Problem 1.

(a) Considering $\mathbb{C}$ as a $\mathbb{R}$-vector space, find a basis for $\mathbb{C}$.

(b) Considering $\mathbb{C}$ as a $\mathbb{C}$-vector space, find a basis for $\mathbb{C}$.

(a) *Proof.* A basis for $\mathbb{C}$ as a $\mathbb{R}$-vector space is $B = \{1, i\}$. This is shown to be a basis by the following: Let $a + bi \in \mathbb{C}$ then consider the linear combination $a(1) + b(i) = a + bi$ as an arbitrary element of $\mathbb{C}$ is a linear combination of the elements of B we have $\text{Span}(B) = \mathbb{C}$. Lastly to show linear independence of B consider the linear combination $a(1) + b(i) = 0 = 0 + 0i$ as two complex numbers are equal if and only if their real parts are equal and imaginary parts are equal we get $a = 0$ and $b = 0$. Thus B is a basis for $\mathbb{C}$ as a $\mathbb{R}$-vector space. $\square$

(b) *Proof.* A basis for $\mathbb{C}$ as a $\mathbb{C}$-vector space is given by $B = \{1 + i\}$. Let $a + bi \in \mathbb{C}$ then consider the linear combination $(\frac{a+b}{2} + \frac{b-a}{2}i)(1 + i) = \frac{a+b}{2} + \frac{a+b}{2}i + \frac{b-a}{2}i - \frac{b-a}{2} = a + bi$ as an arbitrary element of $\mathbb{C}$ is a linear combination of the element of B we have $\text{Span}(B) = \mathbb{C}$. *Note that $\frac{a+b}{2} + \frac{b-a}{2}i \in C$ is a scalar as this is a $\mathbb{C}$-vector space.* Lastly to show linear independence of B consider the linear combination $a + bi \in \mathbb{C}$ and $1 + i \in B$ we have $(a + bi)(1 + i) = 0$ if and only if $a + bi = 0$ as $\mathbb{C}$ is a field hence no zero divisors. $\square$

Problem 2. Let V be a $\mathbb{K}$-vector space, and suppose that S_1, S_2 are subsets of V satisfying the following conditions:

- S_1 and S_2 are both finite.
- $S_1 \cap S_2 = \emptyset$.
- $S_1 \cup S_2$ is a linearly independent set.

(a) Prove that $\text{Span}(S_1 \cup S_2) = \text{Span}(S_1) \oplus \text{Span}(S_2)$.

(b) What would change about the claim in (a) if $S_1 \cup S_2$ was not assumed to be linearly independent?

(a) *Proof.* Assume that V is a $\mathbb{K}$-vector space, and that S_1, S_2 are as described. First I will show that $\text{Span}(S_1) + \text{Span}(S_2)$ is a direct sum. As $\text{Span}(S_1), \text{Span}(S_2)$ are both vector spaces we have that $\vec{0} \in \text{Span}(S_1) + \text{Span}(S_2) \neq \emptyset$. Now assume that $\vec{v} \in \text{Span}(S_1) \cap \text{Span}(S_2)$ then $\vec{v} \in \text{Span}(S_1)$ and $\vec{v} \in \text{Span}(S_2)$ so $v = k_1\vec{s_1} + ... + k_n\vec{s_n}$ where $k_i \in \mathbb{K}$ and $\vec{s} \in S_1$ and $v = c_1\vec{u_1} + ... + c_n\vec{u_n}$ where $c_i \in \mathbb{K}$ and $\vec{u_i} \in S_2$. Then $c_1\vec{u_1}+...+c_n\vec{u_n}+-(k_1\vec{s_1}+...+k_n\vec{s_n}) = c_1\vec{u_1}+...+c_n\vec{u_n}+(-k_1\vec{s_1})+...+(-k_n\vec{s_n}) = \vec{0}$ but as $S_1 \cup S_2$ is linear independent and the previous equation is a linear combination of $S_1 \cup S_2$ we have that for each $c_i, k_i \in \mathbb{K}$ that $c_i = k_i = 0$ hence only $\vec{0} \in \text{Span}(S_1) \cap \text{Span}(S_2)$ which implies $\text{Span}(S_1) + \text{Span}(S_2)$ is a direct sum.

Now suppose $\vec{v} \in \text{Span}(S_1 \cup S_2)$ then $\vec{v} = k_1\vec{s_1}+...+k_n\vec{s_n}+c_1\vec{u_1}+...+c_n\vec{c_n}$ where $k_i, c_i \in \mathbb{K}$ and $\vec{s_i} \in S_1$ and $\vec{u_i} \in S_2$. As $k_1\vec{s_1} + ... + k_n\vec{s_n} \in \text{Span}(S_1)$ and $c_1\vec{u_1} + ... + c_n\vec{u_n} \in S_2$ $\square$

(b)

Problem 3. Let $V = \mathbb{Q}^4$, considered as a $\mathbb{Q}$-vector space, and let U be the subspace

$$U = \operatorname{Span}\big(\mathbf{u_1} = (1, -1, 2, 1),\ \mathbf{u_2} = (2, -3, 6, 3)\big).$$

Extend the set $\{\mathbf{u_1}, \mathbf{u_2}\}$ into a basis for V. That is, find two vectors $\mathbf{v_1}, \mathbf{v_2} \in V$ so that

$$\{\mathbf{u_1}, \mathbf{u_2}, \mathbf{v_1}, \mathbf{v_2}\}$$

is a basis for V.

Proof. □

Problem 4. Let $V = \mathcal{P}_3(\mathbb{R})$ be the $\mathbb{R}$-vector space of polynomials of degree 3 or less. Let U be the subspace (you can take this for granted)

$$U = \{p(x) \in \mathcal{P}_3(\mathbb{R}) \,:\, p'(7) = 0\}$$

where $p'(x)$ is the derivative of $p(x)$ and $p'(7)$ is the derivative evaluated at $x = 7$. Find a basis for U.

Proof. $\square$

Problem 5. The classical "Inclusion-Exclusion Principle" states that, for two finite sets A_1, A_2, the cardinality of the union satisfies:

$$|A_1 \cup A_2| = |A_1| + |A_2| - |A_1 \cap A_2|.$$

Notice that we have a similar formula for vector spaces V_1, V_2:

$$\dim(V_1 + V_2) = \dim(V_1) + \dim(V_2) - \dim(V_1 \cap V_2).$$

For three sets, A_1, A_2, A_3, the Inclusion-Exclusion Principle says

$$\begin{aligned} |A_1 \cup A_2 \cup A_3| = & |A_1| + |A_2| + |A_3| \\ & - |A_1 \cap A_2| - |A_1 \cap A_3| - |A_2 \cap A_3| \\ & + |A_1 \cap A_2 \cap A_3|. \end{aligned}$$

Give an example showing that, sadly, the following analogous formula does not hold for vector spaces V_1, V_2, V_3:

$$\begin{aligned} \dim(V_1 + V_2 + V_3) = & \dim(V_1) + \dim(V_2) + \dim(V_3) \\ & - \dim(V_1 \cap V_2) - \dim(V_1 \cap V_3) - \dim(V_2 \cap V_3) \\ & + \dim(V_1 \cap V_2 \cap V_3). \end{aligned}$$

Hint: consider subspaces of a familiar low-dimensional vector space.