Survey of Difference Approximations

Finite difference approximations are rarely applied to one value of x and no others. Typically we are interested in differentiating an entire domain of x values, or an entire set of experimental measurements, or in repeating an approximation to solve a differential equation. Thus the following nomenclature is commonly used instead of that which we have developed so far:

- Instead of talking about points such as x, x-Δx, x+Δx, and so on, we talk about ***nodal points*** (or simply ***nodes***) x_i, x_{i-1}, x_{i+1}, and the like. It is understood that x_{i-1} is one node to the "left" of the current x_i, or at x-Δx. Likewise x_{i+1} is one node to the "right" of the current x_i, or at x+Δx. By extension x_{i+2} is at x+2Δx, and so on.
- Δx is represented with the variable h instead, purely for simplicity.

With these changes, the following is a list of common finite difference approximations.

Forward Difference Formulas

$$\frac{df}{dx}(x_i) = \frac{f(x_{i+1}) - f(x_i)}{h} + O(h) \qquad \textit{(2-point)}$$

$$\frac{df}{dx}(x_i) = \frac{-f(x_{i+2}) + 4f(x_{i+1}) - 3f(x_i)}{2h} + O(h^2) \qquad \textit{(3-point)}$$

$$\frac{d^2f}{dx^2}(x_i) = \frac{f(x_{i+2}) - 2f(x_{i+1}) + f(x_i)}{h^2} + O(h) \qquad \textit{(3-point)}$$

$$\frac{d^2f}{dx^2}(x_i) = \frac{-f(x_{i+3}) + 4f(x_{i+2}) - 5f(x_{i+1}) + 2f(x_i)}{h^2} + O(h^2) \qquad \textit{(4-point)}$$

Centered Difference Formulas

$$\frac{df}{dx}(x_i) = \frac{f(x_{i+1}) - f(x_{i-1})}{2h} + O(h^2) \qquad \textit{(3-point)}$$

$$\frac{df}{dx}(x_i) = \frac{-f(x_{i+2}) + 8f(x_{i+1}) - 8f(x_{i-1}) + f(x_{i-2})}{12h} + O(h^4) \qquad \textit{(5-point)}$$

$$\frac{d^2f}{dx^2}(x_i) = \frac{f(x_{i+1}) - 2f(x_i) + f(x_{i-1})}{h^2} + O(h^2) \qquad \textit{(3-point)}$$

$$\frac{d^2f}{dx^2}(x_i) = \frac{-f(x_{i+2}) + 16f(x_{i+1}) - 30f(x_i) + 16f(x_{i-1}) - f(x_{i-2})}{12h^2} + O(h^4) \qquad \textit{(5-point)}$$

Backward Difference Formulas

$$\frac{df}{dx}(x_i) = \frac{f(x_i) - f(x_{i-1})}{h} + O(h) \qquad \text{(2-point)}$$

$$\frac{df}{dx}(x_i) = \frac{3f(x_i) - 4f(x_{i-1}) + f(x_{i-2})}{2h} + O(h^2) \qquad \text{(3-point)}$$

$$\frac{d^2f}{dx^2}(x_i) = \frac{f(x_i) - 2f(x_{i-1}) + f(x_{i-2})}{h^2} + O(h) \qquad \text{(3-point)}$$

$$\frac{d^2f}{dx^2}(x_i) = \frac{2f(x_i) - 5f(x_{i-1}) + 4f(x_{i-2}) - f(x_{i-3})}{h^2} + O(h^2) \qquad \text{(4-point)}$$