Diversification and Mean–Variance Optimization

FINM 36700 – Portfolio & Risk Management (Lecture 1 TA Review)

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1 Diversification: Two Assets

Let two risky assets have means μ_1, μ_2, standard deviations σ_1, σ_2, and correlation $\rho \in [-1, 1]$.
Let the weight on asset 1 be $w \in \mathbb{R}$ and on asset 2 be $1 - w$.
The portfolio return $r_p = wr_1 + (1 - w)r_2$ has:

$$\text{(Linear additivity of mean)} \quad \mu_p = \mathbb{E}[r_p] = w\mu_1 + (1 - w)\mu_2, \tag{1}$$

$$\text{(Quadratic aggregation of risk)} \quad \sigma_p^2 = \mathrm{Var}(r_p) = w^2\sigma_1^2 + (1 - w)^2\sigma_2^2 + 2w(1 - w)\rho\,\sigma_1\sigma_2. \tag{2}$$

Risk is sub-additive (diversification). Since $\rho \le 1$,

$$\sigma_p^2 \;\le\; w^2\sigma_1^2 + (1 - w)^2\sigma_2^2 + 2w(1 - w)\,\sigma_1\sigma_2 = \big(w\sigma_1 + (1 - w)\sigma_2\big)^2,$$

hence

$$\sigma_p \;\le\; w\sigma_1 + (1 - w)\sigma_2,$$

with equality if and only if $\rho = 1$ (or $w \in \{0, 1\}$).
Returns add linearly.
Risk adds sub-additively unless assets are perfectly positively correlated. (Discount on portfolio vol)

Correlation cases

- $\rho = 1$ **(co-movement):** $\sigma_p = |w\sigma_1 + (1-w)\sigma_2|$. No diversification benefit. The risk adds linearly.
- $\rho = 0$ **(orthogonal shocks):** $\sigma_p^2 = w^2\sigma_1^2 + (1-w)^2\sigma_2^2$. Risk mixes as a *strictly convex* function of w unless $\sigma_1 = \sigma_2$.
- $\rho < 1$ **(generic diversification):** strict sub-additivity $\sigma_p < w\sigma_1 + (1-w)\sigma_2$ for $w \in (0,1)$. The σ_p curve is bowed in.
- $\rho = -1$ **(perfect hedge):**

$$\sigma_p \;=\; |\, w\sigma_1 - (1-w)\sigma_2 \,|, \quad \Rightarrow \quad \sigma_p = 0 \text{ at } \; w^\star = \frac{\sigma_2}{\sigma_1 + \sigma_2}.$$

This yields a *riskless* portfolio from two risky assets (interesting and rare in practice). Its expected return is $\mu_p^\star = w^\star\mu_1 + (1-w^\star)\mu_2$.
This is not as exciting it is almost never the case that we have correlation = -1. Even if we did, the mean excess returns is probably 0 (Not necessarily an arbitrage)

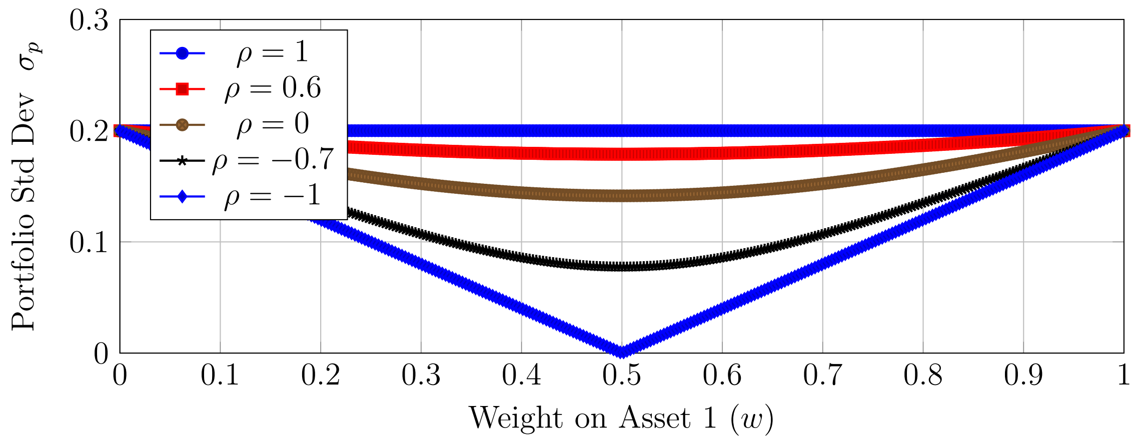


Figure 1: Two-asset diversification: σ_p vs weight w for several correlations.

2 n Assets: Allocation, Decomposition, and Limits

Let $\boldsymbol{w} \in \mathbb{R}^n$ with $\boldsymbol{w}^\top \mathbf{1} = 1$, means $\boldsymbol{\mu} \in \mathbb{R}^n$, and covariance $\Sigma \in \mathbb{R}^{n\times n}$.

$$\mu_p = \boldsymbol{w}^\top\boldsymbol{\mu}, \qquad \sigma_p^2 = \boldsymbol{w}^\top\Sigma\boldsymbol{w}.$$

Equal-weight portfolio and the covariance decomposition

For $w_i = \frac{1}{n}$, define the average variance $\overline{\sigma^2} = \frac{1}{n}\sum_i \sigma_i^2$ and the average covariance $\overline{\sigma_{ij}} = \frac{1}{n(n-1)}\sum_{i\neq j}\sigma_{ij}$. Then

$$\sigma_p^2 = \frac{1}{n}\,\overline{\sigma^2} + \frac{n-1}{n}\,\overline{\sigma_{ij}} \quad \xrightarrow[n\to\infty]{} \quad \overline{\sigma_{ij}}.$$

Idiosyncratic risk vanishes at rate $1/n$. What remains is *systematic (average covariance) risk*.

Note: Idiosyncratic risk can be eliminated through diversification, and is the diversifiable part of portfolio variance. Systematic risk cannot be eliminated through diversification.

Note: In the optimal portfolio, the marginal risk is the covariance with other assets in my portfolio. It is the important part of the risk. The volatility by itself is not the most important.

Uniform σ and uniform ρ. If all $\sigma_i = \sigma$ and all pairwise correlations $\rho_{ij} = \rho$,

$$\sigma_p^2 = \sigma^2\left(\frac{1}{n} + \frac{n-1}{n}\rho\right) \quad \Rightarrow \quad \sigma_p \approx \sigma\sqrt{\rho} \text{ as } n \to \infty.$$

Zero-variance condition (equicorrelation). This equals 0 iff $\rho = -\frac{1}{n-1}$. Examples: $n = 2 \Rightarrow \rho = -1$; $n = 3 \Rightarrow \rho = -\frac{1}{2}$; $n = 5 \Rightarrow \rho = -\frac{1}{4}$.

Note: The correlation required to construct a riskless portfolio approaches 0 as the number of assets grows infinitely. In practice, with a large number of assets, a nearly riskless portfolio can be achieved if correlation = 0. So we can say that the cross correlation to diversify a portfolio of n-assets is a function of the number of assets.

Correlation regimes

- **$\rho = 1$:** No diversification. $\sigma_p = \sigma$ for equal weights.
- **$\rho = 0$:** $\sigma_p = \sigma/\sqrt{n}$; idiosyncratic risk diversifies away.
- **$0 < \rho < 1$:** diversification helps but hits a floor $\sigma\sqrt{\rho}$ as $n \to \infty$.
- **$\rho < 0$:** Stronger risk reduction. a riskless portfolio exists only if $\rho \leq -\frac{1}{n-1}$ (equicorrelation case).

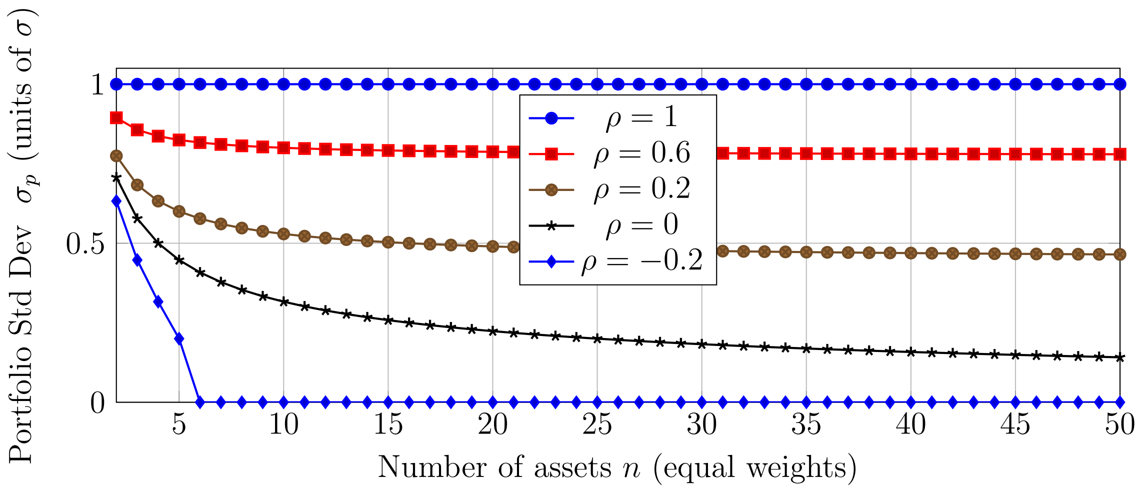


Figure 2: Equal-weight diversification: σ_p/σ vs n for several ρ. As $n \to \infty$, the limit is $\sqrt{\rho}$ (floors at 0 when $\rho \leq 0$).

3 From Diversification to the Mean–Variance Frontier

Diversification $\to$ frontier shape. Covariance drives the bowing-in of the feasible set. The efficient frontier is the upper envelope of variance-minimizers at each μ_p.

3.1 Mean Variance Optimization

- In mean-variance optimization, we aim to reduce risk while maintaining high returns. In this context, we use variance (or volatility) as a proxy for risk. Thus, the optimization involves two variables.
- When combining "n" assets, we can create a mean-variance frontier, represented as a "parabola."
- Every efficient portfolio is a combination of two funds (GMV and origin-tangency) in the no-rf case. With rf, everyone mixes rf and the tangency portfolio.
- The parabola contains means higher than those of any individual asset due to the ability to short assets with lower returns and go long on assets with higher returns.
- A mean-variance investor seeks to be on the upper part of the frontier, where the variance (and volatility) is minimized for each level of mean return.

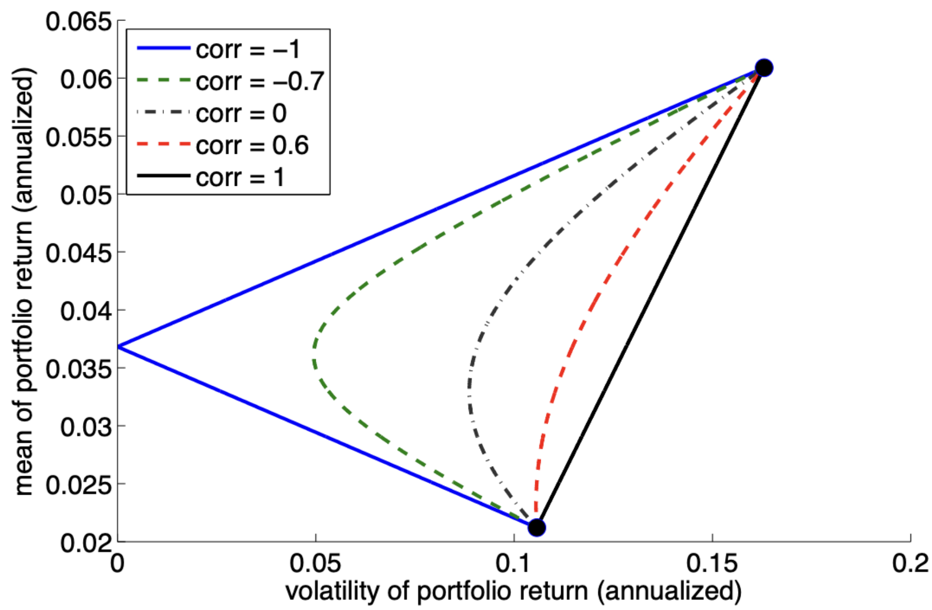


Figure 3: Mean-volatility space of diversification between two assets.

3.2 The risk set and the efficient frontier

Consider n risky assets with expected return vector $\boldsymbol{\mu} \in \mathbb{R}^n$ and covariance matrix $\Sigma \succ 0$. A portfolio with weights $\boldsymbol{w}$ (budget $\boldsymbol{w}^\top \mathbf{1} = 1$) has

$$\mu_p = \boldsymbol{w}^\top \boldsymbol{\mu}, \qquad \sigma_p^2 = \boldsymbol{w}^\top \Sigma \boldsymbol{w}.$$

As $\boldsymbol{w}$ ranges over the simplex, the set of attainable (σ_p, μ_p) is convex. The *efficient frontier* is the upper envelope of portfolios with minimum variance for a given μ_p.

"Marginal risk = covariance." In the optimal portfolio, assets are penalized by their *covariance with the portfolio*, not their standalone σ. This is a reason why low-Sharpe diversifiers can still earn weight.

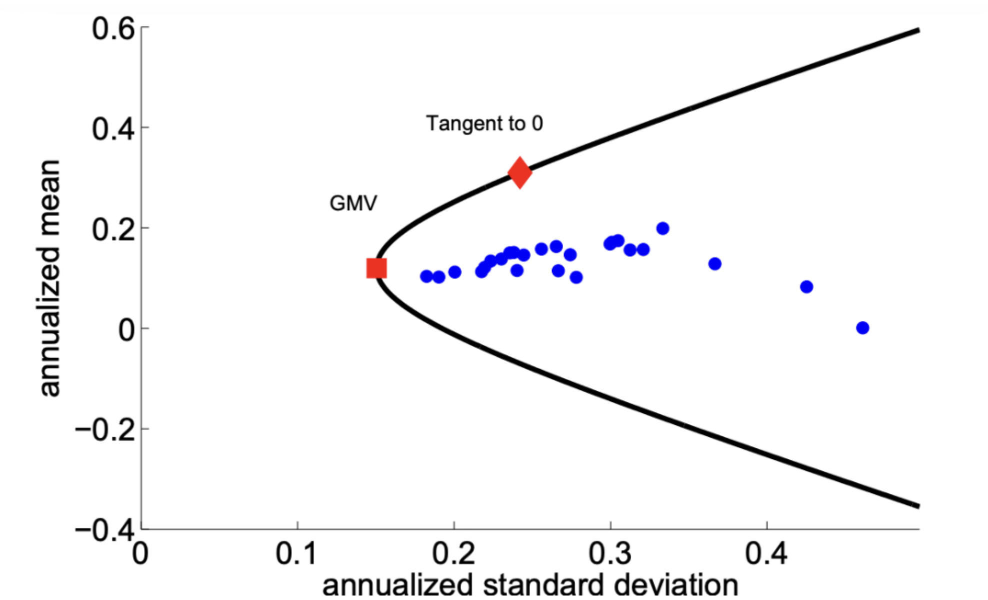


Figure 4: Efficient frontier (no risk-free asset): bowed curve. Special points GMV and "origin-tangency."

3.3 Mean–variance program (no risk-free asset)

For any target mean μ_p,

$$\min_{\boldsymbol{w}} \ \boldsymbol{w}^\top \Sigma \boldsymbol{w} \quad \text{s.t.} \quad \boldsymbol{w}^\top \boldsymbol{\mu} = \mu_p, \ \boldsymbol{w}^\top \mathbf{1} = 1. \tag{3}$$

This convex constraint set leads to a straightforward optimization, where weights are characterized by a first-order solution.

Lagrangian first-order conditions yield

$$\boldsymbol{w}^* = \alpha\, \Sigma^{-1} \boldsymbol{\mu} + \beta\, \Sigma^{-1} \mathbf{1}.$$

Define the handy scalars

$$A = \mathbf{1}^\top \Sigma^{-1} \mathbf{1}, \qquad B = \mathbf{1}^\top \Sigma^{-1} \boldsymbol{\mu}, \qquad C = \boldsymbol{\mu}^\top \Sigma^{-1} \boldsymbol{\mu}, \qquad \Delta = AC - B^2 > 0.$$

Solving for (α, β) under the constraints gives the frontier in closed form:

$$\sigma_p^2(\mu_p) \;=\; \frac{A\, \mu_p^2 - 2B\, \mu_p + C}{\Delta},$$

a parabola in mean–variance space. Every efficient portfolio is a linear combination of two special MV portfolios:

$$\boldsymbol{w}^* = \delta\, \boldsymbol{w}_t + (1-\delta)\, \boldsymbol{w}_v, \qquad \delta \in \mathbb{R},$$

(*two-fund separation* without r_f), where

$$\boldsymbol{w}_v = \frac{\Sigma^{-1}\mathbf{1}}{\mathbf{1}^\top \Sigma^{-1}\mathbf{1}} \quad \text{(GMV)}, \qquad \boldsymbol{w}_t = \frac{\Sigma^{-1}\boldsymbol{\mu}}{\mathbf{1}^\top \Sigma^{-1}\boldsymbol{\mu}} \quad \text{(origin-tangency)}.$$

Useful GMV facts: $\mu_v = \frac{B}{A}$ and $\sigma_v^2 = \frac{1}{A}$.

Therefore, if an investor wants to achieve a higher expected return than the tangency portfolio's return, they should go long on the tangency portfolio with more than 100% of their capital and short the minimum variance portfolio

Interpretation. $\boldsymbol{w}_v$ buys the cheapest variance per unit of budget. The direction of $\Sigma^{-1}\boldsymbol{\mu}$ maximizes slope through the origin. Any target μ_p on the efficient arc is an interpolation between the two.

As an asset manager, if the markets are frictionless and we have the best forecasts for the mean, the efficient frontier is the menu. If you want to be of the top right of the curve, buy more tangency portfolio and short GMV. These two products offer the best part of the entire menu.

3.4 Formula for Tangency Portfolio

The tangency portfolio allocates more weight to assets that:

- 1. Have higher mean returns.
- 2. Have lower volatility (variance).
- 3. Have lower covariance with other assets.

Points (1) and (2) relate directly to Sharpe ratios: intuitively, the higher an asset's Sharpe ratio, the more you should hold of that asset.
Point (3), however, highlights the importance of covariance. Even an asset with a poor Sharpe ratio can be valuable in a portfolio if it has a low correlation with other assets. This asset may be optimized with a higher weight than assets with higher Sharpe ratios but higher correlations.

3.5 Adding a risk-free asset

Let r_f be available and write excess returns $\tilde{\boldsymbol{\mu}} = \boldsymbol{\mu} - r_f \mathbf{1}$. Allocate $\boldsymbol{w}$ to risky assets. The residual goes to r_f (so we *drop* $\boldsymbol{w}^\top \mathbf{1} = 1$ in the risky block). For any target excess mean $\tilde{\mu}_p$,

Why excess returns? They align payoffs to the opportunity cost r_f and make the tangency solution analytic and economically interpretable (Sharpe as slope).

$$\min_{\boldsymbol{w}} \ \boldsymbol{w}^\top \Sigma \boldsymbol{w} \quad \text{s.t.} \quad \boldsymbol{w}^\top \tilde{\boldsymbol{\mu}} = \tilde{\mu}_p. \tag{4}$$

FOCs give $\boldsymbol{w}^* = \lambda \Sigma^{-1} \tilde{\boldsymbol{\mu}}$. The *unique* risky portfolio is the **tangency** portfolio

$$\boldsymbol{w}_t \ = \ \frac{\Sigma^{-1}\tilde{\boldsymbol{\mu}}}{\mathbf{1}^\top \Sigma^{-1} \tilde{\boldsymbol{\mu}}} \quad \text{(normalized so that } \boldsymbol{w}_t^\top \mathbf{1} = 1).$$

Any efficient allocation is a mix of $\boldsymbol{w}_t$ and r_f:

$$\text{Hold } \alpha \text{ in } \boldsymbol{w}_t \text{ and } (1-\alpha) \text{ at } r_f \quad \Rightarrow \quad \mu_p = r_f + \alpha\, \boldsymbol{w}_t^\top \tilde{\boldsymbol{\mu}}, \quad \sigma_p = |\alpha| \sqrt{\boldsymbol{w}_t^\top \Sigma \boldsymbol{w}_t}.$$

The efficient set becomes a *line* through $(0, r_f)$, the **Capital Market Line** (CML):

$$\mu_p \ = \ r_f + \mathrm{SR}^* \, \sigma_p, \qquad \mathrm{SR}^* \ = \ \sqrt{\tilde{\boldsymbol{\mu}}^\top \Sigma^{-1} \tilde{\boldsymbol{\mu}}} \ = \ \frac{\boldsymbol{w}_t^\top \tilde{\boldsymbol{\mu}}}{\sqrt{\boldsymbol{w}_t^\top \Sigma \boldsymbol{w}_t}}.$$

Two-fund separation with r_f. All MV investors hold only two funds: the risk-free asset and the tangency portfolio. Individual risk appetite sets the mix (lend vs borrow at r_f).

Borrow/lend: If you want more (less) risk than $\boldsymbol{w}_t$, lever up (lend) along the CML. All points share the *same* maximal Sharpe.

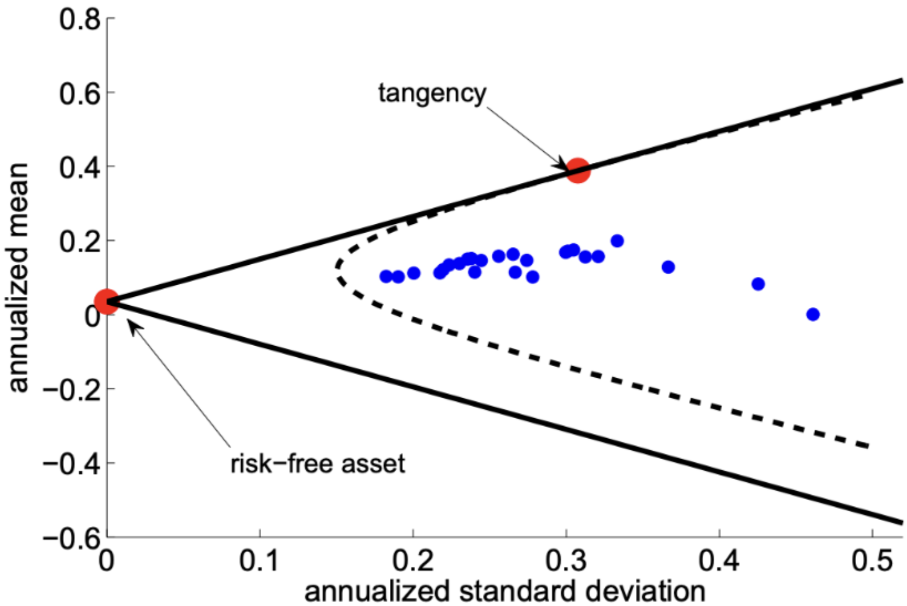


Figure 5: With a risk-free asset: efficient set is the CML. The risky mix is uniquely $\boldsymbol{w}_t$.

3.6 MV Portfolios

No r_f: Write the Lagrangian for (3), differentiate, show $\boldsymbol{w}^* = \alpha\Sigma^{-1}\boldsymbol{\mu} + \beta\Sigma^{-1}\mathbf{1}$, then solve for (α, β) to get the parabola $\sigma_p^2(\mu_p)$ and the representation $\boldsymbol{w}^* = \delta\boldsymbol{w}_t + (1-\delta)\boldsymbol{w}_v$.

With r_f: Write the Lagrangian for (4). Show $\boldsymbol{w}^* \parallel \Sigma^{-1}\tilde{\boldsymbol{\mu}}$; normalize to obtain $\boldsymbol{w}_t$. Draw the CML through $(0, r_f)$ tangent to the risky frontier at $\boldsymbol{w}_t$.

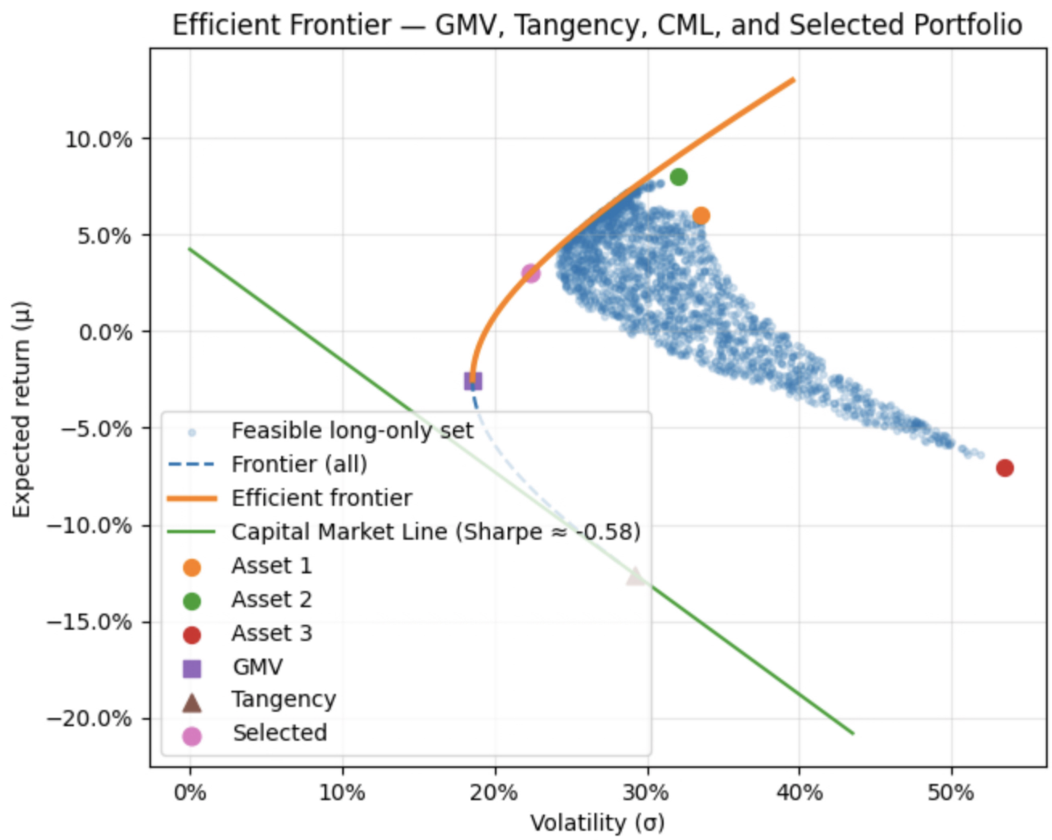
3.7 Will Everyone Choose the Same Portfolio in Mean-Variance Optimization?

- If a risk-free rate is unavailable, investors will choose different portfolios based on their risk preferences.
- However, if a risk-free rate is available, all investors should hold (1) the tangency portfolio and (2) the risk-free rate to create any portfolio with an expected return that maximizes the Sharpe Ratio.
- If an investor seeks a return greater than the tangency portfolio's expected return, they should go long on the tangency portfolio with more than 100% of their assets and short the risk-free rate.

- Conversely, if an investor seeks a lower return and lower risk, they should allocate less than 100% to the tangency portfolio and invest the remainder in the risk-free rate.

3.8 Tangency on Inefficient Frontier

Does it matter if the tangency is on the inefficient (bottom) portion of the frontier?



No. We're not saying the tangency portfolio has any special significance beyond its geometry. It makes for a useful point (along with the risk-free rate or GMV) in building out the frontier. If the tangency is on the inefficient portion, then investors will always choose to short it, which is fine. (The tangency weights already involve many short positions, so shorting the tangency still just leads to a mix of long-and-short positions.)

The solution to the optimization is numerically very sensitive to the specifications of mu and Sigma. We may find that the total-return tangency is on the efficient (upper) portion of the frontier. But subtracting the risk-free rate is just enough of a shift down to cause the excess-return tangency to be on the inefficient (lower) portion of the frontier.