Calculus BC - Worksheet on Integration with Data

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Work the following on **notebook paper**. Give decimal answers correct to **three** decimal places.

1. A tank contains 120 gallons of oil at time $t = 0$ hours. Oil is being pumped into the tank at a rate $R(t)$, where $R(t)$ is measured in gallons per hour and t is measured in hours. Selected values of $R(t)$ are given in the table below.

t (hours)	0	3	5	9	12
$R(t)$ (gallons per hour)	8.9	6.8	6.4	5.9	5.7

(a) Estimate the number of gallons of oil in the tank at $t = 12$ hours by using a trapezoidal approximation with four subintervals and values from the table. Show the computations that led to your answer.

$\int_0^{12} R(t)\,dt \approx 120 + \sum_{n=1}^{4} R(t)\Delta t =$

$120 + \frac{1}{2}\left(3(8.9+6.8) + 2(6.8+6.4) + 4(6.4+5.9) + 3(5.9+5.7)\right) = 120 + 78.75 = 198.75$ gallons.

(b) A model for the rate at which oil is being pumped into the tank is given by the function $G(t) = 3 + \frac{10}{1+\ln(t+2)}$, where $G(t)$ is measured in gallons per hour and t is measured in hours. Use the model to find the number of gallons of oil in the tank at $t = 12$ hours.

$120 + \int_0^{12} G(t)\,dt = 120 + 3\int_0^{12} dt + 10\int_0^{12} \frac{dt}{1+\ln(t+2)} = 120 + 36 + 41.975 = 197.975$ gallons.

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2. A hot cup of coffee is taken into a classroom and set on a desk to cool, The table shows the rate $R(t)$ at which temperature of the coffee is dropping at various times over an eight minute period, where $R(t)$ is measured in degrees Fahrenheit per minute and t is measured in minutes. When $t = 0$, the temperature of the coffee is 113°F.

t (minutes)	0	3	5	8
$R(t)$ (°F/min.)	5.5	2.7	1.6	0.8

(a) Estimate the temperature of the coffee at $t = 8$ minutes by using a left Riemann sum with three subintervals and values from the table. Show the computations that lead to your answer.

$\int_0^{8} R(t)\,dt \approx 113 - \sum_{n=1}^{3} R(t)\Delta t = 3(5.5) + 2(2.7) + 3(1.6) = 113 - 26.7 = 86.3$°F.

(b) Use values from the table to estimate the average rate of change of $R(t)$ over the eight minute period. Show the computations that led to your answer.

$R_{avg} = \frac{R(8)-R(0)}{8-0} = \frac{0.8-5.5}{8} = -\frac{4.7}{8} = -0.588$°F/min.

(c) A model for the rate at which temperature of the coffee is dropping is given by the function $y(t) = 7e^{-0.3t}$, where $y(t)$ is measured in degrees Fahrenheit per minute and t is measured in minutes. Use the model to find the temperature of the coffee at $t = 8$ minutes,
$\int_0^{8} 7e^{-0.3t}\,dt = 113 - [-\frac{7}{0.3e^{0.3t}}]_0^8 = 113 - \frac{7}{0.3} - \frac{7}{0.3e^{2.4}} = 113 - 21.217 = 91.783$°F

(d) Use the model given in (c) to find the average rate at which the temperature of the coffee is dropping over the eight minute period.

$y_{avg} = \frac{\int_0^8 7e^{-0.3t}\,dt}{8-0} = -\frac{21.217}{8} = -2.652$°F/min.

3. (Modification of 2001 AB 2/ BC 2)
The temperature, in degrees Celsius (°C), of the water in a pond is a differentiable function W of time t. The table below shows the water temperature as recorded every 3 days over a 15-day period.

t (days)	0	3	6	9	12	15
$W(t)$ (°C)	20	31	28	24	22	21

(a) Approximate the average temperature, in degrees Celsius, of the water over the time interval $0 \leq t \leq 15$ days by using a trapezoidal approximation with subintervals of length $\Delta t = 3$ days and values from the table. Show the computations that led to your answer.

$\int_0^{15} W(t)\,dt = \sum_{n=1}^{5} 3W(t) = \frac{3}{2}((20+31)+(31+28)+(28+24)+(24+22)+(22+21)) = 376.5$°C

$W_{avg} = \frac{\int_0^{15} W(t)\,d}{15-0} = \frac{376.5}{15} = 25.1$°C/min.

(b) A student proposes the function P, given by $P(t) = 20 + 10te^{-\frac{t}{3}}$, as a model for the temperature of the water in the pond at time t, where t is measured in days and $P(t)$ is measured in degrees Celsius. Use the function P to find the average value, in degrees Celsius, of $P(t)$ over the time interval $0 \leq t \leq 15$ days.

$\int_0^{15} P(t)\,dt = 20\int_0^{15} dt + 10\int_0^{15} \frac{t}{e^{\frac{t}{3}}}\,dt = [20t]_0^{15} + 10[-3e^{-\frac{t}{3}}t - 9e^{-\frac{t}{3}}]_0^{15} = 390 - 540e^{-5} = 386.36$°C

$P_{avg} = \frac{\int_0^{15} P(t)\,dt}{15-0} = \frac{386.36}{15} = 25.757$°C/min.

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4. (Modification of 2004 Form B AB 3/ BC 3)
A test plane flies in a straight line with positive velocity $v(t)$, in miles per minute at time t minutes, where v is a differentiable function of t. Selected values of $v(t)$ for $0 \leq v(t) \leq 40$ are shown in the table below.

t (min)	0	5	10	15	20	25	30	35	40
$v(t)$ (mpm)	7.0	9.2	9.5	7.0	4.5	2.4	2.4	4.3	7.2

(a) Use a midpoint Riemann sum with four subintervals of equal length and values from the table to approximate $\int_0^{40} v(t)\,dt$. Show the computations that lead to your answer. Using correct units, explain the meaning of $\int_0^{40} v(t)\,dt$ in terms of the plane's flight.

$\int_0^{40} v(t)\,dt \approx \sum_{n=1}^{4} 10v(t) = 10(9.2 + 7.0 + 2.4 + 4.3) = 229$ miles.

The test plane flew a horizontal distance of 229 miles in the span of 40 minutes.

(b) The function f, defined by $f(t) = 6 + \cos\left(\frac{t}{10}\right) + 3\sin\left(\frac{7t}{40}\right)$, is used to model the velocity of the plane, in miles per minute, for $0 \leq t \leq 40$. According to this model, what is the average velocity of the plane, in miles per minute over the time interval $0 \leq v(t) \leq 40$?

$\int_0^{40} f(t)\,dt = 6\int_0^{40} dt + \int_0^{40} \cos\frac{t}{10}\,dt + 3\int_0^{40} \sin\frac{7t}{40}\,dt = 6[t]_0^{40} + 10[\sin\frac{t}{10}]_0^{40} - \frac{120}{7}[\cos\frac{7t}{40}]_0^{40} =$

236.651 miles

$f_{avg} = \frac{\int_0^{40} f(t)\,dt}{40-0} = \frac{236.651}{40} = 5.916$ miles per minute, on average.

5. (Modification of 2005 AB 3/ BC 3)
A metal wire of length 8 centimeters is heated at one end. The table below gives selected values of the temperature $T(x)$, in degrees Celsius, of the wire x cm from the heated end.

Distance x (cm)	0	1	5	6	8
Temperature $T(x)$ (°C)	100	93	70	62	55

(a) Estimate $T'(7)$. Show the work that leads to your answer. Indicate units of measure.

$T'(7) = \frac{55-62}{8-6} = -\frac{7}{2} = -3.5$°C/cm.

(b) Write an integral expression in terms of $T(x)$ for the average temperature of the wire. Estimate the average temperature of the wire using a trapezoidal sum with the four subintervals indicated by the data in the table. Indicate units of measure.

$\int_0^8 T(x)\,dx \approx \sum_{n=1}^{4} T(x)\Delta x = \frac{1}{2}\left((100+93)+4(93+70)+(70+62)+2(62+55)\right) = 600.5$°C

$T_{avg} = \frac{\int_0^8 T(x)\,dx}{8-0} \approx \frac{\sum_{n=1}^{4} T(x)\Delta x}{8} = \frac{605.5}{8} = 75.688$°C, on average.

(c) Find $\int_0^8 T'(x)\,dx$, and indicate units of measure. Explain the meaning of $\int_0^8 T'(x)\,dx$ in terms of the temperature of the wire.

$\int_0^8 T'(x)\,dx = T(8) - T(0) = 55 - 100 = -45$ °C

The temperature difference of the heated end of the wire to the other end is 45°C.

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6. (Modification of 2006 AB 4/ BC 4)
Rocket A has positive velocity $v(t)$ after being launched upward from an initial height of 0 feet at time $t = 0$ seconds. The velocity of the rocket is recorded for selected values of t over the interval $0 \leq t \leq 80$ seconds, as shown in the table below.

t (seconds)	0	10	20	30	40	50	60	70	80
$v(t)$ (ft per second)	5	14	22	29	35	40	44	47	49

(a) Using correct units, explain the meaning of $\int_{10}^{70} v(t)\,dt$ in terms of the rocket's flight. Use a midpoint Riemann sum with 3 subintervals of equal length to approximate $\int_{10}^{70} v(t)\,dt$.

$\int_{10}^{70} v(t)\,dt \approx \sum_{n=1}^{3} 20v(t) = 20(22) + 20(35) + 20(44) = 2020$ ft.

Rocket A has a vertical displacement of 2020 feet within the span of one minute.

(b) Rocket B is launched upward with an acceleration of $a(t) = \frac{3}{\sqrt{t+1}}$ feet per second per second. At time $t = 0$ seconds, the initial height of the rocket is 0 feet, and the initial velocity is 2 feet per second. Which of the two rockets is traveling faster at $t = 80$ seconds? Explain your answer.

$v_{y_A}(80) = 49$ ft/sec.

$v_{y_B}(80) = v_{y_{B_0}} + a_{y_B}t = 2 + 3\int_0^{80} \frac{dx}{\sqrt{t+1}} = 2 + 6\sqrt{80+1} - 6\sqrt{0+1} = 50$ ft/sec.

Rocket B is traveling faster than Rocket A at $t = 80$ seconds.