Problem 1. Let V be a finite-dimensional $\mathbb{K}$-vector space with basis $\{\mathbf{v_1}, \dots, \mathbf{v_n}\}$. The vector space $\mathcal{L}(V, \mathbb{K})$ is called the **dual space** of V, and is denoted V'. Last time you proved that $\{\varphi_1, \dots, \varphi_n\}$ was a basis for V' (this basis is called the **dual basis**).

Let W be another finite-dimensional $\mathbb{K}$-vector space with basis $\{\mathbf{w_1}, \dots, \mathbf{w_m}\}$, and dual basis $\{\omega_1, \dots, \omega_m\}$. Given a map $T \in \mathcal{L}(V, W)$, there is another map $T' \in \mathcal{L}(W', V')$ defined by $T'(\psi) = \psi \circ T$. (The notation is a bit strange, but $T'(\psi)$ is a function in $\mathcal{L}(V, \mathbb{K})$, and for every $\mathbf{v} \in V$, we define $T'(\psi)(\mathbf{v}) = \psi(T(\mathbf{v})$.)

Show that $\mathcal{M}(T') = (\mathcal{M}(T))^t$.

Proof. Assume that V, W are both $\mathbb{K}$ vector spaces with basis $\{\mathbf{v_1}, ..., \mathbf{v_n}\}$ and $\{\mathbf{w_1}, ..., \mathbf{w_m}\}$ respectively. Assume that the basis for V' is $\{\varphi_1, ..., \varphi_n\}$ and the basis for W' is $\{\omega_1, ..., \omega_m\}$. Now consider two arbitrary linear transformations $T \in \mathcal{L}(V, W)$ and $T' \in \mathcal{L}(W', V')$. The we have $A = \mathcal{M}(T)$ and $B = \mathcal{M}(T')$. Then we have the entires of B are given by

Then by the definition of T' we get

$$T'\omega_k(\mathbf{v_j}) = \omega_k \circ T(\mathbf{v_j})$$

where $1 \leq k \leq m$ and $1 \leq j \leq n$.

We also get

$$T'\omega_k = \sum_{i=1}^{n} B_{i,k}\varphi_i$$

substituting this into the equation above we get

$$\sum_{i=1}^{n} B_{i,k}\varphi_i(\mathbf{v_j}) = \omega_k \circ \sum_{c=1}^{m} A_{c,j}\mathbf{w_c}$$

which follows from the definition of matrix of a linear map.

Then we have $B_{k,j} = \sum_{i=1}^{n} B_{i,k}\varphi_i(\mathbf{v_j})$ by the definition of matrix of a linear map. We also get $\omega_k \circ \sum_{c=1}^{m} A_{c,j}\mathbf{w_c} = \sum_{c=1}^{m} A_{c,j}\omega_k(\mathbf{w_c})$ which follows due to ω_k being linear. Then by the definition of dual basis we get $\sum_{c=1}^{m} A_{c,j}\omega_k(\mathbf{w_c}) = A_{k,j}$ this implies $\mathcal{M}(T') = (\mathcal{M}(T))^t$ $\square$

Problem 2. Let $D : \mathcal{P}_4(\mathbb{R}) \to \mathcal{P}_3(\mathbb{R})$ be the derivative map $D(p(x)) = \frac{dp}{dx}$. When using the standard polynomial bases $\{1, x, x^2, x^3, x^4\}$ and $\{1, x, x^2, x^3\}$, the matrix $\mathcal{M}(D)$ is

$$\mathcal{M}(D) = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 0 & 4 \end{pmatrix}.$$

Find bases $\mathcal{B}$ for $\mathcal{P}_4(\mathbb{R})$ and $\mathcal{C}$ for $\mathcal{P}_3(\mathbb{R})$ so that

$$\mathcal{M}(D, \mathcal{B}, \mathcal{C}) = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix}.$$

We have that the basis $\mathcal{B} = \{\mathbf{v_1} = x^4, \mathbf{v_2} = x^3, \mathbf{v_3} = x^2, \mathbf{v_2} = x, \mathbf{v_1} = 1\}$ and $\mathcal{C} = \{\mathbf{w_1} = 4x^3, \mathbf{w_2} = 3x^2, \mathbf{w_3} = 2x, \mathbf{w_4} = 1\}$ work. First to show that these are basis $\mathcal{B}$ is the standard basis for $\mathcal{P}_4(\mathbb{R})$ hence it is a basis. Now for $\mathcal{C}$ consider $\alpha_1 4x^3 + \alpha_2 3x^2 + \alpha_3 2x + \alpha_4 1 = 0$ where each α_i is an arbitrary scalar as each $\alpha_1, \alpha_2, \alpha_3, \alpha_4$ is a coefficient for a unique degree polynomial we get $\alpha_1 = \alpha_2 = \alpha_3 = \alpha_4 = 0$ hence $\mathcal{C}$ is linear independent and as $\dim \mathcal{P}_3(\mathbb{R}) = 4$ we get it is a basis.

Now computing $\mathcal{M}(D, \mathcal{B}, \mathcal{C})$ we have

$$D(x^4) = 1 \cdot 4x^3 + 0 \cdot 3x^2 + 0 \cdot 2x + 0 \cdot 1$$

$$\begin{aligned} D(x^3) &= 0 \cdot 4x^3 + 1 \cdot 3x^2 + 0 \cdot 2x + 0 \cdot 1 \\ D(x^2) &= 0 \cdot 4x^3 + 0 \cdot 3x^2 + 1 \cdot 2x + 0 \cdot 1 \\ D(x^1) &= 0 \cdot 4x^3 + 0 \cdot 3x^2 + 0 \cdot 2x + 1 \cdot 1 \\ D(x^0) &= 0 \cdot 4x^3 + 0 \cdot 3x^2 + 0 \cdot 2x + 0 \cdot 1 \end{aligned}$$

Using the definition matrix as a linear map we get the desired matrix.

Problem 3. Let $\mathcal{B} = \{\mathbf{b_1} = (1,-1,0), \mathbf{b_2} = (1,0,2), \mathbf{b_3} = (0,2,-1)\}$ be a basis for $\mathbb{K}^3$ and let $\mathcal{E}$ denote the standard basis for $\mathbb{K}^3$.

(a) Find scalars k_1, k_2, k_3 satisfying $k_1\mathbf{b_1} + k_2\mathbf{b_2} + k_3\mathbf{b_3} = (3,5,1)$.

(b) Find the change of basis matrix $\mathcal{M}(\mathrm{Id}, \mathcal{B}, \mathcal{E})$.

(c) Compute the following matrix product. How does this relate to your work in part (a)?

$$\mathcal{M}(\mathrm{Id}, \mathcal{B}, \mathcal{E}) \begin{pmatrix} 1 & 1 & 0 & 3 \\ -1 & 0 & 2 & 5 \\ 0 & 2 & -1 & 1 \end{pmatrix}$$

(a) The scalars $k_1 = 1, k_2 = 2, k_3 = 3$ work this is shown by computing $1(1,-1,0) + 2(1,0,2) + 3(0,2,-1) = (3, -1+6, 4-3) = (3,5,1)$

(b) In this case it is easier to find the change of basis matrix $(\mathrm{Id}, \mathcal{E}, \mathcal{B})$ and then compute it's inverse. We have

$$\mathcal{M}(\mathrm{Id}, \mathcal{E}, \mathcal{B}) = \begin{pmatrix} 1 & 1 & 0 \\ -1 & 0 & 2 \\ 0 & 2 & -1 \end{pmatrix}$$

Then

$$\mathcal{M}(\mathrm{Id}, \mathcal{E}, \mathcal{B})^{-1} = \begin{pmatrix} \frac{4}{5} & \frac{-1}{5} & \frac{-2}{5} \\ \frac{1}{5} & \frac{1}{5} & \frac{2}{5} \\ \frac{2}{5} & \frac{2}{5} & \frac{-1}{5} \end{pmatrix}$$

Computing the transformations of each of the basis of $\mathcal{B}$ we get $\mathcal{M}(\mathrm{Id}, \mathcal{E}, \mathcal{B})^{-1} \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} = \begin{pmatrix} 4/5 + 1/5 \\ 1/5 - 1/5 \\ 0 \end{pmatrix} = \mathbf{b_1}$.

Doing the same calculation for $\mathbf{b_2}$ we get $\mathcal{M}(\mathrm{Id}, \mathcal{E}, \mathcal{B})^{-1} \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix} = \begin{pmatrix} 4/5 - 4/5 \\ 1/5 + 4/5 \\ 2/5 - 2/5 \end{pmatrix} = \mathbf{b_2}$

Lastly with $\mathbf{b_3}$ we get $\mathcal{M}(\mathrm{Id}, \mathcal{E}, \mathcal{B})^{-1} \begin{pmatrix} 0 \\ 2 \\ -1 \end{pmatrix} = \begin{pmatrix} -2/5 + 2/5 \\ 2/5 - 2/5 \\ 4/5 + 1/5 \end{pmatrix} = \mathbf{b_3}$. Therefore we have found the basis transformation.

(c)

$$\begin{pmatrix} \frac{4}{5} & \frac{-1}{5} & \frac{-2}{5} \\ \frac{1}{5} & \frac{1}{5} & \frac{2}{5} \\ \frac{2}{5} & \frac{2}{5} & \frac{-1}{5} \end{pmatrix} \begin{pmatrix} 1 & 1 & 0 & 3 \\ -1 & 0 & 2 & 5 \\ 0 & 2 & -1 & 1 \end{pmatrix} = \begin{pmatrix} \frac{4}{5}+\frac{1}{5} & \frac{-4}{5}-\frac{4}{5} & \frac{-2}{5}+\frac{2}{5} & \frac{12}{5}-1-\frac{2}{5} \\ \frac{1}{5}-\frac{1}{5} & \frac{1}{5}+\frac{4}{5} & \frac{2}{5}-\frac{2}{5} & \frac{3}{5}+1+\frac{2}{5} \\ \frac{2}{5}-\frac{2}{5} & \frac{2}{5}-\frac{2}{5} & \frac{4}{5}+\frac{1}{5} & \frac{-6}{5}+2-\frac{1}{5} \end{pmatrix}$$
$$= \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 3 \end{pmatrix}$$

I don't see a direct connection with part (a) but I do see one for part (b) which is the first 3 columns of the 4 column matrix is the inverse of $\mathcal{M}(\mathrm{Id}, \mathcal{E}, \mathcal{B})$ so the resulting matrix being the identity matrix (for the first 3) columns was to be expected.

Problem 4. Have a lovely Spring Break!