Calculus BC - Worksheet 2 on 8.1 – 8.3

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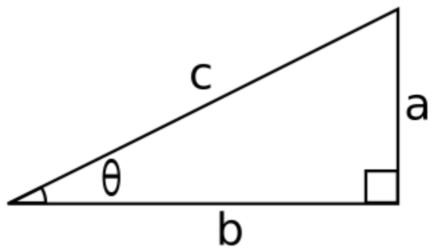
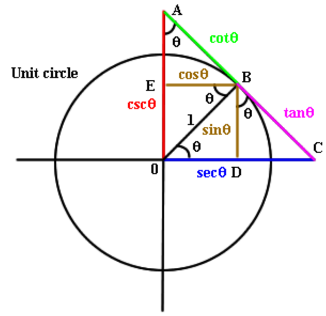
Relevant Formulas:





$$\sin^2\theta + \cos^2\theta = x^2 + y^2 = 1$$

$$\frac{\sin^2\theta + \cos^2\theta}{\cos^2\theta} = \frac{1}{\cos^2\theta} \rightarrow \tan^2\theta + 1 = \sec^2\theta$$

$$\frac{\sin^2\theta + \cos^2\theta}{\sin^2\theta} = \frac{1}{\sin^2\theta} \rightarrow 1 + \cot^2\theta = \csc^2\theta$$

$$\cos(\alpha \pm \beta) = \cos\alpha\cos\beta \mp \sin\alpha\sin\beta \rightarrow \cos(2\theta) = \cos^2\theta - \sin^2\theta = \left(1 - \sin^2\theta\right) - \sin^2\theta = \cos^2\theta - \left(1 - \cos^2\theta\right)$$

$$\frac{1}{2}(\cos(\alpha-\beta) + \cos(\alpha+\beta)) = \frac{1}{2}((\cos\alpha\cos\beta + \sin\alpha\sin\beta) + (\cos\alpha\cos\beta - \sin\alpha\sin\beta)) = \frac{2}{2}(\cos\alpha\cos\beta)$$

$$\frac{1}{2}(\cos(\alpha-\beta) - \cos(\alpha+\beta)) = \frac{1}{2}((\cos\alpha\cos\beta + \sin\alpha\sin\beta) - (\cos\alpha\csc\beta - \sin\alpha\sin\beta)) = \frac{2}{2}(\sin\alpha\sin\beta)$$

$$\cos(2\theta) = \left(1 - \sin^2\theta\right) - \sin^2\theta = 1 - 2\sin^2\theta \therefore \sin^2\theta = \frac{1 - \cos(2x)}{2}$$

$$\cos(2\theta) = \cos^2\theta - \left(1 - \cos^2\theta\right) = 2\cos^2\theta - 1 \therefore \cos^2\theta = \frac{1 + \cos(2x)}{2}$$

$$\sin(\alpha \pm \beta) = \sin\alpha\cos\beta \pm \cos\alpha\sin\beta$$

$$\frac{1}{2}(\sin(\alpha-\beta) + \sin(\alpha+\beta)) = \frac{1}{2}((\sin\alpha\cos\beta - \cos\alpha\sin\beta) + (\sin\alpha\cos\beta + \cos\alpha\sin\beta)) = \frac{2}{2}(\sin\alpha\cos\beta)$$

Work the following on **notebook paper**. **No calculator.**

1. $\int \sec^6(4x)\tan(4x)\,dx$

   Let $\theta = 4x \therefore d\theta = 4dx$

   $\int \sec^6(4x)\tan(4x)\,dx = \frac{1}{4}\int \sec^4\theta\tan\theta\,d\theta$

   Let $u = \sec\theta \therefore du = \sec\theta\tan\theta d\theta$

   $\frac{1}{4}\int \sec^6\theta\tan\theta\,d\theta = \frac{1}{4}\int u^5\,du = \frac{1}{24}u^6 + C = \frac{1}{24}\sec^6(4x) + C$

2. $\int \tan^5(3x)\sec^2(3x)\,dx$

   Let $\theta = 3x \therefore d\theta = 3dx$

   $\int \tan^5(3x)\sec^2(3x)\,dx = \frac{1}{3}\int \tan^5\theta\sec^2\theta\,d\theta$

   Let $u = \tan\theta \therefore du = \sec^2\theta$

   $\frac{1}{3}\int \tan^5\theta\sec^2\theta\,d\theta = \frac{1}{3}\int u^5\,du = \frac{1}{18}u^6 + C = \frac{1}{18}\tan^6(3x) + C$

3. $\int \cos^3(2x)\sin^2(2x)\,dx$

   $\int \cos^3(2x)\sin^2(2x)\,dx = \int \cos^2(2x)\sin^2(2x)\cos(2x)\,dx = \int (1-\sin^2(2x))\sin^2(2x)\cos(2x)\,dx$

   Let $u = \sin(2x) \therefore du = 2\cos(2x)dx$

   $\int (1-\sin^2(2x))\sin^2(2x)\cos(2x)\,dx = \frac{1}{2}\int (1-u^2)u^2\,du = \frac{1}{2}\left(\int u^2\,du - \int u^4\,du\right) = \frac{1}{6}u^3 - \frac{1}{10}u^5 + C =$

   $\frac{1}{6}\sin^3(2x) - \frac{1}{10}\sin^5(2x) + C$

4. $\int \frac{2x-3}{x^2-10x+41}\,dx$

   Let $u = x^2 - 10x + 41 \therefore du = (2x-10)dx$

   $\int \frac{2x-3}{x^2-10x+41}\,dx = \int \frac{du}{u} - 7\int \frac{dx}{x^2-10x+41} = \int \frac{du}{u} - 7\int \frac{dx}{(x-5)^2+16} = \ln|u| - \frac{7}{4}\arctan\left(\frac{x-5}{4}\right) + C =$

   $\ln|x^2-10x+41| - \frac{7}{4}\arctan\left(\frac{x-5}{4}\right) + C$

5. $\int x^2 \sin(3x)\, dx$

Let $u = x^2 \therefore du = 2xdx$ and let $v = -\frac{1}{3}\cos(3x) \therefore dv = \sin(3x)dx$

$\int x^2 \sin(3x)\, dx = \int u\, dv = \frac{2}{3}\int x\cos(3x)\, dx - \frac{1}{3}x^2\cos(3x)$

Let $\alpha = x \therefore d\alpha = dx$ and let $\beta = \sin 3x \therefore d\beta = 3\cos(3x)dx$

$\frac{2}{3}\int x\cos(3x)\, dx - \frac{1}{3}x^2\cos(3x) = \frac{2}{3}\int \alpha\, d\beta - \frac{1}{3}x^2\cos(3x) = \frac{2}{3}\left(x\sin(3x) - \int \sin(3x)\, dx\right) - \frac{1}{3}x^2\cos(3x) =$

$\frac{2}{3}\left(x\sin(3x) + \frac{1}{3}\cos(3x)\right) - \frac{1}{3}x^2\cos(3x) + C$

6. $\int \arcsin(3x)\, dx$

Let $u = \arcsin(3x) \therefore du = \frac{3}{\sqrt{1-9x^2}}dx$ and let $v = x \therefore du = dx$

$\int \arcsin(3x)\, dx = \int u\, dv = x\arcsin(3x) - \int \frac{3x}{\sqrt{1-9x^2}}\, dx$

Let $\alpha = 1 - 9x^2 \therefore d\alpha = -18xdx$

$x\arcsin(3x) - \int \frac{3x}{\sqrt{1-9x^2}}\, dx = x\arcsin(3x) + \frac{1}{6}\int \frac{du}{\sqrt{u}} = x\arcsin(3x) + \frac{1}{3}\sqrt{u} + C =$

$x\arcsin(3x) + \frac{1}{3}\sqrt{1-9x^2} + C$

7. $\int \sin(3x)\cos(2x)\, dx$

$\int \sin(3x)\cos(2x)\, dx = \frac{1}{2}\left(\int \sin x\, dx + \int \sin(5x)\, dx\right) = -\frac{1}{2}\cos x - \frac{1}{10}\cos(5x) + C$

8. $\int \sec^3(7x)\tan(7x)\, dx$

Let $\theta = 7x \therefore d\theta = 7dx$

$\int \sec^3(7x)\tan(7x)\, dx = \frac{1}{7}\int \sec^3\theta\tan\theta\, d\theta$

Let $u = \sec\theta \therefore du = \sec\theta\tan\theta d\theta$

$\frac{1}{7}\int u^2\, du = \frac{1}{21}u^3 + C = \frac{1}{21}\sec^3(7x) + C$

9. $\int \sin^2(5x)\cos^2(5x)\, dx$

$\int \sin^2(5x)\cos^2(5x)\, dx = \int \left(\frac{1-\cos(10x)}{2}\right)\left(\frac{1+\cos(10x)}{2}\right) dx = \frac{1}{4}\left(\int\, dx - \int \cos^2(10x)\, dx\right) =$

$\frac{1}{4}\left(\int\, dx - \frac{1}{2}\left(\int\, dx + \int \cos(20x)\, dx\right)\right) = \frac{1}{8}x - \frac{1}{160}\sin(20x) + C$

10. $\int e^{3x} \cos x \, dx$

    Let $u = e^{3x} \therefore du = 3e^{3x} dx$ and let $v = \sin x \therefore dv = \cos x dx$

    $\int e^{3x} \cos x \, dx = \int u \, dv = e^{3x} \sin x - 3 \int e^{3x} \sin x \, dx$

    Let $\alpha = -\cos x \therefore d\alpha = \sin x dx$

    $e^{3x} \sin x - 3 \int e^{3x} \sin x \, dx = e^{3x} \sin x - 3 \int u \, d\alpha = e^{3x} \sin x - 3 \left(3 \int e^{3x} \cos x \, dx - e^{3x} \cos x\right)$

    Let $\beta = \int e^{3x} \cos x \, dx$

    $\beta = e^{3x} \sin x - 9\beta + 3e^{3x} \cos x \to 10\beta = e^{3x} \sin x + 3e^{3x} \cos x \to \beta = \frac{1}{10} e^{3x} \sin x + \frac{3}{10} e^{3x} \cos x$

11. $\int_0^{\frac{\pi}{6}} x \cos(2x) \, dx$

    Let $u = x \therefore du = dx$ and let $v = \frac{1}{2} \sin(2x) \therefore dv = \cos(2x) dx$

    $\int_0^{\frac{\pi}{6}} x \cos(2x) \, dx = \int u \, dv = \frac{1}{2} [x \sin(2x)]_0^{\frac{\pi}{6}} - \frac{1}{2} \int_0^{\frac{\pi}{6}} \sin(2x) \, dx = \frac{1}{2} [x \sin(2x)]_0^{\frac{\pi}{6}} + \frac{1}{4} [\cos(2x)]_0^{\frac{\pi}{6}} =$

    $\frac{\sqrt{3}\pi - 3}{24}$

12. $\int \cos(5x) \cos(4x) \, dx$

    $\int \cos(5x) \cos(4x) \, dx = \frac{1}{2} \left(\int \cos(9x) \, dx + \int \cos(x) \, dx\right) = \frac{1}{18} \sin(9x) + \frac{1}{2} \sin x + C$

13. (2004 Form B - AB 2) (Calc)
For $0 \leq t \leq 31$, the rate of change of the number of mosquitoes on Tropical Island at time t days is modeled by $R(t) = 5\sqrt{t}\cos\left(\frac{t}{5}\right)$ mosquitoes per day. There are 1000 mosquitoes on Tropical Island at time $t = 0$.

(a) Show that the number of mosquitoes is increasing at time $t = 6$.

$R(6) = 5\sqrt{6}\cos\left(\frac{6}{5}\right) = 4.438$

The number of mosquitoes is increasing at time $t = 6$ because $R(t)$ is positive at that value of t.

(b) At time $t = 6$, is the number of mosquitoes increasing at an increasing rate, or is the number of mosquitoes increasing at a decreasing rate? Give a reason for your answer.

$\frac{dR}{dt} = \frac{5\cos\left(\frac{x}{5}\right)}{2\sqrt{x}} - \sqrt{x}\sin\left(\frac{x}{5}\right)$

$R'(6) = \frac{5\cos\left(\frac{6}{5}\right)}{2\sqrt{6}} - \sqrt{6}\sin\left(\frac{6}{5}\right) = -1.913$

The number of mosquitoes are increasing at a decreasing rate because $R'(t)$ is negative at $t = 6$.

(c) According to the model, how many mosquitoes will be on the island at time $t = 31$? Round your answer to the nearest whole number.

$\int_0^{31} 5\sqrt{t}\cos\left(\frac{t}{5}\right)dt = 35.665$

$100035.665 \approx 964$ mosquitoes.

964 mosquitoes will be on the island at $t = 31$.

(d) To the nearest whole number, what is the maximum number of mosquitoes for $0 \leq t \leq 31$? Show the analysis that leads to your conclusion.

$R(t) = 0$ at $t = 0\,, t = 2.5\pi\,, t = 7.5\pi$