An (brief) introduction to Bayesian Statistics through Astronomical Applications (Lecture 2)

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The Coin Example

The Importance (or not) of Priors

* The prior probability reflects our knowledge or ignorance on the problem
* In practice, for many applications the posterior is dominated by the likelihood
* If radically different priors are thought to be acceptable and the 'answer' depends strongly on the choice of the prior, it just means the data is not constraining enough! (see Jaynes 2003, D'Agostini 1998)

The Coin Example

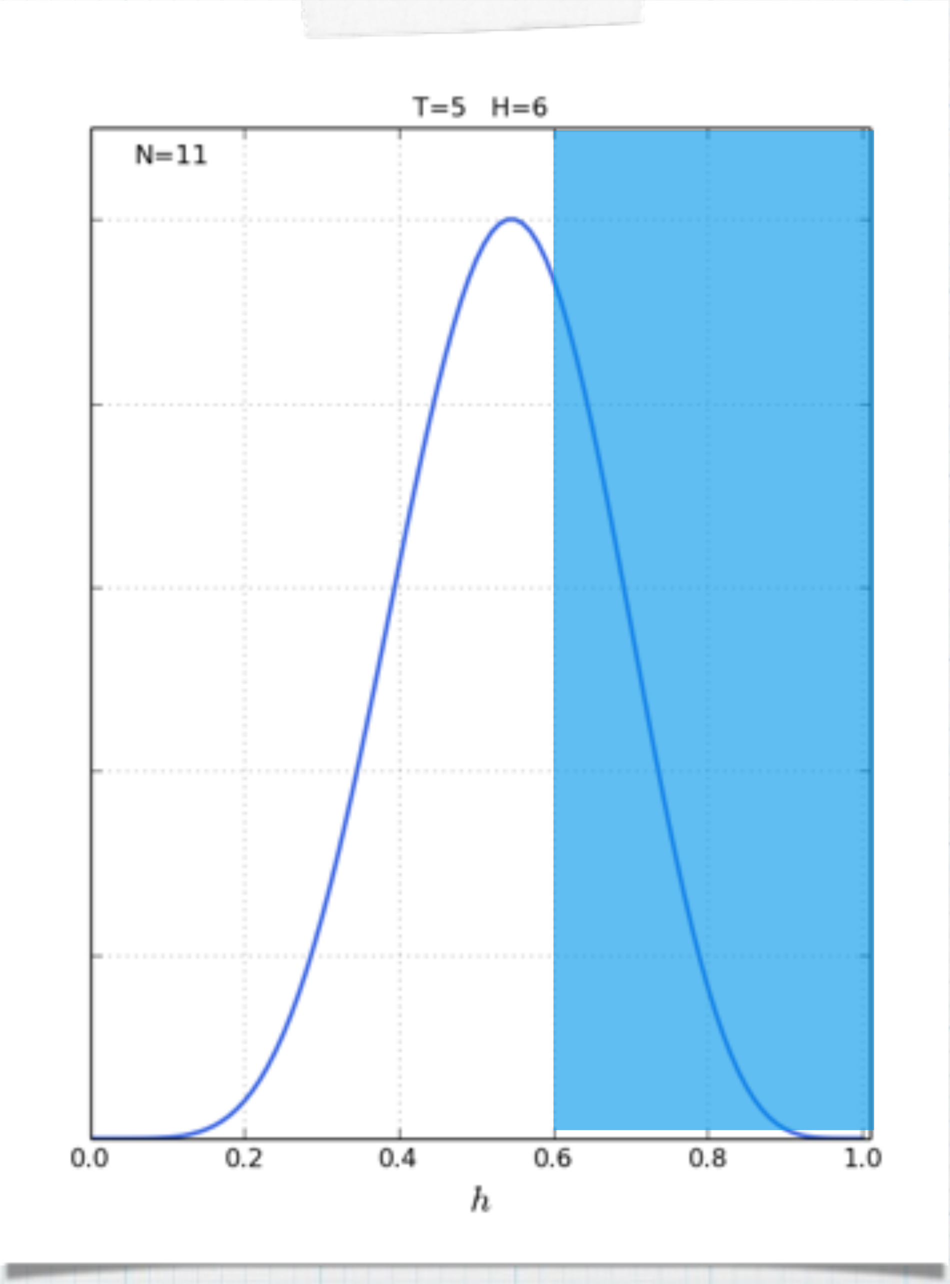
The full posterior IS the answer to our problem

$$P(h|N,N_h,I) = C h^{N_h}(1-h)^{N-N_h}\, P(h|I)$$

* anything else we may want can be calculated from it, e.g.
  * the most probable value of h
  * credible regions (Bayesian term for confidence intervals)
  * The probability that p>0.5
    * int P(p|N,n,I) dp
  * ... more on this tomorrow ...

Marginalization

* We want to compute the probability that the coin is biased towards heads
* Lets say by this, we mean h>0.6
* We have to integrate the posterior over the desired range of h

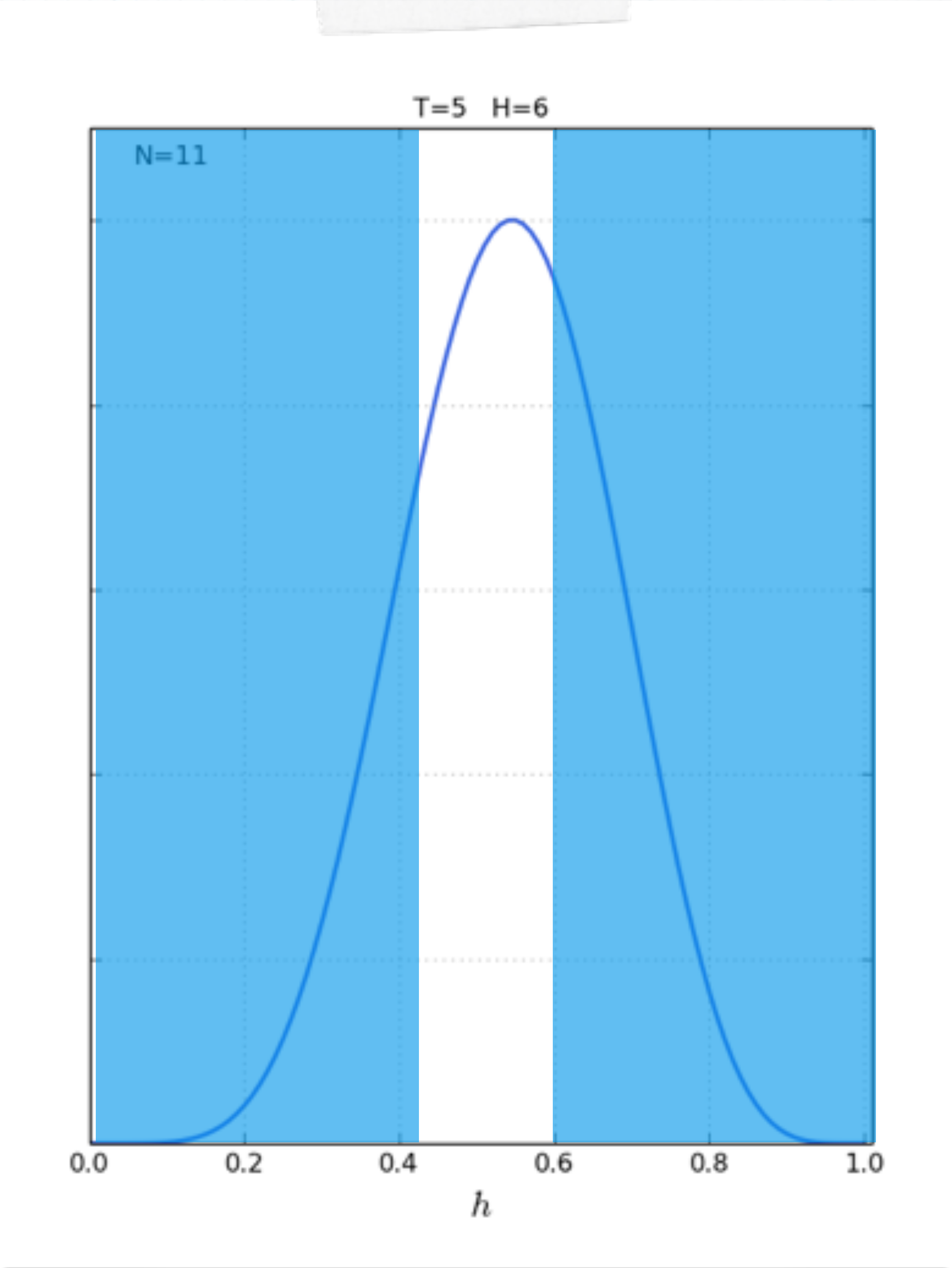


$$P(h > 0.6|N_h, N) = \int_{0.6}^{1} P(h|N_h, N)dh$$

Marginalization

* Now, lets compute the probability that the coin is fair
* Lets say by fair we mean h=0.5 ± x, where x could be e.g. x=0.05

$$P(|h-0.5| < x | N_h, N) = \int_{0.5-x}^{0.5+x} P(h|N_h, N)dh$$



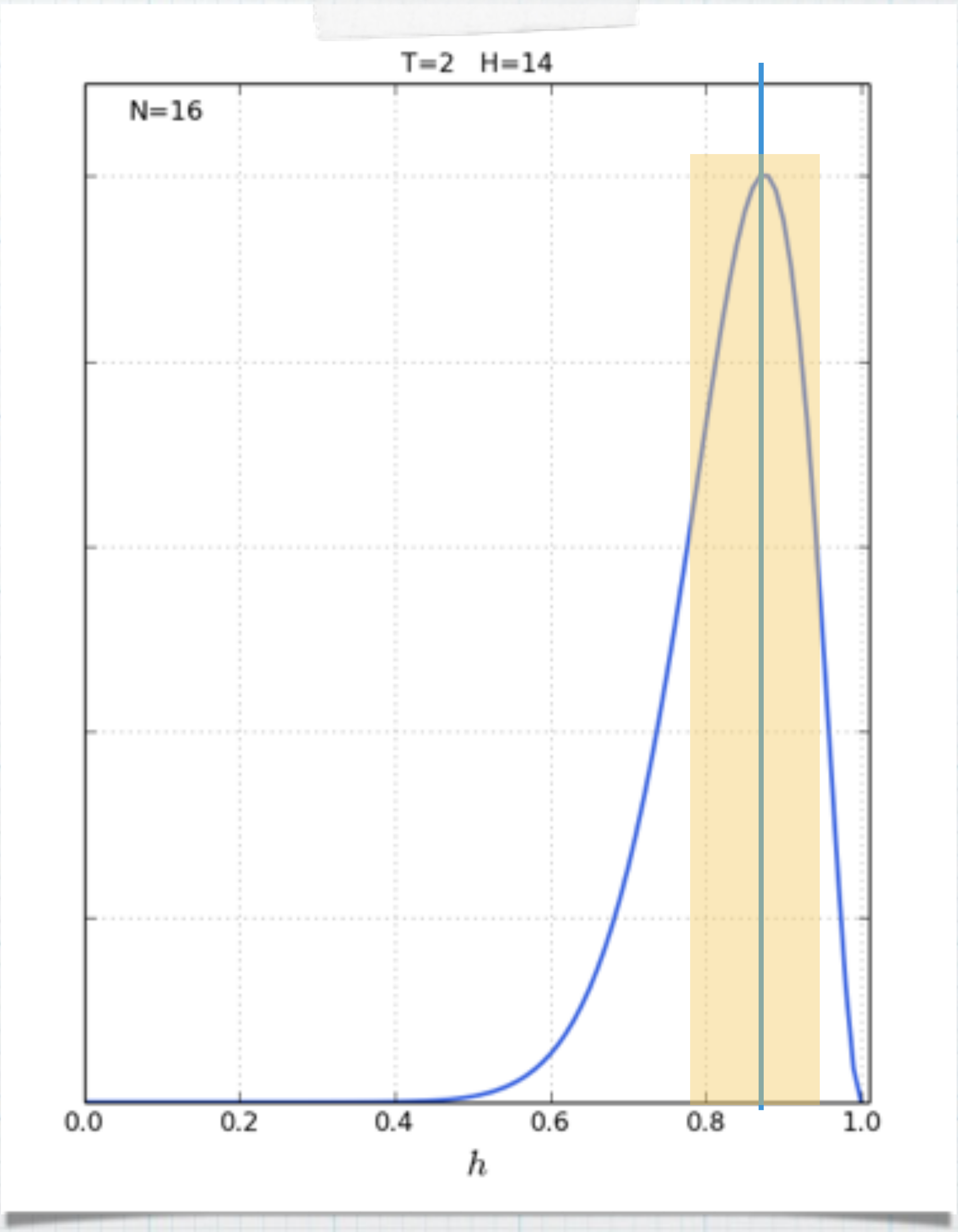
More Things to compute

* h_o = Most probable value of h, i.e. h where $P(h|N_h, N)$ is maximum

* Credible regions:

    * An X% credible region contains X% of the area of the posterior

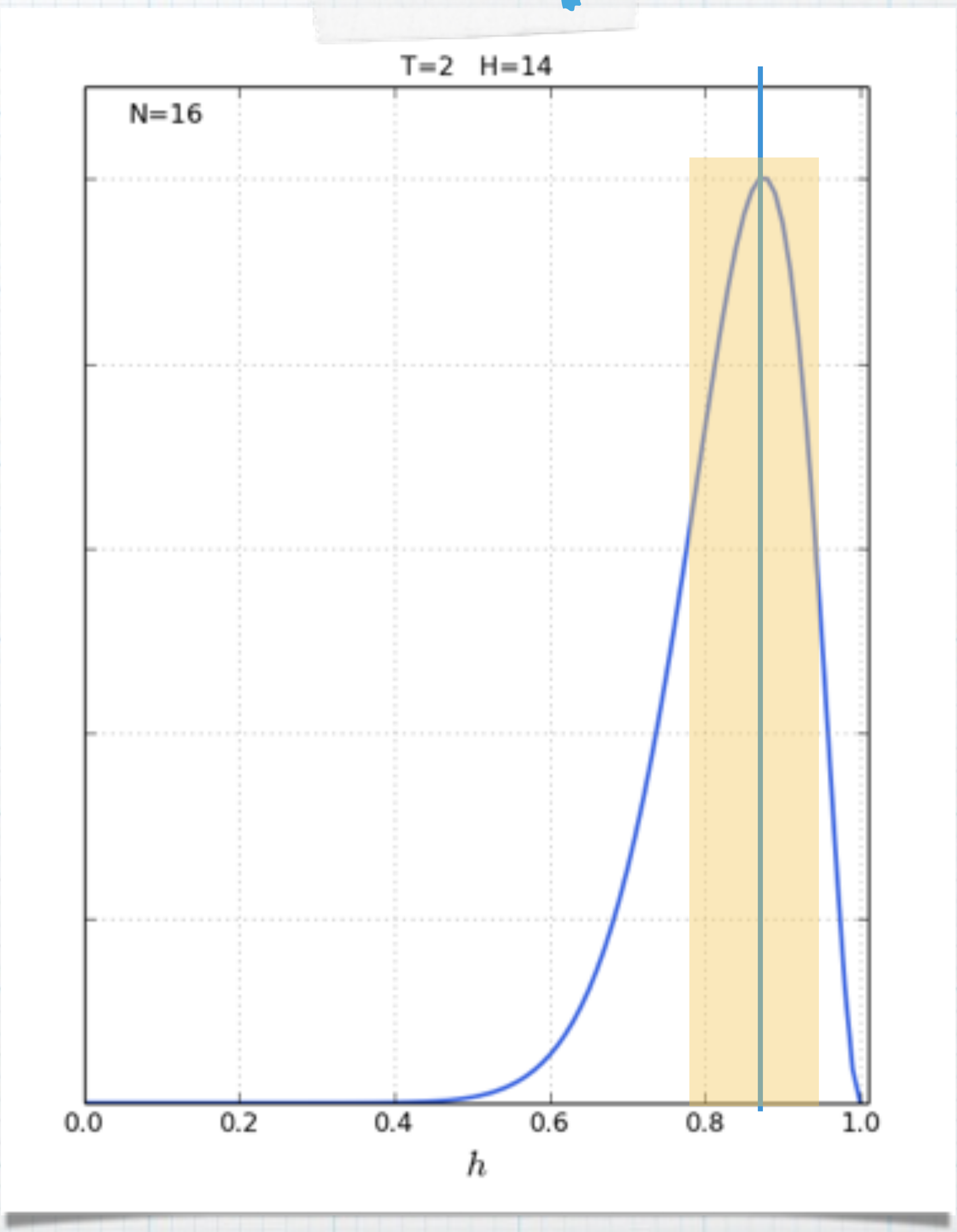
    * e.g. 1σ intervals are a 68% credible region for a gaussian posterior



More Things to compute

* Lets report as an error bar, the 68% credible region

* The most probable value is Nh/N, the usual answer, but there's a natural way of computing the error bars



* This is specially important for extremely low or extremely high values of h

The Coin Example in an astrophysical context

Disk Fractions

* The Coin is just one example of a Binomial problem
* This describes anything that can be expressed as a two-state problem, a 'success' occurring with probability p and 'failure' with probability (1-p), for example p could be:
    * The fraction of radio-loud quasars in a sample
    * The fraction of stars having disks
    * The fraction of early/late type galaxies
    * ....

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A simple Real Life example

* Lets take an example from Downes et al. (in prep)
* We have a sample with a total of N_{VLMS}=77 very low mass stars (VLMS) and N_{BD}=16 brown dwarfs (BD) from the 25 Ori cluster (~10 Myr)
* Out of these 6 VLMS and 4 BDs have disks (infrared excesses observed)
* The key scientific question is

Do VLMS and BDs have the same disk fraction?

VLMS and BD disk Fractions

* The data are conditionally independent, thus from the product rule we have

$$P(f_{disk}^{VLMS}, f_{disk}^{BD}|data) = P(f_{disk}^{VLMS}|data)P(f_{disk}^{BD}|data)$$

i.e. the multiplication of the disk fraction posteriors for VLMS and BDs. Each of these is given by the Binomial distribution as in the coin example

$$P(f_{disk}|data) = f_{disk}^{N_{disk}}(1 - f_{disk})^{N-N_{disk}}$$

in this case we have assumed a uniform prior

$$P(f_{disk}^{VLMS}, f_{disk}^{BD}|data) = 1$$

* In this case our posterior is a two-dimensional function that depends upon $f^{VL}{}_{disk}$ and $f^{BD}{}_{disk}$

$$P(f_{disk}^{VLMS}, f_{disk}^{BD}|data)$$

again, remember the posterior is the 'Holy Grail'

Marginalization

* If we want the posterior dependent upon just one of the parameters we need to marginalise over the other one

* For f^{BD}_{disk} we get

$$P(f^{BD}_{disk}|data) = \int P(f^{VLMS}_{disk}, f^{BD}_{disk}|data) df^{VLMS}_{disk}$$

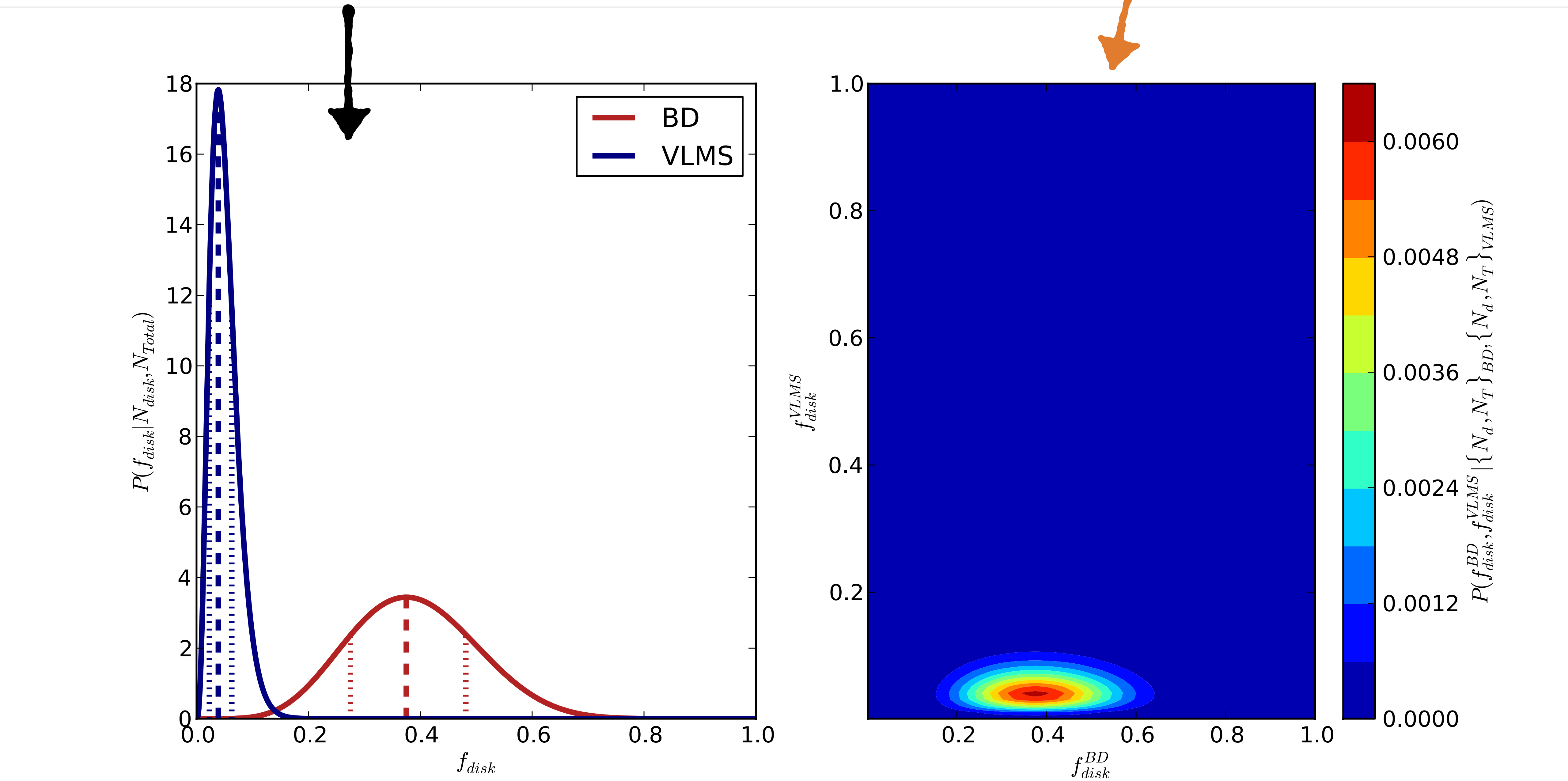
* similarly for f^{VL}_{disk}

$$P(f^{VLMS}_{disk}|data) = \int P(f^{VLMS}_{disk}, f^{BD}_{disk}|data) df^{BD}_{disk}$$

VLMS and BD disk Fractions

Marginal posteriors (integrated upon $f^{VL}{}_{disk}$ or $f^{BD}{}_{disk}$)

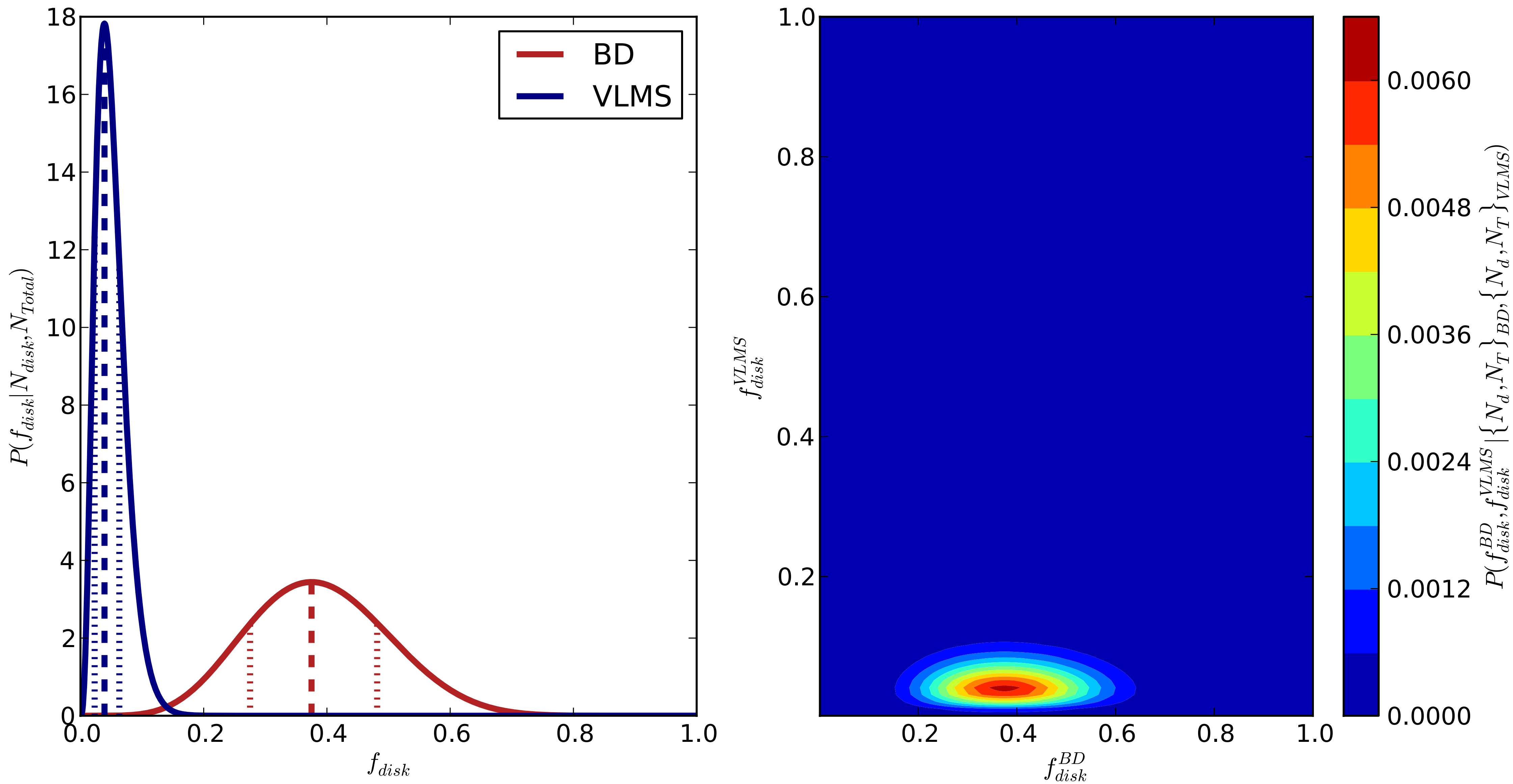
$P(f^{VLMS}_{disk}, f^{BD}_{disk}|data)$



$$N_{VLMS} = 77, N^{disk}_{VLMS} = 4; N^{disk}_{BD} = 16, N^{disk}_{BD} = 6$$

VLMS and BD disk Fractions

How do we answer the initial question: do VLMS and BDs have the same disk fraction?

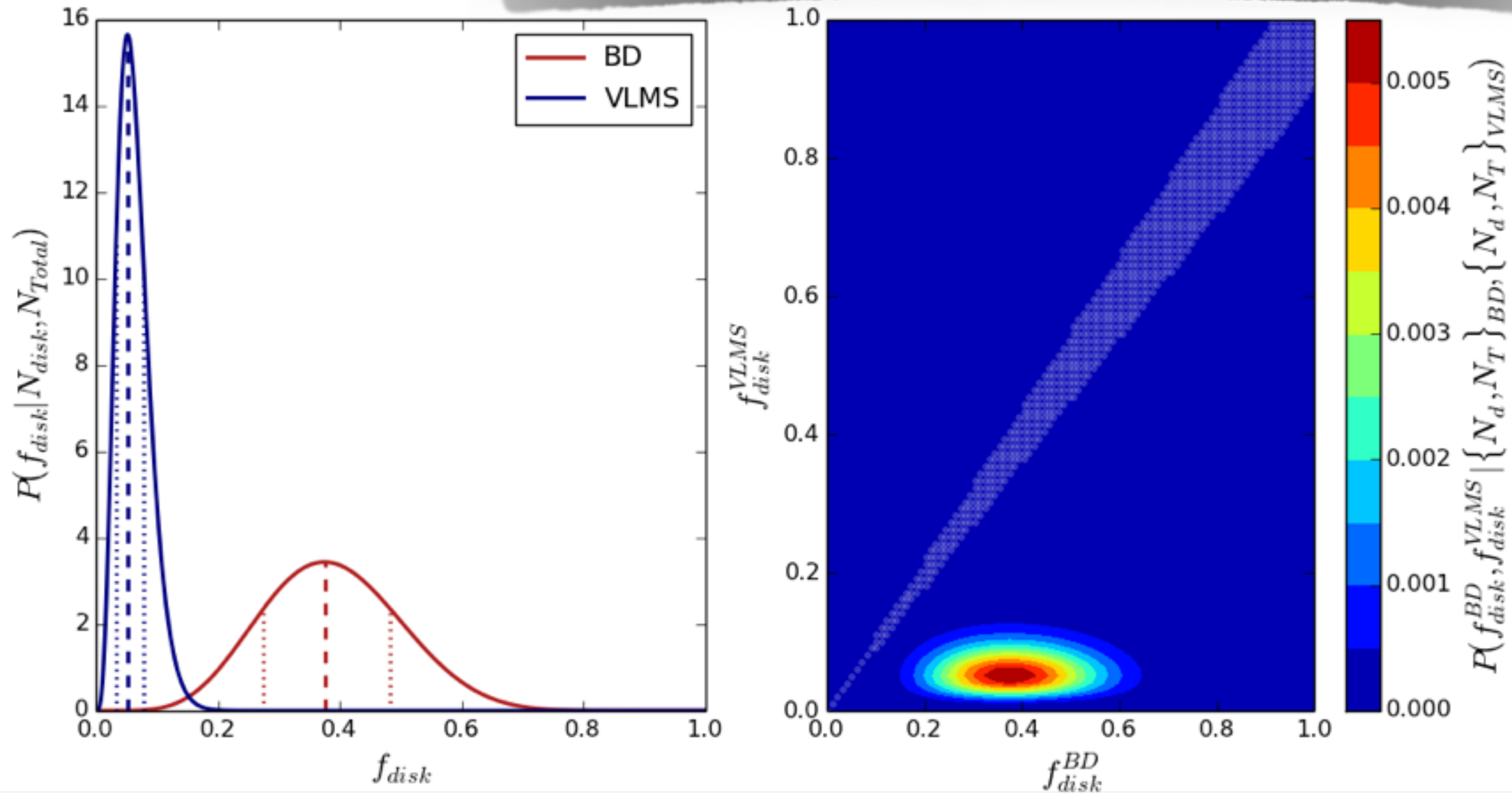


VLMS and BD disk Fractions

Integrate the posterior in the region where

$f^{VL}{}_{disk} \neq f^{BD}{}_{disk}$

this gives the probability that the two disk fractions are different!

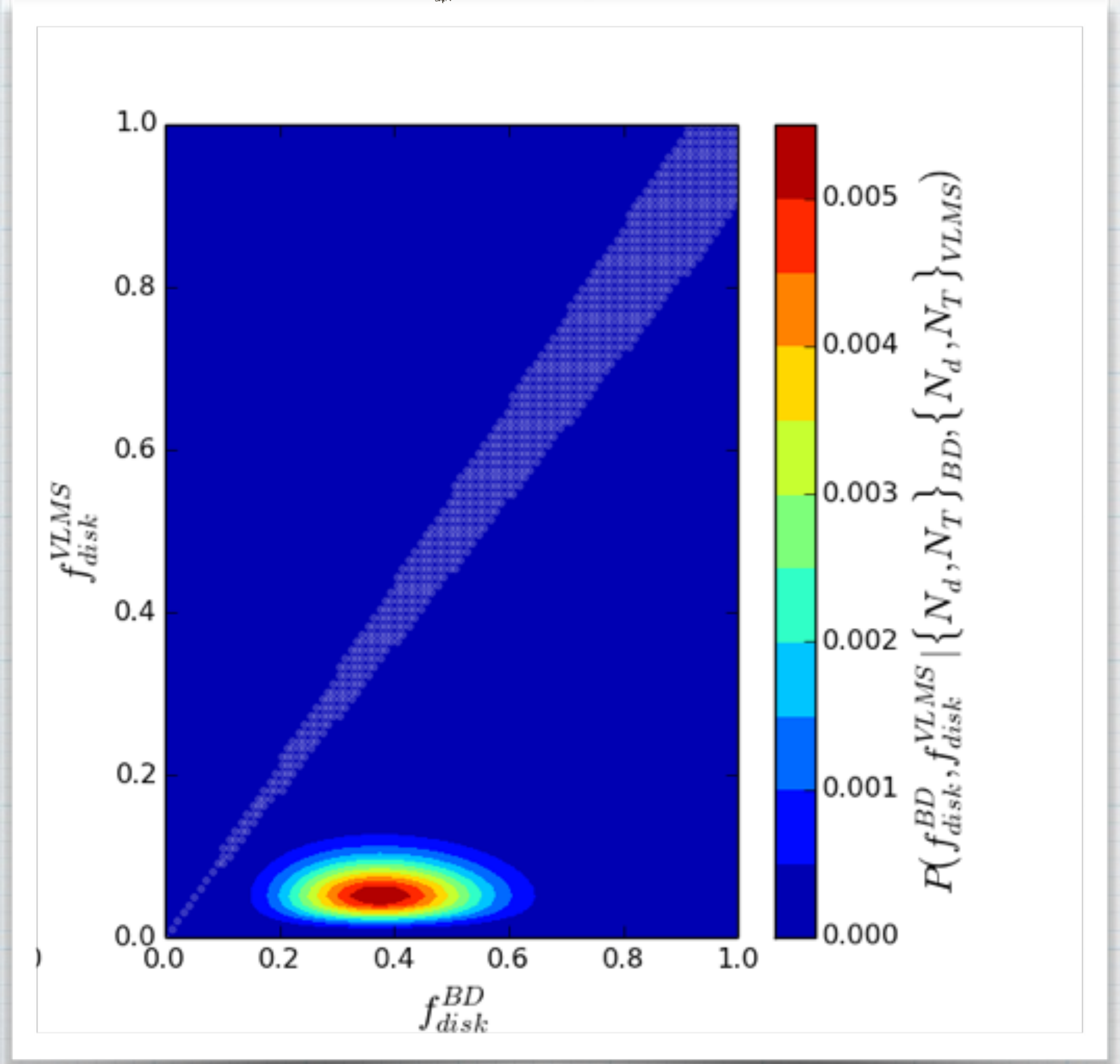


VLMS and BD disk Fractions

Better to integrate the posterior in the region where the disk fractions differ by 10% e.g.

In this example we get that the probability that the two disk fractions are different is

$P_{10\%}$=99.95%

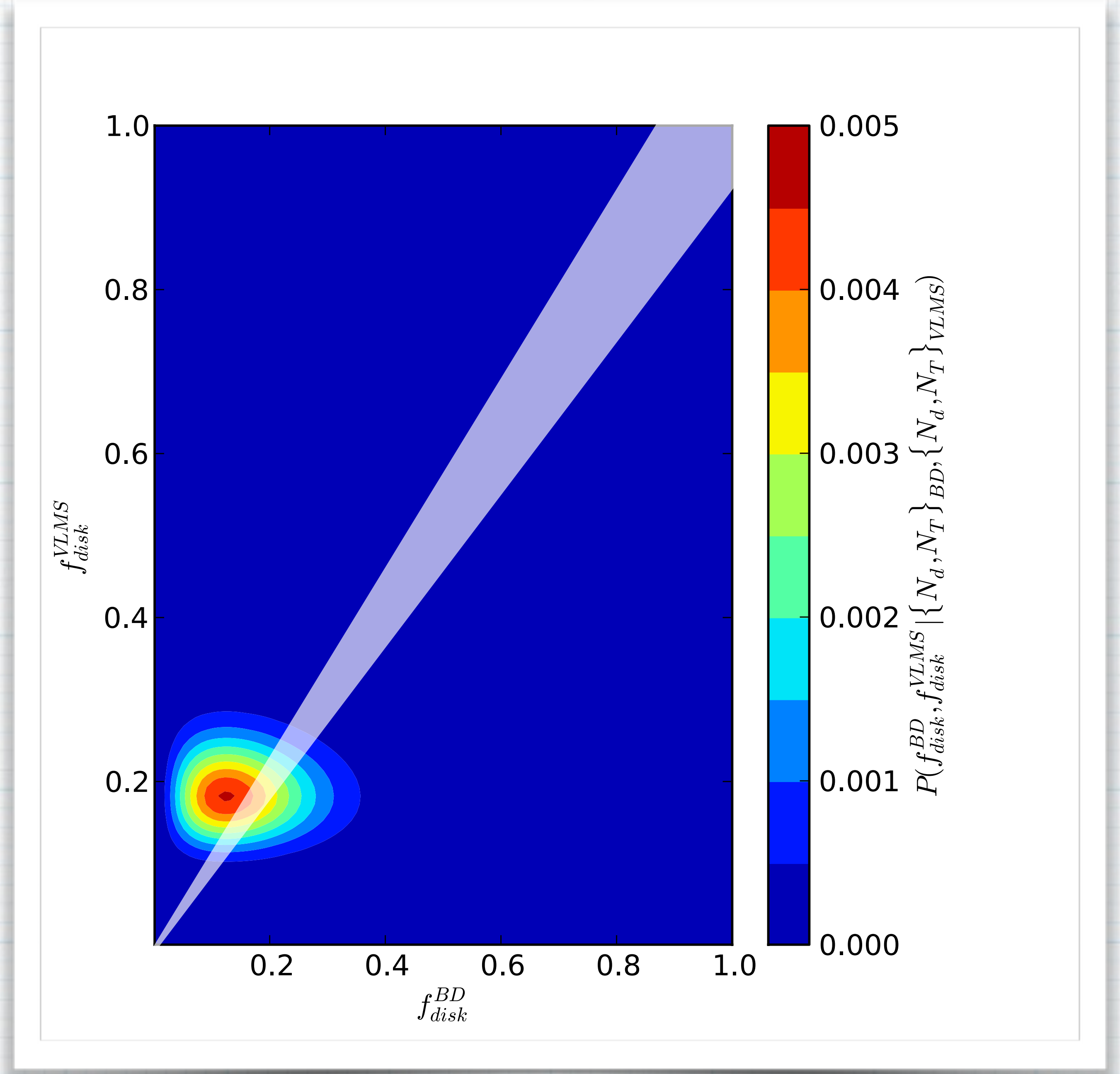


VLMS and BD disk Fractions

This is even more useful in more uncertain cases, e.g. for transitional disks fractions in this same cluster

Here we get that the probability that the two disk fractions are different is

$P_{10\%}$=86.5%



Very short conclusions: Some Motivations for

* Bayesian Statistics provides a clear framework for Inference - Hypothesis testing

* Probability is related to the state of uncertainty in a physical variable/model/theory, not only on the outcome of repeated experiments

* Our prior knowledge, assumptions, prejudices or lack thereof, must be stated explicitly in our model

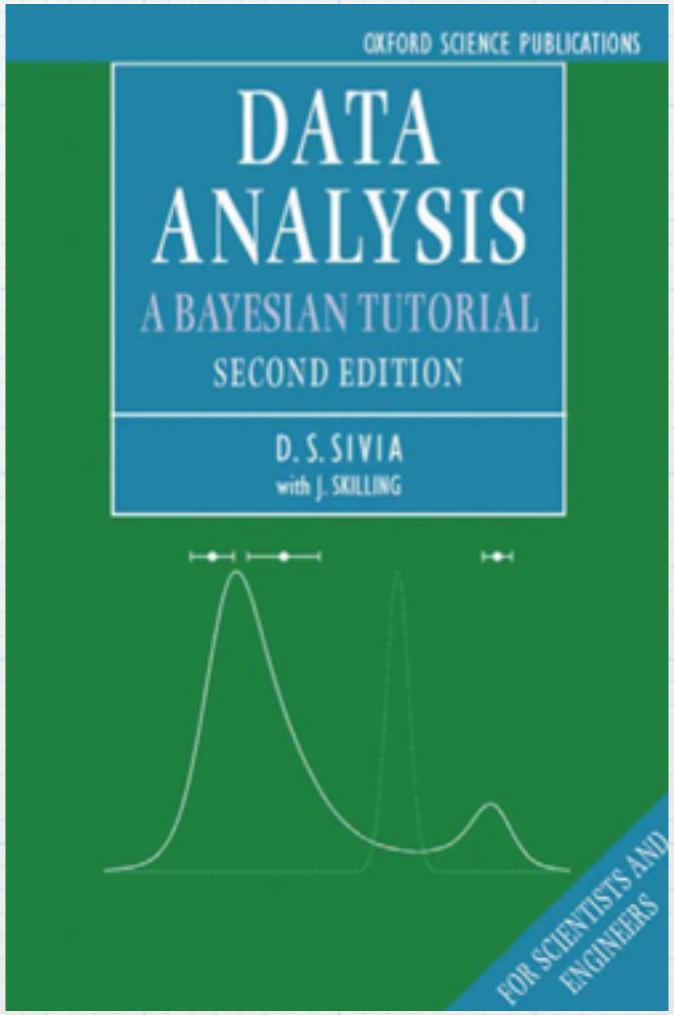
* Propagation of uncertainties follows naturally

* many more...

Very short bibliography

* Highly recommended introductory bibliography:
    * Sivia & Skilling book
    * Giulio D'Agostini's notes available at Tom Loredo's BIPS web page:

http://www.astro.cornell.edu/staff/loredo/bayes/



"Though there be no such thing as Chance in the world; our ignorance of the real cause of any event has the same influence on the understanding"

-David Hume (1748)