Problem 1. Determine (with brief explanation only; no proof required) whether or not each of the following subsets is a subspace of $\mathbb{R}^3$.

a. $\{(x_1, x_2, x_3) \in \mathbb{R}^3 \ : \ x_1 + 2x_2 + 3x_3 = 0\}$

b. $\{(x_1, x_2, x_3) \in \mathbb{R}^3 \ : \ x_1 x_2 x_3 = 0\}$

c. $\{(x_1, x_2, x_3) \in \mathbb{R}^3 \ : \ x_1 = x_3\}$

a. Yes: This is a subspace as it is non empty as $\vec{0}$ is an element. The equation being equal to zero would imply any two added together would be equal to zero and any being multiplied by a scalar would be equal to zero as well.

b. No: Consider the two vectors $(1, 0, 0), (0, 1, 1) \in \mathbb{R}^3$ both would be in the set but adding them together would give $(1, 1, 1)$ which is not in the set.

c. Yes: Nonempty as $\vec{0}$ is an element. As the set would be equal to $\{(x, y, x) \in \mathbb{R}^3\}$ any two added together would be of the form $(a, b, a) + (c, d, c) = (a + c, b + d, a + c)$. As scalar multiplication is distributed the 1st and last elements of any vector would still be equal.

Problem 2. Let V be an arbitrary $\mathbb{K}$-vector space and U_1, U_2 be arbitrary subspaces of V. Prove or disprove each of the following.

(a) $U_1 \cap U_2$ is a subspace of V.

(b) $U_1 \cup U_2$ is a subspace of V.

(a) *Proof.* Suppose V is an arbitrary $\mathbb{K}-$vector space and U_1, U_2 are arbitrary subspaces of V. Consider the set $U_1 \cap U_2$ as both subspaces contain the zero vector we have that $\vec{0} \in U_1 \cup U_2$. Now $\forall \vec{x}, \forall \vec{y} \in U_1 \cap U_2$ we have $\vec{x}, \vec{y} \in U_1$ and $\vec{x}, \vec{y} \in U_2$ as U_1 is a subspace we have $\vec{x} + \vec{y} \in U_1$ by the same reasoning $\vec{x} + \vec{y} \in U_2$ hence $\vec{x} + \vec{y} \in U_1 \cap U_2$. Let $\lambda \in \mathbb{K}$ and $\vec{v} \in U_1 \cap U_2$ as U_1, U_2 are both subspaces we have $\lambda\vec{x} \in U_1$ and $\lambda\vec{x} \in U_2$ hence $\lambda\vec{x} \in U_1 \cap U_2$. $\square$

(b) *Disproof.* Let $V = \mathbb{R}^2$ be a $\mathbb{R}$-vector space. Consider the two sets $U_1 = \{(x, 0) \in \mathbb{R}^2\}$ and $U_2 = \{(0, x) \in \mathbb{R}^2\}$ these are both subspaces of $\mathbb{R}^2$ however $U_1 \cup U_2 = \{(x, 0) \in \mathbb{R}^2 \text{ or } (0, y) \in \mathbb{R}^2\}$. Adding the two vectors $(1, 0), (0, 1) \in U_1 \cup U_2$ we have $(1, 0) + (0, 1) = (1, 1)$ but $(1, 1) \notin U_1 \cup U_2$. $\square$

Problem 3. Given two vectors $\mathbf{u} = (x_1, \ldots, x_n)$ and $\mathbf{v} = (v_1, \ldots, v_n)$ in $\mathbb{R}^n$, recall that the *dot product* is defined as

$$\mathbf{u} \cdot \mathbf{v} = x_1 y_1 + \cdots + x_n y_n.$$

Let U and N be the following subspaces of $\mathbb{R}^3$:

$$U = \{(x, y, x + y) \,:\, x, y \in \mathbb{R}\}$$
$$N = \left\{\mathbf{v} \in \mathbb{R}^3 \,:\, \mathbf{u} \boldsymbol{\cdot} \mathbf{v} = 0 \text{ for every } \mathbf{u} \in U\right\}$$

Find the missing vector entry that makes the following statement true (and provide a proof):

$$N = \left\{(x, x, -x) \in \mathbb{R}^3 \,:\, x \in \mathbb{R}\right\}.$$

Proof. The set should be $N = \{(x, x, -x) \in \mathbb{R}^3 \,:\, x \in \mathbb{R}\}$. Let $(x, y, x + y) \in U$ and let $(a, a, -a) \in N$ the $(x, y, x+y) \cdot (a, a, -a) = ax + ay + -a(x+y) = ax + ay + -ax + -ay = 0$. Now to show that N is a subspace. We have that $(0, 0, 0) \in N$. Now let $(x, x, -x), (y, y, -y) \in N$ adding the vectors yields $(x, x, -x) + (y, y, -y) = \big(x + y, x + y, -(x + y)\big)$ as $x + y \in \mathbb{R}$ we get that $\big(x + y, x + y, -(x + y)\big) \in \mathbb{R}^3$. Let $\lambda \in \mathbb{R}$ and $(x, x, -x) \in N$ as $\lambda x \in \mathbb{R}$ we get $(\lambda x, \lambda x, -\lambda x) \in N$ therefore N is a subspace.

□

Problem 4. Let U_1 and U_2 be the following subspaces of $\mathbb{Q}^4$:

$$U_1 = \{(x, y, z, y) \ : \ x, y, z \in \mathbb{Q}\}$$
$$U_2 = \{(0, x, 0, -x) \ : \ x \in \mathbb{Q}\}$$

Prove that $\mathbb{Q}^4 = U_1 \oplus U_2$.

Proof. Let U_1 and U_2 be as defined above. Let $(x, y+a, z, y-a) \in U_1 \oplus U_2$ then we have $x, y+a, z, y-a \in \mathbb{Q}$ which implies $(x, y+a, z, y-a) \in \mathbb{Q}^4$ so we have $U_1 \oplus U_2 \subseteq \mathbb{Q}^4$. Now let $(a, b, c, d) \in \mathbb{Q}^4$ consider the two vectors $(a, \frac{b+d}{2}, c, \frac{b+d}{2}) \in U_1$ and $(0, \frac{b-d}{2}, 0, -\frac{b-d}{2}) \in U_2$ adding these two vectors we get

$$(a, \frac{b+d}{2}, c, \frac{b+d}{2}) + (0, \frac{b-d}{2}, 0, -\frac{b-d}{2}) = (a, b, c, d)$$

therefore we have $\mathbb{Q}^4 \subseteq U_1 \oplus U_2$ which implies $\mathbb{Q}^4 = U_1 \oplus U_2$.

Suppose $(x, y, z, w) \in U_1 \cap U_2$ then we have $x = z = 0$, and $w = -y$ by the definition of the vectors in U_1 we also get $y = -y$ which implies $y = 0$ therefore $U_1 \cap U_2 = \{\vec{0}\}$.

□

Problem 5. Recall that $\mathcal{P}_m(\mathbb{K})$ is the $\mathbb{K}$-vector space of polynomials of degree (at most) m and with coefficients in $\mathbb{K}$.

a. Find a list of four distinct, nonzero polynomials that span $\mathcal{P}_2(\mathbb{R})$.

b. Prove that the polynomials found in part (a) is linearly dependent.

a. The four polynomials that span $\mathcal{P}_2(\mathbb{R})$ are $S = \{x^1 + 1, x^2 - 1, x, 1\}$. This is shown to span $\mathcal{P}_2(\mathbb{R})$ by letting $ax^2+bx+c \in \mathcal{P}_2(\mathbb{R})$ we have $a(x^2+1)+bx+(c-a)1 = ax^2+bc+c$ as an arbitrary element of $\mathcal{P}_2(\mathbb{R})$ was equal to a linear combination of the elements of S we have $\operatorname{span}(S) = \mathcal{P}_2(\mathbb{R})$.

b. *Proof.* We have $(x^2+1)+(-1)\cdot(x^2-1)+0x+-2\cdot(1) = 0$ as the coefficients are not all zero then we have it is linearly dependent. $\square$