

Problem 1. Let U be the subset of $\mathcal{P}_3(\mathbb{R})$ given by

$$U = \{p \in \mathcal{P}_3(\mathbb{R}) : p(0) = p(1)\}.$$

Define the function

$$\begin{aligned}\alpha : U \times U &\rightarrow \mathbb{R} \\ \alpha(p, q) &= \int_0^1 p(x)q'(x) dx\end{aligned}$$

1. Prove that α is an alternating bilinear form.
2. Find the matrix $\mathcal{M}(\alpha, \mathcal{B})$ for α with respect to the following basis

$$\mathcal{B} = \{x^3 - x, x^2 - x, 1\}$$

Proof. First I will show that α (as defined above) is a bilinear form.

First fix some $p \in U$ then for any $q_1, q_2 \in U$ and $\lambda \in \mathbb{R}$ we have

$$\begin{aligned}\alpha(p, \lambda q_1 + q_2) &= \int_0^1 p(x)(\lambda q_1 + q_2)'(x) dx \\ &= \int_0^1 p(x)\lambda q_1'(x) + p(x)q_2'(x) dx \\ &= \lambda \int_0^1 p(x)q_1'(x) dx + \int_0^1 p(x)q_2'(x) dx \\ &= \lambda \alpha(p, q_1) + \alpha(p, q_2)\end{aligned}$$

Now fix some $q \in U$ then for any $p_1, p_2 \in U$ and $\lambda \in \mathbb{R}$ we have

$$\alpha(\lambda p_1 + p_2, q) = \int_0^1 (\lambda p_1 + p_2)(x)q'(x) dx$$

Using the same logic as above we get $\alpha(\lambda p_1 + p_2, q) = \lambda \alpha(p_1, q) + \alpha(p_2, q)$. Hence we have that α is a bilinear form.

Now let $p \in U$ then we have

$$\begin{aligned}\alpha(p, p) &= \int_0^1 p(x)p'(x) dx \\ &= \frac{1}{2}p(x)^2 \Big|_0^1 \\ &= \frac{1}{2}(p(1)^2 - p(0)^2) \\ &= 0\end{aligned}$$

Hence we get that it is an alternating bilinear form.

□

Now finding the matrix $\mathcal{M}(\alpha, \mathcal{B})$ I will say that $b_1 = x^3 - x$, $b_2 = x^2 - x$, $b_3 = 1$. We have immediately that $\alpha(b_i, b_3) = 0$ for $i \in \{1, 2, 3\}$ additionally as this is an alternating form we get $\alpha(b_i, b_i) = 0$ and we also get $\alpha(b_i, b_j) = -\alpha(b_j, b_i)$ by Theorem 9.16. Hence we only need to examine the values of $\alpha(b_1, b_2)$

$$\begin{aligned}\alpha(b_1, b_2) &= \int_0^1 (x^3 - x)(x^2 - x)' dx \\ &= \int_0^1 (x^3 - x)(2x - 1) dx \\ &= \int_0^1 2x^4 - x^3 - 2x^2 + x dx \\ &= \left. \frac{2}{5}x^5 - \frac{1}{4}x^4 - \frac{2}{3}x^3 + \frac{1}{2}x^2 \right|_0^1 \\ &= \frac{2}{5} - \frac{1}{4} - \frac{2}{3} + \frac{1}{2} \\ &= -\frac{1}{60}\end{aligned}$$

Hence we get the matrix

$$\mathcal{M}(\alpha, \mathcal{B}) = \begin{pmatrix} 0 & -\frac{1}{60} & 0 \\ \frac{1}{60} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Problem 2. If V and W are \mathbb{K} -vector spaces, observe that the Cartesian $V \times W$ is a \mathbb{K} -vector space with the following addition and scalar multiplication operations:

$$(\mathbf{v}_1, \mathbf{w}_1) + (\mathbf{v}_2, \mathbf{w}_2) = (\mathbf{v}_1 + \mathbf{v}_2, \mathbf{w}_1 + \mathbf{w}_2) \quad \text{and} \quad k(\mathbf{v}, \mathbf{w}) = (k\mathbf{v}, k\mathbf{w}).$$

Show that, in general, a bilinear form $\beta \in V^{(2)}$ is not a linear functional, $\mathcal{L}(V \times V, \mathbb{K})$.

Let $\beta \in V^{(2)}$ and $(a_1, b_1), (a_2, b_2) \in V \times V$ then if β were linear we would have

$$\begin{aligned} \beta((a_1, b_1) + (a_2, b_2)) &= \beta(a_1 + a_2, b_1 + b_2) \\ &= \beta(a_1 + a_2, b_1) + \beta(a_1 + a_2, b_2) \\ &= \beta(a_1, b_1) + \beta(a_2, b_1) + \beta(a_1, b_2) + \beta(a_2, b_2) \end{aligned}$$

We only get the equality $\beta((a_1, b_1) + (a_2, b_2)) = \beta((a_1, b_1)) + \beta((a_2, b_2))$ if and only if $\beta(a_2, b_1) + \beta(a_1, b_2) = 0$.

Now let $(a, b) \in V \times V$ and $\lambda \in \mathbb{K}$ then we have

$$\beta(\lambda(a, b)) = \beta(\lambda a, \lambda b) = \lambda \beta(a, \lambda b) = \lambda^2 \beta(a, b)$$

Hence we only get the equality $\beta(\lambda(a, b)) = \lambda \beta((a, b))$ if and only if $\lambda = \lambda^2$ this doesn't hold for all the scalars hence it is not linear.

Problem 3. The notion of a bilinear form can be extended to a **bilinear map** in the following way: Let U, V, W be \mathbb{K} -vector spaces. The function $\Gamma : V \times W \rightarrow U$ is a bilinear map if it satisfies the following: for all scalars k and vectors \mathbf{v}, \mathbf{w} :

$$\Gamma(\mathbf{v}_1 + \mathbf{v}_2, \mathbf{w}) = \Gamma(\mathbf{v}_1, \mathbf{w}) + \Gamma(\mathbf{v}_2, \mathbf{w}) \quad \text{and} \quad \Gamma(k\mathbf{v}, \mathbf{w}) = k\Gamma(\mathbf{v}, \mathbf{w}),$$

$$\Gamma(\mathbf{v}, \mathbf{w}_1 + \mathbf{w}_2) = \Gamma(\mathbf{v}, \mathbf{w}_1) + \Gamma(\mathbf{v}, \mathbf{w}_2) \quad \text{and} \quad \Gamma(\mathbf{v}, k\mathbf{w}) = k\Gamma(\mathbf{v}, \mathbf{w}).$$

1. Go find your old multivariable calculus textbook and look up the definition of the cross product on \mathbb{R}^3 .
2. Prove that $\Gamma : \mathbb{R}^3 \times \mathbb{R}^3 \rightarrow \mathbb{R}^3$ given by

$$\Gamma(\mathbf{v}, \mathbf{w}) = \underbrace{\mathbf{v} \times \mathbf{w}}_{\text{cross product}}$$

is a bilinear map.

3. A bilinear map $\Gamma : V \times V \rightarrow U$ is said to be **alternating** if $\Gamma(\mathbf{v}, \mathbf{v}) = \mathbf{0}$ for all \mathbf{v} . Prove that the cross product map above is alternating.

(1) Stewart's Calculus book defines it as

If $\mathbf{a} = \langle a_1, a_2, a_3 \rangle$ and $\mathbf{b} = \langle b_1, b_2, b_3 \rangle$, then the **cross product** of \mathbf{a} and \mathbf{b} is the vector

$$\mathbf{a} \times \mathbf{b} = \langle a_2b_3 - a_3b_2, a_3b_1 - a_1b_3, a_1b_2 - a_2b_1 \rangle$$

(2) Let $a = (a_1, a_2, a_3), b = (b_1, b_2, b_3), c = (c_1, c_2, c_3) \in \mathbb{R}^3$ and Γ be the cross product map. Then we have

$$\Gamma((a_1, a_2, a_3) + (b_1, b_2, b_3), (c_1, c_2, c_3)) =$$

$$((a_2 + b_2)c_3 - (a_3 + b_3)c_2, (a_3 + b_3)c_1 - (a_1 + b_1)c_3, (a_1 + b_1)c_2 - (a_2 + b_2)c_1)$$

distributing the c_i 's we get the equality

$$(a_2c_3 - a_3c_2, a_3c_1 - a_1c_3, a_1c_2 - a_2c_1) + (b_2c_3 - b_3c_2, b_3c_1 - b_1c_3, b_1c_2 - b_2c_1)$$

which is equal to $\Gamma(a, c) + \Gamma(b, c)$ hence we get the first condition.

Now

$$\Gamma((a_1, a_2, a_3), (b_1, b_2, b_3) + (c_1, c_2, c_3)) =$$

$$(a_2(b_3 + c_3) - a_3(b_2 + c_2), a_3(b_1 + c_1) - a_1(b_3 + c_3), a_1(b_2 + c_2) - a_2(b_1 + c_1))$$

distributing each a_i we get the equality

$$(a_2b_3 - a_3b_2, a_3b_1 - a_1b_3, a_1b_2 - a_2b_1) + (a_2c_3 - a_3c_2, a_3c_1 - a_1c_3, a_1c_2 - a_2c_1)$$

which is equal to $\Gamma(a, b) + \Gamma(a, c)$ hence we have the second condition satisfied.

Now let $k \in K$ then we have

$$\Gamma(k(a_1, a_2, a_3), (b_1, b_2, b_3)) = (ka_2b_3 - ka_3b_2, ka_3b_1 - ka_1b_3, ka_1b_2 - ka_2b_1) \quad (1)$$

$$= k(a_2b_3 - a_3b_2, a_3b_1 - a_1b_3, a_1b_2 - a_2b_1) \quad (2)$$

$$= k\Gamma((a_1, a_2, a_3), (b_1, b_2, b_3)) \quad (3)$$

Now as $\Gamma(a, kb) = (ka_2b_3 - ka_3b_2, ka_3b_1 - ka_1b_3, ka_1b_2 - ka_2b_1)$ we get both $\Gamma(ka, b) = k\Gamma(a, b) = \Gamma(a, kb)$ hence we have completed all the conditions and so Γ is a bilinear map.