Hand in Friday, April 19.

Definition. Let $f : S \to C$ be a continuous map from a circle in the plane to a circle in the plane. Define the **degree** of f to be the winding number of this map around the point $\vec{c}$ at the center of C. If you prefer to think of winding numbers in terms of continuous maps from intervals, give the name θ to a variable running through the interval $[0, 2\pi]$, let $\gamma : [0, 2\pi] \to S$ parametrize S by $\gamma(\theta) = (x_0 + r\cos\theta, y_0 + r\sin\theta)$ for appropriate x_0, y_0, and r, and define the degree of f to be the winding number of $f \circ \gamma$ around $\vec{c}$. [from the textbook by Fulton]

1. Show that, for an f as in the above definition, if f is not surjective, then the degree of f equals zero.

Proof. Using the parametrization $\gamma : [0, 2\pi] \to S$ by $\gamma(\theta) = (x_0 + r\cos\theta, y_0 + r\sin\theta)$ where (x_0, y_0) is the center of S and r is the radius of S. Then we have that f is a closed curve as

$$f \circ \gamma(0) = f(x_0 + r\cos 0, y_0 + r\sin 0) = f(x_0 + r\cos 2\pi, y_0 + r\sin 2\pi) = f \circ \gamma(2\pi)$$

Now let $\vec{p_1} \in C$ be a point that is not in the image of f. Then we have a point $\vec{p_2} \in C$ that is colinear with the line intersecting (x_0, y_0) and $\vec{p_1}$.

We have the constant curve $g : S \to C$ given by the equation $g(\vec{x}) = \vec{p_2}$ for all $\vec{x} \in S$.

Then we create the homotopy $H : [0, 2\pi] \times [0, 1] \to \mathbb{R}^2 \setminus \{(x_0, y_0)\}$ given by

$$H(\theta, s) = f(\gamma(\theta))(1 - s) + s \cdot \vec{p_2}$$

for all $\theta \in [0, 2\pi]$ and $s \in [0, 1]$.

We have $H(\theta, 0) = f(\gamma(\theta)) + 0 \cdot \vec{p_2} = f(\gamma(\theta))$, and $H(\theta, 1) = f(\gamma(\theta)) \cdot 0 + 1 \cdot \vec{p_2} = g(\gamma(\theta))$.

We have that H is continuous as it is the sum of two weighted continuous functions.

Additionally we have that the image of H is contained in $\mathbb{R}^2 \setminus \{(x_0, y_0)\}$ as for any $\theta \in [0, 2\pi]$ we have for all $s \in [0, 1]$ that $H(\theta, s) \neq (x_0, y_0)$ as the only point colinear with (x_0, y_0) and $\vec{p_2}$ is $\vec{p_1}$ and by our assumption that $\vec{p_1}$ is not in the image of f. Then we have that H is a homotopy between f and g.

Then we have that f and g are homotopic and thus have the same winding number. We have that the winding number of g is zero as it is a constant curve. Then we have that the winding number of f is zero. □

2. Calculate the degree of each of the following maps from the unit circle centered at the origin to the unit circle centered at the origin.

a. $f(x, y) = (x, y)$

Using the parametrization $\gamma(\theta) = (\cos\theta, \sin\theta)$, for $\theta \in [0, 2\pi]$. With the four sectors

$$U_1\{(x, y) : x, y \in \mathbb{R} \text{ with } x > 0\}$$
$$U_2 = \{(x, y) : x, y \in \mathbb{R} \text{ with } y > 0\}$$
$$U_3 = \{(x, y) : x, y \in \mathbb{R} \text{ with } x < 0\}$$
$$U_4 = \{(x, y) : x, y \in \mathbb{R} \text{ with } y < 0\}$$

Then we have the the six subdivisions $t_0 = 0, t_1 = \pi/4, t_2 = 3\pi/4, t_3 = 5\pi/4, t_4 = 7\pi/4, t_5 = 2\pi$. With the angle function being for θ_1 being its angle in the range $(-\pi/2, \pi/2)$ for θ_2 being its angle in the range $(0, \pi)$ and for θ_3 being its angle in the range $(\pi/2, 3\pi/2)$ and θ_4 being its angle in the range $(\pi, 2\pi)$.

Then

$W(f \circ \gamma, \vec{0}) = \frac{1}{2\pi}\big(\theta_1(f(\gamma(t_1))) - \theta_1(f(\gamma(t_1))) + \theta_2(f(\gamma(t_2))) - \theta_2(f(\gamma(t_1))) + \theta_3(f(\gamma(t_3))) - \theta_3(f(\gamma(t_2))) + \theta_4(f(\gamma(t_4))) - \theta_4(f(\gamma(t_3))) + \theta_1(f(\gamma(t_5))) - \theta_1(f(\gamma(t_4)))\big)$.

Then $W(f \circ \gamma, \vec{0}) = \frac{1}{2\pi}\left(\frac{\pi}{4} - 0 + \frac{3\pi}{4} - \frac{\pi}{4} + \frac{5\pi}{4} - \frac{3\pi}{4} + \frac{7\pi}{4} - \frac{5\pi}{4} + 0 + \frac{\pi}{4}\right) = \frac{1}{2\pi}\left(\frac{7\pi}{4} + \frac{\pi}{4}\right) = 1$.

b. $g(x, y) = (-x, -y)$

Using the same parametrization with γ as in **a.** With the four sectors

$$U_1 = \{(x, y) : x, y \in \mathbb{R} \text{ with } x < 0\}$$

$$U_2 = \{(x,y) : x,y \in \mathbb{R} \text{ with } y < 0\}$$

$$U_3 = \{(x,y) : x,y \in \mathbb{R} \text{ with } x > 0\}$$

$$U_4 = \{(x,y) : x,y \in \mathbb{R} \text{ with } y > 0\}$$

With the same subdivisions as in **a.** and the angle functions θ_1 being its angle in the range $(\frac{\pi}{2}, \frac{3\pi}{2})$ for θ_2 being its angle in the range $(\pi, 2\pi)$ and for θ_3 being its angle in the range $(\frac{3\pi}{2}, \frac{5\pi}{2})$ and θ_4 being its angle in the range $(2\pi, 3\pi)$.

Then $W(g\circ\gamma, \vec{0}) = \frac{1}{2\pi}\big(\theta_1(g(\gamma(t_1)))-\theta_1(g(\gamma(t_0)))+\theta_2(g(\gamma(t_2)))-\theta_2(g(\gamma(t_3)))+\theta_3(g(\gamma(t_3)))-\theta_3(g(\gamma(t_2)))+\theta_4(g(\gamma(t_4))) - \theta_4(g(\gamma(t_3))) + \theta_1(g(\gamma(t_5))) - \theta_1(g(\gamma(t_4)))\big)$

Then $W(g \circ \gamma, \vec{0}) = \frac{1}{2\pi}\big(\frac{5\pi}{4} - \pi + \frac{7\pi}{4} - \frac{5\pi}{4} + \frac{9\pi}{4} - \frac{7\pi}{4} + 3\pi - \frac{9\pi}{4}\big) = \frac{1}{2\pi}\big(3\pi - \pi\big) = 1$

c. $h(x,y) = (x, -y)$

Using the same parametrization with γ, and the following four sectors.

$$U_1 = \{(x,y) : x,y \in \mathbb{R} \text{ with } x > 0\}$$
$$U_2 = \{(x,y) : x,y \in \mathbb{R} \text{ with } y < 0\}$$
$$U_3 = \{(x,y) : x,y \in \mathbb{R} \text{ with } x < 0\}$$
$$U_4 = \{(x,y) : x,y \in \mathbb{R} \text{ with } y > 0\}$$

With the same subdivisions as in **a.**. With the following angle functions θ_1 being it angle in the range $(-\pi/2, \pi/2)$ θ_2 being its angle in the range $(-\pi, 0)$ θ_3 being its angle in the range $(-3\pi/2, -\pi/2)$ and θ_4 being its angle in the range $(-2\pi, -\pi)$.

Then $W(h\circ\gamma, \vec{0}) = \frac{1}{2\pi}\big(\theta_1(h(\gamma(t_1)))-\theta_1(h(\gamma(t_0)))+\theta_2(h(\gamma(t_2)))-\theta_2(h(\gamma(t_1)))+\theta_3(h(\gamma(t_3))-\theta_3(h(\gamma(t_2)))+\theta_4(h(\gamma(t_4))) - \theta_4(h(\gamma(t_3))) + \theta_1(h(\gamma(t_5))) - \theta_1(h(\gamma(t_4)))\big)$

Then $W(h \circ \gamma, \vec{0}) = \frac{1}{2\pi}\big(- \frac{\pi}{4} - 0 + -\frac{3\pi}{4} + \frac{\pi}{4} - \frac{5\pi}{4} + \frac{3\pi}{4} - \frac{7\pi}{4} + \frac{5\pi}{4} - 2\pi + \frac{7\pi}{4}\big) = -1$

d. $k(\cos(\theta), \sin(\theta)) = (\cos(n\theta), \sin(n\theta))$, where n is an arbitrary integer. Breaking into cases.

If $n = 0$ then k would just be a constant curve hence it would have winding number 0.

Now if $n > 0$ then consider the sectors

$$U_1 = \{(x,y) : x,y \in \mathbb{R} \text{ with } x > 0\}$$
$$U_2 = \{(x,y) : x,y \in \mathbb{R} \text{ with } y > 0\}$$
$$U_3 = \{(x,y) : x,y \in \mathbb{R} \text{ with } x < 0\}$$
$$U_4 = \{(x,y) : x,y \in \mathbb{R} \text{ with } y < 0\}$$

With the subdivisions being $t_0 = 0, t_{4n+1} = 2\pi$ and $t_i = (\frac{(i-1)\pi}{2} + \frac{\pi}{4})$, for $i \in \{1, ..., 4n\}$. With the angle functions being θ_1 being its angle in the range $(-\pi/2, \pi/2)$ and θ_2 being its angle in the range $(0, \pi)$ and θ_3 being its angle in the range $(\pi/2, 3\pi/2)$ and θ_4 being its angle in the range $(\pi, 2\pi)$.

Then on each trip around the circle we have that all but terms cancel but $\frac{7\pi}{4} + \frac{\pi}{4} = 1$ as this happens n times we have that the winding number is n.

Now if $n < 0$ then consider the sectors

$$U_1 = \{(x,y) : x,y \in \mathbb{R} \text{ with } x > 0\}$$
$$U_2 = \{(x,y) : x,y \in \mathbb{R} \text{ with } y < 0\}$$
$$U_3 = \{(x,y) : x,y \in \mathbb{R} \text{ with } x < 0\}$$
$$U_4 = \{(x,y) : x,y \in \mathbb{R} \text{ with } y > 0\}$$

With the same subdivisions as in the positive case and the angle functions being θ_1 being its angle in the range $(-\pi/2, \pi/2)$ and θ_2 being its angle in the range $(-\pi, 0)$ and θ_3 being its angle in the range $(-3\pi/2, -\pi/2)$ and θ_4 being its angle in the range $(-2\pi, -\pi)$.

Then on each trip around the circle we have that all but terms cancel but $-\frac{\pi}{4} - \frac{7\pi}{4} = -1$ as this happens n times we have that the winding number is $-n$.

Definition. If Y is a topological subspace of a topological space X, a **retraction** from X to Y is a continuous map $r : X \to Y$ that satisfies, for all $y \in Y$, $r(y) = y$. When such a retraction exists, we call Y a **retract** of X. [from the textbook by Fulton]

3. Show that, if Y is a retract of X and if every continuous map from X to X has a fixed point, then every continuous map from Y to Y has a fixed point. **Hint.** Start with an arbitrary continuous map $f : Y \to Y$. How can you make a continuous map $g : X \to X$ whose behavior has the needed implications for f's behavior?

Proof. Let Y be a retraction of X given by the map $r : X \to Y$ and $f : Y \to Y$ be an arbitrary continuous map. Then consider the function $g : X \to X$ where $g = j \circ f \circ r$ where j is the inclusion map given by (Munkres Theorem 18.2). We have as g is a continuous and $g : X \to X$ that there exists a fixed point in $x_0 \in X$. Then we have that $r(x_0) = y_0$ for some $y_0 \in Y$ and hence we get $g(x_0) = j(f(r(x_0))) = j(f(y_0)) = f(y_0) = x_0$ as j is the inclusion map this implies that $r \circ g(x_0) = y_0$ but hence we get $r(j(f(y_0))) = y_0$ which implies that $f(y_0) = y_0$ as the codomain of f is Y and r is a retraction and j is the inclusion map.

□

4. Let B be the open unit disk in $\mathbb{R}^2$ and let D be the closed unit disk in $\mathbb{R}^2$. Show that, for any $\vec{p} \in B$, the unit circle C in $\mathbb{R}^2$ is a retract of $D \setminus \{\vec{p}\}$. **Hint.** When $\vec{p}$ is the origin, the map $\vec{x} \mapsto \frac{\vec{x}}{|\vec{x}|}$ is the retraction. When $\vec{p}$ is more general, consider solving $|\vec{p} + t(\vec{x} - \vec{p})| = 1$ for t.

Proof. Consider the function $r : D \setminus \{p\} \to C$ defined by the equation $r(\vec{x}) = \vec{p} + t(\vec{x} - \vec{p})$ such that t is the positive solution to $|\vec{p} + t(\vec{x} - \vec{p})| = 1$ such a solution is guaranteed to exist as the line parametrized by $\vec{p} + t(\vec{x} - \vec{p})$ will have a point intersecting the unit circle based on the fact that $\vec{p} \in B$. Now suppose that $\vec{x} \in C$ then we have that the line parametrized by $\vec{p} + t(\vec{x} - \vec{p})$ only intersects the unit circle with $t > 0$ when $t = 1$ hence $\vec{p} + \vec{x} - \vec{p} = \vec{x}$. Then we have that $r(\vec{x}) = \vec{x}$ and hence r is a retraction.

□