CS 4124
Solutions to Homework Assignment 3
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[25] 1. Let

$$M_1 \quad = \quad (Q^{(\text{poll,acc})}, Q^{(\text{poll,rej})}, Q^{(\text{aut})}, \Sigma, \Gamma, \delta, q_0)$$

be the OTM with the following algebraic specification

$$\begin{aligned} Q^{(\text{poll,acc})} &= \{q_0\} \\ Q^{(\text{poll,rej})} &= \{q_1\} \\ Q^{(\text{aut})} &= \emptyset \\ \Sigma &= \{0,1\} \\ \Gamma &= \{0,1,\boxed{\text{B}}\}, \end{aligned}$$

where the specification of δ is given in a transition table; see Table 2.

A. On input $w = 01100$, give the computation (sequence of configurations) that M_1 goes through.

B. Let L_1 be the language accepted by M_1. What is L_1? Explain.

The sequence of configurations is

$$C_0^{M_1}(01100) = \langle \varepsilon, \boxed{\text{B}}q_0\boxed{\text{B}}\boxed{\text{B}}\rangle$$
$$C_1^{M_1}(01100) = \langle 0, \boxed{\text{B}}0q_1\boxed{\text{B}}\rangle$$
$$C_2^{M_1}(01100) = \langle 01, \boxed{\text{B}}01q_0\boxed{\text{B}}\rangle$$
$$C_3^{M_1}(01100) = \langle 011, \boxed{\text{B}}011q_1\boxed{\text{B}}\rangle$$
$$C_4^{M_1}(01100) = \langle 0110, \boxed{\text{B}}0110q_0\boxed{\text{B}}\rangle$$
$$C_5^{M_1}(01100) = \langle 01100, \boxed{\text{B}}01100q_1\boxed{\text{B}}\rangle$$

The language that M_1 accepts is given by

$$L_1 = \{w \in \{0,1\}^\star : w \text{ has even length}\}$$

q	$\delta((q,0),0)$	$\delta((q,1),0)$	$\delta((q,0),1)$	$\delta((q,1),1)$	$\delta((q,0),\boxed{\text{B}})$	$\delta((q,1),\boxed{\text{B}})$
q_0	$(q_1,0,R)$	$(q_1,1,R)$	$(q_1,0,R)$	$(q_1,1,R)$	$(q_1,0,R)$	$(q_1,1,R)$
q_1	$(q_0,0,R)$	$(q_0,1,R)$	$(q_0,0,R)$	$(q_0,1,R)$	$(q_0,0,R)$	$(q_0,1,R)$

Table 1: Transition table for Problem 1.

. We have no autonomous states and all the state transition functions only ever move right hence we only have to examine when the final state is q_0. Looking at the transition table we see that on any input and any tape symbol the state goes from q_0 to q_1 or from q_1 to q_0 informally the state flips at each step in the configuration. As the initial state is q_0 then for us to end up in q_0 again we need to have flipped an even number of times which implies that the word was even length.

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[25] 2. Let $L_2 \subseteq \{a, b\}^\star$ be the language

$$L_2 \quad = \quad \{(aab)^i \mid i \geq 0\}.$$

Three example strings in L_2 are ϵ, $aabaab$, and $aabaabaabaabaab$. Three example strings not in L_2 are baa, $aaaab$, and $aabaa$.

Design an OTM M_2 that accepts L_2. Give the complete algebraic specification for M_2:

$$M_2 \quad = \quad (Q^{(\text{poll,acc})}, Q^{(\text{poll,rej})}, Q^{(\text{aut})}, \Sigma, \Gamma, \delta, q_0).$$

You may want to put δ in a transition table; otherwise, you can specify all the transitions as equations in an `eqnarray*` or a `displaymath` environment.

HINT: **You will need a dead state that is in $Q^{(\text{poll,rej})}$.**

Let $M_2 = (Q^{(\text{poll,acc})}, Q^{(\text{poll,rej})}, Q^{(\text{aut})}, \Sigma, \Gamma, \delta)$ be an OTM with the following algebraic specification.

$$Q^{(\text{pull,acc})} = \{q_0, q_3\}$$

$$Q^{(\text{pull,rej})} = \{q_1, q_2, q_4\}$$

$$Q^{(\text{aut})} = \emptyset$$

$$\Sigma = \{a, b\}$$

$$\Gamma = \{a, b, \boxed{\text{B}}\}$$

Where δ is given by the state transition Table 2. Note each of the transition functions will be doing the same thing regardless of what is on the tape so for the sake of readability I will construct the table with an arbitrary tape element $d \in \Gamma$.

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q	$\delta((q, a), d)$	$\delta((q, b), d)$
q_0	(q_1, a, R)	(q_4, b, R)
q_1	(q_2, a, R)	(q_4, b, R)
q_2	(q_4, a, R)	(q_3, b, R)
q_3	(q_1, a, R)	(q_4, b, R)
q_4	(q_4, a, R)	(q_4, b, R)

Table 2: Transition table for Problem 2 with $d \in \Gamma$

[20] 3. Let D be a set of cards from a standard deck, 52 cards in all. Let C be a set of 50 pennies, each with a different year so they are easily distinguished.

Prove by contradiction that there is no injection

$$h : D \to C.$$

HINT: **Assume that such a injection h exists and derive a contradiction. Think about the Pigeonhole Principle.**

Assume that D, C are sets as described above. Also assume that there exists an injective function $h : D \to C$. Then from the definition of injective we get for all $x, y \in D$ with $x \neq y$ we have $h(x) \neq h(y)$ but as $|D| = 52$ and $|C| = 50$ we have by the Pigeonhole Principle that there exists two distinct elements $a, b \in D$ where $h(a) = h(b)$ this contradicts the assumption that h is injective hence we get that no such injective function exists.

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[30] 4. Let A be any nonempty set. Let $\mathcal{P}(A)$ be the power set of A, the set of all subsets of A; see Page 492.

Prove by contradiction (diagonalization) that there is no bijection

$$h : A \to \mathcal{P}(A).$$

HINT: **Assume that such a bijection h exists and derive a contradiction. Be careful with your argument.**

Let A be an arbitrary set and assume that such a bijection $h : A \to \mathcal{P}(A)$ exists. Then we create the set $\{x : x \notin h(x)\}$ now as we claim that h is bijection we have that for some $a \in A$ that $h(a) = \{x : x \notin h(x)\}$. We have two cases $a \in h(a)$ or $a \notin h(a)$ if $a \in h(a)$ then would imply $a \in \{x : x \notin h(x)\}$ but from the definition we get $a \notin h(a)$ this is a contradiction. Now in the other case if $a \notin h(a)$ then from the definition of the set we get $a \in h(a)$ again a contradiction. This implies that the assumption that there exists some $a \in A$ such that $h(a) = \{x : x \notin h(x)\}$ is not correct. Hence we have that h can not be surjective which is sufficient to prove no such bijection exists.
