A brief introduction to Bayesian Statistics through Astronomical Applications (Lecture 3)

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More Examples

Coordinate Transformations

- Lets say we have a problem in which we know how to write our posterior P(x) in a variable x, but in reality we are interested in knowing the posterior P(y) for variable y=f(x)

- We know the integral of the probability must be conserved

$$\int P(x)dx = \int P(y)dy \qquad\qquad P(x)dx = P(y)dy$$

- This makes it easy, we know how to change variables inside an integral

$$P(y) = \left|\frac{dx}{dy}\right| P(x)$$

Jacobian of the transformation f(x)

The Parallax Example

- Lets illustrate this with an example. Say we have N measurements of the parallax for a star from Gaia

- Lets assume we have an error model and CU7/DPAC gives us an estimate on the parallax error and tells us that its safe to assume these errors are gaussian

- What we'd ultimately like to get is the posterior on the distance to the star

The Parallax Example

- Lets write the posterior on the parallax first

$$P(\varpi|\{\varpi_i\}) = P(\{\varpi_i\}|\varpi)P(\varpi) \qquad |I$$

- Our likelihood is

$$P(\{\varpi_i\}|\varpi) = \prod_{i=1}^{N} e^{-\frac{(\varpi-\varpi_i)^2}{2\sigma_\varpi^2}}$$

- So the posterior is

$$P(\varpi|\{\varpi_i\}) = \prod_{i=1}^{N} e^{-\frac{(\varpi-\varpi_i)^2}{2\sigma_\varpi^2}} P(\varpi) \qquad |I$$

no coordinate transformations yet...

The Parallax Example

- Now we want the posterior for the distance D=1/**ϖ**

$$P(\varpi|\{\varpi_i\})d\varpi = P(D|\{\varpi_i\})dD$$

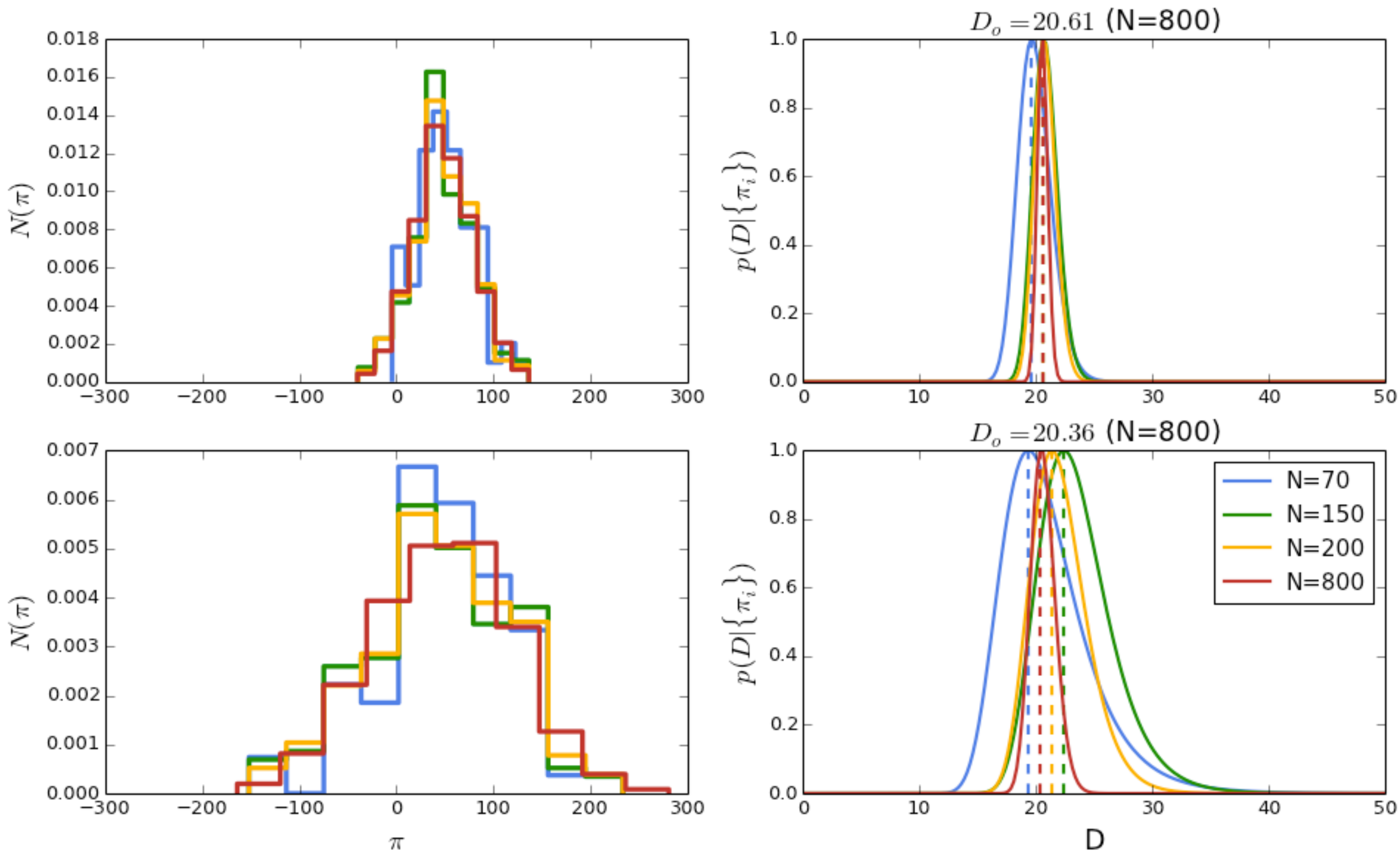
$$P(D|\{\varpi_i\}) = P(\varpi|\{\varpi_i\})\left|\frac{d\varpi}{dD}\right|$$

$$P(\varpi|\{\varpi_i\}) = \prod_{i=1}^{N} e^{-\frac{(\varpi-\varpi_i)^2}{2\sigma_\varpi^2}} P(\varpi)$$

$$P(D|\{\varpi_i\}) = P(\varpi|\{\varpi_i\})\frac{1}{D^2}$$

$$P(D|\{\varpi_i\}) = \prod_{i=1}^{N} e^{-\frac{(1/D-\varpi_i)^2}{2\sigma_\varpi^2}} \frac{1}{D^2} P(\varpi(D))$$

The Parallax Example



Error Propagation

Error Propagation

- Lets say we know the PDF for two variables X and Y

$$P(X) \qquad p(Y) \qquad\qquad |data, I$$

- Now we'd like to know what is the PDF for Z, where

$$Z = X + Y$$

- First, lets look at the case when X and Y are conditionally independent. If this is the case, the joint probability P(X,Y)=P(X)p(Y), from the Marginalization rule this is:

$$P(Z) = \iint_{\substack{\text{for all X,Y} \\ \text{t.q. Z=X+y}}} P(X)p(Y)dXdY \qquad |data, I$$

Error Propagation

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$$P(Z) = \iint P(X)p(Y)dXdY \qquad |data, I$$

for all X,Y
t.q. Z=X+Y

$$P(Z) = \iint P(X)p(Y)\delta(Z - [X+Y])dXdY$$

$$P(Z) = \int P(X)p(Y = Z - X)dX$$

note this is the convolution of P(X) and p(Y)

Error Propagation

- If P(X) and p(y) are gaussians, e.g. lets say like yesterday we have a model with gaussian uncertainties and a uniform prior,

$$P(X) = \frac{1}{\sqrt{2\pi}\sigma_X} e^{-\frac{(X-X_o)^2}{2\sigma_X^2}} \qquad P(Y) = \frac{1}{\sqrt{2\pi}\sigma_Y} e^{-\frac{(Y-Y_o)^2}{2\sigma_Y^2}}$$

$$|data, I$$

then

$$P(Z) = \int P(X) p(Y = Z - X) dX$$

is the convolution of these two gaussians

$$P(Z) = \frac{1}{2\pi\sigma_X\sigma_Y} \int_{-\infty}^{+\infty} e^{-\frac{(X-X_o)^2}{2\sigma_X^2}} e^{-\frac{(Z-X-Y_o)^2}{2\sigma_Y^2}} dX$$

Error Propagation

- After completing squares and simplifying we get

$$P(Z) = \frac{1}{2\pi\sigma_z} e^{-\frac{(Z-Z_o)^2}{2\sigma_z^2}}$$

where

$$Zo = X_o + Y_o \qquad \text{and} \qquad \sigma_Z^2 = \sigma_X^2 + \sigma_Y^2$$

so, the sum in quadrature rule is derived

- Lets go back to the general problem, where Z=f(X,Y)

Error Propagation

- In general, i.e. without assuming cond. independence of X,Y we have

$$P(Z) = \iint_{\substack{\text{for all X,Y} \\ \text{t.q.} \\ \text{Z=f(X,Y)}}} P(X,Y)\,dX\,dY$$

- Since Z=f(X,Y) we have

$$P(Z) = \iint P(X,Y)\delta(Z - f(X,Y))\,dX\,dY$$

The Parallax Example

- Now we want the posterior for the distance D=1/ϖ

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More Examples

Fitting a density profile to data

- Another example. Lets say you have a catalogue of a particular stellar tracer, e.g. RR Lyrae stars, Red Clump stars, Cepheids or whatever.

- We'd like to use these tracers to study how stars are distributed in space in the different components in the Galaxy, i.e. their density profile ρ

- How do we compare and fit this function to a sample of stars?

Thick Disc and Halo RRL Density Profiles

- Thick Disc density profile

$$\rho_{\rm DG} = C_{\rm DG} e^{-\frac{R-R}{h_R}} e^{-\frac{|z|}{h_z}}$$

- Halo density profile

$$\rho_{\rm H} = \frac{C_{\rm H}}{R_\odot^n}\left[R^2 + \left(\frac{z}{q}\right)^2\right]^{n/2}$$

... but the directly observable quantity is not the density ρ, but the number N_{RR} of RRLs in the survey volume V_S

$$N_{RR} = \iiint\limits_{V_S} \rho(\vec{r})dV = \iiint_{V_S} [\rho_{\rm H}(R,z) + \rho_{\rm DG}(R,z)]R\,dR\,dz\,d\varphi$$

Density Profiles: A Bayesian approach

- Our free parameters are:

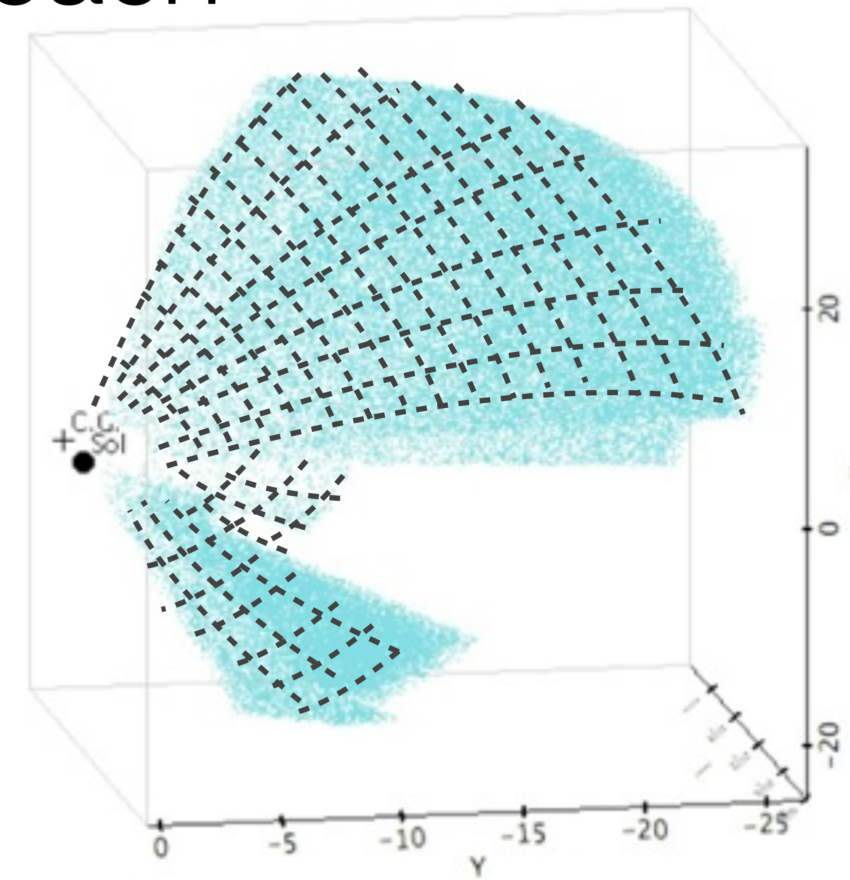
$$\vec{\theta} = (h_z, h_r, C_{tkd}, n, C_h)$$

- We build an imaginary grid restricted *only* to the survey volume Vs

- Our likelihood function is then

$$L \equiv p(\{\eta\}|\vec{\theta}) = \prod_{i \in V_S} p(\eta_i|\vec{\theta}) = \prod_{i \in V_S} \frac{\mu_i^{\eta_i} e^{-\mu_i}}{\eta_i!}$$

$$\ln L = \sum_{i \in V_S} \eta_i \ln \mu_i - \mu_i$$



η_i - ***observed*** number or RRLS on i-th bin

μ_i - ***predicted*** number or RRLS on i-th bin

- If now we make the grid cell size tend to 0

$$\mu_i(\vec{\theta}, \vec{r}_i^{RRLS}) \rightarrow \rho(\vec{\theta}, \vec{r}_i^{RRLS})$$

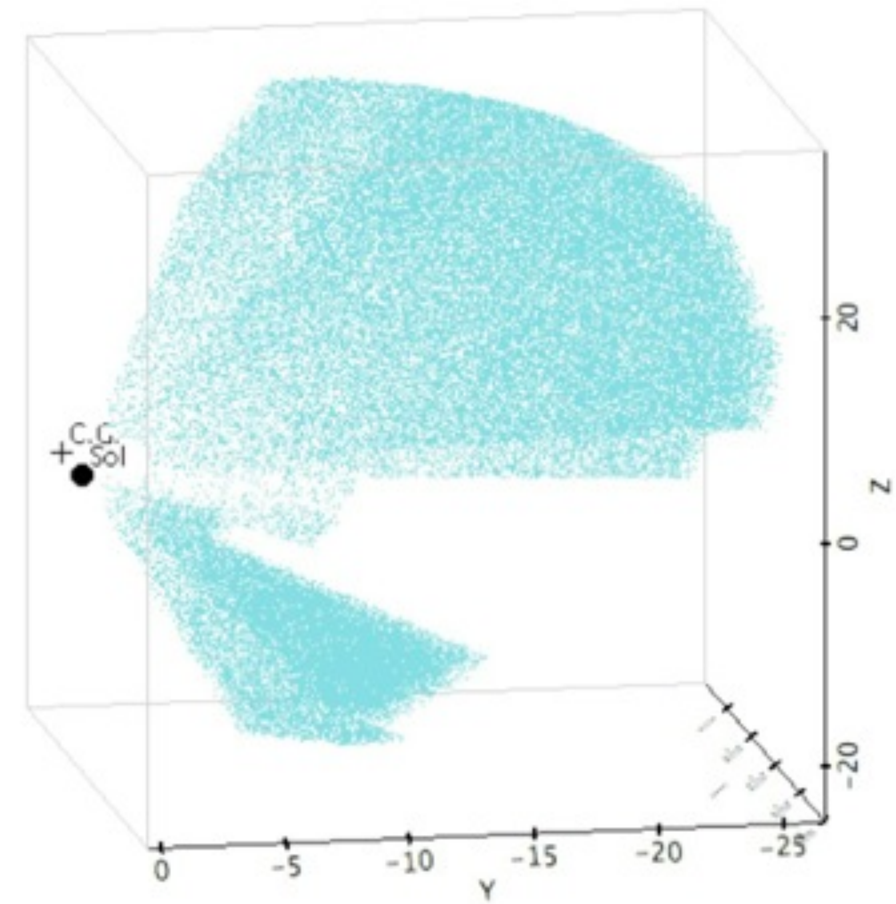
- We finally get

$$\ln L = \sum_{i=1}^{N_{obs}^{RRL}} \ln \rho(\vec{\theta}, \vec{r}_i^{RRLS}) - N_{model}^{RRL}(\vec{\theta})$$

Lombardi et al. (2013)

Density Profiles: A Bayesian approach

- This framework allows us to account for the inhomogeneities of the survey volume due to the variable extinction
- We could also include an incompleteness function for example

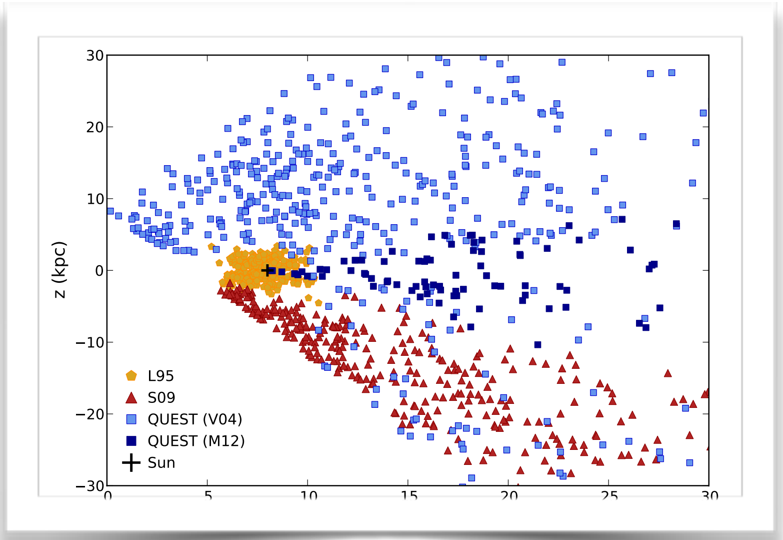


$$\ln L = \sum_{i=1}^{N_{obs}^{RRL}} \ln \rho(\vec{\theta}, \vec{r}_i^{RRLS}) - N_{model}^{RRL}(\vec{\theta})$$

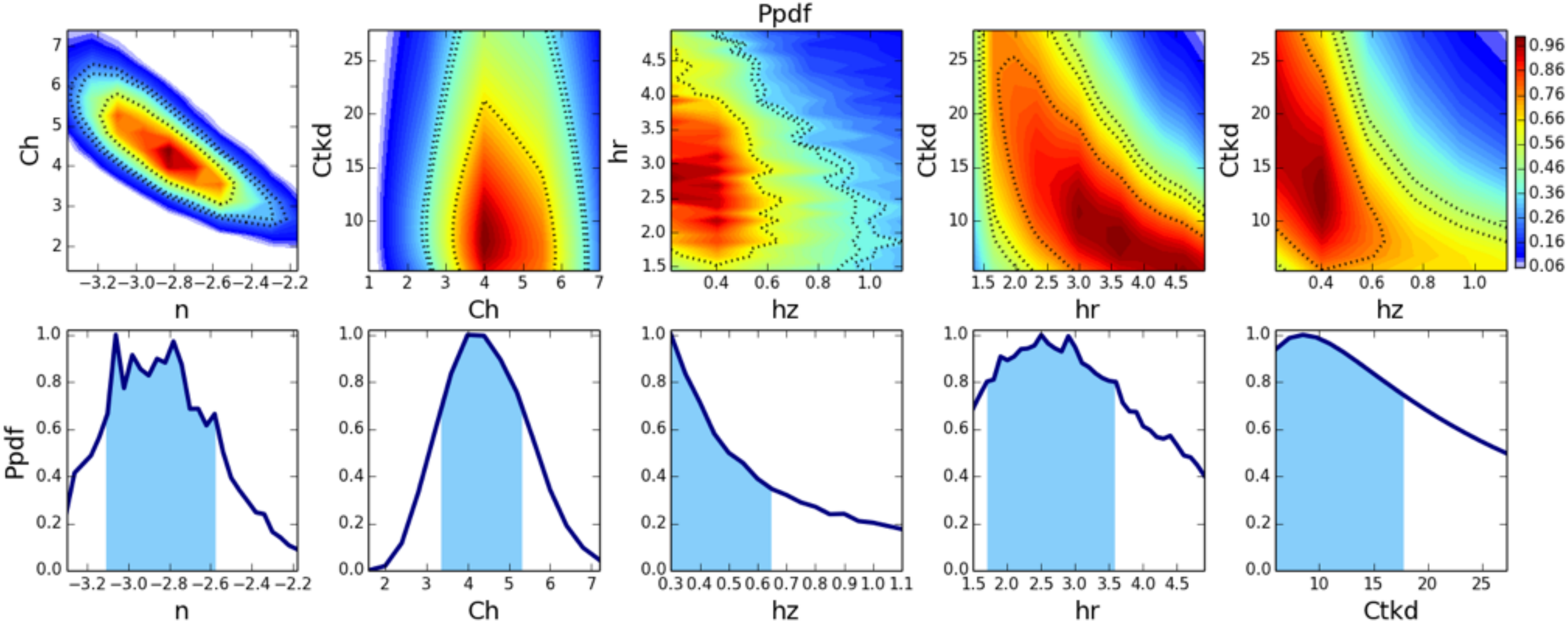
This is the computationally intensive part

$$N_{RR} = \iiint\limits_{V_S} \rho(\vec{r})dV = \iiint_{V_S} [\rho_{\mathrm{H}}(R,z) + \rho_{\mathrm{DG}}(R,z)]R\,dR\,dz\,d\varphi$$

Density Profiles: One sample at a time



The marginal posteriors taking only the **QUEST sample** are as follows:



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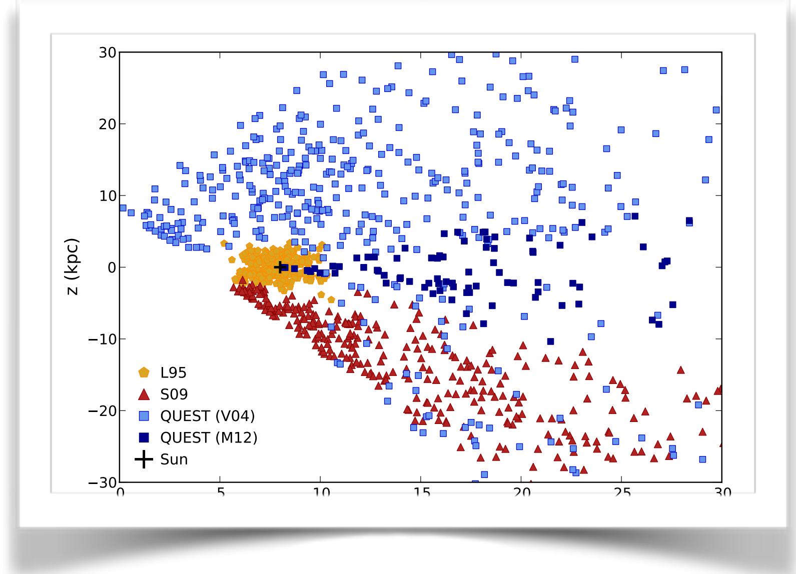
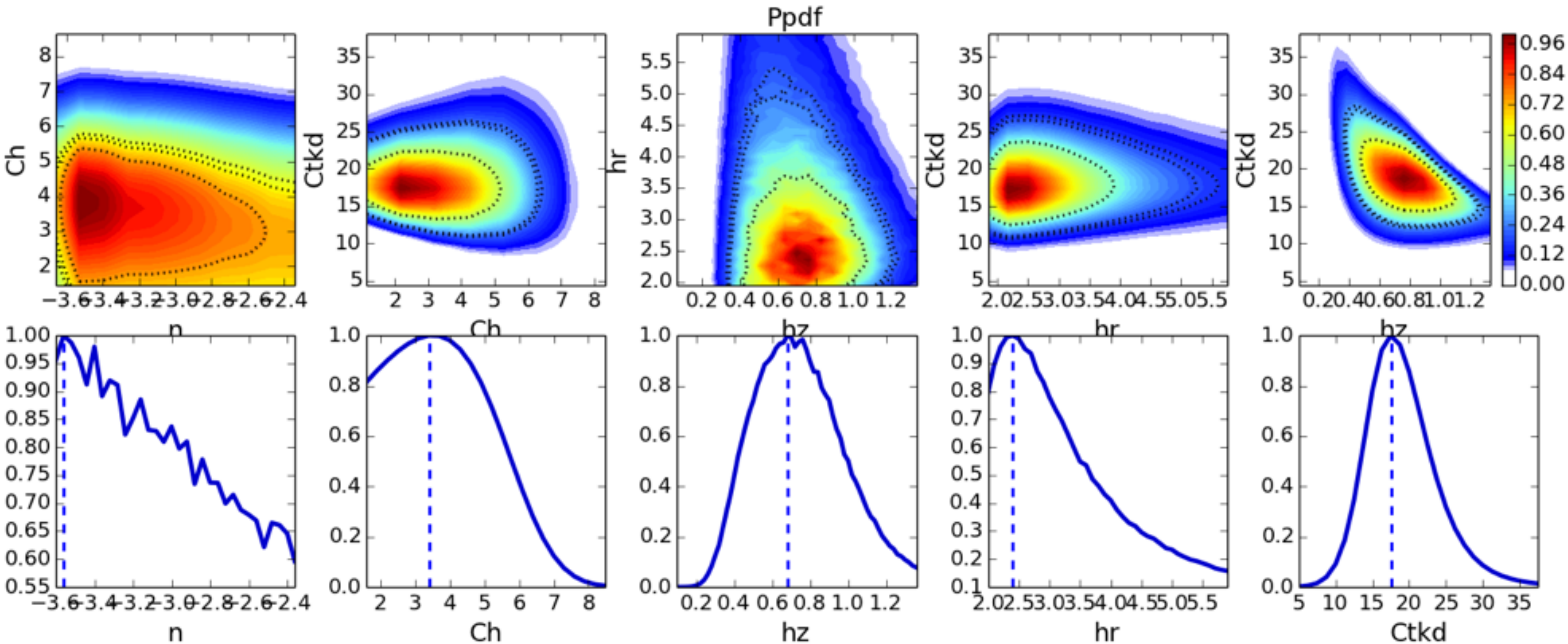
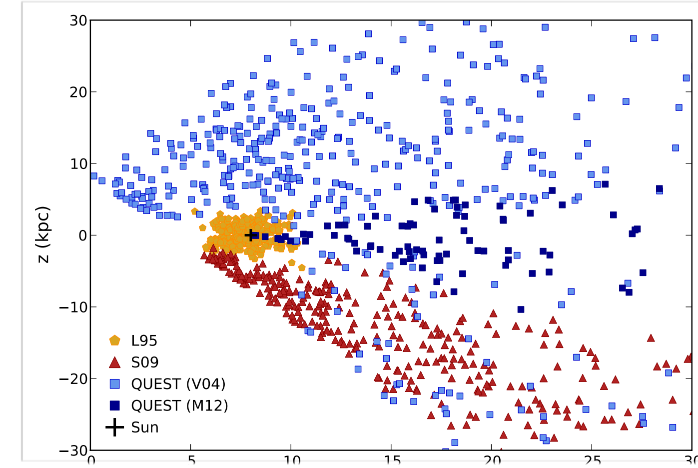


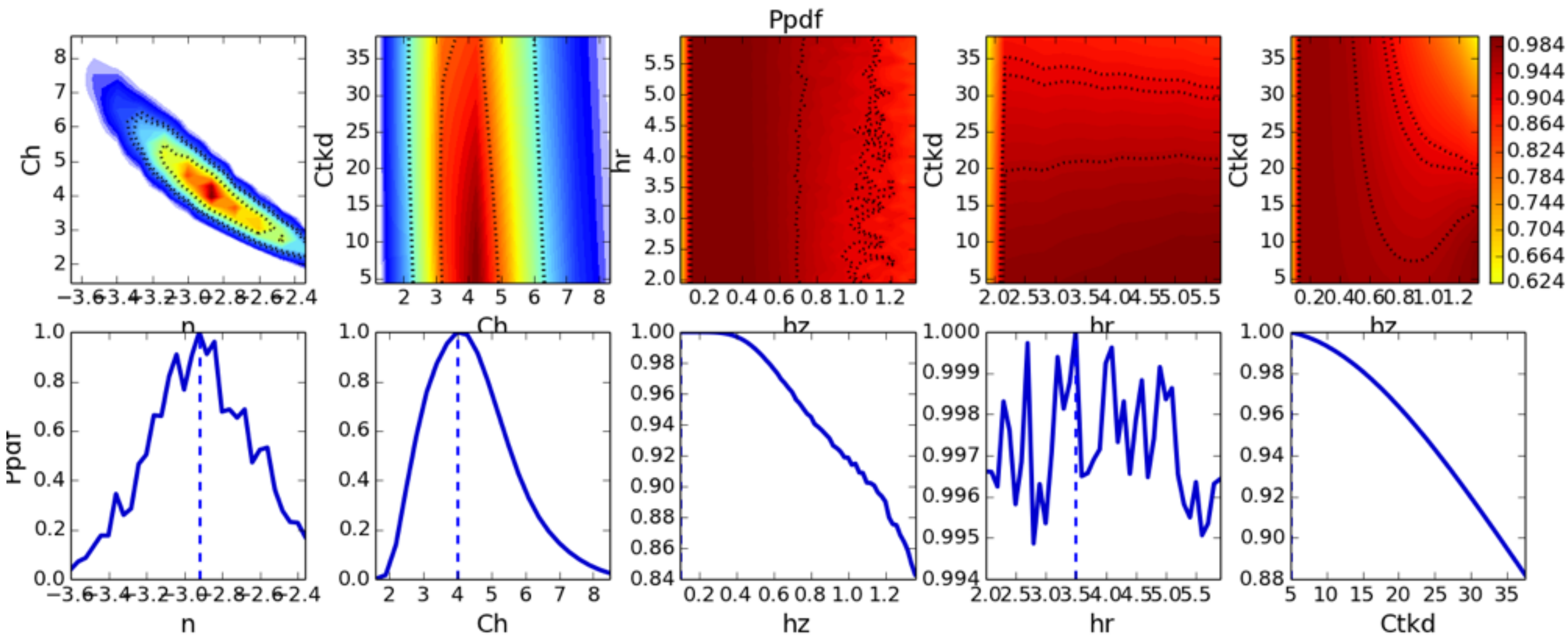
The marginal posteriors taking only the **Layden sample** are as follows:



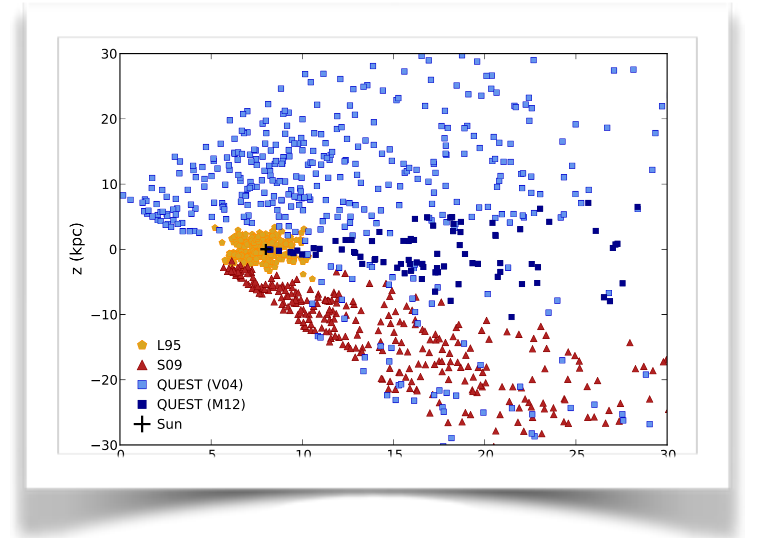
Density Profiles: One sample at a time



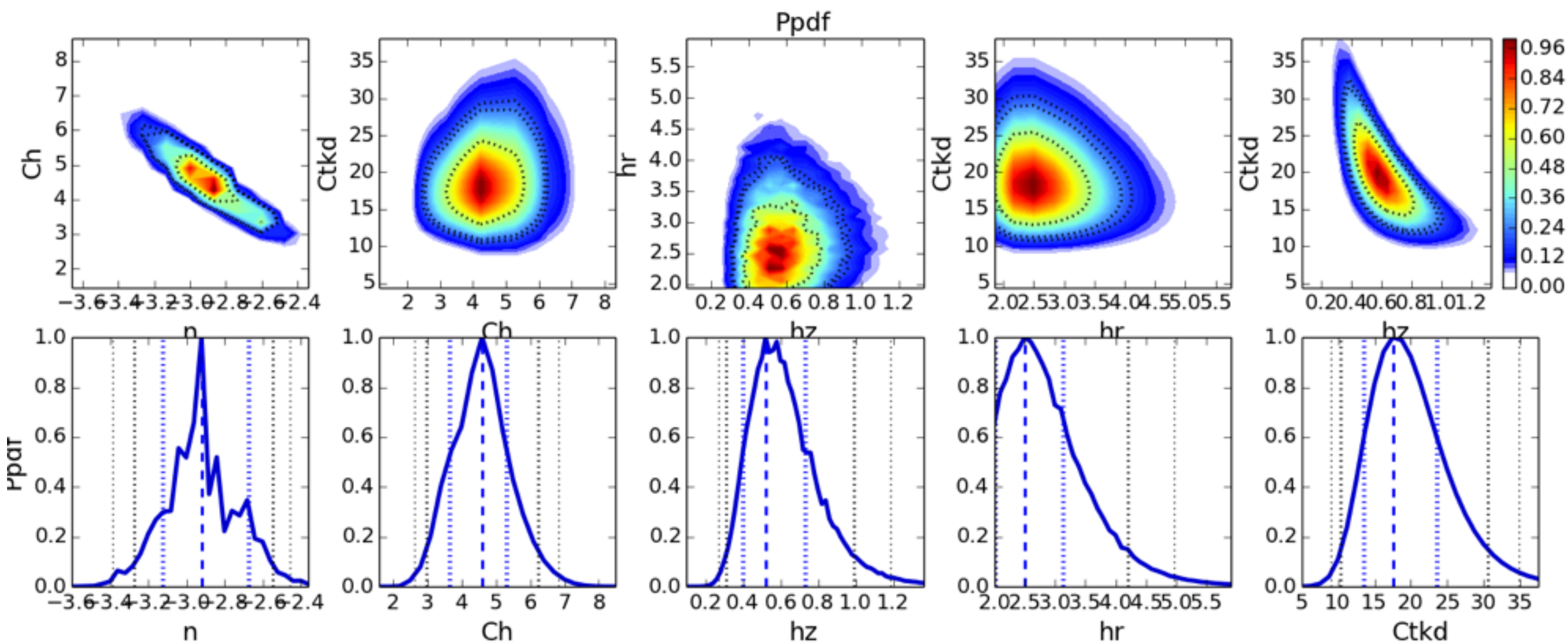
The marginal posteriors taking only the **Sesar sample** are as follows:



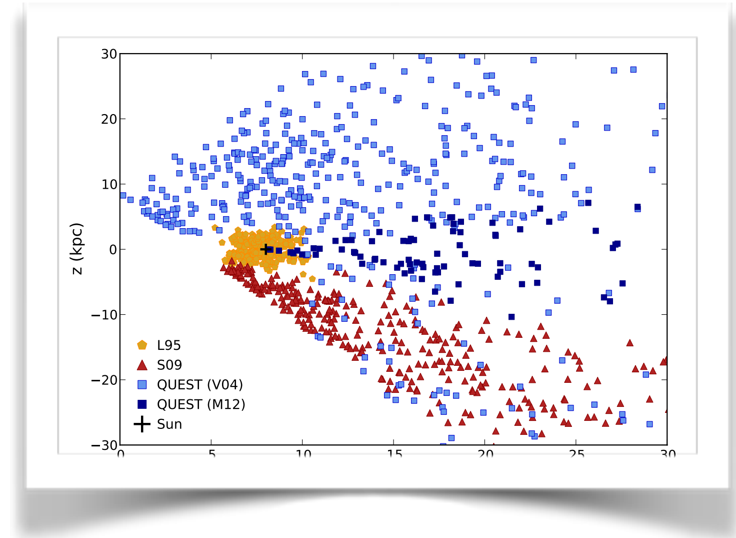
Density Profiles: Combined samples



Combining three different samples we find these marginal posteriors (remember its the product of the individual pdfs in 5-D space, and then marginalizing):



Density Profiles: Combined samples



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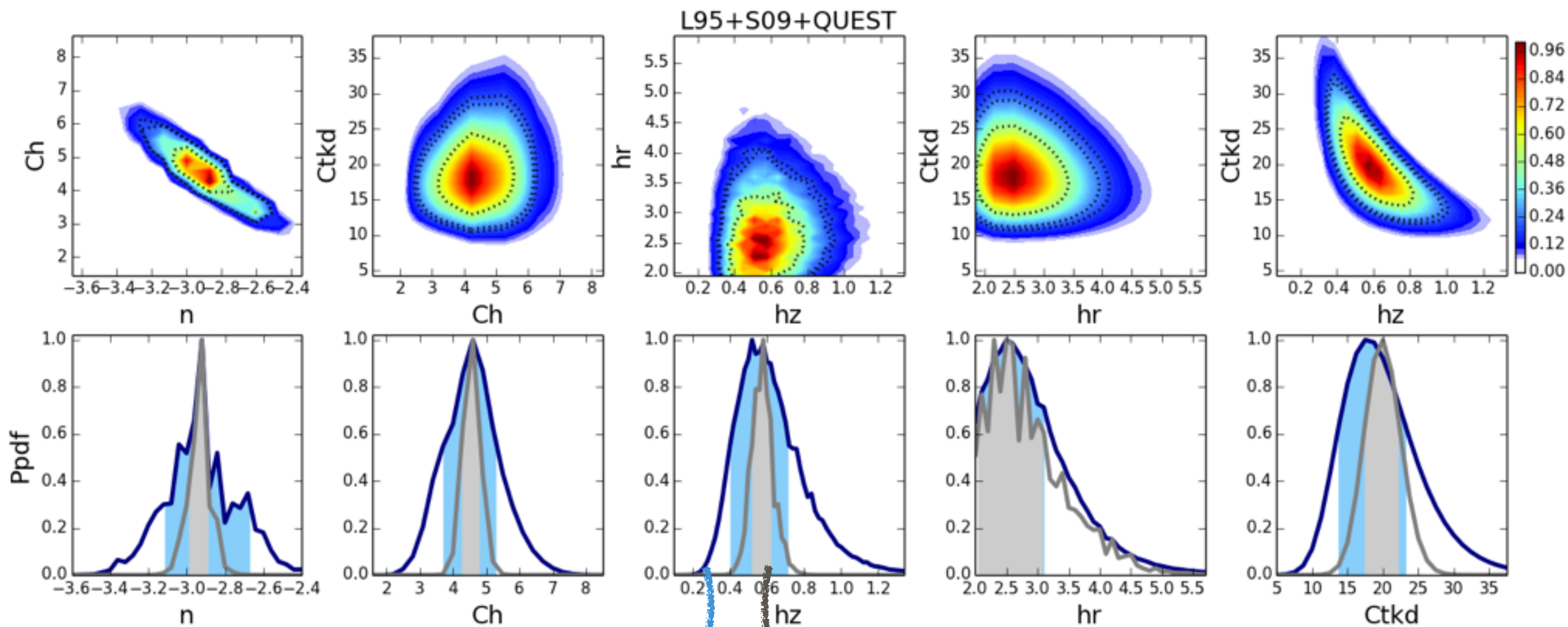

marginal distribution

joint distribution