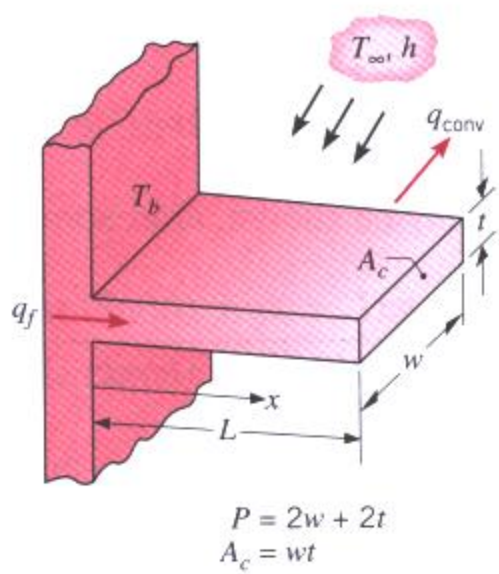
Finite Differences & Matrix Math Example: Temperature Distribution in a Rectangular Fin

Fins are extended surfaces used to enhance heat transfer, like those on an automotive radiator assembly, or on high-power computer chips. Consider the rectangular fin shown in the figure below, described by the geometric dimensions of length L, width w, thickness t, perimeter P, and cross-sectional area A_c; made of a material with thermal conductivity k; subjected to convective heat transfer from the surface characterized by a heat transfer coefficient h and ambient temperature T_∞; and maintained at a base temperature (at $x=0$) T_b and tip temperature (at $x=L$) of T_L. Following a one-dimensional heat transfer assumption, the temperature distribution in the fin is governed by the following ordinary differential equation (ODE):

$$\frac{d^2T}{dx^2} - \frac{hP}{kA_c}(T - T_\infty) = 0 \tag{1}$$



Because the fin is subjected to known temperatures at each end, this comprises a **boundary value problem** for solution. To simplify matters, consider the following substitutions:

$$\theta(x) \equiv T(x) - T_\infty\,, \quad m^2 \equiv \frac{hP}{kA_c} \tag{2}$$

With these in hand, the governing ODE (1) becomes one for $\theta(x)$ instead:

$$\frac{d^2\theta}{dx^2} - m^2\theta = 0 \tag{3}$$

The boundary conditions established by T_b and T_L must also be transformed into values for θ_b and θ_L as well. By appropriate methods, the exact solution to (3) and the specified boundary conditions becomes:

$$\theta(x) = \frac{\theta_L \sinh(mx) + \theta_b \sinh(m(L-x))}{\sinh(mL)} \tag{4}$$

Alternatively, a finite difference approach can be used to **approximate** a solution for $\theta(x)$. Consider setting up a series of $N+1$ equally-spaced nodal points x_i which span the entire length of the fin from x_0 at $x=0$ to x_N at $x=L$, separated by the distance Δx. At each x_i, (3) can be approximated with a 3-point finite difference equation as (refer to the handout *Survey of Difference Approximations* for more details):

$$\frac{\theta(x_{i+1}) - 2\theta(x_i) + \theta(x_{i-1})}{(\Delta x)^2} - m^2 \cdot \theta(x_i) = 0 \tag{5}$$

Note that the approximation in (5) is **not** applied at x_0 nor x_N, since in this case the values of θ are already known there. Thus, by applying (5) to all N - 1 **interior** nodes yields N - 1 linear simultaneous equations in the N - 1 unknown values of θ.

Consider a rectangular fin with the following conditions:

$L = 500$ mm
$w = 62$ mm
$t = 4$ mm
$h = 65$ W/m^2·K
$k = 35$ W/m·K
$T_\infty = 20$ ºC
$T_b = 98$ ºC
$T_L = 35$ ºC

Using the approach outlined above, set up and solve for the approximate temperature distribution $\theta(x)$ with 50 divisions ($N = 50$) in x. Compare the results with those given by the exact solution in (4).