

## Eigenvalues and Eigenvectors

### 1. State-Space Models

Dynamic behaviour of systems can be modeled by differential equations. In general, nonlinear differential equations are required to model actual dynamic systems. In many cases, however, linear approximations can be obtained to describe the dynamic behaviour of such systems in the vicinity of some nominal operating point. The mathematical models of some simple linear dynamic systems are described below.

#### 1.1 A Series RLC Electrical Circuit

Figure 1 shows a series circuit connecting a resistance  $R$ , an inductance  $L$  and a capacitance  $C$ . A voltage  $v_i(t)$  is applied to the circuit which results a loop current  $i(t)$ . The dynamic response of this circuit,  $i(t)$  can be described by a differential equation using **Kirchhoff's voltage law**. According to Kirchhoff's law, voltage applied in a closed loop at any instant must be equal to the sum of all voltage drops around that loop.

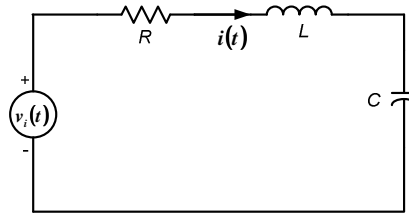


Figure E 1.1 A series RLC circuit

Therefore,

$$v_i(t) = v_R(t) + v_L(t) + v_C(t) \quad (\text{E } 1.1)$$

Where

$v_R(t)$  = voltage drop in the resistance,  $R$  at time  $t$   
 $v_L(t)$  = voltage drop in the inductance,  $L$  at time  $t$  and  
 $v_C(t)$  = voltage drop in the capacitance,  $C$  at time  $t$ .

$$v_i(t) = Ri(t) + L \frac{di(t)}{dt} + \frac{1}{C} \int i(t) dt \quad (\text{E } 1.2)$$

Differentiate Eqn. (E.2) with respect to time. The mathematical model of the system is:

$$\frac{dv_i(t)}{dt} = R \frac{di(t)}{dt} + L \frac{d^2i(t)}{dt^2} + \frac{i(t)}{C} \quad (\text{E } 1.3)$$

#### 1.2 An Automobile Suspension System

The spring-mass-damper system shown in Figure E 1.2 can be used to represent a simplified suspension system for each wheel of an automobile.

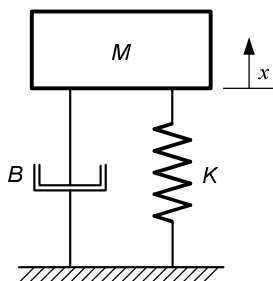


Figure E 1.2 An automobile suspension system

A force applied to the mass,  $M$  in the direction of  $x$  will be balanced by three other forces in the opposite direction of  $x$ . The balancing forces are the gravitational force on the mass, the damping force due to the shock absorber and the spring force.

The damping force is:  $F_B(t) = B \frac{dx(t)}{dt}$  (E 1.4)

The spring force is:  $F_K(t) = Kx(t)$  (E 1.5)

The gravitational force is:  $Mg$

Applying Newton's Second Law we can write

$$-Mg - F_K(t) - F_B(t) = M \frac{d^2 x(t)}{dt^2}$$

or,  $M \frac{d^2 x(t)}{dt^2} + B \frac{dx(t)}{dt} + Kx(t) = -Mg$  (E 1.6)

### 1.3 A Two-tank Liquid Level System

Consider the two-tank liquid level system shown in Figure E 1.3.

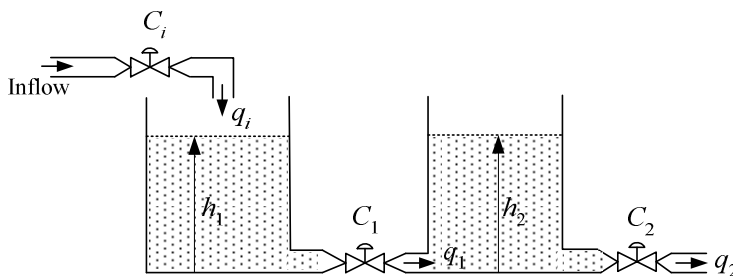


Figure E 1.3 A two-tank liquid level system

Assume linearized flow rates. Therefore, the flow rates can be considered proportional to the corresponding liquid heights. If  $R_1$  and  $R_2$  are constant resistance factors of the control valves  $C_1$  and  $C_2$  then

$$q_1(t) = \frac{h_1(t) - h_2(t)}{R_1} \quad (\text{E 1.7})$$

$$q_2(t) = \frac{h_2(t)}{R_2} \quad (\text{E 1.8})$$

If  $V_1(t)$  and  $V_2(t)$  are volumes of liquid in tanks 1 and 2 respectively and  $q_i(t)$  is the inflow in the first tank then

$$\frac{dV_1(t)}{dt} = \frac{d\{A_1 h_1(t)\}}{dt} = q_i(t) - q_1(t) \quad (\text{E 1.9})$$

$$\frac{dV_2(t)}{dt} = \frac{d\{A_2 h_2(t)\}}{dt} = q_1(t) - q_2(t) \quad (\text{E 1.10})$$

where  $A_1$  and  $A_2$  are respective surface areas of tank 1 and 2.

Substituting Eqn. (E 1.7) in Eqn. (E 1.9) we get

$$A_1 \frac{dh_1(t)}{dt} = q_i(t) - \left\{ \frac{h_1(t) - h_2(t)}{R_1} \right\} \quad (\text{E 1.11})$$

Substituting Eqns. (E 1.7) and (E 1.8) in Eqn. (E 1.10) we get

$$A_2 \frac{dh_2(t)}{dt} = \left\{ \frac{h_1(t) - h_2(t)}{R_1} \right\} - \frac{h_2(t)}{R_2} \quad (\text{E 1.12})$$

Eqns. (E 1.11) and (E 1.12) mathematically model the dynamic behaviour this two-tank liquid level system.

## 2. State-Space Representations

The dynamic behaviour of many physical systems can be modeled with the help of a set of  $n$  first-order differential equations involving  $n$  state variables. These  $n$  state variables completely define the internal behaviour of the system. The state variables are usually denoted as  $x_1(t), x_2(t), \dots, x_n(t)$ . The state model of a system is not unique, but depends on the choice of a set of state variables.

The general state-space model of a linear time-invariant system can be expressed as:

$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \\ \vdots \\ \dot{x}_n(t) \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} & \cdots & A_{1n} \\ A_{21} & A_{22} & \cdots & A_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ A_{n1} & A_{n2} & \cdots & A_{nn} \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_n(t) \end{bmatrix} + \begin{bmatrix} B_{11} & \cdots & B_{1m} \\ B_{21} & \cdots & B_{2m} \\ \vdots & \ddots & \vdots \\ B_{n1} & \cdots & B_{nm} \end{bmatrix} \begin{bmatrix} u_1(t) \\ u_2(t) \\ \vdots \\ u_m(t) \end{bmatrix} \quad (\text{E 2.1})$$

Eqn. (E 2.1) can be written in a concise form as

$$[\dot{x}(t)] = [A][x(t)] + [B][u(t)] \quad (\text{E 2.2})$$

where  $[A]$  is the  $n \times n$  state matrix and  $[B]$  is the  $n \times m$  input matrix.

It is important to note at this point that although the entire  $n$ -dimensional state vector  $[x(t)]$  characterizes the complete internal behaviour of a state-space system, it is usually impossible to directly measure or observe all  $n$  components of  $[x(t)]$ . The external output vector can be expressed as

$$\begin{bmatrix} y_1(t) \\ y_2(t) \\ \vdots \\ y_p(t) \end{bmatrix} = \begin{bmatrix} C_{11} & \cdots & C_{1n} \\ C_{21} & \cdots & C_{2n} \\ \vdots & \ddots & \vdots \\ C_{p1} & \cdots & C_{pn} \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_n(t) \end{bmatrix} \quad (\text{E 2.3})$$

Eqn. (E 2.3) written in a concise form as

$$[y(t)] = [C][x(t)] \quad (\text{E 2.4})$$

where  $[C]$  is the  $p \times n$  output matrix.

Eqns. (E 2.2) and (E 2.4) are the state-space representation of a linear, time-invariant system.

### 3. Transfer Function Matrices and Stability

Transfer function is utilized to determine the stability of a dynamic system. For an  $n$ -dimensional dynamic system the transfer function matrix can be found by applying Laplace transform to Eqns. (E 2.2) and (E 2.4). The Laplace transform of a vector is the vector of the Laplace transform of its elements. The Laplace transform of  $[x(t)]$  and  $[\dot{x}(t)]$  are as follows:

$$[X(s)] = L[x(t)] = \begin{bmatrix} L\{x_1(t)\} \\ L\{x_2(t)\} \\ \vdots \\ L\{x_n(t)\} \end{bmatrix} = \begin{bmatrix} X_1(s) \\ X_2(s) \\ \vdots \\ X_n(s) \end{bmatrix} \quad L[\dot{x}(t)] = \begin{bmatrix} L\{\dot{x}_1(t)\} \\ L\{\dot{x}_2(t)\} \\ \vdots \\ L\{\dot{x}_n(t)\} \end{bmatrix}$$

where

$$L\{\dot{x}_i(t)\} = sX_i(s) - x_i(0)$$

Therefore, the Laplace transform of  $[\dot{x}(t)] = [A][x(t)] + [B][u(t)]$  is  $s[X(s)] - [x_0] = [A][X(s)] + [B][U(s)]$

$$\text{or, } (s[I] - [A])[X(s)] = [B][U(s)] + [x_0] \quad (\text{E 3.1})$$

where  $[I]$  is the unit matrix.

The Laplace transform of  $[y(t)] = [C][x(t)]$  is  $[Y(s)] = [C][X(s)]$

Premultiplying both side of Eqn. (E 3.1) by  $(s[I] - [A])^{-1}$  we get

$$[X(s)] = (s[I] - [A])^{-1} [B] U(s) + (s[I] - [A])^{-1} [x_0]$$

The output in Laplace domain is

$$[Y(s)] = [C](s[I] - [A])^{-1} [B] U(s) + [C](s[I] - [A])^{-1} [x_0]$$

$$[Y(s)] = [G(s)] U(s) + [C](s[I] - [A])^{-1} [x_0]$$

where  $[G(s)] = [C](s[I] - [A])^{-1} [B]$  is called the transfer function matrix.

A transfer function matrix relates the transforms of the input and output vectors for zero initial conditions.

The inverse of matrix  $(s[I] - [A])$  can be expressed as

$$(s[I] - [A])^{-1} = \frac{\text{adj}(s[I] - [A])}{|s[I] - [A]|} \quad (\text{E 3.2})$$

Using Eqn. (E 3.2), the transfer function matrix can be written as

$$[G(s)] = \frac{[C] \text{adj}(s[I] - [A]) [B]}{|s[I] - [A]|} \quad (\text{E 3.3})$$

The dynamic system modeled by Eqns. (E 2.2) and (E 2.4) is stable if all the roots of the characteristic equation  $|s[I] - [A]| = 0$  lie on the left half of the s-plane. The roots of the characteristic equation,  $|s[I] - [A]| = 0$  are called the **eigenvalues** of the system coefficient matrix (system matrix)  $[A]$ . That means the eigenvalues of a stable system are all negative.

**Example:** Consider a 2-dimensional system with the following system and input matrices.

$$[A] = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix} \quad [B] = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad \text{and} \quad [C] = [1 \quad 0]$$

$$(s[I] - [A]) = \begin{bmatrix} s & -1 \\ 2 & s+3 \end{bmatrix}$$

$$(s[I] - [A])^{-1} = \frac{\text{adj}(s[I] - [A])}{|s[I] - [A]|} = \frac{1}{s^2 + 3s + 2} \begin{bmatrix} s+3 & 1 \\ -2 & s \end{bmatrix}$$

The transfer function of the system is

$$[G(s)] = [C](s[I] - [A])^{-1} [B]$$

$$[G(s)] = \frac{[1 \quad 0]}{s^2 + 3s + 2} \begin{bmatrix} s+3 & 1 \\ -2 & s \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \frac{s+3}{(s+1)(s+2)}$$

The denominator of the transfer function is

$$|s[I] - [A]| = s^2 + 3s + 2 = (s + 1)(s + 2)$$

The roots of the characteristic equation, i.e., the eigenvalues are -1 and -2 and therefore, the system is stable.

#### 4. Coordinate Transformation and Eigenvalues

It has been mentioned earlier that the state model of a dynamic system is not unique. The model depends on the choice of a set of state variables. It is often beneficial to define a new state vector  $[z]$  with the help of a coordinate transformation  $[x] = [T][z]$ .

The state model of the system described by  $[\dot{x}] = [A][x] + [B][u]$  and  $[y] = [C][x]$  will be transformed by this coordinate transformation in the following manner.

$$[T][\dot{z}] = [A][T][z] + [B][u] \quad (\text{E 4.1})$$

$$[y] = [C][T][z] \quad (\text{E 4.2})$$

Premultiply Eqn. (E 4.1) by  $[T]^{-1}$ .

$$[T]^{-1}[T][\dot{z}] = [T]^{-1}[A][T][z] + [T]^{-1}[B][u]$$

$$[\dot{z}] = [\hat{A}][z] + [\hat{B}][u]$$

The transformed system (with new state vector  $[z]$ ) is

$$[\dot{z}] = [\hat{A}][z] + [\hat{B}][u] \quad [y] = [\hat{C}][z]$$

where  $[\hat{A}] = [T]^{-1}[A][T]$  and  $[\hat{C}] = [C][T]$

The use of the new state vector (internal state variables) should not affect the stability (i.e., eigenvalues) and the input-output relation of the system. However, this can be verified by evaluating the characteristic equation and the transfer function matrix of the transformed system in the following manner.

Characteristic equation:

$$|s[I] - [\hat{A}]| = |s[I] - [T]^{-1}[A][T]| = |s[T]^{-1}[T][I] - [T]^{-1}[A][T]|$$

Because,  $[I]$  is the unit matrix we can write

$$|s[I] - [\hat{A}]| = |s[T]^{-1}[I][T] - [T]^{-1}[A][T]| = |[T]^{-1}(s[I] - [A])[T]|$$

Remember the properties of determinants.

$$|s[I] - [\hat{A}]| = |[T]^{-1}(s[I] - [A])[T]| = |[T]^{-1}|(s[I] - [A])|[T]|$$

$$|s[I] - [\hat{A}]| = |(s[I] - [A])|$$

So the characteristic equation remains unchanged.

Transfer function matrix:

The transfer function matrix of the transformed system is

$$\begin{aligned}
 [\hat{G}(s)] &= [\hat{C}][s[I] - [\hat{A}]]^{-1}[\hat{B}] = [C][T](s[I] - [T^{-1}[A][T]])^{-1}[T]^{-1}[B] \\
 &= [C][T]\{[T]^{-1}(s[I] - [A])[T]\}^{-1}[T]^{-1}[B] \\
 &= [C][T][T]^{-1}(s[I] - [A])^{-1}[T][T]^{-1}[B] \\
 &= [C](s[I] - [A])^{-1}[B] = [G]
 \end{aligned}$$

The transfer function remains unaffected by the transformation.

The question is, what kind of transformation would be helpful? The particular transformation that we are interested is the one for which  $[\hat{A}]$  would be a diagonal matrix. Why we want a coefficient matrix that is diagonal? To answer this question, let us have another look at the system state equations, particularly Eqn. (E 2.1).

$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \\ \vdots \\ \dot{x}_n(t) \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} & \cdots & A_{1n} \\ A_{21} & A_{22} & \cdots & A_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ A_{n1} & A_{n2} & \cdots & A_{nn} \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_n(t) \end{bmatrix} + \begin{bmatrix} B_{11} & \cdots & B_{1m} \\ B_{21} & \cdots & B_{2m} \\ \vdots & \ddots & \vdots \\ B_{n1} & \cdots & B_{nm} \end{bmatrix} \begin{bmatrix} u_1(t) \\ u_2(t) \\ \vdots \\ u_m(t) \end{bmatrix}$$

In order to solve for each state, we have to integrate each equation. Integrating the second term on the right-hand side of the equation should be done with little difficulty. However, the integration of the first term on the right-hand side cannot be done that easily. Because, we will have to wait for the determination of other state variables. Now consider a coefficient matrix,  $[A]$  with only diagonal elements, all other elements being zero.

$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \\ \vdots \\ \dot{x}_n(t) \end{bmatrix} = \begin{bmatrix} A_{11} & 0 & \cdots & 0 \\ 0 & A_{22} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & A_{nn} \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_n(t) \end{bmatrix} + \begin{bmatrix} B_{11} & \cdots & B_{1m} \\ B_{21} & \cdots & B_{2m} \\ \vdots & \ddots & \vdots \\ B_{n1} & \cdots & B_{nm} \end{bmatrix} \begin{bmatrix} u_1(t) \\ u_2(t) \\ \vdots \\ u_m(t) \end{bmatrix}$$

Each individual equation as shown in the matrix form can be expressed in terms of its self term only. Therefore, each equation can be integrated with little difficulty.

$$\begin{aligned}
 \dot{x}_1(t) &= A_{11}x_1(t) + B_{11}u_1(t) + B_{12}u_2(t) + \cdots + B_{1m}u_m(t) \\
 \dot{x}_2(t) &= A_{22}x_2(t) + B_{21}u_1(t) + B_{22}u_2(t) + \cdots + B_{2m}u_m(t) \\
 \dot{x}_3(t) &= A_{33}x_3(t) + B_{31}u_1(t) + B_{32}u_2(t) + \cdots + B_{3m}u_m(t) \\
 &\vdots \\
 \dot{x}_n(t) &= A_{nn}x_n(t) + B_{n1}u_1(t) + B_{n2}u_2(t) + \cdots + B_{nm}u_m(t)
 \end{aligned} \tag{E 4.3}$$

The set of Equations (E 4.3) represents a system model that contains only the self terms. All mutual terms are zero. This type of system is often called a decoupled system. It is relatively very straightforward to carry out mathematical analysis of a decoupled system.

We are, therefore, interested in a particular type of coordinate transformation that would transform our coupled system into a decoupled system. We have already discussed that the transfer function matrix and the characteristic equation would remain unchanged even after this transformation. Assume that this transformation can be achieved with the help of a transformation matrix  $[M]$ .

$$\text{Therefore, } [x] = [M][z] \text{ and } [\hat{A}] = [M]^{-1}[A][M] = [\Lambda] = \text{diag}\{\lambda_1, \lambda_2, \dots, \lambda_n\}$$

The characteristic equation of the system is

$$\begin{aligned} |s[I] - [A]| &= |s[I] - [\Lambda]| = \det\{\text{diag}[(s - \lambda_1), (s - \lambda_2), \dots, (s - \lambda_n)]\} \\ &= (s - \lambda_1)(s - \lambda_2) \cdots (s - \lambda_n) \end{aligned}$$

The elements of the diagonal matrix  $[\Lambda]$  are the roots of the characteristic equation

$$|s[I] - [A]| = |s[I] - [\Lambda]| = 0 \text{ and are therefore the eigenvalues of the system.}$$

Now the question is what would be the elements of such a transformation matrix  $[M]$ ?

$$[M] = \begin{bmatrix} m_{11} & m_{12} & \cdots & m_{1n} \\ m_{21} & m_{22} & \cdots & m_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ m_{n1} & m_{n2} & \cdots & m_{nn} \end{bmatrix}$$

$$[M]^{-1}[A][M] = \Lambda$$

$$[M][M]^{-1}[A][M] = [M][\Lambda]$$

$$[A][M] = [M][\Lambda]$$

$$\begin{bmatrix} A_{11} & A_{12} & \cdots & A_{1n} \\ A_{21} & A_{22} & \cdots & A_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ A_{n1} & A_{n2} & \cdots & A_{nn} \end{bmatrix} \begin{bmatrix} m_{11} & m_{12} & \cdots & m_{1n} \\ m_{21} & m_{22} & \cdots & m_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ m_{n1} & m_{n2} & \cdots & m_{nn} \end{bmatrix} = \begin{bmatrix} m_{11} & m_{12} & \cdots & m_{1n} \\ m_{21} & m_{22} & \cdots & m_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ m_{n1} & m_{n2} & \cdots & m_{nn} \end{bmatrix} \begin{bmatrix} \lambda_1 & 0 & 0 & 0 \\ 0 & \lambda_2 & 0 & 0 \\ 0 & 0 & \ddots & 0 \\ 0 & 0 & 0 & \lambda_n \end{bmatrix}$$

Equate columns on both sides. Equating the 2nd column we can write

$$\begin{bmatrix} A_{11} & A_{12} & \cdots & A_{1n} \\ A_{21} & A_{22} & \cdots & A_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ A_{n1} & A_{n2} & \cdots & A_{nn} \end{bmatrix} \begin{bmatrix} m_{12} \\ m_{22} \\ \vdots \\ m_{n2} \end{bmatrix} = \lambda_2 \begin{bmatrix} m_{12} \\ m_{22} \\ \vdots \\ m_{n2} \end{bmatrix}$$



In the same manner equating the  $i$ th column we can write

$$\begin{bmatrix} A_{11} & A_{12} & \cdots & A_{1n} \\ A_{21} & A_{22} & \cdots & A_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ A_{n1} & A_{n2} & \cdots & A_{nn} \end{bmatrix} \begin{bmatrix} m_{1i} \\ m_{2i} \\ \vdots \\ m_{ni} \end{bmatrix} = \lambda_i \begin{bmatrix} m_{1i} \\ m_{2i} \\ \vdots \\ m_{ni} \end{bmatrix} \quad (\text{E 4.4})$$

$$[A][m^i] = \lambda_i [m^i] \quad (\text{E 4.5})$$

$$\lambda_i [m^i] - [A][m^i] = 0$$

$$(\lambda_i [I] - [A])[m^i] = 0 \quad (\text{E 4.6})$$

Eqn. (E 4.6) can be solved to find the  $i$ th column of the transformation matrix  $[M]$ . Observe that associated with each eigenvalue there is a corresponding column or vector for the transformation matrix  $[M]$ . These vectors are called the eigenvectors of the system. A physical (geometric) interpretation of an eigenvector can be derived from Eqn. (E 4.5). The operation by the system matrix on an eigenvector is equivalent of changing the magnitude and direction of the vector by a factor of its corresponding eigenvalue. If the eigenvalue is positive, the pre-multiplication operation simply changes the magnitude of the vector. If the eigenvalue is negative, the operation in addition to changing the magnitude reverses the direction of the resulting vector. The magnitude is amplified by a factor equal to the absolute value of the corresponding eigenvalue. From Eqn. (E 4.6) it becomes obvious that any multiple of the eigenvector  $[m^i]$  will satisfy the equation. Therefore, any multiple of  $[m^i]$  is also an eigenvector.

The transformation matrix  $[M]$  that diagonalizes the system matrix  $[A]$  is called the modal matrix.

**Example:** Find the eigenvalues and eigenvectors

$$[A] = \begin{bmatrix} -3 & 2 \\ -1 & 0 \end{bmatrix}$$

Characteristic equation is

$$|\lambda[I] - [A]| = 0$$

$$\begin{vmatrix} \lambda & 0 \\ 0 & \lambda \end{vmatrix} - \begin{vmatrix} -3 & 2 \\ -1 & 0 \end{vmatrix} = 0$$

$$\begin{vmatrix} \lambda + 3 & -2 \\ 1 & \lambda \end{vmatrix} = 0$$

$$\lambda^2 + 3\lambda + 2 = 0$$

$$(\lambda + 2)(\lambda + 1) = 0$$

The eigenvalues are  $\lambda_1 = -2$  and  $\lambda_2 = -1$

The eigenvector  $[m^1]$  corresponding to the eigenvalue  $\lambda_1 = -2$  is related in the following manner.

$$\begin{bmatrix} -3 & 2 \\ -1 & 0 \end{bmatrix} [m^1] = \lambda_1 [m^1]$$

$$\begin{bmatrix} -3 & 2 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} m_{11} \\ m_{21} \end{bmatrix} = -2 \begin{bmatrix} m_{11} \\ m_{21} \end{bmatrix}$$

$$m_{11} = 2m_{21}$$

$$[m^1] = \begin{bmatrix} m_{11} \\ m_{21} \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

The eigenvector  $[m^2]$  corresponding to the eigenvalue  $\lambda_2 = -1$  is related in the following manner.

$$\begin{bmatrix} -3 & 2 \\ -1 & 0 \end{bmatrix} [m^2] = \lambda_2 [m^2]$$

$$\begin{bmatrix} -3 & 2 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} m_{12} \\ m_{22} \end{bmatrix} = -1 \begin{bmatrix} m_{12} \\ m_{22} \end{bmatrix}$$

$$m_{12} = m_{22}$$

$$[m^2] = \begin{bmatrix} m_{12} \\ m_{22} \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

The modal matrix is

$$[M] = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$$

Now we will discuss another method of finding eigenvectors. For this method we will use Eqn. (E 4.6) again.

$$(\lambda_i [I] - [A]) [m^i] = 0$$

We know that  $[K][K]^{-1} = [I]$  and also  $[K]^{-1} = \frac{\text{adj}[K]}{|K|}$

Therefore,  $[K][K]^{-1} = [K] \frac{\text{adj}[K]}{|K|} = [I]$

$$[K]adj[K] = |K|[I]$$

$$(\lambda_i[I] - [A])adj(\lambda_i[I] - [A]) = |(\lambda_i[I] - [A])|[I]$$

However,  $|(\lambda_i[I] - [A])|$  is equal to zero since  $\lambda_i$  is an eigenvalue.

Therefore, we can write

$$(\lambda_i[I] - [A])adj(\lambda_i[I] - [A]) = \mathbf{0} \quad (\text{E 4.7})$$

A comparison of Eqns. (E 4.6) and (E 4.7) show that the eigenvector  $[m^i]$  is proportional to any nonzero column of  $adj(\lambda_i[I] - [A])$ .

We can, therefore, use  $adj(\lambda_i[I] - [A])$  to obtain the modal matrix.

### Example:

Find the eigenvalues and eigenvectors for the system with a system matrix A.

$$A := \begin{pmatrix} -5 & 1 & 0 \\ 0 & -4 & 1 \\ -13 & 1 & 0 \end{pmatrix}$$

$$\text{eigenvals}(A) = \begin{pmatrix} -6.1 \\ -2.339 \\ -0.561 \end{pmatrix} \quad \lambda := \text{eigenvals}(A) \quad \lambda = \begin{pmatrix} -6.1 \\ -2.339 \\ -0.561 \end{pmatrix}$$

$$D_1 := \lambda_1 \cdot \text{identity}(3) - A \quad D_2 := \lambda_2 \cdot \text{identity}(3) - A \quad D_3 := \lambda_3 \cdot \text{identity}(3) - A$$

$$\lambda_1 \cdot \text{identity}(3) - A = \begin{pmatrix} -1.1 & -1 & 0 \\ 0 & -2.1 & -1 \\ 13 & -1 & -6.1 \end{pmatrix} \quad adj(D_1) := |D_1| \cdot D_1^{-1}$$

$$adj(D_1) = \begin{pmatrix} 11.554 & -5.966 & 0.978 \\ -12.714 & 6.566 & -1.076 \\ 26.705 & -13.791 & 2.261 \end{pmatrix} \quad adj(D_2) = \begin{pmatrix} -4.825 & -2.31 & 0.988 \\ -12.84 & -6.147 & 2.628 \\ -21.328 & -10.211 & 4.366 \end{pmatrix}$$

$$adj(D_3) = \begin{pmatrix} -3.332 & -0.638 & 1.138 \\ -14.79 & -2.832 & 5.051 \\ -50.867 & -9.739 & 17.37 \end{pmatrix}$$

The modal matrix  $[M]$  is

$$[M] = \begin{bmatrix} \mathbf{0.978} & \mathbf{0.988} & \mathbf{1.138} \\ \mathbf{-1.076} & \mathbf{2.628} & \mathbf{5.051} \\ \mathbf{2.261} & \mathbf{4.366} & \mathbf{17.37} \end{bmatrix}$$

$$M^{-1} A \cdot M = \begin{pmatrix} -6.1 & 0 & 0 \\ 0 & -2.339 & 0 \\ 0 & 0 & -0.561 \end{pmatrix}$$

**References:**

Feedback Control Systems, John Van de Vegte, Prentice Hall

Automatic Control Systems: Basic Analysis and Design, William A. Wolovich, Saunders College Publishing