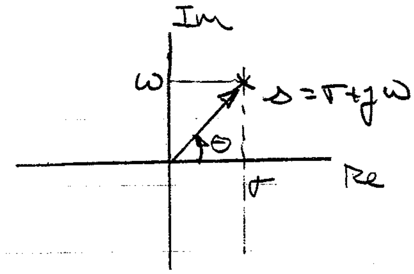
Laplace Transforms:



Recall: s-plane $\quad s = \underset{Re}{\sigma} + j\underset{Im}{\omega}$

$$|s| = \left[Re^2 + Im^2\right]^{1/2} = \sqrt{\sigma^2 + \omega^2}$$

$$\theta = \measuredangle s = \tan^{-1}\left[\frac{Im}{Re}\right] = \tan^{-1}(\omega/\sigma)$$

For single-sided Laplace transforms:

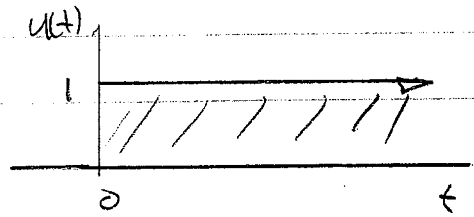
given: $f(t)$ such that $\int_0^\infty \left| f(t)\, e^{-\sigma t} \right| dt < \infty$

for σ = finite, real

then:

$$\mathcal{L}[f(t)] \equiv F(s) = \int_{0^-}^{\infty} f(t)\, e^{-st}\, dt$$

Example: unit step: $\quad f(t) = u(t) = \begin{cases} 1 & t > 0 \\ 0 & t < 0 \end{cases}$



$$\therefore F(s) = \int_0^\infty u(t)\, e^{-st}\, dt$$

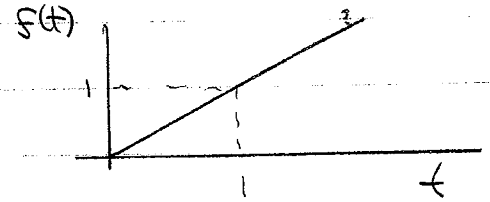
$$= \int_{0^+}^\infty 1\, e^{-st}\, dt$$

$$= \frac{-1}{s} e^{-st} \Big|_0^\infty = 1/s$$

$$\therefore \boxed{\mathcal{L}[u(t)] = 1/s}$$

<u>Example</u>: unit ramp $f(t) = t \quad t \geq 0$

$= t\,u(t)$



$$\therefore F(s) = \int_0^\infty t\, e^{-st}\, dt$$

integrate by parts:

$$\int u\,dv = uv\Big| - \int v\,du$$

<u>Recall</u>: $\int u\,dv = uv\Big| - \int v\,du$

$\therefore$ Let $u = t \Rightarrow du = dt$

$dv = e^{-st}dt \Rightarrow v = -\frac{1}{s}e^{-st}$

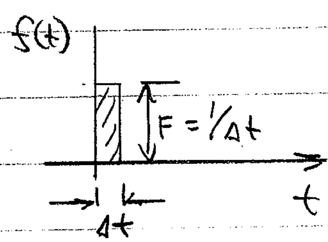
$$\therefore F(s) = \int_0^\infty t\,e^{-st}\,dt = t\cdot\left(-\frac{1}{s}e^{-st}\right)\Big|_{t=0}^{\infty} - \int_0^\infty -\frac{1}{s}e^{-st}\,dt$$

$$= -\frac{1}{s}\left[\underbrace{\infty\cdot e^{-\infty} - 0\cdot e^{0}}_{0}\right] - -\frac{1}{s}\int e^{-st}\,dt$$

$$\boxed{\mathcal{L}[t\,u(t)] = 1/s^2} \qquad = -\frac{1}{s^2}e^{-st}\Big|_0^\infty = 1/s^2$$

and in general $\mathcal{L}[t\,f(t)] = -\frac{d}{ds}F(s)$

<u>Example</u>: unit impulse $f(t) = \delta(t)$



where: $\int_0^{\Delta t} \delta(t)\,dt = \lim_{\Delta t\to 0} F\,\Delta t = \lim_{\Delta t \to 0}\int_0^{\Delta t} F\,dt \equiv 1$

(ie, area under impulse always = 1)

$$F(s) = \int_0^\infty \delta(t)\,e^{-st}\,dt$$

<u>but</u>: $\delta(t) = F = 1/\Delta t$ for $0 < t < \Delta t$ and $\Delta t \to 0$

$$= \lim_{\Delta t\to 0}\int_0^{\Delta t}\frac{1}{\Delta t}e^{-st}\,dt = \frac{1}{\Delta t}\int_0^{\Delta t} e^{-st}\,dt = \left(\frac{-1}{s\,\Delta t}\right)e^{-st}\Big|_0^{\Delta t}$$

$$= \frac{-1}{s\,\Delta t}\left[e^{-s\Delta t} - 1\right] = \frac{1 - e^{-s\Delta t}}{s\,\Delta t}$$

(con't)

Now as $\Delta t \to 0$ $\quad \mathcal{L}[\delta(t)] = \lim\limits_{\Delta t \to 0} \dfrac{1-e^{-s\Delta t}}{s\,\Delta t} = 0/0$!

∴ use L'Hopital's Rule:

$$\lim_{\Delta t \to 0} \frac{1-e^{-s\Delta t}}{s\Delta t} = \lim_{\Delta t \to 0} \frac{\frac{d}{d\Delta t}(1-e^{-s\Delta t})}{\frac{d}{d\Delta t}(s\Delta t)} = \lim_{\Delta t \to 0} \frac{--s\,e^{-s\Delta t}}{s} = 1$$

$$\therefore \boxed{\mathcal{L}[\delta(t)] = 1}$$

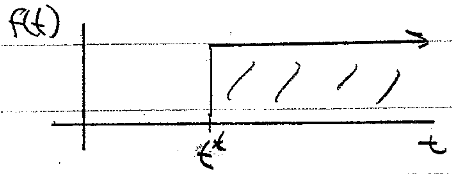
<u>Scale Factor</u>: $\mathcal{L}[a f(t)] = ?$

look at step: $\mathcal{L}[a\,u(t)] = \int_0^\infty a\,u(t)\,e^{-st}\,dt = \dots = a/s$

$$\therefore \boxed{\mathcal{L}[a f(t)] = a F(s)}$$

<u>Time shift</u>: $\mathcal{L}[f(t-t^*)] = ?$

look at step (shifted): $\mathcal{L}[u(t-t^*)] = \int_0^\infty f(t)\,e^{-st}\,dt$



$$= \underbrace{\int_0^{t^*} 0\,e^{-st}\,dt}_{0} + \int_{t^*}^\infty 1\,e^{-st}\,dt$$

$f(t) = 0 \quad t < t^*$
$ = 1 \quad t > t^*$

Let $\tau = t - t^*$, $\begin{cases} t = t^* \to \tau = 0 \\ t = \infty \to \tau = \infty \end{cases}$

$\therefore t = \tau + t^*$

$$\therefore \int_{t^*}^\infty 1\,e^{-st}\,dt = \int_0^\infty e^{-s(\tau + t^*)}\,d\tau$$
$$= \int_0^\infty e^{-s\tau}\cdot e^{-st^*}\,dt = e^{-st^*}\,\frac{1}{s}$$

$$\boxed{\mathcal{L}[f(t-t^*)] = e^{-st^*} F(s)}$$

Example: exponential $g(t) = e^{at} \cdot u(t)$

$$G(s) = \int_0^\infty e^{at} e^{-st} dt = \int_0^\infty e^{-(s-a)t} dt$$

$$= \frac{-1}{(s-a)} e^{-(s-a)t} \Big|_0^\infty = \frac{1}{s-a}$$

$$\boxed{\mathcal{L}[e^{\pm at}] = \frac{1}{s \mp a}}$$

In general:
$$\mathcal{L}[f(t) e^{-at}] = F(s+a)$$

Example: derivatives $g(t) = \frac{df(t)}{dt}$

$$G(s) = \mathcal{L}\left[\frac{df(t)}{dt}\right] = \int_0^\infty \frac{df(t)}{dt} e^{-st}$$

by parts $uv - \int v du$

let:
$dv = \frac{df}{dt} dt = df$
$v = f$
$u = e^{-st}$ $du = -s e^{-st} dt$

$$= \int e^{-st} \frac{df}{dt} dt$$

$$= e^{-st} f \Big|_0^\infty - \int_0^\infty -s e^{-st} f dt$$

$$= -f(0) + s \int_0^\infty f e^{-st} dt = -f(0) + s F(s)$$

$$\boxed{\mathcal{L}\left[\frac{df}{dt}\right] = s F(s) - f(0)}$$

also $\boxed{\mathcal{L}\left[\int f dt\right] = \frac{1}{s} F(s)}$

for $\mathcal{L}(\sin\omega t)$:

Recall linearity property :

$$\mathcal{L}[a f(t) + g f(t)] = a\mathcal{L}[f(t)] + b\mathcal{L}[g(t)]$$
$$= a F(s) + b G(s)$$

also $\mathcal{L}[e^{at}] = \frac{1}{s-a}$

hence if $a = j\omega$ then

$$\mathcal{L}[e^{j\omega t}] = \frac{1}{s-j\omega}$$

$$\therefore \mathcal{L}[e^{j\omega t}] = \frac{1}{s-j\omega}\frac{(s+j\omega)}{(s+j\omega)} = \frac{s+j\omega}{s^2+\omega^2} = \frac{s}{s^2+\omega^2} + j\frac{\omega}{s^2+\omega^2} \quad (1)$$

but $e^{j\omega t} = \cos\omega t + j\sin\omega t$ by definition

hence $\mathcal{L}[e^{j\omega t}] = \mathcal{L}[\cos\omega t + j\sin\omega t]$

$$= \mathcal{L}[\cos\omega t] + j\mathcal{L}[\sin\omega t] \quad (2)$$

$\therefore$ equating (1) & (2) term-wise :

$$\mathcal{L}[\cos\omega t] = \frac{s}{s^2+\omega^2} \quad ; \quad \mathcal{L}[\sin\omega t] = \frac{\omega}{s^2+\omega^2}$$