Calculus BC - Worksheet on 7.2

Volume by Cross Sections

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Relevant Formulas and Notes:

$$V = \int_a^b A(x)\,dx,\ A(x) = (f(x) - g(x))(\text{Given Values of Other Properties})$$

$$V = \int_a^b A(y)\,dy,\ A(y) = (f(y) - g(y))(\text{Given Values of Other Properties})$$

Properties of Shapes:

Circles:

$$r = \frac{d}{2},\ A = \pi r^2$$

Squares (where h is hypotenuse):

$$s = \sqrt{\frac{h^2}{2}} = \frac{\sqrt{2}h}{2},\ A = s^2$$

Equilateral Triangles:

$$b = s,\ A = \frac{\sqrt{3}s^2}{4} = \frac{\sqrt{3}}{4}A_{\text{square}}$$

Semiellipses:

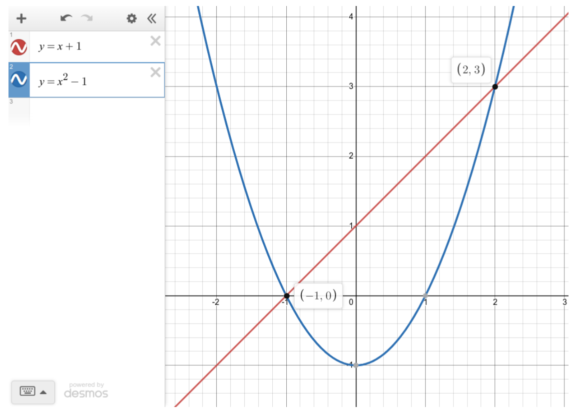
$$ab = rh,\ A = \frac{1}{2}\pi ab$$

Work the following on **notebook paper**. For each problem, draw a figure, set up an integral, and then evaluate on your calculator. Give decimal answers correct to three decimal places.

1. Find the volume of the solid whose base is bounded by the graphs of $y = x + 1$ and $y = x^2 - 1$, with the indicated cross sections taken perpendicular to the x-axis.

   Intersections of Graphs (Bounds): $a = -1, b = 2$

   $f(x) - g(x) = x + 1 - (x^2 - 1) = x + 2 - x^2$



   (a) Squares

   $V = \int_a^b s^2\,dx = \int_{-1}^2 \left(x + 2 - x^2\right)^2 dx = \int_{-1}^2 x^4 - 2x^3 - 3x^2 + 4x + 4\,dx =$

   $[\frac{1}{5}x^5 - \frac{1}{2}x^4 - x^3 + 2x^2 + 4x]_{-1}^2 = \frac{32+1}{5} - \frac{16-1}{2} - (8+1) + 2(4-1) + 4(2+1) = \frac{81}{10} = 8.1$

   Going further, let $I = \int_a^b s^2\,dx = \int_{-1}^2 \left(x + 2 - x^2\right)^2 dx = 8.1$.

   (b) Rectangles of height 1

   $V = \int_a^b bh\,dx = \int_{-1}^2 (x + 2 - x^2)(1)\,dx = [\frac{1}{2}x^2 + 2x - \frac{1}{3}x^3]_{-1}^2 = \frac{4-1}{2} + 2(2+1) - \frac{8+1}{3} = \frac{9}{2} = 4.5$

   Going further, let $J = \int_a^b bh\,dx = \int_{-1}^2 (x + 2 - x^2)\,dx = 4.5$.

   (c) Semiellipses of height 2 (The area of an ellipse is given by the formula $A = \pi ab$, where a and b are the distances from the center to the ellipse to the endpoints of the axes of the ellipse.)

   $V = \frac{\pi}{2} \int_a^b \frac{dh}{2}\,dx = \frac{J\pi}{2} \approx 7.069$

   (d) Equilateral triangles

   $V = \int_a^b \frac{\sqrt{3}s^2}{4}\,dx = \frac{\sqrt{3}I}{4} \approx 3.507$

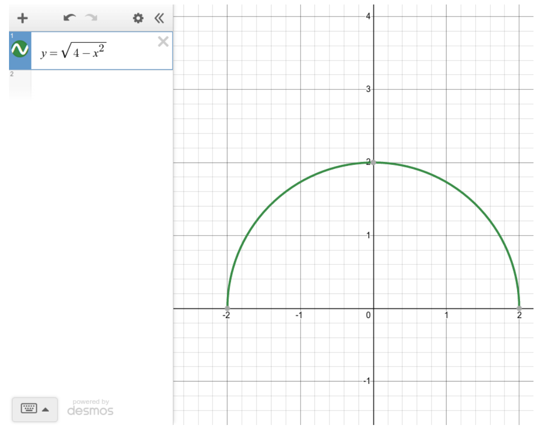
2. Find the volume of the solid whose base is bounded by the circle $x^2 + y^2 = 4$ with the indicated cross sections taken perpendicular to the x-axis.

$x^2 + y^2 = 4 \to y = \sqrt{4 - x^2}$

Zeroes (Bounds): $a = -2, b = 2$

Because our function results in a semicircle when solving for y, let each base be doubled.

(Base $= 2\sqrt{4 - x^2}$)



(a) Squares

$V = \int_a^b (2s)^2\,dx = 4\int_{-2}^{2} 4 - x^2\,dx = 4[4x - \frac{1}{3}x^3]_{-2}^{2} = 4(8+8) - 4\left(\frac{8+8}{3}\right) = \frac{128}{3} \approx 42.667$

Going further, let $I = \int_a^b (2s)^2\,dx = \int_{-2}^{2} (2\sqrt{4 - x^2})^2\,dx = 42.667$

(b) Equilateral Triangles

$V = \int_a^b \frac{\sqrt{3}(2s)^2}{4}\,dx = \frac{\sqrt{3}I}{4} \approx 18.475$

(c) Semicircles

$V = \int_a^b \frac{1}{2}\pi r^2\,dx = \frac{\pi}{2}\int_{-2}^{2} \left(\frac{2\sqrt{4-x^2}}{2}\right)^2 dx = \frac{I\pi}{8} \approx 16.755$

(d) Isosceles right triangles with the hypotenuse as the base of the solid

$V = \int_a^b \frac{1}{2}(2s)^2\,dx = \frac{I}{2} \approx 21.333$

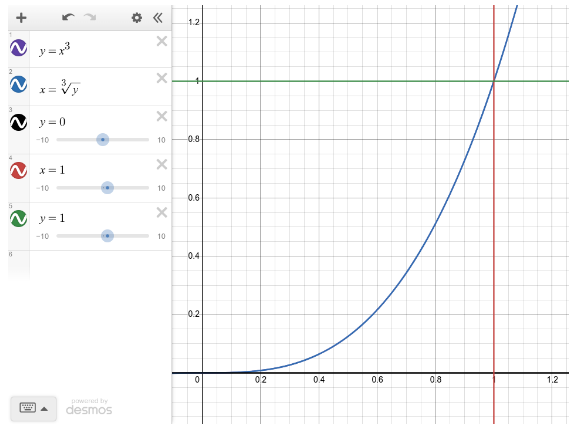
3. The base of a solid is bounded by $y = x^3$, $y = 0$, and $x = 1$. Find the volume of the solid for each of the following cross sections taken perpendicular to the y-axis.

$y = x^3 \rightarrow x = \sqrt[3]{y}$

$x = 1 \rightarrow y = x^3 = 1$

Bounds: $a = 0$, $b = 1$

$f(x) - g(x) = 1 - \sqrt[3]{y}$



(a) Squares

$V = \int_a^b s^2\,dy = \int_0^1 \left(1 - \sqrt[3]{y}\right)^2 dy = [y - \frac{3}{2}y^{\frac{4}{3}} + \frac{3}{5}y^{\frac{5}{3}}]_0^1 = 1 - \frac{3}{2} + \frac{3}{5} = \frac{1}{10} = 0.1$

Going forward, let $I = \int_a^b s^2\,dy = \int_0^1 \left(1 - \sqrt[3]{y}\right)^2 dy = 0.1$.

(b) Semicircles

$V = \int_a^b \frac{1}{2}\pi r^2\,dy = \frac{\pi}{2}\int_0^1 \left(\frac{1-\sqrt[3]{y}}{2}\right)^2 = \frac{I\pi}{8} = 0.039$

(c) Equilateral Triangles

$V = \int_a^b \frac{\sqrt{3}s^2}{4} = \frac{\sqrt{3}I}{4} = 0.043$

(d) Semiellipses whose heights are twice the length of their bases

$V = \int_a^b \frac{1}{2}\pi ab\,dy = \frac{\pi}{2}\int_0^1 \left(\frac{1-\sqrt[3]{y}}{2}\right)\left(2\left(1 - \sqrt[3]{y}\right)\right) dy = \frac{I\pi}{2} \approx 0.157$