Calculus BC - Worksheet on 5.4

Craig Cabrera

26 January 2022

Work the following on **notebook paper** No calculator. Evaluate.

1. $\int \frac{e^x + e^{-x}}{e^x - e^{-x}}\, dx$

   Let $u = e^x - e^{-x} \therefore du = e^x + e^{-x} dx$

   $-\int \frac{1}{u}\, du = \ln|u| + C = \ln|e^x - e^{-x}| + C$

2. $\int \frac{5 - e^x}{e^{2x}}\, dx$

   Let $u = e^x \therefore du = e^x dx$

   $\int \frac{5-u}{u^3}\, du = 5\int u^{-3}\, du - \int \frac{1}{u^2}\, du = \frac{1}{u} - \frac{5}{2u^2} + C = \frac{1}{e^x} - \frac{5}{2e^{2x}} + C$

3. $\int e^{-x} \tan(e^{-x})$

   Let $\theta = e^{-x} \therefore d\theta = -e^{-x} dx$

   $-\int \tan\theta\, d\theta = -\int \frac{\sin\theta}{\cos\theta}\, d\theta$

   Let $u = \cos\theta \therefore du = -\sin\theta d\theta$

   $\int \frac{1}{u}\, du = \ln|u| = \ln|\cos\theta| + C = \ln|\cos e^{-x}| + C$

4. $\int_0^1 e^{-2x}\, dx$

   Let $u = -2x \therefore du = -2dx$

   $-\frac{1}{2}\int_0^{-2} e^u\, du = \frac{1}{2}\int_{-2}^0 e^u\, du = [\frac{1}{2}e^u]_{-2}^0 = \frac{1}{2}(1 - e^{-2}) = \frac{e^2 - 1}{2e^2}$

5. $\int_0^1 xe^{-x^2}\, dx$

   Let $u = -x^2 \therefore du = -2xdx$

   $-\frac{1}{2}\int_0^{-1} e^u\, du = \frac{1}{2}\int_{-1}^0 e^u\, du = [\frac{e^u}{2}]_{-1}^0 = \frac{1}{2}\left(\frac{1}{e} - 1\right) = \frac{e-1}{2e}$

6. $\int_1^3 \frac{e^{\frac{3}{x}}}{x^2}\, dx$

   Let $u = \frac{3}{x} \therefore du = -\frac{3}{x^2} dx$

   $-\frac{1}{3}\int_3^1 e^u\, du = \frac{1}{3}\int_1^3 e^u\, du = \frac{1}{3}[e^u]_1^3 = \frac{e^3 - e}{3}$

7. $\int_0^3 \frac{2e^{2x}}{1+e^{2x}}\,dx$

   Let $u = 1 + e^{2x} \therefore du = 2e^{2x}dx$

   $\int_1^{e^6} \frac{1}{u}\,du$

   $\int_2^{e^6+1} \frac{1}{u}\,du = [\ln|u|]_2^{e^6+1} = \ln\left(\frac{e^6+1}{2}\right)$

8. $\int_0^{\frac{\pi}{2}} e^{\sin(\pi x)} \cos(\pi x)\,dx$

   Let $\theta = \pi x \therefore d\theta = \pi dx$

   $\frac{1}{\pi}\int_0^{\frac{\pi^2}{2}} e^{\sin\theta}\cos\theta\,d\theta$

   Let $u = \sin\theta \therefore du = \cos\theta d\theta$

   $\frac{1}{\pi}\int_0^{-1} e^u = -\frac{1}{\pi}\int_{-1}^{0} e^u = -\frac{1}{\pi}[e^u]_{-1}^{0} = -\frac{1}{\pi}(1-e^u) = -\frac{1}{\pi}\left(1 - e^{\sin\frac{\pi^2}{2}}\right)$

---

9. Solve: $\frac{dy}{dx} = xe^{ax^2}$ (a is a parameter)

   $y = \int xe^{ax^2}$

   Let $u = ax^2 \therefore du = 2axdx$

   $y = \frac{1}{2a}\int e^u = \frac{e^u}{2a} + C = \frac{1}{2a}e^{ax^2} + C$

10. Solve for $f'(x)$ and $f(x)$, given $f''(x) = \frac{1}{2}(e^x + e^{-x})$, $f'(0) = 0$, $f(0) = 1$

    $f'(x) = \int \frac{1}{2}(e^x + e^{-x})\,dx = \frac{1}{2}\int e^x\,dx + \frac{1}{2}\int e^{-x}\,dx = \frac{1}{2}e^x - \frac{1}{2e^x} + C = \frac{1}{2}(e^x - e^{-x}) + C$

    $f'(0) = \frac{1}{2}(e^x - e^{-x}) + C = \frac{1}{2}(1-1) + C = 0 + C \to C = 0 \therefore f'(x) = \frac{1}{2}(e^x - e^{-x})$

    $f(x) = \int \frac{1}{2}(e^x - e^{-x}) = \frac{1}{2}\int e^x\,dx - \frac{1}{2}e^{-x}\,dx = \frac{1}{2}(e^x + e^{-x}) + C$

    $f(0) = \frac{1}{2}(1+1) + C = 1 \to C = 0 \therefore f(x) = \frac{1}{2}(e^x + e^{-x})$

Multiple Choice. All work must be shown.

11. If $x^3 + 3xy + 2y^3 = 17$, then in terms of x and y, $\frac{dy}{dx} =$

   (a) $-\frac{x^2+y}{x+2y^2}$
   (b) $-\frac{x^2+y}{x+y^2}$
   (c) $-\frac{x^2+y}{x+2y}$
   (d) $-\frac{x^2+y}{2y^2}$
   (e) $\frac{-x^2}{1+2y^2}$

   $\frac{d}{dx}\left(x^3 + 3xy + 2y^3 = 17\right) \to 3x^2 + 3x\frac{dy}{dx} + 6y^2\frac{dy}{dx} + 3y = 0$

   $3(x^2 + y) = 3x^2 + 3y = -3x\frac{dy}{dx} - 6y^2\frac{dy}{dx} = -3\frac{dy}{dx}(-2y^2 - x)$

   $\frac{x^2+y}{2y^2+x} = -\frac{dy}{dx}$

   $-\frac{x^2+y}{2y^2+x} = \frac{dy}{dx}$

---

12. $\int \frac{3x^2}{\sqrt{x^3+1}}\,dx =$

   (a) $2\sqrt{x^3+1} + C$
   (b) $\frac{3}{2}\sqrt{x^3+1} + C$
   (c) $\sqrt{x^3+1} + C$
   (d) $\ln\sqrt{x^3+1} + C$
   (e) $\ln(x^3+1) + C$

   Let $u = x^3 + 1 \therefore du = 3x^2dx$

   $\int \frac{1}{\sqrt{u}}\,du = 2\sqrt{u} + C = 2\sqrt{x^3+1} + C$

13. For what value of x does the function $f(x) = (x-2)(x-3)^2$ have a relative maximum?

    (a) -3
    (b) $-\frac{7}{3}$
    (c) $-\frac{5}{2}$
    (d) $\frac{7}{3}$
    (e) $\frac{5}{2}$

    Let us conduct an intervals test for the following function. We will first solve for our critical points (the values in which $f'(x) = 0$.)

    $f'(x) = \frac{d}{dx} f(x) = 3x^2 - 16x + 21$

    $\frac{16 \pm \sqrt{256-252}}{6} = \frac{16 \pm 2}{6} = \frac{7}{3}$ and 3

    Now that we have our critical points, let us solve for the values in between.

Interval:	$\left(-\infty, \frac{7}{3}\right)$	$x = \frac{7}{3}$	$\left(\frac{7}{3}, 3\right)$	$x = 3$	$(3, \infty)$
Direction:	+	0	-	0	+

    Because $f'(x)$ changes from positive to negative values on the x value of $\frac{7}{3}$, we can conclude that $f(x)$ has a relative maximum at $x = \frac{7}{3}$.

---

14. If $f(x) = (x-1)^2 \sin x$, then $f'(0) =$

    (a) -2
    (b) -1
    (c) 0
    (d) 1
    (e) 2

    $f'(x) = \frac{d}{dx} f(x) = 2(x-1)\sin x + (x-1)^2 \cos x$

    $f'(0) = 2(0-1)\sin 0 + (-1)^2 \cos 0 = 0 + 1\cos 0 = 1$

---

15. The acceleration of a particle moving along the x-axis at time t is given by $a(t) = 6t - 2$. If the velocity is 25 when $t = 3$ and the position is 10 when $t = 1$, then the position $x(t) =$

    (a) $9t^2 + 1$
    (b) $3t^2 - 2t + 4$
    (c) $t^3 - t^2 + 4t + 6$
    (d) $t^3 - t^2 + 9t - 20$
    (e) $36t^3 - 4t^2 - 77t + 55$

    $v(t) = \int a(t)\,dt = 3t^2 - 2t + C$

    $v(3) = 27 - 6 + C = 25 \rightarrow C = 4 \therefore v(t) = 3t^2 - 2t + 4$

    $x(t) = \int v(t)\,dt = t^3 - t^2 + 4t + C$

    $x(1) = 1 - 1 + 4 + C = 10 \rightarrow C = 6 x(t) = t^3 - t^2 + 4t + 6$