

1.

*Proof.* Suppose  $X$  is a compact topological space and  $f : X \rightarrow \mathbb{R}$  is as described. As  $X$  is compact we have the open cover  $\{U_x\}_{x \in X}$  where for each  $U_x \in \{U_x\}_{x \in X}$  we have that there exists a constant  $M_x$  such that for all  $z \in U_x$  that  $|f(z)| \leq M_x$ .

Then as  $X$  is compact we have that the collection has a finite subcover  $\{U_{x_1}, \dots, U_{x_n}\}$ . Each of these in the finite subcover has a corresponding constant  $\{M_{x_1}, \dots, M_{x_n}\}$  with the property that for all  $z \in U_{x_i}$  that  $|f(z)| \leq M_{x_i}$  where  $i = 1, \dots, n$ . Let  $M = \max\{M_{x_1}, \dots, M_{x_n}\}$  then for any  $z \in X$  we have that  $z \in U_{x_i}$  for some  $i = 1, \dots, n$ . Hence  $|f(z)| \leq M_{x_i} \leq M$  so this choice of  $M$  works for all  $z \in X$ .  $\square$

2.

*Proof.* Suppose that  $X$  is a topological space  $C$  is a connected subset of  $X$  and  $C_\alpha$  is a connected subset of  $X$ . With for all  $\alpha$  that  $C_\alpha \cap C \neq \emptyset$ . Suppose that  $C \cup (\cup_{\alpha \in A} C_\alpha)$  is not connected. Then there exists a separation  $U \cup V$  of  $C \cup (\cup_{\alpha \in A} C_\alpha)$ .

Then as  $C$  is connected we have either  $C \subset U$  with  $C \cap V = \emptyset$  or  $C \subset V$  with  $C \cap U = \emptyset$ . If  $C$  wasn't fully contained in only one then  $(C \cap U), (C \cap V)$  would be a separation of  $C$  but  $C$  is connected. For each  $\alpha \in A$  we get that  $C_\alpha$  is contained in exactly one  $U$  or  $V$ .

WLOG suppose  $C \subset U$  then as  $C \cap C_\alpha \neq \emptyset$  we get that  $C_\alpha \subset U$ . Hence we get  $C \cup (\cup_{\alpha \in A} C_\alpha) \subset U$  and  $(C \cup (\cup_{\alpha \in A} C_\alpha)) \cap V = \emptyset$  hence  $V = \emptyset$  so this separation can not exist. So we have  $C \cup (\cup_{\alpha \in A} C_\alpha)$  is connected.  $\square$

3.

- (a) We have that a set is closed if and only if it contains its limit points. As any singleton in  $\{1/n : n \in \mathbb{N}\}$  is open. Which is shown as we have for any  $1/n \in \{1/n : n \in \mathbb{N}\}$  we have that  $(1/n - \epsilon, 1/n + \epsilon)$  for  $\epsilon > 0$  is open in  $\mathbb{R}$  choosing a sufficiently small  $\epsilon > 0$  we get  $(1/n - \epsilon, 1/n + \epsilon) \cap \{1/n : n \in \mathbb{N}\} = \{1/n\}$ . As the choice of  $1/n$  was arbitrary we get that all singletons are open.

Hence any  $B \subset \{1/n : n \in \mathbb{N}\}$  we get that  $B' = \emptyset$  as for any  $b \in \{1/n : n \in \mathbb{N}\}$  we have the neighborhood  $\{b\}$  that doesn't intersect  $B$  at any place other than possibly itself. Hence  $B$  is closed as  $B$  is arbitrary we have that all subsets are closed.

- (b) We have that all singletons other than  $\{0\}$  are open using the same reasoning as above. Hence  $\{1/n : n \in \mathbb{N}\}$  is open so we get  $\{1/n : n \in \mathbb{N}\}^c = \{0\}$  is closed. We have that all singletons are closed as any singleton other than  $\{0\}$  don't have any limit points. As finite unions of closed sets are closed then all finite subsets are closed.

Lastly any infinite set containing 0 is closed. This is shown as let  $B \subset \{0\} \cup \{1/n : n \in \mathbb{N}\}$  be an infinite set containing 0. Then we have  $B^c$  is a union of singletons each of which are open hence  $B^c$  is open so  $B$  is closed.

Any infinite set not containing 0 is not closed as 0 is a limit point of this set.