Appendix 1

Useful Formulae

BASIC SYMBOLS AND UNITS*

ω_n = undamped natural frequency (rad/s)
$\omega_d = \omega$ = damped natural frequency (rad/s)
ζ = damping ratio
β = damped frequency ratio
τ = time constant (s)
λ = period of sinusoid (s)
γ = frequency ratio
ϕ = phase angle (rad)
a = decay coefficient (1/s)
r = first order term (rad/s)

SYSTEM ELEMENTS AND UNITS

M = mass (kg or $N \cdot s^2/m$)
D = damper ($N \cdot s/m$)
K = spring (N/m)
J = mass moment of inertia ($kg \cdot m$ or $N \cdot m \cdot s^2$)
D_t = torsional damper ($N \cdot m \cdot s/rad$)
K_t = torsional spring ($N \cdot m/rad$)
C = electrical capacitance (F or farad)
R = electrical resistance (Ω or ohm)
L = electrical inductance (H or henry)

*_Note_: In Appendices 1 and 3, whenever the symbol ω appears without a subscript, subscript d will be understood. If no units are shown, the term is dimensionless.

$1/C_f$ = reciprocal fluid capacitance (N/m^5)
R'_f = fluid resistance ($N \cdot s/m^5$)
I = inertance ($N \cdot s^2/m^5$)

VIBRATION FORMULAE*

1. $\omega_n^2 = K/M = K_t/J = 1/LC = 1/C_f I$
2. $\zeta = D/D_c = D/2M\omega_n = D/2J\omega_n = R/2L\omega_n = R_f/2I\omega_n$
3. $a = \zeta\omega_n = D/2M = D_t/2J = R/2L = R_f/2I$
4. $\omega = \omega_d = \omega_n\beta$
5. $\beta = \sqrt{1 - \zeta^2}$, $\beta^2 + \zeta^2 = 1$
6. $\omega_n^2 = a^2 + \omega^2$
7. $r^2 = (c - a)^2 + \omega^2$
8. $\tau = 1/a = 1/\zeta\omega_n$
9. $\lambda = 2\pi/\omega$
10. $\gamma = \omega/\omega_n$
11. e^{-at}, decay envelope
12. $s^2 + 2as + \omega_n^2$, underdamped quadratic ($1/s^2$)

PHASE ANGLES

13. $\phi = \sin^{-1}\beta = \cos^{-1}\zeta = \tan^{-1}\omega/a$
14. $\phi_r = \sin^{-1}\omega/r$
15. $\sin\phi = \beta = \omega/\omega_n$
16. $\sin 2\phi = 2\beta\zeta = 2a\omega/\omega_n^2$
17. $\sin 3\phi = \beta(4\zeta^2 - 1)$
18. $\sin 4\phi = 4\zeta\beta(2\zeta^2 - 1)$
19. $\cos\phi = a/\omega_n = \zeta$
20. $\cos 2\phi = 2\zeta^2 - 1 = (a^2 - \omega^2)/\omega_n^2$
21. $\cos 3\phi = \zeta(4\zeta^2 - 3)$
22. $\cos 4\phi = 1 + 8\zeta^2(\zeta^2 - 1)$
23. $\tan\phi = \omega/a = \beta/\zeta$
24. $\tan(\pi - \phi) = \omega/-a$
25. $\tan 2\phi = \dfrac{2a\omega}{a^2 - \omega^2} = \dfrac{2\beta\zeta}{2\zeta^2 - 1}$
26. $\sin\phi_r = \omega/r$
27. $\cos\phi_r = (c - a)/r$
28. $\tan\phi_r = \omega/(c - a)$

Note: See footnotes on page 679.

EULER'S IDENTITIES

29. $e^{jx} = \cos x + j \sin x$
30. $e^{-jx} = \cos x - j \sin x$
31. $\cos x = \dfrac{e^{jx} + e^{-jx}}{2}$
32. $\sin x = \dfrac{e^{jx} - e^{-jx}}{2j}$
33. $e^{x} = \cosh x + \sinh x$
34. $e^{-x} = \cosh x - \sinh x$
35. $\cosh x = \dfrac{e^{x} + e^{-x}}{2}$
36. $\sinh x = \dfrac{e^{x} - e^{-x}}{2}$

Appendix 3

Laplace Transform Pairs

For definition of terms and symbols, see Appendix 1. The entry numbers in the following table of Laplace transform pairs are arranged as follows:

hundreds place: family or type of system;

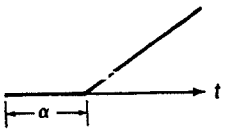
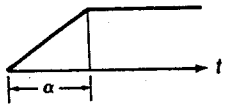
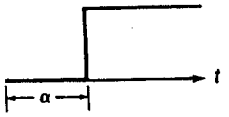
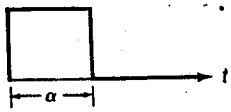
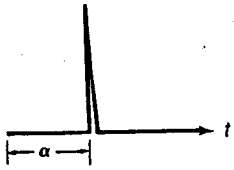
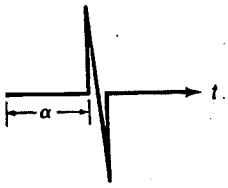
tens place: group within a family;

units place: ascending numerical value indicates derivative, descending value indicates an integration.

Note that certain entries have been deliberately omitted to provide exercises for the student.

	Laplace domain	Time domain
100	*General Operators*	
110	Basics	
111	F	$\int_0^\infty e^{-st} f(t)\,dt$ Definition
112	$F + G$	$f(t) + g(t)$ Sum
113	aF	$af(t)$ Multiplication by constant
114	$F(s + a)$	$e^{-at} f(t)$ Shifting theorem
115	$F(s)G(s)$	$\int_0^t f(t - \tau)g(\tau)\,d\tau$ Convolution theorem $= \int_0^t f(\tau)g(t - \tau)\,d\tau$
116	$\lim_{s \to 0} sF$	$\lim_{t \to \infty} f(t)$ Final value theorem
117	$\lim_{s \to \infty} sF$	$\lim_{t \to 0} f(t)$ Initial value theorem

	Laplace domain	Time domain
120	Integrals and Derivatives in the Time Domain	
122	$\frac{F}{s^n}$	$\int\int^n \int_{0^+}^t f(t)\, dt^n$
123	$\frac{F}{s^2}$	$\int_{0^+}^t \int_{0^+}^t f(t)\, dt^2$
124	$\frac{F}{s}$	$\int_{0^+}^t f(t)\, dt$
125	F	$f(t)$ (also listed as entry 111)
126	$sF - f(0^-)$	$df(t)/dt$
127	$s^2F - sf(0^-) - \dot{f}(0^-)$	$d^2f(t)/dt^2$
128	$s^nF - \sum_{k=1}^{n} s^{n-k} \frac{d^{k-1}f(0^-)}{dt^{k-1}}$	$\frac{d^nf(t)}{dt^n}$
130	Integrals and Derivatives in the Laplace Domain	
134	$\int_s^\infty F\, ds$	$\frac{1}{t}f(t)$
135	F	$f(t)$ (also listed as entry 111)
136	$\frac{dF}{ds}$	$-tf(t)$
137	$\frac{d^2F}{ds^2}$	$t^2f(t)$
138	$\frac{d^nF}{ds^n}$	$(-1)^n t^n f(t)$
140	Jump Functions	
141	$\frac{1}{s^{n+1}}$	$\frac{t^n}{n!}$ $n = 0, 1, 2, \ldots$
142	$\frac{1}{s^4}$	$\frac{t^3}{3!}$
143	$\frac{1}{s^3}$	$\frac{t^2}{2!}$ parabolic function
144	$\frac{1}{s^2}$	t ramp function
145	$\frac{1}{s}$	1 step function
146	1	$\delta(t)$ impulse function
147	s	$\delta'(t)$ doublet impulse
148	s^2	$\delta''(t)$ triplet impulse

	Laplace domain	Time domain	
150	Delayed Functions		
151	$e^{-\alpha s}F(s)$	$f(t-\alpha)$ Function delayed by time α (also listed as entry 115)	
152	$\frac{e^{-\alpha s}}{s^2}$	$(t-\alpha)v(t-\alpha)$ Delayed ramp	
153	$\frac{1-e^{-\alpha s}}{s^2}$	$vt-\{(t-\alpha)v(t-\alpha)\}$ Ramp step	
154	$\frac{e^{-\alpha s}}{s}$	$U(t-\alpha)$ Delayed step	
155	$\frac{1-e^{-\alpha s}}{s}$	$U(t)-U(t-\alpha)$ Rectangular pulse	
156	$e^{-\alpha s}$	$\delta(t-\alpha)$ Delayed impulse	
157	$se^{-\alpha s}$	$\delta'(t-\alpha)$ Delayed doublet	
200	*First Order Factors*		
210	Single Factor		
211	$\frac{a^5}{s^5(s+a)}$	$\frac{a^4t^4}{4!}-\frac{a^3t^3}{3!}+\frac{a^2t^2}{2!}-at+1-e^{-at}$	
212	$\frac{a^4}{s^4(s+a)}$	$\frac{a^3t^3}{3!}-\frac{a^2t^2}{2!}+at-1+e^{-at}$	
213	$\frac{a^3}{s^3(s+a)}$	$\frac{a^2t^2}{2!}-at+1-e^{-at}$	

	Laplace domain	Time domain
214	$\frac{a^2}{s^2(s+a)}$	$at - 1 + e^{-at}$
215	$\frac{a}{s(s+a)}$	$1 - e^{-at}$
216	$\frac{1}{s+a}$	e^{-at}
217	$\frac{s}{a(s+a)}$	$\frac{\delta}{a} - e^{-at}$
218	$\frac{s^2}{a^2(s+a)}$	$\frac{\delta'}{a^2} - \frac{\delta}{a} + e^{-at}$
219	$\frac{s^3}{a^3(s+a)}$	$\frac{\delta''}{a^3} - \frac{\delta'}{a^2} + \frac{\delta}{a} - e^{-at}$
220	Two First Order Factors (If the damping ratio ζ of a quadratic is greater than unity, the quadratic *must* be factored into two first order factors). $a \neq b$.	
222	$\frac{a^3b^3}{s^3(s+a)(s+b)}$	$\frac{a^2b^2t^2}{2!} - \frac{(b^2-a^2)abt}{b-a} + \frac{b^3-a^3}{b-a} - \left[\frac{b^3e^{-at} - a^3e^{-bt}}{b-a}\right]$
223	$\frac{a^2b^2}{s^2(s+a)(s+b)}$	$abt - \frac{b^2-a^2}{b-a} + \left[\frac{b^2e^{-at} - a^2e^{-bt}}{b-a}\right]$
224	$\frac{ab}{s(s+a)(s+b)}$	$1 - \left[\frac{be^{-at} - ae^{-bt}}{b-a}\right]$
225	$\frac{1}{(s+a)(s+b)}$	$\frac{e^{-at} - e^{-bt}}{b-a}$
226	$\frac{s}{(s+a)(s+b)}$	$-\left[\frac{ae^{-at} - be^{-bt}}{b-a}\right]$
227	$\frac{s^2}{(s+a)(s+b)}$	$\delta + \left[\frac{a^2e^{-at} - b^2e^{-bt}}{b-a}\right]$
228	$\frac{s^3}{(s+a)(s+b)}$	$\delta' - \frac{(b^2-a^2)\delta}{b-a} - \left[\frac{a^3e^{-at} + b^3e^{-bt}}{b-a}\right]$
229	$\frac{s^4}{(s+a)(s+b)}$	$\delta'' - \frac{(b^2-a^2)\delta'}{b-a} + \frac{(b^3-a^3)\delta}{b-a} + \left[\frac{a^4e^{-at} - b^4e^{-bt}}{b-a}\right]$

	Laplace domain	Time domain
240	Three First Order Factors The cubic can be factored into first and quadratic factors. (If the damping ratio ζ of the quadratic is greater than unity, then it *must* be factored. In that case, there will be three first order factors.) $a \neq b \neq c$.	
243	$\dfrac{abc}{s(s+a)(s+b)(s+c)}$	$1 - \dfrac{bce^{-at}}{(b-a)(c-a)} - \dfrac{ace^{-bt}}{(a-b)(c-b)} - \dfrac{abe^{-ct}}{(a-c)(b-c)}$
244	$\dfrac{1}{(s+a)(s+b)(s+c)}$	$\dfrac{e^{-at}}{(b-a)(c-a)} + \dfrac{e^{-bt}}{(a-b)(c-b)} + \dfrac{e^{-ct}}{(a-c)(b-c)}$
245	$\dfrac{s}{(s+a)(s+b)(s+c)}$	$-\dfrac{ae^{-at}}{(b-a)(c-a)} - \dfrac{be^{-bt}}{(a-b)(c-b)} - \dfrac{ce^{-ct}}{(a-c)(b-c)}$
246	$\dfrac{s^2}{(s+a)(s+b)(s+c)}$	$\dfrac{a^2e^{-at}}{(b-a)(c-a)} + \dfrac{b^2e^{-bt}}{(a-b)(c-b)} + \dfrac{c^2e^{-ct}}{(a-c)(b-c)}$
247	$\dfrac{s^3}{(s+a)(s+b)(s+c)}$	$-\delta - \dfrac{a^3e^{-at}}{(b-a)(c-a)} - \dfrac{b^3e^{-bt}}{(a-b)(c-b)} - \dfrac{c^3e^{-ct}}{(a-c)(b-c)}$
260	Two Repeated First Order Factors *Note*: If the damping ratio ζ is equal to unity, the quadratic is a perfect square, yielding two identical or repeated factors.	
261	$\dfrac{a^5}{s^4(s+a)^2}$	$\dfrac{a^3t^3}{3!} - a^2t^2 + 3at - 4 + (at+4)e^{-at}$
262	$\dfrac{a^4}{s^3(s+a)^2}$	$\dfrac{a^2t^2}{2} - 2at + 3 - (at+3)e^{-at}$
263	$\dfrac{a^3}{s^2(s+a)^2}$	$at - 2 + (at+2)e^{-at}$
264	$\dfrac{a^2}{s(s+a)^2}$	$1 - (at+1)e^{-at}$

	Laplace domain	Time domain
265	$\frac{a}{(s+a)^2}$	ate^{-at}
266	$\frac{s}{(s+a)^2}$	$-(at-1)e^{-at}$
267	$\frac{s^2}{a(s+a)^2}$	$\frac{\delta}{a}+(at-2)e^{-at}$
268	$\frac{s^3}{a^2(s+a)^2}$	$\frac{\delta'}{a^2}-\frac{2\delta}{a}-(at-3)e^{-at}$
269	$\frac{s^4}{a^3(s+a)^2}$	$\frac{\delta''}{a^3}-\frac{2\delta'}{a^2}+\frac{3\delta}{a}+(at-4)e^{-at}$
280	N Repeated First Order Factors	
285	$\frac{a^n}{(s+a)^{n+1}}$	$\frac{a^n t^n e^{-at}}{n!} \quad n>-1$
286	$\frac{s}{(s+a)^{n+1}}$	$\frac{(n-at)t^{n-1}e^{-at}}{n!} \quad n>0$
300	*Undamped Quadratics*	
310	Single Quadratic	
311	$\frac{\omega^5}{s^4(s^2+\omega^2)}$	$\frac{\omega^3 t^3}{3!}-\omega t+\sin\omega t$
312	$\frac{\omega^4}{s^3(s^2+\omega^2)}$	$\frac{\omega^2 t^2}{2!}-1+\cos\omega t$
313	$\frac{\omega^3}{s^2(s^2+\omega^2)}$	$\omega t-\sin\omega t$
314	$\frac{\omega^2}{s(s^2+\omega^2)}$	$1-\cos\omega t$
315	$\frac{\omega}{s^2+\omega^2}$	$\sin\omega t$
316	$\frac{s}{s^2+\omega^2}$	$\cos\omega t$
317	$\frac{s^2}{\omega(s^2+\omega^2)}$	$\frac{\delta}{\omega}-\sin\omega t$
318	$\frac{s^3}{\omega^2(s^2+\omega^2)}$	$\frac{\delta'}{\omega^2}-\cos\omega t$
319	$\frac{s^4}{\omega^3(s^2+\omega^2)}$	$\frac{\delta''}{\omega^3}-\frac{\delta}{\omega}+\sin\omega t$

	Laplace domain	Time domain
320	Two Undamped Quadratics, $\alpha \neq \omega$	
321	$\dfrac{\alpha^3\omega^3}{s^2(s^2+\omega^2)(s^2+\alpha^2)}$	$t - \left[\dfrac{\alpha^3 \sin \omega t - \omega^3 \sin \alpha t}{\alpha^2 - \omega^2}\right]$
322	$\dfrac{\alpha^2\omega^2}{s(s^2+\omega^2)(s^2+\alpha^2)}$	$1 - \left[\dfrac{\alpha^2 \cos \omega t - \omega^2 \cos \alpha t}{\alpha^2 - \omega^2}\right]$
323	$\dfrac{\alpha\omega}{(s^2+\omega^2)(s^2+\alpha^2)}$	$\dfrac{\alpha \sin \omega t - \omega \sin \alpha t}{\alpha^2 - \omega^2}$
324	$\dfrac{s}{(s^2+\omega^2)(s^2+\alpha^2)}$	$\dfrac{\cos \omega t - \cos \alpha t}{\alpha^2 - \omega^2}$
325	$\dfrac{s^2}{(s^2+\omega^2)(s^2+\alpha^2)}$	$-\left[\dfrac{\omega \sin \omega t - \alpha \sin \alpha t}{\alpha^2 - \omega^2}\right]$
326	$\dfrac{s^3}{(s^2+\omega^2)(s^2+\alpha^2)}$	$-\left[\dfrac{\omega^2 \cos \omega t - \alpha^2 \cos \alpha t}{\alpha^2 - \omega^2}\right]$
327	$\dfrac{s^4}{(s^2+\omega^2)(s^2+\alpha^2)}$	$\delta + \left[\dfrac{\omega^3 \sin \omega t - \alpha^3 \sin \alpha t}{\alpha^2 - \omega^2}\right]$
328	$\dfrac{s^5}{(s^2+\omega^2)(s^2+\alpha^2)}$	$\delta' + \left[\dfrac{\omega^4 \cos \omega t - \alpha^4 \cos \alpha t}{\alpha^2 - \omega^2}\right]$
329	$\dfrac{s^6}{(s^2+\omega^2)(s^2+\alpha^2)}$	$\delta'' + \dfrac{(\omega^4 - \alpha^4)\delta}{\alpha^2 - \omega^2} - \left[\dfrac{\omega^5 \sin \omega t - \alpha^5 \sin \alpha t}{\alpha^2 - \omega^2}\right]$
330	Repeated Undamped Quadratics	
332	$\dfrac{2\omega^5}{s^2(s^2+\omega^2)^2}$	$2\omega t + \omega t \cos \omega t - 3 \sin \omega t$
333	$\dfrac{2\omega^4}{s(s^2+\omega^2)^2}$	$2 - \omega t \sin \omega t - 2 \cos \omega t$
334	$\dfrac{2\omega^3}{(s^2+\omega^2)^2}$	$-\omega t \cos \omega t + \sin \omega t$
335	$\dfrac{2\omega^2 s}{(s^2+\omega^2)^2}$	$\omega t \sin \omega t$

	Laplace domain	Time domain
336	$\frac{2\omega s^2}{(s^2+\omega^2)^2}$	$\omega t \cos \omega t + \sin \omega t$
337	$\frac{2s^3}{(s^2+\omega^2)^2}$	$-\omega t \sin \omega t + 2\cos \omega t$
338	$\frac{2s^4}{\omega(s^2+\omega^2)^2}$	$\frac{2\delta}{\omega} + \omega t \cos \omega t - 3 \sin \omega t$
340	Undamped Modulation	
341	$\frac{4\omega\alpha}{s^2\{s^2+(\omega+\alpha)^2\}\{s^2+(\omega-\alpha)^2\}}$	$-\frac{4\omega\alpha t}{(\omega^2-\alpha^2)^2} + \frac{\sin(\omega+\alpha)t}{(\omega+\alpha)^3} - \frac{\sin(\omega-\alpha)t}{(\omega-\alpha)^3}$
342	$\frac{4\omega\alpha}{s\{s^2+(\omega+\alpha)^2\}\{s^2+(\omega-\alpha)^2\}}$	$-\frac{4\omega\alpha}{(\omega^2-\alpha^2)^2} + \frac{\cos(\omega+\alpha)t}{(\omega+\alpha)^2} - \frac{\cos(\omega-\alpha)t}{(\omega-\alpha)^2}$
343	$\frac{4\omega\alpha}{\{s^2+(\omega+\alpha)^2\}\{s^2+(\omega-\alpha)^2\}}$	$-\frac{\sin(\omega+\alpha)t}{(\omega+\alpha)} + \frac{\sin(\omega-\alpha)t}{(\omega-\alpha)}$
344	$\frac{4\omega\alpha s}{\{s^2+(\omega+\alpha)^2\}\{s^2+(\omega-\alpha)^2\}}$	$-\cos(\omega+\alpha)t + \cos(\omega-\alpha)t$
345	$\frac{4\omega\alpha s^2}{\{s^2+(\omega+\alpha)^2\}\{s^2+(\omega-\alpha)^2\}}$	$(\omega+\alpha)\sin(\omega+\alpha)t - (\omega-\alpha)\sin(\omega-\alpha)t$
346	$\frac{4\omega\alpha s^3}{\{s^2+(\omega+\alpha)^2\}\{s^2+(\omega-\alpha)^2\}}$	$(\omega+\alpha)^2\cos(\omega+\alpha)t - (\omega-\alpha)^2\cos(\omega-\alpha)t$
347	$\frac{4\omega\alpha s^4}{\{s^2+(\omega+\alpha)^2\}\{s^2+(\omega-\alpha)^2\}}$	$4\omega\alpha\delta - (\omega+\alpha)^3\sin(\omega+\alpha)t + (\omega-\alpha)^3\sin(\omega-\alpha)t$

	Laplace domain	Time domain
348	$\dfrac{4\omega\alpha s^5}{\left\{s^2+(\omega+\alpha)^2\right\}\left\{s^2+(\omega-\alpha)^2\right\}}$	$4\omega\alpha\delta' - (\omega+\alpha)^4\cos(\omega+\alpha)t + (\omega-\alpha)^4\cos(\omega-\alpha)t$
400	*Underdamped Quadratics* *Note*: The 400 family can be used only if the damping ratio ζ is greater than zero but less than unity. For other values of ζ, see groups 220, 260, and 310. The term a is a special constant; $a = \zeta\omega_n$. See Appendix 1 for definitions of symbols.	
410	Numerator is Single Term	
412	$\dfrac{\omega_n^4}{s^3\left(s^2+2as+\omega_n^2\right)}$	$\dfrac{\omega_n^2t^2}{2} - 2at + (4\zeta^2-1) - \dfrac{\omega_n}{\omega_d}e^{-at}\sin(\omega_d t+3\phi)$
413	$\dfrac{\omega_n^3}{s^2\left(s^2+2as+\omega_n^2\right)}$	$\omega_n t - 2\zeta + \dfrac{\omega_n}{\omega_d}e^{-at}\sin(\omega_d t+2\phi)$
414	$\dfrac{\omega_n^2}{s\left(s^2+2as+\omega_n^2\right)}$	$1 - \dfrac{\omega_n}{\omega_d}e^{-at}\sin(\omega_d t+\phi)$
415	$\dfrac{\omega_n}{s^2+2as+\omega_n^2}$	$\dfrac{\omega_n}{\omega_d}e^{-at}\sin\omega_d t$
416	$\dfrac{s}{s^2+2as+\omega_n^2}$	$-\dfrac{\omega_n}{\omega_d}e^{-at}\sin(\omega_d t-\phi)$
417	$\dfrac{s^2}{\omega_n\left(s^2+2as+\omega_n^2\right)}$	$\dfrac{\delta}{\omega_n} + \dfrac{\omega_n}{\omega_d}e^{-at}\sin(\omega_d t-2\phi)$
418	$\dfrac{s^3}{\omega_n^2\left(s^2+2as+\omega_n^2\right)}$	$\dfrac{\delta'}{\omega_n^2} - \dfrac{2\zeta\delta}{\omega_n} - \dfrac{\omega_n}{\omega_d}e^{-at}\sin(\omega_d t-3\phi)$
419	$\dfrac{s^4}{\omega_n^3\left(s^2+2as+\omega_n^2\right)}$	$\dfrac{\delta''}{\omega_n^3} - \dfrac{2\zeta\delta'}{\omega_n^2} + (4\zeta^2-1)\dfrac{\delta}{\omega_n} + \dfrac{\omega_n}{\omega_d}e^{-at}\sin(\omega_d t-4\phi)$
420	Numerator Contains First Two Terms of Quadratic (See notes for entry 400.)	
422	$\dfrac{\omega_n^2(s+2a)}{s^2\left(s^2+2as+\omega_n^2\right)}$	$2at - (4\zeta^2-1) + \dfrac{\omega_n}{\omega_d}e^{-at}\sin(\omega_d t+3\phi)$
423	$\dfrac{\omega_n(s+2a)}{s\left(s^2+2as+\omega_n^2\right)}$	$2\zeta - \dfrac{\omega_n}{\omega_d}e^{-at}\sin(\omega_d t+2\phi)$
424	$\dfrac{s+2a}{s^2+2as+\omega_n^2}$	$\dfrac{\omega_n}{\omega_d}e^{-at}\sin(\omega_d t+\phi)$

	Laplace domain	Time domain
425	$\frac{s(s+2a)}{\omega_n(s^2+2as+\omega_n^2)}$	$\frac{\delta}{\omega_n}-\frac{\omega_n}{\omega_d}e^{-at}\sin\omega_d t$
426	$\frac{s^2(s+2a)}{\omega_n^2(s^2+2as+\omega_n^2)}$	$\frac{\delta'}{\omega_n^2}+\frac{\omega_n}{\omega_d}e^{-at}\sin(\omega_d t-\phi)$
427	$\frac{s^3(s+2a)}{\omega_n^3(s^2+2as+\omega_n^2)}$	$\frac{\delta''}{\omega_n^3}-\frac{\delta}{\omega_n}-\frac{\omega_n}{\omega_d}e^{-at}\sin(\omega_d t-2\phi)$
430	Numerator Contains First and Last Terms of Quadratic (See notes for entry 400.)	
432	$\frac{\omega_n^4(s^2+\omega_n^2)}{2as^4(s^2+2as+\omega_n^2)}$	$\frac{\omega_n^4t^3}{(2a)3!}-\frac{\omega_n^2t^2}{2}+2at-(4\zeta^2-1)+\frac{\omega_n}{\omega_d}e^{-at}\sin(\omega_d t+3\phi)$
433	$\frac{\omega_n^3(s^2+\omega_n^2)}{2as^3(s^2+2as+\omega_n^2)}$	$\frac{\omega_n^3t^2}{4a}-\omega_n t+\frac{2a}{\omega_n}-\frac{\omega_n}{\omega_d}e^{-at}\sin(\omega_d t+2\phi)$
434	$\frac{\omega_n^2(s^2+\omega_n^2)}{2as^2(s^2+2as+\omega_n^2)}$	$\frac{\omega_n^2t}{2a}-1+\frac{\omega_n}{\omega_d}e^{-at}\sin(\omega_d t+\phi)$
435	$\frac{\omega_n(s^2+\omega_n^2)}{2as(s^2+2as+\omega_n^2)}$	$\frac{\omega_n}{2a}-\frac{\omega_n}{\omega_d}e^{-at}\sin\omega_d t$
436	$\frac{s^2+\omega_n^2}{2a(s^2+2as+\omega_n^2)}$	$\frac{\delta}{2a}+\frac{\omega_n}{\omega_d}e^{-at}\sin(\omega_d t-\phi)$
437	$\frac{s(s^2+\omega_n^2)}{2a\omega_n(s^2+2as+\omega_n^2)}$	$\frac{\delta'}{2a\omega_n}-\frac{\delta}{\omega_n}-\frac{\omega_n}{\omega_d}e^{-at}\sin(\omega_d t-2\phi)$
438	$\frac{s^2(s^2+\omega_n^2)}{2a\omega_n^2(s^2+2as+\omega_n^2)}$	$\frac{\delta''}{2a\omega_n^2}-\frac{\delta'}{\omega_n^2}+\frac{2\zeta\delta}{\omega_n}+\frac{\omega_n}{\omega_d}e^{-at}\sin(\omega_d t-3\phi)$
440	Numerator Contains Last Two Terms of Quadratic (See notes for entry 400.)	
442	$\frac{\omega_n^4(2as+\omega_n^2)}{s^5(s^2+2as+\omega_n^2)}$	$\frac{\omega_n^4t^4}{4!}-\frac{\omega_n^2t^2}{2!}+2at-(4\zeta^2-1)+\frac{\omega_n}{\omega_d}e^{-at}\sin(\omega_d t+3\phi)$
443	$\frac{\omega_n^3(2as+\omega_n^2)}{s^4(s^2+2as+\omega_n^2)}$	$\frac{\omega_n^3t^3}{3!}-\omega_n t+2\zeta-\frac{\omega_n}{\omega_d}e^{-at}\sin(\omega_d t+2\phi)$

	Laplace domain	Time domain
444	$\frac{\omega_n^2(2as+\omega_n^2)}{s^3(s^2+2as+\omega_n^2)}$	$\frac{\omega_n^2 t^2}{2!} - 1 + \frac{\omega_n}{\omega_d} e^{-at}\sin(\omega_d t + \phi)$
445	$\frac{\omega_n(2as+\omega_n^2)}{s^2(s^2+2as+\omega_n^2)}$	$\omega_n t - \frac{\omega_n}{\omega_d} e^{-at}\sin\omega_d t$
446	$\frac{2as+\omega_n^2}{s(s^2+2as+\omega_n^2)}$	$1 + \frac{\omega_n}{\omega_d} e^{-at}\sin(\omega_d t - \phi)$
447	$\frac{2as+\omega_n^2}{\omega_n(s^2+2as+\omega_n^2)}$	$-\frac{\omega_n}{\omega_d} e^{-at}\sin(\omega_d t - 2\phi)$
448	$\frac{s(2as+\omega_n^2)}{\omega_n^2(s^2+2as+\omega_n^2)}$	$\frac{2\zeta\delta}{\omega_n} + \frac{\omega_n}{\omega_d} e^{-at}\sin(\omega_d t - 3\phi)$
449	$\frac{s^2(2as+\omega_n^2)}{\omega_n^3(s^2+2as+\omega_n^2)}$	$\frac{2\zeta\delta'}{\omega_n^2} - \frac{(4\zeta^2-1)\delta}{\omega_n} - \frac{\omega_n}{\omega_d} e^{-at}\sin(\omega_d t - 4\phi)$
460	Numerator Contains s Plus Half-Damping (See notes for entry 400.) Note that, generally, damping produces the term $2a$. Thus, the appearance of the term a represents half-damping. See Appendix 1 for definition of symbols.	
462	$\frac{\omega_n^3(s+a)}{s^3(s^2+2as+\omega_n^2)}$	$\frac{a\omega_n t^2}{2} - (2\zeta^2-1)\omega_n t + \zeta(4\zeta^2-3) - e^{-at}\cos(\omega t + 3\phi)$
463	$\frac{\omega_n^2(s+a)}{s^2(s^2+2as+\omega_n^2)}$	$at - (2\zeta^2-1) + e^{-at}\cos(\omega t + 2\phi)$
464	$\frac{\omega_n(s+a)}{s(s^2+2as+\omega_n^2)}$	$\zeta - e^{-at}\cos(\omega t + \phi)$
465	$\frac{s+a}{s^2+2as+\omega_n^2}$	$e^{-at}\cos\omega t$
466	$\frac{s(s+a)}{\omega_n(s^2+2as+\omega_n^2)}$	$\frac{\delta}{\omega_n} - e^{-at}\cos(\omega t - \phi)$
467	$\frac{s^2(s+a)}{\omega_n^2(s^2+2as+\omega_n^2)}$	$\frac{\delta'}{\omega_n^2} + \frac{\zeta\delta}{\omega_n} + e^{-at}\cos(\omega t - 2\phi)$
468	$\frac{s^3(s+a)}{\omega_n^3(s^2+2as+\omega_n^2)}$	$\frac{\delta''}{\omega_n^2} + \frac{\zeta\delta'}{\omega_n^2} + \frac{(2\zeta^2-1)\delta}{\omega_n} - e^{-at}\cos(\omega t$

	Laplace domain	Time domain
470	Half-Damping Plus ω_n^2 in Numerator (See notes for entry 400.) See note for entry 460. See Appendix 1 for definition of symbols.	
472	$\frac{\omega_n^3(as+\omega_n^2)}{s^4(s^2+2as+\omega_n^2)}$	$\frac{\omega_n^3 t^3}{3!} - \frac{a\omega_n t^2}{2} + (2\zeta^2-1)\omega_n t - \zeta(4\zeta^2-3) + e^{-at}\cos(\omega t + 3\phi)$
473	$\frac{\omega_n^2(as+\omega_n^2)}{s^3(s^2+2as+\omega_n^2)}$	$\frac{\omega_n^2 t^2}{2!} - at + (2\zeta^2-1) - e^{-at}\cos(\omega t + 2\phi)$
474	$\frac{\omega_n(as+\omega_n^2)}{s^2(s^2+2as+\omega_n^2)}$	$\omega_n t - \zeta + e^{-at}\cos(\omega t + \phi)$
475	$\frac{as+\omega_n^2}{s(s^2+2as+\omega_n^2)}$	$1 - e^{-at}\cos\omega t$
476	$\frac{as+\omega_n^2}{\omega_n(s^2+2as+\omega_n^2)}$	$e^{-at}\cos(\omega t - \phi)$
477	$\frac{s(as+\omega_n^2)}{\omega_n^2(s^2+2as+\omega_n^2)}$	$-\frac{\zeta\delta}{\omega_n} - e^{-at}\cos(\omega t - 2\phi)$
478	$\frac{s^2(as+\omega_n^2)}{\omega_n^3(s^2+2as+\omega_n^2)}$	$-\frac{\zeta\delta'}{\omega_n^2} + \frac{(2\zeta^2-1)\delta}{\omega_n} + e^{-at}\cos(\omega t - 3\phi)$
490	General First Degree Factor in Numerator *Note*: The constant c is not related to any other constant in the system. For the special constant a, see notes for entry 400.	
493	$\frac{\omega_n^2(s+c)}{s^2(s^2+2as+\omega_n^2)}$	$ct + \left[2\zeta^2 - 1 + 2\zeta\frac{(c-a)}{\omega}\right] + \frac{r}{\omega_d}e^{-at}\sin(\omega t + \phi_r + 2\phi)$
494	$\frac{\omega_n(s+c)}{s(s^2+2as+\omega_n^2)}$	$\frac{c}{\omega_n} - \frac{r}{\omega_d}e^{-at}\sin(\omega t + \phi_r + \phi)$
495	$\frac{s+c}{s^2+2as+\omega_n^2}$	$\frac{r}{\omega_d}e^{-at}\sin(\omega t + \phi_r)$
496	$\frac{s(s+c)}{\omega_n(s^2+2as+\omega_n^2)}$	$\frac{\delta}{\omega_n} - \frac{r}{\omega_d}e^{-at}\sin(\omega t + \phi_r - \phi)$
497	$\frac{s^2(s+c)}{\omega_n^2(s^2+2as+\omega_n^2)}$	$\frac{\delta'}{\omega_n^2} + \left[\frac{(c-a)}{\omega} - \zeta\right]\frac{\delta}{\omega_n} + \frac{r}{\omega_d}e^{-at}\sin(\omega t + \phi_r - 2\phi)$

	Laplace domain	Time domain
500	*Underdamped Cubics* *Note*: The 500 family can be used only if the damping ratio ζ of the quadratic factor is greater than zero but less than unity (see notes for entry 400). For overdamped cubics, see group 240. $\omega = \omega_d$	
510	Numerator is Single Term	
513	$\dfrac{\omega_n^2 r^2}{s^2(s+c)(s^2+2as+\omega_n^2)}$	$\dfrac{r^2}{\omega_n^2} + \dfrac{c^2+2(a^2-\omega^2)}{\omega_n^2} - \dfrac{\omega_n^2}{c^2} + \dfrac{\omega_n^2 e^{-ct}}{c^2} + \dfrac{r}{\omega_d} e^{-at}\sin(\omega t - \phi_r + 2\phi)$
514	$\dfrac{\omega_n r^2}{s(s+c)(s^2+2as+\omega_n^2)}$	$\dfrac{r^2}{c\omega_n} - \dfrac{\omega_n e^{-ct}}{c} - \dfrac{r}{\omega_d} e^{-at}\sin(\omega t - \phi_r + \phi)$
515	$\dfrac{r^2}{(s+c)(s^2+2as+\omega_n^2)}$	$e^{-ct} + \dfrac{r}{\omega_d} e^{-at}\sin(\omega t - \phi_r)$
516	$\dfrac{r^2 s}{\omega_n(s+c)(s^2+2as+\omega_n^2)}$	$-\dfrac{ce^{-ct}}{\omega_n} - \dfrac{r}{\omega_d} e^{-at}\sin(\omega t - \phi_r - \phi)$
517	$\dfrac{r^2 s^2}{\omega_n^2(s+c)(s^2+2as+\omega_n^2)}$	$\dfrac{c^2 e^{-ct}}{\omega_n^2} + \dfrac{r}{\omega_d} e^{-at}\sin(\omega t - \phi_r - 2\phi)$
518	$\dfrac{r^2 s^3}{\omega_n^3(s+c)(s^2+2as+\omega_n^2)}$	$\dfrac{(c-\omega_n)^2\delta}{\omega_n^3} - \dfrac{c^3 e^{-ct}}{\omega_n^3} - \dfrac{r}{\omega_d} e^{-at}\sin(\omega t - \phi_r - 3\phi)$
520	Numerator Contains First Two Terms of Quadratic (See notes for entry 400.)	
524	$\dfrac{r^2(s+2a)}{\omega_n(s+c)(s^2+2as+\omega_n^2)}$	$\dfrac{2r^2}{c\omega_n} - \dfrac{(\omega_n^2+r^2)e^{-ct}}{c\omega_n} - \dfrac{r}{\omega_d} e^{-at}\sin(\omega t - \phi_r + \phi)$
525	$\dfrac{r^2 s(s+2a)}{\omega_n^2(s+c)(s^2+2as+\omega_n^2)}$	$2e^{-ct} + \dfrac{r}{\omega_d} e^{-at}\sin(\omega t - \phi_r)$
526	$\dfrac{r^2 s^2(s+2a)}{\omega_n^3(s+c)(s^2+2as+\omega_n^2)}$	$\dfrac{r^2\delta}{\omega_n^3} - \dfrac{c(\omega_n^2+r^2)e^{-ct}}{\omega_n^3} - \dfrac{r}{\omega_d} e^{-at}\sin(\omega t - \phi_r - \phi)$

	Laplace domain	Time domain
530	Numerator Contains First and Last Terms of Quadratic (See notes for entry 400.)	
534	$\dfrac{\omega_n r^2(s^2+\omega_n^2)}{2as^2(s+c)(s^2+2as+\omega_n^2)}$	$\dfrac{\omega_n r^2 t}{2ac} - \dfrac{r^2(\omega_n^2-2ac)}{2ac^2\omega_n} + \dfrac{\omega_n(r^2-2ac)e^{-ct}}{2ac^2} - \dfrac{r}{\omega_d}e^{-at}\sin(\omega t-\phi_r+\phi)$
535	$\dfrac{r^2(s^2+\omega_n^2)}{2as(s+c)(s^2+2as+\omega_n^2)}$	$\dfrac{r^2}{2ac} - \dfrac{(r^2-2ac)e^{-ct}}{2ac} + \dfrac{r}{\omega_d}e^{-at}\sin(\omega t-\phi_r)$
536	$\dfrac{r^2(s^2+\omega_n^2)}{2a\omega_n(s+c)(s^2+2as+\omega_n^2)}$	$\dfrac{(r^2-2ac)e^{-ct}}{2a\omega_n} - \dfrac{r}{\omega_d}e^{-at}\sin(\omega t-\phi_r-\phi)$
540	Numerator Contains Last Two Terms of Quadratic (See notes for entry 400.)	
544	$\dfrac{\omega_n r^2(2as+\omega_n^2)}{s^3(s+c)(s^2+2as+\omega_n^2)}$	$\dfrac{\omega_n r^2 t^2}{2c} - \dfrac{\omega_n r^2 t}{c^2} + \dfrac{r^2(\omega_n^2+c^2)}{c^3\omega_n} - \dfrac{\omega_n(r^2+c^2)e^{-ct}}{c^3} - \dfrac{r}{\omega_d}e^{-at}\sin(\omega t-\phi_r+\phi)$
545	$\dfrac{r^2(2as+\omega_n^2)}{s^2(s+c)(s^2+2as+\omega_n^2)}$	$\dfrac{r^2 t}{c} - \dfrac{r^2}{c^2} + \dfrac{(r^2+c^2)e^{-ct}}{c^2} + \dfrac{r}{\omega_d}e^{-at}\sin(\omega t-\phi_r)$
546	$\dfrac{r^2(2as+\omega_n^2)}{\omega_n s(s+c)(s^2+2as+\omega_n^2)}$	$\dfrac{r^2}{c\omega_n} - \dfrac{(r^2+c^2)e^{-ct}}{c\omega_n} - \dfrac{r}{\omega_d}e^{-at}\sin(\omega t-\phi_r-\phi)$

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