OpenGeoProver Output for conjecture "Polar 001"

Wu's method used

February 12, 2012

1 Validation of Construction Protocol

Construction steps:

- Free point O
- Free point M
- Circle c with center O and one point M
- Free point P
- Random point polarA1980 from circle c
- Random point polarB1863 from circle c
- Line polara525 through two points P and polarA1980
- Line polarb490 through two points P and polarB1863
- Intersection point polarA2760 of point sets c and polara525
- Intersection point polarB2879 of point sets c and polarb490
- Tangent line polara1655 through point polarA1980 of set of points c
- Tangent line polara2302 through point polarA2760 of set of points c
- Tangent line polarb1602 through point polarB1863 of set of points c
- Tangent line polarb2910 through point polarB2879 of set of points c
- Intersection point polarA444 of point sets polara1655 and polara2302
- Intersection point polarB67 of point sets polarb1602 and polarb2910
- Line p through two points polarA444 and polarB67
- Random point A from circle c
- Line s through two points P and A
- Intersection point B of point sets s and c
- Intersection point Q of point sets s and p

Theorem statement:

- Pair of points A and B is in harmonic conjunction with pair of points P and Q

Validation result: Construction protocol is valid.

2 Transformation of Construction Protocol to algebraic form

Transformation of Construction steps

2.1 Transformation of point O:

- Point O has been assigned following coordinates: $(0, 0)$

2.2 Transformation of point M:

- Point M has been assigned following coordinates: $(0, u_1)$

2.3 Transformation of point P:

- Point P has been assigned following coordinates: (u_2, u_3)

2.4 Transformation of point polarA1980:

- Point polarA1980 has been assigned following coordinates: (u_4, x_1)
- Polynomial that point polarA1980 has to satisfy is:

$$p \;=\; x_1^2 + (u_4^2 - u_1^2)$$

- Processing of polynomial

$$p \;=\; x_1^2 + (u_4^2 - u_1^2)$$

Info: Polynomial

$$p \;=\; x_1^2 + (u_4^2 - u_1^2)$$

added to system of polynomials that represents the constructions

- New polynomial added to system of hypotheses

2.5 Transformation of point polarB1863:

- Point polarB1863 has been assigned following coordinates: (u_5, x_2)
- Polynomial that point polarB1863 has to satisfy is:

$$p \;=\; x_2^2 + (u_5^2 - u_1^2)$$

- Processing of polynomial

$$p \quad = \quad x_2^2 + (u_5^2 - u_1^2)$$

Info: Polynomial

$$p \quad = \quad x_2^2 + (u_5^2 - u_1^2)$$

added to system of polynomials that represents the constructions

- New polynomial added to system of hypotheses

2.6 Transformation of point polarA2760:

- Point polarA2760 has been assigned following coordinates: (x_3, x_4)

- Polynomial that point polarA2760 has to satisfy is:

$$p \quad = \quad x_4^2 + x_3^2 - u_1^2$$

- Processing of polynomial

$$p \quad = \quad x_4^2 + x_3^2 - u_1^2$$

Info: Polynomial

$$p \quad = \quad x_4^2 + x_3^2 - u_1^2$$

added to system of polynomials that represents the constructions

- New polynomial added to system of hypotheses

- Polynomial that point polarA2760 has to satisfy is:

$$p \quad = \quad (u_4 - u_2)x_4 - x_3x_1 + u_3x_3 + u_2x_1 - u_4u_3$$

- Processing of polynomial

$$p \quad = \quad (u_4 - u_2)x_4 - x_3x_1 + u_3x_3 + u_2x_1 - u_4u_3$$

Info: Polynomial

$$p \quad = \quad (u_4 - u_2)x_4 - x_3x_1 + u_3x_3 + u_2x_1 - u_4u_3$$

added to system of polynomials that represents the constructions

- New polynomial added to system of hypotheses

2.7 Transformation of point polarB2879:

- Point polarB2879 has been assigned following coordinates: $(x_5,\ x_6)$

- Polynomial that point polarB2879 has to satisfy is:

$$p \quad = \quad x_6^2 + x_5^2 - u_1^2$$

- Processing of polynomial

$$p \quad = \quad x_6^2 + x_5^2 - u_1^2$$

  Info: Polynomial

$$p \quad = \quad x_6^2 + x_5^2 - u_1^2$$

  added to system of polynomials that represents the constructions

- New polynomial added to system of hypotheses

- Polynomial that point polarB2879 has to satisfy is:

$$p \quad = \quad (u_5 - u_2)x_6 - x_5x_2 + u_3x_5 + u_2x_2 - u_5u_3$$

- Processing of polynomial

$$p \quad = \quad (u_5 - u_2)x_6 - x_5x_2 + u_3x_5 + u_2x_2 - u_5u_3$$

  Info: Polynomial

$$p \quad = \quad (u_5 - u_2)x_6 - x_5x_2 + u_3x_5 + u_2x_2 - u_5u_3$$

  added to system of polynomials that represents the constructions

- New polynomial added to system of hypotheses

2.8 Transformation of point polarA444:

- Point polarA444 has been assigned following coordinates: $(x_7,\ x_8)$

- Polynomial that point polarA444 has to satisfy is:

$$p \quad = \quad x_8x_1 + u_4x_7 - x_1^2 - u_4^2$$

- Processing of polynomial

$$p \quad = \quad x_8x_1 + u_4x_7 - x_1^2 - u_4^2$$

  Info: Polynomial

$$p \quad = \quad x_8x_1 + u_4x_7 - x_1^2 - u_4^2$$

  added to system of polynomials that represents the constructions

- New polynomial added to system of hypotheses

- Polynomial that point polarA444 has to satisfy is:

$$p \quad = \quad x_8 x_4 + x_7 x_3 - x_4^2 - x_3^2$$

- Processing of polynomial

$$p \quad = \quad x_8 x_4 + x_7 x_3 - x_4^2 - x_3^2$$

Info: Polynomial

$$p \quad = \quad x_8 x_4 + x_7 x_3 - x_4^2 - x_3^2$$

added to system of polynomials that represents the constructions

- New polynomial added to system of hypotheses

2.9   Transformation of point polarB67:

- Point polarB67 has been assigned following coordinates: $(x_9,\ x_{10})$

- Polynomial that point polarB67 has to satisfy is:

$$p \quad = \quad x_{10} x_2 + u_5 x_9 - x_2^2 - u_5^2$$

- Processing of polynomial

$$p \quad = \quad x_{10} x_2 + u_5 x_9 - x_2^2 - u_5^2$$

Info: Polynomial

$$p \quad = \quad x_{10} x_2 + u_5 x_9 - x_2^2 - u_5^2$$

added to system of polynomials that represents the constructions

- New polynomial added to system of hypotheses

- Polynomial that point polarB67 has to satisfy is:

$$p \quad = \quad x_{10} x_6 + x_9 x_5 - x_6^2 - x_5^2$$

- Processing of polynomial

$$p \quad = \quad x_{10} x_6 + x_9 x_5 - x_6^2 - x_5^2$$

Info: Polynomial

$$p \quad = \quad x_{10} x_6 + x_9 x_5 - x_6^2 - x_5^2$$

added to system of polynomials that represents the constructions

- New polynomial added to system of hypotheses

2.10 Transformation of point A:

- Point A has been assigned following coordinates: (u_6, x_{11})

- Polynomial that point A has to satisfy is:

$$p \quad = \quad x_{11}^2 + (u_6^2 - u_1^2)$$

- Processing of polynomial

$$p \quad = \quad x_{11}^2 + (u_6^2 - u_1^2)$$

  Info: Polynomial

$$p \quad = \quad x_{11}^2 + (u_6^2 - u_1^2)$$

  added to system of polynomials that represents the constructions

- New polynomial added to system of hypotheses

2.11 Transformation of point B:

- Point B has been assigned following coordinates: (x_{12}, x_{13})

- Polynomial that point B has to satisfy is:

$$p \quad = \quad (u_6 - u_2)x_{13} - x_{12}x_{11} + u_3x_{12} + u_2x_{11} - u_6u_3$$

- Processing of polynomial

$$p \quad = \quad (u_6 - u_2)x_{13} - x_{12}x_{11} + u_3x_{12} + u_2x_{11} - u_6u_3$$

  Info: Polynomial

$$p \quad = \quad (u_6 - u_2)x_{13} - x_{12}x_{11} + u_3x_{12} + u_2x_{11} - u_6u_3$$

  added to system of polynomials that represents the constructions

- New polynomial added to system of hypotheses

- Polynomial that point B has to satisfy is:

$$p \quad = \quad x_{13}^2 + x_{12}^2 - u_1^2$$

- Processing of polynomial

$$p \quad = \quad x_{13}^2 + x_{12}^2 - u_1^2$$

  Info: Polynomial

$$p \quad = \quad x_{13}^2 + x_{12}^2 - u_1^2$$

  added to system of polynomials that represents the constructions

- New polynomial added to system of hypotheses

2.12   Transformation of point Q:

- Point Q has been assigned following coordinates: (x_{14}, x_{15})

- Polynomial that point Q has to satisfy is:

$$p \quad = \quad (u_6 - u_2)x_{15} - x_{14}x_{11} + u_3x_{14} + u_2x_{11} - u_6u_3$$

- Processing of polynomial

$$p \quad = \quad (u_6 - u_2)x_{15} - x_{14}x_{11} + u_3x_{14} + u_2x_{11} - u_6u_3$$

Info: Polynomial

$$p \quad = \quad (u_6 - u_2)x_{15} - x_{14}x_{11} + u_3x_{14} + u_2x_{11} - u_6u_3$$

added to system of polynomials that represents the constructions

- New polynomial added to system of hypotheses

- Polynomial that point Q has to satisfy is:

$$p \quad = \quad x_{15}x_9 - x_{15}x_7 - x_{14}x_{10} + x_{14}x_8 + x_{10}x_7 - x_9x_8$$

- Processing of polynomial

$$p \quad = \quad x_{15}x_9 - x_{15}x_7 - x_{14}x_{10} + x_{14}x_8 + x_{10}x_7 - x_9x_8$$

Info: Polynomial

$$p \quad = \quad x_{15}x_9 - x_{15}x_7 - x_{14}x_{10} + x_{14}x_8 + x_{10}x_7 - x_9x_8$$

added to system of polynomials that represents the constructions

- New polynomial added to system of hypotheses

Transformation of Theorem statement

- Polynomial for theorem statement:

$$p \quad = \quad x_{14}x_{12} + (u_6 - 2u_2)x_{14} + (-2u_6 + u_2)x_{12} + u_6u_2$$

Time spent for transformation of Construction Protocol to algebraic form

- 1.888 seconds

3   Invoking the theorem prover

The used proving method is Wu's method.

The input system is:

$$
\begin{aligned}
p_1 &= x_1^2 + (u_4^2 - u_1^2) \\
p_2 &= x_2^2 + (u_5^2 - u_1^2) \\
p_3 &= x_4^2 + x_3^2 - u_1^2 \\
p_4 &= (u_4 - u_2)x_4 - x_3 x_1 + u_3 x_3 + u_2 x_1 - u_4 u_3 \\
p_5 &= x_6^2 + x_5^2 - u_1^2 \\
p_6 &= (u_5 - u_2)x_6 - x_5 x_2 + u_3 x_5 + u_2 x_2 - u_5 u_3 \\
p_7 &= x_8 x_1 + u_4 x_7 - x_1^2 - u_4^2 \\
p_8 &= x_8 x_4 + x_7 x_3 - x_4^2 - x_3^2 \\
p_9 &= x_{10} x_2 + u_5 x_9 - x_2^2 - u_5^2 \\
p_{10} &= x_{10} x_6 + x_9 x_5 - x_6^2 - x_5^2 \\
p_{11} &= x_{11}^2 + (u_6^2 - u_1^2) \\
p_{12} &= (u_6 - u_2)x_{13} - x_{12} x_{11} + u_3 x_{12} + u_2 x_{11} - u_6 u_3 \\
p_{13} &= x_{13}^2 + x_{12}^2 - u_1^2 \\
p_{14} &= (u_6 - u_2)x_{15} - x_{14} x_{11} + u_3 x_{14} + u_2 x_{11} - u_6 u_3 \\
p_{15} &= x_{15} x_9 - x_{15} x_7 - x_{14} x_{10} + x_{14} x_8 + x_{10} x_7 - x_9 x_8
\end{aligned}
$$

3.1   Triangulation, step 1

Choosing variable: Trying the variable with index 15.

Variable x_{15} selected: The number of polynomials with this variable, with indexes from 1 to 15, is 2.

Minimal degrees: 2 polynomial(s) with degree 1.

Polynomial with linear degree: Removing variable x_{15} from all other polynomials by reducing them with polynomial p_{14} from previous step.

Finished a triangulation step, the current system is:

$$
\begin{aligned}
p_1 &= x_1^2 + (u_4^2 - u_1^2) \\
p_2 &= x_2^2 + (u_5^2 - u_1^2) \\
p_3 &= x_4^2 + x_3^2 - u_1^2 \\
p_4 &= (u_4 - u_2)x_4 - x_3 x_1 + u_3 x_3 + u_2 x_1 - u_4 u_3 \\
p_5 &= x_6^2 + x_5^2 - u_1^2 \\
p_6 &= (u_5 - u_2)x_6 - x_5 x_2 + u_3 x_5 + u_2 x_2 - u_5 u_3 \\
p_7 &= x_8 x_1 + u_4 x_7 - x_1^2 - u_4^2 \\
p_8 &= x_8 x_4 + x_7 x_3 - x_4^2 - x_3^2 \\
p_9 &= x_{10} x_2 + u_5 x_9 - x_2^2 - u_5^2 \\
p_{10} &= x_{10} x_6 + x_9 x_5 - x_6^2 - x_5^2 \\
p_{11} &= x_{11}^2 + (u_6^2 - u_1^2) \\
p_{12} &= (u_6 - u_2)x_{13} - x_{12} x_{11} + u_3 x_{12} + u_2 x_{11} - u_6 u_3
\end{aligned}
$$

$$
\begin{aligned}
p_{13} &= x_{13}^2 + x_{12}^2 - u_1^2 \\
p_{14} &= x_{14}x_{11}x_9 - x_{14}x_{11}x_7 + (-u_6 + u_2)x_{14}x_{10} - u_3x_{14}x_9 + \\
&\quad (u_6 - u_2)x_{14}x_8 + u_3x_{14}x_7 - u_2x_{11}x_9 + u_2x_{11}x_7 + \\
&\quad (u_6 - u_2)x_{10}x_7 + (-u_6 + u_2)x_9x_8 + u_6u_3x_9 - u_6u_3x_7 \\
p_{15} &= (u_6 - u_2)x_{15} - x_{14}x_{11} + u_3x_{14} + u_2x_{11} - u_6u_3
\end{aligned}
$$

3.2 Triangulation, step 2

Choosing variable: Trying the variable with index 14.

Variable x_{14} selected: The number of polynomials with this variable, with indexes from 1 to 14, is 1.

Single polynomial with chosen variable: Chosen polynomial is p_{14}. No reduction needed.

The triangular system has not been changed.

3.3 Triangulation, step 3

Choosing variable: Trying the variable with index 13.

Variable x_{13} selected: The number of polynomials with this variable, with indexes from 1 to 13, is 2.

Minimal degrees: 1 polynomial(s) with degree 1 and 1 polynomial(s) with degree 2.

Polynomial with linear degree: Removing variable x_{13} from all other polynomials by reducing them with polynomial p_{12} from previous step.

Finished a triangulation step, the current system is:

$$
\begin{aligned}
p_1 &= x_1^2 + (u_4^2 - u_1^2) \\
p_2 &= x_2^2 + (u_5^2 - u_1^2) \\
p_3 &= x_4^2 + x_3^2 - u_1^2 \\
p_4 &= (u_4 - u_2)x_4 - x_3x_1 + u_3x_3 + u_2x_1 - u_4u_3 \\
p_5 &= x_6^2 + x_5^2 - u_1^2 \\
p_6 &= (u_5 - u_2)x_6 - x_5x_2 + u_3x_5 + u_2x_2 - u_5u_3 \\
p_7 &= x_8x_1 + u_4x_7 - x_1^2 - u_4^2 \\
p_8 &= x_8x_4 + x_7x_3 - x_4^2 - x_3^2 \\
p_9 &= x_{10}x_2 + u_5x_9 - x_2^2 - u_5^2 \\
p_{10} &= x_{10}x_6 + x_9x_5 - x_6^2 - x_5^2 \\
p_{11} &= x_{11}^2 + (u_6^2 - u_1^2) \\
p_{12} &= x_{12}^2x_{11}^2 - 2u_3x_{12}^2x_{11} + \\
&\quad (u_6^2 - 2u_6u_2 + u_3^2 + u_2^2)x_{12}^2 - 2u_2x_{12}x_{11}^2 + \\
&\quad (2u_6u_3 + 2u_3u_2)x_{12}x_{11} - 2u_6u_3^2x_{12} + u_2^2x_{11}^2
\end{aligned}
$$

$$
\begin{aligned}
&\quad\quad -2u_6u_3u_2x_{11} + \\
&\quad\quad (u_6^2u_3^2 - u_6^2u_1^2 + 2u_6u_2u_1^2 - u_2^2u_1^2) \\
p_{13} &= (u_6 - u_2)x_{13} - x_{12}x_{11} + u_3x_{12} + u_2x_{11} - u_6u_3 \\
p_{14} &= x_{14}x_{11}x_9 - x_{14}x_{11}x_7 + (-u_6 + u_2)x_{14}x_{10} - u_3x_{14}x_9 + \\
&\quad\quad (u_6 - u_2)x_{14}x_8 + u_3x_{14}x_7 - u_2x_{11}x_9 + u_2x_{11}x_7 + \\
&\quad\quad (u_6 - u_2)x_{10}x_7 + (-u_6 + u_2)x_9x_8 + u_6u_3x_9 - u_6u_3x_7 \\
p_{15} &= (u_6 - u_2)x_{15} - x_{14}x_{11} + u_3x_{14} + u_2x_{11} - u_6u_3
\end{aligned}
$$

3.4 Triangulation, step 4

Choosing variable: Trying the variable with index 12.

Variable x_{12} selected: The number of polynomials with this variable, with indexes from 1 to 12, is 1.

Single polynomial with chosen variable: Chosen polynomial is p_{12}. No reduction needed.

The triangular system has not been changed.

3.5 Triangulation, step 5

Choosing variable: Trying the variable with index 11.

Variable x_{11} selected: The number of polynomials with this variable, with indexes from 1 to 11, is 1.

Single polynomial with chosen variable: Chosen polynomial is p_{11}. No reduction needed.

The triangular system has not been changed.

3.6 Triangulation, step 6

Choosing variable: Trying the variable with index 10.

Variable x_{10} selected: The number of polynomials with this variable, with indexes from 1 to 10, is 2.

Minimal degrees: 2 polynomial(s) with degree 1.

Polynomial with linear degree: Removing variable x_{10} from all other polynomials by reducing them with polynomial p_9 from previous step.

Finished a triangulation step, the current system is:

$$
\begin{aligned}
p_1 &= x_1^2 + (u_4^2 - u_1^2) \\
p_2 &= x_2^2 + (u_5^2 - u_1^2) \\
p_3 &= x_4^2 + x_3^2 - u_1^2 \\
p_4 &= (u_4 - u_2)x_4 - x_3x_1 + u_3x_3 + u_2x_1 - u_4u_3
\end{aligned}
$$

$$
\begin{aligned}
p_5 &= x_6^2 + x_5^2 - u_1^2 \\
p_6 &= (u_5 - u_2)x_6 - x_5 x_2 + u_3 x_5 + u_2 x_2 - u_5 u_3 \\
p_7 &= x_8 x_1 + u_4 x_7 - x_1^2 - u_4^2 \\
p_8 &= x_8 x_4 + x_7 x_3 - x_4^2 - x_3^2 \\
p_9 &= -u_5 x_9 x_6 + x_9 x_5 x_2 - x_6^2 x_2 + x_6 x_2^2 + u_5^2 x_6 - x_5^2 x_2 \\
p_{10} &= x_{10} x_2 + u_5 x_9 - x_2^2 - u_5^2 \\
p_{11} &= x_{11}^2 + (u_6^2 - u_1^2) \\
p_{12} &= x_{12}^2 x_{11}^2 - 2u_3 x_{12}^2 x_{11} + \\
&\quad (u_6^2 - 2u_6 u_2 + u_3^2 + u_2^2)x_{12}^2 - 2u_2 x_{12} x_{11}^2 + \\
&\quad (2u_6 u_3 + 2u_3 u_2)x_{12} x_{11} - 2u_6 u_3^2 x_{12} + u_2^2 x_{11}^2 \\
&\quad -2u_6 u_3 u_2 x_{11} + \\
&\quad (u_6^2 u_3^2 - u_6^2 u_1^2 + 2u_6 u_2 u_1^2 - u_2^2 u_1^2) \\
p_{13} &= (u_6 - u_2)x_{13} - x_{12} x_{11} + u_3 x_{12} + u_2 x_{11} - u_6 u_3 \\
p_{14} &= x_{14} x_{11} x_9 - x_{14} x_{11} x_7 + (-u_6 + u_2)x_{14} x_{10} - u_3 x_{14} x_9 + \\
&\quad (u_6 - u_2)x_{14} x_8 + u_3 x_{14} x_7 - u_2 x_{11} x_9 + u_2 x_{11} x_7 + \\
&\quad (u_6 - u_2)x_{10} x_7 + (-u_6 + u_2)x_9 x_8 + u_6 u_3 x_9 - u_6 u_3 x_7 \\
p_{15} &= (u_6 - u_2)x_{15} - x_{14} x_{11} + u_3 x_{14} + u_2 x_{11} - u_6 u_3
\end{aligned}
$$

3.7   Triangulation, step 7

Choosing variable: Trying the variable with index 9.

Variable x_9 selected: The number of polynomials with this variable, with indexes from 1 to 9, is 1.

Single polynomial with chosen variable: Chosen polynomial is p_9. No reduction needed.

The triangular system has not been changed.

3.8   Triangulation, step 8

Choosing variable: Trying the variable with index 8.

Variable x_8 selected: The number of polynomials with this variable, with indexes from 1 to 8, is 2.

Minimal degrees: 2 polynomial(s) with degree 1.

Polynomial with linear degree: Removing variable x_8 from all other polynomials by reducing them with polynomial p_7 from previous step.

Finished a triangulation step, the current system is:

$$
\begin{aligned}
p_1 &= x_1^2 + (u_4^2 - u_1^2) \\
p_2 &= x_2^2 + (u_5^2 - u_1^2) \\
p_3 &= x_4^2 + x_3^2 - u_1^2
\end{aligned}
$$

$$
\begin{aligned}
p_4 &= (u_4 - u_2)x_4 - x_3x_1 + u_3x_3 + u_2x_1 - u_4u_3 \\
p_5 &= x_6^2 + x_5^2 - u_1^2 \\
p_6 &= (u_5 - u_2)x_6 - x_5x_2 + u_3x_5 + u_2x_2 - u_5u_3 \\
p_7 &= -u_4x_7x_4 + x_7x_3x_1 - x_4^2x_1 + x_4x_1^2 + u_4^2x_4 - x_3^2x_1 \\
p_8 &= x_8x_1 + u_4x_7 - x_1^2 - u_4^2 \\
p_9 &= -u_5x_9x_6 + x_9x_5x_2 - x_6^2x_2 + x_6x_2^2 + u_5^2x_6 - x_5^2x_2 \\
p_{10} &= x_{10}x_2 + u_5x_9 - x_2^2 - u_5^2 \\
p_{11} &= x_{11}^2 + (u_6^2 - u_1^2) \\
p_{12} &= x_{12}^2x_{11}^2 - 2u_3x_{12}^2x_{11} + \\
&\quad (u_6^2 - 2u_6u_2 + u_3^2 + u_2^2)x_{12}^2 - 2u_2x_{12}x_{11}^2 + \\
&\quad (2u_6u_3 + 2u_3u_2)x_{12}x_{11} - 2u_6u_3^2x_{12} + u_2^2x_{11}^2 \\
&\quad -2u_6u_3u_2x_{11} + \\
&\quad (u_6^2u_3^2 - u_6^2u_1^2 + 2u_6u_2u_1^2 - u_2^2u_1^2) \\
p_{13} &= (u_6 - u_2)x_{13} - x_{12}x_{11} + u_3x_{12} + u_2x_{11} - u_6u_3 \\
p_{14} &= x_{14}x_{11}x_9 - x_{14}x_{11}x_7 + (-u_6 + u_2)x_{14}x_{10} - u_3x_{14}x_9 + \\
&\quad (u_6 - u_2)x_{14}x_8 + u_3x_{14}x_7 - u_2x_{11}x_9 + u_2x_{11}x_7 + \\
&\quad (u_6 - u_2)x_{10}x_7 + (-u_6 + u_2)x_9x_8 + u_6u_3x_9 - u_6u_3x_7 \\
p_{15} &= (u_6 - u_2)x_{15} - x_{14}x_{11} + u_3x_{14} + u_2x_{11} - u_6u_3
\end{aligned}
$$

3.9 Triangulation, step 9

Choosing variable: Trying the variable with index 7.

Variable x_7 selected: The number of polynomials with this variable, with indexes from 1 to 7, is 1.

Single polynomial with chosen variable: Chosen polynomial is p_7. No reduction needed.

The triangular system has not been changed.

3.10 Triangulation, step 10

Choosing variable: Trying the variable with index 6.

Variable x_6 selected: The number of polynomials with this variable, with indexes from 1 to 6, is 2.

Minimal degrees: 1 polynomial(s) with degree 1 and 1 polynomial(s) with degree 2.

Polynomial with linear degree: Removing variable x_6 from all other polynomials by reducing them with polynomial p_6 from previous step.

Finished a triangulation step, the current system is:

$$
\begin{aligned}
p_1 &= x_1^2 + (u_4^2 - u_1^2) \\
p_2 &= x_2^2 + (u_5^2 - u_1^2) \\
p_3 &= x_4^2 + x_3^2 - u_1^2 \\
p_4 &= (u_4 - u_2)x_4 - x_3 x_1 + u_3 x_3 + u_2 x_1 - u_4 u_3 \\
p_5 &= x_5^2 x_2^2 - 2u_3 x_5^2 x_2 + \\
&\quad (u_5^2 - 2u_5 u_2 + u_3^2 + u_2^2)x_5^2 - 2u_2 x_5 x_2^2 + \\
&\quad (2u_5 u_3 + 2u_3 u_2)x_5 x_2 - 2u_5 u_3^2 x_5 + u_2^2 x_2^2 \\
&\quad -2u_5 u_3 u_2 x_2 + \\
&\quad (u_5^2 u_3^2 - u_5^2 u_1^2 + 2u_5 u_2 u_1^2 - u_2^2 u_1^2) \\
p_6 &= (u_5 - u_2)x_6 - x_5 x_2 + u_3 x_5 + u_2 x_2 - u_5 u_3 \\
p_7 &= -u_4 x_7 x_4 + x_7 x_3 x_1 - x_4^2 x_1 + x_4 x_1^2 + u_4^2 x_4 - x_3^2 x_1 \\
p_8 &= x_8 x_1 + u_4 x_7 - x_1^2 - u_4^2 \\
p_9 &= -u_5 x_9 x_6 + x_9 x_5 x_2 - x_6^2 x_2 + x_6 x_2^2 + u_5^2 x_6 - x_5^2 x_2 \\
p_{10} &= x_{10} x_2 + u_5 x_9 - x_2^2 - u_5^2 \\
p_{11} &= x_{11}^2 + (u_6^2 - u_1^2) \\
p_{12} &= x_{12}^2 x_{11}^2 - 2u_3 x_{12}^2 x_{11} + \\
&\quad (u_6^2 - 2u_6 u_2 + u_3^2 + u_2^2)x_{12}^2 - 2u_2 x_{12} x_{11}^2 + \\
&\quad (2u_6 u_3 + 2u_3 u_2)x_{12} x_{11} - 2u_6 u_3^2 x_{12} + u_2^2 x_{11}^2 \\
&\quad -2u_6 u_3 u_2 x_{11} + \\
&\quad (u_6^2 u_3^2 - u_6^2 u_1^2 + 2u_6 u_2 u_1^2 - u_2^2 u_1^2) \\
p_{13} &= (u_6 - u_2)x_{13} - x_{12} x_{11} + u_3 x_{12} + u_2 x_{11} - u_6 u_3 \\
p_{14} &= x_{14} x_{11} x_9 - x_{14} x_{11} x_7 + (-u_6 + u_2)x_{14} x_{10} - u_3 x_{14} x_9 + \\
&\quad (u_6 - u_2)x_{14} x_8 + u_3 x_{14} x_7 - u_2 x_{11} x_9 + u_2 x_{11} x_7 + \\
&\quad (u_6 - u_2)x_{10} x_7 + (-u_6 + u_2)x_9 x_8 + u_6 u_3 x_9 - u_6 u_3 x_7 \\
p_{15} &= (u_6 - u_2)x_{15} - x_{14} x_{11} + u_3 x_{14} + u_2 x_{11} - u_6 u_3
\end{aligned}
$$

3.11 Triangulation, step 11

Choosing variable: Trying the variable with index 5.

Variable x_5 selected: The number of polynomials with this variable, with indexes from 1 to 5, is 1.

Single polynomial with chosen variable: Chosen polynomial is p_5. No reduction needed.

The triangular system has not been changed.

3.12 Triangulation, step 12

Choosing variable: Trying the variable with index 4.

Variable x_4 selected: The number of polynomials with this variable, with indexes from 1 to 4, is 2.

Minimal degrees: 1 polynomial(s) with degree 1 and 1 polynomial(s) with degree 2.

Polynomial with linear degree: Removing variable x_4 from all other polynomials by reducing them with polynomial p_4 from previous step.

Finished a triangulation step, the current system is:

$$
\begin{aligned}
p_1 &= x_1^2 + (u_4^2 - u_1^2) \\
p_2 &= x_2^2 + (u_5^2 - u_1^2) \\
p_3 &= x_3^2 x_1^2 - 2u_3 x_3^2 x_1 + \\
&\quad (u_4^2 - 2u_4 u_2 + u_3^2 + u_2^2)x_3^2 - 2u_2 x_3 x_1^2 + \\
&\quad (2u_4 u_3 + 2u_3 u_2)x_3 x_1 - 2u_4 u_3^2 x_3 + u_2^2 x_1^2 \\
&\quad -2u_4 u_3 u_2 x_1 + \\
&\quad (u_4^2 u_3^2 - u_4^2 u_1^2 + 2u_4 u_2 u_1^2 - u_2^2 u_1^2) \\
p_4 &= (u_4 - u_2)x_4 - x_3 x_1 + u_3 x_3 + u_2 x_1 - u_4 u_3 \\
p_5 &= x_5^2 x_2^2 - 2u_3 x_5^2 x_2 + \\
&\quad (u_5^2 - 2u_5 u_2 + u_3^2 + u_2^2)x_5^2 - 2u_2 x_5 x_2^2 + \\
&\quad (2u_5 u_3 + 2u_3 u_2)x_5 x_2 - 2u_5 u_3^2 x_5 + u_2^2 x_2^2 \\
&\quad -2u_5 u_3 u_2 x_2 + \\
&\quad (u_5^2 u_3^2 - u_5^2 u_1^2 + 2u_5 u_2 u_1^2 - u_2^2 u_1^2) \\
p_6 &= (u_5 - u_2)x_6 - x_5 x_2 + u_3 x_5 + u_2 x_2 - u_5 u_3 \\
p_7 &= -u_4 x_7 x_4 + x_7 x_3 x_1 - x_4^2 x_1 + x_4 x_1^2 + u_4^2 x_4 - x_3^2 x_1 \\
p_8 &= x_8 x_1 + u_4 x_7 - x_1^2 - u_4^2 \\
p_9 &= -u_5 x_9 x_6 + x_9 x_5 x_2 - x_6^2 x_2 + x_6 x_2^2 + u_5^2 x_6 - x_5^2 x_2 \\
p_{10} &= x_{10} x_2 + u_5 x_9 - x_2^2 - u_5^2 \\
p_{11} &= x_{11}^2 + (u_6^2 - u_1^2) \\
p_{12} &= x_{12}^2 x_{11}^2 - 2u_3 x_{12}^2 x_{11} + \\
&\quad (u_6^2 - 2u_6 u_2 + u_3^2 + u_2^2)x_{12}^2 - 2u_2 x_{12} x_{11}^2 + \\
&\quad (2u_6 u_3 + 2u_3 u_2)x_{12} x_{11} - 2u_6 u_3^2 x_{12} + u_2^2 x_{11}^2 \\
&\quad -2u_6 u_3 u_2 x_{11} + \\
&\quad (u_6^2 u_3^2 - u_6^2 u_1^2 + 2u_6 u_2 u_1^2 - u_2^2 u_1^2) \\
p_{13} &= (u_6 - u_2)x_{13} - x_{12} x_{11} + u_3 x_{12} + u_2 x_{11} - u_6 u_3 \\
p_{14} &= x_{14} x_{11} x_9 - x_{14} x_{11} x_7 + (-u_6 + u_2)x_{14} x_{10} - u_3 x_{14} x_9 + \\
&\quad (u_6 - u_2)x_{14} x_8 + u_3 x_{14} x_7 - u_2 x_{11} x_9 + u_2 x_{11} x_7 + \\
&\quad (u_6 - u_2)x_{10} x_7 + (-u_6 + u_2)x_9 x_8 + u_6 u_3 x_9 - u_6 u_3 x_7 \\
p_{15} &= (u_6 - u_2)x_{15} - x_{14} x_{11} + u_3 x_{14} + u_2 x_{11} - u_6 u_3
\end{aligned}
$$

3.13 Triangulation, step 13

Choosing variable: Trying the variable with index 3.

Variable x_3 selected: The number of polynomials with this variable, with indexes from 1 to 3, is 1.

Single polynomial with chosen variable: Chosen polynomial is p_3. No reduction needed.

The triangular system has not been changed.

3.14 Triangulation, step 14

Choosing variable: Trying the variable with index 2.

Variable x_2 selected: The number of polynomials with this variable, with indexes from 1 to 2, is 1.

Single polynomial with chosen variable: Chosen polynomial is p_2. No reduction needed.

The triangular system has not been changed.

3.15 Triangulation, step 15

Choosing variable: Trying the variable with index 1.

Variable x_1 selected: The number of polynomials with this variable, with indexes from 1 to 1, is 1.

Single polynomial with chosen variable: Chosen polynomial is p_1. No reduction needed.

The triangular system has not been changed.

The triangular system is:

$$
\begin{aligned}
p_1 &= x_1^2 + (u_4^2 - u_1^2) \\
p_2 &= x_2^2 + (u_5^2 - u_1^2) \\
p_3 &= x_3^2 x_1^2 - 2u_3 x_3^2 x_1 + \\
&\quad (u_4^2 - 2u_4 u_2 + u_3^2 + u_2^2)x_3^2 - 2u_2 x_3 x_1^2 + \\
&\quad (2u_4 u_3 + 2u_3 u_2)x_3 x_1 - 2u_4 u_3^2 x_3 + u_2^2 x_1^2 \\
&\quad -2u_4 u_3 u_2 x_1 + \\
&\quad (u_4^2 u_3^2 - u_4^2 u_1^2 + 2u_4 u_2 u_1^2 - u_2^2 u_1^2) \\
p_4 &= (u_4 - u_2)x_4 - x_3 x_1 + u_3 x_3 + u_2 x_1 - u_4 u_3 \\
p_5 &= x_5^2 x_2^2 - 2u_3 x_5^2 x_2 + \\
&\quad (u_5^2 - 2u_5 u_2 + u_3^2 + u_2^2)x_5^2 - 2u_2 x_5 x_2^2 + \\
&\quad (2u_5 u_3 + 2u_3 u_2)x_5 x_2 - 2u_5 u_3^2 x_5 + u_2^2 x_2^2 \\
&\quad -2u_5 u_3 u_2 x_2 + \\
&\quad (u_5^2 u_3^2 - u_5^2 u_1^2 + 2u_5 u_2 u_1^2 - u_2^2 u_1^2) \\
p_6 &= (u_5 - u_2)x_6 - x_5 x_2 + u_3 x_5 + u_2 x_2 - u_5 u_3 \\
p_7 &= -u_4 x_7 x_4 + x_7 x_3 x_1 - x_4^2 x_1 + x_4 x_1^2 + u_4^2 x_4 - x_3^2 x_1 \\
p_8 &= x_8 x_1 + u_4 x_7 - x_1^2 - u_4^2 \\
p_9 &= -u_5 x_9 x_6 + x_9 x_5 x_2 - x_6^2 x_2 + x_6 x_2^2 + u_5^2 x_6 - x_5^2 x_2
\end{aligned}
$$

$$
\begin{aligned}
p_{10} &= x_{10}x_2 + u_5 x_9 - x_2^2 - u_5^2 \\
p_{11} &= x_{11}^2 + (u_6^2 - u_1^2) \\
p_{12} &= x_{12}^2 x_{11}^2 - 2u_3 x_{12}^2 x_{11} + \\
&\quad (u_6^2 - 2u_6 u_2 + u_3^2 + u_2^2)x_{12}^2 - 2u_2 x_{12}x_{11}^2 + \\
&\quad (2u_6 u_3 + 2u_3 u_2)x_{12}x_{11} - 2u_6 u_3^2 x_{12} + u_2^2 x_{11}^2 \\
&\quad -2u_6 u_3 u_2 x_{11} + \\
&\quad (u_6^2 u_3^2 - u_6^2 u_1^2 + 2u_6 u_2 u_1^2 - u_2^2 u_1^2) \\
p_{13} &= (u_6 - u_2)x_{13} - x_{12}x_{11} + u_3 x_{12} + u_2 x_{11} - u_6 u_3 \\
p_{14} &= x_{14}x_{11}x_9 - x_{14}x_{11}x_7 + (-u_6 + u_2)x_{14}x_{10} - u_3 x_{14}x_9 + \\
&\quad (u_6 - u_2)x_{14}x_8 + u_3 x_{14}x_7 - u_2 x_{11}x_9 + u_2 x_{11}x_7 + \\
&\quad (u_6 - u_2)x_{10}x_7 + (-u_6 + u_2)x_9 x_8 + u_6 u_3 x_9 - u_6 u_3 x_7 \\
p_{15} &= (u_6 - u_2)x_{15} - x_{14}x_{11} + u_3 x_{14} + u_2 x_{11} - u_6 u_3
\end{aligned}
$$

4 Final Remainder

4.1 Final remainder for conjecture Polar 001

Calculating final remainder of the conclusion:

$$
g = x_{14}x_{12} + (u_6 - 2u_2)x_{14} + (-2u_6 + u_2)x_{12} + u_6 u_2
$$

with respect to the triangular system.

1. Pseudo remainder with p_{15} over variable x_{15}:

$$
g = x_{14}x_{12} + (u_6 - 2u_2)x_{14} + (-2u_6 + u_2)x_{12} + u_6 u_2
$$

2. Pseudo remainder with p_{14} over variable x_{14}:

$$
\begin{aligned}
g = &\; (-2u_6 + 2u_2)x_{12}x_{11}x_9 + (2u_6 - 2u_2)x_{12}x_{11}x_7 + \\
&\; (-u_6 + u_2)x_{12}x_{10}x_7 + (2u_6^2 - 3u_6 u_2 + u_2^2)x_{12}x_{10} + \\
&\; (u_6 - u_2)x_{12}x_9 x_8 + (u_6 u_3 - u_3 u_2)x_{12}x_9 + \\
&\; (-2u_6^2 + 3u_6 u_2 - u_2^2)x_{12}x_8 + (-u_6 u_3 + u_3 u_2)x_{12}x_7 + \\
&\; (2u_6 u_2 - 2u_2^2)x_{11}x_9 + (-2u_6 u_2 + 2u_2^2)x_{11}x_7 + \\
&\; (-u_6^2 + 3u_6 u_2 - 2u_2^2)x_{10}x_7 + (-u_6^2 u_2 + u_6 u_2^2)x_{10} + \\
&\; (u_6^2 - 3u_6 u_2 + 2u_2^2)x_9 x_8 + (-u_6^2 u_3 + u_6 u_3 u_2)x_9 + \\
&\; (u_6^2 u_2 - u_6 u_2^2)x_8 + (u_6^2 u_3 - u_6 u_3 u_2)x_7
\end{aligned}
$$

3. Pseudo remainder with p_{13} over variable x_{13}:

$$
g = (-2u_6 + 2u_2)x_{12}x_{11}x_9 + (2u_6 - 2u_2)x_{12}x_{11}x_7 +
$$

$$(-u_6 + u_2)x_{12}x_{10}x_7 + (2u_6^2 - 3u_6u_2 + u_2^2)x_{12}x_{10} +$$
$$(u_6 - u_2)x_{12}x_9x_8 + (u_6u_3 - u_3u_2)x_{12}x_9 +$$
$$(-2u_6^2 + 3u_6u_2 - u_2^2)x_{12}x_8 + (-u_6u_3 + u_3u_2)x_{12}x_7 +$$
$$(2u_6u_2 - 2u_2^2)x_{11}x_9 + (-2u_6u_2 + 2u_2^2)x_{11}x_7 +$$
$$(-u_6^2 + 3u_6u_2 - 2u_2^2)x_{10}x_7 + (-u_6^2u_2 + u_6u_2^2)x_{10} +$$
$$(u_6^2 - 3u_6u_2 + 2u_2^2)x_9x_8 + (-u_6^2u_3 + u_6u_3u_2)x_9 +$$
$$(u_6^2u_2 - u_6u_2^2)x_8 + (u_6^2u_3 - u_6u_3u_2)x_7$$

4. Pseudo remainder with p_{12} over variable x_{12}:

$$
\begin{aligned}
g \;=\; & (-2u_6 + 2u_2)x_{12}x_{11}x_9 + (2u_6 - 2u_2)x_{12}x_{11}x_7 + \\
& (-u_6 + u_2)x_{12}x_{10}x_7 + (2u_6^2 - 3u_6u_2 + u_2^2)x_{12}x_{10} + \\
& (u_6 - u_2)x_{12}x_9x_8 + (u_6u_3 - u_3u_2)x_{12}x_9 + \\
& (-2u_6^2 + 3u_6u_2 - u_2^2)x_{12}x_8 + (-u_6u_3 + u_3u_2)x_{12}x_7 + \\
& (2u_6u_2 - 2u_2^2)x_{11}x_9 + (-2u_6u_2 + 2u_2^2)x_{11}x_7 + \\
& (-u_6^2 + 3u_6u_2 - 2u_2^2)x_{10}x_7 + (-u_6^2u_2 + u_6u_2^2)x_{10} + \\
& (u_6^2 - 3u_6u_2 + 2u_2^2)x_9x_8 + (-u_6^2u_3 + u_6u_3u_2)x_9 + \\
& (u_6^2u_2 - u_6u_2^2)x_8 + (u_6^2u_3 - u_6u_3u_2)x_7
\end{aligned}
$$

5. Pseudo remainder with p_{11} over variable x_{11}:

$$
\begin{aligned}
g \;=\; & (-2u_6 + 2u_2)x_{12}x_{11}x_9 + (2u_6 - 2u_2)x_{12}x_{11}x_7 + \\
& (-u_6 + u_2)x_{12}x_{10}x_7 + (2u_6^2 - 3u_6u_2 + u_2^2)x_{12}x_{10} + \\
& (u_6 - u_2)x_{12}x_9x_8 + (u_6u_3 - u_3u_2)x_{12}x_9 + \\
& (-2u_6^2 + 3u_6u_2 - u_2^2)x_{12}x_8 + (-u_6u_3 + u_3u_2)x_{12}x_7 + \\
& (2u_6u_2 - 2u_2^2)x_{11}x_9 + (-2u_6u_2 + 2u_2^2)x_{11}x_7 + \\
& (-u_6^2 + 3u_6u_2 - 2u_2^2)x_{10}x_7 + (-u_6^2u_2 + u_6u_2^2)x_{10} + \\
& (u_6^2 - 3u_6u_2 + 2u_2^2)x_9x_8 + (-u_6^2u_3 + u_6u_3u_2)x_9 + \\
& (u_6^2u_2 - u_6u_2^2)x_8 + (u_6^2u_3 - u_6u_3u_2)x_7
\end{aligned}
$$

6. Pseudo remainder with p_{10} over variable x_{10}:

$$
\begin{aligned}
g \;=\; & (-2u_6 + 2u_2)x_{12}x_{11}x_9x_2 + (2u_6 - 2u_2)x_{12}x_{11}x_7x_2 + \\
& (u_6 - u_2)x_{12}x_9x_8x_2 + (u_6u_5 - u_5u_2)x_{12}x_9x_7 + \\
& (u_6u_3 - u_3u_2)x_{12}x_9x_2 + \\
& (-2u_6^2u_5 + 3u_6u_5u_2 - u_5u_2^2)x_{12}x_9 + \\
& (-2u_6^2 + 3u_6u_2 - u_2^2)x_{12}x_8x_2 + (-u_6 + u_2)x_{12}x_7x_2^2 + \\
& (-u_6u_3 + u_3u_2)x_{12}x_7x_2 + (-u_6u_5^2 + u_5^2u_2)x_{12}x_7 + \\
& (2u_6^2 - 3u_6u_2 + u_2^2)x_{12}x_2^2 + \\
& (2u_6^2u_5^2 - 3u_6u_5^2u_2 + u_5^2u_2^2)x_{12} +
\end{aligned}
$$

$$(2u_6u_2 - 2u_2^2)x_{11}x_9x_2 + (-2u_6u_2 + 2u_2^2)x_{11}x_7x_2 +$$
$$(u_6^2 - 3u_6u_2 + 2u_2^2)x_9x_8x_2 +$$
$$(u_6^2u_5 - 3u_6u_5u_2 + 2u_5u_2^2)x_9x_7 +$$
$$(-u_6^2u_3 + u_6u_3u_2)x_9x_2 + (u_6^2u_5u_2 - u_6u_5u_2^2)x_9 +$$
$$(u_6^2u_2 - u_6u_2^2)x_8x_2 + (-u_6^2 + 3u_6u_2 - 2u_2^2)x_7x_2^2 +$$
$$(u_6^2u_3 - u_6u_3u_2)x_7x_2 +$$
$$(-u_6^2u_5^2 + 3u_6u_5^2u_2 - 2u_5^2u_2^2)x_7 +$$
$$(-u_6^2u_2 + u_6u_2^2)x_2^2 +$$
$$(-u_6^2u_5^2u_2 + u_6u_5^2u_2^2)$$

7. Pseudo remainder with p_9 over variable x_9:

   Polynomial too big for output (text size is 3013 characters, number of terms is 50)

8. Pseudo remainder with p_8 over variable x_8:

   Polynomial too big for output (text size is 4983 characters, number of terms is 70)

9. Pseudo remainder with p_7 over variable x_7:

   Polynomial too big for output (text size is 11653 characters, number of terms is 140)

10. Pseudo remainder with p_6 over variable x_6:

    Polynomial too big for output (number of terms is 258)

11. Pseudo remainder with p_5 over variable x_5:

    Polynomial too big for output (number of terms is 294)

12. Pseudo remainder with p_4 over variable x_4:

    Polynomial too big for output (number of terms is 478)

13. Pseudo remainder with p_3 over variable x_3:

    Polynomial too big for output (number of terms is 526)

14. Pseudo remainder with p_2 over variable x_2:

    Polynomial too big for output (text size is greater than 2000 characters, number of terms is 192)

15. Pseudo remainder with p_1 over variable x_1:

    Polynomial too big for output (text size is greater than 2000 characters, number of terms is 64)

5 Prover results

Status: Theorem can't be neither proved nor disproved.

Space Complexity: The biggest polynomial obtained during prover execution contains 526 terms.

Time Complexity: Time spent by the prover is 5.978 seconds.