Calculus BC - Worksheet on Integration by Parts

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29 January 2022

Work the following on notebook paper. **No calculator.** Evaluate.

1. $\int xe^{2x}\,dx$

   Let $\alpha = 2x \to x = \frac{\alpha}{2} \therefore d\alpha = 2dx$

   $\int \frac{\alpha}{2} * \frac{1}{2} e^{\alpha}\,d\alpha = \frac{1}{4}\int \alpha e^{\alpha}\,d\alpha$

   Let $u = \alpha \therefore du = d\alpha$ and let $v = e^{\alpha} \therefore dv = e^{\alpha} d\alpha$

   $\frac{1}{4}\int \alpha e^{\alpha}\,d\alpha = \frac{1}{4}\int u\,dv = \frac{1}{4}\left(\alpha e^{\alpha} - e^{\alpha}\right) + C = \frac{1}{4}(2xe^{2x} - e^{2x}) + C = \frac{e^{2x}}{4}(2x-1) + C$

2. $\int x\sec^2 x\,dx$

   Let $u = x \therefore du = dx$ and let $v = \tan x \therefore dv = \sec^2 x dx$

   $\int x\sec^2 x\,dx = \int u\,dv = x\tan x - \int \tan x\,dx = x\tan x - \int \frac{\sin x}{\cos x}\,dx$

   Let $\alpha = \cos x \therefore d\alpha = -\sin x dx$

   $x\tan x - \int \frac{\sin x}{\cos x}\,dx = x\tan x + \int \frac{1}{\alpha}\,dx = x\tan x + \ln|\alpha| + C = x\tan x + \ln|\cos x| + C$

3. $\int x^2\sin x\,dx$

   Let $u = x^2 \therefore du = 2xdx$ and let $v = -\cos x \therefore dv = \sin x\,dx$

   $\int x^2\sin x\,dx = \int u\,dv = 2\int x\cos x\,dx - x^2\cos x$

   Let $\alpha = x \therefore d\alpha = dx$ and let $\beta = \sin x \therefore d\beta = \cos x dx$

   $2\int x\cos x\,dx - x^2\cos x = 2\int \alpha\,d\beta - x^2\cos x = 2(x\sin x - \int \sin x\,dx) - x^2\cos x =$

   $2(x\sin x + \cos x) - x^2\cos x + C$

4. $\int x^3\ln x\,dx$

   Let $u = \ln x \therefore du = \frac{dx}{x}$ and let $v = \frac{x^4}{4} \therefore dv = x^3 dx$

   $\int x^3\ln x\,dx = \int u\,dv = \frac{1}{4}x^4\ln x - \frac{1}{4}\int x^3\,dx = \frac{1}{4}x^4\ln x - \frac{1}{16}x^4 + C = \frac{x^4}{4}\left(\ln x - \frac{1}{4}\right) + C$

5. $\int \frac{x}{e^{3x}}\,dx$

   Let $u = x \therefore du = dx$ and let $v = -\frac{1}{3e^{3x}} \therefore dv = \frac{1}{e^{3x}}dx$

   $\int \frac{x}{e^{3x}}\,dx = \int u\,dv = \frac{1}{3}\int \frac{1}{e^{3x}}\,dx - \frac{x}{3e^{3x}} = -\frac{1}{9e^{3x}} - \frac{x}{3e^{3x}} + C = -\frac{1}{3e^{3x}}\left(\frac{1}{3} + x\right) + C$

6. $\int \arctan(2x)\, dx$

   Let $u = \arctan 2x \therefore du = \frac{2}{4x^2+1}dx$ and let $v = x \therefore dv = dx$

   $\int \arctan 2x\, dx = \int u\, dv = x \arctan 2x - \int \frac{2x}{4x^2+1}\, dx$

   Let $\alpha = 4x^2 + 1 \therefore d\alpha = 8xdx$

   $x \arctan 2x - \int \frac{2x}{4x^2+1}\, dx = x \arctan 2x - \int \frac{du}{4u} = x \arctan 2x - \frac{1}{4} \int \frac{du}{u} = x \arctan 2x - \frac{1}{4} \ln|\alpha| + C =$

   $x \arctan 2x - \frac{1}{4} \ln|4x^2 + 1| + C$

7. $\int e^{4x} \sin x\, dx$

   Let $u = e^{4x} \therefore du = 4e^{4x}dx$ and let $v = -\cos x \therefore dv = \sin x dx$

   $\int e^{4x} \sin x\, dx = \int u\, dv = 4 \int e^{4x} \cos x\, dx - e^{4x} \cos x$

   Let $\beta = \sin x \therefore d\beta = \cos x dx$

   $4 \int e^{4x} \cos x\, dx - e^{4x} \cos x = 4 \int u\, d\beta - e^{4x} \cos x = 4 \left(e^{4x} \sin x - 4 \int e^{4x} \sin x\, dx\right) - e^{4x} \cos x$

   Let $\gamma = \int e^{4x} \sin x\, dx$

   $\gamma = 4 \left(e^{4x} \sin x - 4 \int e^{4x} \sin x\, dx\right) - e^{4x} \cos x = 4 \left(e^{4x} \sin x - 4\gamma\right) - e^{4x} \cos x = 4e^{4x} \sin x - 16\gamma - e^{4x} \cos x$

   $17\gamma = 4e^{4x} \sin x - e^{4x} \cos x$

   $\gamma = \frac{4e^{4x} \sin x}{17} - \frac{e^{4x} \cos x}{17}$

8. $\int_0^1 xe^{-5x}\, dx$

   Let $u = x \therefore du = dx$ and let $v = -\frac{1}{5e^{5x}} \therefore dv = \frac{1}{e^{5x}}dx$

   $\int_0^1 xe^{-5x}\, dx = \int_0^1 u\, dv = \int_0^1 \frac{1}{5e^{5x}}\, dx - [\frac{x}{5e^{5x}}]_0^1 = -[\frac{1}{25e^{5x}}]_0^1 - [\frac{x}{5e^{5x}}]_0^1 = \frac{1}{25} - \frac{1}{5e^5} - \frac{1}{25e^5}$

9. $\int_{\sqrt{e}}^{e} x \ln x\, dx$

   Let $u = \ln x \therefore du = \frac{dx}{x}$ and let $v = \frac{x^2}{2} \therefore dv = xdx$

   $\int_{\sqrt{e}}^{e} x \ln x\, dx = \int_{\sqrt{e}}^{e} u\, dv = [\frac{1}{2}x^2 \ln x]_{\sqrt{e}}^{e} - \frac{1}{2} \int_{\sqrt{e}}^{e} \frac{x^2}{x}\, dx = [\frac{1}{2}x^2 \ln x]_{\sqrt{e}}^{e} - \frac{1}{2} \int_{\sqrt{e}}^{e} x\, dx =$

   $[\frac{1}{2}x^2 \ln x]_{\sqrt{e}}^{e} - \frac{1}{4}[x^2]_{\sqrt{e}}^{e} = \left(\frac{e^2}{2} - \frac{e}{4}\right) - \frac{1}{4}\left(e^2 - e\right) = \frac{1}{4}e^2$

10. $\int \arcsin(3x)\,dx$

    Let $u = \arcsin 3x \therefore du = \frac{3}{\sqrt{1-9x^2}}dx$ and let $v = x \therefore dv = dx$

    $\int \arcsin(3x)\,dx = \int u\,dv = x \arcsin 3x - \int \frac{3x}{\sqrt{1-9x^2}}dx$

    Let $\alpha = 1 - 9x^2 \therefore d\alpha = -18xdx$

    $x \arcsin 3x - \int \frac{3x}{\sqrt{1-9x^2}}dx = x \arcsin 3x + \int \frac{3}{18\sqrt{\alpha}}\,d\alpha = x \arcsin 3x + \frac{1}{6}\int \frac{1}{\sqrt{\alpha}}\,d\alpha = x \arcsin 3x + \frac{\sqrt{1-9x^2}}{3} + C$

11. $\int x^3 e^{2x}\,dx$

    Let $\alpha = 2x \to x = \frac{\alpha}{2} \therefore d\alpha = 2dx$

    $\int \left(\frac{a}{2}\right)^3 * \frac{1}{2}e^{\alpha}\,d\alpha = \frac{1}{16}\int \alpha^3 e^{\alpha}\,d\alpha$

    Let $u = \alpha^3 \therefore du = 3\alpha^2 d\alpha$ and let $v = e^{\alpha} \therefore dv = e^{\alpha}d\alpha$

    $\frac{1}{16}\int \alpha^3 e^{\alpha}\,d\alpha = \frac{1}{16}\int u\,dv = \frac{1}{16}\left(\alpha^3 e^{\alpha} - 3\int \alpha^2 e^{\alpha}\,d\alpha\right)$

    Let $\beta = \alpha^2 \therefore d\beta = 2\alpha d\alpha$

    $\frac{1}{16}\left(\alpha^3 e^{\alpha} - 3\int \alpha^2 e^{\alpha}\,d\alpha\right) = \frac{1}{16}\left(\alpha^3 e^{\alpha} - 3\int v\,d\beta\right) = \frac{1}{16}\left(\alpha^3 e^{\alpha} - 3\left(e^{\alpha}\alpha^2 - 2\int \alpha e^{\alpha}\right)\right)$

    Let $\gamma = \int \alpha e^{\alpha}$ (Refer to Question 1 for Solution)

    $\frac{1}{16}\left(\alpha^3 e^{\alpha} - 3\left(e^{\alpha}\alpha^2 - 2\int \alpha e^{\alpha}\right)\right) = \frac{1}{16}\left(\alpha^3 e^{\alpha} - 3\left(e^{\alpha}\alpha^2 - 2\left(\alpha e^{\alpha} - e^{\alpha}\right)\right)\right) =$

    $\frac{1}{16}\alpha^3 e^{\alpha} - \frac{3}{16}\alpha^2 e^{\alpha} + \frac{3}{8}\alpha e^{\alpha} - \frac{3}{8}e^{\alpha} + C = \frac{1}{2}x^3 e^{2x} - \frac{3}{4}x^2 e^{2x} + \frac{3}{4}x e^{2x} - \frac{3}{8}e^{2x} + C = \frac{e^{2x}}{2}\left(x^3 - \frac{3}{2}\left(x^2 - x + \frac{1}{2}\right)\right) + C$

12. $\int \ln(x^2+1)\,dx$

    Let $u = \ln(x^2+1) \therefore du = \frac{2x}{x^2+1}dx$ and let $v = x \therefore dv = dx$

    $\int \ln(x^2+1)\,dx = \int u\,dv = x\ln(x^2+1) - 2\int \frac{x^2}{x^2+1}\,dx = x\ln(x^2+1) - 2\int \frac{x^2+1-1}{x^2+1}\,dx =$

    $x\ln(x^2+1) - 2\int 1 - \frac{x^2}{x^2+1}\,dx = x\ln(x^2+1) - 2\left(\int\,dx - \int \frac{x^2}{x^2+1}\,dx\right) = x\ln(x^2+1) - 2x + 2\arctan x + C$

13. (2003 AB 5) (No calc)
A coffeepot has the shape of a cylinder with radius 5 inches, as shown in the figure. Let h be the depth of the the coffee in the pot, measured in inches, where h is a function of time t, measured in seconds. The volume V of coffee in the pot is changing at a rate of $-5\pi\sqrt{h}$ cubic inches per second. (The volume V of a cylinder with radius r and height h is $V = \pi r^2 h$.)

- $r = 5$ in.
- $h(t) = \frac{V}{\pi r^2} = \frac{V}{25\pi}$
- $\frac{dV}{dt} = -5\pi\sqrt{h}$

(a) Show that $\frac{dh}{dt} = -\frac{\sqrt{h}}{5}$.

$\frac{dh}{dt} = \frac{1}{25\pi}\frac{dV}{dt} = -\frac{5\pi\sqrt{h}}{25\pi} = -\frac{\sqrt{h}}{5}$ as required.

(b) Given that $h = 17$ at time $t = 0$, solve the differential equation $\frac{dh}{dt} = -\frac{\sqrt{h}}{5}$ for h as a function of t.

$\frac{dh}{dt} = -\frac{\sqrt{h}}{5}$

$dh = -\frac{\sqrt{h}}{5}dt$

$\frac{dh}{\sqrt{h}} = -\frac{1}{5}dt$

$\int \frac{dh}{\sqrt{h}} = -\frac{1}{5}\int dt$

$2\sqrt{h} = -\frac{1}{5}t + C$

$\sqrt{h} = -\frac{1}{10}t + C \to \sqrt{17} = -\frac{0}{10} + C \to C = \sqrt{17}$

$h = \left(-\frac{1}{10}t + \sqrt{17}\right)^2$

$17 = \left(-\frac{0}{10} + \sqrt{17}\right)^2 = (\sqrt{17})^2 = 17$

(c) At what time t is the coffeepot empty?

$\sqrt{h} = -\frac{1}{10}t + \sqrt{17}$

$\sqrt{h} - \sqrt{17} = -\frac{1}{10}t$

$10\sqrt{17} - 10\sqrt{h} = t$

$10\sqrt{17} - 10\sqrt{0} = 10\sqrt{17} = t$

The coffeepot is empty at time $t = 10\sqrt{17}$ seconds.