Research Statement

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I want to understand the mathematical structures that underpin contemporary techniques in optimization and computer science. Progress in the applied sciences outpaces our collective mathematical knowledge – as toy examples, we don't quite know why ReLU works in neural nets, or why the simplex algorithm is so fast in practice. I am most inspired when math combines the possibility of real world impact with the beauty that can come from abstraction. My training is primarily in combinatorics, convex geometry, and optimization, and I sometimes recruit tools from functional analysis and real algebraic geometry. Computational experiments and illustrative examples help me build intuition and an understanding of what is possible. This has the benefit of providing students a hands-on way to gain familiarity with a research problem. What follows are the directions of my past and current research.

1 **Polynomial SAGE optimization** – new hierarchies for convex relaxations
2 **Boundary structure of non-polyhedral convex bodies** – beyond polytopes and $\mathbb{R}^n$
3 **Graph k-sparsifiers** – a new way to remove edges while preserving graph characteristics
4 **Graphical designs & eigenpolytopes** – sampling on graphs and resulting combinatorics

After describing the project and some context, I discuss my contributions and then future directions of research. The sections can be read independently.

1 Polynomial SAGE optimization

It is a central and notably difficult problem in theoretical and applied mathematics to certify that a polynomial is nonnegative. Sums-of-squares (SOS) techniques provide a well-studied and theoretically tractable answer, as SOS relies on semi-definite programming (SDP) (see [14]); it also cannot be implemented at scale for this reason. Moreover, SOS certificates of nonnegativity are not sparsity preserving. That is, given a polynomial $f(x) = \sum_{\alpha \in \mathcal{A}} c_\alpha x^\alpha$ with *support* $\operatorname{supp}(f) = \mathcal{A}$, an SOS decomposition of f may require (many) terms outside of $\mathcal{A}$. Enter sums-of-arithmetic-geometric-exponentials (SAGE) programming, which relies on the arithmetic-geometric mean inequality to certify nonnegativity [42, 43, 16]. SAGE programming, which applies to the broader class of *signomials*, is sparsity preserving and efficiently implementable. We note that polynomials with SAGE certificates of nonnegativity are exactly the class of *SONC polynomials* [26, 37]. To date, constrained SAGE optimization typically relies on tractably presented convex constraints [38].

Theta bodies provide a hierarchy of SOS relaxations for the convex hull of a real variety $\operatorname{conv}(\mathcal{V}_{\mathbb{R}}(I))$, which is intertwined with the problem of certifying nonnegativity modulo a polynomial ideal [23]. In particular, theta bodies are tighter than the *Laserre hierarchy* [28, 29, 30] in that they only see linear functions which are SOS modulo an ideal, rather than all polynomials. Theta bodies answered a question of Lovász regarding an SDP relaxation of the max stable set problem on a graph [35], which is the first theta body of an ideal associated to a graph.

Current work: SAGE bodies

I am working with Venkat Chandrasekaran and Greg Blekherman to develop "SAGE bodies", a SAGE hierarchy for optimization akin to theta bodies. In fact, we have shown a more general story. What follows will be posted to the arxiv hopefully within a few months.

Let $\mathcal{C} \subset \{f(x) : f(x) \geq 0, x = (x_1, \ldots, x_n)\}$ be a cone of polynomials (ie, closed under positive scaling), and let $\mathcal{A} \subseteq \mathbb{N}^n$ index a set of monomials. Let $\mathcal{A}_k = \{\sum_{\alpha \in \mathcal{A}} m_\alpha \alpha : m_\alpha \in \mathbb{N}, \sum_{\alpha \in \mathcal{A}} m_\alpha \leq k\}$ denote the k-th level of the lattice generated by $\mathcal{A}$. Let $I \subset \mathbb{R}[x]$ be an ideal.

Definition 1.1 1. *We call $f \in \mathbb{R}[x]$* k-cone mod I *if there exists $g \in \mathcal{C}$ such that* $\operatorname{supp}(g) \subseteq \mathcal{A}_k$ *and $f \equiv g \mod I$. We denote the cone of such polynomials by $\mathcal{C}^{\mathcal{A}_k}(I)$.*

2. *The* k-th cone body *of I is*

$$\mathrm{CB}_{\mathcal{A}_k}(I) = \{x \in \mathbb{R}^n : f(x) \geq 0 \textit{ for every linear } f \textit{ that is } k\textit{-cone mod } I\}.$$

If $\mathrm{CB}_{\mathcal{A}_k}(I) = \operatorname{conv}(\mathcal{V}_{\mathbb{R}}(I))$, we say that the ideal is $\mathcal{A}_k$ exact*.*

Traditional theta bodies fall into this setting, where $\mathcal{A} = 0 \cup \{e_i\}_{i=1}^n$ provides the usual degree grading and $\mathcal{C}$ is the SOS cone, which is dual to the cone of *combinatorial moment matrices* [31]. A solid understanding of the dual cone is crucial to using our conic generalization.

Theorem 1.2 $\mathrm{CB}_{\mathcal{A}_k}(I)$ *is the closure of a projection of a slice of the dual cone $(\mathcal{C}^{\mathcal{A}_k}(I))^*$.*

In the case of SAGE optimization, the dual cone is well understood in terms of the *relative entropy cone* [16], hence we have hope of actually using SAGE bodies in computations. See Fig 1 for a small univariate example where SAGE improves on SOS. We still have details to nail down, and in particular, here are our most pressing questions.

Question 1 *Can we find a more compelling example where SAGE bodies outperform SOS techniques? This will likely exploit sparsity. Further, can we find an example where the SAGE hierarchy terminates in finitely many steps, but the SOS hierarchy does not?*

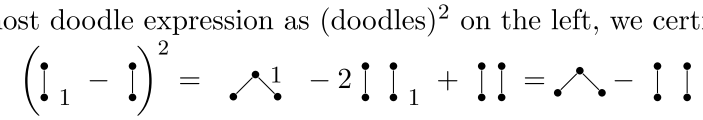
Question 2 *Can we characterize when the first SAGE body is exactly the closure of the variety in question? That is, when does the hierarchy terminate at the first step?*

Question 3 *Can SAGE bodies provide better relaxations of optimization problems that are independently interesting? For instance, theta bodies provided new hierarchies for the max cut and max stable set problems [23].*

Figure 1: Let $I = \langle x^{2d} - 1 \rangle$. Its real variety is $\mathcal{V}_{\mathbb{R}}(I) = \{\pm 1\}$ (black dots). We can compute the SAGE body $\mathrm{CB}_{\{0,1,2d\}} = [-1, 1]$ (cyan highlight); that is, this ideal is $\{0, 1, 2d\}$-exact. The theta body hierarchy provides the trivial relaxation until the d-th step, where it becomes exact. Further, the SOS certificates involved at the d-th step are not sparse – they use $\log d$ terms.

Future directions: Extremal combinatorics and flag algebras

One of my motivations to understand the algebra of polynomial SAGE optimization comes from extremal combinatorics. *Flag algebras* [41] are a powerful computational tool to prove inequalities among *graph homomorphism densities* (see [34]). The first Blakley-Roy inequality provides a simple example [13]. Let P_k denote a path with k edges. In words, a graph has at least as many P_2's as it has pairs of P_1's. This is a comparison of the densities of the path subgraphs P_1 and P_2 within a larger host graph. By what appears like magic or nonsense, one can verify this statement with the following annotated doodles, which concisely package the power of flag algebras [34, pp 28-29]. By expressing the rightmost doodle expression as (doodles)^2 on the left, we certify its nonnegativity.



A series of papers by Raymond, Thomas, Singh and coauthors [40, 39, 15] connect Razborov's flag algebra calculus and symmetry-adapted polynomial optimization modulo an ideal [21], and in particular show that sums-of-squares techniques are insufficient to tackle the full range of combinatorial questions posed by graph densities [15]. Concretely, SOS techniques provide no information about the smallest open case of Sidarenko's conjecture [15, Cor. 1.5].

Question 4 *Can SAGE programming prove new results in extremal combinatorics through flag algebras? Can it provide novel or shorter proofs of existing results?*

2 Boundary structure of non-polyhedral convex bodies

The facial structure of a polytope distills its key combinatorial and geometric attributes. A deeper understanding of this boundary structure has in turn led to applications and insights in the context of linear programming and computational complexity theory (e.g. [12] as a contemporary example). Modern optimization techniques, such as semidefinite programming, sums-of-squares methods (see [14]), hyperbolic programs [25], and SAGE relaxations [16], often rely on non-polyhedral feasible sets, which has spurred research on classes of non-polyhedral convex bodies.

Current work + Continuing directions

AIM SQuaRE: the PSD cone and beyond

Venkat Chandrasekaran, Isabelle Shankar, and Amy Wiebe, and I are seeking to understand non-polyhedral sets in $\mathbb{R}^n$ through the structure of their exposed faces. Gale diagrams [20, 24] are a powerful tool in polyhedral combinatorics which exactly capture the facial structure of a polytope through a dual picture, see Fig 2 for an example. If the polytope has not many more vertices than its dimension, then these diagrams are low-dimensional, and hence easier to understand than the polytope itself. We are focusing on convex bodies which arise as the linear image of the slice of a proper cone in $\mathbb{R}^n$, in analogy with polytopes being the linear image of a slice of $\mathbb{R}^n_+$. Before I joined the project, the other three had generalized Gale diagrams to this broader conic setting. While Gale diagrams can be understood purely through convex geometry, the statement of polytopal Gale duality is succinctly captured through oriented matroid duality ([49, Ch. 6], see also [11]). (Oriented) matroids are a combinatorial abstraction that capture the heart of many discrete structures, including graphs, polytopes, triangulations, hyperplane arrangements, matchings, and algebraic independence. Thus a major task at the first meeting of our square was to establish a generalization of matroids built on the facial structure of a cone, rather than subsets of $[n]$. We have arrived at a definition that reduces to ordinary matroids when the cone is $\mathbb{R}^n_+$ and agrees with the structure implied by our Gale diagrams for slices of the positive semidefinite cone. For the sake of notation, let $K \subset \mathbb{R}^n$ be a proper convex cone, $S(K)$ be the slice, and $A : \mathbb{R}^n \to \mathbb{R}^d$ be the linear map. That is, the convex body is $A(S(K))$. Following classic polytope results, it's natural to ask about $A(S(K))$:

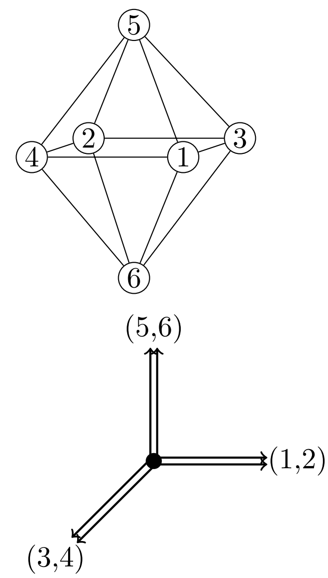


Figure 2: The octahedron and its Gale diagram. Note that if $J \subseteq [6]$ indexes a face of the polytope, then 0 (black dot) lies in the convex hull of the dual vectors $[6] \setminus J$. Check that, say, $\{1, 4, 5\}$ is a facet, and $0 \in \mathrm{conv}(\{2, 3, 6\})$.

Question 5 *Can Gale diagrams classify convex bodies* $A(S(K))$ *if* $\ker(A)$ *is low dimensional?*

Question 6 *Can Gale diagrams prove the existence of bizarre convex bodies?*

Question 7 *At a high level, the simplex algorithm runs by traversing the facial structure of a polytope in a smart way. Can we use our understanding of the exposed faces of* $A(S(K))$ *to design simplex-like algorithms for non-polyhedral convex bodies?*

In the direction of matroids, we hope to answer the following.

Question 8 *Is there a natural way to* orient *these conic matroids? An orientation is key for recovering a polytope from a matroid.*

Question 9 *Oriented matroids for point configurations in $\mathbb{R}^n$ are intimately tied to triangulations. Will our matroidal viewpoint allow us to 'triangulate' nonpolyhedral convex bodies?*

Extension to Hilbert Spaces

Independently of this SQuaRE project, I have been thinking about whether a similar duality structure persists in ℓ^2. My initial motivation was separate; Gale duality was a key ingredient in my work on graphical designs for finite graphs [7]. Here, the polytope and Gale diagram came from the eigenvectors of the graph Laplacian. *Graphons* are an analytic limiting object of graphs (see [34] and Fig 3). Instead of an adjacency matrix, one has a symmetric measurable function $W : [0,1]^2 \to [0,1]$. By the spectral theorem, the Laplacian-like operator $f(x) \mapsto \int W(x,y)f(y)dy$ has an orthonormal basis of eigenfunctions for $\ell^2([0,1])$. Letting K be the positive span of an orthonormal basis of $\ell^2([0,1])$, I have shown the desired geometric result:

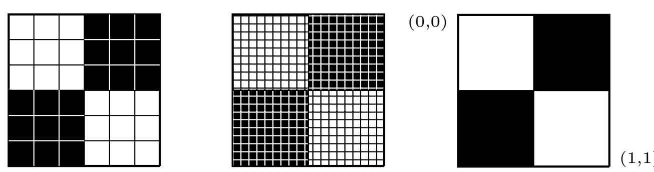
Theorem 2.1 *The face structure of $A(S(K))$ corresponds to the combinatorics of a dual picture.*

The question remains,

Question 10 *Will 'Gale duality' in Hilbert spaces provide a meaningful framework for a theory of designs on graphons?*

$$\begin{bmatrix} 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 \end{bmatrix}$$

(a)



(b) (c) (d)

Figure 3: (A) shows the adjacency matrix of the complete bipartite graph $K_{3,3}$, (B) shows its depiction as a unit square chopped into a 6×6 grid, where white is 0 and black is 1. (C) illustrates $K_{n,n}$ in this manner for n large. (D) shows the limiting graphon $W(x,y)$.

3 Graph k-sparsifiers

Graph sparsification is the process of modifying the edge weights of a graph (or removing them altogether) while preserving 'essential' properties of the original graph. This raises the question: which properties are essential, and how can we best preserve them? Spielman and Teng proposed *spectral sparsfiers* [8, 9, 44, 45], which were key in the proof of the Kadison-Singer conjecture [27, 36]. These are sparse subgraphs which preserve the graph Laplacian as a quadratic form within an ε tolerance. This is a natural definition for several reasons, including as a continuous relaxation of the *cut sparsification* problem [10]. Perhaps more importantly, the spectrum of the graph Laplacian provides control over connectivity, cuts, clustering, and mixing time of random walks, to name a few properties, most famously through Cheeger's Inequality [2, 17, 32] . Thus if one can (approximately) preserve the Laplacian, one (approximately) preserves these graph attributes.

Prior work

Stefan Steinerberger, Rekha Thomas, and I proposed a new regime for graph sparsification in [6], motivated by the observation that these desirable graph properties are controlled more directly by the low frequency part of the Laplacian spectrum, rather than the entire quadratic form. To this end, our *k-sparsifiers* require that the smallest k Laplacian eigenvalues and eigenvectors are preserved exactly, rather than approximately, but we place no other restrictions on the higher frequencies. In particular, this *exactly* preserves our control of graph attributes that are controlled by λ_2, such as cuts and clustering. We begin by describing the geometry of the set of k-sparsifiers, which is the intersection of a polyhedron and a *spectrahedron* (see Figure 4). A spectrahedron is the intersection of the PSD cone with a linear subspace. This allows us to establish some preliminary bounds and deduce a natural heuristic which guides intuition regarding how far a graph may sparsify. We illustrate our construction and the utility of our bounds on several families of graphs.

A reasonable intermediary between k-sparsifiers and spectral sparsifiers would preserve the first k eigenvalues and eigenvectors within an ε tolerance. We consider this entirely reasonable, but were pleased enough with the exact case that we restricted our attention there.

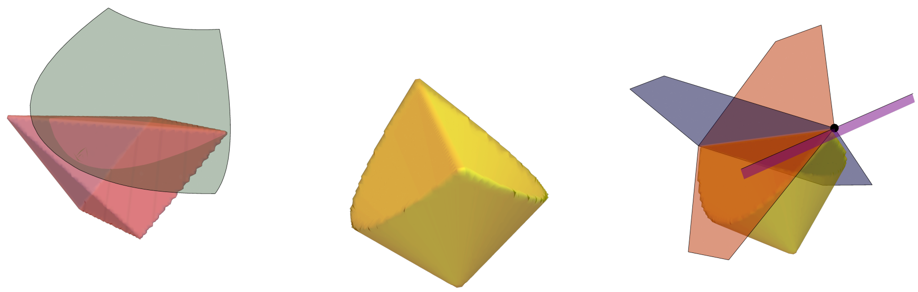


Figure 4: We first see the intersection of the polyhedral and spectrahedral parts of a set of 3−sparsifiers of the complete graph K_5. The resulting convex body is in the middle. Right shows the sparsest 3-sparsifier at the black dot, where three hyperplanes from the polyhedron intersect.

Future directions

Computation and algorithms

A major benefit of Spielman-Teng spectral sparsification is that every graph can be sparsified near optimally in polynomial time [8]. On the other hand, our k-sparsfiers seek a sparse vector in a convex set, which is handed to us as the intersection of a polyhedron and a spectrahedron. Semidefinite programming is technically polynomial time using, for instance, interior point methods, but in practice, SDPs cannot be solved at scale. More concerningly, finding the sparsest vector in a subspace is NP-hard [18, 48]. One would expect that finding a sparsest k-sparsifier is also NP-hard.

Question 11 *How hard is it to find a minimal k-sparsifier?*

Assuming the problem is as difficult as it appears, some follow up questions include finding approximate solutions and identifying graph structures which allow for efficient algorithms.

Other related models

Given the probable intractability of our first exact model, additional wiggle room in the spectrum may be needed to create practical algorithms. We also briefly considered seeking a graph whose Laplacian quadratic form acts identically to that of the initial graph on the subspace spanned by the first k eigenvectors of the intial graph, see [6, Section 6]. Broadly,

Question 12 *Are there other models of k-sparsifiers which strike a balance between computability, utility in applications, and mathematical intuition?*

4 Graphical designs and eigenpolytopes

A *graphical design* is a *quadrature rule* for a weighted graph $G = ([n], E, w)$. On smooth domains, quadrature rules are a classical way to approximate the integral of a 'nice' function by sampling the function value at finitely many points. Steinerberger first proposed graphical designs in [46], mimicking the construction of *spherical t-designs* [19]. Since graphs are already finite, we approximate averages by sampling at a proper subset of graph vertices. We ask that a graphical design be able to *average* a basis of eigenvectors for the *low frequency* eigenspaces of the graph Laplacian $L = D - A$. A subset of vertices $S \subset [n]$ and weights $(a_s \in \mathbb{R} : s \in S)$ averages a vector $\varphi \in \mathbb{R}^n$ if

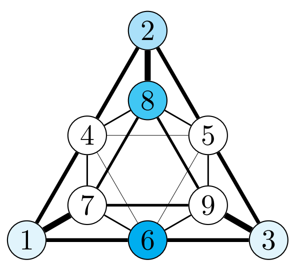
$$\frac{1}{n}\sum_{i=1}^{n}\varphi(i) = \frac{1}{|S|}\sum_{s\in S} a_s\varphi(s).$$

We call an eigenvector (or eigenspace) of L low frequency if its eigenvalue is small.

Prior work

My PhD was primarily devoted to graphical designs. In my master's thesis [3], I refined the initial definition of graphical designs to resolve the ambiguity posed by eigenspaces with multiplicity, established connections between unweighted graphical designs on the hypercube graph and error correcting codes, and clarified the distinction between graphical designs and existing related concepts of extremal designs, t-designs, and maximum stable sets.

In[7], Rekha Thomas and I established a bijection between positively weighted graphical designs and the faces of corresponding *eigenpolytopes* associated to a graph [22], see Fig 5 for an example. This bijection is afforded through Gale duality, or more broadly, oriented matroid duality (see [49, Ch 6]). Briefly, oriented matroids are a combinatorial abstraction that unify many discrete structures, including graphs, polytopes, triangulations, hyperplane arrangements, matchings, and algebraic field extensions. This connection allows us to organize positively weighted designs in the structure of the face poset of the polytope, and in some cases provides an easier method to compute designs. This bijection also established the first universal existence result for graphical designs – every polytope has faces, hence every graph has positively weighted graphical designs.



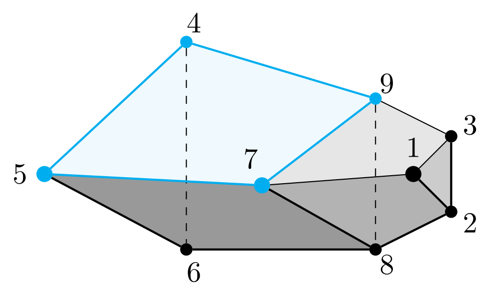


Figure 5: A graphical design and its corresponding face on the Gale dual eigenpolytope. A more saturated color on a graph vertex corresponds to a larger quadrature weight.

The polytope connection hinted at a path forward to computational complexity results for graphical designs. David Shiroma, an undergraduate Rekha and I had recruited, and I showed that every combinatorial type of polytope appears as the eigenpolytope of a weighted graph [5]. Our construction is strongly polynomial time, so we can port over hardness results about polytopes. In particular it is strongly NP-complete to determine if there is a graphical design smaller than a simple geometric upper bound, it is NP-hard to find a smallest graphical design, and it is #P-complete to count the number of support-minimal graphical designs [5].

Given these complexity results, we needed a new approach to make graphical designs amenable to actual applications. To this end, J. Carlos Martínez Mori, Hessa Al-Thani, and I developed a linear program which uses a fixed budget of vertices and seeks to minimize error [47]. We showcase this method on actual data of the taxicab system in Manhattan (Fig 6).

Figure 6: Data from the Manhattan taxicab system and a graphical design computed using our methods. Larger nodes mean more weight. This design averages the first 274 eigenvectors.

I was fortunate to have the opportunity to popularize graphical designs with a "What is...?" article in the AMS notices[4]. It has been awesome to see the article's reach. I've gotten cold emails from eager undergrads all the way up to a compliment from Donald Knuth!

Future directions

Extension to simplicial complexes

A natural generalization of graphical designs would come from higher order Laplacians of simplicial complexes (see [1, 33]). These are matrices, which have eigenvectors, hence one can play the same game. While an ultimate goal may be to study abstract simplicial complexes, a more concrete starting place may be to look at the *clique complex* of a graph. Once a definition is established and verified reasonable by examples, the question is then:

Question 13 *What additional information do higher order designs carry?*

A selling point of graphical designs is their promise to compress data on a graph with minimal loss. While the problem in full is NP hard, our work in [47] describes a linear programming adaptation of graphical designs suitable for applications.

Question 14 *Can we create efficient algorithms which find good higher order designs?*

Classifying eigenpolytopes

A primary result of our work in [5] shows that any combinatorial type of polytope appears as the eigenpolytope of some positively weighted graph. There is space to probe further in this direction.

Question 15 *Can every polytope appear as the eigenpolytope of an unweighted graph?*

Some intermediate questions include restricting to special classes of polytopes, such as 0-1 polytopes, determining combinatorial or algebraic conditions which suffice to show (or prevent) eigenpolytopeness, or showing that every polytope appears as the face of an eigenpolytope. Eigenpolytopes allow for interior points and repeated vertices, thus the number of graph vertices and the number of polytope vertices need not track in these constructions. The minimum size of an unweighted graph needed to realize a given polytope could therefore serve as some measure of complexity.

Question 16 *How does the complexity of embedding a polytope in an unweighted graph relate to existing notions of extension complexity?*

To the best of my knowledge, no one has studied the eigenpolytopes of higher order Laplacians.

Question 17 *Is there a combinatorial hierarchy which organizes all the higher order eigenpolytopes of a simplicial complex?*

Quantum error correcting codes

Given the existing connections between graphical designs, coding theory, and association schemes, and the zeitgeist around all things quantum, one might ask

Question 18 *Is there a family of graphs where quantum error correcting codes connect to designs?*

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