Calculus BC - Worksheet on 8.1–8.2

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30 January 2022

Work the following on **notebook paper**. **No calculator.**

1. $\int \frac{2x}{x-4}\,dx$

   Let $u = x - 4 \to x = u + 4 \therefore du = dx$

   $\int \frac{2x}{x-4}\,dx = 2\int \frac{u+4}{u}\,du = 2\left(\int du + 4\int \frac{1}{u}\,du\right) = 2u + 4\ln|u| + C = 2x - 8 + 4\ln|x-4| + C$

2. $\int \frac{x+1}{x^2+2x-4}\,dx$

   Let $u = x^2 + 2x - 4 \therefore du = (2x+2)dx$

   $\int \frac{x+1}{x^2+2x-4}\,dx = \int \frac{du}{u} * \frac{x+1}{2(x+1)} = \int \frac{du}{2u} = \frac{1}{2}\ln|u| + C = \frac{1}{2}\ln|x^2+2x-4| + C$

3. $\int xe^{-3x}\,dx$

   Let $u = x \therefore du = dx$ and let $v = -\frac{1}{3e^{3x}} \therefore dv = \frac{1}{e^{3x}}dx$

   $\int \frac{x}{e^{3x}}\,dx = \int u\,dv = \frac{1}{3}\int \frac{1}{e^{3x}}\,dx - \frac{x}{3e^{3x}} = -\frac{1}{9e^{3x}} - \frac{x}{3e^{3x}} + C = -\frac{1}{3e^{3x}}\left(\frac{1}{3} + x\right) + C$

4. $\int \sec(4x)\,dx$

   Let $\alpha = 4x \therefore d\alpha = 4dx$

   $\int \sec(4x)\,dx = \frac{1}{4}\int \sec\alpha\,d\alpha = \frac{1}{4}\int \sec\alpha\left(\frac{\sec\alpha+\tan\alpha}{\sec\alpha+\tan\alpha}\right)d\alpha = \frac{1}{4}\int \frac{\sec^2\alpha+\sec\alpha\tan\alpha}{\sec\alpha+\tan\alpha}\,d\alpha$

   Let $u = \sec\alpha + \tan\alpha \therefore du = \sec^2\alpha + \sec\alpha\tan\alpha$

   $\frac{1}{4}\int \frac{\sec^2\alpha+\sec\alpha\tan\alpha}{\sec\alpha+\tan\alpha}\,d\alpha = \frac{1}{4}\int \frac{du}{u} = \frac{1}{4}\ln|u| + C = \frac{1}{4}\ln|\sec\alpha + \tan\alpha| + C = \frac{1}{4}\ln|\sec 4x + \tan 4x| + C$

5. $\int \frac{\ln x}{x^2}\,dx$

   Let $u = \ln x \therefore du = \frac{dx}{x}$ and let $v = -\frac{1}{x} \therefore dv = \frac{dx}{x^2}$

   $\int \frac{\ln x}{x^2}\,dx = \int u\,dv = \int \frac{dx}{x^2} - \frac{\ln x}{x} = -\frac{1}{x} - \frac{\ln x}{x} + C = -\frac{1}{x}(1 + \ln x) + C$

6. $\int \frac{\sin x}{\sqrt{\cos x}}\,dx$

   Let $u = \cos x \therefore du = -\sin x dx$

   $\int \frac{\sin x}{\sqrt{\cos x}}\,dx = -\int \frac{1}{\sqrt{u}}\,du = -2\sqrt{u} + C = -2\sqrt{\cos x} + C$

7. $\int_0^{\pi} x\sin{(2x)}\,dx$

   Let $u = x \therefore du = dx$ and let $v = -\frac{1}{2}\cos 2x \therefore dv = \sin 2x dx$

   $\int_0^{\pi} x\sin{(2x)}\,dx = \int_0^{\pi} u\,dv = \frac{1}{2}\int_0^{\pi}\cos 2x\,dx - [\frac{1}{2}x\cos 2x]_0^{\pi} = [\frac{1}{2}\sin 2x]_0^{\pi} - [\frac{1}{2}x\cos 2x]_0^{\pi} =$

   $\frac{1}{2}(0-0) - \frac{1}{2}(\pi - 0) = -\frac{\pi}{2}$

8. $\int \frac{1}{\sqrt{2-2x-x^2}}\,dx$

   $\int \frac{dx}{\sqrt{2-2x-x^2}} = \int \frac{dx}{\sqrt{3-(x+1)^2}} = \arcsin\left(\frac{x+1}{\sqrt{3}}\right) + C$

9. $\int \arctan{(3x)}\,dx$

   Let $u = \arctan 2x \therefore du = \frac{3}{9x^2+1}dx$ and let $v = x \therefore dv = dx$

   $\int \arctan{(3x)}\,dx = \int u\,dv = x\arctan{(3x)} - \int \frac{3x}{9x^2+1}\,dx$

   Let $\alpha = 9x^2 + 1 \therefore d\alpha = 18xdx$

   $x\arctan{(3x)} - \int \frac{3x}{9x^2+1}\,dx = x\arctan{(3x)} - \int \frac{du}{6u} = x\arctan{(3x)} - \frac{1}{6}\int\frac{du}{u} = x\arctan{(3x)} - \frac{1}{6}\ln|u| + C =$

   $x\arctan{(3x)} - \frac{1}{6}\ln|9x^2+1| + C$

10. $\int_0^1 e^x \sin x\,dx$

    Let $u = e^x \therefore du = e^x dx$ and let $v = -\cos x \therefore du = \sin x dx$

    $\int_0^1 e^x\sin x\,dx = \int_0^1 u\,dv = \int_0^1 e^x\cos x\,dx - [e^x\cos x]_0^1$

    Let $\alpha = \sin x \therefore d\alpha = \cos x dx$

    $\int_0^1 e^x\cos x\,dx - [e^x\cos x]_0^1 = \int_0^1 u\,d\alpha - [e^x\cos x]_0^1 = \left([e^x\sin x]_0^1 - \int_0^1 e^x\sin x\,dx\right) - [e^x\cos x]_0^1$

    Let $\beta = \int_0^1 e^x\sin x\,dx$

    $\beta = [e^x\sin x]_0^1 - \beta - [e^x\cos x]_0^1 \rightarrow 2\beta = [e^x\sin x - e^x\cos x]_0^1 = e\sin 1 - e\cos 1 + 1$

    $\beta = \frac{e\sin 1 - e\cos 1 + 1}{2}$

11. $\int \frac{2x-5}{x^2+2x+2}\,dx$

$\int \frac{2x-5}{x^2+2x+2}\,dx = \int \frac{2x-5}{(x+1)^2+1}\,dx$

Let $u = x+1 \to x = u-1 \therefore du = dx$

$\int \frac{2x-5}{(x+1)^2+1}\,dx = \int \frac{2u-7}{u^2+1}\,du = \int \frac{2u}{u^2+1}\,du - 7\int \frac{du}{u^2+1}$

Let $v = u^2+1 \therefore dv = 2udu$

$\int \frac{2u}{u^2+1}\,du - 7\int \frac{du}{u^2+1} = \int \frac{dv}{v} - 7\int \frac{du}{u^2+1} = \ln|v| - 7\arctan u + C = \ln|x^2+2x+2| - 7\arctan(x+1) + C$

12. $\int \arcsin(5x)\,dx$

Let $u = \arcsin 5x \therefore du = \frac{5}{1-25x^2}dx$ and let $v = x \therefore dv = dx$

$\int \arcsin(5x)\,dx = \int u\,dv = x\arcsin 5x - \int \frac{5x}{\sqrt{1-25x^2}}\,dx$

Let $\alpha = 1-25x^2 \therefore d\alpha = -50xdx$

$x\arcsin 5x - \int \frac{5x}{\sqrt{1-25x^2}}\,dx = x\arcsin 5x - \int \frac{d\alpha}{10\sqrt{u}} = x\arcsin 5x - \frac{1}{10}\int \frac{d\alpha}{\sqrt{\alpha}} = x\arcsin 5x - \frac{\sqrt{\alpha}}{5} + C =$

$x\arcsin 5x - \frac{\sqrt{1-25x^2}}{5} + C$

13. $\int \frac{x^3}{x^2+4}\,dx$

Let $u = x^2+4 \to x = \sqrt{u-4} \therefore du = 2xdx$

$\int \frac{x^3}{x^2+4}\,dx = \frac{1}{2}\int \frac{u-4}{u}\,du = \frac{1}{2}\left(\int du - 4\int \frac{du}{u}\right) = \frac{1}{2}(u - 4\ln|u|) + C = \frac{1}{2}u - 2\ln|u| + C =$

$\frac{x^2+4}{2} - 2\ln|x^2+4| + C$

14. $\int_0^1 x^2e^x\,dx$

Let $u = x^2 \therefore du = 2xdx$ and let $v = e^x \therefore du = e^xdx$

$\int_0^1 x^2e^x\,dx = \int_0^1 u\,dv = [x^2e^x]_0^1 - 2\int_0^1 xe^x\,dx$

Let $\alpha = x \therefore d\alpha = dx$

$[x^2e^x]_0^1 - 2\int_0^1 xe^x\,dx = [x^2e^x]_0^1 - 2\int_0^1 v\,d\alpha = [x^2e^x]_0^1 - 2\left([xe^x]_0^1 - [e^x]_0^1\right) = [x^2e^x - 2xe^x + 2e^x]_0^1 = e-2$

Multiple Choice. All work must be shown.

15. If $f(x) = \sin\left(\frac{x}{2}\right)$, then there exists a number c in the interval $\frac{\pi}{2} < x < \frac{3\pi}{2}$ that satisfies the conclusion of the Mean Value Theorem. Which of the following could be c?

    (a) $\frac{2\pi}{3}$
    (b) $\frac{3\pi}{4}$
    (c) $\frac{5\pi}{6}$
    (d) π
    (e) $\frac{3\pi}{2}$

Let us first check the conditions of the Mean Value Theorem:

- $f(x)$ is continuous on $[\frac{\pi}{2}, \frac{3\pi}{2}]$.
- $f(x)$ is differentiable on $(\frac{\pi}{2}, \frac{3\pi}{2})$.

The conditions are met. Let us solve for our value of c.

$f'(c) = \frac{f(b)-f(a)}{b-a}$

$\frac{1}{2}\cos\left(\frac{x}{2}\right) = \frac{\frac{\sqrt{2}}{2}-\frac{\sqrt{2}}{2}}{\frac{\pi}{2}-\frac{3\pi}{2}} = 0$ @ $x = \pi + 2\pi n$

The value of c listed that satisfies the conclusion of the Mean Value Theorem is π.

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16. If $f(x) = (x-1)^2 \sin x$, then $f'(0) =$

    (a) -2
    (b) -1
    (c) 0
    (d) 1
    (e) 2

Apply the Product Rule:

$$\frac{d}{dx}(gh) = hg' + gh'$$

- $g(x) = (x-1)^2 = x^2 - 2x + 1$
- $g'(x) = 2x - 2$
- $h(x) = \sin x$
- $h'(x) = \cos x$

$f'(x) = hg' + gh' = (2x-2)\sin x + (x-1)^2 \cos x$

$f'(0) = (-2)\sin 0 + (-1)^2 \cos 0 = (-2)(0) + (1)(1) = 1$

17. The acceleration of a particle moving along the x-axis at time t is given by $a(t) = 6t - 2$. If the velocity is 25 when $t = 3$ and the position is 10 when $t = 1$, then the position $x(t) =$

    (a) $9t^2 + 1$
    (b) $3t^2 - 2t + 4$
    (c) $t^3 - t^2 + 4t + 6$
    (d) $t^3 - t^2 + 9t - 20$
    (e) $36t^3 - 4t^2 - 77t + 55$

    Note the relation between acceleration, velocity, and time.

    $$a(t) = \frac{dv}{dt} = \frac{d^2x}{dt^2}$$

    $v_x(3) = v_{0x} + \int_0^3 a_x\,dt = [3t^2 - 2t + v_{0x}]_0^3 = 21 + v_{0x} = 25 \rightarrow v_{0x} = 4$

    $x(1) = x_0 + \int_0^1 v_x\,dt = [t^3 - t^2 + 4t + x_0]_0^1 = 4 + x_0 = 10 \rightarrow x_0 = 6$

    $\therefore x(t) = t^3 - t^2 + 4t + 6$

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18. $\frac{d}{dx}\int_0^x \cos(2\pi u)\,du$ is

    (a) 0
    (b) $\frac{1}{2\pi}\sin x$
    (c) $\frac{1}{2\pi}\cos(2\pi x)$
    (d) $\cos(2\pi x)$
    (e) $2\pi\cos(2\pi x)$

    Apply the Second Fundamental Theorem of Calculus:

    $$\frac{d}{dx}\int_a^x f(t)\,dt = f(x)$$

    $\frac{d}{dx}\int_0^x \cos(2\pi u)\,du = \cos(2\pi x)$