Calculus BC – Worksheet on the Integral Test

Craig Cabrera

7 March 2022

Relevant Formulas and Notes:

Integral Test:

Suppose that, for all $x < 1$, the function $a(x)$ is continuous, positive, and decreasing. Consider the series and the integral

$$\sum_{k=1}^{\infty} a_k \text{ and } \int_1^{\infty} a(x)\,dx,$$

where $a_k = a(k)$ for integers $k \geq 1$.

- If either diverges, so does the other.

- If either converges, so does the other. In this case, we have

  $\int_1^{\infty} a(x)\,dx < \sum_{n=1}^{\infty} a_k < a_1 + \int_1^{\infty} a(x)\,dx$

  and $R_n = \sum_{k=n+1}^{\infty} a_k \leq \int_n^{\infty} a(x)\,dx$.

  - PERSONAL NOTE: The summation is more than the initial integration in the above bullet because, since $a(x)$ is positive but decreasing, a summation approximation will result in an overestimate of the true value under the curve.

Work the following on **notebook paper**.
Use the Integral Test to determine whether each of the given **series** converges or diverges. Justify your answers.

1. $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$

   Conditions for Integral Test:

   - $\frac{1}{\sqrt{x}}$ is continuous for all $x \geq 1$.
   - $\frac{1}{\sqrt{x}}$ is positive for all $x \geq 1$.
   - $\frac{d}{dx}\left(\frac{1}{\sqrt{x}}\right) = -\frac{1}{2x^{\frac{3}{2}}} \to f'(x)$ is negative for all $x \geq 1 \to f(x)$ is decreasing for all $x \geq 1$.

   $\lim_{c\to\infty} \int_1^c \frac{1}{\sqrt{x}}\, dx = \lim_{c\to\infty} [2\sqrt{x}]_1^c = 2\sqrt{\infty} - 2\sqrt{1} = \infty \therefore$ diverges by the Integral Test.

2. $\sum_{n=1}^{\infty} \frac{1}{n\sqrt{n}}$

   Conditions for Integral Test:

   - $\frac{1}{x\sqrt{x}}$ is continuous for all $x \geq 1$.
   - $\frac{1}{x\sqrt{x}}$ is positive for all $x \geq 1$.
   - $\frac{d}{dx}\left(\frac{1}{x\sqrt{x}}\right) = -\frac{3}{2x^{\frac{5}{2}}} \to f'(x)$ is negative for all $x \geq 1 \to f(x)$ is decreasing for all $x \geq 1$.

   $\lim_{c\to\infty} \int_1^c \frac{1}{x\sqrt{x}}\, dx = \lim_{c\to\infty} \left[-\frac{2}{\sqrt{x}}\right]_1^c = \frac{2}{\sqrt{1}} - \frac{2}{\sqrt{\infty}} = 2 \therefore$ converges by the Integral Test.

3. $\sum_{n=2}^{\infty} \frac{\ln n}{n}$

   Conditions for Integral Test:

   - $\frac{\ln x}{x}$ is continuous for all $x \geq 1$.
   - $\frac{\ln x}{x}$ is positive for all $x \geq 1$.
   - $\frac{d}{dx}\left(\frac{\ln x}{x}\right) = \frac{1-\ln(x)}{x^2} \to f'(x)$ is negative for all $x \geq e \to f(x)$ is decreasing for all $x \geq e$. (This condition is on odd ground, but can still be applied to this function)

   Let $u = \ln x \therefore du = \frac{dx}{x}$

   $\lim_{c\to\infty} \int_2^c \frac{\ln x}{x}\, dx = \lim_{c\to\infty} \int_{\ln 2}^{\ln c} u\, du = \lim_{c\to\infty} \left[\frac{1}{2}u^2\right]_{\ln 2}^{\ln c} = \infty \therefore$ diverges by the Integral Test.

4. $\sum_{n=1}^{\infty} \frac{1}{2n+5}$

   Conditions for Integral Test:

   - $\frac{1}{2x+5}$ is continuous for all $x \geq 1$.
   - $\frac{1}{2x+5}$ is positive for all $x \geq 1$.
   - $\frac{d}{dx}\left(\frac{1}{2x+5}\right) = -\frac{2}{(2x+5)^2} \rightarrow f'(x)$ is negative for all $x \geq 1 \rightarrow f(x)$ is decreasing for all $x \geq 1$.

   Let $u = 2x + 5 \therefore du = 2dx$

   $\lim_{c\to\infty} \int_1^c \frac{dx}{2x+5} = \lim_{c\to\infty} \frac{1}{2} \int_7^c \frac{du}{u} = \lim_{c\to\infty} \frac{1}{2}[\ln|u|]_7^c = \frac{1}{2} \ln\left|\frac{\infty}{7}\right| = \infty \therefore$ diverges by the Integral Test.

5. $\sum_{n=1}^{\infty} \frac{1}{n^2+1}$

   Conditions for Integral Test:

   - $\frac{1}{x^2+1}$ is continuous for all $x \geq 1$.
   - $\frac{1}{x^2+1}$ is positive for all $x \geq 1$.
   - $\frac{d}{dx}\left(\frac{1}{x^2+1}\right) = -\frac{2x}{(x^2+1)^2} \rightarrow f'(x)$ is negative for all $x \geq 1 \rightarrow f(x)$ is decreasing for all $x \geq 1$.

   $\lim_{c\to\infty} \int_1^c \frac{dx}{x^2+1} = \lim_{c\to\infty}[\arctan x]_1^c = \arctan\infty - \arctan 1 = \frac{\pi}{4} \therefore$ converges by the Integral Test.

6. $\sum_{n=1}^{\infty} \frac{1}{n^4}$

   Conditions for Integral Test:

   - $\frac{1}{x^4}$ is continuous for all $x \geq 1$.
   - $\frac{1}{x^4}$ is positive for all $x \geq 1$.
   - $\frac{d}{dx}\left(\frac{1}{x^4}\right) = -\frac{4}{x^5} \rightarrow f'(x)$ is negative for all $x \geq 1 \rightarrow f(x)$ is decreasing for all $x \geq 1$.

   $\lim_{c\to\infty} \int_1^c \frac{1}{x^4} = \lim_{c\to\infty} \left[-\frac{1}{3x^3}\right]_1^c = \frac{1}{3} - \frac{1}{3\infty^3} = \frac{1}{3} \therefore$ converges by the Integral Test.

7. Use the Integral Test to find an upper and lower bound on the limit of the series $\sum_{n=1}^{\infty} \frac{1}{n^2+1}$.

   $a_1 = \frac{1}{1^2+1} = \frac{1}{2}$

   $\lim_{c\to\infty} \int_1^c \frac{dx}{x^2+1} = \frac{\pi}{4}$

   $\int_1^{\infty} \frac{1}{n^2+1}\, dx = \frac{1}{2} < \sum_{n=1}^{\infty} \frac{1}{n^2+1} < a_1 + \int_1^{\infty} \frac{1}{n^2+1}\, dx = \frac{\pi+2}{4}$

   Lower Bound: $\frac{1}{2}$

   Upper Bound: $\frac{\pi+2}{4}$

8. Use the Integral Test to find an upper and lower bound on the limit of the series $\sum_{n=1}^{\infty} \frac{1}{n^4}$.

   $a_1 = \frac{1}{1^4} = 1$

   $\lim_{c\to\infty} \int_1^c \frac{1}{x^4} = \frac{1}{3}$

   $\int_1^{\infty} \frac{1}{n^4}\, dx = 1 < \sum_{n=1}^{\infty} \frac{1}{n^4} < a_1 + \int_1^{\infty} \frac{1}{n^4}\, dx = \frac{4}{3}$

   Lower Bound: $\frac{1}{3}$

   Upper Bound: $\frac{4}{3}$

Multiple Choice

9. If $\lim_{b\to\infty} \int_1^{\infty} \frac{dx}{x^p}$ is finite, then which of the following must be true?

   (a) $\sum_{n=1}^{\infty} \frac{1}{n^p}$ converges

   (b) $\sum_{n=1}^{\infty} \frac{1}{n^p}$ diverges

   (c) $\sum_{n=1}^{\infty} \frac{1}{n^{p-2}}$ converges

   (d) $\sum_{n=1}^{\infty} \frac{1}{n^{p-1}}$ converges

   (e) $\sum_{n=1}^{\infty} \frac{1}{n^{p+1}}$ diverges

   Mix of Integral Test and p-series Test: If $\lim_{b\to\infty} \int_1^{\infty} \frac{dx}{x^p}$ is finite, then $p > 1$, and by extension any summation of $p_{\text{init}} \leq p_{\text{final}}$ will converge.

10. Let f be a positive, continuous, and decreasing function such that $a_n = f(n)$. If $\sum_{n=1}^{\infty} a_n$ converges to k, which of the following must be true?

    (a) $\lim_{n\to\infty} a_n = k$

    (b) $\int_1^n f(x)\,dx = k$

    (c) $\int_1^\infty f(x)\,dx$ diverges

    (d) $\int_1^\infty f(x)\,dx$ converges

    (e) $\int_1^\infty f(x)\,dx = k$

    Definition of the Integral Test. Because $\sum_{n=1}^{\infty} a_n$ converges, then $\int_1^\infty f(x)\,dx$ must also converge. A does not follow the Integral Test, n is not defined in B, C is just wrong, and in E the integral would actually be less than the convergence value of the summation because the summation is an overestimation.

---

11. If $\lim_{b\to\infty} \int_1^b \frac{dx}{x^p} = 3$, then which of the following must be true?

    (a) $\sum_{n=1}^{\infty} \frac{1}{n^p} = 3$

    (b) $\sum_{n=1}^{\infty} \frac{1}{n^p} < 3$

    (c) $\sum_{n=1}^{\infty} \frac{1}{n^p} > 3$

    (d) $\sum_{n=1}^{\infty} \frac{1}{n^p}$ diverges.

    (e) No conclusion can be reached.

    $\lim_{b\to\infty} \int_1^b \frac{dx}{x^p} < \sum_{n=1}^{\infty} \frac{1}{n^p} \therefore \sum_{n=1}^{\infty} \frac{1}{n^p} > 3$

12. (2010 BC 5 – No calculator)
Consider the differential equation $\frac{dy}{dx} = 1-y$. Let $y = f(x)$ be the particular solution to this differential equation with the initial condition $f(1) = 0$. For this particular solution, $f(x) < 1$ for all values of x.

(a) Use Euler's method, starting at $x = 1$ with two steps of equal size, to approximate $f(0)$. Show the work that leads to your answer.

$f(0.5) = 0 + 1(-0.5) = -0.5$

$f(0) = -0.5 + 1.5(-0.5) = -1.25$

(b) Find $\lim_{x\to 1} \frac{f(x)}{x^3-1}$. Show the work that leads to your answer.

$\lim_{x\to 1} \frac{f(x)}{x^3-1} = \frac{f(1)}{1^3-1} = \frac{0}{0} \therefore$ L'Hospital's Rule Applies.

$\lim_{x\to 1} \frac{f(x)}{x^3-1} = \lim_{x\to 1} \frac{f'(x)}{\frac{d}{dx}(x^3-1)} = \frac{f'(1)}{3(1)^2} = \frac{1}{3}$

(c) Find the particular solution $y = f(x)$ to the differential equation $\frac{dy}{dx} = 1 - y$ with the initial condition $f(1) = 0$.

$\frac{dy}{1-y} = dx$

$-\ln|1-y| = x + C \to -\ln(1-0) = 1 + C \to C = -1$

$\ln|1-y| = -x - C = 1 - x$

$1 - y = e^{1-x}$

$-y = e^{1-x} - 1$

$y = 1 - e^{1-x}$

Reflection: I need to further study questions such as the second part, as I was not sure how to solve for the value of $f'(1)$ within the question, instead opting for confirming my answer with the third part ($\frac{d}{dx}(1 - e^{1-x}) = e^{1-x} \therefore f'(1) = 1$). Other than this, I believe my work can be fixed with better simplification of work and justifying statements.