Calculus BC - Worksheet on Average Value

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Work the following on **notebook paper**. Use your calculator on problems 3-6, and give decimal answers correct to **three** decimal places.

Mean Value Theorem for Integrals: If the Fundamental Theorem of Calculus holds (i.e. $F(x)$ is differentiable on (a,b) or $f(x)$ is continuous on $[a,b]$), then there exists some value c such that:

$$f(c) = \frac{F(b)-F(a)}{b-a} = \frac{\int_a^b f(x)}{b-a}\,dx \rightarrow \int_a^b f(x)\,dx = f(c)(b-a)$$

1. $f(x) = (x-3)^2, [2,5]$

   Test of Continuity: $\lim_{x\to 2} f(x) = f(2) = 1, \lim_{x\to 5} f(x) = f(5) = 4$
   $\therefore f(x)$ is continuous on $[2,5]$ (Fundamental Theorem of Calculus holds.)

   (a) Find the average value of f on the given interval.

   $f_{AVE} = \frac{\int_2^5 f(x)\,dx}{5-2} = \frac{\int_2^5 x^2\,dx - 6\int_2^5 x\,dx + 9\int_2^5\,dx}{5-2} = \frac{[\frac{125-8}{3}]-6[\frac{25-4}{2}]+9[5-2]}{5-2} = \frac{39-63+27}{3} = 13-21+9 = 1$

   (b) Find the value of c such that $f_{AVE} = f(c)$

   $\int_2^5 f(x)\,dx = 1(5-2) = 3 \therefore f(c) = 1$

   $(x-3)^2 = x^2 - 6x + 9 = 1 \rightarrow x^2 - 6x + 8 = (x-2)(x-4) = 0 \therefore x = 2, 4$

2. $f(x) = \sqrt{x}, [0,4]$

   Test of Continuity: $\lim_{x\to 0} f(x) = f(0) = 0, \lim_{x\to 4} f(x) = f(4) = 2$
   $\therefore f(x)$ is continuous on $[0,4]$ (Fundamental Theorem of Calculus holds.)

   (a) Find the average value of f on the given interval.

   $f_{AVG} = \frac{\int_0^4 f(x)\,dx}{4-0} = \frac{[\frac{2}{3}x^{\frac{3}{2}}]_0^4}{4} = \frac{[\frac{16}{3}]}{4} = \frac{4}{3} \approx 1.333$

   (b) Find the value of c such that $f_{AVE} = f_c$

   $\int_2^5 f(x)\,dx = \frac{4}{3}(4-0) = \frac{16}{3} \therefore f(c) = \frac{4}{3}$

   $\sqrt{x} = \frac{4}{3} \rightarrow x = \frac{4^2}{3^2} = \frac{16}{9} \approx 1.778$

3. The table below gives values of a continuous function. Use a midpoint Riemann Sum with three equal subintervals to estimate the average value of f on $[20, 50]$.

x	20	25	30	35	40	45	50
$f(x)$	42	38	31	29	35	48	60

Test of Continuity: All values listed exist within the function
$\therefore f(x)$ is continuous on $[0.12]$ and differentiable on $(0, 12)$ (Fundamental Theorem of Calculus holds.)

$f_{avg} \approx \frac{\sum_{i=1}^{3} 10f(x)}{50-20} = \frac{10(38+29+48)}{30} = \frac{1150}{30} = \frac{115}{3} \approx 38.333$

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4. The velocity graph of an accelerating car is shown on the first page.

   $v(t)$ is continuous on $[0, 12]$ (Fundamental Theorem of Calculus holds.)

   (a) Estimate the average velocity of the car during the first 12 seconds by using a midpoint Riemann sum with three equal subintervals.

   $v_{avg} \approx \frac{\sum_{i=1}^{3} 4v(t)}{12-0} = \frac{4(20+50+65)}{12} = \frac{540}{12} = \frac{135}{3} = 45$ kilometres/hour on average

   (b) At what time was the instantaneous velocity equal to the average velocity?

   $\int_0^{12} v(t)\,dt = 45(12-0) \approx 540$ km $\therefore f(c) = 45$ kilometres/hour on average

   $v(t) = 45$ km/hr when $t \approx 5$ seconds

   (Graph is a bit confusing on units; x-axis labeled in seconds whereas y-axis is labeled in kilometres per hour.)

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5. In a certain city, the temperature, in °F, t hours after 9 AM was modeled by the function $T(t) = 50 + 14\sin\left(\frac{\pi t}{12}\right)$. Find the average temperature during the period from 9 AM to 9 PM.

   Test of Continuity:
   $\lim_{t\to 0} T(t) = T(0) = \lim_{t\to 12} T(t) = T(12) = 50 \therefore T(t)$ is continuous on $[0, 12]$ (Fundamental Theorem of Calculus holds.)

   $T_{avg} = \frac{\int_0^{12} T(t)\,dt}{12-0} = \frac{50\int_0^{12}\,dt + 14\int_0^{12}\sin\left(\frac{\pi t}{12}\right)dt}{12} = \frac{50[x]_0^{12} + 14[-\frac{12}{\pi}\cos\left(\frac{\pi t}{12}\right)]_0^{12}}{12} = \frac{50[12] - \frac{168}{\pi}[-2]}{12} = \frac{50[12] - \frac{168}{\pi}[-2]}{12} =$

   $\frac{600 + \frac{336}{\pi}}{12} = 50 + \frac{28}{\pi} \approx 58.913$ °F on average

6. If a cup of coffee has temperature 95°C in a room where the temperature is 20°C, then, according to Newton's Law of Cooling, the temperature of the coffee after t minutes is given by the function $T(t) = 20 + 75e^{-\frac{t}{50}}$. What is the average temperature of the coffee during the first half hour?

   Test of Continuity:
   $\lim_{t\to 0} T(t) = T(0) = 95, \lim_{t\to 30} T(t) = T(30) \approx 61.161$
   $\therefore T(t)$ is continuous on $[0, 30]$ (Fundamental Theorem of Calculus holds.)

   $f_{avg} = \frac{\int_0^{30} T'(t)\,dt}{30-0} = \frac{20\int_0^{30}\,dt + 75\int_0^{30} e^{-\frac{t}{50}}}{30} = \frac{20[x]_0^{30} + 75[-50e^{-\frac{t}{50}}]_0^{30}}{30} \approx \frac{20[30]-3750[-0.451]}{30} = 20 - 125(-0.451) =$

   $20 + 56.399 = 76.399$ °C, on average.

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7. Suppose the $C(t)$ represents the daily cost of heating your house, measured in dollars per day, where t is time measured in days and $t = 0$ corresponds to January 1, 2010. Interpret $\int_0^{90} C(t)\,dt$ and $\frac{1}{90-0}\int_0^{90} C(t)\,dt$.

   - $\int_0^{90} C(t)\,dt$ is an expression representing the estimated total cost in dollars of heating your home 90 days into 2010 (or in other terms, from 1 January 2010 to 1 April 2010).

   - $\frac{1}{90-0}\int_0^{90} C(t)\,dt$ is an expression representing the estimated average cost per day of heating your home 90 days into 2010 (or in other terms, from 1 January 2010 to 1 April 2010).

8. Using the figure on the second page,

   $f(x)$ is continuous on $[1, 6]$. It does not need to be differentiable as we are integrating the function.

   (a) Find $\int_1^6 f(x)\,dx$.

   $\int_1^6 f(x)\,dx \approx \sum_{i=1}^{4} f(x)\Delta x = bh + bh + \frac{bh}{2} + \frac{bh}{2} = \frac{2*1}{2} + \frac{1*1}{2} + 5 + 2 = 1 + \frac{1}{2} + 5 + 2 = 8.5$

   (b) What is the average value of f on $[1, 6]$?

   $f_{avg} = \frac{\int_1^6 f(x)\,dx}{6-1} = \frac{8.5}{5} = 1.7$

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9. The average value of $y = f(x)$ equals 4 for $1 \le x \le 6$ and equals 5 for $6 \le x \le 8$.

   What is the average value of $f(x)$ for $1 \le x \le 8$?

   Because average values for each interval are valid, we can assume the Fundamental Theorem of Calculus holds.

   $\int_1^8 f(x)\,dx = \int_1^6 f(x)\,dx + \int_8^6 f(x)\,dx = 4*(6-1) + 5*(8-6) = 10 = 30$

   $f_{avg} = \frac{\int_1^8 f(x)\,dx}{8-1} = \frac{30}{7}$

In problems 10-11, find the average value of the function on the given integral without integrating.

Hint: Use Geometry. (No calculator)

10. $f(x) = \begin{cases} x+4, -4 \leq x \leq -1 \\ -x+2, -1 \leq x \leq 2 \end{cases}$ on $[-4, 2]$

Test of Continuity:

$\lim_{x \to -4} f(x) = f(-4) = 0, \lim_{x \to -1} f(x) = f(-1) = 3, \lim_{x \to 2} f(x) = f(2) = 0$
$\therefore f(x)$ is continuous on $[-4, 2]$ (Fundamental Theorem of Calculus holds.)

$f_{avg} = \frac{\frac{bh}{2}}{2+4} = \frac{\frac{6*3}{2}}{6} = \frac{9}{6} = 1.5$

11. $f(x) = 1 - \sqrt{1 - x^2}$ $[-1, 1]$

Test of Continuity:

$\lim_{x \to -1} f(x) = f(-1) = \lim_{x \to 1} f(x) = f(1) = 0$
$\therefore f(x)$ is continuous on $[-1, 1]$ (Fundamental Theorem of Calculus holds.)

$f_{avg} = \frac{bh - \frac{\pi r^2}{2}}{1+1} = \frac{2*1 - \frac{\pi}{2}}{2} = 1 - \frac{\pi}{4} \approx 0.215$