RESEARCH STATEMENT

Introduction. My dissertation research concerns graphical designs, which are quadrature rules for a graph. Quadrature rules approximate the integral of a function over a given domain by sampling the function value at finitely many points. On smooth domains, quadrature rules are classical and even ancient, dating back to Newton [37], Gauss [19], Pythagoras, and the Babylonians. S. Steinerberger first proposed graphical designs in 2020 [43], mimicking the construction of *spherical t-designs* [13]. Since graphs are already finite, a graphical design approximates the average of a function on a graph by sampling at a proper subset of graph vertices. Graphical designs have since been connected to extremal combinatorics [23] and random walks on graphs [33, 44]. My own work has connected them to error-correcting codes [3] and *eigenpolytopes* [22] through Gale duality [5, 6].

There are many directions in which to extend the existing work on graphical designs, but I am certainly interested in other topics and projects as well. My mathematical interests and training include combinatorics, discrete geometry, (spectral) graph theory, algorithms and complexity, and optimization – particularly combinatorial, polynomial, semidefinite, sums of squares, and convex. The rest of this research statement will focus on my past work and future ideas for graphical designs.

Graphical designs provide a method for graph sampling. Modern data is often modeled by graphs, such as online communities, power grids, and transit networks. As our data evolves, so must our data processing tools. The relatively new field of graph signal processing [38, 39] translates traditional signal processing techniques to graphs. Graph sampling is a major challenge studied in this field – [44] provides evidence that graphical designs sample effectively. Some research goals include establishing further connections with real world applications in graph sampling and optimization, better understanding the mathematical structure of graphical designs, solving related algorithmic and computational problems, and extending to the more general domain of simplicial complexes.

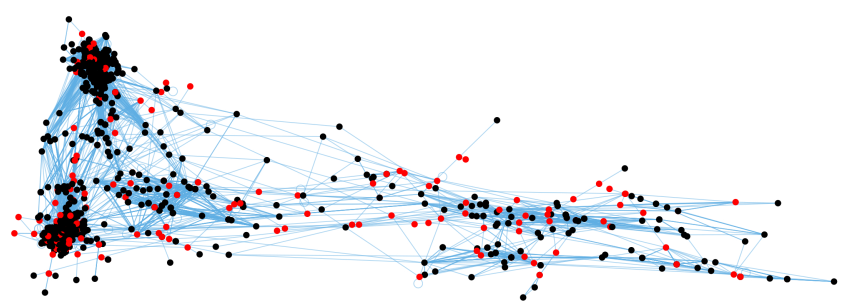

Figure 1. The red vertices are the support of a graphical design on the graph of Wikipedia pages hyperlinked to the page "Chameleons" [1, 4]

Past Accomplishments. My thesis work clarifies and expands the existing work on graphical designs. I have also increased access to these ideas through a growing database of examples https://sites.math.washington.edu/~GraphicalDesigns/, code for computations at https://github.com/cmbabecki/GraphicalDesigns, and a "What is...?" article in the AMS *Notices* [4].

Definitions and Scope. In [43], graphical designs were defined for unweighted graphs by the *low frequency* eigenvectors of a normalized graph Laplacian, $AD^{-1} - I$. The spectrum of $AD^{-1} - I$

is contained in $[-2,0]$; high frequency eigenvectors have an eigenvalue near -1, and low frequency eigenvectors have an eigenvalue near -2 or 0. It was left open how to resolve the ambiguity created by eigenspaces with multiplicity, an issue which is not of major concern in the continuous case. My masters thesis [3] resolved this by redefining graphical designs in terms of entire eigenspaces. In [6], R. R. Thomas and I considered other orderings of eigenspaces. In [5], D. Shiroma and I extended graphical designs to positively weighted graphs $G = ([n], E, w)$ and other graph operators, focusing on the combinatorial graph Laplacian $D - A$. In [3], I also distinguished graphical designs in the frequency order from other seemingly related objects, in particular t-designs in Q-polynomial association schemes [12]. Note also *extremal designs* [23], which average all but one eigenspace.

Definition 1. Suppose a diagonalizable graph operator $L \in \mathbb{R}^{n\times n}$ of $G = ([n], E, w)$ has m distinct eigenspaces ordered as $\Lambda_1 < \ldots < \Lambda_m$. A *$k$-graphical design* of G is a subset $S \subseteq [n]$ and real weights $(a_s \neq 0 : s \in S)$ such that for any $\varphi \in \Lambda_1, \ldots, \Lambda_k$,

$$(1) \qquad \sum_{s\in S} a_s \varphi(s) = \frac{1}{n}\sum_{i=1}^{n} \varphi(i)$$

There are three types of quadrature weights of interest: arbitrary ($a_s \neq 0$), positive ($a_s > 0$), and combinatorial ($a_s = a_{s'}$ for all $s, s' \in S$). In classical numerical integration, negative weights are undesirable as they can lead to divergent solutions and numerical instability [27]. Hence we focus on positively weighted graphical designs. I will now explain my major connections and results.

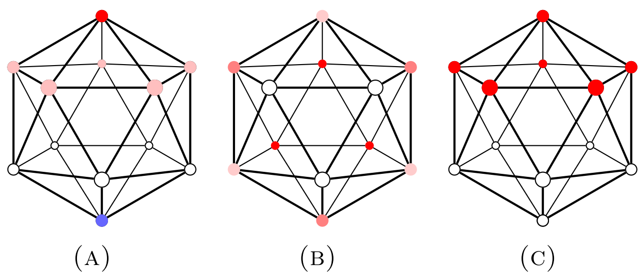

FIGURE 2. Three graphical designs of the icosahedral graph in a fixed eigenspace ordering. Lighter colors represent weights of smaller magnitude, red indicates positive and blue is negative. (A) shows an arbitrarily weighted 3-graphical design, (B) shows a positively weighted 3-graphical design, and (C) shows a combinatorial 2-graphical design.

Connections to Coding Theory. In [3], I showed that linear error correcting codes [35] provide combinatorial designs on the d-dimensional cube graph Q_d. In particular, the Hamming code H_r provides extremal designs in the frequency ordering for Q_{2^r-1}, Q_{2^r} and Q_{2^r+1} which average many eigenspaces for their size. This result was generalized in [6] to find small combinatorial extremal designs in the frequency order on any Q_d with $d \not\equiv 2 \mod 4$.

Polyhedral Structure. From Definition 1, it is not obvious that there is further structure on the graphical designs of a graph. In [6], R. R. Thomas and I connected graphical designs for regular graphs to the *eigenpolytopes* [22] of a graph through Gale duality [16, 24, 48], which is a consequence of oriented matroid duality. This proves the existence of k-graphical designs with positive quadrature weights for any k, which answered an open question of [43] and sparked the work in [44]. This connection organizes graphical designs on the faces of eigenpolytopes and provides an alternative method to compute graphical designs using polyhedral software such as Polymake [20]. In [5], D. Shiroma and I extended the Gale duality framework to undirected positively weighted graphs. Using Gale duality on families of graphs, R. R. Thomas and I classified all graphical designs of cocktail party graphs and the positively weighted extremal designs in the frequency order for cycles and Q_d for $d \equiv 2 \mod 4$ in [6].

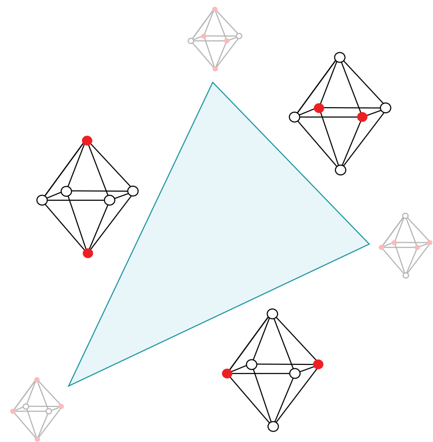

FIGURE 3. Positively weighted 2-graphical designs of the octohedral graph arranged on the corresponding eigenpolytope faces.

Computational Complexity. It is important to understand the computational complexity of graphical designs if we are to find efficient algorithms for practical applications. To this end, D. Shiroma and I proved the following are hard problems: determining if a simple bound arising from the eigenpolytope connection is tight, and counting the number of support-minimal positively weighted graphical designs [5]. To prove these results, we show that the spectral information of positively weighted graphs is sufficiently general, which we then use to show the universality of eigenpolytopes. That is, any combinatorial polytope can appear as an eigenpolytope of a positively weighted graphs. These are each a type of inverse eigenvalue problem [7], a widely studied class of problems due to their utility in applications.

Future Directions. I outline below several directions for future work.

Algorithms and Complexity. To date, [5] has the only complexity results for graphical designs. In particular, the complexity of finding support-minimum or support-minimal graphical designs is unknown. We expect that these problems will not be polynomial-time, as they are closely related to the known hard problems of finding a smallest circuit in a matroid [47], finding a sparsest nonzero vector in a subspace [9, 36], and integer programming [28], as well as the general convex hull problem [2], which is open but widely expected to be NP-hard. In this case, a next step is to find efficient approximations or relaxations of the graphical design problem. Possibilities include determining the approximation ratio of the linear programming relaxation, defining graphical designs through low-rank approximations of the graph operator, and loosening the condition in Equation (1) to $\sum_{s\in S} a_s\varphi(s) \approx \frac{1}{n}\sum_{i=1}^{n}\varphi(s)$ using a least-squares relaxation or other methods.

Applications. Preliminary connections to graph sampling on regular graphs were established in [44]. I plan to explore deeper connections to graph sampling for more general graphs. We are optimistic that graphical designs are connected to spectral clustering and effectively sample graphs in a quantifiable way. This can be further applied to sampling on a manifold through existing techniques in manifold learning that rely on approximations by graphs. Graphical designs may also be able to improve the analysis of distributive algorithms, where one tries to optimization a function over the nodes of a network using only local information from a sample [32, 41]. In this manner, graphical designs may provide more efficient combinatorial optimization computations.

Representation Theory and Symmetry. Every element in the orbit of a graphical design under $\mathrm{Aut}(G)$ is also a graphical design, and [22] shows that $\mathrm{Aut}(G)$ maps into the symmetry groups of eigenpolytopes. This begs the question, can we speed up computations to find a graphical design by exploiting the symmetry of a graph? Symmetry reduction [18, 29, 46] has been successful in many applications of semi-definite programming, including polynomial optimization [18, 40], coding theory [21, 30, 42], and some graph problems [25, 29]. Particularly if the given graph has additional structure and symmetry, such as a Cayley graph, we are optimistic that symmetry reduction techniques can be applied to the graphical design problem for faster computations.

Analysis. The definition of graphical designs is rooted in classical quadrature rules, but beyond that there is no further verification of how the two concepts relate. It would speak to the legitimacy of graphical designs as a method for numerical integration and sampling if the limiting behavior of graphical designs on certain graphs recovers natural quadrature rules on known smooth domains. Examples of such graph families and limiting domains include the path graph P_n and the line segment $[-1,1]$, a grid graph $(P_n)^{\square m}$ and the region $[-1,1]^m$, the cycle graph C_n and the circle, and triangulations of a manifold and the manifold itself.

Combinatorics. Simplicial complexes generalize graphs by allowing certain types of higher dimensional edges, providing a natural place to generalize graphical designs. The additional structure encodes higher-order interactions, which can be crucial for complex data such as in neural networks [10]. Simplicial complexes are also used to triangulate manifolds [31] and point configurations [11]. A first step is to define a sensible analogue of graphical designs for simplicial complexes. There are several existing combinatorial Laplacians for simplicial complexes to consider [8, 14, 15, 17, 26, 34, 45]. Computational experiments will play a key role in deciding and illustrating these choices. It will then be important to show that the resulting designs have desirable properties for numerical integration, sampling, or other applications.

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