i,\ i \in \mathbb{N}_{[1:q]}$ are known.

In order to ensure the robust constraint satisfaction, the FPT are required to satisfy for all $k \in \mathbb{N}_{N-1}$:

$$X_k \subseteq \mathbb{X} \text{ and } U_k \subseteq \mathbb{U}, \tag{30}$$

while, to induce invariance and stability, the additional constraints are given, as in (Raković et al., 2010a,b), by:

$$X_N \subseteq \mathbb{X}_f \text{ and } e_{(0,0)} = e(x, \mathbf{d}_{(M,N)}) \in \mathbb{X}_f. \tag{31}$$

As demonstrated in (Raković et al., 2010a,b), the constraint $e_{(0,0)} = e(x, \mathbf{d}_{(M,N)}) \in \mathbb{X}_f$ does not affect the size of the domain of attraction while together with our reparameterization allows for a convenient selection of the cost function associated with the FPT. As also noted in (Raković et al., 2010a,b), the actual computation is highly tractable since it utilizes effectively basic properties of the support function (Rockafellar, 1970), based on which the set of constraints $\forall k \in \mathbb{N}_{N-1},\ X_k \subseteq \mathbb{X}$ in (30) is satisfied if and only if, for all $l \in \mathbb{N}_{[1:p]}$ and all $k \in \mathbb{N}_{N-1}$,:

$$F_l^T(z_{(0,k)} + e_{(0,k)}) + \sum_{j=1}^{N} f_{(l,j,k)} \leq 1,\ \text{and}, \forall j \in \mathbb{N}_{[1:N]},$$
$$\forall i \in \mathbb{N}_{[1:q]},\ F_l^T(z_{(i,j,k)} + e_{(i,j,k)}) \leq f_{(l,j,k)}, \tag{32}$$

for some scalars $f_{(l,j,k)},\ l \in \mathbb{N}_{[1:p]},\ j \in \mathbb{N}_{[1:N]},\ k \in \mathbb{N}_{N-1}$. By the same token, the constraints $\forall k \in \mathbb{N}_{N-1},\ U_k \subseteq \mathbb{U}$ in (30) and $X_N \subseteq \mathbb{X}_f$ in (31) are satisfied if and only if, for all $l \in \mathbb{N}_{[1:r]}$ and all $k \in \mathbb{N}_{N-1}$, it holds that:

$$G_l^T(v_{(0,k)} + Ke_{(0,k)}) + \sum_{j=1}^{N} g_{(l,j,k)} \leq 1,\ \text{and}, \forall j \in \mathbb{N}_{[1:N]},$$
$$\forall i \in \mathbb{N}_{[1:q]},\ G_l^T(v_{(i,j,k)} + Ke_{(i,j,k)}) \leq g_{(l,j,k)}, \tag{33}$$

for some scalars $g_{(l,j,k)},\ l \in \mathbb{N}_{[1:p]},\ j \in \mathbb{N}_{[1:N]},\ k \in \mathbb{N}_{N-1}$, and, for all $l \in \mathbb{N}_{[1:t]}$, it holds that:

$$H_l^T(z_{(0,N)} + e_{(0,N)}) + \sum_{j=1}^{N} h_{(l,j,N)} \leq 1,\ \text{and}, \forall j \in \mathbb{N}_{[1:N]},$$
$$\forall i \in \mathbb{N}_{[1:q]},\ H_l^T(z_{(i,j,N)} + e_{(i,j,N)}) \leq h_{(l,j,N)}, \tag{34}$$

for some scalars $h_{(l,j,N)}$, $l \in \mathbb{N}_{[1:p]}$, $j \in \mathbb{N}_{[1:N]}$. Finally, the constraints $e_{(0,0)} = e(x, \mathbf{d}_{(M,N)}) \in \mathbb{X}_f$ in (31) is satisfied if and only if, for all $l \in \mathbb{N}_{[1:t]}$, it holds that:

$$H_l^T(x - \sum_{j=0}^{N} z_{(1,j,0)}) \leq 1. \tag{35}$$

Let $\mathbf{d}_{(S,N)}$ denote the vectorized form of the set of slack variables $f_{(l,j,k)}$ and $g_{(l,j,k)}$ with $l \in \mathbb{N}_{[1:p]}$, $j \in \mathbb{N}_{[1:N]}$, $k \in \mathbb{N}_{N-1}$, and $h_{(l,j,N)}$ with $l \in \mathbb{N}_{[1:p]}$, $j \in \mathbb{N}_{[1:N]}$ so that $\mathbf{d}_{(S,N)} \in \mathbb{R}^{N_S}$ where $N_S := (p+r)N^2 + tN$. With the equivalences above, any admissible FPT for a given $x \in \mathbb{X}$ is induced from the overall decision variable $\mathbf{d}_N := (\mathbf{d}_{(M,N)}^T, \mathbf{d}_{(S,N)}^T)^T \in \mathbb{R}^{N_M+N_S}$ and the set of admissible decision variables for a given $x \in \mathbb{X}$ is the value of the set–valued map $\mathcal{D}_N(x)$ given, for all $x \in \mathbb{X}$, by:

$$\begin{aligned}\mathcal{D}_N(x) := \{\mathbf{d}_N \in \mathbb{R}^{N_M+N_S} \ :\ &\text{relations (21a), (23a),}\\ &\text{(24a), (24b), (32), (33), (34) and (35) hold}\},\end{aligned} \tag{36}$$

where the relations (28) and (29) have been utilized to eliminate/evaluate the values of $e_{(0,k)}$, $k \in \mathbb{N}_N$ and $e_{(i,j,k)}$, $i \in \mathbb{N}_{[1:q]}$, $j \in \mathbb{N}_{[1:N]}, k \in \mathbb{N}_N$. We note that the graph of the set–valued map $\mathcal{D}_N(\cdot)$, namely, the set $D_N \subseteq \mathbb{R}^{n+N_M+N_S}$ given by:

$$D_N := \{(x, \mathbf{d}_N) \ :\ x \in \mathbb{X},\ \mathbf{d}_N \in \mathcal{D}_N(x)\}, \tag{37}$$

is a closed polyhedral set in $\mathbb{R}^{n+N_M+N_S}$ since $\mathbb{X}$ is a polytope and all relationships involved in the definition of the set–valued map $\mathcal{D}_N(\cdot)$ take the form of linear/affine equalities and inequalities whose number is a quadratic function of the horizon length $N$ (hence, the set $D_N$ is a tractable set from the computational point of view).

A sensible cost associated with the FPT is obtained by utilizing the partial stage and terminal cost functions, $\ell(\cdot,\cdot)$ and $V_f(\cdot)$, given, for all $z \in \mathbb{R}^n$ and $v \in \mathbb{R}^m$ by:

$$\ell(z,v) := |z|_{\mathcal{Q}} + |v|_{\mathcal{R}},\ \text{and,}\ V_f(z) := |z|_{\mathcal{P}}, \tag{38}$$

where the sets $\mathcal{Q}$, $\mathcal{R}$ and $\mathcal{P}$ satisfy the Lyapunov condition:

*Assumption 4.1.* The sets $\mathcal{Q} \subseteq \mathbb{R}^n$, $\mathcal{R} \subseteq \mathbb{R}^m$ and $\mathcal{P} \subseteq \mathbb{R}^n$ are symmetric $PC$–polytopic sets in $\mathbb{R}^n$, $\mathbb{R}^m$, and $\mathbb{R}^n$, respectively, and are such that, for all $z \in \mathbb{R}^n$,:

$$|(A+BK)z|_{\mathcal{P}} - |z|_{\mathcal{P}} \leq -(|z|_{\mathcal{Q}} + |Kz|_{\mathcal{R}}). \tag{39}$$

With any decision variable $\mathbf{d}_N \in \mathbb{R}^{N_M+N_S}$ we associate partial cost functions $V_{(j,N)}(\cdot) \ :\ \mathbb{R}^{N_M+N_S} \to \mathbb{R}_+$ with $j \in \mathbb{N}_N$ given by:

$$V_{(0,N)}(\mathbf{d}_N) := \sum_{k=0}^{N-1} \ell(z_{(0,k)}, v_{(0,k)}) + V_f(z_{(0,N)}), \tag{40}$$

and, for all $j \in \mathbb{N}_{[1:N]}$, by:

$$V_{(j,N)}(\mathbf{d}_N) := \sum_{i=1}^{q}(\sum_{k=0}^{N-1} \ell(z_{(i,j,k)}, v_{(i,j,k)}) + V_f(z_{(i,j,N)})). \tag{41}$$

The cost function $V_N(\cdot) \ :\ \mathbb{R}^{N_M+N_S} \to \mathbb{R}_+$ associated with any decision variable $\mathbf{d}_N \in \mathbb{R}^{N_M+N_S}$ is specified by:

$$V_N(\mathbf{d}_N) := \sum_{j=0}^{N} V_{(j,N)}(\mathbf{d}_N). \tag{42}$$

The fully parameterized tube optimal control (FPTOC) problem $\mathbb{P}_N(x)$ is specified, for all $x \in \mathbb{X}$, by:

$$V_N^0(x) := \min_{\mathbf{d}_N}\{V_N(\mathbf{d}_N) \ :\ (x, \mathbf{d}_N) \in D_N\}, \tag{43a}$$

$$\mathbf{d}_N^*(x) := \arg\min_{\mathbf{d}_N}\{V_N(\mathbf{d}_N) \ :\ (x, \mathbf{d}_N) \in D_N\}, \tag{43b}$$

where the function $V_N(\cdot)$ and set $D_N$ are given, respectively, by (42) and (37). The effective domain of the value function $V_N^0(\cdot)$ is referred to as $N$–step FPT controllability set and is, clearly, given by:

$$\mathcal{X}_N := \text{Proj}_{\mathbb{R}^n}(D_N),\ \text{and,}\ \mathcal{X}_0 := \mathbb{X}_f. \tag{44}$$

The relevant topological properties of the (FPTOC) problem $\mathbb{P}_N(x)$ are summarized by the following result, the proof of which can be found in (Raković et al., 2011).

*Theorem 4.1.* Suppose Assumptions 2.1–4.1 hold. Then: $(i)$ The $N$–step FPT controllability set $\mathcal{X}_N$ is a $PC$–polytopic set in $\mathbb{R}^n$ such that $\mathbb{X}_f \subseteq \mathcal{X}_N \subseteq \mathbb{X}$; $(ii)$ The value function $V_N^0(\cdot) \ :\ \mathcal{X}_N \to \mathbb{R}_+$ is convex, piecewise affine and continuous function such that $\forall x \in \mathbb{X}_f,\ V_N^0(x) = 0$ and $\forall x \in \mathcal{X}_N \setminus \mathbb{X}_f,\ V_N^0(x) > 0$; $(iii)$ There exists a piecewise affine and continuous function $\mathbf{d}_N^0(\cdot) \ :\ \mathcal{X}_N \to \mathbb{R}^{N_S}$ such that, $\forall x \in \mathcal{X}_N,\ \mathbf{d}_N^0(x) = (\mathbf{d}_{(M,N)}^0(x), \mathbf{d}_{(S,N)}^0(x)) \in \mathbf{d}_N^*(x)$ and, furthermore, $\forall x \in \mathbb{X}_f,\ \mathbf{d}_{(M,N)}^0(x) = 0$ and $\forall x \in \mathcal{X}_N \setminus \mathbb{X}_f,\ |\mathbf{d}_{(M,N)}^0(x)|_{L_M} > 0$.

*Remark 4.1.* For any fixed $x \in \mathcal{X}_N$ the FPTOC problem $\mathbb{P}_N(x)$, $x \in \mathcal{X}_N$ reduces to a tractable linear programming problem, the solution of which yields the values of the value function $V_N^0(x)$ and the optimal main and slack decision variables $\mathbf{d}_{(M,N)}^0(x)$ and $\mathbf{d}_{(S,N)}^0(x)$. The particular value of the optimal main decision variable $\mathbf{d}_{(M,N)}^0(x)$ can then be utilized in conjunction with algebraic relations provided in Sections 3 and 4 to obtain the optimal FPT $\mathbf{X}_N^0(x)$ and $\mathbf{U}_{N-1}^0(x)$ which enjoy the properties established in Propositions 3.1 and 3.2 and Theorem 3.1.

## 5. FULLY PARAMETERIZED TMPC

We consider the FTMPC law $\kappa_N^0(\cdot) \ :\ \mathcal{X}_N \to \mathbb{U}$ given by:

$$\kappa_N^0(x) = \sum_{j=0}^{N} v_{(j,0)}^0(x) + K(x - \sum_{j=0}^{N} z_{(j,0)}^0(x)), \tag{45}$$

where, for notational simplicity, for $j \in \mathbb{N}_{[1:N]}$, $v_{(j,0)}^0(x) := v_{(1,j,0)}^0(x)$ and $z_{(j,0)}^0(x) := z_{(1,j,0)}^0(x)$ and these values are obtained from $\mathbf{d}_{(M,N)}^0(x)$. Theorem 4.1 implies that the control function $\kappa_N^0(\cdot) \ :\ \mathcal{X}_N \to \mathbb{U}$ is a piecewise affine and continuous function. The FPTMPC law $\kappa_N^0(\cdot) \ :\ \mathcal{X}_N \to \mathbb{U}$ induces the controlled, uncertain, dynamics given, for all $x \in \mathcal{X}_N$, by:

$$x^+ \in \mathcal{F}(x) := \{Ax + B\kappa_N^0(x) + w \ :\ w \in \mathbb{W}\}. \tag{46}$$

We note that, by Theorem 4.1, for all $x \in \mathbb{X}_f$ it holds that $\kappa_N^0(x) = Kx$ and $\mathcal{F}(x) = (A+BK)x \oplus \mathbb{W}$.

The invariance properties induced by the FPTMPC law $\kappa_N^0(\cdot)$ are established by our next result, the proof of which is given in (Raković et al., 2011).

*Proposition 5.1.* Suppose Assumptions 2.1–4.1 hold. Then, the $N$–step FPT controllability set $\mathcal{X}_N$ is robust positively invariant set for the system $x^+ = Ax + B\kappa_N^0(x) + w$ and constraint set $(\mathbb{X}_{\kappa_N^0}, \mathbb{W})$, that is $\mathcal{X}_N \subseteq \mathbb{X}$, and for $x \in \mathcal{X}_N$, $\kappa_N^0(x) \in \mathbb{U}$ and $\mathcal{F}(x) \subseteq \mathcal{X}_N$ where $\kappa_N^0(\cdot)$ and $\mathcal{F}(\cdot)$ are given by (45) and (46).

*Remark 5.1.* We note that Propositions 3.2 and 5.1 imply that, for all $N \in \mathbb{N}$, it holds that $\mathbb{X}_f \subseteq \mathcal{X}_N \subseteq \mathcal{X}_{N+1} \subseteq \mathbb{X}$.

Let, for all $x \in \mathcal{X}_N$,:

$$\ell_z^0(x) := |z^0_{(0,0)}(x)|_{\mathcal{Q}} + \sum_{i=1}^{q}\sum_{j=1}^{N} |z^0_{(i,j,0)}(x)|_{\mathcal{Q}},$$

$$\ell_v^0(x) := |v^0_{(0,0)}(x)|_{\mathcal{R}} + \sum_{i=1}^{q}\sum_{j=1}^{N} |v^0_{(i,j,0)}(x)|_{\mathcal{R}}, \text{ and,}$$

$$\ell^0(x) := \ell_z^0(x) + \ell_v^0(x), \tag{47}$$

and note that, by arguments similar to those of (Raković et al., 2010a,b), there exists a scalar $c_1 \in (0,1)$ such that for all $x \in \mathcal{X}_N$ it holds that $c_1\ell_z^0(x) \le \operatorname{dist}(\mathcal{Q}, x, \mathbb{X}_f) \le \ell_z^0(x)$. The desired Lyapunov property of the value function $V_N^0(\cdot)$ is verified by our next result, the proof of which is reported in (Raković et al., 2011).

*Proposition 5.2.* Suppose Assumptions 2.1–4.1 hold. Then there exists a scalar $c_2 \in [1,\infty)$ such that, for all $x \in \mathcal{X}_N$,:

$$\ell_z^0(x) \le V_N^0(x) \le c_2\ell_z^0(x),\ \ell^0(x) \le V_N^0(x) \le c_2\ell^0(x), \text{ and,}$$

$$\text{for all } x^+ \in \mathcal{F}(x),\ V_N^0(x^+) - V_N^0(x) \le -\ell^0(x), \tag{48}$$

where $\mathcal{F}(\cdot)$ is given by (46).

Proposition 5.2 implies that, under Assumptions 2.1–4.1, there exists a scalar pair $(\bar{a}_N, \bar{b}_N) \in [0,1) \times (0,\infty)$ such that, for all $k \in \mathbb{N}$, the inequalities:

$$V_N^0(x_k) \le \bar{a}_N^k V_N^0(x_0),$$

$$\ell^0(x_k) \le \bar{a}_N^k \bar{b}_N \ell^0(x_0) \text{ and } \ell_z^0(x_k) \le \bar{a}_N^k \bar{b}_N \ell_z^0(x_0), \tag{49}$$

hold true for any $x_0 \in \mathcal{X}_N$ and any corresponding state sequence $\{x_k\}_{k\in\mathbb{N}}$ generated by (46). Hence, we have:

*Theorem 5.1.* Suppose Assumptions 2.1–4.1 hold. Then: $(i)$ There exists a scalar pair $(a_N, b_N) \in [0,1)\times(0,\infty)$ such that, for all $k \in \mathbb{N}$, the relationships $\operatorname{dist}(\mathcal{Q}, x_k, \mathcal{X}_N) = 0$ and $\operatorname{dist}(\mathcal{Q}, x_k, \mathbb{X}_f) \le a_N^k b_N \operatorname{dist}(\mathcal{Q}, x_0, \mathbb{X}_f)$, as well as $\operatorname{dist}(\mathcal{R}, \kappa_N^0(x_k), \mathbb{U}) = 0$ and $\operatorname{dist}(\mathcal{R}, \kappa_N^0(x_k), K\mathbb{X}_f) \le a_N^k b_N$, hold true for any $x_0 \in \mathcal{X}_N$ and any corresponding state sequence $\{x_k\}_{k\in\mathbb{N}}$ generated by (46); and $(ii)$ The set $\mathbb{X}_f$ is robustly exponentially stable for the system $x^+ = Ax + B\kappa_N^0(x) + w$ and the constraint set $(\mathbb{X}_{\kappa_N^0}, \mathbb{W})$ with the basin of attraction being equal to the $N$–step FPT controllability set $\mathcal{X}_N$.

Stronger stability and convergence related property follows from (Raković et al., 2010a,b, Corollary 3):

*Corollary 5.1.* Suppose Assumptions 2.1–4.1 hold. Then, the minimal robust positively invariant set $X_\infty$ for the system $x^+ = (A + BK)x + w$, given by $X_\infty = \bigoplus_{i=0}^{\infty}(A + BK)^i\mathbb{W}$, is the minimal set (with respect to the set inclusion) which is robustly exponentially stable for the system $x^+ = Ax + B\kappa_N^0(x) + w$ and the constraint set $(\mathbb{X}_{\kappa_N^0}, \mathbb{W})$ with the basin of attraction being equal to the $N$–step FPT controllability set $\mathcal{X}_N$.

*Remark 5.2.* Robust MPC can be very demanding and for this reason use of dynamic programming and min–max feedback MPC are ruled out from the class of practicable algorithms. FPTMPC deploys piecewise affine control strategies which are more general than the affine in the disturbance MPC. Indeed, since FPTMPC includes as a special case the PTMPC with upper triangular parameterization of (Raković et al., 2010a,b) and since the latter includes as a special case the affine in the disturbance MPC, it follows that FPTMPC will, in general give, better performance and a larger region of attraction than the affine in the disturbance MPC. Significantly however, this is achieved at comparable computational cost.

## 6. CONCLUSIONS

In this paper, we introduced a FPTMPC synthesis method which utilizes the FPT and the corresponding control policy, and permits their online optimization via a tractable linear programming problem. The developed method is, under rather mild assumptions, computationally efficient while it guarantees strong system theoretic properties for the controlled uncertain dynamics.

## REFERENCES

Bank, B., Guddat, J., Klatte, D., Kummer, B., and Tammer, K. (1983). *Non–linear Parametric Optimization*. Birkhäuser, Basel, Boston, Stuttgart.

Bertsekas, D.P. (2007). *Dynamic Programming and Optimal Control*. Athena Scientific. Volumes I and II – 3rd Edition.

Blanchini, F. (1990). Control synthesis of discrete time systems with control and state bounds in the presence of disturbance. *Journal of Optimization Theory and Applications*, 65(1), 29–40.

Goulart, P.J., Kerrigan, E.C., and Maciejowski, J.M. (2006). Optimization over state feedback policies for robust control with constraints. *Automatica*, 42(4), 523–533.

Kolmanovsky, I.V. and Gilbert, E.G. (1998). Theory and computation of disturbance invariant sets for discrete time linear systems. *Mathematical Problems in Engineering: Theory, Methods and Applications*, 4, 317–367.

Lee, Y.I., Kouvaritakis, B., and Cannon, M. (2002). Constrained receding horizon predictive control for nonlinear systems. *Automatica*, 38(12), 2093–2102.

Löfberg, J. (2003). *Minimax approaches to robust model predictive control*. Ph.D. thesis, Department of Electrical Engineering, Linköping University, Linköping , Sweden.

Mayne, D.Q., Seron, M., and Raković, S.V. (2005). Robust model predictive control of constrained linear systems with bounded disturbances. *Automatica*, 41, 219–224.

Raković, S.V. (2007). Minkowski Algebra and Banach Contraction Principle in Set Invariance for Linear Discrete Time Systems. In *Proceedings of 46th IEEE Conference on Decision and Control, CDC 2007*. New Orleans, LA, USA.

Raković, S.V., Kouvaritakis, B., Cannon, M., Panos, C., and Findeisen, R. (2010a). Parameterized tube model predictive control. *IEEE Transactions on Automatic Control*. Under Review.

Raković, S.V., Kouvaritakis, B., Cannon, M., Panos, C., and Findeisen, R. (2010b). Parameterized tube model predictive control. Technical Report OUEL 2315/10, University of Oxford, Department of Engineering Science.

Raković, S.V., Kouvaritakis, B., Cannon, M., Panos, C., and Findeisen, R. (2011). Fully parameterized tube model predictive control. Technical Report OUEL 2324/11, University of Oxford, Department of Engineering Science.

Raković, S.V., Kouvaritakis, B., Findeisen, R., and Cannon, M. (2010c). Exact scaling dynamics in homothetic tube mpc. *Automatica*. Under Review.

Raković, S.V., Kouvaritakis, B., Findeisen, R., and Cannon, M. (2010d). Homothetic tube model predictive control. *Automatica*. Under Review.

Rawlings, J.B. and Mayne, D.Q. (2009). *Model Predictive Control: Theory and Design*. Nob Hill Publishing, Madison.

Rockafellar, R.T. (1970). *Convex Analysis*. Princeton University Press, USA.

Scokaert, P.O.M. and Mayne, D.Q. (1998). Min–max feedback model predictive control for constrained linear systems. *IEEE Transactions on Automatic Control*, 43, 1136–1142.