Math 4324 Order Topology and Introduction to Continuity

Hand in Friday, March 1.

1. Recall that a map $f : X \to Y$ between topological spaces is said to be continuous at a point $x_0 \in X$ if and only if, for each open set V that contains $f(x_0)$, there is an open U satisfying $x_0 \in U \subset f^{-1}(V)$. Let $\mathcal{B}_X$ be a basis for the topology of X and $\mathcal{B}_Y$ a basis for the topology of Y. Show that f is continuous at $x_0 \in X$ if and only if, for each $W \in \mathcal{B}_Y$ that contains $f(x_0)$, there is a $\widetilde{W} \in \mathcal{B}_X$ satisfying $x_0 \in \widetilde{W} \subset f^{-1}(W)$.

Proof. Suppose that $f : X \mapsto Y$ is continuous at $x_0 \in X$ then we have for all neighborhoods $V_{f(x_0)}$ of $f(x_0)$ that there exists an open neighborhood U_{x_0} of x_0 with $x_0 \in U_{x_0} \subset f^{-1}(V_{f(x_0)})$ but as U_{x_0} is open then it is the union of bases elements hence there exists a basis element U_b with $x_0 \in U_b \subset f^{-1}(V_{f(x_0)})$ as $V_{f(x_0)}$ is an arbitrary neighborhood of $f(x_0)$ then this implies that the above is true for all neighborhoods $V_b \in \mathcal{B}_Y$ of $f(x_0)$ which completes the forward direction.

Assume that for $f : X \mapsto Y$ we have for each $x_0 \in X$ that for each $W \in \mathcal{B}_Y$ that contains $f(x_0)$, there is a $\widetilde{W} \in \mathcal{B}_X$ with $x_0 \in \widetilde{W} \subset f^{-1}(W)$. Now for any $x_0 \in X$ consider an arbitrary neighborhood V of $f(x_0)$. Then we have $V = \bigcup U_\alpha$ where $U_\alpha \in \mathcal{B}_Y$ which implies for some $U'_\alpha \in \mathcal{B}_y$ that $f(x_0) \in U'_\alpha \subset V$ this implies (by the assumption) that there exists $\widetilde{W} \in \mathcal{B}_X$ with $x_0 \in \widetilde{W} \subset f^{-1}(U'_\alpha) \subset f^{-1}(\bigcup U_\alpha) = f^{-1}(V)$ which shows that f is continuous at $x_0 \in X$.

□

2. When X is a metric space, we know that the set of open balls, $\{B(x, r) : x \in X \text{ and } r > 0\}$ is a basis for X's metric topology $\mathcal{T}$.

a. Show that $\{B(x, r) : x \in X \text{ and } 0 < r < 1\}$ is a basis for a topology on X. Call this topology $\mathcal{T}_1$.

Proof. Let $x \in X$ then we have $x \in B(x, 0.5)$ hence we have the first condition for being a basis satisfied. Now suppose that we have $x \in B(y_1, r_1) \cap B(y_2, r_2)$ where $y_1, y_2 \in X$ and $0 < r_1 < 1$ and $0 < r_2 < 1$. Then let $r_3 = \min(r_1 - d(x, y_1), r_2 - d(x, y_2))$ then consider the set $B(x, r_3)$. Then for any $x_0 \in B(x, r_3)$ we have $d(x_0, y_1) \leq d(x_0, x) + d(x, y_1) < r_3 + d(x, y_1) \leq r_1 - d(x, y_1) + d(x, y_1) \leq r_1$ not that the strict inequality follows from $d(x_0, x) < r_3$ and I used the triangle inequality for the first inequality and the third inequality follows due to the choice of r_3. Using the similar reasoning we get $d(x_0, y_2) < r_2$ hence we get $B(x, r_3) \subset B(y_1, r_1)$ and $B(x, r_3) \subset B(y_2, r_2)$ hence we get $x \in B(x, r_3) \subset B(y_1, r_1) \cap B(y_2, r_2)$. Which shows that it is a basis. □

b. Show that $\mathcal{T} = \mathcal{T}_1$.

Proof. Suppose $B(x_0, r_0) \in \{B(x, r) : x \in X \text{ and } 0 < r < 1\}$ then as $x \in X$ and $0 < r_0$ we get $B(x_0, r_0) \in \{B(x, r) : x \in X \text{ and } r > 0\}$ hence $\{B(x, r) : x \in X \text{ and } 0 < r < 1\} \subset \{B(x, r) : x \in X \text{ and } r > 0\}$ which implies $\mathcal{T}_1 \subset \mathcal{T}$.

Now using Munkres Lemma 13.3 for an arbitrary $x \in X$ and an arbitrary $B(x_0, r_0) \in \{B(x, r) : x \in X \text{ and } 0 < r < 1\}$ with $x \in B(x_0, r_0)$ then we have $B(x_0, r_0) \in \{B(x, r) : x \in X \text{ and } 0 < r\}$ then as $x \in B(x_0, r_0) \subset B(x_0, r_0)$ we get $\mathcal{T} \supset \mathcal{T}_1$ as we have double inclusion we get $\mathcal{T}_1 = \mathcal{T}$ □

3. Let X be $\{0\} \bigcup \{1/k : k \in \mathbb{N}\}$ be ordered by the usual "less than" $<$, i.e. the order it gets from "less than" when X is regarded as a subset of $[0, 1]$. Give X the associated order topology. In the associated product topology on $X \times X$, show that every open set that contains $(0, 1)$ contains infinitely many points with second coordinate 1 and that there is an open set that contains $(0, 1)$ and in which every point has second coordinate 1.

Proof. Let $X = \{0\} \bigcup \{1/k : k \in \mathbb{N}\}$ with the order topology. Now let $A \times B$ be an arbitrary open set in the product topology with $(0, 1) \in A \times B$. Then as this is the product topology we have A, B are open sets of X with $0 \in A$ and $1 \in B$. Then we have two basis elements of the form $[0, b) \subset A$ where $b = 1/n$ for some

$n \in \mathbb{N}$ and $(a, 1]$ where $a = 1/k$ for some $k \in \mathbb{K}$. Then we have $(1/j, 1) \in [0, b) \times (a, 1] \subset A \times B$ where $j \in \mathbb{N}$ and $j \geq n$ this shows that there is an infinite number of points where the second coordinate is 1.

We have $(1/2, 1] = \{1\}$ and $[0, 1/2)$ are open in X and as this is the product topology we have that $[0, 1/2) \times \{1\}$ is open in $X \times X$ but $[0, 1/2) \times \{1\} = \{(a, 1) : a = 0 \text{ or } a = 1/n \text{ where } n > 2\}$ hence every element has 2nd coordinate 1. □

If we start with the same order on X and give $X \times X$ the associated dictionary order, then in the topology $X \times X$ gets from the dictionary order, show that, for every $y \in X$, every open set that contains $(0, 1)$ contains infinitely many points with second coordinate y.

Proof. Assume that $X \times X$ has the dictionary order and the order topology. Let $A \times B$ be an open set with $(0, 1) \in A \times B$. Then we have a basis element of the form $\big((a, b), (c, d)\big) = \{(x, z) \in X \times X : (a, b) < (x, z) \text{ and } (x, z) < (c, d)\}$ where $a, b, c, d \in X$ with $(0, 1) \in \big((a, b), (c, d)\big) \subset A \times B$. Then as $(0, 1) \in \big((a, b), (c, d)\big)$ we have $(a, b) < (0, 1) < (c, d)$. This implies that $a = 0$ and $b < 1$ as this is the order topology we get that $0 < c$ as if it where not then we would have $c = 0$ and $d = 1$ which would imply that the basis is $\big((0, 1), (0, 1)\big)$ which is not an element of the order topology. Hence we get the strict inequality $0 < c$ then we have for all $n \in \mathbb{N}$ with $1/n < c$ that for all $y \in X$ the inequality $(a, b) < (1/n, y) < (c, d)$ as there is an infinite number of natural numbers $k \in \mathbb{N}$ such that $0 = a < 1/k < c$ this completes the proof. □

The discussion immediately below and the three problems that follow it are extra credit problems. This assignment is worth thirty points, which you can earn on problems 1 - 3. Any points you score on problems 4 - 6 will be extra, added to your homework score, possibly to exceed "possible" points. I believe that problems 4 - 6, and particularly the sections of Munkres they connect to, introduce those who work through them to an intriguing blend of topology and fairly advanced set theory, but this material is not a required part of the course, and it will not appear on the next test or on the final exam.

Background for problems 4 - 6 is in Munkres Section 10 for well-ordered sets and in Munkres Section 7 for countable and uncountable sets; but I'll try to provide in this assignment the information you need to do the assignment. Of course you can also speak to me about topics on which you want more information.

A set is called countable if it is finite or in bijective correspondence with $\mathbb{N}$. The sets in bijective correspondence with $\mathbb{N}$ are infinite and, when we want to distinguish them from finite sets, we call them countably infinite. There are infinite sets that are not countable (also called uncountable). The set of points on the real line is an example, as is the set of subsets of $\mathbb{N}$. A subset of a countable set is countable, the union of countably many countable sets is countable, and a product of finitely many sets, with each of the factors countable, is countable.

A well-ordered set is a linearly ordered set in which every nonempty subset has a least element. That apparently innocuous property has many implications. Note that, in the usual "less than" order $<$, $\mathbb{Z}$ and $\mathbb{R}$ are not well-ordered, but every subset of $\mathbb{N}$, including $\mathbb{N}$ itself, is well-ordered. Other examples include $\mathbb{N} \bigcup \{\omega\}$, where we add to $\mathbb{N}$ an element that we consider larger than every element of $\mathbb{N}$. Unless otherwise specified, the topology on a well-ordered set will be the order topology. In a well-ordered set, I'll use interval notations, including or excluding endpoints, to have their usual meaning in the presence of a "less than" order $<$.

If x is an element of a well-ordered set X, the section S_x refers to $\{y \in X : y < x\}$. If x is an element of a well-ordered set, we call y the immediate predecessor of x if y is the greatest element that is less than x. In $\mathbb{N} \bigcup \{\omega\}$, each $n > 1$ has the immediate predecessor $n - 1$, but ω has no immediate predecessor.

An example with interesting properties is the uncountable well-ordered set S_Ω with the property that, for all $x \in S_\Omega$, the section S_x is countable. $\overline{S}_\Omega$ is a notation used for $S_\Omega \bigcup \{\Omega\}$, which is well-ordered by considering the added element Ω to be larger than every element in S_Ω. In this context, the notation for the set S_Ω is then consistent with the notation for the section of all elements of $\overline{S}_\Omega$ that are less than Ω. In a problem below, you will check that the overline notation in $\overline{S}_\Omega$ is consistent with our usual notation for the closure of S_Ω in $\overline{S}_\Omega$'s order topology.

If Y is a subset of a topological space X, then the expression **open cover** of Y refers to a collection $\{U_\alpha : \alpha \in A\}$ of open subsets in X that satisfies $Y \subset \bigcup_{\alpha \in A} U_\alpha$.

4. a. Show that, if X is a well-ordered set, then every subset of X that has an upper bound has a least upper bound (also known as a supremum or sup).

b. Show that, if Y is a countable subset of S_Ω, then there is a $b \in S_\Omega$ for which $Y \subset [a_0, b]$. (**Hint.** Is $(\bigcup_{y \in Y} S_y) \bigcup Y$ countable?)

c. Show that, in $S_\Omega \bigcup \Omega$, no sequence (y_n) of elements of S_Ω converges to Ω.

d. Show that, in $S_\Omega \bigcup \Omega$, Ω is in the closure of S_Ω.

Remark. The last two parts of the above problem show that a point can be in the closure of a set even if no sequence in the set converges to the point. This does not happen in metric spaces.

5. Let S_Ω and $\overline{S}_\Omega$ be as above. For either set call the least element a_0.

a. Show that, for all $x \in S_\Omega$, there is a $y \in S_\Omega$ that satisfies $x < y$.

b. Show that there is a collection $\{U_\gamma : \gamma \in \Gamma\}$ of open subsets of S_Ω satisfying the following condition: $S_\Omega \subset \bigcup_{\gamma \in \Gamma} U_\gamma$ but for every countable subcollection $\{U_{\gamma_j} : j \in J\}$ of the U_γ's, there is an $x \in S_\Omega$ for which $x \notin \bigcup_{j \in J} U_{\gamma_j}$. Here J can be either all of $\mathbb{N}$ or a finite subset of $\mathbb{N}$.

c. Suppose that $\{V_\beta : \beta \in B\}$ is a collection of open subsets of $\overline{S}_\Omega$ that satisfies $\overline{S}_\Omega \subset \bigcup_{\beta \in B} V_\beta$. Show that there is a finite subcollection $\{V_{\beta_1}, \ldots, V_{\beta_n}\}$ of the V_β's for which $\overline{S}_\Omega \subset \bigcup_{j=1}^n V_{\beta_j}$. (**Hint.** Consider the set of elements x of $\overline{S}_\Omega$ for which no finite subcollection $\{V_{\beta_1}, \ldots, V_{\beta_n}\}$ satisfies $[a_0, x] \subset \bigcup_{j=1}^n V_{\beta_j}$.)

6. Let S_Ω be as above. Call its least element a_0.

a. Show that, for each $z \in S_\Omega \bigcup \Omega$, the single-element set $\{z\}$ is a closed subset of $S_\Omega \bigcup \Omega$.

b. Suppose that W is a countably infinite subset of S_Ω. Show that W has a limit point in S_Ω. (**Hint.** By the result of problem 4.b., we may choose a $b \in S_\Omega$ for which $W \subset [a_0, b]$. If V is a subset of $[a_0, b]$ for which each point c in $[a_0, b]$ has a neighborhood U_c satisfying $(U_c \setminus \{c\}) \bigcap V = \emptyset$, make an open cover $\{U_c : c \in [a_0, b]\} \bigcup \{(b, \Omega]\}$ of $S_\Omega \bigcup \Omega$. What does the result of problem 5.c. imply about V?)

c. Suppose that W is a countably infinite subset of S_Ω. Show that, for some $c \in S_\Omega$, every open neighborhood of c contains infinitely many elements of W. (**Hint.** If d has an open neighborhood that has finite intersection with W, does the result of problem 6.a. permit d to be a limit point of W? Combine the answer to that question with the result of problem 6.b.)

Remark. The reasoning used at the end of the preceding answer applies to limit points in any topological space in which every single-element set is closed.