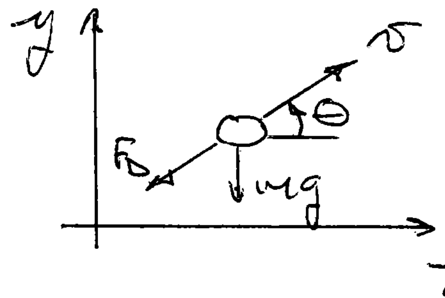
ODE Solution

Recall our projectile motion (2-D) problem with aerodynamic drag which was formulated using 2-coupled equations of motion (EOM) in x & y:



Here: $F_D = \frac{1}{2} C_D \rho A v^2$

and $\vec{v} = v_x \hat{i} + v_y \hat{j}$ vectoral

$= v\cos\theta\,\hat{i} + v\sin\theta\,\hat{j}$

and $v = |\vec{v}| = \sqrt{v_x^2 + v_y^2}$

so that $\cos\theta = \frac{v_x}{v}$, $\sin\theta = v_y/v$

after $\Sigma F_x = m a_x$ we obtained with $a_x = dv_x/dt$

$$\underline{EOM_x}: \quad \frac{dv_x}{dt} = -\frac{c}{m}\sqrt{v_x^2 + v_y^2} \cdot v_x \qquad (1)$$

after $\Sigma F_y = m a_y$ we obtained with $a_y = dv_y/dt$

$$\underline{EOM_y}: \quad \frac{dv_y}{dt} = -\frac{c}{m}\sqrt{v_x^2 + v_y^2} \cdot v_y - g \qquad (2)$$

<u>where</u>: $c = \frac{1}{2} C_D \rho A$ for object in motion (Ball)

also via calculus:

$$\frac{d}{dt}(\text{displacement}) = \text{velocity} \quad \begin{cases} \dfrac{dx}{dt} = v_x & (3) \\[2ex] \dfrac{dy}{dt} = v_y & (4) \end{cases}$$

<u>note</u>: that eqns (1) & (2) are coupled requiring simultaneous solution

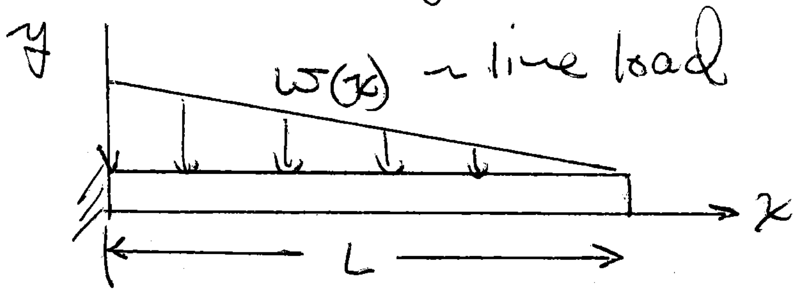
Note also: that eqns (1)-(4) are all 1st order diff eqns with (presumably) constant coefficients

$\Longrightarrow$ so called "state equations" of systems theory

But all of eqns (1) $\to$ (4) represent gradients which dictate how the "state variables" (here v_x, v_y, x, y) evolve over time

Recall: Engineering Mechanics / Strength of Matls

Cantilever Beam deflection:



$\Rightarrow$ yields $M(x)$ bending moment

from Euler-Bernoulli Beam theory: $\frac{d^2y}{dx^2} = -\frac{M(x)}{EI}$
(Beam curvature)

we define Beam slope $\triangleq \frac{dy}{dx} = g(x)$ (5)

Hence curvature could be rewritten as $\frac{dg}{dx} = -\frac{M(x)}{EI}$ (6)

$\therefore$ these slope & curvature eqns are also "state eqns" in the independent variable x

General solution scheme

(*) note that for the projectile [eqns (1) → (4)] or for the beam [eqns (5) & (6)] require the integration of their respective state eqns in order to obtain the solution variables:

→ $v_x(t)$, $v_y(t)$ and $x(t)$, $y(t)$ for projectile

→ $y(x)$, $\theta(x)$ for beam

(*) each solution has its respective independent variable (t ~ projectile, x ~ beam) but this is largely application specific, as are the dependent variables

(*) as the respective solutions evolve (play out) the right-hand-sides of eqns (1)–(6) dictate the rate of variable evolution. These rates need-not be constant, hence the solution will depend non-linearly on the independent variable and not-necessarily via some polynomial (as exponentials, sinusoids, etc. may be involved)

(*) so to solve our state eq'ns we need an accurate way of integrating (all of) our state eq'ns:

<u>to start</u>: presume that a Taylor (McLaurin) Series expansion would yield a (near) exact solution for a generic problem with $y(x)$ being the solution variable, and x being the independent variable

<u>say we are given</u>: our state eq'n $\frac{dy}{dx} = f(x,y)$ $\Delta x \triangleq h$

subject to boundary (initial) condition y_{IC} @ x_{IC}

<u>Taylor series</u> (discrete data assumed) approximation

$$y_{i+1} = y_i + y_i' \cdot h + \frac{y_i''}{2!} h^2 + \cdots + \frac{y_i^{(n)}}{n!} h^n + \text{remainder } O(h^{n+1})$$

1st derivative $\left.\frac{dy}{dx}\right|_i$ — 2nd derivative $\left.\frac{d^2y}{dx^2}\right|_i$ — n-th derivative $\left.\frac{d^ny}{dx^n}\right|_i$

but by our state eq'n $\to f(x_i, y_i)$ $\to f'(x_i, y_i)$ $\to f^{(n-1)}(x_i, y_i)$

(*) each successive term of the Taylor series yields a more accurate approximation to the 'true' value of y_{i+1}

Big caveat: (Q) do we have values for $f, f', f'' \ldots$ evaluated at our (x_i, y_i) data pair ??

(A) if not, then any "truncation" of series worsens the approximation, but might make series implementation "easier"

$\therefore$ Accuracy vs ease-of-use trade-offs exist naturally

Easiest Method: Euler's Method

$$y_{i+1} = y_i + f(x_i, y_i) \cdot h$$

1st order Method (since $f \sim \frac{dy}{dx}$)

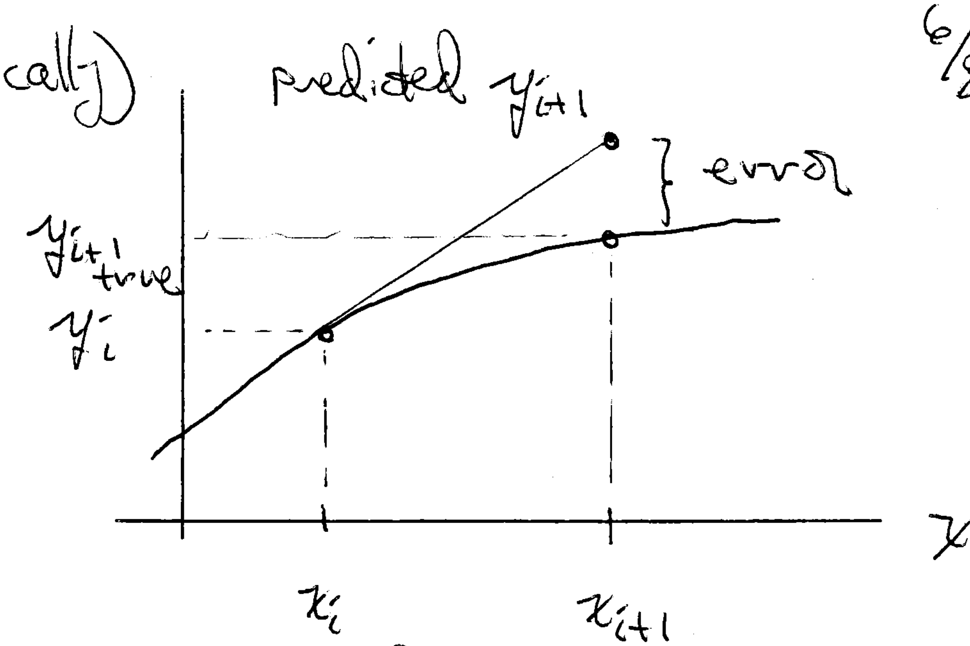
Note: missing all terms above 1st order $\sim n=1$

$\therefore$ local (@ this point) truncation error $O(h^{n+1}) = O(h^2)$

and global (cummulative) error $O(h^n) = O(h^1)$

Euler (graphically)

(text Fig 25.2)



(note: Euler's Method is what we have been using for projectile & HW1)

Improved Euler (AKA Heun's Method)

notice that Euler's Method only requires data at the current location (i) to predict the state value at ~~i+1~~ the next location ($i+1$); since we 'know' (x_i, y_i) & can calculate $f(x_i, y_i)$, then

$y_{i+1} = y_i + f(x_i, y_i) \cdot h$ is Explicit

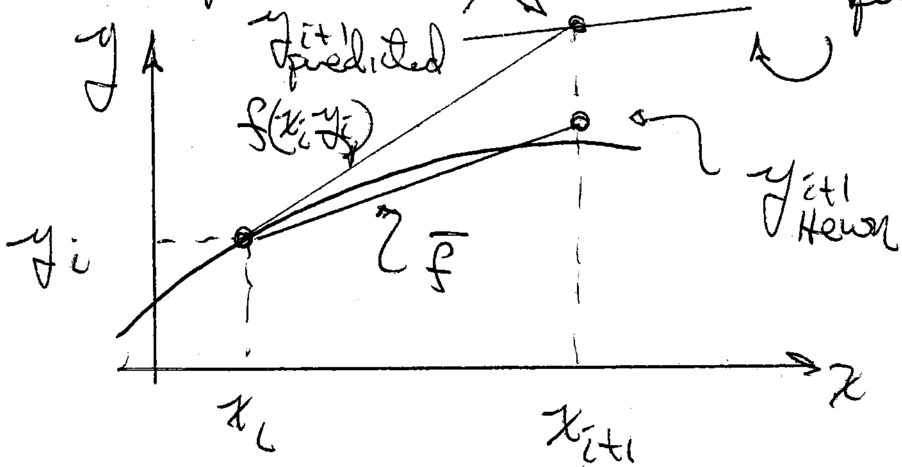
However, as can be seen in the Figure (25.2), the likelyhood over 'overshooting' where y_{i+1} should be (namely $y_{i+1 \text{ true}}$) is high since $y_{i+1 \text{ predicted}}$ the slope $f(x, y)$ is changing as x progresses, and is lower (as shown) @ $x = x_{i+1}$

so Heun proposed a predictor-corrector approach to get a better y_{i+1}:

predict: $y_{i+1_{predicted}} = y_i + f(x_i, y_i) \cdot h$ a la Euler

↳ text calls this y^0_{i+1}

estimate slope @ $i+1$ via $f(x_{i+1}, y_{i+1_{predicted}})$



now get a "corrected slope" via averaging

$$\bar{f} = \left[\frac{f(x_i, y_i) + f(x_{i+1}, y_{i+1_{predicted}})}{2}\right]$$

so that a "corrected" estimate can be made:

$$y_{i+1_{Heun}} = y_i + \left[\frac{f(x_i, y_i) + f(x_{i+1}, y_{i+1_{predicted}})}{2}\right] \cdot h$$

thus $y_{i+1_{Heun}}$ is implicit due to use of predicted y_{i+1}

Note also that the "corrected gradient" term $\bar{f}$ when multiplied by the interval h is essentially a trapezoidal rule for integration of acceleration $\frac{d^2y}{dx^2} = f'(x,y)$

Hence the remainder terms are related to $f''(h^3)$ so that:

$$\text{truncation errors for Heun} \simeq O(h^3) = O(h^{n+1})$$

$$\text{global errors for Heun} \simeq O(h^2) = O(h^n)$$

with $n=2$

$\therefore$ Heun is said to be a 2nd order method

and $\downarrow h$ causes more rapid Reduction in error over Euler's Method

Text Reading:

Ch 25.1 – 25.2 ODE (Euler, Heun)
Ch 3.1 – 3.4 Errors, Roundoff, Truncation
Ch 4.1 Taylor series