Calculus BC – Worksheet on 9.1 - 9.6

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Relevant Formulas and Notes:

Ratio Test:

Suppose we have the series $\sum_{n=0}^{\infty} a_n$. Consider:

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right|$$

Then, the series is convergent if $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| < 1$. It is divergent if the limit is greater than one, and must be proven with another method if the limit equals one.

Root Test:

Suppose we have the series $\sum_{n=0}^{\infty} a_n$. Consider:

$$\lim_{n\to\infty} \sqrt[n]{|a_n|} = \lim_{n\to\infty} |a_n|^{\frac{1}{n}}$$

Then, the series is convergent if $\lim_{n\to\infty} |a_n|^{\frac{1}{n}} < 1$. It is divergent if the limit is greater than one, and must be proven with another method if the limit equals one.

Work the following on **notebook paper**. Use the Ratio Test to determine the convergence or divergence of the series.

1. $\sum_{n=0}^{\infty} \frac{n!}{3^n}$

   Let $a_n = \frac{n!}{3^n}$ and $a_{n+1} = \frac{(n+1)!}{3^{n+1}}$

   $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left|\frac{\frac{(n+1)!}{3^{n+1}}}{\frac{n!}{3^n}}\right| = \lim_{n\to\infty} \left|\frac{3^n(n+1)!}{3^{n+1}n!}\right| = \lim_{n\to\infty} \left|\frac{n+1}{3}\right| = \infty$

   $\therefore \sum_{n=0}^{\infty} \frac{n!}{3^n}$ diverges by the Ratio Test.

2. $\sum_{n=1}^{\infty} \frac{n}{4^n}$

   Let $a_n = \frac{n}{4^n}$ and $a_{n+1} = \frac{n+1}{4^{n+1}}$

   $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left|\frac{\frac{n+1}{4^{n+1}}}{\frac{n}{4^n}}\right| = \lim_{n\to\infty} \left|\frac{4^n(n+1)}{n4^{n+1}}\right| = \lim_{n\to\infty} \left|\frac{n+1}{4n}\right| = \lim_{n\to\infty} \left|\frac{1+\frac{1}{n}}{4}\right| = \frac{1}{4}$

   $\therefore \sum_{n=1}^{\infty} \frac{n}{4^n}$ converges by the Ratio Test.

3. $\sum_{n=0}^{\infty} \frac{(-1)^n 2^n}{n!}$

   Let $a_n = \frac{(-1)^n 2^n}{n!}$ and $a_{n+1} = \frac{(-1)^{n+1}2^{n+1}}{(n+1)!}$

   $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left|\frac{\frac{(-1)^{n+1}2^{n+1}}{(n+1)!}}{\frac{(-1)^n 2^n}{n!}}\right| = \lim_{n\to\infty} \left|\frac{(-1)^{n+1}2^{n+1}n!}{(-1)^n 2^n(n+1)!}\right| = \lim_{n\to\infty} \left|-\frac{2}{n+1}\right| = 0$

   $\therefore \sum_{n=0}^{\infty} \frac{(-1)^n 2^n}{n!}$ converges by the Ratio Test.

Use the Root Test to determine the convergence or divergence of the series.

4. $\sum_{n=1}^{\infty} \left(\frac{n}{2n+1}\right)^n$

   $\lim_{n\to\infty} \sqrt[n]{|a_n|} = \lim_{n\to\infty} \sqrt[n]{\left|\left(\frac{n}{2n+1}\right)^n\right|} = \lim_{n\to\infty} \left|\frac{n}{2n+1}\right| = \lim_{n\to\infty} \left|\frac{1}{2+\frac{1}{n}}\right| = \frac{1}{2}$

   $\therefore \sum_{n=1}^{\infty} \left(\frac{n}{2n+1}\right)^n$ converges by the Root Test.

5. $\sum_{n=2}^{\infty} \frac{(-1)^n}{(\ln n)^n}$

   $\lim_{n\to\infty} \sqrt[n]{|a_n|} = \lim_{n\to\infty} \sqrt[n]{\left|\frac{(-1)^n}{(\ln n)^n}\right|} = \lim_{n\to\infty} \left|-\frac{1}{\ln n}\right| = 0$

   $\therefore \sum_{n=2}^{\infty} \frac{(-1)^n}{(\ln n)^n}$ converges by the Root Test.

Determine whether each of the given **series** converges or diverges/ You must use each of the ten tests at least once. Show all work, and **justify** your answers.

6. $\sum_{n=1}^{\infty} \frac{1}{n^{\frac{3}{2}}}$

7.