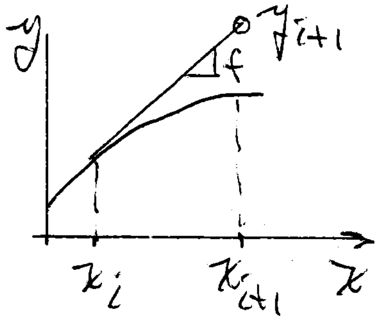
ODE Solution (con't)

Previously we noted $\frac{dy}{dx} \triangleq y' \triangleq f(x, y)$

a) Euler Integrator $y_{i+1} = y_i + f(x_i, y_i)$



1st order: since only uses $f = y' = \frac{d^1 y}{dx^1}$

Explicit: uses 'known' info @ x_i, y_i to 'shoot' solution fwd

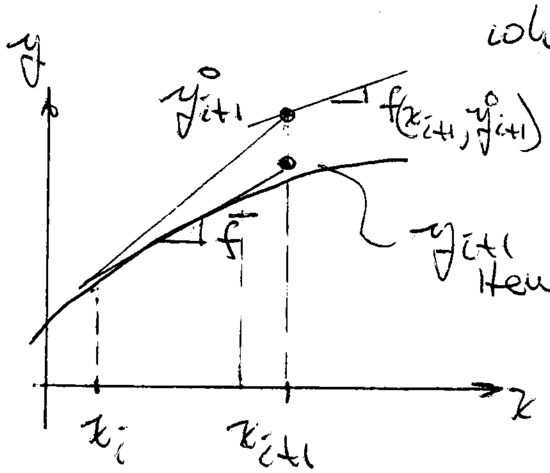
b) Heun's Method $y_{i+1} = y_i + \underbrace{\left(\frac{f(x_i, y_i) + f(x_{i+1}, y^0_{i+1})}{2}\right)}_{\bar{f}} h$

AKA: "corrector eqn"

which uses $y^0_{i+1} = y_i + f(x_i, y_i) \cdot h$

Euler

to predict y_{i+1} so it can be used to estimate slope @ (x_{i+1}) which is then averaged with Euler slope @ x_i

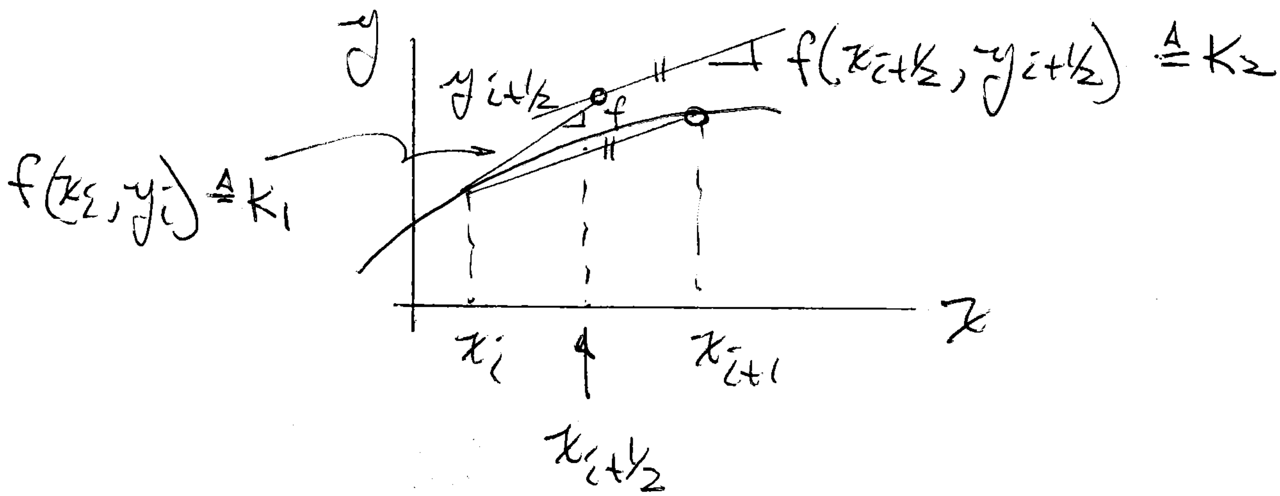


2nd order since corrector eqn slope $\bar{f}$ is mean of $f(x_i, y_i)$ and $f(x_{i+1}, y^0_{i+1})$

Implicit: since y^0_{i+1} is a 'guesstimate' but could be improved with successive iterations now try to do Better:

c) Modified Euler (midpoint) Method

since Heun's Method shows promise, why not just use Euler to shoot ½-way between x_i & x_{i+1}, and then predict $y'(x_{i+1/2}, y_{i+1/2})$ and use this pseudo-mean slope to shoot the entire interval from x_i to x_{i+1}



$$\therefore y_{i+1} = y_i + f(x_{i+1/2}, y_{i+1/2}) \cdot h$$

<u>2nd order</u> since $f(x_{i+1/2}, y_{i+1/2})$ is essentially a centered finite difference slope calculation

~~Implicit~~ This is also a <u>predictor-corrector</u> approach ~~which~~

✱ note: Heun may overestimate y_{i+1}

midpoint may underestimate y_{i+1}

✱ So why not work some combination ⇒ <u>Runge-Kutta</u> to improve accuracy

d) <u>Classical 4th order Runge-Kutta</u> (many variations)

premise: use a weighted combination of slope estimates a la Euler, Heun, & midpoint to get effective <u>4th order approximation</u> for y_{i+1} with <u>truncation errors</u> $O(h^5)$
<u>global errors</u> $O(h^4)$

d1) use Euler @ $x_i \xrightarrow{\text{to get}} f(x_i, y_i) \triangleq k_1$ rename for convenience

d2) use midpoint @ $x_{i+1/2} = x_i + \frac{1}{2}h \longrightarrow f(x_{i+1/2}, y_{i+1/2})$
$\triangleq k_2$

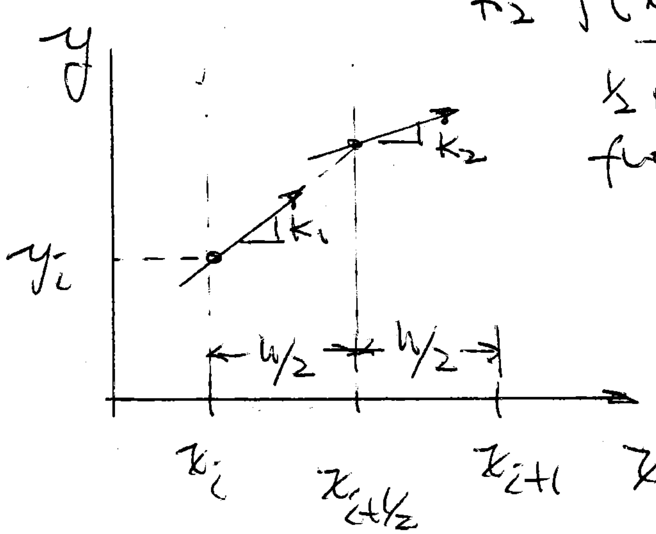
<u>note</u>: a more explicit notation has emmerged in the text

$k_1 = f(x_i, y_i)$ ~ slope evaluated @ (x_i, y_i) a/a Euler

$k_2 = f(\underline{x_i + \frac{1}{2}h}, y_i + \frac{1}{2}\underline{k_1 h})$

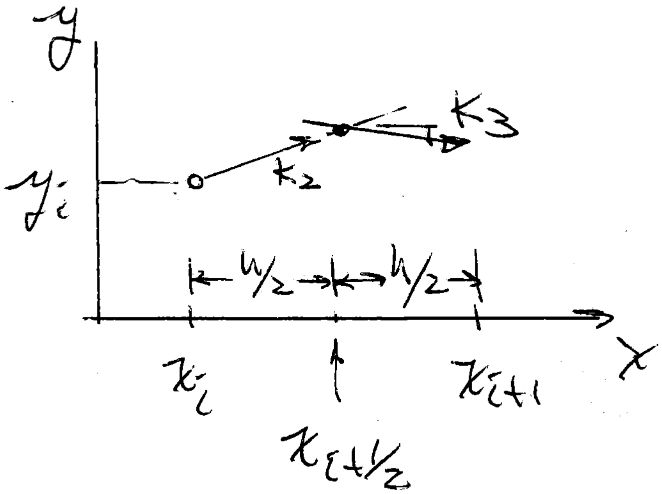
½ interval from x_i

Euler slope shot ½ way across interval (i.e. $k_1 \cdot \frac{h}{2}$)



d3) use midpoint again @ $x_{i+1/2}$
but this time use k_2 as slope to shoot ½ way
"~~pseudo Euler~~"

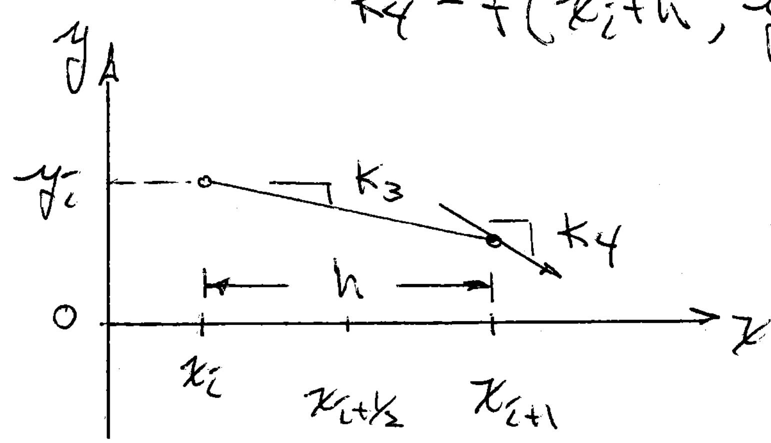
$$k_3 = f\left(x_i + \tfrac{1}{2}h,\ y_i + k_2 \tfrac{h}{2}\right)$$



midpoint slope used to shoot ½ way across interval

d4) use pseudo-Euler to shoot across entire interval ~~to~~ using k_3 and estimate slope at end of interval

$$k_4 = f(x_i + h,\ y_i + k_3 \cdot h)$$



now assemble each prediction of y_{i+1}:
with weights

d1) $[y_{i+1} = y_i + k_1 h] * \cancel{w_1} \rightarrow 1$ 1st order

d2) $[y_{i+1} = y_i + k_2 h] * \cancel{w_2} \rightarrow 2$ 2nd order

d3) $[y_{i+1} = y_i + k_3 h] * \cancel{w_3} \rightarrow 2$ 3rd order

d4) $[y_{i+1} = y_i + k_4 h] * \cancel{w_4} \rightarrow 1$ 4th order

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$$d_1+d_2+d_3+d_4 \Rightarrow 6y_{i+1} = 6y_i + \{k_1+2k_2+2k_3+k_4\}\, h$$

normalize:

$$y_{i+1} = y_i + \frac{h}{6}\{k_1+2k_2+2k_3+k_4\}$$

<u>4th order</u> local truncation error ~ $O(h^5)$

global error ~ $O(h^4)$