

**Problem 1.** Let  $V$  be a finite-dimensional  $\mathbb{K}$ -vector space with basis  $\{\mathbf{v}_1, \dots, \mathbf{v}_n\}$ . Let  $\{\varphi_1, \dots, \varphi_n\}$  be a set of linear maps in  $\mathcal{L}(V, \mathbb{K})$  satisfying

$$\varphi_j(\mathbf{v}_k) = \begin{cases} 1 & \text{when } j = k \\ 0 & \text{otherwise.} \end{cases}$$

Prove that  $\{\varphi_1, \dots, \varphi_n\}$  is a basis for  $\mathcal{L}(V, \mathbb{K})$ .

The proof Lemma 3D.8 might be useful to you.

**Problem 2.** Give an example of two  $2 \times 2$  matrices  $A$  and  $B$  for which  $AB \neq BA$ .

Let  $A = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$  and let  $B = \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix}$ . Then  $AB = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$  and

$$BA = \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 2 & 1 \end{bmatrix}$$

**Problem 3.** Recall that the Fibonacci sequence  $\{F_n\} = \{0, 1, 1, 2, 3, 5, 8, 13, \dots\}$  is recursively-defined:

$$F_0 = 0$$

$$F_1 = 1$$

$$F_n = F_{n-1} + F_{n-2} \text{ for } n \geq 2.$$

Prove that, for all  $n \geq 0$ , the  $3 \times 3$  matrix  $A_n = \begin{pmatrix} F_n & F_{n+1} & F_{n+2} \\ F_{n+3} & F_{n+4} & F_{n+5} \\ F_{n+6} & F_{n+7} & F_{n+8} \end{pmatrix}$  formed by consecutive Fibonacci terms cannot have rank 3.

*Challenge.* Can you prove that  $A_n$  must always have rank 2?

*Proof.* To prove that the rank cannot be 3 it suffices to show that the nullity is always strictly greater than 0. So for any  $n \in \mathbb{N}$  (0 is a natural number) we have  $\begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} \in \text{null}A_n$  this is

$$\text{shown as computing } A_n \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} = \begin{bmatrix} F_n + F_{n+1} - F_{n+2} \\ F_{n+3} + F_{n+4} - F_{n+5} \\ F_{n+6} + F_{n+7} - F_{n+8} \end{bmatrix} = \begin{bmatrix} F_{n+2} - F_{n+2} \\ F_{n+5} - F_{n+5} \\ F_{n+8} - F_{n+8} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

As we have the null space is a vector space and  $\text{Span}\left(\begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}\right) = 1$  and the  $\text{Span}\left(\begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}\right)$  is a subspace of  $\text{null}A_n$  then we have  $3 \geq \text{nullity} \geq 1$  by the fundamental theorem of linear algebra we get  $\text{rank}A_n = 2$

□

**Problem 4.** Let  $\mathbb{K}^{m,n}$  and  $\mathbb{K}^{n,m}$  denote the vector spaces of  $m \times n$  and  $n \times m$ , respectively. Prove or disprove the following

$$\begin{aligned} T : \mathbb{K}^{m,n} &\rightarrow \mathbb{K}^{n,m} \\ T(A) &= A^t \end{aligned}$$

is a linear map.

*Proof.* First to show scalar multiplication. Let  $\lambda \in \mathbb{K}, A \in \mathbb{K}^{m,n}$  then we have  $T(\lambda A) = (\lambda A)^t = \lambda A^t = \lambda T(A)$ . Now let  $A, B \in \mathbb{K}^{m,n}$  then  $T(A + B) = (A + B)^t = A^t + B^t = T(A) + T(B)$  hence we have  $T$  is linear.

□

**Problem 5.** Give an example of linear maps  $T \in \mathcal{L}(\mathbb{R}^3, \mathbb{R}^2)$  and  $S \in \mathcal{L}(\mathbb{R}^2, \mathbb{R}^3)$  for which exactly one of  $TS$  or  $ST$  is invertible.