Calculus BC – Worksheet 1 on Differential Equations

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Relevant Formulas and Notes:

Let us define some common notational rules in the following table:

Formulas	Lagrange Notation	Leibniz Notation
Function	$f(x)$ or y	f or y
First Derivative	$f'(x)$ or y'	$\frac{df}{dx}$ or $\frac{dy}{dx}$ or $\frac{d}{dx}f(x)$
Second Derivative	$f''(x)$ or y''	$\frac{d^2f}{dx^2}$ or $\frac{d^2y}{dx^2}$ or $\frac{d^2}{dx^2}f(x)$
Other Derivatives	$f^{(n)}(x)$ or $y^{(n)}$	$\frac{d^nf}{dx^n}$ or $\frac{d^ny}{dx^n}$ or $\frac{d^n}{dx^n}f(x)$

For this worksheet, we will convert all forms of Lagrange Notation to Leibniz Notation to simplify integration technique.

Direct Proportionality of y to variable t:

$$y = kt$$

Inverse Proportionality of y to variable t:

$$y = \frac{k}{t}$$

Joint Proportionality of y to variables x, t:

$$y = kxt$$

Work the following on **notebook paper**.

1. $\frac{dy}{dx} = \frac{x-3}{y}$ and $y(2) = -5$

   $y\frac{dy}{dx} = x - 3$

   $ydy = (x-3)dx$

   $\int y\,dy = \int (x-3)\,dx$

   $\frac{1}{2}y^2 = \frac{1}{2}x^2 - 3x + C \to \frac{1}{2}\left(-5\right)^2 = \frac{1}{2}\left(2\right)^2 - 3\left(2\right) + C \to C = \frac{25-4+12}{2} = \frac{33}{2}$

   $y^2 = x^2 - 6x + 2C$

   $y = \pm\sqrt{x^2 - 6x + 2C} = \pm\sqrt{x^2 - 6x + 33}$

   Because the negative value is the only one that satisfies the initial condition $y(2) = -5$, $y = -\sqrt{x^2 - 6x + 33}$.

2. $y' = 2x\sqrt{y}$ and $y(2) = 25$

   $\frac{1}{\sqrt{y}}\frac{dy}{dx} = 2x$

   $\frac{dy}{\sqrt{y}} = 2xdx$

   $\int \frac{dy}{\sqrt{y}} = \int 2x\,dx$

   $2\sqrt{y} = x^2 + C \to 2\sqrt{25} = 2^2 + C \to C = 10 - 4 = 6$

   $\sqrt{y} = \frac{1}{2}x^2 + \frac{1}{2}C$

   $y = \left(\frac{1}{2}x^2 + \frac{1}{2}C\right) = \left(\frac{1}{2}x^2 + 3\right) = \frac{1}{4}x^4 + 3x^2 + 9$

3. $\frac{dy}{dx} = 4y^2\sec^2\left(2x\right)$ and $y\left(\frac{\pi}{8}\right) = 1$

   $\frac{1}{y^2}\frac{dy}{dx} = 4\sec^2\left(2x\right)$

   $\frac{dy}{y^2} = 4\sec^2\left(2x\right)dx$

   $\int \frac{dy}{y^2} = 4\int \sec^2\left(2x\right)dx$

   $-\frac{1}{y} = 2\tan\left(2x\right) + C \to -1 = 2\tan\left(\frac{\pi}{4}\right) + C \to C = -2 - 1 = -3$

   $\frac{1}{y} = -2\tan\left(2x\right) - C$

   $y = -\frac{1}{2\tan\left(2x\right)+C} = -\frac{1}{2\tan\left(2x\right)-3}$

4. $xy\frac{dy}{dx} = \ln x$ and $y(1) = 2$

   $y\frac{dy}{dx} = \frac{\ln x}{x}$

   $ydy = \frac{\ln x}{x}dx$

   $\int y\,dy = \int \frac{\ln x}{x}\,dx$

   $\frac{1}{2}y^2 = \frac{1}{2}\ln^2 x + C \rightarrow \frac{1}{2}\left(2\right)^2 = \frac{1}{2}\ln^2 1 + C \rightarrow C = 2$

   $y^2 = \ln^2 x + 2C$

   $y = \pm\sqrt{\ln^2 x + 2C} = \pm\sqrt{\ln^2 x + 4}$

   Because the positive value is the only one that satisfies the initial condition $y(1) = 2$, $y = \sqrt{\ln^2 x + 4}$.

5. $y' = 2x\sec y$ and $y(2) = -\frac{\pi}{2}$

   $\frac{1}{\sec y}\frac{dy}{dx} = 2x$

   $\frac{dy}{\sec y} = 2xdx$

   $\cos ydy = 2xdx$

   $\int \cos y\,dy = \int 2x\,dx$

   $\sin y = x^2 + C \rightarrow \sin\left(-\frac{\pi}{2}\right) = 2^2 + C \rightarrow C = -1 - 4 = -5$

   $y = \arcsin x^2 + C = \arcsin x^2 - 5$

6. $y' - xe^y = 2e^y$ and $y(0) = 0$

$\frac{dy}{dx} = 2e^y + xe^y = (2+x)\,e^y$

$\frac{1}{e^y}\frac{dy}{dx} = 2 + x$

$\frac{dy}{e^y} = (2+x)\,dx$

$\int \frac{dy}{e^y} = \int (2+x)\ dx$

$-\frac{1}{e^y} = \frac{1}{2}x^2 + 2x + C$

$-\frac{1}{e^y} = \frac{1}{2}x^2 + 2x - C \to \frac{1}{e^0} = \frac{1}{2} * 0^2 - 2(0) - C \to C = -1$

$e^y = -\frac{1}{\frac{1}{2}x^2+2x+1}$

$y = \ln\left(-\frac{1}{\frac{1}{2}x^2-2x+1}\right) = -\ln\left(-\frac{1}{2}x^2 - 2x + 1\right)$

7. $\frac{dy}{dx} = 2xy^3 \sin\left(x^2\right)$ and $y(0) = -1$

$\frac{1}{y^3}\frac{dy}{dx} = 2x\sin\left(x^2\right)$

$\int \frac{dy}{y^3} = 2\int x\sin\left(x^2\right)dx$

$-\frac{1}{2y^2} = -\cos\left(x^2\right) + C$

$\frac{1}{y^2} = 2\cos\left(x^2\right) - 2C \to \frac{1}{(-1)^2} = 2\cos\left(0^2\right) - 2C \to C = \frac{1}{2}$

$y^2 = \frac{1}{2\cos\left(x^2\right)-1}$

$y = \pm\sqrt{\frac{1}{2\cos\left(x^2\right)-1}}$

Because the negative value is the only one that satisfies the initial condition $y(0) = -1$, $y = -\sqrt{\frac{1}{2\cos\left(x^2\right)-1}}$.

8. $\frac{dy}{dx} = \frac{1}{y^2}$ and $y(0) = 4$

$y^2 dy = dx$

$\int y^2\,dy = \int\ dx$

$\frac{1}{3}y^3 = x + C \to \frac{64}{3} = C$

$y^3 = 3x + 64$

$y = \sqrt[3]{3x+64}$

9. Find a curve in the xy-plane that passes through the point $(0, 3)$ and whose tangent line at a point (x, y) has slope $\frac{2x}{y^2}$.

   $\frac{dy}{dx} = \frac{2x}{y^2}$ and $y(0) = 3$

   $y^2 dy = 2x dx$

   $\int y^2\, dy = \int 2x\, dx$

   $\frac{1}{3}y^3 = x^2 + C \to 9 = C$

   $y^3 = 3x^2 + 27$

   $y = \sqrt[3]{3x^2 + 27}$

Write a differential equation to represent the following.

10. The rate of change of a population y, with respect to time t, is proportional to t.

    $\frac{dy}{dt} = kt$

11. The rate of change of a population P, with respect to time t, is proportional to the cube of the population.

    $\frac{dP}{dt} = k\sqrt[3]{P}$

12. Let $P(t)$ represent the number of wolves in a population at time t years, where $t \geq 0$. The rate of change of the population $P(t)$, with respect to t, is directly proportional to $500 - P(t)$

    $\frac{dP}{dt} = k(500 - P(t))$

13. Water leaks out of a barrel at a rate proportional to the square root of the depth of the water at that time.

    $\frac{dh}{dt} = k\sqrt{h(t)}$

14. Oil leaks out of a tank at a rate inversely proportional to the amount of oil in the tank.

    $\frac{dV}{dt} = \frac{k}{V}$