Calculus BC - Worksheet on 8.1 – 8.5

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Relevant Formulas:

For integrals using $\sqrt{a^2 - u^2}$:

$$\sin\theta = \frac{u}{a} \to u = a\sin\theta$$

$$\cos\theta = \frac{\sqrt{a^2 - u^2}}{a} \to \sqrt{a^2 - u^2} = a\cos\theta$$

For integrals using $\sqrt{a^2 + u^2}$:

$$\tan\theta = \frac{u}{a} \to u = a\tan\theta$$

$$\sec\theta = \frac{\sqrt{a^2 + u^2}}{a} \to \sqrt{a^2 + u^2} = a\sec\theta$$

For integrals using $\sqrt{u^2 - a^2}$:

$$\sec\theta = \frac{u}{a} \to u = a\sec\theta$$

$$\tan\theta = \frac{\sqrt{u^2 - a^2}}{a} \to \sqrt{u^2 - a^2} = a\tan\theta$$

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Volume of solid rotated around the x-axis:

$$V = \pi\int_a^b f^2(x)\,dx \therefore \int_a^b f(x)\,dx = r$$

Work the following on **notebook paper**.

1. $\int_2^5 \frac{dx}{x^2-1}$

   $\int_2^5 \frac{dx}{x^2-1} = \int_2^5 \frac{dx}{(x+1)(x-1)} = \int_2^5 \frac{A}{x+1}\,dx + \int_2^5 \frac{B}{x-1}\,dx = \int_2^5 \frac{A(x-1)+B(x+1)}{x^2-1}\,dx$

   $x^2 - 1 = 0 \to x = \pm 1$

   $1 = A(x-1) + B(x+1)$

   $1 = A(1-1) + B(1+1) = 2B \therefore B = \frac{1}{2}$

   $1 = A(-1-1) + B(-1+1) = -2A \therefore A = -\frac{1}{2}$

   $\therefore \int_2^5 \frac{A}{x+1}\,dx + \int_2^5 \frac{B}{x-1}\,dx = \int_2^5 \frac{dx}{2x-2} - \int_2^5 \frac{dx}{2x+2} = \left[\ln\left|\frac{2x-2}{2x+2}\right|\right]_2^5 = \left[\ln\left|\frac{2}{3}\right| - \ln\left|\frac{1}{3}\right|\right] = \ln\left(\frac{\frac{2}{3}}{\frac{1}{3}}\right) = \ln 2$

2. $\int \frac{5-x}{2x^2+x-1}\,dx$

   $\int \frac{5-x}{2x^2+x-1}\,dx = \int \frac{A}{2x-1}\,dx + \int \frac{B}{x+1}\,dx = \int \frac{A(x+1)+B(2x-1)}{2x^2+x-1}\,dx$

   $2x^2 + x - 1 = (2x-1)(x+1) = 0$ at $x = -1, x = \frac{1}{2}$

   $5 - x = A(x+1) + B(2x-1)$

   $5 - (-1) = 6 = A(-1+1) + B(2(-1)-1) = -3B \therefore B = -2$

   $5 - \frac{1}{2} = \frac{9}{2} = A\left(\frac{1}{2}+1\right) + B\left(2\left(\frac{1}{2}\right)-1\right) = \frac{3}{2}A \therefore 3 = A$

   Let $u = 2x - 1 \therefore du = 2dx$

   $\therefore \int \frac{A}{2x-1}\,dx + \int \frac{B}{x+1}\,dx = 3\int \frac{dx}{2x-1} - 2\int \frac{dx}{x+1} = \frac{3}{2}\int \frac{du}{u}\,dx - 2\int \frac{dx}{x+1} = \frac{3}{2}\ln|u| - 2\ln|x+1| + C =$

   $\frac{3}{2}\ln|2x-1| - 2\ln|x+1| + C$

3. $\int_0^{\frac{\sqrt{3}}{2}} \frac{x^2}{(1-x^2)^{\frac{3}{2}}}\,dx$

   $\cos\theta = \frac{\sqrt{a^2-u^2}}{a} = \frac{\sqrt{1-x^2}}{1} = \sqrt{1-x^2} \therefore \cos^3\theta = \left(1-x^2\right)^{\frac{3}{2}}$

   $\sin\theta = \frac{u}{a} = x \to \theta = \arcsin x \therefore dx = \cos\theta d\theta$

   $\int_0^{\frac{\sqrt{3}}{2}} \frac{x^2}{(1-x^2)^{\frac{3}{2}}}\,dx = \int_0^{\frac{\pi}{3}} \frac{\sin^2\theta}{\cos^3\theta}\cos\theta\,d\theta = \int_0^{\frac{\pi}{3}} \tan^2\theta\,d\theta = \int_0^{\frac{\pi}{3}} \sec^2\theta\,d\theta - \int_0^{\frac{\pi}{3}}\,d\theta = [\tan\theta - \theta]_0^{\frac{\pi}{3}} = \sqrt{3} - \frac{\pi}{3}$

4. $\int \sin^3(6x)\,dx$

   Let $\theta = 6x \therefore d\theta = 6dx$

   $\int \sin^3(6x)\,dx = \frac{1}{6}\int \sin^3\theta\,d\theta = \frac{1}{6}\int \sin^2\theta\sin\theta\,d\theta = \frac{1}{6}\int \sin\theta\left(1-\cos^2\theta\right)d\theta = \frac{1}{6}\left(\int \sin\theta\,d\theta - \int \cos^2\theta\sin\theta\,d\theta\right)$

   Let $u = \cos\theta \therefore d\theta = -\sin\theta d\theta$

   $\frac{1}{6}\left(\int \sin\theta\,d\theta - \int \cos^2\theta\sin\theta\,d\theta\right) = \frac{1}{6}\left(\int u^2\,du - \cos\theta\right) = \frac{1}{18}u^3 - \frac{1}{6}\cos\theta + C = \frac{1}{18}\cos^3(6x) - \frac{1}{6}\cos(6x) + C$

5. $\int \frac{7x^2-16x+5}{x^3-2x^2+x}\,dx$

   $\int \frac{7x^2-16x+5}{x^3-2x^2+x}\,dx = \int \frac{7x^2-16x+5}{x(x-1)^2}\,dx = \int \frac{A}{x}\,dx + \int \frac{B}{x-1}\,dx + \int \frac{C}{(x-1)^2}\,dx = \int \frac{A(x-1)^2+Bx(x-1)+Cx}{x^3-2x^2+x}$

   $x^3 - 2x^2 + x = 0 \rightarrow x = 0, 1$

   $7x^2 - 16x + 5 = A\left(x-1\right)^2 + Bx(x-1) + Cx$

   $7(0)^2 - 16(0) + 5 = 5 = A\left(0-1\right)^2 + B(0)((0)-1) + C(0) = -A \therefore A = 5$

   $7(1)^2 - 16(1) + 5 = -4 = A\left(1-1\right)^2 + B(1)(1-1) + C(1) = C \therefore C = -4$

   $7(2)^2 - 16(2) + 5 = 1 = 5\left(2-1\right)^2 + B(2)(2-1) - 4(2) = 2B - 3 \therefore B = 2$

   $\therefore \int \frac{A}{x}\,dx + \int \frac{B}{x-1}\,dx + \int \frac{C}{(x-1)^2}\,dx = 5\int \frac{dx}{x} + 2\int \frac{dx}{x-1} - 4\int \frac{dx}{(x-1)^2} = 5\ln|x| + 2\ln|x-1| + \frac{4}{x-1} + C$

6. $\int \frac{1}{(x^2+3)^{\frac{3}{2}}}\,dx$

   $\tan\theta = \frac{u}{a} = \frac{x}{\sqrt{3}} \rightarrow x = \sqrt{3}\tan\theta \therefore dx = \sqrt{3}\sec^2\theta d\theta$

   $\sec\theta = \frac{\sqrt{u^2+a^2}}{a} = \frac{\sqrt{x^2+3}\sqrt{3}}{\rightarrow}(x^2+3)^{\frac{3}{2}} = 3\sqrt{3}\sec^3\theta$

   $\int \frac{1}{(x^2+3)^{\frac{3}{2}}}\,dx = \int \frac{\sqrt{3}\sec^2\theta}{3\sqrt{3}\sec^3\theta}\,d\theta = \frac{1}{3}\int \cos\theta\,d\theta = \frac{1}{3}\sin\theta + C = \frac{x}{3\sqrt{x^2+3}} + C$

7. $\int \arctan(5x)\,dx$

   Let $u = \arctan(5x) \therefore du = \frac{5}{1+25x^2}$ and let $v = x \therefore dv = dx$

   $\int \arctan(5x)\,dx = \int u\,dv = x\arctan(5x) - \int \frac{5x}{1+25x^2}\,dx = x\arctan(5x) - \frac{1}{10}\int \frac{du}{u} =$

   $x\arctan(5x) - \frac{1}{10}\ln\left|1+25x^2\right| + C$

8. $\int_1^2 \frac{x+1}{x(x^2+1)}\,dx$

   $\int_1^2 \frac{x+1}{x(x^2+1)}\,dx = \int_1^2 \frac{A}{x}\,dx + \int_1^2 \frac{Bx+C}{x^2+1}$

   $x(x^2+1) = 0 \to x = 0$

   $x+1 = A(x^2+1) + Bx^2 + Cx$

   $0+1 = 1 = A(0^2+1) + B(0)^2 + C(0) = A \therefore 1 = A$

   $(-1)+1 = 0 = 1((-1)^2+1) + B(-1)^2 + C(-1) = 2 + B - C \therefore C = 2 + B$

   $1+1 = 2 = 1(1^2+1) + B(1)^2 + (2+B)(1) = 4 + 2B \therefore 2 = 4 + 2B \to B = -1 \therefore C = 1$

   $\therefore \int_1^2 \frac{A}{x}\,dx + \int_1^2 \frac{Bx+C}{x^2+1} = \int_1^2 \frac{1}{x}\,dx + \int_1^2 \frac{-x+1}{x^2+1}\,dx = \int_1^2 \frac{1}{x}\,dx - \int_1^2 \frac{x}{x^2+1}\,dx + \int_1^2 \frac{dx}{x^2+1} =$

   $[\ln x]_1^2 - \frac{1}{2}[\ln|x^2+1|]_1^2 + [\arctan x]_1^2 = (\ln 2) - \frac{1}{2}(\ln 5 - \ln 2) + (\arctan 2 - \arctan 1) =$

   $-\frac{1}{2}\ln 5 + \frac{3}{2}\ln 2 + \arctan 2 - \frac{\pi}{4}$

9. $\int e^x \cos(2x)\,dx$

   Let $u = \cos(2x) \therefore du = -2\sin(2x)dx$ and let $v = e^x \therefore du = e^x dx$ and let $\alpha = \sin(2x) \therefore d\alpha = 2\cos(2x)dx$

   $\int e^x \cos(2x)\,dx = \int u\,dv = 2\int e^x \sin(2x)\,dx - 2e^x\cos(2x) = 2\int \alpha\,dv - 2e^x\cos(2x) = 2\left(e^x\sin(2x) - 2\int e^x\cos(2x)\,dx\right) - 2e^x\cos(2x)$

   Let $\beta = \int e^x \cos(2x)\,dx$

   $\beta = 2e^x\sin(2x) - 4\beta - 2e^x\cos(2x) \to 5\beta = 2e^x\sin(2x) - 2e^x\cos(2x) \to \beta = \frac{2}{5}e^x\sin(2x) - \frac{2}{5}e^x\cos(2x)$

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10. Given the region bounded by the graphs of $y = \cos\left(\frac{x}{2}\right)$, $y = 0$, $x = 0$, and $x = \pi$, Find the volume of the solid generated by revolving the region about the x-axis.

    $V = \pi\int_0^\pi \cos^2\left(\frac{x}{2}\right)dx = \frac{\pi}{2}\int_0^\pi\,dx + \frac{\pi}{2}\int_0^\pi \cos x\,dx = \frac{\pi}{2}[x + \sin x]_0^\pi = \frac{\pi}{2}\left[(\pi - \sin\pi) - (0 - \sin 0)\right] =$

    $\frac{\pi}{2}(\pi) = \frac{\pi^2}{2}$

Find the derivative.

11. $f(x) = \arcsin(3x)$

    $\frac{d}{dx}(\arcsin(3x)) = \frac{(3x)'}{\sqrt{1-(3x)^2}} = \frac{3}{\sqrt{1-9x^2}}$

12. $y = \cos^{-1}(5x^2)$

$\frac{d}{dx}\left(\arccos(5x^2)\right) = -\frac{(5x^2)'}{\sqrt{1-(5x^2)^2}} = -\frac{10x}{\sqrt{1-25x^4}}$

13. $y = \arctan(e^x)$

$\frac{d}{dx}(\arctan(e^x)) = \frac{(e^x)'}{1^2+(e^x)^2} = \frac{e^x}{1+e^{2x}}$

14. $f(x) = \sin(\arccos(2x))$

$\frac{d}{dx}(\sin(\arccos(2x))) = \frac{d}{dx}\left(\sqrt{1-4x^2}\right) = \frac{\left(1-4x^2\right)}{2\sqrt{1-4x^2}} = -\frac{4x}{\sqrt{1-4x^2}}$

Multiple Choice. All work must be shown.

15. An antiderivtive for $\frac{1}{x^2-2x+2}$ is

(a) $-\left(x^2-2x+2\right)^{-2}$
(b) $\ln(x^2-2x+2)$
(c) $\ln|\frac{x-2}{x+1}|$
(d) $\operatorname{arcsec}(x-1)$
(e) $\arctan(x-1)$

$\int \frac{1}{x^2-2x+2}\,dx = \int \frac{1}{(x-1)^2+1}\,dx = \frac{1}{1}\arctan\left(\frac{x-1}{1}\right) + C = \arctan(x-1)$

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16. The region enclosed by the x-axis, the line $x = 3$, and the curve $y = \sqrt{x}$ is rotated about the x-axis. What is the volume of the solid generated?

(a) 3π
(b) $3\sqrt{3}\pi$
(c) $\frac{9}{2}\pi$
(d) 9π
(e) $\frac{36\sqrt{3}}{5}\pi$

$V = \pi\int_0^3 \left(\sqrt{x}\right)^2 dx = \frac{\pi}{2}[x^2]_0^3 = \frac{9\pi}{2}$

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17. $\int_0^{\sqrt{3}} \frac{dx}{\sqrt{4-x^2}} =$

(a) $\frac{\pi}{3}$
(b) $\frac{\pi}{4}$
(c) $\frac{\pi}{6}$
(d) $\frac{1}{2}\ln 2$
(e) $-\ln 2$

$\int_0^{\sqrt{3}} \frac{dx}{\sqrt{4-x^2}} = [\arcsin\left(\frac{x}{2}\right)]_0^{\sqrt{3}} = \frac{\pi}{3} - 0 = \frac{\pi}{3}$