Laplace Solution:

$$\ddot{x} + 2\zeta\omega_n \dot{x} + \omega_n^2 x = f(t)$$

where: $f(t) = A\,u(t)$ $\qquad x(0) = x_{IC}$, $\dot{x}(0) = 0$

Recall: $\mathcal{L}[f(t)] = F(s) = A/s$

$$\mathcal{L}[\dot{x}] = s\,X(s) - x(0)$$

$$\mathcal{L}[\ddot{x}] = \mathcal{L}[\dot{d}] \qquad \text{where } d \triangleq \dot{x}$$

$$= s\,D(s) - d(0) \qquad \text{but } D(s) = \mathcal{L}[d] = \mathcal{L}[\dot{x}]$$

$$= s\left[sX(s) - x(0)\right] - \dot{x}(0) = s^2 X(s) - s\,x(0) - \dot{x}(0)$$

$$\therefore \mathcal{L}\{\text{diff eqn}\} \Rightarrow s^2 X(s) - s\,x(0) - \dot{x}(0) + 2\zeta\omega_n\{s\,X(s) - x(0)\} + \omega_n^2 X(s) = A/s$$

$$X(s)\left[s^2 + 2\zeta\omega_n s + \omega_n^2\right] = A/s + \underbrace{\dot{x}(0)}_{0} + \underbrace{x(0)}_{x_{IC} = \text{const}}\left[s + 2\zeta\omega_n\right]$$

$$\therefore X(s) = \frac{\frac{A}{s} + x_{IC}\left[s + 2\zeta\omega_n\right]}{s^2 + 2\zeta\omega_n s + \omega_n^2} \qquad \underline{\text{assume } \zeta < 1}$$

$$\therefore \text{poles @ } s_{1,2} = \frac{-2\zeta\omega_n}{2} \pm \frac{1}{2}\sqrt{(2\zeta\omega_n)^2 - 4\omega_n^2}$$

$$= -\underbrace{\zeta\omega_n}_{a} \pm j\underbrace{\omega_n\sqrt{1-\zeta^2}}_{\omega_d} \qquad \text{if } \zeta < 1$$

$$\therefore s_{1,2} = -a \pm j\omega_d$$

Hence: $X(s) = \dfrac{\frac{A}{s} + x_{IC}[s+2a]}{(s^2+2as+\omega_n^2)} = \dfrac{A}{s(\quad)} + \dfrac{x_{IC}[s+2a]}{(\quad)}$

Note: from Tables: (Cochin & Plass)

$$(414): \quad \frac{\omega_n^2}{s(s^2+2as+\omega_n^2)} \underset{\mathcal{L}}{\overset{\mathcal{L}^{-1}}{\Longleftrightarrow}} 1 - \frac{\omega_n}{\omega_d} e^{-at} \sin(\omega_d t + \phi) \cdot u(t)$$

$$(424): \quad \frac{s+2a}{s^2+2as+\omega_n^2} \Longleftrightarrow \frac{\omega_n}{\omega_d} e^{-at} \sin(\omega_d t + \phi) \cdot u(t)$$

where: $\phi = \tan^{-1}(\omega/a)$

$\omega = \omega_d = \omega_n\sqrt{1-\zeta^2}$

Now: $\dfrac{A}{s(\quad)} = \dfrac{A}{\omega_n^2} \cdot \dfrac{\omega_n^2}{s(\quad)} \Longleftrightarrow \dfrac{A}{\omega_n^2}\left[1 - \dfrac{\omega_n}{\omega_d} e^{-at} \sin(\omega_d t + \phi)\right]$

$$\frac{x_{IC}[s+2a]}{(\quad)} = x_{IC} \cdot \frac{s+2a}{(\quad)} \Longleftrightarrow x_{IC} \frac{\omega_n}{\omega_d} e^{-at} \sin(\omega_d t + \phi)$$

$$\therefore x(t) = \underbrace{\frac{A}{\omega_n^2}\left[1 - \frac{\omega_n}{\omega_d} e^{-at} \sin(\omega_d t + \phi)\right] u(t)}_{\text{forced (step) response}} + \underbrace{x_{IC} \frac{\omega_n}{\omega_d} e^{-at} \sin(\omega_d t + \phi) \cdot u(t)}_{\text{initial value response}}$$

$$\Updownarrow \qquad\qquad\qquad \Updownarrow$$

$$X(s) = \frac{A}{s(s^2+2as+\omega_n^2)} + x_{IC} \frac{(s+2a)}{(s^2+2as+\omega_n^2)}$$

Note the respective "duals"

<u>Note</u>: what if input were different forcing function?

(A) we used: $f(t) = A\,u(t) \iff F(s) = A/s$ (step)

with $X(s)\Big|_{\substack{\text{step}\\ \text{in}}} = \dfrac{A/s}{(s^2 + 2as + \omega_n^2)}$

(B) if $f(t) = A\,t\,u(t) \iff F(s) = A/s^2$ (ramp)

$\therefore X(s)\Big|_{\substack{\text{ramp}\\ \text{in}}} = \dfrac{A}{s^2(s^2+2as+\omega_n^2)} = \dfrac{A}{\omega_n^3} \cdot \dfrac{\omega_n^3}{s^2(\quad)}$ → Form 413

$\therefore x(t)\Big|_{\substack{\text{ramp}\\ \text{in}}} = \dfrac{A}{\omega_n^3}\left[\omega_n t - 2\zeta + \dfrac{\omega_n}{\omega_d} e^{-at} \sin(\omega_d t + 2\phi)\right] u(t)$

<u>look</u>: $\dfrac{A}{\omega_n^3}\omega_n t = \dfrac{A}{\omega_n^2} t$ ⇝ ramp out

$\dfrac{A}{\omega_n^3}(-2\zeta) = \dfrac{-A\zeta}{\omega_n^3}$ ⇝ amplitude shift

$\dfrac{A}{\omega_n^3} \dfrac{\omega_n}{\omega_d} e^{-at} \sin(\omega_d t + 2\phi)$ → transient.

(C) if $f(t) = A\,\delta(t) \iff F(s) = A$ (impulse)

$X(s)\Big|_{\substack{\text{impulse}\\ \text{in}}} = \dfrac{A}{(s^2+2as+\omega_n^2)} = \dfrac{A}{\omega_n} \cdot \dfrac{\omega_n}{(s^2+2as+\omega_n^2)}$ → Form 415

$x(t)_{\text{impulse in}} = \dfrac{A}{\omega_n} \cdot \dfrac{\omega_n}{\omega_d} e^{-at} \sin \omega_d t$ ⇝ like initial value response

Laplace Solution

MJK 1/3

Note on sinusoidal inputs: (Van de Vegte)

consider a system: $G(s)$ with input $R(s)$ and output $C(s)$

$$r(t) \longrightarrow \boxed{\text{system}} \longrightarrow c(t) \quad \text{or} \quad R(s) \longrightarrow \boxed{G(s)} \longrightarrow C(s)$$

Let $r(t) = A \sin \omega t \Rightarrow R(s) = \dfrac{A\omega}{s^2+\omega^2}$

Now: $\dfrac{C(s)}{R(s)} = G(s)$ ~ Transfer function approach

and $C(s) = G(s) \cdot R(s) = G(s)\left(\dfrac{A\omega}{s^2+\omega^2}\right) = \underbrace{\dfrac{k_1}{s+j\omega} + \dfrac{k_2}{s-j\omega}}_{\text{from } R(s)} + \underbrace{\circ\circ\circ}_{\text{from } G(s)}$

$$k_1 = \left[(s+j\omega)\, C(s)\right]\Big|_{s=-j\omega} = \frac{A\omega\, G(s)}{s-j\omega}\Big|_{s=-j\omega} = \frac{A}{-2j}\, G(-j\omega)$$

$$K_2 = k_1^* = \frac{A}{2j}\, G(j\omega)$$

Recall: $\dfrac{1}{-2j} = \dfrac{1}{2} e^{\pi/2\, j}$

$\dfrac{1}{2j} = \dfrac{1}{2} e^{-\pi/2\, j}$

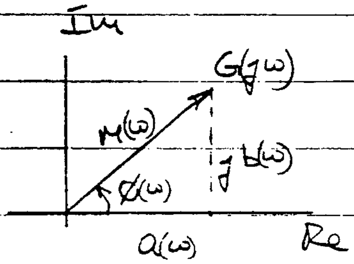
$\therefore$ Forced Response $\triangleq c_{ss}(t) = \mathcal{L}^{-1}\left[\dfrac{k_1}{s+j\omega} + \dfrac{k_2}{s-j\omega}\right]$

$$= \mathcal{L}^{-1}\left[\frac{A}{2j}\left\{\frac{-G(-j\omega)}{s+j\omega} + \frac{G(j\omega)}{s-j\omega}\right\}\right]$$

$$\boxed{c_{ss}(t) = \frac{A}{2j}\left[-G(-j\omega)\, e^{-j\omega t} + G(j\omega)\, e^{j\omega t}\right]}$$

Note: $G(j\omega)$ = complex variable $= a(\omega) + j\,b(\omega)$

$= M(\omega)\, e^{j\phi(\omega)}$



where: $M(\omega) = |G(j\omega)| = \sqrt{a^2(\omega) + b^2(\omega)}$

$\phi = \tan^{-1}\left(\frac{b(\omega)}{a(\omega)}\right)$

$$G(-j\omega) = G^*(j\omega) = M(\omega)\, e^{-j\phi(\omega)}$$

Hence: $C_{ss}(t) = \frac{A}{2j}\Big[\underbrace{-M(\omega)e^{-j\phi(\omega)}}_{-G(-j\omega)} e^{-j\omega t} + \underbrace{M(\omega)e^{j\phi(\omega)}}_{G(j\omega)} e^{j\omega t}\Big]$

$$= A\,M(\omega)\,\frac{\left[e^{j(\omega t+\phi)} - e^{-j(\omega t+\phi)}\right]}{2j}$$

$$\boxed{C_{ss}(t) = A\,M(\omega)\,\sin(\omega t + \phi)}$$

where: $M(\omega) = \sqrt{Re^2[G(j\omega)] + Im^2[G(j\omega)]} = \sqrt{a^2+b^2}$

$\phi(\omega) = \tan^{-1}\left[\frac{Im[G(j\omega)]}{Re[G(j\omega)]}\right] = \tan^{-1}\left(\frac{b}{a}\right)$

$\therefore$ if $G(s) = \frac{1}{s^2 + 2\zeta\omega_n s + \omega_n^2} \Rightarrow G(j\omega) = \frac{1}{(j\omega)^2 + 2\zeta\omega_n j\omega + \omega_n^2}$

$$= \frac{1}{(\omega_n^2 - \omega^2) + j\,2\zeta\omega_n\omega}$$

$$= \frac{1/\omega_n^2}{1 - (\omega/\omega_n)^2 + j\,(2\zeta\omega/\omega_n)}$$

$$\therefore\ G(j\omega) = \frac{1/\omega_n^2}{x + j\,y} \cdot \frac{x - j\,y}{x - j\,y} = \frac{(1/\omega_n^2)\,x - j\,y}{x^2 + y^2}$$

$x = 1 - (\omega/\omega_n)^2$

$y = 2\zeta\omega/\omega_n$

$$G(j\omega) = \underbrace{\frac{1/\omega_n^2\ x}{x^2+y^2}}_{Re} - j\,\underbrace{\frac{1/\omega_n^2\ y}{x^2+y^2}}_{Im}$$

$$\therefore\ M(\omega) = |G(j\omega)| = \sqrt{Re^2 + Im^2} = \left[\frac{(1/\omega_n^2)^2 x^2}{(x^2+y^2)^2} + \frac{(1/\omega_n^2)^2 y^2}{(\quad)^2}\right]^{1/2}$$

$$= \left[\frac{(1/\omega_n^2)^2\,\cancel{[x^2+y^2]}}{(x^2+y^2)^{\cancel{2}}}\right]^{1/2}$$

ANS $$M(\omega) = \frac{1/\omega_n^2}{\sqrt{[1-(\omega/\omega_n)^2]^2 + (2\zeta\omega/\omega_n)^2}}$$

ANS $$\phi(\omega) = \tan^{-1}\left[\frac{Im}{Re}\right] = \tan^{-1}\left[-y/x\right] = \tan^{-1}\left[\frac{-2\zeta\omega/\omega_n}{1-(\omega/\omega_n)^2}\right]$$

$$= -\tan^{-1}\left[\frac{2\zeta\omega/\omega_n}{1-(\omega/\omega_n)^2}\right]$$