# Calculus BC - Worksheet on Second Fundamental Theorem
# &
# Functions Defined by Integrals

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1. Evaluate.

   (a) $\frac{d}{dx}\int_3^x \frac{\sin t}{t}\,dt$

   $\frac{d}{dx}\int_3^x \frac{\sin t}{t}\,dt = \frac{\sin x}{x}$

   (b) $\frac{d}{dx}\int_\pi^x e^{-t^2}\,dt$

   $\frac{d}{dx}\int_\pi^x e^{-t^2}\,dt = e^{-x^2}$

   (c) $\frac{d}{dx}\int_1^{\cos x} \frac{1}{t}\,dt$

   $\frac{d}{dx}\int_1^{\cos x} \frac{1}{t}\,dt = \left(\frac{1}{\cos x}\right)(-\sin x) = -\tan x$

   (d) $\frac{d}{dx}\int_x^2 \ln(t^2)\,dt$

   $\frac{d}{dx}\int_x^2 \ln(t^2)\,dt = -\frac{d}{dx}\int_2^x \ln(t^2)\,dt = -2\ln x$

   (e) $\frac{d}{dx}\int_{-5}^{x^2} \cos(t^3)\,dt$

   $\frac{d}{dx}\int_{-5}^{x^2} \cos(t^3)\,dt = \left(\cos x^6\right)(2x) = 2x\cos x^6$

   (f) $\frac{d}{dx}\int_{\tan x}^{17} \sin(t^4)\,dt$

   $\frac{d}{dx}\int_{\tan x}^{17} \sin(t^4)\,dt = -\frac{d}{dx}\int_{17}^{\tan x} \sin(t^4)\,dt = \left(-\sin\left(\tan^4(x)\right)\right)\left(\sec^2 x\right) = -\sec^2 x\sin(tan^4 x)$

2. The graph of a function $f$ consists of a semicircle and two line segments as shown. Let $g$ be the function given by $g(x) = \int_0^x f(t)\,dt$.

   (a) Find $g(0)$, $g(3)$, $g(-2)$, and $g(5)$.

   $g(0) = \int_0^0 f(t)\,dt = 0$

   $g(3) = \int_0^3 f(t)\,dt = \int_0^2 f(t)\,dt + \int_2^3 f(t)\,dt = \frac{\pi\,r^2}{4} + \frac{bh}{2} = \frac{\pi\,(2)^2}{4} - \frac{1*1}{2} = \pi - \frac{1}{2}$

   $g(-2) = \int_0^{-2} f(t)\,dt = -\int_{-2}^0 f(t)\,dt = -\int_0^2 f(t)\,dt = -\pi$

   $g(5) = \int_0^5 f(t)\,dt = \int_0^3 f(t)\,dt + \int_3^4 f(t)\,dt + \int_4^5 f(t)\,dt = g(3) + \frac{bh}{2} + \frac{bh}{2} = \left(\pi - \frac{1}{2}\right) - \frac{1*1}{2} + \frac{1*1}{2} =$

   $\pi - \frac{1}{2} + \frac{1}{2} - \frac{1}{2} = \pi - \frac{1}{2}$

   (b) Find all values of $x$ on the open interval $(-2, 5)$ at which $g$ has a relative maximum. Justify your answers.

   $g$ has a relative maximum at $x = 2$ because $f$ changes from positive to negative, indicating a change on $g$ from increasing to decreasing.

   (c) Find the absolute minimum value of $g$ on the closed interval $[-2, 5]$ and the value of $x$ at which it occurs. Justify your answer.

   Let us conduct a Candidate Test using our critical points (the values where $f(x) = 0$) by graphing our $x$-values and their respective values on $g(x)$.

| $x$ | -2 | 2 | 4 |
|---|---|---|---|
| $g(x)$ | $-\pi$ | $\pi$ | $\pi - 1$ |

   The absolute minimum value of $g$ on the closed interval $[-2, 5]$ is $-\pi$ at $x = -2$ by the Candidate Test.

   (d) Write an equation for the line tangent to the graph of $g$ at $x = 3$.

   $y - g(x_0) = f(x_0)(x - x_0) \rightarrow y - g(3) = f(3)(x - 3)$

   $y - (\pi - \frac{1}{2}) = -(x - 3) \rightarrow y = -x + \frac{5}{2} + \pi$

   (e) Find the $x$-coordinate of each point of inflection of the graph of $g$ on the open interval $(-2, 5)$. Justify your answer.

   The function $g$ has an inflection point at $x = 0$ because $f$ changes from increasing to decreasing at this value.

   The function also has an inflection point at $x = 3$ because $f$ changes from dereasing to increasing on this value.

   (f) Find the range of $g$.

   We currently know the lower bound of the range of $g$ because we have solved for an absolute minimum. Our previous Candidate Test also shows an upper bound of the range of $g$, and using this table we can conclude the range of $g$ is $[-\pi, \pi]$.

3. Let $g(x) = \int_0^x f(t)\,dt$, where $f$ is the function whose graph is shown.

   (a) Evaluate $g(0)$, $g(1)$, $g(2)$, and $g(7)$.

   $g(0) = \int_0^0 f(t)\,dt = 0$

   $g(1) = \int_0^1 f(t)\,dt = bh = 2 * 1 = 2$

   $g(2) = \int_0^2 f(t)\,dt = \int_0^1 f(t)\,dt + \int_1^2 f(t)\,dt = g(1) + \frac{b_1+b_2}{2}h = 2 + \frac{2+4}{2} * 1 = 2 + 3 = 5$

   $g(7) = \int_0^7 f(t)\,dt = \int_0^2 f(t)\,dt + \int_2^3 f(t)\,dt + \int_3^5 f(t)\,dt + \int_5^6 f(t)\,dt + \int_6^7 f(t)\,dt =$

   $g(2) + \frac{bh}{2} + \frac{bh}{2} + bh + \frac{bh}{2} = 5 + \frac{4*1}{2} - \frac{2*2}{2} - (2*1) - \frac{2*1}{2} = 5 + 2 - 2 - 2 - 1 = 2$

   (b) Write an equation for the line tangent to the graph of $g$ at $x = 4$.

   $y - g(x_0) = f(x_0)(x - x_0) \rightarrow y - g(4) = f(4)(x - 4)$

   $y - 6 = -(x - 4) \rightarrow y = -x + 10$

   (c) On what intervals is $g$ increasing? Decreasing? Justify your answer.

   $g$ is increasing on the interval $(0, 3)$ because all values of $g' = f$ are positive within this interval, and $g$ is decreasing on the interval $(3, 7)$ because all values of $g' = f$ are negative within this interval.

   (d) Find all values of $x$ on the open interval $0 < x < 7$ at which $g$ has a relative maximum. Justify your answer.

   $g$ has a relative maximum at $x = 3$ because $f$ changes from positive to negative, indicating a change on $g$ from increasing to decreasing.

   (e) Where does $g$ have its absolute maximum value? What is the maximum value? Justify your answer.

   For this question as well as the following one, let us conduct a Candidate Test using our critical points (the values where $f(x) = 0$) by tabulating our $x$-values and their respective values on $g(x)$.

   | $x$ | 3 | 7 |
   |---|---|---|
   | $g(x)$ | 7 | 2 |

   From this table, we can determine that the absolute maximum value of $g$ is 7 at $x = 3$ by the Candidate Test.

   (f) Where does $g$ have its absolute minimum value? What is the minimum value? Justify your answer.

   The absolute minimum value of $g$ is 2 at $x = 7$ by the Candidate Test.

4. Let $g(x) = \int_0^x f(t)\,dt$, where $f$ is the function whose graph is shown.

   (a) On what intervals is $g$ decreasing? Justify.

   $g$ is decreasing on the intervals $(1, 2.5) \cup (4, \infty)$ because all values of $f$ are negative within this interval.

   (b) For what value(s) of $x$ does $g$ have a relative maximum? Justify.

   $g$ has a relative maximum at $x = 0.5$ and $x = 3.25$ because $f$ changes from positive to negative, indicating a change on $g$ from increasing to decreasing.

   (c) On what intervals is $g$ concave down? Justify.

   $g$ is concave down on the intervals $(0.5, 1.75) \cup (3.25, \infty)$ because $f$ is decreasing on these intervals, indication downwards concavity on $g$.

   (d) At what values of $x$ does $g$ have an inflection point? Justify.

   $g$ has inflection points at $x = 0.5$, $x = 1.75$, and $x = 3.25$ because $f$ changes from increasing to decreasing or decreasing to increasing, indicating a value of $f' = 0$ at these points.

5. The graph of the function $f$, consisting of three line segments, is shown on the second page. Let $g(x) = \int_1^x f(t)\,dt$,

   (a) Find $g(2)$, $g(4)$, and $g(-2)$.

   $g(2) = \int_1^2 f(t)\,dt = \frac{b_1+b_2}{2}h = \frac{3+1}{2} * 1 = 2$

   $g(4) = \int_1^2 f(t)\,dt + \int_2^3 f(t)\,dt + \int_3^4 f(t)\,dt = g(2) + \frac{bh}{2} + \frac{bh}{2} = 2 + \frac{1*1}{2} - \frac{1*1}{2} = 2$

   $g(-2) = \int_1^{-2} f(t)\,dt = -\int_{-2}^1 f(t)\,dt = -\frac{bh}{2} = -\frac{3*3}{2} = -\frac{9}{2}$

   (b) Find $g'(0)$ and $g'(3)$.

   $g'(x) = \frac{d}{dx}\int_1^x f(t)\,dt = f(x)$

   $g'(0) = f(0) = 2$

   $g'(3) = f(3) = 0$

   (c) Find the instantaneous rate of change of $g$ with respect to $x$ at $x = 2$.

   $g'(2) = f(2) = 1$

   (d) Find the absolute maximum value of $g$ on the interval $[-2, 4]$. Justify.

   For this question, let us conduct a Candidate Test using our critical points (the values where $f(x) = 0$) by tabulating our $x$-values and their respective values on $g(x)$.

| $x$ | -2 | 3 |
|---|---|---|
| $g(x)$ | $-\frac{9}{2}$ | $\frac{5}{2}$ |

   Based upon these values, the absolute maximum value of $g$ on the interval $[-2, 4]$ is $\frac{5}{2}$ at $x = 3$.

   (e) The second derivative of $g$ is not defined at $x = 1$ and $x = 2$. Which of these values are $x$-coordinates of points of inflection of the graph of $g$? Justify.

   Of the two $x$-values, $x = 1$ contains a point of inflection of the graph of $g$, because the function $f$ changes from increasing to decreasing, indicating a change in $f'$ from positive to negative, or a moment where $x = 0$. $x = 2$ is not a point of inflection as it continually decreases, indicating negative values on the function $f'$.