Everything you should have remembered from Differential Equations

Purpose

It may have been some time since you formally studied the solution to first-order and second-order ordinary differential equations with constant coefficients. Knowledge of these analytical solution techniques is vital to the study of System Dynamics. Hence, the current laboratory lecture and homework is provided to help the student refresh important differential equation solution skills.

Background

First-order Systems: Consider the general, first-order, non-homogeneous differential equation

$$\alpha \frac{dX}{dt} + \beta X = \gamma \qquad (1)$$

in which $\alpha = \alpha(t)$, $\beta = \beta(t)$, $\gamma = \gamma(t)$, and $X = X(t)$. Equation (1) is usually written in *standard* form by dividing both sides by $\alpha(t)$ to obtain

$$\frac{dX}{dt} + aX = b \qquad (2)$$

where $a(t) = \beta(t)/\alpha(t)$ and $b(t) = \gamma(t)/\alpha(t)$.

Recall from your previous study of differential equations that equations (1) and (2) are *linear* and can be solved using such techniques as *variation of parameters* or *integrating factors*. The solution of the *homogeneous* (i.e. $\gamma(t) = b(t) = 0$) formulation can be easily solved using *separation of variables* to yield

$$X(t) = \phi e^{-\int a(t)\,dt} \qquad (3)$$

where ϕ is a constant of integration. Note that equation (3) is *exponential* regardless of the form of $a(t)$.

Now consider the solution of the non-homogeneous equation (2) by use of the integrating factor $e^{\int a(t)\,dt}$. Multiplication of both sides of equation (2) first by dt and then by the integrating factor yields

$$e^{\int a(t)dt}\left[dX + a(t)Xdt\right] = e^{\int a(t)dt}b(t)dt \tag{4}$$

where the left side of equation (4) is the exact differential

$$d\left(Xe^{\int a(t)dt}\right) \tag{5}$$

Substitution of equation (5) into (4) and subsequent integration gives

$$Xe^{\int a(t)dt} = \int e^{\int a(t)dt}b(t)dt + \phi \tag{6}$$

and hence

$$X(t) = e^{-\int a(t)dt}\int e^{\int a(t)dt}b(t)dt + \phi e^{-\int a(t)dt} \tag{7}$$

Note that the second term on the right hand side of equation (7) is the homogeneous solution given above in equation (3), while the first term is the *particular* solution. The constant of integration ϕ is evaluated from the initial condition on X (i.e., $X(t = 0)$).

Consider a modification of equation (2) in which a and b are *constant*. Its solution (7) can therefore be simplified to the following form

$$X(t) = \frac{b}{a} + \phi e^{-at} \tag{8}$$

If we assume $X(t = 0) = c$ (where c is a constant) then equation (8) becomes

$$X(t) = \frac{b}{a} + \left(c - \frac{b}{a}\right)e^{-at} \tag{9}$$

Note that in equation (9) the initial condition is preserved, while the final or *steady-state* solution is $X(t = \infty) = b/a$.

Second-order Systems: Consider the general, second-order, non-homogeneous differential equation

$$\alpha\frac{d^2X}{dt^2} + \beta\frac{dX}{dt} + \gamma X = \delta \tag{10}$$

in which $\alpha = \alpha(t)$, $\beta = \beta(t)$, $\gamma = \gamma(t)$, $\delta = \delta(t)$ and $X = X(t)$. Equation (10) is usually written in *standard* form by dividing both sides by $\alpha(t)$ to obtain

$$\frac{d^2X}{dt^2} + a\frac{dX}{dt} + bX = f \tag{11}$$

where $a(t) = \beta(t)/\alpha(t)$, $b(t) = \gamma(t)/\alpha(t)$, and $f(t) = \delta(t)/\alpha(t)$.

Recall that equations (10) and (11) are *linear*, with solutions that take the form

$$X = X_c + X_p \tag{12}$$

where $X_c(t)$ represents the *complementary* solution of the reduced (or homogeneous) form of equation (11) (i.e., with $f(t) = 0$), and X_p represents any *particular solution* which satisfies the complete form of equation (11). The complementary solution takes the general form

$$X_c = c_1X_1 + c_2X_2 \tag{13}$$

where c_1 and c_2 are constants determined from prescribed boundary conditions, and $X_1(t)$ and $X_2(t)$ are *linearly independent* solutions to the homogeneous form of equation (11).

If the coefficients a and b in equation (11) are *constant*, the complimentary solution can be easily solved once the roots of the *characteristic equation* are determined. Using equation (11) with $f(t) = 0$ we obtain the reduced equation (with constant coefficients)

$$\frac{d^2X}{dt^2} + a\frac{dX}{dt} + bX = 0 \tag{14}$$

The characteristic equation is of the form

$$s^2 + as + b = 0 \tag{15}$$

with roots

$$s_1, s_2 = -\frac{a}{2} \pm \sqrt{\frac{a^2}{4} - b} \tag{16}$$

Equation (13) therefore takes one of *three* distinct formulations based on whether the roots of equation (15) are

1. real and unequal

$$X_c(t) = c_1e^{s_1t} + c_2e^{s_2t} \tag{17}$$

2. real and equal

$$X_c(t) = c_1 e^{s_1 t} + c_2 t e^{s_2 t} \tag{18}$$

3. or complex conjugate

$$X_c(t) = e^{\sigma t}\left(c_1 \cos\omega_{nd} t + c_2 \sin\omega_{nd} t\right) \tag{19}$$

where $\sigma = -a/2$ and $\omega_{nd} = \sqrt{b - a^2/4}$

The formulation expressed in equation (19) can be re-written (after algebraic manipulation) as

$$X_c(t) = Ae^{\sigma t} \sin\left(\omega_{nd} t + \phi\right) \tag{20}$$

where $A = \sqrt{c_1^2 + c_2^2}$ and $\phi = \arctan(c_1/c_2)$. Note that ϕ merely represents a shift in the time-axis intercept of $X_c(t)$ (i.e., $X_c = 0$ when $t = -\phi/\omega_{nd}$).

The *particular solution* $X_p(t)$ of the non-homogeneous equation (11) can be determined by the method of *variation of parameters* if the coefficients a and b are time-dependent (i.e. non-constant), or by either *variation of parameters* or the *method of undetermined coefficients* if a and b are constant in time. Assuming that a and b are constant, a simple determination of X_p is possible if $f(t) = constant$. Under such a scenario, we note that the long-time solution of (11) must approach that prescribed by the constant *forcing function* f. Hence, if we simply neglect the time-differential terms in equation (11) (since a constant long time-time solution has zero time derivatives) we obtain $bX = f$ or

$$X_p = \frac{f}{b} \equiv X_{ss} \tag{21}$$

where X_{ss} is known as the steady-state solution of $X(t)$. Use of the appropriate form for $X_c(t)$ (i.e., one of equations (17) through (20)) and equation (21) uniquely determines the complete solution to the non-homogeneous constant-coefficient, second-order differential equation postulated in equation (11) subject to a constant forcing function f.