Problem 1. Let V be a finite-dimensional $\mathbb{K}$-vector space with basis $\{\mathbf{v_1}, \ldots, \mathbf{v_n}\}$. Let $\{\varphi_1, \ldots, \varphi_n\}$ be a set of linear maps in $\mathcal{L}(V, \mathbb{K})$ satisfying

$$\varphi_j(\mathbf{v_k}) = \begin{cases} 1 & \text{when } j = k \\ 0 & \text{otherwise.} \end{cases}$$

Prove that $\{\varphi_1, \ldots, \varphi_n\}$ is a basis for $\mathcal{L}(V, \mathbb{K})$.

The proof Lemma 3D.8 might be useful to you.

Proof. Assume V is a finite $\mathbb{K}$-vector space with basis $\{v_1, ..., v_n\}$. Assume that $\{\varphi_1, ..., \varphi_n\}$ are as described. We have that $\dim(\mathcal{L}(V, \mathbb{K})) = \dim(V)\dim(\mathbb{K}) = \dim(V) = n$. Hence we just have to show that $\{\varphi_1, ..., \varphi_n\}$ is linear independent and we can apply the theorem linear independent set of the right length is a basis.

Now assume for $\alpha_1, ..., \alpha_n \in \mathbb{K}$ we have the map $Z \in \mathcal{L}(V, \mathbb{K})$ where

$$Z(v) = \alpha_1\varphi_1(v) + ... + \alpha_n\varphi_n(v) = 0$$

but then evaluating the function at each of the basis elements we get

$$Z(v_i) = \alpha_1\varphi_1(v_i) + ... + \alpha_i\varphi_i(v_i) + ... + \alpha_n\varphi_n(v_i) = \alpha_i = 0$$

hence we get $\alpha_1 = ... = \alpha_n = 0$ which shows that $\{\varphi_1, ..., \varphi_n\}$ is a linear independent set. Then as $\dim(\mathcal{L}(V, \mathbb{K})) = |\{\varphi_1, ..., \varphi_n\}|$ we get that we get that $\{\varphi_1, ..., \varphi_n\}$ is a basis. $\square$

Problem 2. Give an example of two 2×2 matrices A and B for which $AB \neq BA$.

Let $A = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$ and let $B = \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix}$. Then $AB = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$ and $BA = \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 2 & 1 \end{bmatrix}$

Problem 3. Recall that the Fibonacci sequence $\{F_n\} = \{0, 1, 1, 2, 3, 5, 8, 13, \ldots\}$ is recursively-defined:

$$\begin{aligned} F_0 &= 0 \\ F_1 &= 1 \\ F_n &= F_{n-1} + F_{n-2} \text{ for } n \geq 2. \end{aligned}$$

Prove that, for all $n \geq 0$, the 3×3 matrix $A_n = \begin{pmatrix} F_n & F_{n+1} & F_{n+2} \\ F_{n+3} & F_{n+4} & F_{n+5} \\ F_{n+6} & F_{n+7} & F_{n+8} \end{pmatrix}$ formed by consecutive Fibonacci terms cannot have rank 3.

Challenge. Can you prove that A_n must always have rank 2?

Proof. To prove that the rank cannot be 3 it suffices to show that the nullity is always strictly greater then 0. So for any $n \in \mathbb{N}$ (0 is a natural number :)) we have $\begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} \in$ null(A_n) this is shown as computing $A_n \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} = \begin{bmatrix} F_n + F_{n+1} - F_{n+2} \\ F_{n+3} + F_{n+4} - F_{n+5} \\ F_{n+6} + F_{n+7} - F_{n+8} \end{bmatrix} = \begin{bmatrix} F_{n+2} - F_{n-2} \\ F_{n+5} - F_{n+5} \\ F_{n+8} - F_{n+8} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$.

As we have the null space is a vector space and Span$(\begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}) = 1$ and the Span$(\begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix})$ is a subspace of null(A_n) then we have $3 \geq$ nullity$(A_n) \geq 1$ by the fundamental theorem of linear algebra we get rank$(A_n) = \dim \mathbb{R}^3 -$ nullity$A_n \leq 3 - 1 \leq 2$ as the n was an arbitrary natural number then the claim is proved.

$\square$

Problem 4. Let $\mathbb{K}^{m,n}$ and $\mathbb{K}^{n,m}$ denote the vector spaces of $m \times n$ and $n \times m$, respectively. Prove or disprove the following

$$T : \mathbb{K}^{m,n} \to \mathbb{K}^{n,m}$$
$$T(A) = A^t$$

is a linear map.

Proof. First to show scalar multiplication. Let $\lambda \in \mathbb{K}, A \in \mathbb{K}^{m,n}$ then we have $T(\lambda A) = (\lambda A)^t = \lambda A^t = \lambda T(A)$. Now let $A, B \in \mathbb{K}^{m,n}$ then $T(A+B) = (A+B)^t = A^t + B^t = T(A) + T(B)$ hence we have T is linear. $\square$

Problem 5. Give an example of linear maps $T \in \mathcal{L}(\mathbb{R}^3, \mathbb{R}^2)$ and $S \in \mathcal{L}(\mathbb{R}^2, \mathbb{R}^3)$ for which exactly one of TS or ST is invertible.

Consider the linear map $T \in (\mathbb{R}^3, \mathbb{R}^2)$ where $T(x, y, z) = (x + z, y + z)$ and the linear map $S \in \mathcal{L}(\mathbb{R}^2, \mathbb{R}^3)$ where $S(x, y) = (x, y, 0)$. Then we have $ST(x, y, z) = (x + z, y + z, 0)$ this is not invertible as it is neither injective or surjective. However we have $TS(x, y) = (x + 0, y + 0) = (x, y)$. We have that TS is just the identity function on $\mathbb{R}^2$ hence it is invertible.