A Note About Numerical Modification to WAVEWATCH III in the Presence of Sea Ice

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1 Wave Energy Equation and the Time Splitting Numerical Solution in the Original WAVEWATCH III

The equation describing the wave action density spectrum N is

$$\frac{D}{Dt}N(\mathbf{x},t,\theta) = \mathscr{S}_{\text{no ice}} + \mathscr{S}_{\text{ice}}, \tag{1}$$

where the total derivative of N is balanced by net sources and sinks for N. The net sources are composed of the effects of non sea ice wave-interactions $\mathscr{S}_{\text{no ice}}$ and sea ice-wave interactions $\mathscr{S}_{\text{ice}}$.

After rewriting Eq. 1 in partial differential form, the numerical solution of the time evolution in WAVEWATCH III proceeds with time splitting, beginning with a series of four steps to compute the spatial propagation and intra-spectral parts that leave only

$$\frac{\partial}{\partial t}N(\mathbf{x},t,\theta) = \mathscr{S}_{\text{no ice}} + \mathscr{S}_{\text{ice}}. \tag{2}$$

But then WAVEWATCH III splits the solution to Eq. 2 into a further pair of steps that seperates the two net source terms on the right-hand-side, with the justification, according to the manual (page 138), that "attenuation and scattering in the ice can be very strong (although they are linear)".

In reality wave energy in sea ice is damped, with some scattering by sea ice floes that can redirect wave propagation. Damping is strongest for high-frequency ($f \gtrsim 0.2$ Hz) waves, commonly known as wind waves. Many studies have described swell in sea ice as low-frequency ($f \lesssim 0.2$ Hz) waves that develop in open ocean areas and survive propagation through the marginal ice zone (e.g., Squire 2020), where the high-frequency waves are dissipated within a few $\sim$10s of km (Ardhuin et al 2020). However, observations and theory show that wind waves can be generated in ice-covered seas, when concentrations are low enough to permit substantial fetch (Squire and Moore, 1980; Masson and Leblond, 1989; Cooper et al 2022).

1.1 Time Spliting with Sea Ice Attenuation

When wind energy input is considerable and sea aice is also strongly attenuating, the split solution method employed by WAVEWATCH III can be highly inaccurate, especially for large global time steps ($\Delta t \sim 1800$ s) typical of coarse resolution models. To understand why this split solution is innaccurate when sea ice is damping, it is helpful to neglect any scattering for now (section 2 will reconsider scattering) and rewrite Eq. 2 as

$$\frac{\partial N}{\partial t} = \mathscr{S}_{\text{no ice}} + \beta N, \tag{3}$$

where $\beta = -a_{\text{ice}} c_g \alpha$ with a_{ice} as the sea ice fractional coverage, c_g as the amplitude of the group velocity and α as the sea ice attenuation coefficient. Finite differencing with time-splitting becomes

$$\begin{aligned} \frac{N^* - N^n}{\Delta t} &= \mathscr{S}_{\text{no ice}}(N^n) + \varepsilon \left[N^* D(N^n) - N^n D(N^n) \right] && (4) \\ N^{n+1} &= N^* e^{\beta \Delta t}, && (5) \end{aligned}$$

where D is the derivative of $\mathscr{S}_{\text{no ice}}$ with respect to N and ε sets the accuracy of the solution and controls whether the effect of the higher-order terms are damping or amplifying. In Eq. 4, $\varepsilon = 0.5$ would give a second-order accurate solution in Δt. However, presently this is not done in WW3. The manual says that $\varepsilon = 1$ is more "appropriate for the large time steps in the equilibrium range of the spectrum ... and it results in a much smoother integration of the spectrum". However, the model's code has $\varepsilon = 1$ only if $D < 0$ and otherwise $\varepsilon = 0$. We'll see below examples of how ε is an important factor that controls accuracy and numerical damping.

The intermediate approximation to the spectrum N^* is

$$N^* = N^n + \frac{\Delta t \mathscr{S}_{\text{no ice}}(N^n)}{1 - \varepsilon D(N^n) \Delta t}. \qquad (6)$$

Though we have primarily raised an alarm and discussed conditions of wind input and sea ice attenuation, we note that the equations in this subsection are general for all non sea ice source terms in WAVEWATCH III and for any sea ice attenuation scheme as well as the simple diffusive scattering option (IS1). However, there are some combinations of sources that might not result in significant inaccuracy. For instance wind input in the presence of strong damping by viscous mud would likely have reasonable accuracy.

2 Comparing Time Split and Merged Numerical Solutions

In partially sea ice covered seas, winds or other factors can generate wave energy in grid cells where sea ice is simultaneously attenuating wave amplitude, in which case in WAVEWATCH III the first step above would raise N and the second would dampen it. These conditions would always result in $\varepsilon = 0$ because $D > 0$. This reduces the accuracy of the time-split solution to first-order and generally over damps the solution. If the wave model is then coupled to a sea ice fracture model at this point, the wave amplitude would be smaller and sea ice would fracture less. How much it matters depends on wave and sea ice conditions and the time step.

If instead we consider the steps merged — eliminating time-splitting — then when sea ice is attenuating β is negative so $D + \beta$ is less positive than D alone. We'll see that eliminating time splitting by merging the steps is nearly always more accurate than splitting the solution into two steps.

Next we consider numerical solutions to four cases in partial sea ice cover, chosen to illustrate plausible conditions where there is sufficient open water and winds to generate wind-waves and enough sea ice to attenuate wave amplitudes. For simplicity, we'll consider a single a single direction and single frequency, though numerical solutions to N in /ww3 discretize the direction and frequency, so N is a matrix at every point in space and time $(\mathbf{x}, t)$.

We'll continue to neglect any scattering for these cases, and let $\mathscr{S}_{\text{no ice}} = \gamma N$, where γ is an arbitrary number representing input that we will vary among the cases. With this simplification, Eq. 3 becomes

$$\frac{\partial N}{\partial t} = (\gamma + \beta) N, \tag{7}$$

which has an exact solution that can be compared with numerical solutions. We'll consider two numerical solutions. First, the original time splitting method in WAVEWATCH III given by Eqs. 4 and 5 and a new solution without time splitting and given our simple assumption of linear ice and no ice sources:

$$N^{n+1} = N^n + \frac{\Delta t\,(\gamma+\beta)\,N^n}{1-\varepsilon\,(\gamma+\beta)\,\Delta t}. \tag{8}$$

We'll consider waves with a period of $T = 5$ s, 50% sea ice cover, and $\alpha = 2 \times 10^{-4}$ (a typicall attenuation rate for WAVEWATCH III with the IC4 option and a 5 s wave). We let the $c_g = 0.8T$, so $\beta = -4 \times 10^{-4}\ \text{s}^{-1}$. We'll find solutions to the three cases as a function of Δt ranging from 20 to 1800 s.

The four cases we consider have

A. Energy input only so waves are growing as there is no sea damping (i.e., $\beta = 0$).

B. No input (i.e., $\gamma = 0$) so there is only wave energy damping by sea ice.

C. Energy input a little less than damping by sea ice (i.e., $\gamma = -0.75\beta$), so the net effect is damping,

D. Energy input a little larger than damping by sea ice (i.e., $\gamma = -1.25\beta$), so the net effect is growth,

The four cases are available in the Python Notebook WW3_improvednumerics_version3.ipynb and will eventually be described here.

3 The Merged Numerical Solutions with Floe-Size Dependent Scattering

A more general way of writing the merged solution to Eq. 2 is (need to find a place to say this and am temporarily doing it here)

$$N^{n+1} = N^n + \frac{\Delta t\,(\mathscr{S}_{noice} + \mathscr{S}_{ice})}{1-\varepsilon\,(D+\beta)\,\Delta t}. \tag{9}$$

Presently, WAVEWATCH III has an option to include more complex floe-size dependent, non isotropic scattering in sea ice (option IS2). In this case, it would be wise to exclude this type of scattering from $\mathscr{S}_{noice}$ and add a third source term in Eq. 2 that is proportional to a scattering matrix representing the scattered direction for each discrete direction of propagation. The influence of this kind of scattering is solved numerically by expanding N in the eigenvalues of the scattering matrix. This solution is best solved as its own step, after merging steps with non sea ice and sea ice attenuation sources.