Calculus BC - Worksheet 2 on Fundamental Theorem of Calculus

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Work the following on **notebook paper**. Use your calculator on problems 3, 8, and 13.

1. If $f(1) = 12$, f' is continuous, and $\int_1^4 f'(x)\,dx = 17$, what is the value of $f(4)$?

   $f'(x)$ is continuous on $[1, 4]$ $\therefore$ $f(x)$ is continuous on $[1, 4]$ and differentiable on $(1, 4)$ (Fundamental Theorem of Calculus holds.)

   $\int_1^4 f'(x)\,dx = f(4) - f(1) = 17 \rightarrow f(4) = 17 + f(1) = 17 + 12 = 29$

2. If $\int_2^5 2f(x) + 3\,dx = 17$, find $\int_2^5 f(x)\,dx$.

   No test of differentiablity is used as all integrals are either simplified into variables or are integrals of continuous constants.

   Let $F = \int_2^5 f(x)\,dx$

   $2F + 3\int_2^5 \,dx = 2F + 3[x]_2^5 = 2F + 3[3] = 17 \rightarrow 2F = 8 \rightarrow F = \int_2^5 f(x)\,dx = 4$

3. Water is pumped out of a holding tank at a rate of $5 - 5e^{-0.12t}$ litres/minute, where t is in minutes since the pump is started. If the holding tank contains 1000 litres of water when the pump is started, how much water does it hold one hour later?

   Test of Differentiability: $\lim_{t\to 0} r(t) = r(0) = 0, \lim_{t\to 60} r(t) = r(60) \approx 4.996$
   $\therefore$ $R(t)$ is continuous on $[0, 60]$ and differentiable on $(0, 60)$ (Fundamental Theorem of Calculus holds.)

   $5\int_0^{60} dt - 5\int_0^{60} e^{-0.12t}\,dt = 5[x]_0^{60} - 5[-\frac{1}{0.12}e^{-0.12t}]_0^{60} \approx 5[60] - 5[-0.006 + 8.333] = 300 - 41.635 =$ 258.365 litres pumped out of a tank in one hour.

   $1000 - 258.364 = 741.636$ litres in the tank one hour later.

4. Given the values of the derivative $f'(x)$ in the table and that $f(0) = 100$, estimate $f(x)$ for $x = 2, 4, 6$. Use a right Riemann sum.

x	0	2	4	6
$f'(x)$	10	18	23	25

   $f'(x)$ is continuous on $[0, 6]$ $\therefore$ $f(x)$ is continuous on $[0, 6]$ and differentiable on $(0, 6)$ (Fundamental Theorem of Calculus holds.)

   - $\sum_{i=1}^{2} f(x_i^*) = 2(18) = 36$

     $\int_0^2 f'(x)\,dx = f(2) - f(0) = 36 \rightarrow f(2) = 36 + f(0) = 36 + 100 = 136$
   - $\sum_{i=1}^{2} f(x_i^*) = 2(23) = 46$

     $\int_2^4 f'(x)\,dx = f(4) - f(2) = 46 \rightarrow f(4) = 136 + f(2) = 136 + 46 = 182$
   - $\sum_{i=1}^{2} f(x_i^*) = 2(25) = 50$

     $\int_4^6 f'(x)\,dx = f(6) - f(4) = 50 \rightarrow f(6) = 50 + f(4) = 50 + 182 = 232$

5. Consider the function f that is continuous on the interval $[-5,5]$ and for which $\int_0^5 f(x)\,dx = 4$. Evaluate:

   $f'(x)$ is continuous on $[-5,5]$ $\therefore$ $f(x)$ is continuous on $[-5,5]$ and differentiable on $(-5,5)$ for all situations (Fundamental Theorem of Calculus holds.)

   (a) $\int_0^5 f(x) + 2\,dx =$

   $\int_0^5 f(x)\,dx + 2\int_0^5\,dx = \int_0^5 f(x) + 2[x]_0^5 = 4 + 10 = 14$

   (b) $\int_{-2}^3 f(x+2)\,dx =$

   Let $u = x + 2 \therefore du = dx$

   $\int_0^5 f(x)\,dx = 4$

   (c) $\int_{-5}^5 f(x)\,dx$ (f is even)=

   Because an even function has reflection symmetry about the y-axis, $\int_{-5}^0 f(x)\,dx = \int_0^5 f(x)\,dx$

   $\int_{-5}^5 f(x)\,dx = \int_{-5}^0 f(x)\,dx + \int_0^5 f(x)\,dx = 4 + 4 = 8$

   (d) $\int_{-5}^5 f(x)\,dx$ (f is odd)=

   Because an odd function has rotational symmetry about the origin, $\int_{-5}^0 f(x)\,dx = -\int_0^5 f(x)\,dx$

   $\int_{-5}^5 f(x)\,dx = \int_{-5}^0 f(x)\,dx + \int_0^5 f(x)\,dx = 4 + (-4) = 0$

6. Use the figure on the first page and the fact that $P(0) = 2$ to find values of P when $t = 1, 2, 3, 4$, and 5.

   $\frac{dP}{dt}$ is continuous on $[0,5]$ $\therefore$ $P(t)$ is continuous on $[0,5]$ and differentiable on $(0,5)$ (Fundamental Theorem of Calculus holds.)

   - $\int_0^1 \frac{dP}{dt}\,dt \approx bh = (1)(-1) = P(1) - P(0) = -1 \rightarrow P(1) = -1 + P(0) = -1 + 2 = 1$
   - $\int_1^2 \frac{dP}{dt}\,dt \approx bh = (1)(-1) = P(2) - P(1) = -1 \rightarrow P(2) = -1 + P(1) = -1 + 1 = 0$
   - $\int_2^3 \frac{dP}{dt}\,dt \approx \frac{bh}{2} = \frac{(1)(-1)}{2} = P(3) - P(2) = -\frac{1}{2} \rightarrow P(3) = P(2) - \frac{1}{2} = -\frac{1}{2}$
   - $\int_3^4 \frac{dP}{dt}\,dt \approx \frac{bh}{2} = \frac{1*1}{2} = P(4) - P(3) = -\frac{1}{2} \rightarrow P(4) = P(3) - \frac{1}{2} = \frac{1-1}{2} = 0$
   - $\int_4^5 \frac{dP}{dt}\,dt \approx bh = (1)(1) = P(5) - P(4) = 1 \rightarrow P(5) = 1 - P(4) = 1$

7. In the figure on the first page, the graph of g is given. Let $G(t)$ be the antiderivative of $g(t)$.

   $g(t)$ is continuous on $[0,5]$ $\therefore$ $G(t)$ is continuous on $[0,5]$ and differentiable on $(0,5)$ (Fundamental Theorem of Calculus holds.)

   (a) Given $G(0) = 5$, find $G(2)$, $G(4)$, and $G(5)$

   - $\int_0^2 g(t)\,dt = G(2) - G(0) = 16 \to G(2) = 16 + G(0) = 16 + 5 = 21$
   - $\int_2^4 g(t)\,dt = G(4) - G(2) = -8 \to G(4) = G(2) - 8 = 21 - 8 = 13$
   - $\int_4^5 g(t)\,dt = G(5) - G(4) = 2 \to G(5) = 2 + G(4) = 2 + 13 = 15$

   (b) Find the intervals where the graph of G is increasing and decreasing. Justify your answer.

   - $g(t)$ is positive on $0 < t < 2 \cup 4 < t < 5 \therefore G(t)$ is increasing on $0 < t < 2 \cup 4 < t < 5$.
   - $g(t)$ is negative on $2 \le t \le 4 \therefore G(t)$ is decreasing on $2 \le t \le 4$.

   (c) Find the intervals where the graph of G is concave up and concave down. Justify your answer.

   - $g(t)$ is increasing on $0 < t < \frac{2}{3} \cup 3 < x < \frac{9}{2} \therefore G(t)$ is concave up on $0 < t < \frac{2}{3} \cup 3 < x < \frac{9}{2}$.
   - $g(t)$ is decreasing on $\frac{2}{3} < t < 3 \cup 4.5 < t < 5 \therefore G(t)$ is concave down on $\frac{2}{3} < t < 3 \cup 4.5 < t < 5$.

   (d) Sketch a graph of an antiderivate $G(t)$. Label each critical point of $G(t)$ with its coordinates.

   Graph on next page.

8. Find the value of $F(1)$, where $F'(1) = e^{-x^2}$ and $F(0) = 2$.

   Test of Differentiability: $\lim_{x\to 0} F'(x) = F'(0) = 1, \lim_{x\to 1} F'(x) = F'(1) \approx 0.368$
   $\therefore F(x)$ is continuous on $[0,1]$ and differentiable on $(0,1)$ (Fundamental Theorem of Calculus holds.)

   $\int_0^1 F'(x)\,dx = F(1) - F(0) \approx 0.747 \to F(1) = 0.747 + F(0) = 2.747$

9. Given $f(x) = \begin{cases} 2x, x \le 1 \\ 2, x > 1 \end{cases}$. Evaluate $\int_{\frac{1}{2}}^5 f(x)\,dx$.

   $f(x)$ is continuous on $[\frac{1}{2}, 5]$ $\therefore$ $F(x)$ is continuous on $[\frac{1}{2}, 5]$ and differentiable on $(\frac{1}{2}, 5)$ (Fundamental Theorem of Calculus holds.)

   $\int_{\frac{1}{2}}^5 f(x)\,dx = [2x]_{x=5} - [x^2]_{x=\frac{1}{2}} = 10 - \frac{1}{4} = \frac{35}{4} = 8.75$

10. A bowl of soup is placed on a kitchen counter to cool. The temperature of the soup is given in the table below.

Time t (minutes)	0	5	8	12
Temperature $T(x)$ (°F)	105	99	97	93

Test of Differentiability: No x or y-value are the same $\therefore \forall x \exists T'(x) \neq 0$
$\therefore T(x)$ is continuous on $[0.12]$ and differentiable on $(0,12)$ (Fundamental Theorem of Calculus holds.)

(a) Find $\int_0^{12} T'(x)\,dx$.

$\int_0^{12} T'(x)\,dx = [T(x)]_0^{12} = T(12) - T(0) = 93 - 105 = -12$°F

(b) Find the average rate of change of $T(x)$ over the time interval $t = 5$ to $t = 8$ minutes.

Average Value Function:

$$\Delta x = \frac{b-a}{n} \to n = \frac{b-a}{\Delta x}$$

$$f_{avg} = \frac{\lim_{n\to\infty}\sum_{i=1}^{n} f(x_i^*)}{n} = \frac{\lim_{n\to\infty}\sum_{i=1}^{n} f(x_i^*)\Delta x}{b-a} = \frac{\int_a^b f(x)\,dx}{b-a}$$

$T_{avg} = \frac{\int_5^8 T'(x)\,dx}{8-5} = \frac{T(8)-T(5)}{8-5} = \frac{97-99}{8-5} = -\frac{2}{3}$ °F/minute.

11. The graph of f' which consists of a line segment and a semicircle, is shown on the second page. Given that $f(1) = 4$, find:

$f'(x)$ is continuous on $[-2,5]$ $\therefore f(x)$ is continuous on $[-2,5]$ and differentiable on $(-2,5)$ (Fundamental Theorem of Calculus holds.)

(a) $f(-2)$

$\int_{-2}^{1} f'(x)\,dx \approx \sum_{i=1}^{2} f(x)\Delta x = \frac{-4*2}{2} + \frac{1*2}{2} = 1 - 4 = -3$

$f(1) - f(-2) = -3 \to f(-2) = 3 + f(1) = 3 + 4 = 7$

(b) $f(5)$

$\int_1^5 f'(x)\,dx \approx \sum_{i=1}^{2} f(x)\Delta x = bh - \frac{\pi * r^2}{2} = (4*2) - \frac{4\pi}{2} = 8 - 2\pi \approx 1.717$

$f(5) - f(1) = 8 - 2\pi \to f(5) = f(1) + 8 - 2\pi = 12 - 2\pi \approx 5.717$

12. (Multiple Choice) If f and g are continuous functions such that $g'(x) = f(x)$ for all x, then $\int_2^3 f(x)\,dx =$

    (a) $g'(2) - g'(3)$
    (b) $g'(3) - g'(2)$
    (c) $g(3) - g(2)$
    (d) $f(3) - f(2)$
    (e) $f'(3) - f'(2)$

    f and g are continuous functions such that $g'(x) = f(x)$ for all x $\therefore$ $F(x) = g(x)$ is continuous and differentiable on $(-\infty, \infty)$ (Fundamental Theorem of Calculus holds.)

    $\int_2^3 f(x)\,dx = F(3) - F(2) = g(3) - g(2)$

13. (Multiple Choice) If the function $f(x)$ is defined by $f(x) = \sqrt{x^3 + 2}$ and g is an antiderivate of f such that $g(3) = 5$, then $g(1) =$

    (a) -3.268
    (b) -1.585
    (c) 1.732
    (d) 6.585
    (e) 11.585

    Test of Differentiability: $\lim_{x\to 1} f(x) = f(1) = \sqrt{3} \approx 1.732, \lim_{x\to 3} f(x) = f(3) = \sqrt{29} \approx 5.385$
    $\therefore g(x)$ is continuous on $[1, 3]$ and differentiable on $(1, 3)$ (Fundamental Theorem of Calculus holds.)

    $\int_1^3 \sqrt{x^3 + 2}\,dx \approx 6.585$

    $g(3) - g(1) = 5 - g(1) = 6.585 \to g(1) = 5 - 6.585 = -1.585$

14. (Multiple Choice) The graph of f is shown in the figure on the second page. If $\int_1^3 f(x)\,dx = 2.3$ and $F'(x) = f(x)$, then $F(3) - F(0) =$

    (a) 0.3
    (b) 1.3
    (c) 3.3
    (d) 4.3
    (e) 5.3

    $f(x)$ is continuous on $[1, 3]$ $\therefore$ $F(x)$ is continuous on $[1, 3]$ and differentiable on $(1, 3)$ (Fundamental Theorem of Calculus holds.)

    $F(3) - F(0) = \int_0^3 f(x)\,dx = \int_0^1 f(x)\,dx + \int_1^3 f(x)\,dx = bh + 2.3 = (2 * 1) + 2.3 = 4.3$