Calculus BC - Worksheet on Riemann Sums and Integration

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Work the following on **notebook paper**.
On problems 1-6, write the expression as a definite integral, given that n is a positive integer.

Definite Integral as the Limit of a Riemann Sum:

$$\lim_{n\to\infty}\sum_{i=1}^{n} f(a+i\Delta x)\Delta x = \lim_{n\to\infty}\sum_{i=1}^{n} f(x_k)\Delta x = \int_a^b f(x)dx$$

Tip for Converting Summation to Integral (no idea what to call this):

$$\lim_{n\to\infty}\sum_{i=1}^{cn}\frac{1}{n}f(a+\frac{i}{n}) = \lim_{\frac{n}{c}\to\infty}\sum_{i=1}^{n}\frac{c}{n}f(a+\frac{ic}{n})$$

where c is some constant.

1. $\lim_{n\to\infty}\frac{1}{n}[(2+\frac{1}{n})^4+(2+\frac{2}{n})^4+...+(2+\frac{5n}{n})^4]$

   $= \lim_{n\to\infty}\frac{1}{n}\sum_{i=1}^{5n}(2+\frac{i}{n})^4 = \lim_{\frac{n}{5}\to\infty}\frac{5}{n}\sum_{i=1}^{n}(2+\frac{5i}{n})^4 = \int_2^7 x^4\,dx = [\frac{x^5}{5}]_2^7 = \frac{16775}{5} = 3355$

2. $\lim_{n\to\infty}\frac{1}{n}[(\frac{1}{n})^3+(\frac{2}{n})^3+...(\frac{5n}{n})^3]$

   $= \lim_{n\to\infty}\frac{1}{n}\sum_{i=1}^{5n}(\frac{i}{n})^3 = \lim_{\frac{n}{5}\to\infty}\frac{5}{n}\sum_{i=1}^{n}(\frac{5i}{n})^3 = \int_0^5 x^3\,dx = [\frac{x^4}{4}]_0^5 = \frac{625}{4} = 156.25$

3. $\lim_{n\to\infty}\frac{1}{n}[\sqrt[3]{1+\frac{1}{n}}+\sqrt[3]{1+\frac{2}{n}}+...+\sqrt[3]{1+\frac{7n}{n}}]$

   $= \lim_{n\to\infty}\sum_{i=1}^{7n}\sqrt[3]{1+\frac{i}{n}} = \lim_{\frac{n}{7}\to\infty}\frac{7}{n}\sum_{i=1}^{n}\sqrt[3]{1+\frac{7i}{n}} = \int_1^8\sqrt[3]{x} = [\frac{3}{4}x^{\frac{4}{3}}]_1^8 = \frac{45}{4} = 11.25$

4. $\lim_{n\to\infty}\frac{3}{n}\sum_{k=1}^{n}(2+\frac{3k}{n})^4$

   $= \int_2^5 x^4\,dx = [\frac{x^5}{5}]_2^5 = \frac{3093}{5} = 618.6$

5. $\lim_{n\to\infty}\frac{4}{n}\sum_{k=1}^{n}e^{-2+\frac{7k}{n}}$

   $= \lim_{n\to\infty}\frac{4}{n}\sum_{k=1}^{n}e^{\frac{7}{4}(-\frac{8}{7}+\frac{4k}{n})} = \int_{-\frac{8}{7}}^{\frac{20}{7}} e^{\frac{7x}{4}}\,dx = [\frac{4}{7}e^{\frac{7x}{4}}]_{-\frac{8}{7}}^{\frac{20}{7}} = \frac{4}{7}[e^5-e^{-2}] \approx 84.730$

6. $\lim_{n\to\infty}\frac{3}{n}\sum_{k=1}^{n}\sin(1+\frac{6k}{n})$

   $= \lim_{n\to\infty}\frac{3}{n}\sum_{k=1}^{n}\sin(2(\frac{1}{2}+\frac{3k}{n})) = \int_{0.5}^{3.5}\sin 2x\,dx = [-\frac{1}{2}\cos 2x]_{0.5}^{3.5} = -\frac{1}{2}[\cos 7-\cos 1] \approx -0.1068$

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7. The closed interval $[c,d]$ is partitioned into n equal subintervals, each of width Δx, by the numbers $c = x_0, x_1, ..., x_n$ where $x_0 < x_1 < x_2 < ... < x_{n-1} = d$. Write $\lim_{n\to\infty}\sum_{i=1}^{n}(x_k)^2\Delta x$ as a definite integral.

   $\lim_{n\to\infty}\sum_{i=1}^{n}(x_k)^2\Delta x = \int_c^d x^2\,dx = \frac{1}{3}[d^3-c^3]$

Evaluate. Do not leave negative exponents or complex fractions in your answers.

8. $\int 12x^3 + 5x^2 - 4 + \frac{6}{x^3}\, dx$

   $12\int x^3\, dx + 5\int x^2\, dx - 4\int\, dx + 6\int x^{-3}\, dx = 3x^4 + \frac{5}{3}x^3 - 4x - \frac{3}{x^2} + C$

9. $\int_{-1}^{2} (3x^2 - 2x + 5)\, dx$

   $3\int_{-1}^{2} x^2\, dx - 2\int_{-1}^{2} x\, dx + 5\int\, dx = [x^3]_{-1}^{2} - [x^2]_{-1}^{2} + 5[x]_{-1}^{2} = [9] - [3] + 5[3] = 21$

10. $\int x^3(x^4+1)^2\, dx$

    Let $u = x^4 + 1 \therefore du = 4x^3dx \to dx = \frac{du}{4x^3}$

    $\frac{1}{4}\int u^2\, du = \frac{1}{12}(x^4+1)^3 + C$

11. $x\sqrt{x^2+5}\, dx$

    Let $u = x^2 + 5 \therefore du = 2xdx \to dx = \frac{du}{2x}$

    $\frac{1}{2}\int \sqrt{u}\, du = \frac{1}{3}(x^2+5)^{\frac{3}{2}} + C$

12. $\int x\sqrt{x+5}\, dx$

    Let $u = x + 5 \to x = u - 5 \therefore du = dx$

    $\int (u-5)\sqrt{u}\, du = \int u^{\frac{3}{2}}\, dx - 5\int \sqrt{u}\, du = \frac{2}{5}(x+5)^{\frac{5}{2}} - \frac{10}{3}(x+5)^{\frac{3}{2}} + C$

13. $\int ((3x+2)(x-1))\, dx$

    $3\int x^2\, dx - \int x\, dx - 2\int\, dx = x^3 - \frac{x^2}{2} - 2x + C$

14. $\int_0^2 \frac{x}{\sqrt{1+2x^2}}\, dx$

    Let $u = 1 + 2x^2 \therefore du = 4xdx \to dx = \frac{du}{4x}$

    $\frac{1}{4}\int_1^9 \frac{1}{\sqrt{u}}\, du = [\frac{\sqrt{u}}{2}]_1^9 = \frac{3-1}{2} = 1$

15. $\int_1^6 \frac{x}{\sqrt[3]{x+2}}\, dx$

    Let $u = x + 2 \to x = u - 2 \therefore du = dx$

    $\int_3^8 \frac{u-2}{\sqrt[3]{u}}\, du = \int_3^8 u^{\frac{2}{3}} - 2\int_3^8 u^{-\frac{1}{3}}\, du = [\frac{3}{5}u^{\frac{5}{3}}]_3^8 - 3[u^{\frac{2}{3}}]_3^8 \approx \frac{3}{5}[25.76] - 3[1.92] = 9.696$

16. $\int x^2 \sin(x^3)\, dx$

    Let $\theta = x^3 \therefore d\theta = 3x^2 dx \rightarrow dx = \frac{d\theta}{3x^2}$

    $\frac{1}{3}\int \sin\theta\, d\theta = -\frac{1}{3}\cos(x^3) + C$

17. $\int \sec(3x)\tan(3x)\, dx$

    Let $\theta = 3x \therefore d\theta = 3dx \rightarrow dx = \frac{d\theta}{3}$

    $\frac{1}{3}\int \sec\theta\tan\theta\, d\theta = \frac{1}{3}\sec(3x) + C$

18. $\int \tan^3(5x)\sec^2(5x)\, dx$

    Let $\theta = 5x \therefore d\theta = 5dx \rightarrow dx = \frac{d\theta}{5}$

    $\int \tan^3\theta \sec^2\theta\, d\theta$

    Also let $u = \tan\theta \therefore du = \sec^2\theta d\theta \rightarrow d\theta = \frac{du}{\sec^2\theta}$

    $\int u^3\, du = \frac{1}{4}\tan^4(5x) + C$

19. $\int_0^{\frac{\pi}{2}} \sin\left(\frac{2x}{3}\right) dx$

    Let $\theta = \frac{2x}{3} \therefore d\theta = \frac{2dx}{3} \rightarrow dx = \frac{3du}{2}$

    $\int_0^{\frac{\pi}{3}} \sin\theta\, d\theta = [-\cos\theta]_0^{\frac{\pi}{3}} = -[\frac{1}{2} - 1] = \frac{1}{2}$

20. $\int_{\frac{\pi}{12}}^{\frac{\pi}{9}} \sin^3(3x)\cos(3x)\, dx$

    Let $\theta = 3x \therefore d\theta = 3dx \rightarrow dx = \frac{d\theta}{3}$

    $\frac{1}{3}\int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \sin^3\theta\cos\theta\, d\theta$

    Also let $u = \sin\theta \therefore du = \cos\theta d\theta \rightarrow d\theta = \frac{du}{\cos\theta}$

    $\frac{1}{3}\int_{\frac{\sqrt{2}}{2}}^{\frac{\sqrt{3}}{2}} u^3\, du = [\frac{1}{12}u^4]_{\frac{\sqrt{2}}{2}}^{\frac{\sqrt{3}}{2}} = \frac{1}{12}[\frac{9-4}{16}] = \frac{5}{192} \approx 0.026$

Use Geometry to evaluate.

21. $\int_0^3 \sqrt{9-x^2}\,dx$

Formula for a Circle:
$$x^2+y^2=r^2 \to y^2=r^2-x^2 \to y=\sqrt{r^2+x^2}$$

Area of a Circle:
$$A=\pi r^2$$

(We will use $\frac{1}{4}$ of the circle due to the layout of the graph

$y=\sqrt{9-x^2} \to y^2=3^2-x^2 \to x^2+y^2=3^2$

$A=\frac{9\pi}{4}=2.25\pi \approx 7.069$

22. $\int_0^8 |2x-10|\,dx$

$\int_0^8 |2x-10|\,dx=\sum_{i=1}^{2} f(x)\Delta x=\frac{bh}{2}+\frac{bh}{2}=\frac{50}{2}+\frac{18}{2}=25+9=34$