# Calculus BC - Worksheet on Algebraic & U-Substitution

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Work the following on **notebook paper**. Do not use your calculator.
Evaluate.

1. $\int x\sqrt{x+2}\,dx$

   Let $u = x+2 \rightarrow x = u-2 \therefore du = dx$

   $\int (u-2)\sqrt{u}\,du = \int u\sqrt{u} - 2\sqrt{u}\,du = \int u^{\frac{3}{2}}\,du - 2\int u^{\frac{1}{2}}\,du = [\frac{2}{5}u^{\frac{5}{2}}] - 2[\frac{2}{3}u^{\frac{3}{2}}] + C = \frac{2}{5}(x+2)^{\frac{5}{2}} - \frac{4}{3}(x+2)^{\frac{3}{2}} + C$

2. $\int x\sqrt{x^2+2}\,dx$

   Let $u = x^2+2 \therefore du = 2xdx \rightarrow dx = \frac{du}{2x}$

   $\int \frac{\sqrt{u}}{2}\,du = \frac{1}{2}\int \sqrt{u}\,du = \frac{1}{2}[\frac{2}{3}u^{\frac{3}{2}}] = \frac{1}{3}(x^2+2)^{\frac{3}{2}} + C$

3. $\int x^2\sqrt{1-x}\,dx$

   Let $u = 1-x \rightarrow x = 1-u \therefore dx = -du$

   $\int -(1-u)^2\sqrt{u}\,du = \int -(u^2-2u+1)\sqrt{u}\,du = -\int u^{\frac{5}{2}}\,du + 2\int u^{\frac{3}{2}}\,du - \int \sqrt{u}\,du =$
   $-\frac{2}{7}(1-x)^{\frac{7}{2}} + \frac{4}{5}(1-x)^{\frac{5}{2}} - \frac{2}{3}(1-x)^{\frac{3}{2}} + C$

4. $\int \frac{x}{\sqrt{x+4}}\,dx$

   Let $u = x+4 \rightarrow x = u-4 \therefore du = dx$

   $\int \frac{u-4}{\sqrt{u}}\,du = \int \sqrt{u}\,du - 4\int u^{-\frac{1}{2}}\,du = \frac{2}{3}(x+4)^{\frac{3}{2}} - 8\sqrt{x+4} + C$

5. $\int \frac{2x}{\sqrt{x^2+4}}\,dx$

   Let $u = x^2+4 \therefore du = 2xdx \rightarrow dx = \frac{du}{2x}$

   $\int \frac{1}{\sqrt{u}}\,du = \int u^{-\frac{1}{2}}\,du = [2u^{\frac{1}{2}}] + C = 2\sqrt{x^2+4} + C$

6. $\int_{-1}^{7} x\sqrt[3]{x+1}\,dx$

   Let $u = x+1 \rightarrow x = u-1 \therefore du = dx$

   $\int_0^8 (u-1)\sqrt[3]{u}\,du = \int_0^8 u^{\frac{4}{3}}\,du - \int_0^8 \sqrt[3]{u}\,du = [\frac{3}{7}(u)^{\frac{7}{3}}]_0^8 - [\frac{3}{4}(u)^{\frac{4}{3}}]_0^8 = \frac{384}{7} - \frac{36}{3} = \frac{300}{7} \approx 42.857$

7. $\int_{-1}^{1} x(x^2+1)^3\,dx$

   Let $u = x^2+1 \rightarrow x = \sqrt{u-1} \therefore du = 2xdx \rightarrow dx = \frac{du}{2x}$

   $\frac{1}{2}\int_2^2 u^3\,du = 0$ because the upper and lower bounds of the domain are equal. $(a = b \therefore \int_a^b f(x)\,dx = 0)$

8. $\int_1^5 \frac{x}{\sqrt{2x-1}}\,dx$

   Let $u = 2x - 1 \rightarrow x = \frac{u+1}{2} \therefore du = 2dx \rightarrow dx = \frac{du}{2}$

   $\int_1^9 \frac{\frac{u+1}{2}}{2\sqrt{u}}\,du = \int_1^9 \frac{u+1}{4\sqrt{u}}\,du = \frac{1}{4}(\int_1^9 \sqrt{u}\,du - \int_1^9 u^{-\frac{1}{2}}\,du) = \frac{[2u^{\frac{3}{2}}]_1^9 + [6\sqrt{u}]_1^9}{4} = \frac{54-2+18-6}{12} = \frac{16}{3} \approx 5.333$

9. $\int_1^2 2x^2\sqrt{x^3+1}\,dx$

   Let $u = x^3 + 1 \rightarrow x = \sqrt[3]{u-1} \therefore du = 3x^2dx \rightarrow dx = \frac{du}{3x^2}$

   $\frac{2}{3}\int_2^9 \sqrt{u}\,du = \frac{2}{3}[\frac{2}{3}u^{\frac{3}{2}}]_2^9 = \frac{4}{9}[27 - 2\sqrt{2}] \approx 10.742$

10. $\int_0^4 \frac{1}{\sqrt{2x+1}}\,dx$

    Let $u = 2x + 1 \rightarrow x = \frac{u-1}{2} \therefore du = 2dx \rightarrow dx = \frac{du}{2}$
    $\frac{1}{2}\int_1^9 u^{-\frac{1}{2}}\,du = \frac{1}{2}[2\sqrt{u}]_1^9 = \frac{4}{2} = 2$

11. $\int_{-4}^3 \frac{x}{\sqrt[3]{x+5}}\,dx$

    Let $u = x + 5 \rightarrow x = u - 5 \therefore du = dx$

    $\int_1^8 u^{\frac{2}{3}}\,du - 5\int_1^8 u^{-\frac{1}{3}}\,du = [\frac{3}{5}u^{\frac{5}{3}}]_1^8 - 5[\frac{3}{2}u^{\frac{2}{3}}]_1^8 = \frac{93}{5} - \frac{45}{2} = \frac{186-225}{10} = -\frac{39}{10} = -3.9$

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12. Find the area bounded by the graph of $y = x\sqrt[3]{x+1}$ and the $x$-axis on the interval $[0, 7]$.

    Let $u = x + 1 \rightarrow x = u - 1 \therefore dx = du$

    $\int_1^8 (u-1)\sqrt[3]{u}\,du = \int_1^8 u^{\frac{4}{3}}\,du - \int_1^8 \sqrt[3]{u}\,du = [\frac{3}{7}u^{\frac{7}{3}}]_1^8 - [\frac{3}{4}u^{\frac{4}{3}}]_1^8 = \frac{384-3}{7} - \frac{48-3}{4} = \frac{1524-315}{28} = \frac{1209}{28} \approx 43.179$

13. Find the area bounded by the graph of $y = x + \cos x$ and the $x$-axis on the interval $[\frac{\pi}{3}, \frac{\pi}{2}]$.

    $\int_{\frac{\pi}{3}}^{\frac{\pi}{2}} x\,dx + \int_{\frac{\pi}{3}}^{\frac{\pi}{2}} \cos x\,dx = [\frac{1}{2}x^2]_{\frac{\pi}{3}}^{\frac{\pi}{2}} + [\sin x]_{\frac{\pi}{3}}^{\frac{\pi}{2}} = \frac{1}{2}[\frac{\pi^2}{4} - \frac{\pi^2}{9}] + [\frac{2-\sqrt{3}}{2}] = \frac{5\pi^2+72-36\sqrt{3}}{72} \approx 0.819$

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14. Solve: $\frac{dy}{dx} = -\frac{48}{(3x+5)^3}$ given that $(-1, 3)$ is a point on the solution curve.

    Let $u = 3x + 5 \rightarrow x = \frac{u-5}{3} \therefore du = 3dx \rightarrow dx = \frac{du}{3}$

    $-16\int u^{-3}\,du = -16[-\frac{1}{2}u^{-2}] + C = \frac{8}{(3x+5)^2} + C$

    $\frac{8}{(3(-1)+5)^2} + C = \frac{8}{4} + C = 3 \rightarrow 2 + C = 3 \rightarrow C = 1 \therefore f(x) = \frac{8}{(3x+5)^2} + 1$

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Given that $f(x)$ is an even function and that $\int_0^2 f(x)\,dx = \frac{8}{3}$, find:

15. $\int_{-2}^0 f(x)\,dx$

    Because an even function has reflection symmetry about the y-axis, $\int_0^2 f(x)\,dx = \int_{-2}^0 f(x)\,dx = \frac{8}{3}$

16. $\int_{-2}^{2} f(x)\,dx$

$\int_{-2}^{2} f(x)\,dx = \int_{0}^{2} f(x)\,dx + \int_{-2}^{0} f(x)\,dx = \frac{16}{3}$

17. $\int_{-2}^{0} 3f(x)\,dx$

$\int_{-2}^{0} 3f(x)\,dx = 3\int_{-2}^{0} f(x)\,dx = \frac{24}{3} = 8$

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Given that $f(x)$ is an odd function and that $\int_{0}^{2} f(x)\,dx = \frac{8}{3}$, find:

18. $\int_{-2}^{0} f(x)\,dx$

Because an odd function has rotational symmetry about the origin, $\int_{-2}^{0} f(x)\,dx = -\int_{0}^{2} f(x)\,dx = -\frac{8}{3}$

19. $\int_{-2}^{2} f(x)\,dx$

$\int_{-2}^{2} f(x)\,dx = \int_{0}^{2} f(x)\,dx + \int_{-2}^{0} f(x)\,dx = 0$

20. $\int_{-2}^{0} 3f(x)\,dx$

$\int_{-2}^{0} 3f(x)\,dx = 3\int_{-2}^{0} f(x)\,dx = -\frac{24}{3} = -8$

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21. Write $\lim_{n\to\infty} \frac{1}{n}[(\frac{1}{n})^3 + (\frac{2}{n})^3 + ... + (\frac{5n}{n})^3]$ as a definite integral, given that $n$ is a positive integer.

Definite Integral as the Limit of a Riemann Sum:

$$\lim_{n\to\infty} \sum_{i=1}^{n} f(a+i\Delta x)\Delta x = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_k)\Delta x = \int_a^b f(x)dx$$

$\lim_{n\to\infty} \frac{1}{n}[(\frac{1}{n})^3 + (\frac{2}{n})^3 + ... + (\frac{5n}{n})^3] = \lim_{\frac{n}{5}\to\infty} \sum_{i=1}^{n} f(0+\frac{5i}{n})\frac{5}{n} = \int_0^5 x^3\,dx = [\frac{1}{4}x^4]_0^5 = \frac{625}{4} = 156.25$

22. The closed interval $[c, d]$ is partitioned into $n$ equal subintervals, each of width $\Delta x$, by the numbers $c = x_0, x_1, ..., x_n$ where $x_0 < x_1 < x_2 < ... < x_{n-1} = d$. Write $\lim_{n\to\infty} \sum_{i=1}^{n} (x_k)^2 \Delta x$ as a definite integral.

Definite Integral as the Limit of a Riemann Sum:

$$\lim_{n\to\infty} \sum_{i=1}^{n} f(a+i\Delta x)\Delta x = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_k)\Delta x = \int_a^b f(x)dx$$

$\lim_{n\to\infty} \sum_{i=1}^{n} (x_k)^2 \Delta x = \int_c^d x^2\,dx = \frac{1}{3}[d^3 - c^3]$