Calculus BC - Worksheet 1 on 8.1 – 8.3

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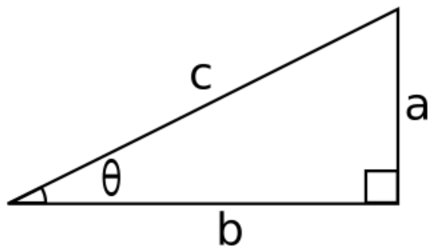
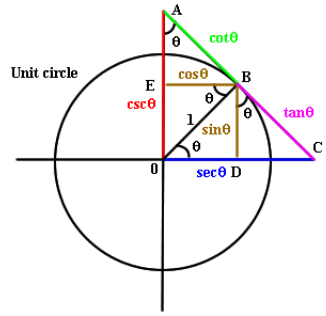
Relevant Formulas:





$$\sin^2\theta+\cos^2\theta=x^2+y^2=1$$

$$\frac{\sin^2\theta+\cos^2\theta}{\cos^2\theta}=\frac{1}{\cos^2\theta}\to\tan^2\theta+1=\sec^2\theta$$

$$\frac{\sin^2\theta+\cos^2\theta}{\sin^2\theta}=\frac{1}{\sin^2\theta}\to 1+\cot^2\theta=\csc^2\theta$$

$$\cos(\alpha\pm\beta)=\cos\alpha\cos\beta\mp\sin\alpha\sin\beta\to\cos(2\theta)=\cos^2\theta-\sin^2\theta=\left(1-\sin^2\theta\right)-\sin^2\theta=\cos^2\theta-\left(1-\cos^2\theta\right)$$

$$\frac{1}{2}(\cos(\alpha-\beta)+\cos(\alpha+\beta))=\frac{1}{2}((\cos\alpha\cos\beta+\sin\alpha\sin\beta)+(\cos\alpha\cos\beta-\sin\alpha\sin\beta))=\frac{2}{2}(\cos\alpha\cos\beta)$$

$$\frac{1}{2}(\cos(\alpha-\beta)-\cos(\alpha+\beta))=\frac{1}{2}((\cos\alpha\cos\beta+\sin\alpha\sin\beta)-(\cos\alpha\cos\beta-\sin\alpha\sin\beta))=\frac{2}{2}(\sin\alpha\sin\beta)$$

$$\cos(2\theta)=\left(1-\sin^2\theta\right)-\sin^2\theta=1-2\sin^2\theta\therefore\sin^2\theta=\frac{1-\cos(2x)}{2}$$

$$\cos(2\theta)=\cos^2\theta-\left(1-\cos^2\theta\right)=2\cos^2\theta-1\therefore\cos^2\theta=\frac{1+\cos(2x)}{2}$$

$$\sin(\alpha\pm\beta)=\sin\alpha\cos\beta\pm\cos\alpha\sin\beta$$

$$\frac{1}{2}(\sin(\alpha-\beta)+\sin(\alpha+\beta))=\frac{1}{2}((\sin\alpha\cos\beta-\cos\alpha\sin\beta)+(\sin\alpha\cos\beta+\cos\alpha\sin\beta))=\frac{2}{2}(\sin\alpha\cos\beta)$$

Work the following on **notebook paper**. **No calculator.**

1. $\int \cos^2(2x)\sin^2(2x)\,dx$

   $\int \cos^2(2x)\sin^2(2x)\,dx = \int \left(\frac{1+cos(4x)}{2}\right)\left(\frac{1-\cos(4x)}{2}\right)\,dx = \frac{1}{4}\int 1 - cos^2(4x)\,dx = \frac{1}{4}\int \sin^2(4x)\,dx =$

   $\frac{1}{4}\int \frac{1-\cos(8x)}{2}\,dx = \frac{1}{8}\left(\int\,dx - \int \cos(8x)\,dx\right) = \frac{1}{8}x - \frac{1}{64}\sin(8x) + C$

2. $\int \frac{2x-1}{x^2-6x+25}\,dx$

   $\int \frac{2x-1}{x^2-6x+25}\,dx = \int \frac{2x-6+5}{x^2-6x+25}\,dx = \int \frac{2x-6}{x^2-6x+25}\,dx + 5\int \frac{dx}{x^2-6x+25}$

   Let $u = x^2 - 6x + 25 \therefore du = (2x-6)dx$

   $\int \frac{2x-6}{x^2-6x+25}\,dx + 5\int \frac{dx}{x^2-6x+25} = \int \frac{du}{u} + 5\int \frac{dx}{(x-3)^2+16} = \ln|u| + \frac{5}{4}\arctan\left(\frac{x-3}{4}\right) + C =$

   $\ln|x^2-6x+25| + \frac{5}{4}\arctan\left(\frac{x-3}{4}\right) + C$

3. $\int x^2\sin(3x)\,dx$

   Let $u = x^2 \therefore du = 2xdx$ and let $v = -\frac{1}{3}\cos(3x) \therefore dv = \sin(3x)dx$

   $\int x^2\sin(3x)\,dx = \int u\,dv = \frac{2}{3}\int x\cos(3x)\,dx - \frac{1}{3}x^2\cos(3x)$

   Let $\alpha = x \therefore d\alpha = dx$ and let $\beta = \sin 3x \therefore d\beta = 3\cos(3x)dx$

   $\frac{2}{3}\int x\cos(3x)\,dx - \frac{1}{3}x^2\cos(3x) = \frac{2}{3}\int \alpha\,d\beta - \frac{1}{3}x^2\cos(3x) = \frac{2}{3}\left(x\sin(3x) - \int \sin(3x)\,dx\right) - \frac{1}{3}x^2\cos(3x) =$

   $\frac{2}{3}\left(x\sin(3x) + \frac{1}{3}\cos(3x)\right) - \frac{1}{3}x^2\cos(3x) + C$

4. $\int \arcsin(4x)\,dx$

   Let $u = \arcsin(4x) \therefore du = \frac{4}{\sqrt{1-16x^2}}dx$ and let $v = x \therefore dv = dx$

   $\int \arcsin(4x)\,dx = \int u\,dv = x\arcsin(4x) - 4\int \frac{x}{\sqrt{1-16x^2}}dx$

   Let $\alpha = 1 - 16x^2 \therefore d\alpha = -32xdx$

   $x\arcsin(4x) - 4\int \frac{x}{\sqrt{1-16x^2}}dx = x\arcsin(4x) + \frac{1}{8}\int \frac{d\alpha}{\sqrt{\alpha}} = x\arcsin(4x) + \frac{1}{4}\sqrt{\alpha} + C =$

   $x\arcsin(4x) + \frac{1}{4}\sqrt{1-16x^2} + C$

5. $\int \sin^2(5x)\,dx$

   $\int \sin^2(5x)\,dx = \int \frac{1-\cos 10x}{2}\,dx = \frac{1}{2}\left(\int\,dx - \int \cos(10x)\,dx\right) = \frac{1}{2}x - \frac{1}{20}\sin(10x) + C$

6. $\int \frac{x^3+3x^2}{x^2+1}\,dx$

   $\int \frac{x^3+3x^2}{x^2+1}\,dx = \int \frac{x^3}{x^2+1}\,dx + 3\int \frac{x^2}{x^2+1}\,dx$

   Let $u = x^2+1 \therefore du = 2xdx$

   $\int \frac{x^3}{x^2+1}\,dx + 3\int \frac{x^2}{x^2+1}\,dx = \frac{1}{2}\int \frac{u-1}{u}\,du + 3\int \frac{(x^2+1)-1}{x^2+1}\,dx = \frac{1}{2}\left(\int \frac{u}{u}\,du - \int \frac{du}{u}\right) + 3\left(\int \frac{x^2+1}{x^2+1}\,dx - \int \frac{dx}{x^2+1}\right) =$

   $\frac{u}{2} - \frac{1}{2}\ln|u| + 3(x - \arctan x) + C = \frac{x^2+1}{2} - \frac{1}{2}\ln\left(\left|x^2+1\right|\right) + 3(x - \arctan x)$

7. $\int \cos^4(6x)\,dx$

   $\int \cos^4(6x)\,dx = \int \left(\cos^2(6x)\right)^2 dx = \int \left(\cos^2(6x)\right)\left(1-\sin^2(6x)\right) dx =$

   $\int \cos^2(6x)\,dx - \int \sin^2(6x)\cos^2(6x)\,dx = \int \frac{1+\cos(12x)}{2}\,dx - \int \left(\frac{1-\cos(12x)}{2}\right)\left(\frac{1+\cos(12x)}{2}\right) dx =$

   $\frac{1}{2}\left(\int\,dx + \int \cos(12x)\,dx\right) - \frac{1}{4}\left(\int\,dx - \int \cos^2(12x)\,dx\right) =$

   $\frac{1}{2}\left(\int\,dx + \int \cos(12x)\,dx\right) - \frac{1}{4}\left(\int\,dx - \frac{1}{2}\left(\int\,dx + \int \cos(24x)\,dx\right)\right) =$

   $\frac{1}{2}\int\,dx + \frac{1}{2}\int \cos(12x)\,dx - \frac{1}{4}\int\,dx + \frac{1}{8}\int\,dx + \frac{1}{8}\int \cos(24x)\,dx =$

   $\frac{3}{8}x + \frac{1}{24}\sin(12x) + \frac{1}{192}\sin(24x) + C$

8. $\int x^2 \ln x\,dx$

   Let $u = \ln x \therefore du = \frac{dx}{x}$ and let $v = \frac{1}{3}x^3 \therefore dv = x^2dx$

   $\int x^2 \ln x\,dx = \int u\,dv = \frac{1}{3}x^3\ln x - \frac{1}{3}\int x^2\,dx = \frac{1}{3}x^3\ln x - \frac{1}{9}x^3 + C$

9. $\int e^{2x} \sin x \, dx$

   Let $u = e^{2x} \therefore du = 2e^{2x}$ and let $v = -\cos x \therefore dv = \sin x$

   $\int e^{2x} \sin x \, dx = \int u \, dv = 2 \int e^{2x} \cos x \, dx - e^{2x} \cos x$

   Let $\alpha = \sin x \therefore d\alpha = \cos x dx$

   $2 \int e^{2x} \cos x \, dx - e^{2x} \cos x = 2 \int u \, d\alpha - e^{2x} \cos x = 2 \left(e^{2x} \sin x - 2 \int e^{2x} \sin x \, dx \right) - e^{2x} \cos x$

   Let $\beta = \int e^{2x} \sin x \, dx$

   $2 \left(e^{2x} \sin x - 2 \int e^{2x} \sin x \, dx \right) - e^{2x} \cos x = 2e^{2x} \sin x - 4 \int e^{2x} \sin x \, dx - e^{2x} \cos x$

   $\beta = 2e^{2x} \sin x - 4\beta - e^{2x} \cos x \rightarrow 5\beta = 2e^{2x} \sin x - e^{2x} \cos x \rightarrow \beta = \frac{2}{5} e^{2x} \sin x - \frac{1}{5} e^{2x} \cos x$

10. $\int \frac{dx}{\sqrt{12+4x-x^2}}$

    $\int \frac{dx}{\sqrt{12+4x-x^2}} = \int \frac{dx}{\sqrt{16-(x-2)^2}} = \arcsin \left(\frac{x-2}{4} \right) + C$

11. $\int_0^{\frac{\pi}{2}} \cos^3 x \, dx$

    $\int_0^{\frac{\pi}{2}} \cos^3 x \, dx = \int_0^{\frac{\pi}{2}} \cos x \left(1 - \sin^2 x \right) \, dx = \int_0^{\frac{\pi}{2}} \cos x \, dx - \int_0^{\frac{\pi}{2}} \cos x \sin^2 x \, dx$

    Let $u = \sin x \therefore du = \cos x dx$

    $\int_0^{\frac{\pi}{2}} \cos x \, dx - \int_0^{\frac{\pi}{2}} \cos x \sin^2 x \, dx = \int_0^{\frac{\pi}{2}} \cos x \, dx - \int_0^1 u^2 \, du = [\sin x]_0^{\frac{\pi}{2}} - \frac{1}{3} [u^3]_0^1 = [1-0] - \frac{1}{3}[1-0] = \frac{2}{3}$

12. $\int_0^{\frac{\pi}{2}} x \cos x \, dx$

    Let $u = x \therefore du = dx$ and let $v = \sin x \therefore du = \cos x dx$

    $\int_0^{\frac{\pi}{2}} x \cos x \, dx = \int_0^{\frac{\pi}{2}} u \, dv = [x \sin x]_0^{\frac{\pi}{2}} - \int_0^{\frac{\pi}{2}} \sin x \, dx = [\frac{\pi}{2} - 0] + [\cos x]_0^{\frac{\pi}{2}} = [\frac{\pi}{2} - 0] + [0 - 1] = \frac{\pi}{2} - 1$

Multiple Choice. All work must be shown.

x	2	5	7	8
$f(x)$	10	30	40	20

13. The function f is continuous on the closed interval $[2,8]$ and has values that are given in the table above. Using the subintervals $[2,5]$, $[5,7]$, and $[7,8]$, what is the trapezoidal approximation of $\int_2^8 f(x)\,dx$?

    (a) 110
    (b) 130
    (c) 160
    (d) 190
    (e) 210

$\int_2^8 f(x)\,dx \approx \sum_{n=1}^{3} f(x)\Delta x = \frac{1}{2}(3*40+2*70+1*60) = \frac{1}{2}(120+140+60) = \frac{320}{2} = 160$

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14. What is the minimum value of $f(x) = x\ln x$?

    (a) $-e$
    (b) -1
    (c) $-\frac{1}{e}$
    (d) 0
    (e) $f(x)$ has no minimum value.

Let us conduct an Intervals Test using the critical points of $f(x)$.

$f'(x) = \ln x + \frac{x}{x} = \ln x + 1 = 0 \rightarrow \ln x = -1 \rightarrow e^{\ln x} = x = e^{-1}$

Intervals	$(-\infty, \frac{1}{e})$	$(\frac{1}{e}, \infty)$
$f'(x)$	-	+
$f(x)$	Decreasing	Increasing

The minimum value of $f(x)$ occurs at $\frac{1}{e}$.

$f(\frac{1}{e}) = \frac{1}{e}\ln(\frac{1}{e}) = \frac{1}{e} * -1 = -\frac{1}{e}$

The minimum value of $f(x) = x\ln x$ is $-\frac{1}{e}$.

15. At what value of x does the graph of $y = \frac{1}{x^2} - \frac{1}{x^3}$ have a point of inflection?
    (a) 0
    (b) 1
    (c) 2
    (d) 3
    (e) At no value of x

Solve for $\frac{d^2y}{dx^2}$.

$\frac{dy}{dx} = \frac{3}{x^4} - \frac{2}{x^3}$

$\frac{d^2y}{dx^2} = \frac{6}{x^4} - \frac{12}{x^5}$

Equal $\frac{d^2y}{dx^2}$ to zero.

$\frac{6}{x^4} - \frac{12}{x^5} = 0$

$\frac{6}{x^4} = \frac{12}{x^5}$

$\frac{1}{x^4} = \frac{2}{x^5}$

$1 = \frac{2}{x}$

$x = 2$