Table 3.1.3 Solution forms.

Equation	Solution form
First order: $\dot{x} + ax = b \quad a \neq 0$	$x(t) = \dfrac{b}{a} + Ce^{-at}$
Second order: $\ddot{x} + a\dot{x} + bx = c \quad b \neq 0$	
1. $(a^2 > 4b)$ distinct, real roots: s_1, s_2	$x(t) = C_1 e^{s_1 t} + C_2 e^{s_2 t} + \dfrac{c}{b}$
2. $(a^2 = \overset{4b}{\cancel{b/4}})$ repeated, real roots: s_1, s_1	$x(t) = (C_1 + tC_2)e^{s_1 t} + \dfrac{c}{b}$
3. $(a = 0, b > 0)$ imaginary roots: $s = \pm j\omega$, $\omega = \sqrt{b}$	$x(t) = C_1 \sin\omega t + C_2 \cos\omega t + \dfrac{c}{b}$
4. $(a \neq 0, a^2 < 4b)$ complex roots: $s = \sigma \pm j\omega$, $\sigma = -a/2, \omega = \sqrt{4b - a^2}/2$	$x(t) = e^{\sigma t}(C_1 \sin\omega t + C_2 \cos\omega t) + \dfrac{c}{b}$

Table 3.3.1 Table of Laplace transform pairs.

	$X(s)$	$x(t),\ t \geq 0$
1.	1	$\delta(t)$, unit impulse
2.	$\dfrac{1}{s}$	$u_s(t)$, unit step
3.	$\dfrac{c}{s}$	constant, c
4.	$\dfrac{e^{-sD}}{s}$	$u_s(t - D)$, shifted unit step
5.	$\dfrac{n!}{s^{n+1}}$	t^n
6.	$\dfrac{1}{s+a}$	e^{-at}
7.	$\dfrac{1}{(s+a)^n}$	$\dfrac{1}{(n-1)!} t^{n-1} e^{-at}$
8.	$\dfrac{b}{s^2 + b^2}$	$\sin bt$
9.	$\dfrac{s}{s^2 + b^2}$	$\cos bt$
10.	$\dfrac{b}{(s+a)^2 + b^2}$	$e^{-at} \sin bt$
11.	$\dfrac{s+a}{(s+a)^2 + b^2}$	$e^{-at} \cos bt$
12.	$\dfrac{a}{s(s+a)}$	$1 - e^{-at}$
13.	$\dfrac{1}{(s+a)(s+b)}$	$\dfrac{1}{b-a}\left(e^{-at} - e^{-bt}\right)$
14.	$\dfrac{s+p}{(s+a)(s+b)}$	$\dfrac{1}{b-a}\left[(p-a)e^{-at} - (p-b)e^{-bt}\right]$
15.	$\dfrac{1}{(s+a)(s+b)(s+c)}$	$\dfrac{e^{-at}}{(b-a)(c-a)} + \dfrac{e^{-bt}}{(c-b)(a-b)} + \dfrac{e^{-ct}}{(a-c)(b-c)}$
16.	$\dfrac{s+p}{(s+a)(s+b)(s+c)}$	$\dfrac{(p-a)e^{-at}}{(b-a)(c-a)} + \dfrac{(p-b)e^{-bt}}{(c-b)(a-b)} + \dfrac{(p-c)e^{-ct}}{(a-c)(b-c)}$

Table 8.1.1 Free, step, and ramp response of $\tau\dot{y} + y = r(t)$.

Free response $[r(t) = 0]$

$y(t) = y(0)e^{-t/\tau}$

$y(\tau) \approx 0.37y(0)$

$y(4\tau) \approx 0.02y(0)$

Step response $[r(t) = Ru_s(t), y(0) = 0]$

$y(t) = R(1 - e^{-t/\tau})$

$y(\infty) = y_{ss} = R$

$y(\tau) \approx 0.63y_{ss}$

$y(4\tau) \approx 0.98y_{ss}$

Ramp response $[r(t) = mt, y(0) = 0]$

$y(t) = m(t - \tau + \tau e^{-t/\tau})$

Table 3.1.2 The exponential function.

Taylor series

$$e^x = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \cdots + \frac{x^n}{n!} + \cdots$$

Euler's identities

$e^{j\theta} = \cos\theta + j\sin\theta$

$e^{-j\theta} = \cos\theta - j\sin\theta$

Limits

$\lim_{x\to\infty} xe^{-x} = 0$ if x is real.

$\lim_{t\to\infty} e^{-st} = 0$ if the real part of s is positive.

If a is real and positive,

$e^{-at} < 0.02$ if $t > 4/a$.

$e^{-at} < 0.01$ if $t > 5/a$.

The *time constant* is $\tau = 1/a$.

Table 8.3.1 Unit step response of a stable second-order model.

Model: $m\ddot{x} + c\dot{x} + kx = u_s(t)$

Initial conditions: $x(0) = \dot{x}(0) = 0$

Characteristic roots: $s = \dfrac{-c \pm \sqrt{c^2 - 4mk}}{2m} = -r_1, -r_2$

1. Overdamped case ($\zeta > 1$): distinct, real roots: $r_1 \neq r_2$

$$x(t) = A_1 e^{-r_1 t} + A_2 e^{-r_2 t} + \frac{1}{k} = \frac{1}{k}\left(\frac{r_2}{r_1 - r_2} e^{-r_1 t} - \frac{r_1}{r_1 - r_2} e^{-r_2 t} + 1\right)$$

2. Critically damped case ($\zeta = 1$): repeated, real roots: $r_1 = r_2$

$$x(t) = (A_1 + A_2 t) e^{-r_1 t} + \frac{1}{k} = \frac{1}{k}[(-r_1 t - 1) e^{-r_1 t} + 1]$$

3. Underdamped case ($0 \leq \zeta < 1$): complex roots: $s = -\zeta\omega_n \pm j\omega_n\sqrt{1 - \zeta^2}$

$$x(t) = Be^{-t/\tau} \sin\left(\omega_n \sqrt{1 - \zeta^2} t + \phi\right) + \frac{1}{k}$$

$$= \frac{1}{k}\left[\frac{1}{\sqrt{1 - \zeta^2}} e^{-\zeta\omega_n t} \sin\left(\omega_n \sqrt{1 - \zeta^2} t + \phi\right) + 1\right]$$

$$\phi = \tan^{-1}\left(\frac{\sqrt{1 - \zeta^2}}{\zeta}\right) + \pi \quad \text{(third quadrant)}$$

Time constant: $\tau = 1/\zeta\omega_n$

Table 10.5.3 Routh-Hurwitz stability conditions.

1. **Second-Order:** $a_2 s^2 + a_1 s + a_0 = 0$
   Stable if and only if a_2, a_1, and a_0 all have the same sign.
2. **Third-Order:** $a_3 s^3 + a_2 s^2 + a_1 s + a_0 = 0$
   Assuming $a_3 > 0$, stable if and only if a_2, a_1, and a_0 are all positive and $a_2 a_1 > a_3 a_0$.
3. **Fourth-Order:** $a_4 s^4 + a_3 s^3 + a_2 s^2 + a_1 s + a_0 = 0$
   Assuming $a_4 > 0$, stable if and only if a_3, a_2, a_1, and a_0 are all positive, $a_2 a_3 > a_1 a_4$, and

$$a_1(a_2 a_3 - a_1 a_4) - a_0 a_3^2 > 0$$

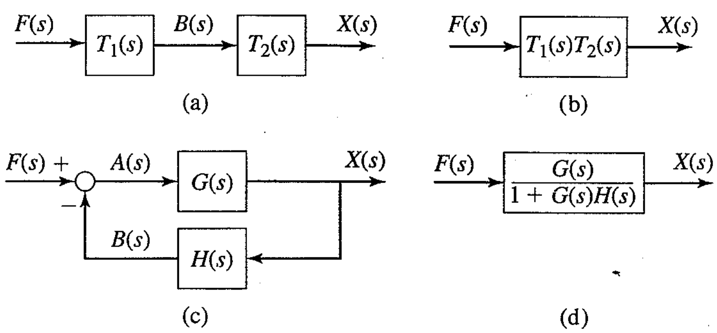
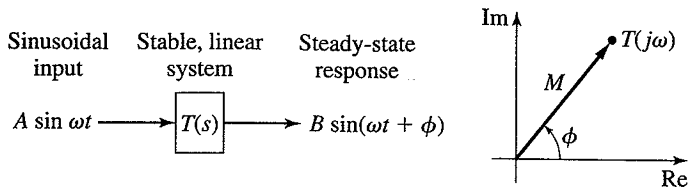


Figure 5.1.4 (a) and (b) Simplification of series blocks. (c) and (d) Simplification of a feedback loop.



$$M = \frac{B}{A} = |T(j\omega)|$$

$$\phi = \measuredangle\, T(j\omega)$$

Figure 9.1.2 Frequency response of a stable linear system.

Table 9.1.2 Frequency response of the model $\tau \dot{y} + y = f(t)$.

$M = \dfrac{\lvert Y \rvert}{\lvert F \rvert} = \dfrac{1}{\sqrt{1 + \omega^2 \tau^2}}$	(1)
$\phi = -\tan^{-1}(\omega \tau)$	(2)

Table 9.2.2 Frequency response of a second-order system.

Model:	$T(s) = \dfrac{\omega_n^2}{s^2 + 2\zeta \omega_n s + \omega_n^2}$	
Resonant frequency:	$\omega_r = \omega_n \sqrt{1 - 2\zeta^2}$	$0 \le \zeta \le 0.707$
Resonant response:	$M_r = \dfrac{1}{2\zeta \sqrt{1 - \zeta^2}}$	$0 \le \zeta \le 0.707$
	$m_r = -20 \log \left(2\zeta \sqrt{1 - \zeta^2}\right)$	$0 \le \zeta \le 0.707$
	$\phi_r = -\tan^{-1} \dfrac{\sqrt{1 - 2\zeta^2}}{\zeta}$	$0 \le \zeta \le 0.707$

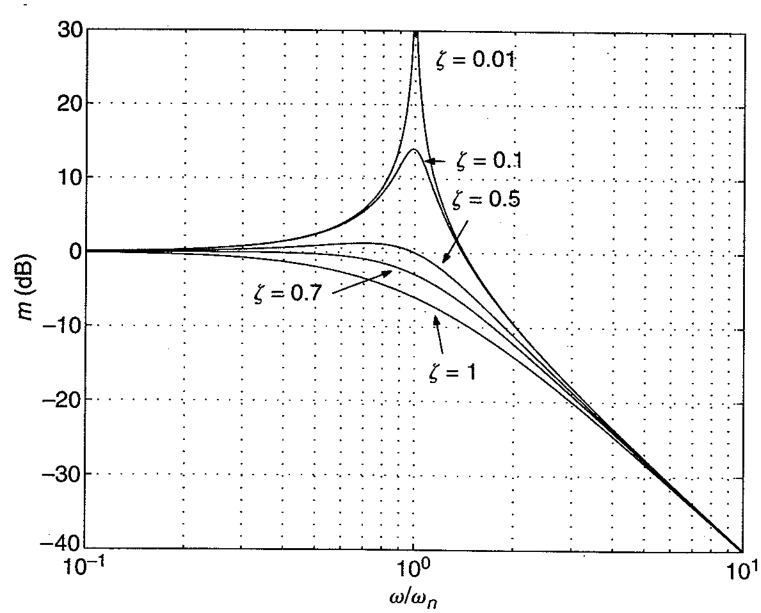


Figure 9.2.4 Semilog plot of log magnitude ratio of the model $\omega_n^2/(s^2 + 2\zeta\omega_n s + \omega_n^2)$.

Table 10.5.2 Useful results for second-order systems.

1. Model: $m\ddot{x} + c\dot{x} + kx = f(t)$
2. Transfer function:

$$\frac{X(s)}{F(s)} = \frac{1}{ms^2 + cs + k}$$

3. Characteristic equation: $ms^2 + cs + k = 0$
4. Characteristic roots:

$$s = \frac{-c \pm \sqrt{c^2 - 4mk}}{2m}$$

5. Damping ratio and undamped natural frequency:

$$\zeta = \frac{c}{2\sqrt{mk}} \qquad \omega_n = \sqrt{\frac{k}{m}}$$

6. Time constant: If $\zeta \le 1$,

$$\tau = \frac{2m}{c}$$

If $\zeta > 1$, the dominant (larger) time constant is

$$\tau_1 = \frac{2m}{c - \sqrt{c^2 - 4mk}}$$

and the secondary (smaller) time constant is

$$\tau_2 = \frac{2m}{c + \sqrt{c^2 - 4mk}}$$

7. Maximum percent overshoot and peak time:

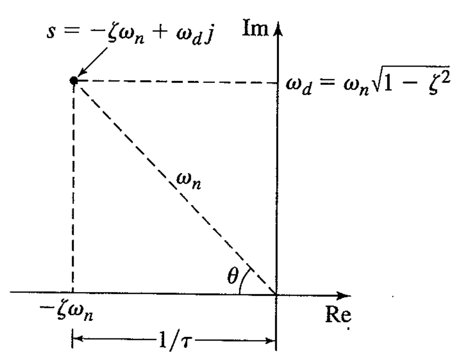
$$M_\% = 100e^{-\pi\zeta/\sqrt{1-\zeta^2}} \qquad t_p = \frac{\pi}{\omega_n\sqrt{1-\zeta^2}}$$

8. The complex root pair $s = -a \pm bj$ corresponds to the characteristic equation

$$(s + a)^2 + b^2 = 0$$

9. The value $\zeta = 0.707$ corresponds to a root pair having equal real and imaginary parts: $s = -a \pm aj$.

Figure 8.2.6 Graphical interpretation of the parameters ζ, τ, ω_n, and ω_d.



$$\zeta = \cos\theta = \cos\left[\tan^{-1}(\tau\omega_d)\right]$$