# Calculus BC - Worksheet on Improper Integrals

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**Relevant Formulas:**

$$\int_a^{\infty} f(x)\,dx = \lim_{b\to\infty} \int_a^b f(x)\,dx$$

$$\int_{-\infty}^{b} f(x)\,dx = \lim_{a\to\infty} \int_a^b f(x)\,dx$$

$$\int_{-\infty}^{\infty} f(x)\,dx = \int_{-\infty}^{c} f(x)\,dx + \int_c^{\infty} f(x)\,dx = \lim_{a\to\infty} \int_a^c f(x)\,dx + \lim_{b\to\infty} \int_c^b f(x)\,dx$$

Work the following on **notebook paper**. No calculator.

1. $\int_0^\infty \frac{2x}{(x^2+1)^2}\,dx$

   $\int_0^\infty \frac{2x}{(x^2+1)^2}\,dx = \lim_{b\to\infty}\int_0^b \frac{2x}{(x^2+1)^2}\,dx$

   Let $u = x^2+1 \therefore du = 2xdx$

   $\lim_{b\to\infty}\int_1^{b^2+1} \frac{2x}{(x^2+1)^2}\,dx = \lim_{b\to\infty}\int_1^{b^2+1} \frac{du}{u^2} = \lim_{b\to\infty}[\frac{-1}{u}]_1^{b^2+1} = \lim_{b\to\infty}(-\frac{1}{b^2+1}+\frac{1}{1}) = (\frac{1}{1}-0) = 1$

   $\therefore \int_0^\infty \frac{2x}{(x^2+1)^2}\,dx$ converges to 1

2. $\int_2^\infty \frac{3}{x^2-x}\,dx$

   $\int_2^\infty \frac{3}{x^2-x}\,dx = \lim_{b\to\infty} 3\int_2^b \frac{dx}{x^2-x} = \lim_{b\to\infty} 3\int_2^b \frac{dx}{x(x-1)} = \lim_{b\to\infty} 3\left(\int_2^b \frac{A}{x}\,dx + \int_2^b \frac{B}{x-1}\,dx\right) =$

   $\lim_{b\to\infty} 3\left(\int_2^b \frac{A(x-1)+Bx}{x(x-1)}\,dx\right)$

   $x(x-1) = 0 \to x = 0, 1$

   $1 = A(x-1) + Bx$

   $1 = A(1-1) + B(1) \therefore B = 1$

   $1 = A(0-1) + B(0) \therefore A = -1$

   $\therefore \lim_{b\to\infty} 3\left(\int_2^b \frac{A}{x}\,dx + \int_2^b \frac{B}{x-1}\,dx\right) = \lim_{b\to\infty} 3\left(\int_2^b \frac{dx}{x-1} - \int_2^b \frac{dx}{x}\right) = \lim_{b\to\infty} 3\left(\left[\ln\left(\frac{x-1}{x}\right)\right]_2^b\right) =$

   $\lim_{b\to\infty} 3\left(\ln\left(\frac{\frac{b}{b}-\frac{1}{b}}{\frac{b}{b}}\right) - \ln\left(\frac{2-1}{2}\right)\right) = 3\ln\left(\frac{1}{\frac{1}{2}}\right) = 3\ln 2$

   $\therefore \int_2^\infty \frac{3}{x^2-x}\,dx$ converges to $3\ln 2 \approx 2.079$

3. $\int_0^1 \frac{dx}{\sqrt{1-x^2}}$

   $\int_0^1 \frac{dx}{\sqrt{1-x^2}} = [\arcsin x]_0^1 = \frac{\pi}{2} - 0 = \frac{\pi}{2}$

   $\therefore \int_0^1 \frac{dx}{\sqrt{1-x^2}}$ converges to $\frac{\pi}{2} \approx 1.571$

4. $\int_0^2 \frac{x+1}{\sqrt{4-x^2}}\,dx$

   $\int_0^2 \frac{x+1}{\sqrt{4-x^2}}\,dx = \int_0^2 \frac{x}{\sqrt{4-x^2}}\,dx + \int_0^2 \frac{dx}{\sqrt{4-x^2}}$

   Let $u = 4 - x^2 \therefore du = -2xdx$

   $\int_0^2 \frac{x}{\sqrt{4-x^2}}\,dx + \int_0^2 \frac{dx}{\sqrt{4-x^2}} = \int_0^2 \frac{dx}{\sqrt{4-x^2}} - \frac{1}{2}\int_4^0 \frac{du}{\sqrt{u}} = \int_0^2 \frac{dx}{\sqrt{4-x^2}} + \frac{1}{2}\int_0^4 \frac{du}{\sqrt{u}} = [\arcsin\left(\frac{x}{2}\right)]_0^2 + \frac{1}{2}[2\sqrt{u}]_0^4 =$

   $[\arcsin 1 - \arcsin 0] + \frac{1}{2}[2\sqrt{4} - 0] = \frac{\pi}{2} + 2$

   $\therefore \int_0^2 \frac{x+1}{\sqrt{4-x^2}}\,dx$ converges to $\frac{\pi}{2} + 2 \approx 3.571$

5. $\int_0^{\ln 2} \frac{e^{\frac{1}{x}}}{x^2}\,dx$

   Let $u = \frac{1}{x} \therefore du = -\frac{1}{x^2}$

   Use the limits identity: $\lim_{\to 0^+} \frac{1}{x} = +\infty$

   $\int_0^{\ln 2} \frac{e^{\frac{1}{x}}}{x^2}\,dx = \lim_{a\to 0^+} -\int_{\frac{1}{a}}^{\frac{1}{\ln 2}} e^u\,du = \lim_{a\to\infty} -\int_a^{\frac{1}{\ln 2}} e^u\,du = \lim_{a\to\infty}\int_{\frac{1}{\ln 2}}^a e^u\,du = \lim_{a\to\infty}[e^u]_{\frac{1}{\ln 2}}^a =$

   $e^\infty - e^{\frac{1}{\ln 2}} = \infty$

   $\therefore \int_0^{\ln 2} \frac{e^{\frac{1}{x}}}{x^2}\,dx$ diverges to $\infty$

6. $\int_{-1}^{\infty} \frac{dx}{x^2+5x+6}$

   $\int_{-1}^{\infty} \frac{dx}{x^2+5x+6} = \lim_{b\to\infty}\int_{-1}^b \frac{dx}{x^2+5x+6} = \lim_{b\to\infty}\int_{-1}^b \frac{dx}{(x+2)(x+3)} = \lim_{b\to\infty}\left(\int_{-1}^b \frac{\alpha}{x+2}\,dx + \int_{-1}^b \frac{\beta}{x+3}\,dx\right)$

   $\frac{\alpha}{(x+3)} + \frac{\beta}{(x+2)} = \frac{\beta(x+3)}{(x+2)(x+3)} + \frac{\alpha(x+2)}{(x+2)(x+3)}$

   $1 = \alpha(x+2) + \beta(x+3) \to 1 = \alpha(-2+2) + \beta(-2+3) \to 1 = \beta$

   $1 = \alpha(-3+2) + \beta(-3+3) \to 1 = -\alpha$

   $\therefore \frac{1}{(x+2)(x+3)} = \frac{\alpha}{(x+3)} + \frac{\beta}{(x+2)} = \frac{1}{(x+2)} - \frac{1}{(x+3)}$

   $\lim_{b\to\infty}\int_{-1}^b \frac{dx}{x^2+5x+6} = \lim_{b\to\infty}\int_{-1}^b \frac{dx}{x+2} - \lim_{b\to\infty}\int_{-1}^b \frac{dx}{x+3} = \lim_{b\to\infty}\left[\ln\left|\frac{x+2}{x+3}\right|\right]_{-1}^b =$

   $\lim_{b\to\infty}\left[\ln\left|\frac{b+2}{b+3}\right| - \ln\left|\frac{2-1}{3-1}\right|\right] = \lim_{b\to\infty}\left[\ln\left|\frac{\frac{b}{b}+\frac{2}{b}}{\frac{b}{b}+\frac{3}{b}}\right| - \ln\frac{1}{2}\right] = \lim_{b\to\infty}\ln\left(2\left(\left|\frac{1+\frac{2}{b}}{1+\frac{3}{b}}\right|\right)\right) = \ln 2$

   $\therefore \int_{-1}^{\infty} \frac{dx}{x^2+5x+6}$ converges to $\ln 2 \approx 0.693$

7. $\int_{-8}^{1} \frac{dx}{\sqrt[3]{x}}$

   $\int_{-8}^{1} \frac{dx}{\sqrt[3]{x}} = \int_{-8}^{1}[\frac{3}{2}x^{\frac{2}{3}}]_{-8}^{1} = \left[\frac{3}{2}\left(1^{\frac{2}{3}} - (-8)^{\frac{2}{3}}\right)\right] = \frac{3}{2}\left(1-4\right) = -\frac{9}{2} = -4.5$

   $\therefore \int_{-8}^{1} \frac{dx}{\sqrt[3]{x}}$ converges to $-4.5$

8. $\int_{-\infty}^{0} xe^{x}\, dx$

   $\int_{-\infty}^{0} xe^{x}\, dx = \lim_{a\to-\infty} \int_{a}^{0} xe^{x}\, dx$

   Let $u = x \therefore du = dx$ and let $v = e^{x} \therefore dv = e^{x}\, dx$

   $\lim_{a\to-\infty} \int_{a}^{0} xe^{x}\, dx = \lim_{a\to-\infty} \left([xe^{x}]_{a}^{0} - [e^{x}]_{a}^{0}\right) = [0 - (-\infty)e^{-\infty}] - [1 - e^{-\infty}] = -1$

   $\therefore \int_{-\infty}^{0} xe^{x}\, dx$ converges to $-1$

9. $\int_{-\infty}^{2} \frac{2}{x^2+4}\, dx$

   $\int_{-\infty}^{2} \frac{2}{x^2+4}\, dx = \lim_{a\to-\infty} 2\int_{a}^{2} \frac{2}{x^2+4}\, dx = \lim_{a\to-\infty} \left[\arctan\left(\frac{x}{2}\right)\right]_{a}^{2} = \left[\arctan\left(\frac{2}{2}\right) - \arctan\left(\frac{-\infty}{2}\right)\right] =$

   $\frac{\pi}{4} + \frac{\pi}{2} = \frac{3\pi}{4}$

   $\therefore \int_{-\infty}^{2} \frac{2}{x^2+4}\, dx$ converges to $\frac{3\pi}{4}$