Calculus BC - Worksheet on Definite Integrals and Area

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Work the following on **notebook paper**. Do not use your calculator except on problem 15.
Evaluate.

1. $\int_0^1 x(x^2+1)^3dx$

   $\int_0^1 x^7dx + 3\int_0^1 x^5dx + 3\int_0^1 x^3dx + \int_0^1 xdx = [\frac{x^8}{8}]_0^1 + 3[\frac{x^6}{6}]_0^1 + 3[\frac{x^4}{4}]_0^1 + [\frac{x^2}{2}]_0^1 = \frac{1}{8} + \frac{1}{2} + \frac{3}{4} + \frac{1}{2} = \frac{15}{8} = 1.875$

2. $\int_0^1 x^2\sqrt{x^3+1}dx$

   Let $u = x^3+1 \therefore du = 3x^2dx \rightarrow dx = \frac{du}{3x^2}$

   $\int_0^1 \frac{x^2\sqrt{u}}{3x^2}du = \frac{1}{3}\int_0^1 \sqrt{u}du = \frac{1}{3}[\frac{2}{3}u^{\frac{3}{2}}]_0^1 = \frac{2}{9}[(x^3+1)^{\frac{3}{2}}]_0^1 = \frac{2}{9}(2)^{\frac{3}{2}} - \frac{2}{9}(1) = \frac{4\sqrt{2}-2}{9} = \frac{5.657-2}{9} = 0.406$

3. $\int_{-2}^{-1} \frac{x}{(x^2+2)^3}dx$

   Let $u = x^2+2 \therefore du = 2xdx \rightarrow dx = \frac{du}{2x}$

   $\int_6^3 \frac{du}{2u^3} = -\frac{1}{2}\int_3^6 \frac{du}{u^3} = -\frac{1}{2}[\frac{u^{-2}}{-2}]_3^6 = \frac{1}{4}[\frac{1}{u^2}]_3^6 = \frac{1}{4}(\frac{1}{36} - \frac{1}{9}) = \frac{-3}{144} \approx -0.021$

4. $\int_{-1}^4 |2x-1|dx$

   Let $u = 2x-1 \therefore du = 2dx \rightarrow dx = \frac{du}{2}$

   $\frac{1}{2}\int_{-3}^7 |u|dx = \frac{1}{2}[\frac{u|u|}{2}]_{-3}^7 = \frac{1}{4}[u|u|]_{-3}^7 = \frac{1}{4}(49+9) = \frac{29}{2} = 14.5$

5. $\int_0^5 \sqrt{25-x^2}dx$

   Formula for a Circle:
   $$x^2 + y^2 = r^2$$

   Area for a Circle:
   $$A = \pi r^2$$

   (We will use 1/4 of a circle due to the layout of the graph.)

   $x^2 + y^2 = 5^2 \rightarrow y^2 = 25 - x^2 \rightarrow y = \sqrt{25-x^2}$

   $A = \frac{1}{4}(25\pi) = \frac{25\pi}{4} \approx 19.635$

6. $\int_0^4 \frac{x}{\sqrt{9+x^2}}dx$

   Let $u = 9+x^2 \therefore du = 2xdx \rightarrow dx = \frac{du}{2x}$

   $\int_9^{25} \frac{1}{2\sqrt{u}}du = \frac{1}{2}\int_9^{25} u^{-\frac{1}{2}}du = \frac{1}{2}[\frac{\sqrt{u}}{\frac{1}{2}}]_9^{25} = [\sqrt{u}]_9^{25} = (5-3) = 2$

7. $\int_0^{\frac{\pi}{6}} \cos(3x)dx$

   Let $u = 3x \therefore du = 3dx \to dx = \frac{du}{3}$

   $\int_0^{\frac{\pi}{2}} \frac{\cos u}{3} du = \frac{1}{3}[\sin u]_0^{\frac{\pi}{2}} = \frac{1}{3}[1-0] = \frac{1}{3} \approx 0.333$

8. $\int_{-\frac{\pi}{12}}^{\frac{\pi}{6}} \sin(2x)dx$

   Let $u = 2x \therefore du = 2dx \to dx = \frac{du}{2}$

   $\frac{1}{2}\int_{-\frac{\pi}{6}}^{\frac{\pi}{3}} \sin(u)du = -\frac{1}{2}[\cos u]_{-\frac{\pi}{6}}^{\frac{\pi}{3}} = -\frac{1}{2}(\frac{1}{2} - \frac{\sqrt{3}}{2}) = \frac{\sqrt{3}-1}{4} \approx 0.183$

9. $\int_0^{\frac{\pi}{2}} \cos(\frac{2x}{3})dx$

   Let $u = \frac{2x}{3} \therefore du = \frac{2}{3}dx \to dx = \frac{3du}{2}$

   $\frac{3}{2}\int_0^{\frac{\pi}{3}} \cos u du = \frac{3}{2}[\sin u]_0^{\frac{\pi}{3}} \frac{3}{2}(\frac{\sqrt{3}}{2}) = \frac{3\sqrt{3}}{4} \approx 1.299$

10. $\int_{\frac{\pi}{6}}^{\frac{\pi}{2}} \sin^3 x \cos x\, dx$

    Let $u = \sin x \therefore du = \cos x dx \to dx = \frac{du}{\cos x}$

    $\int_{\frac{1}{2}}^{1} u^3 du = [\frac{u^4}{4}]_{\frac{1}{2}}^{1} = \frac{1}{4} - \frac{1}{64} = \frac{15}{64} \approx 0.234$

11. $\int_0^{\frac{\pi}{4}} \sqrt{\tan x} \sec^2 x\, dx$

    Let $u = \tan x \therefore du = \sec^2 x dx \to dx = \frac{du}{\sec^2 x}$

    $\int_0^1 \sqrt{u} du = [\frac{u^{3/2}}{3/2}]_0^1 = [\frac{2}{3}u^{\frac{3}{2}}]_0^1 = \frac{2}{3}(1) - \frac{2}{3}(0) = \frac{2}{3} \approx 0.667$

12. $\int_0^{\frac{\pi}{6}} \sec(2x)\tan(2x)\, dx$

    Let $u = 2x \therefore du = 2dx \to dx = \frac{du}{2}$

    $\frac{1}{2}\int_0^{\frac{\pi}{3}} \sec u \tan u\, du = \frac{1}{2}[\sec u]_0^{\frac{\pi}{3}} = \frac{1}{2}[\frac{1}{\cos u}]_0^{\frac{\pi}{3}} = \frac{1}{2}(\frac{1}{\frac{1}{2}} - \frac{1}{1}) = \frac{1}{2}(2-1) = \frac{1}{2} = 0.5$

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13. Find the area bounded by the graph of $f(x) = 2\sin x + \sin(2x)$ and the x-axis on the interval $[0, \pi]$.

    $\int_0^{\pi} 2\sin x + \sin(2x)\, dx = 2\int_0^{\pi} \sin x\, dx + \int_0^{\pi} \sin 2x\, dx = -2[\cos x]_0^{\pi} - [\cos 2x]_0^{\pi} = -2(-1-1) - (1-1) = 4$

14. Find the area bounded by the graph of $f(x) = \sec^2\left(\frac{x}{2}\right)$ and the x-axis on the interval $[\frac{\pi}{2}, \frac{2\pi}{3}]$.

    Let $u = \frac{x}{2} \therefore du = \frac{dx}{2} \rightarrow dx = 2du$

    $2\int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \sec^2 udu = 2[\tan u]_{\frac{\pi}{4}}^{\frac{\pi}{3}} = 2[\frac{\sin u}{\cos u}]_{\frac{\pi}{4}}^{\frac{\pi}{3}} = 2[\sqrt{3} - 1] \approx 1.464$

Use your calculator on problem 15.

15. The rate at which water is being pumped into a tank is given by the function $R(t)$. A table of selected values of $R(t)$, for the time interval $0 \leq t \leq 20$, is shown below.

t (min.)	0	4	9	17	20
$R(t)$ (gal/min)	25	28	33	42	46

   (a) Use data from the table and four subintervals to find a left Riemann sum to approximate the value of $\int_0^{20} R(t)dt$.

   $\sum_{n=1}^{4} R(t)\Delta x = 4(25) + 5(28) + 8(33) + 3(42) = 100 + 140 + 264 + 126 = 630$ gallons

   (b) Use data from the table and four subintervals to find a right Riemann sum to approximate the value of $\int_0^{20} R(t)dt$.

   $\sum_{n=1}^{4} R(t)\Delta x = 4(28) + 5(33) + 8(42) + 3(46) = 112 + 165 + 336 + 138 = 751$ gallons

   (c) A model for the rate at which water is being pumped into the tank is given by the function $W(t) = 25e^{0.03t}$, where t is measured in minutes and $W(t)$ is measured in gallons per minute.

   Use the model to find the value of $\int_0^{20} W(t)dt$

   $\int_0^{20} W(t)dt = 25\int_0^{20} e^{0.03t}dt = 25[\frac{1}{0.03}e^{0.03t}]_0^{20} = \frac{2500}{3}(e^{0.6} - 1) \approx \frac{2500(0.822)}{3} = \frac{2055}{3} = 685$ gallons