

# OpenGeoProver Output for conjecture “Chou 011”

Wu’s method used

February 12, 2012

## 1 Validation of Construction Protocol

### Construction steps:

- General Conic Section  $c$
- Random point  $A_0$  from general conic  $c$
- Random point  $A_1$  from general conic  $c$
- Random point  $A_2$  from general conic  $c$
- Random point  $A_3$  from general conic  $c$
- Random point  $A_4$  from general conic  $c$
- Random point  $A_5$  from general conic  $c$
- Line  $A_0A_1$  through two points  $A_0$  and  $A_1$
- Line  $A_2A_3$  through two points  $A_2$  and  $A_3$
- Intersection point  $A$  of point sets  $A_0A_1$  and  $A_2A_3$
- Line  $A_4A_5$  through two points  $A_4$  and  $A_5$
- Intersection point  $B$  of point sets  $A_0A_1$  and  $A_4A_5$
- Line  $A_0A_2$  through two points  $A_0$  and  $A_2$
- Line  $A_1A_4$  through two points  $A_1$  and  $A_4$
- Intersection point  $C$  of point sets  $A_0A_2$  and  $A_1A_4$
- Line  $A_0A_3$  through two points  $A_0$  and  $A_3$
- Line  $A_1A_5$  through two points  $A_1$  and  $A_5$
- Intersection point  $D$  of point sets  $A_0A_3$  and  $A_1A_5$
- Line  $A_0A_4$  through two points  $A_0$  and  $A_4$
- Line  $A_1A_2$  through two points  $A_1$  and  $A_2$

- Intersection point E of point sets A0A4 and A1A2
- Line A0A5 through two points A0 and A5
- Line A1A3 through two points A1 and A3
- Intersection point F of point sets A0A5 and A1A3
- General Conic Section c1 which contains points A, B, C, D and E

**Theorem statement:**

- Point F lies on set of points c1

**Validation result:** Construction protocol is valid.

## 2 Transformation of Construction Protocol to algebraic form

### Transformation of Construction steps

#### 2.1 Transformation of general conic section c:

List of parametric points

- Point Ac has been assigned following coordinates:  $(u_1, 0)$
- Point Bc has been assigned following coordinates:  $(u_2, 0)$
- Point Cc has been assigned following coordinates:  $(u_3, 0)$
- Point Dc has been assigned following coordinates:  $(u_4, 0)$
- Point Ec has been assigned following coordinates:  $(u_5, 0)$
- Condition for point  $X(x_1, x_2)$  to belong to this conic section is following equation:

$$p = u_3x_2^2 + u_2x_2x_1 + u_5x_2 + u_1x_1^2 + u_4x_1$$

#### 2.2 Transformation of point A0:

- Point A0 has been assigned following coordinates:  $(0, 0)$

#### 2.3 Transformation of point A1:

- Point A1 has been assigned following coordinates:  $(0, x_1)$
- Polynomial that point A1 has to satisfy is:

$$p = u_3x_1^2 + u_5x_1$$

- Processing of polynomial

$$p = u_3x_1^2 + u_5x_1$$

**Info:** Polynomial

$$p = u_3x_1^2 + u_5x_1$$

added to system of polynomials that represents the constructions

- New polynomial added to system of hypotheses

## 2.4 Transformation of point A2:

- Point A2 has been assigned following coordinates:  $(u_6, x_2)$
- Polynomial that point A2 has to satisfy is:

$$p = u_3x_2^2 + (u_6u_2 + u_5)x_2 + (u_6^2u_1 + u_6u_4)$$

- Processing of polynomial

$$p = u_3x_2^2 + (u_6u_2 + u_5)x_2 + (u_6^2u_1 + u_6u_4)$$

**Info:** Polynomial

$$p = u_3x_2^2 + (u_6u_2 + u_5)x_2 + (u_6^2u_1 + u_6u_4)$$

added to system of polynomials that represents the constructions

- New polynomial added to system of hypotheses

## 2.5 Transformation of point A3:

- Point A3 has been assigned following coordinates:  $(u_7, x_3)$
- Polynomial that point A3 has to satisfy is:

$$p = u_3x_3^2 + (u_7u_2 + u_5)x_3 + (u_7^2u_1 + u_7u_4)$$

- Processing of polynomial

$$p = u_3x_3^2 + (u_7u_2 + u_5)x_3 + (u_7^2u_1 + u_7u_4)$$

**Info:** Polynomial

$$p = u_3x_3^2 + (u_7u_2 + u_5)x_3 + (u_7^2u_1 + u_7u_4)$$

added to system of polynomials that represents the constructions

- New polynomial added to system of hypotheses

## 2.6 Transformation of point A4:

- Point A4 has been assigned following coordinates:  $(u_8, x_4)$
- Polynomial that point A4 has to satisfy is:

$$p = u_3x_4^2 + (u_8u_2 + u_5)x_4 + (u_8^2u_1 + u_8u_4)$$

- Processing of polynomial

$$p = u_3x_4^2 + (u_8u_2 + u_5)x_4 + (u_8^2u_1 + u_8u_4)$$

**Info:** Polynomial

$$p = u_3x_4^2 + (u_8u_2 + u_5)x_4 + (u_8^2u_1 + u_8u_4)$$

added to system of polynomials that represents the constructions

- New polynomial added to system of hypotheses

## 2.7 Transformation of point A5:

- Point A5 has been assigned following coordinates:  $(u_9, x_5)$
- Polynomial that point A5 has to satisfy is:

$$p = u_3x_5^2 + (u_9u_2 + u_5)x_5 + (u_9^2u_1 + u_9u_4)$$

- Processing of polynomial

$$p = u_3x_5^2 + (u_9u_2 + u_5)x_5 + (u_9^2u_1 + u_9u_4)$$

**Info:** Polynomial

$$p = u_3x_5^2 + (u_9u_2 + u_5)x_5 + (u_9^2u_1 + u_9u_4)$$

added to system of polynomials that represents the constructions

- New polynomial added to system of hypotheses

## 2.8 Transformation of point A:

- Point A has been assigned following coordinates:  $(x_6, x_7)$
- Polynomial that point A has to satisfy is:

$$p = x_6x_1$$

- Processing of polynomial

$$p = x_6x_1$$

**Info:** Polynomial

$$p = x_6x_1$$

added to system of polynomials that represents the constructions

- New polynomial added to system of hypotheses
- Polynomial that point A has to satisfy is:

$$p = (u_7 - u_6)x_7 - x_6x_3 + x_6x_2 + u_6x_3 - u_7x_2$$

- Processing of polynomial

$$p = (u_7 - u_6)x_7 - x_6x_3 + x_6x_2 + u_6x_3 - u_7x_2$$

**Info:** Polynomial

$$p = (u_7 - u_6)x_7 - x_6x_3 + x_6x_2 + u_6x_3 - u_7x_2$$

added to system of polynomials that represents the constructions

- New polynomial added to system of hypotheses

## 2.9 Transformation of point B:

- Point B has been assigned following coordinates:  $(x_8, x_9)$
- Polynomial that point B has to satisfy is:

$$p = x_8x_1$$

- Processing of polynomial

$$p = x_8x_1$$

**Info:** Polynomial

$$p = x_8x_1$$

added to system of polynomials that represents the constructions

- New polynomial added to system of hypotheses
- Polynomial that point B has to satisfy is:

$$p = (u_9 - u_8)x_9 - x_8x_5 + x_8x_4 + u_8x_5 - u_9x_4$$

- Processing of polynomial

$$p = (u_9 - u_8)x_9 - x_8x_5 + x_8x_4 + u_8x_5 - u_9x_4$$

**Info:** Polynomial

$$p = (u_9 - u_8)x_9 - x_8x_5 + x_8x_4 + u_8x_5 - u_9x_4$$

added to system of polynomials that represents the constructions

- New polynomial added to system of hypotheses

## 2.10 Transformation of point C:

- Point C has been assigned following coordinates:  $(x_{10}, x_{11})$
- Polynomial that point C has to satisfy is:

$$p = u_6x_{11} - x_{10}x_2$$

- Processing of polynomial

$$p = u_6x_{11} - x_{10}x_2$$

**Info:** Polynomial

$$p = u_6x_{11} - x_{10}x_2$$

added to system of polynomials that represents the constructions

- New polynomial added to system of hypotheses
- Polynomial that point C has to satisfy is:

$$p = u_8x_{11} - x_{10}x_4 + x_{10}x_1 - u_8x_1$$

- Processing of polynomial

$$p = u_8x_{11} - x_{10}x_4 + x_{10}x_1 - u_8x_1$$

**Info:** Polynomial

$$p = u_8x_{11} - x_{10}x_4 + x_{10}x_1 - u_8x_1$$

added to system of polynomials that represents the constructions

- New polynomial added to system of hypotheses

## 2.11 Transformation of point D:

- Point D has been assigned following coordinates:  $(x_{12}, x_{13})$
- Polynomial that point D has to satisfy is:

$$p = u_7x_{13} - x_{12}x_3$$

- Processing of polynomial

$$p = u_7x_{13} - x_{12}x_3$$

**Info:** Polynomial

$$p = u_7x_{13} - x_{12}x_3$$

added to system of polynomials that represents the constructions

- New polynomial added to system of hypotheses

- Polynomial that point D has to satisfy is:

$$p = u_9x_{13} - x_{12}x_5 + x_{12}x_1 - u_9x_1$$

- Processing of polynomial

$$p = u_9x_{13} - x_{12}x_5 + x_{12}x_1 - u_9x_1$$

**Info:** Polynomial

$$p = u_9x_{13} - x_{12}x_5 + x_{12}x_1 - u_9x_1$$

added to system of polynomials that represents the constructions

- New polynomial added to system of hypotheses

## 2.12 Transformation of point E:

- Point E has been assigned following coordinates:  $(x_{14}, x_{15})$
- Polynomial that point E has to satisfy is:

$$p = u_8x_{15} - x_{14}x_4$$

- Processing of polynomial

$$p = u_8x_{15} - x_{14}x_4$$

**Info:** Polynomial

$$p = u_8x_{15} - x_{14}x_4$$

added to system of polynomials that represents the constructions

- New polynomial added to system of hypotheses
- Polynomial that point E has to satisfy is:

$$p = u_6x_{15} - x_{14}x_2 + x_{14}x_1 - u_6x_1$$

- Processing of polynomial

$$p = u_6x_{15} - x_{14}x_2 + x_{14}x_1 - u_6x_1$$

**Info:** Polynomial

$$p = u_6x_{15} - x_{14}x_2 + x_{14}x_1 - u_6x_1$$

added to system of polynomials that represents the constructions

- New polynomial added to system of hypotheses

### 2.13 Transformation of point F:

- Point F has been assigned following coordinates:  $(x_{16}, x_{17})$
- Polynomial that point F has to satisfy is:

$$p = u_9x_{17} - x_{16}x_5$$

- Processing of polynomial

$$p = u_9x_{17} - x_{16}x_5$$

**Info:** Polynomial

$$p = u_9x_{17} - x_{16}x_5$$

added to system of polynomials that represents the constructions

- New polynomial added to system of hypotheses
- Polynomial that point F has to satisfy is:

$$p = u_7x_{17} - x_{16}x_3 + x_{16}x_1 - u_7x_1$$

- Processing of polynomial

$$p = u_7x_{17} - x_{16}x_3 + x_{16}x_1 - u_7x_1$$

**Info:** Polynomial

$$p = u_7x_{17} - x_{16}x_3 + x_{16}x_1 - u_7x_1$$

added to system of polynomials that represents the constructions

- New polynomial added to system of hypotheses

### 2.14 Transformation of general conic section c1:

List of parametric points

- Point Bc1 has been assigned following coordinates:  $(x_{18}, 0)$
- Point Cc1 has been assigned following coordinates:  $(x_{19}, 0)$
- Point Dc1 has been assigned following coordinates:  $(x_{20}, 0)$
- Point Ec1 has been assigned following coordinates:  $(x_{21}, 0)$
- Point Fc1 has been assigned following coordinates:  $(x_{22}, 0)$
- Condition for point X( $x_1, x_2$ ) to belong to this conic section is following equation:

$$p = x_{22} + x_{21}x_2 + x_{20}x_1 + x_{19}x_2^2 + x_{18}x_2x_1 + x_1^2$$



- Polynomial condition for point A to belong to conic section c1 is:

$$p = x_{22} + x_{21}x_7 + x_{20}x_6 + x_{19}x_7^2 + x_{18}x_7x_6 + x_6^2$$

- Polynomial condition for point B to belong to conic section c1 is:

$$p = x_{22} + x_{21}x_9 + x_{20}x_8 + x_{19}x_9^2 + x_{18}x_9x_8 + x_8^2$$

- Polynomial condition for point C to belong to conic section c1 is:

$$p = x_{22} + x_{21}x_{11} + x_{20}x_{10} + x_{19}x_{11}^2 + x_{18}x_{11}x_{10} + x_{10}^2$$

- Polynomial condition for point D to belong to conic section c1 is:

$$p = x_{22} + x_{21}x_{13} + x_{20}x_{12} + x_{19}x_{13}^2 + x_{18}x_{13}x_{12} + x_{12}^2$$

- Polynomial condition for point E to belong to conic section c1 is:

$$p = x_{22} + x_{21}x_{15} + x_{20}x_{14} + x_{19}x_{15}^2 + x_{18}x_{15}x_{14} + x_{14}^2$$

### Transformation of Theorem statement

- Polynomial for theorem statement:

$$p = x_{22} + x_{21}x_{17} + x_{20}x_{16} + x_{19}x_{17}^2 + x_{18}x_{17}x_{16} + x_{16}^2$$

### Time spent for transformation of Construction Protocol to algebraic form

- 0.141 seconds

## 3 Invoking the theorem prover

The used proving method is Wu's method.

The input system is:

$$\begin{aligned} p_1 &= u_3x_1^2 + u_5x_1 \\ p_2 &= u_3x_2^2 + (u_6u_2 + u_5)x_2 + (u_6^2u_1 + u_6u_4) \\ p_3 &= u_3x_3^2 + (u_7u_2 + u_5)x_3 + (u_7^2u_1 + u_7u_4) \\ p_4 &= u_3x_4^2 + (u_8u_2 + u_5)x_4 + (u_8^2u_1 + u_8u_4) \\ p_5 &= u_3x_5^2 + (u_9u_2 + u_5)x_5 + (u_9^2u_1 + u_9u_4) \\ p_6 &= x_6x_1 \\ p_7 &= (u_7 - u_6)x_7 - x_6x_3 + x_6x_2 + u_6x_3 - u_7x_2 \\ p_8 &= x_8x_1 \\ p_9 &= (u_9 - u_8)x_9 - x_8x_5 + x_8x_4 + u_8x_5 - u_9x_4 \\ p_{10} &= u_6x_{11} - x_{10}x_2 \\ p_{11} &= u_8x_{11} - x_{10}x_4 + x_{10}x_1 - u_8x_1 \end{aligned}$$

$$\begin{aligned}
p_{12} &= u_7x_{13} - x_{12}x_3 \\
p_{13} &= u_9x_{13} - x_{12}x_5 + x_{12}x_1 - u_9x_1 \\
p_{14} &= u_8x_{15} - x_{14}x_4 \\
p_{15} &= u_6x_{15} - x_{14}x_2 + x_{14}x_1 - u_6x_1 \\
p_{16} &= u_9x_{17} - x_{16}x_5 \\
p_{17} &= u_7x_{17} - x_{16}x_3 + x_{16}x_1 - u_7x_1 \\
p_{18} &= x_{22} + x_{21}x_7 + x_{20}x_6 + x_{19}x_7^2 + x_{18}x_7x_6 + x_6^2 \\
p_{19} &= x_{22} + x_{21}x_9 + x_{20}x_8 + x_{19}x_9^2 + x_{18}x_9x_8 + x_8^2 \\
p_{20} &= x_{22} + x_{21}x_{11} + x_{20}x_{10} + x_{19}x_{11}^2 + x_{18}x_{11}x_{10} + x_{10}^2 \\
p_{21} &= x_{22} + x_{21}x_{13} + x_{20}x_{12} + x_{19}x_{13}^2 + x_{18}x_{13}x_{12} + x_{12}^2 \\
p_{22} &= x_{22} + x_{21}x_{15} + x_{20}x_{14} + x_{19}x_{15}^2 + x_{18}x_{15}x_{14} + x_{14}^2
\end{aligned}$$

### 3.1 Triangulation, step 1

**Choosing variable:** Trying the variable with index 22.

**Variable  $x_{22}$  selected:** The number of polynomials with this variable, with indexes from 1 to 22, is 5.

**Minimal degrees:** 5 polynomial(s) with degree 1.

**Polynomial with linear degree:** Removing variable  $x_{22}$  from all other polynomials by reducing them with polynomial  $p_{18}$  from previous step.

Finished a triangulation step, the current system is:

$$\begin{aligned}
p_1 &= u_3x_1^2 + u_5x_1 \\
p_2 &= u_3x_2^2 + (u_6u_2 + u_5)x_2 + (u_6^2u_1 + u_6u_4) \\
p_3 &= u_3x_3^2 + (u_7u_2 + u_5)x_3 + (u_7^2u_1 + u_7u_4) \\
p_4 &= u_3x_4^2 + (u_8u_2 + u_5)x_4 + (u_8^2u_1 + u_8u_4) \\
p_5 &= u_3x_5^2 + (u_9u_2 + u_5)x_5 + (u_9^2u_1 + u_9u_4) \\
p_6 &= x_6x_1 \\
p_7 &= (u_7 - u_6)x_7 - x_6x_3 + x_6x_2 + u_6x_3 - u_7x_2 \\
p_8 &= x_8x_1 \\
p_9 &= (u_9 - u_8)x_9 - x_8x_5 + x_8x_4 + u_8x_5 - u_9x_4 \\
p_{10} &= u_6x_{11} - x_{10}x_2 \\
p_{11} &= u_8x_{11} - x_{10}x_4 + x_{10}x_1 - u_8x_1 \\
p_{12} &= u_7x_{13} - x_{12}x_3 \\
p_{13} &= u_9x_{13} - x_{12}x_5 + x_{12}x_1 - u_9x_1 \\
p_{14} &= u_8x_{15} - x_{14}x_4 \\
p_{15} &= u_6x_{15} - x_{14}x_2 + x_{14}x_1 - u_6x_1 \\
p_{16} &= u_9x_{17} - x_{16}x_5 \\
p_{17} &= u_7x_{17} - x_{16}x_3 + x_{16}x_1 - u_7x_1 \\
p_{18} &= x_{21}x_9 - x_{21}x_7 + x_{20}x_8 - x_{20}x_6 + x_{19}x_9^2 - x_{19}x_7^2 +
\end{aligned}$$

$$\begin{aligned}
& x_{18}x_9x_8 - x_{18}x_7x_6 + x_8^2 - x_6^2 \\
p_{19} &= x_{21}x_{11} - x_{21}x_7 + x_{20}x_{10} - x_{20}x_6 + x_{19}x_{11}^2 - x_{19}x_7^2 + \\
& x_{18}x_{11}x_{10} - x_{18}x_7x_6 + x_{10}^2 - x_6^2 \\
p_{20} &= x_{21}x_{13} - x_{21}x_7 + x_{20}x_{12} - x_{20}x_6 + x_{19}x_{13}^2 - x_{19}x_7^2 + \\
& x_{18}x_{13}x_{12} - x_{18}x_7x_6 + x_{12}^2 - x_6^2 \\
p_{21} &= x_{21}x_{15} - x_{21}x_7 + x_{20}x_{14} - x_{20}x_6 + x_{19}x_{15}^2 - x_{19}x_7^2 + \\
& x_{18}x_{15}x_{14} - x_{18}x_7x_6 + x_{14}^2 - x_6^2 \\
p_{22} &= x_{22} + x_{21}x_7 + x_{20}x_6 + x_{19}x_7^2 + x_{18}x_7x_6 + x_6^2
\end{aligned}$$

### 3.2 Triangulation, step 2

**Choosing variable:** Trying the variable with index 21.

**Variable  $x_{21}$  selected:** The number of polynomials with this variable, with indexes from 1 to 21, is 4.

**Minimal degrees:** 4 polynomial(s) with degree 1.

**Polynomial with linear degree:** Removing variable  $x_{21}$  from all other polynomials by reducing them with polynomial  $p_{18}$  from previous step.

Finished a triangulation step, the current system is:

$$\begin{aligned}
p_1 &= u_3x_1^2 + u_5x_1 \\
p_2 &= u_3x_2^2 + (u_6u_2 + u_5)x_2 + (u_6^2u_1 + u_6u_4) \\
p_3 &= u_3x_3^2 + (u_7u_2 + u_5)x_3 + (u_7^2u_1 + u_7u_4) \\
p_4 &= u_3x_4^2 + (u_8u_2 + u_5)x_4 + (u_8^2u_1 + u_8u_4) \\
p_5 &= u_3x_5^2 + (u_9u_2 + u_5)x_5 + (u_9^2u_1 + u_9u_4) \\
p_6 &= x_6x_1 \\
p_7 &= (u_7 - u_6)x_7 - x_6x_3 + x_6x_2 + u_6x_3 - u_7x_2 \\
p_8 &= x_8x_1 \\
p_9 &= (u_9 - u_8)x_9 - x_8x_5 + x_8x_4 + u_8x_5 - u_9x_4 \\
p_{10} &= u_6x_{11} - x_{10}x_2 \\
p_{11} &= u_8x_{11} - x_{10}x_4 + x_{10}x_1 - u_8x_1 \\
p_{12} &= u_7x_{13} - x_{12}x_3 \\
p_{13} &= u_9x_{13} - x_{12}x_5 + x_{12}x_1 - u_9x_1 \\
p_{14} &= u_8x_{15} - x_{14}x_4 \\
p_{15} &= u_6x_{15} - x_{14}x_2 + x_{14}x_1 - u_6x_1 \\
p_{16} &= u_9x_{17} - x_{16}x_5 \\
p_{17} &= u_7x_{17} - x_{16}x_3 + x_{16}x_1 - u_7x_1 \\
p_{18} &= -x_{20}x_{11}x_8 + x_{20}x_{11}x_6 + x_{20}x_{10}x_9 - x_{20}x_{10}x_7 - x_{20}x_9x_6 + \\
& x_{20}x_8x_7 + x_{19}x_{11}^2x_9 - x_{19}x_{11}^2x_7 - x_{19}x_{11}x_9^2 + \\
& x_{19}x_{11}x_7^2 + x_{19}x_9^2x_7 - x_{19}x_9x_7^2 + x_{18}x_{11}x_{10}x_9 \\
& - x_{18}x_{11}x_{10}x_7 - x_{18}x_{11}x_9x_8 + x_{18}x_{11}x_7x_6 + x_{18}x_9x_8x_7
\end{aligned}$$

$$\begin{aligned}
& -x_{18}x_9x_7x_6 - x_{11}x_8^2 + x_{11}x_6^2 + x_{10}^2x_9 - x_{10}^2x_7 \\
& -x_9x_6^2 + x_8^2x_7 \\
p_{19} = & -x_{20}x_{13}x_8 + x_{20}x_{13}x_6 + x_{20}x_{12}x_9 - x_{20}x_{12}x_7 - x_{20}x_9x_6 + \\
& x_{20}x_8x_7 + x_{19}x_{13}^2x_9 - x_{19}x_{13}^2x_7 - x_{19}x_{13}x_9^2 + \\
& x_{19}x_{13}x_7^2 + x_{19}x_9^2x_7 - x_{19}x_9x_7^2 + x_{18}x_{13}x_{12}x_9 \\
& -x_{18}x_{13}x_{12}x_7 - x_{18}x_{13}x_9x_8 + x_{18}x_{13}x_7x_6 + x_{18}x_9x_8x_7 \\
& -x_{18}x_9x_7x_6 - x_{13}x_8^2 + x_{13}x_6^2 + x_{12}^2x_9 - x_{12}^2x_7 \\
& -x_9x_6^2 + x_8^2x_7 \\
p_{20} = & -x_{20}x_{15}x_8 + x_{20}x_{15}x_6 + x_{20}x_{14}x_9 - x_{20}x_{14}x_7 - x_{20}x_9x_6 + \\
& x_{20}x_8x_7 + x_{19}x_{15}^2x_9 - x_{19}x_{15}^2x_7 - x_{19}x_{15}x_9^2 + \\
& x_{19}x_{15}x_7^2 + x_{19}x_9^2x_7 - x_{19}x_9x_7^2 + x_{18}x_{15}x_{14}x_9 \\
& -x_{18}x_{15}x_{14}x_7 - x_{18}x_{15}x_9x_8 + x_{18}x_{15}x_7x_6 + x_{18}x_9x_8x_7 \\
& -x_{18}x_9x_7x_6 - x_{15}x_8^2 + x_{15}x_6^2 + x_{14}^2x_9 - x_{14}^2x_7 \\
& -x_9x_6^2 + x_8^2x_7 \\
p_{21} = & x_{21}x_9 - x_{21}x_7 + x_{20}x_8 - x_{20}x_6 + x_{19}x_9^2 - x_{19}x_7^2 + \\
& x_{18}x_9x_8 - x_{18}x_7x_6 + x_8^2 - x_6^2 \\
p_{22} = & x_{22} + x_{21}x_7 + x_{20}x_6 + x_{19}x_7^2 + x_{18}x_7x_6 + x_6^2
\end{aligned}$$

### 3.3 Triangulation, step 3

**Choosing variable:** Trying the variable with index 20.

**Variable  $x_{20}$  selected:** The number of polynomials with this variable, with indexes from 1 to 20, is 3.

**Minimal degrees:** 3 polynomial(s) with degree 1.

**Polynomial with linear degree:** Removing variable  $x_{20}$  from all other polynomials by reducing them with polynomial  $p_{18}$  from previous step.

Finished a triangulation step, the current system is:

$$\begin{aligned}
p_1 &= u_3x_1^2 + u_5x_1 \\
p_2 &= u_3x_2^2 + (u_6u_2 + u_5)x_2 + (u_6^2u_1 + u_6u_4) \\
p_3 &= u_3x_3^2 + (u_7u_2 + u_5)x_3 + (u_7^2u_1 + u_7u_4) \\
p_4 &= u_3x_4^2 + (u_8u_2 + u_5)x_4 + (u_8^2u_1 + u_8u_4) \\
p_5 &= u_3x_5^2 + (u_9u_2 + u_5)x_5 + (u_9^2u_1 + u_9u_4) \\
p_6 &= x_6x_1 \\
p_7 &= (u_7 - u_6)x_7 - x_6x_3 + x_6x_2 + u_6x_3 - u_7x_2 \\
p_8 &= x_8x_1 \\
p_9 &= (u_9 - u_8)x_9 - x_8x_5 + x_8x_4 + u_8x_5 - u_9x_4 \\
p_{10} &= u_6x_{11} - x_{10}x_2 \\
p_{11} &= u_8x_{11} - x_{10}x_4 + x_{10}x_1 - u_8x_1 \\
p_{12} &= u_7x_{13} - x_{12}x_3
\end{aligned}$$

$$\begin{aligned}
p_{13} &= u_9x_{13} - x_{12}x_5 + x_{12}x_1 - u_9x_1 \\
p_{14} &= u_8x_{15} - x_{14}x_4 \\
p_{15} &= u_6x_{15} - x_{14}x_2 + x_{14}x_1 - u_6x_1 \\
p_{16} &= u_9x_{17} - x_{16}x_5 \\
p_{17} &= u_7x_{17} - x_{16}x_3 + x_{16}x_1 - u_7x_1 \\
p_{18} &= \dots \\
p_{19} &= \dots \\
p_{20} &= -x_{20}x_{11}x_8 + x_{20}x_{11}x_6 + x_{20}x_{10}x_9 - x_{20}x_{10}x_7 - x_{20}x_9x_6 + \\
&\quad x_{20}x_8x_7 + x_{19}x_{11}^2x_9 - x_{19}x_{11}^2x_7 - x_{19}x_{11}x_9^2 + \\
&\quad x_{19}x_{11}x_7^2 + x_{19}x_9^2x_7 - x_{19}x_9x_7^2 + x_{18}x_{11}x_{10}x_9 \\
&\quad - x_{18}x_{11}x_{10}x_7 - x_{18}x_{11}x_9x_8 + x_{18}x_{11}x_7x_6 + x_{18}x_9x_8x_7 \\
&\quad - x_{18}x_9x_7x_6 - x_{11}x_8^2 + x_{11}x_6^2 + x_{10}^2x_9 - x_{10}^2x_7 \\
&\quad - x_9x_6^2 + x_8^2x_7 \\
p_{21} &= x_{21}x_9 - x_{21}x_7 + x_{20}x_8 - x_{20}x_6 + x_{19}x_9^2 - x_{19}x_7^2 + \\
&\quad x_{18}x_9x_8 - x_{18}x_7x_6 + x_8^2 - x_6^2 \\
p_{22} &= x_{22} + x_{21}x_7 + x_{20}x_6 + x_{19}x_7^2 + x_{18}x_7x_6 + x_6^2
\end{aligned}$$

### 3.4 Triangulation, step 4

**Choosing variable:** Trying the variable with index 19.

**Variable  $x_{19}$  selected:** The number of polynomials with this variable, with indexes from 1 to 19, is 2.

**Minimal degrees:** 2 polynomial(s) with degree 1.

**Polynomial with linear degree:** Removing variable  $x_{19}$  from all other polynomials by reducing them with polynomial  $p_{18}$  from previous step.

Finished a triangulation step, the current system is:

$$\begin{aligned}
p_1 &= u_3x_1^2 + u_5x_1 \\
p_2 &= u_3x_2^2 + (u_6u_2 + u_5)x_2 + (u_6^2u_1 + u_6u_4) \\
p_3 &= u_3x_3^2 + (u_7u_2 + u_5)x_3 + (u_7^2u_1 + u_7u_4) \\
p_4 &= u_3x_4^2 + (u_8u_2 + u_5)x_4 + (u_8^2u_1 + u_8u_4) \\
p_5 &= u_3x_5^2 + (u_9u_2 + u_5)x_5 + (u_9^2u_1 + u_9u_4) \\
p_6 &= x_6x_1 \\
p_7 &= (u_7 - u_6)x_7 - x_6x_3 + x_6x_2 + u_6x_3 - u_7x_2 \\
p_8 &= x_8x_1 \\
p_9 &= (u_9 - u_8)x_9 - x_8x_5 + x_8x_4 + u_8x_5 - u_9x_4 \\
p_{10} &= u_6x_{11} - x_{10}x_2 \\
p_{11} &= u_8x_{11} - x_{10}x_4 + x_{10}x_1 - u_8x_1 \\
p_{12} &= u_7x_{13} - x_{12}x_3 \\
p_{13} &= u_9x_{13} - x_{12}x_5 + x_{12}x_1 - u_9x_1
\end{aligned}$$

$$\begin{aligned}
p_{14} &= u_8x_{15} - x_{14}x_4 \\
p_{15} &= u_6x_{15} - x_{14}x_2 + x_{14}x_1 - u_6x_1 \\
p_{16} &= u_9x_{17} - x_{16}x_5 \\
p_{17} &= u_7x_{17} - x_{16}x_3 + x_{16}x_1 - u_7x_1 \\
p_{18} &= \dots \\
p_{19} &= \dots \\
p_{20} &= -x_{20}x_{11}x_8 + x_{20}x_{11}x_6 + x_{20}x_{10}x_9 - x_{20}x_{10}x_7 - x_{20}x_9x_6 + \\
&\quad x_{20}x_8x_7 + x_{19}x_{11}^2x_9 - x_{19}x_{11}^2x_7 - x_{19}x_{11}x_9^2 + \\
&\quad x_{19}x_{11}x_7^2 + x_{19}x_9^2x_7 - x_{19}x_9x_7^2 + x_{18}x_{11}x_{10}x_9 \\
&\quad - x_{18}x_{11}x_{10}x_7 - x_{18}x_{11}x_9x_8 + x_{18}x_{11}x_7x_6 + x_{18}x_9x_8x_7 \\
&\quad - x_{18}x_9x_7x_6 - x_{11}x_8^2 + x_{11}x_6^2 + x_{10}^2x_9 - x_{10}^2x_7 \\
&\quad - x_9x_6^2 + x_8^2x_7 \\
p_{21} &= x_{21}x_9 - x_{21}x_7 + x_{20}x_8 - x_{20}x_6 + x_{19}x_9^2 - x_{19}x_7^2 + \\
&\quad x_{18}x_9x_8 - x_{18}x_7x_6 + x_8^2 - x_6^2 \\
p_{22} &= x_{22} + x_{21}x_7 + x_{20}x_6 + x_{19}x_7^2 + x_{18}x_7x_6 + x_6^2
\end{aligned}$$

### 3.5 Triangulation, step 5

**Choosing variable:** Trying the variable with index 18.

**Variable  $x_{18}$  selected:** The number of polynomials with this variable, with indexes from 1 to 18, is 1.

**Single polynomial with chosen variable:** Chosen polynomial is  $p_{18}$ . No reduction needed.

The triangular system has not been changed.

### 3.6 Triangulation, step 6

**Choosing variable:** Trying the variable with index 17.

**Variable  $x_{17}$  selected:** The number of polynomials with this variable, with indexes from 1 to 17, is 2.

**Minimal degrees:** 2 polynomial(s) with degree 1.

**Polynomial with linear degree:** Removing variable  $x_{17}$  from all other polynomials by reducing them with polynomial  $p_{16}$  from previous step.

Finished a triangulation step, the current system is:

$$\begin{aligned}
p_1 &= u_3x_1^2 + u_5x_1 \\
p_2 &= u_3x_2^2 + (u_6u_2 + u_5)x_2 + (u_6^2u_1 + u_6u_4) \\
p_3 &= u_3x_3^2 + (u_7u_2 + u_5)x_3 + (u_7^2u_1 + u_7u_4) \\
p_4 &= u_3x_4^2 + (u_8u_2 + u_5)x_4 + (u_8^2u_1 + u_8u_4) \\
p_5 &= u_3x_5^2 + (u_9u_2 + u_5)x_5 + (u_9^2u_1 + u_9u_4)
\end{aligned}$$

$$\begin{aligned}
p_6 &= x_6x_1 \\
p_7 &= (u_7 - u_6)x_7 - x_6x_3 + x_6x_2 + u_6x_3 - u_7x_2 \\
p_8 &= x_8x_1 \\
p_9 &= (u_9 - u_8)x_9 - x_8x_5 + x_8x_4 + u_8x_5 - u_9x_4 \\
p_{10} &= u_6x_{11} - x_{10}x_2 \\
p_{11} &= u_8x_{11} - x_{10}x_4 + x_{10}x_1 - u_8x_1 \\
p_{12} &= u_7x_{13} - x_{12}x_3 \\
p_{13} &= u_9x_{13} - x_{12}x_5 + x_{12}x_1 - u_9x_1 \\
p_{14} &= u_8x_{15} - x_{14}x_4 \\
p_{15} &= u_6x_{15} - x_{14}x_2 + x_{14}x_1 - u_6x_1 \\
p_{16} &= u_7x_{16}x_5 - u_9x_{16}x_3 + u_9x_{16}x_1 - u_9u_7x_1 \\
p_{17} &= u_9x_{17} - x_{16}x_5 \\
p_{18} &= \dots \\
p_{19} &= \dots \\
p_{20} &= -x_{20}x_{11}x_8 + x_{20}x_{11}x_6 + x_{20}x_{10}x_9 - x_{20}x_{10}x_7 - x_{20}x_9x_6 + \\
&\quad x_{20}x_8x_7 + x_{19}x_{11}^2x_9 - x_{19}x_{11}^2x_7 - x_{19}x_{11}x_9^2 + \\
&\quad x_{19}x_{11}x_7^2 + x_{19}x_9^2x_7 - x_{19}x_9x_7^2 + x_{18}x_{11}x_{10}x_9 \\
&\quad - x_{18}x_{11}x_{10}x_7 - x_{18}x_{11}x_9x_8 + x_{18}x_{11}x_7x_6 + x_{18}x_9x_8x_7 \\
&\quad - x_{18}x_9x_7x_6 - x_{11}x_8^2 + x_{11}x_6^2 + x_{10}^2x_9 - x_{10}^2x_7 \\
&\quad - x_9x_6^2 + x_8^2x_7 \\
p_{21} &= x_{21}x_9 - x_{21}x_7 + x_{20}x_8 - x_{20}x_6 + x_{19}x_9^2 - x_{19}x_7^2 + \\
&\quad x_{18}x_9x_8 - x_{18}x_7x_6 + x_8^2 - x_6^2 \\
p_{22} &= x_{22} + x_{21}x_7 + x_{20}x_6 + x_{19}x_7^2 + x_{18}x_7x_6 + x_6^2
\end{aligned}$$

### 3.7 Triangulation, step 7

**Choosing variable:** Trying the variable with index 16.

**Variable  $x_{16}$  selected:** The number of polynomials with this variable, with indexes from 1 to 16, is 1.

**Single polynomial with chosen variable:** Chosen polynomial is  $p_{16}$ . No reduction needed.

The triangular system has not been changed.

### 3.8 Triangulation, step 8

**Choosing variable:** Trying the variable with index 15.

**Variable  $x_{15}$  selected:** The number of polynomials with this variable, with indexes from 1 to 15, is 2.

**Minimal degrees:** 2 polynomial(s) with degree 1.

**Polynomial with linear degree:** Removing variable  $x_{15}$  from all other polynomials by reducing them with polynomial  $p_{14}$  from previous step.

Finished a triangulation step, the current system is:

$$\begin{aligned}
p_1 &= u_3x_1^2 + u_5x_1 \\
p_2 &= u_3x_2^2 + (u_6u_2 + u_5)x_2 + (u_6^2u_1 + u_6u_4) \\
p_3 &= u_3x_3^2 + (u_7u_2 + u_5)x_3 + (u_7^2u_1 + u_7u_4) \\
p_4 &= u_3x_4^2 + (u_8u_2 + u_5)x_4 + (u_8^2u_1 + u_8u_4) \\
p_5 &= u_3x_5^2 + (u_9u_2 + u_5)x_5 + (u_9^2u_1 + u_9u_4) \\
p_6 &= x_6x_1 \\
p_7 &= (u_7 - u_6)x_7 - x_6x_3 + x_6x_2 + u_6x_3 - u_7x_2 \\
p_8 &= x_8x_1 \\
p_9 &= (u_9 - u_8)x_9 - x_8x_5 + x_8x_4 + u_8x_5 - u_9x_4 \\
p_{10} &= u_6x_{11} - x_{10}x_2 \\
p_{11} &= u_8x_{11} - x_{10}x_4 + x_{10}x_1 - u_8x_1 \\
p_{12} &= u_7x_{13} - x_{12}x_3 \\
p_{13} &= u_9x_{13} - x_{12}x_5 + x_{12}x_1 - u_9x_1 \\
p_{14} &= u_6x_{14}x_4 - u_8x_{14}x_2 + u_8x_{14}x_1 - u_8u_6x_1 \\
p_{15} &= u_8x_{15} - x_{14}x_4 \\
p_{16} &= u_7x_{16}x_5 - u_9x_{16}x_3 + u_9x_{16}x_1 - u_9u_7x_1 \\
p_{17} &= u_9x_{17} - x_{16}x_5 \\
p_{18} &= \dots \\
p_{19} &= \dots \\
p_{20} &= -x_{20}x_{11}x_8 + x_{20}x_{11}x_6 + x_{20}x_{10}x_9 - x_{20}x_{10}x_7 - x_{20}x_9x_6 + \\
&\quad x_{20}x_8x_7 + x_{19}x_{11}^2x_9 - x_{19}x_{11}^2x_7 - x_{19}x_{11}x_9^2 + \\
&\quad x_{19}x_{11}x_7^2 + x_{19}x_9^2x_7 - x_{19}x_9x_7^2 + x_{18}x_{11}x_{10}x_9 \\
&\quad - x_{18}x_{11}x_{10}x_7 - x_{18}x_{11}x_9x_8 + x_{18}x_{11}x_7x_6 + x_{18}x_9x_8x_7 \\
&\quad - x_{18}x_9x_7x_6 - x_{11}x_8^2 + x_{11}x_6^2 + x_{10}^2x_9 - x_{10}^2x_7 \\
&\quad - x_9x_6^2 + x_8^2x_7 \\
p_{21} &= x_{21}x_9 - x_{21}x_7 + x_{20}x_8 - x_{20}x_6 + x_{19}x_9^2 - x_{19}x_7^2 + \\
&\quad x_{18}x_9x_8 - x_{18}x_7x_6 + x_8^2 - x_6^2 \\
p_{22} &= x_{22} + x_{21}x_7 + x_{20}x_6 + x_{19}x_7^2 + x_{18}x_7x_6 + x_6^2
\end{aligned}$$

### 3.9 Triangulation, step 9

**Choosing variable:** Trying the variable with index 14.

**Variable  $x_{14}$  selected:** The number of polynomials with this variable, with indexes from 1 to 14, is 1.

**Single polynomial with chosen variable:** Chosen polynomial is  $p_{14}$ . No reduction needed.

The triangular system has not been changed.



### 3.10 Triangulation, step 10

**Choosing variable:** Trying the variable with index 13.

**Variable  $x_{13}$  selected:** The number of polynomials with this variable, with indexes from 1 to 13, is 2.

**Minimal degrees:** 2 polynomial(s) with degree 1.

**Polynomial with linear degree:** Removing variable  $x_{13}$  from all other polynomials by reducing them with polynomial  $p_{12}$  from previous step.

Finished a triangulation step, the current system is:

$$\begin{aligned}
p_1 &= u_3x_1^2 + u_5x_1 \\
p_2 &= u_3x_2^2 + (u_6u_2 + u_5)x_2 + (u_6^2u_1 + u_6u_4) \\
p_3 &= u_3x_3^2 + (u_7u_2 + u_5)x_3 + (u_7^2u_1 + u_7u_4) \\
p_4 &= u_3x_4^2 + (u_8u_2 + u_5)x_4 + (u_8^2u_1 + u_8u_4) \\
p_5 &= u_3x_5^2 + (u_9u_2 + u_5)x_5 + (u_9^2u_1 + u_9u_4) \\
p_6 &= x_6x_1 \\
p_7 &= (u_7 - u_6)x_7 - x_6x_3 + x_6x_2 + u_6x_3 - u_7x_2 \\
p_8 &= x_8x_1 \\
p_9 &= (u_9 - u_8)x_9 - x_8x_5 + x_8x_4 + u_8x_5 - u_9x_4 \\
p_{10} &= u_6x_{11} - x_{10}x_2 \\
p_{11} &= u_8x_{11} - x_{10}x_4 + x_{10}x_1 - u_8x_1 \\
p_{12} &= -u_7x_{12}x_5 + u_9x_{12}x_3 + u_7x_{12}x_1 - u_9u_7x_1 \\
p_{13} &= u_7x_{13} - x_{12}x_3 \\
p_{14} &= u_6x_{14}x_4 - u_8x_{14}x_2 + u_8x_{14}x_1 - u_8u_6x_1 \\
p_{15} &= u_8x_{15} - x_{14}x_4 \\
p_{16} &= u_7x_{16}x_5 - u_9x_{16}x_3 + u_9x_{16}x_1 - u_9u_7x_1 \\
p_{17} &= u_9x_{17} - x_{16}x_5 \\
p_{18} &= \dots \\
p_{19} &= \dots \\
p_{20} &= -x_{20}x_{11}x_8 + x_{20}x_{11}x_6 + x_{20}x_{10}x_9 - x_{20}x_{10}x_7 - x_{20}x_9x_6 + \\
&\quad x_{20}x_8x_7 + x_{19}x_{11}^2x_9 - x_{19}x_{11}^2x_7 - x_{19}x_{11}x_9^2 + \\
&\quad x_{19}x_{11}x_7^2 + x_{19}x_9^2x_7 - x_{19}x_9x_7^2 + x_{18}x_{11}x_{10}x_9 \\
&\quad - x_{18}x_{11}x_{10}x_7 - x_{18}x_{11}x_9x_8 + x_{18}x_{11}x_7x_6 + x_{18}x_9x_8x_7 \\
&\quad - x_{18}x_9x_7x_6 - x_{11}x_8^2 + x_{11}x_6^2 + x_{10}^2x_9 - x_{10}^2x_7 \\
&\quad - x_9x_6^2 + x_8^2x_7 \\
p_{21} &= x_{21}x_9 - x_{21}x_7 + x_{20}x_8 - x_{20}x_6 + x_{19}x_9^2 - x_{19}x_7^2 + \\
&\quad x_{18}x_9x_8 - x_{18}x_7x_6 + x_8^2 - x_6^2 \\
p_{22} &= x_{22} + x_{21}x_7 + x_{20}x_6 + x_{19}x_7^2 + x_{18}x_7x_6 + x_6^2
\end{aligned}$$

### 3.11 Triangulation, step 11

**Choosing variable:** Trying the variable with index 12.

**Variable  $x_{12}$  selected:** The number of polynomials with this variable, with indexes from 1 to 12, is 1.

**Single polynomial with chosen variable:** Chosen polynomial is  $p_{12}$ . No reduction needed.

The triangular system has not been changed.

### 3.12 Triangulation, step 12

**Choosing variable:** Trying the variable with index 11.

**Variable  $x_{11}$  selected:** The number of polynomials with this variable, with indexes from 1 to 11, is 2.

**Minimal degrees:** 2 polynomial(s) with degree 1.

**Polynomial with linear degree:** Removing variable  $x_{11}$  from all other polynomials by reducing them with polynomial  $p_{10}$  from previous step.

Finished a triangulation step, the current system is:

$$\begin{aligned} p_1 &= u_3x_1^2 + u_5x_1 \\ p_2 &= u_3x_2^2 + (u_6u_2 + u_5)x_2 + (u_6^2u_1 + u_6u_4) \\ p_3 &= u_3x_3^2 + (u_7u_2 + u_5)x_3 + (u_7^2u_1 + u_7u_4) \\ p_4 &= u_3x_4^2 + (u_8u_2 + u_5)x_4 + (u_8^2u_1 + u_8u_4) \\ p_5 &= u_3x_5^2 + (u_9u_2 + u_5)x_5 + (u_9^2u_1 + u_9u_4) \\ p_6 &= x_6x_1 \\ p_7 &= (u_7 - u_6)x_7 - x_6x_3 + x_6x_2 + u_6x_3 - u_7x_2 \\ p_8 &= x_8x_1 \\ p_9 &= (u_9 - u_8)x_9 - x_8x_5 + x_8x_4 + u_8x_5 - u_9x_4 \\ p_{10} &= -u_6x_{10}x_4 + u_8x_{10}x_2 + u_6x_{10}x_1 - u_8u_6x_1 \\ p_{11} &= u_6x_{11} - x_{10}x_2 \\ p_{12} &= -u_7x_{12}x_5 + u_9x_{12}x_3 + u_7x_{12}x_1 - u_9u_7x_1 \\ p_{13} &= u_7x_{13} - x_{12}x_3 \\ p_{14} &= u_6x_{14}x_4 - u_8x_{14}x_2 + u_8x_{14}x_1 - u_8u_6x_1 \\ p_{15} &= u_8x_{15} - x_{14}x_4 \\ p_{16} &= u_7x_{16}x_5 - u_9x_{16}x_3 + u_9x_{16}x_1 - u_9u_7x_1 \\ p_{17} &= u_9x_{17} - x_{16}x_5 \\ p_{18} &= \dots \\ p_{19} &= \dots \\ p_{20} &= -x_{20}x_{11}x_8 + x_{20}x_{11}x_6 + x_{20}x_{10}x_9 - x_{20}x_{10}x_7 - x_{20}x_9x_6 + \\ &\quad x_{20}x_8x_7 + x_{19}x_{11}^2x_9 - x_{19}x_{11}^2x_7 - x_{19}x_{11}x_9^2 + \end{aligned}$$

$$\begin{aligned}
& x_{19}x_{11}x_7^2 + x_{19}x_9^2x_7 - x_{19}x_9x_7^2 + x_{18}x_{11}x_{10}x_9 \\
& - x_{18}x_{11}x_{10}x_7 - x_{18}x_{11}x_9x_8 + x_{18}x_{11}x_7x_6 + x_{18}x_9x_8x_7 \\
& - x_{18}x_9x_7x_6 - x_{11}x_8^2 + x_{11}x_6^2 + x_{10}^2x_9 - x_{10}^2x_7 \\
& - x_9x_6^2 + x_8^2x_7 \\
p_{21} = & x_{21}x_9 - x_{21}x_7 + x_{20}x_8 - x_{20}x_6 + x_{19}x_9^2 - x_{19}x_7^2 + \\
& x_{18}x_9x_8 - x_{18}x_7x_6 + x_8^2 - x_6^2 \\
p_{22} = & x_{22} + x_{21}x_7 + x_{20}x_6 + x_{19}x_7^2 + x_{18}x_7x_6 + x_6^2
\end{aligned}$$

### 3.13 Triangulation, step 13

**Choosing variable:** Trying the variable with index 10.

**Variable  $x_{10}$  selected:** The number of polynomials with this variable, with indexes from 1 to 10, is 1.

**Single polynomial with chosen variable:** Chosen polynomial is  $p_{10}$ . No reduction needed.

The triangular system has not been changed.

### 3.14 Triangulation, step 14

**Choosing variable:** Trying the variable with index 9.

**Variable  $x_9$  selected:** The number of polynomials with this variable, with indexes from 1 to 9, is 1.

**Single polynomial with chosen variable:** Chosen polynomial is  $p_9$ . No reduction needed.

The triangular system has not been changed.

### 3.15 Triangulation, step 15

**Choosing variable:** Trying the variable with index 8.

**Variable  $x_8$  selected:** The number of polynomials with this variable, with indexes from 1 to 8, is 1.

**Single polynomial with chosen variable:** Chosen polynomial is  $p_8$ . No reduction needed.

The triangular system has not been changed.

### 3.16 Triangulation, step 16

**Choosing variable:** Trying the variable with index 7.

**Variable  $x_7$  selected:** The number of polynomials with this variable, with indexes from 1 to 7, is 1.

**Single polynomial with chosen variable:** Chosen polynomial is  $p_7$ . No reduction needed.

The triangular system has not been changed.

### 3.17 Triangulation, step 17

**Choosing variable:** Trying the variable with index 6.

**Variable  $x_6$  selected:** The number of polynomials with this variable, with indexes from 1 to 6, is 1.

**Single polynomial with chosen variable:** Chosen polynomial is  $p_6$ . No reduction needed.

The triangular system has not been changed.

### 3.18 Triangulation, step 18

**Choosing variable:** Trying the variable with index 5.

**Variable  $x_5$  selected:** The number of polynomials with this variable, with indexes from 1 to 5, is 1.

**Single polynomial with chosen variable:** Chosen polynomial is  $p_5$ . No reduction needed.

The triangular system has not been changed.

### 3.19 Triangulation, step 19

**Choosing variable:** Trying the variable with index 4.

**Variable  $x_4$  selected:** The number of polynomials with this variable, with indexes from 1 to 4, is 1.

**Single polynomial with chosen variable:** Chosen polynomial is  $p_4$ . No reduction needed.

The triangular system has not been changed.

### 3.20 Triangulation, step 20

**Choosing variable:** Trying the variable with index 3.

**Variable  $x_3$  selected:** The number of polynomials with this variable, with indexes from 1 to 3, is 1.

**Single polynomial with chosen variable:** Chosen polynomial is  $p_3$ . No reduction needed.

The triangular system has not been changed.

### 3.21 Triangulation, step 21

**Choosing variable:** Trying the variable with index 2.

**Variable  $x_2$  selected:** The number of polynomials with this variable, with indexes from 1 to 2, is 1.

**Single polynomial with chosen variable:** Chosen polynomial is  $p_2$ . No reduction needed.

The triangular system has not been changed.

### 3.22 Triangulation, step 22

**Choosing variable:** Trying the variable with index 1.

**Variable  $x_1$  selected:** The number of polynomials with this variable, with indexes from 1 to 1, is 1.

**Single polynomial with chosen variable:** Chosen polynomial is  $p_1$ . No reduction needed.

The triangular system has not been changed.

The triangular system is:

$$\begin{aligned}
p_1 &= u_3x_1^2 + u_5x_1 \\
p_2 &= u_3x_2^2 + (u_6u_2 + u_5)x_2 + (u_6^2u_1 + u_6u_4) \\
p_3 &= u_3x_3^2 + (u_7u_2 + u_5)x_3 + (u_7^2u_1 + u_7u_4) \\
p_4 &= u_3x_4^2 + (u_8u_2 + u_5)x_4 + (u_8^2u_1 + u_8u_4) \\
p_5 &= u_3x_5^2 + (u_9u_2 + u_5)x_5 + (u_9^2u_1 + u_9u_4) \\
p_6 &= x_6x_1 \\
p_7 &= (u_7 - u_6)x_7 - x_6x_3 + x_6x_2 + u_6x_3 - u_7x_2 \\
p_8 &= x_8x_1 \\
p_9 &= (u_9 - u_8)x_9 - x_8x_5 + x_8x_4 + u_8x_5 - u_9x_4 \\
p_{10} &= -u_6x_{10}x_4 + u_8x_{10}x_2 + u_6x_{10}x_1 - u_8u_6x_1 \\
p_{11} &= u_6x_{11} - x_{10}x_2 \\
p_{12} &= -u_7x_{12}x_5 + u_9x_{12}x_3 + u_7x_{12}x_1 - u_9u_7x_1 \\
p_{13} &= u_7x_{13} - x_{12}x_3 \\
p_{14} &= u_6x_{14}x_4 - u_8x_{14}x_2 + u_8x_{14}x_1 - u_8u_6x_1 \\
p_{15} &= u_8x_{15} - x_{14}x_4 \\
p_{16} &= u_7x_{16}x_5 - u_9x_{16}x_3 + u_9x_{16}x_1 - u_9u_7x_1 \\
p_{17} &= u_9x_{17} - x_{16}x_5 \\
p_{18} &= \dots \\
p_{19} &= \dots \\
p_{20} &= -x_{20}x_{11}x_8 + x_{20}x_{11}x_6 + x_{20}x_{10}x_9 - x_{20}x_{10}x_7 - x_{20}x_9x_6 + \\
&\quad x_{20}x_8x_7 + x_{19}x_{11}^2x_9 - x_{19}x_{11}^2x_7 - x_{19}x_{11}x_9^2 +
\end{aligned}$$

$$\begin{aligned}
& x_{19}x_{11}x_7^2 + x_{19}x_9^2x_7 - x_{19}x_9x_7^2 + x_{18}x_{11}x_{10}x_9 \\
& - x_{18}x_{11}x_{10}x_7 - x_{18}x_{11}x_9x_8 + x_{18}x_{11}x_7x_6 + x_{18}x_9x_8x_7 \\
& - x_{18}x_9x_7x_6 - x_{11}x_8^2 + x_{11}x_6^2 + x_{10}^2x_9 - x_{10}^2x_7 \\
& - x_9x_6^2 + x_8^2x_7 \\
p_{21} = & x_{21}x_9 - x_{21}x_7 + x_{20}x_8 - x_{20}x_6 + x_{19}x_9^2 - x_{19}x_7^2 + \\
& x_{18}x_9x_8 - x_{18}x_7x_6 + x_8^2 - x_6^2 \\
p_{22} = & x_{22} + x_{21}x_7 + x_{20}x_6 + x_{19}x_7^2 + x_{18}x_7x_6 + x_6^2
\end{aligned}$$

## 4 Final Remainder

### 4.1 Final remainder for conjecture Chou 011

Calculating final remainder of the conclusion:

$$g = x_{22} + x_{21}x_{17} + x_{20}x_{16} + x_{19}x_{17}^2 + x_{18}x_{17}x_{16} + x_{16}^2$$

with respect to the triangular system.

1. Pseudo remainder with  $p_{22}$  over variable  $x_{22}$ :

$$\begin{aligned}
g = & x_{21}x_{17} - x_{21}x_7 + x_{20}x_{16} - x_{20}x_6 + x_{19}x_{17}^2 - x_{19}x_7^2 + \\
& x_{18}x_{17}x_{16} - x_{18}x_7x_6 + x_{16}^2 - x_6^2
\end{aligned}$$

2. Pseudo remainder with  $p_{21}$  over variable  $x_{21}$ :

$$\begin{aligned}
g = & -x_{20}x_{17}x_8 + x_{20}x_{17}x_6 + x_{20}x_{16}x_9 - x_{20}x_{16}x_7 - x_{20}x_9x_6 + \\
& x_{20}x_8x_7 + x_{19}x_{17}^2x_9 - x_{19}x_{17}^2x_7 - x_{19}x_{17}x_9^2 + \\
& x_{19}x_{17}x_7^2 + x_{19}x_9^2x_7 - x_{19}x_9x_7^2 + x_{18}x_{17}x_{16}x_9 \\
& - x_{18}x_{17}x_{16}x_7 - x_{18}x_{17}x_9x_8 + x_{18}x_{17}x_7x_6 + x_{18}x_9x_8x_7 \\
& - x_{18}x_9x_7x_6 - x_{17}x_8^2 + x_{17}x_6^2 + x_{16}^2x_9 - x_{16}^2x_7 \\
& - x_9x_6^2 + x_8^2x_7
\end{aligned}$$

3. Pseudo remainder with  $p_{20}$  over variable  $x_{20}$ :

*Polynomial too big for output (text size is 4292 characters, number of terms is 134)*

4. Pseudo remainder with  $p_{19}$  over variable  $x_{19}$ :

*Polynomial too big for output (number of terms is 2612)*

5. Pseudo remainder with  $p_{18}$  over variable  $x_{18}$ :

## 5 Prover results

**Status:** Proving failed - Space limit has been reached.

**Space Complexity:** The biggest polynomial obtained during prover execution contains 180808 terms.

**Time Complexity:** Time spent by the prover is 69.874 seconds.