Calculus BC - Worksheet 2 on Functions Defined by Integrals

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23 January 2022

Work the following on **notebook paper**.

1. Find the equation of the tangent line to the curve $y = F(x)$ where $F(x) = \int_1^x \sqrt[3]{t^2+7}\,dt$ at the point on the curve where $x = 1$.

   $y - F(1) = F'(1)(x-1)$

   $y - \int_1^1 \sqrt[3]{t^2+7}\,dt = \sqrt[3]{t^2+7}(x-1)$

   $y = \sqrt[3]{8}(x-1)$

   $y = 2x - 2$

2. Suppose that $5x^3 + 40 = \int_c^x f(t)\,dt$.

   (a) What is $f(x)$?

   $f(x) = \frac{d}{dx}\int_c^x f(t)\,dt = \frac{d}{dx}(5x^3+40) = 15x^2$

   (b) Find the value of c.

   $\int_c^x f(t)\,dt = [5x^3]_c^x = 5x^3 - 5c^3 = 5x^3 + 40$

   $5c^3 = -40$

   $c^3 = -8$

   $c = -2$

3. If $F(x) = \int_{-4}^x (t-1)^2(t+3)\,dt$, for what values of x is F decreasing? Justify your answer.

   To determine where our initial function is decreasing, we must find where our derivative is less than zero, indicating a negative slope.

   $\frac{d}{dx}\int_{-4}^x (t-1)^2(t+3)\,dt = (x-1)^2(x+3)$

   $x - 1 < 0 \to x < 1$

   $x + 3 < 0 \to x < -3$

   $\therefore$ Because $F'(x) < 0$ at $x < -3$, $F(x)$ is decreasing at $x < -3$.

4. The function F is defined for all x by $F(x) = \int_0^{x^2} \sqrt{t^2+8}\,dt$.

   (a) Find $F'(x)$.

   $F'(x) = \frac{d}{dx} \int_0^{x^2} \sqrt{t^2+8}\,dt = 2x\sqrt{x^4+8}$

   (b) Find $F'(1)$.

   $F'(1) = 2\sqrt{1^4+8} = 2\sqrt{9} = 6$

   (c) Find $F''(x)$.

   $F''(x) = \frac{d}{dx} 2x\sqrt{x^4+8} = 2\sqrt{x^4+8} + 2x\left(\frac{2x^3}{\sqrt{x^4+8}}\right) = 2\sqrt{x^4+8} + \frac{4x^4}{\sqrt{x^4+8}}$

   (d) Find $F''(1)$.

   $F''(1) = 2\sqrt{1^4+8} + \frac{4(1)^4}{\sqrt{1^4+8}} = 2\sqrt{9} + \frac{4}{\sqrt{9}} = 6 + \frac{4}{3} = 7\frac{1}{3} = \frac{22}{3}$

5. The function F is defined for all x by $F(x) = \int_0^x f(t)\,dt$, where f is the function graphed in the figure. The graph of f is made up of straight lines and a semicircle.

   (a) For what values of x is F decreasing? Justify your answer.

   Because the graph of the derivative f is negative at $(-5, -3.5) \cup (2, 5)$, we can determine that F is decreasing on $(-5, -3.5) \cup (2, 5)$

   (b) For what values of x does F have a local maximum? A local minimum? Justify your answer.

   Because f changes from positive to negative at $x = 2$, we can determine that F has a local maximum at $x = 2$. Because f changes from negative to positive at $x = -3.5$, we can determine that F has a local minimum at $x = -3.5$

   (c) Evaluate $F(2)$, $F'(2)$, and $F''(2)$.

   $F(2) = \int_0^2 f(t)\,dt = \int_0^1 f(t)\,dt + \int_1^2 f(t)\,dt = \frac{b_1+b_2}{2}h + \frac{bh}{2} = \frac{2+3}{2} + \frac{3*1}{2} = \frac{5+3}{2} = 4$

   $F'(x) = \frac{d}{dx}\int_0^x f(t)\,dt = f(x) \rightarrow F'(2) = f(2) = 0$

   $F''(x) = \frac{d}{dx}f(x) = \frac{y_2-y_1}{x_2-x_1} = \frac{-3-3}{3-1} = -\frac{6}{2} = -3$

   (d) Write an equation of the line tangent to the graph of F at $x = 4$.

   $y - F(4) = f(4)(x-4)$

   $y - \int_0^4 f(t)\,dt = f(4)(x-4)$

   $y = -2x + 8$

   (e) For what values of x does F have an inflection point? Justify your answer.

   Because the graph of f changes from increasing to decreasing or decreasing to increasing at $x = -3, x = -2, x = 1$ and , $x = 3$, we can determine that F has inflection points at $x = -3, x = -2, x = 1$ and , $x = 3$.

6. The graph of a function f consists of a semicircle and two line segments as shown on the right. Let $g(x) = \int_1^x f(t)\,dt$.

(a) Find $g(1)$, $g(3)$, $g(-1)$.

$g(1) = \int_1^1 f(t)\,dt = 0$

$g(3) = \int_1^3 f(t)\,dt = \frac{bh}{2} = \frac{-1*2}{2} = -1$

$g(-1) = \int_1^{-1} f(t)\,dt = -\int_{-1}^1 f(t)\,dt = -\frac{\pi r^2}{4} = -\frac{4\pi}{4} = -\pi$

(b) On what interval(s) of x is g decreasing? Justify your answer.

Because the graph of the derivative f is negative on $(1, 3)$, we can determine that g is decreasing on $(1, 3)$.

(c) Find all values of x on the open interval $(-3, 4)$ at which g has a relative minimum. Justify your answer.

Because f changes from negative to positive at $x = 3$, we can determine that g has a local minimum at $x = 3$ on the interval $(-3, 4)$.

(d) Find the absolute maximum value of g on the interval $[-3, 4]$ at the value of x at which it occurs. Justify your answer.

Let us conduct a Candidate Test using our critical points (the values where $f(x) = 0$) by graphing our x-values and their respective values on $g(x)$.

x	-3	1	3
$g(x)$	-2π	0	-1

The absolute maximum value of g on the closed interval $[-3, 4]$ is 0 at $x = 1$ by the Candidate Test.

(e) On what interval(s) of x is g concave up? Justify your answer.

To determine where a function is concave up, we must find on the graph where f is increasing, denoting where f' is positive. Because f is increasing on $(-3, -1) \cup (2, 4)$, we can determine that g is concave up on $(-3, -1) \cup (2, 4)$.

(f) For what value(s) of x does the graph of g have an inflection point? Justify your answer.

Because the graph of f changes from increasing to decreasing or decreasing to increasing at $x = -1$ and $x = 2$, we can determine that g has inflection points at $x = -1$ and $x = 2$.

(g) Write an equation for the line tangent to the graph of g at $x = -1$.

$y - g(-1) = f(-1)(x + 1)$

$y + \pi = 2x + 2$

$y = 2x + 2 - \pi$

7. The graph of the velocity $v(t)$, in ft/sec, of a car traveling on a straight road, for $0 \leq t \leq 35$, is shown in the figure.

   (a) Find the average acceleration of the car, in ft / sec^2, over the interval $0 \leq t \leq 35$.

   $a(t) = [\frac{dv}{dt}]_0^{35}$

   $a(35) = \frac{v_2 - v_1}{t_2 - t_1} = \frac{30}{35} = \frac{6}{7}$ ft/sec^2

   (b) Find an approximation for the acceleration of the car, in ft / sec^2, at $t = 20$. Show your computations.

   $a(20) = \frac{v_2 - v_1}{t_2 - t_1} = \frac{30-40}{25-20} = -\frac{10}{5} = -2$ ft / sec^2

   (c) Approximate $\int_5^{35} v(t)\, dt$ with a Riemann sum, using the midpoints of three subintervals of equal length. Explain the meaning of this integral.

   $\int_5^{35} v(t)\, dt \approx \sum_{i=1}^{3} 10v(t) = 10(30 + 40 + 20) = 900$ ft / sec^2

   The integral represents the approximate distance the car has traveled between 5 and 35 seconds.