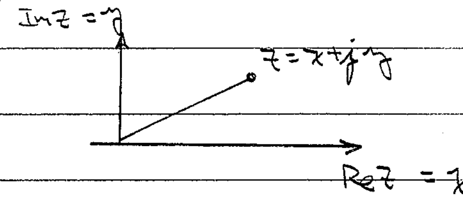
Complex Variables:

Let $z = x + jy$ Represent a (general) complex # $j \equiv \sqrt{-1}$

where: $\text{Re}\, z \equiv x$

$\text{Im}\, z \equiv y$



Now suppose I have a pair of complex #'s:

Algebraic operations:

$$\left.\begin{aligned} z_1 &= x_1 + j y_1 \\ z_2 &= x_2 + j y_2 \end{aligned}\right\}$$

a) $z_1 = z_2$ iff $x_1 = x_2$ and $y_1 = y_2$

b) $z_1 \pm z_2 = (x_1 \pm x_2) + j(y_1 \pm y_2)$

c) $z_1 z_2 = (x_1 + j y_1)(x_2 + j y_2)$

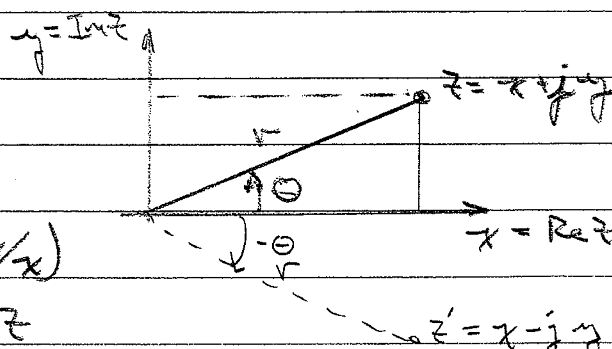
d) $z_3 = z_1/z_2 = \dfrac{x_1 + j y_1}{x_2 + j y_2}$ multiply top & bottom by complex conjugate of z_2

(i.e., c.c. of $z_2 \equiv x_2 - j y_2$)

$$\therefore z_1/z_2 = \frac{(x_1 + j y_1)}{(x_2 + j y_2)} \frac{(x_2 - j y_2)}{(x_2 - j y_2)} = \frac{(x_1 + j y_1)(x_2 - j y_2)}{(x_2^2 - \cancel{j x_2 y_2} + \cancel{j x_2 y_2} + y_2^2)}$$

$$= \underbrace{\frac{x_1 x_2 + y_1 y_2}{x_2^2 + y_2^2}}_{\text{Re}\, z_1/z_2} + j \underbrace{\frac{x_2 y_1 - x_1 y_2}{x_2^2 + y_2^2}}_{\text{Im}\, z_1/z_2}$$

Polar / Trigonometric Form



$$\left.\begin{aligned} x &= r\cos\theta \\ y &= r\sin\theta \end{aligned}\right\} \quad \begin{aligned} r &= \sqrt{x^2 + y^2} \\ \theta &= \arctan(y/x) \\ &\equiv \arg z \end{aligned}$$

Hence:
$$\boxed{\begin{aligned} z &= x+jy = r(\cos\theta + j\sin\theta) \\ z' &= x-jy = r(\cos\theta - j\sin\theta) \end{aligned}} \quad (1)$$

where: $|z| = \sqrt{x^2+y^2} = r$

Exponential Formulation

Recall! $e^{x_1+x_2} = e^{x_1}e^{x_2}$ and $\frac{d}{dx}(e^x) = e^x$

$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$ Maclaurin (Taylor) Series
$\left(e^x = \sum_{n=0}^{\infty} x^n/n!\right)$

Now! If $z = x + jy$ (as before)

$$e^z = e^{x+jy} = e^x e^{jy} \quad (2)$$

$$e^z = 1 + z + \frac{z^2}{2!} + \cdots = \sum_{n=0}^{\infty}\frac{z^n}{n!} \quad \text{(Maclaurin Series)} \quad (3)$$

If $z = jy$ in (3): $e^{jy} = \sum_{n=0}^{\infty}\frac{(jy)^n}{n!} = \underbrace{\sum_{k=0}^{\infty}(-1)^k\frac{y^{2k}}{(2k)!}}_{\text{Maclaurin series for: } \cos y} + j\underbrace{\sum_{k=0}^{\infty}(-1)^k\frac{y^{2k+1}}{(2k+1)!}}_{\sin y}$

$\therefore \boxed{e^{jy} = \cos y + j\sin y}$ (4) Euler Formula

also $\boxed{e^{-jy} = \cos y - j\sin y}$

$\therefore$ (2) becomes $e^z = e^x(\cos y + j\sin y) = e^x e^{jy}$

$\therefore e^{\pm jy} = \cos y \pm j\sin y$ ⊛

Hence (1) can be rewritten as one of the following!

$$\boxed{\begin{aligned} z = x+jy &= r(\cos\theta + j\sin\theta) = re^{j\theta} \\ r = \sqrt{x^2+y^2} \quad & \quad \theta = \arctan y/x \end{aligned}} \quad (5)$$

also note from (4), that $\left|e^{jy}\right| = \sqrt{\cos^2 y + \sin^2 y} = 1$

$$\arg e^{z} = y$$

and that $e^{2\pi j} = \cancel{\cos}^{1}\, 2\pi + j\, \cancel{\sin}^{0}\, 2\pi = 1$

$$e^{\pi j} = e^{-\pi j} = -1 \qquad e^{\frac{\pi}{2} j} = j \qquad e^{-\frac{\pi}{2} j} = -j$$

and $\frac{1}{j} = j^{-1} = e^{-\frac{\pi}{2} j}$ (A)

Hence $e^{z \pm 2\pi j} = e^{z} e^{\pm 2\pi j} = e^{z}$

$\therefore$ in general $\boxed{e^{z \pm 2\pi n j} = e^{z} \qquad n = 0, 1, 2, \ldots}$ (6)

$\therefore e^{z}$ is periodic w/ Imaginary period $2\pi j$

<u>other useful formulae:</u>

α in radians ; $j \equiv \sqrt{-1}$

$$e^{\pm j\alpha} = \cos\alpha \pm j \sin\alpha$$

$$\boxed{\begin{aligned} \sin\alpha &= \frac{e^{j\alpha} - e^{-j\alpha}}{2j} \qquad \cos\alpha = \frac{e^{j\alpha} + e^{-j\alpha}}{2} \\ \tan\alpha &= -j\left(\frac{e^{j\alpha} - e^{-j\alpha}}{e^{j\alpha} + e^{-j\alpha}}\right) = -j\left(\frac{e^{2j\alpha} - 1}{e^{2j\alpha} + 1}\right) \end{aligned}} \quad (7)$$

Note:

$$2j \sin\alpha = e^{j\alpha} - e^{-j\alpha}$$
$$= \cos\alpha + j\sin\alpha - e^{-j\alpha}$$

$\therefore e^{-j\alpha} = \cos\alpha + \cancel{j\sin\alpha} - \cancel{2}j \sin\alpha$ $\quad \therefore \boxed{e^{-j\alpha} = \cos\alpha - j\sin\alpha}$ (8)

Euler formulas: $e^{\pm j\theta} = \cos\theta \pm j\sin\theta$

Note: $|e^{\pm j\theta}| = |\cos\theta \pm j\sin\theta|$

$= \sqrt{\cos^2\theta + \sin^2\theta} = 1$

Recall: $\sin(a \pm b) = \sin a \cos b \pm \cos a \sin b$

$\cos(a \pm b) = \cos a \cos b \mp \sin a \sin b$

let: $e^{+j\theta} = \cos\theta + j\sin\theta$ (1)

$e^{-j\theta} = \cos\theta - j\sin\theta$ (2)

now add eqns (1) & (2): $e^{+j\theta} + e^{-j\theta} = 2\cos\theta$

$\therefore \boxed{\cos\theta = \dfrac{e^{+j\theta} + e^{-j\theta}}{2}}$ (3)

Also subtract (2) from (1): $e^{+j\theta} - e^{-j\theta} = 2j\sin\theta$

$\therefore \boxed{\sin\theta = \dfrac{e^{+j\theta} - e^{-j\theta}}{2j}}$ (4)

but note: $\boxed{j = e^{\pi/2\, j}} = \underbrace{\cos \pi/2}_{0} + j\underbrace{\sin \pi/2}_{1}$ (5)

and $\boxed{1/j = j^{-1} = e^{-\pi/2\, j}}$ (6)

also $\boxed{1 = e^{\pm 2\pi j}} = \underbrace{\cos 2\pi}_{1} + j\underbrace{\sin 2\pi}_{0}$ (7)

$\boxed{-1 = e^{-\pi j} = e^{\pi j}}$ (8)

Now look at eqn (4)

$$\sin\theta = \frac{e^{j\theta} - e^{-j\theta}}{2j} = \frac{(e^{j\theta} - e^{-j\theta})}{2} \cdot j^{-1}$$

$$= \frac{(e^{j\theta} - e^{-j\theta})}{2} \cdot e^{-\pi/2\, j}$$

$$= \frac{1}{2}\left[e^{j\theta}e^{-\pi/2\, j} - e^{-j\theta}e^{-\pi/2\, j}\right]$$

but $-1 = e^{\pi j}$

$$= \frac{1}{2}\left[e^{j\theta}e^{-\pi/2\, j} + e^{\pi j}e^{-j\theta}e^{-\pi/2\, j}\right]$$

$$= \frac{1}{2}\left[e^{j(\theta - \pi/2)} + e^{-j(\theta + \pi/2 - \pi)}\right]$$

$\therefore$

$$\boxed{\sin\theta = \frac{1}{2}\left[e^{j(\theta - \pi/2)} + e^{-j(\theta - \pi/2)}\right]}$$

using (3)
by comparison

$$\boxed{\sin\theta = \cos(\theta - \pi/2)}$$

check $\cos(\theta - \pi/2) = \cos\theta \underbrace{\cos\pi/2}_{0} + \sin\theta \underbrace{\sin\pi/2}_{1} = \sin\theta$

QED