Math 4324 Topologies

Hand in Friday, February 2. Homework due on a class day should be submitted as an email attachment sent to me by the beginning of class.

1. Suppose that $\{U_\alpha\}_{\alpha\in A}$ is a collection of sets satisfying the property

$$\forall a, b \in A, \quad U_a \bigcap U_b \in \{U_\alpha\}_{\alpha\in A}.$$

a. Show that for any finite collection $\{U_1, ..., U_n\}$ of sets from this collection, $\displaystyle\bigcap_{j=1}^{n} U_j$ is in the collection.

Proof. Using mathematical induction we have the base case $n = 1$ then $\bigcap_{j=1}^{1} U_j = U_1$ as $U_1 \in \{U_\alpha\}_{\alpha\in A}$ the base case is complete. Now assume that there exists $n \in \mathbb{N}$ with $n > 1$ where $\bigcap_{i=1}^{n} U_i \in \{U_\alpha\}_{\alpha\in A}$. Then $\bigcap_{i=1}^{n+1} U_i = (\bigcap_{i=1}^{n} U_i) \cap U_{n+1}$ based on the induction hypothesis we have $\bigcap_{i=1}^{n} U_i \in \{U_\alpha\}_{\alpha\in A}$ therefore for some $\beta \in A$ we have $U_\beta = \bigcap_{i=1}^{n} U_i$. This implies $(\bigcap_{i=1}^{n} U_i) \cap U_{n+1} = U_\beta \cap U_{n+1}$ which by the assumption of the collection of sets we get $U_\beta \cap U_{n+1} \in \{U_\alpha\}_{\alpha\in A}$ which completes the proof. □

b. Give an example of a collection that satisfies the property stated at the beginning of this problem and in which there is an infinite collection of sets whose intersection is not in the collection. As always, prove that your example has the asserted properties.

Proof. Consider the collection $\{U_\alpha\}_{\alpha\in\mathbb{N}\setminus\{0\}}$ of sets of the form $U_n = (0, \frac{1}{n})$ where $n = \mathbb{N} \setminus \{0\}$. We have this satisfying the property at the beginning as given any two positive integers a, b we have $U_a \bigcap U_b = U_{\min(\{a,b\})}$ as $\min(\{a, b\}) \in \mathbb{N} \setminus \{0\}$ we have $U_a \cap U_b \in \{U_\alpha\}_{\alpha\in\mathbb{N}\setminus\{0\}}$.

Now take the intersection $\bigcap_{n=1}^{\infty} U_n$ we have that this intersection is just the empty set. This is proven by assuming that it is non empty so there is a real number $r \in \bigcap_{n=1}^{\infty} U_n$ this implies that there exists a real number $0 < r < \frac{1}{n}$ for all positive integers n. Rearranging the inequality we get $0 < n < \frac{1}{r}$ which would imply that the positive integers are bounded above a contradiction. Hence the intersection is empty. Now to show that $\emptyset \notin \bigcap_{n=1}^{\infty} U_n$ assume that $\emptyset \in \bigcap_{n=1}^{\infty} U_n$ then we would have for some $n \in \mathbb{N} \setminus \{0\}$ the following is true $\frac{1}{n} = 0$ which would imply $1 = 0$ a contradiction.

□

c. In the setting described at the beginning of this problem, assume that A_1 and A_2 are subsets of A. Show that

$$\left(\bigcup_{\beta\in A_1} U_\beta\right) \bigcap \left(\bigcup_{\gamma\in A_2} U_\gamma\right) = \bigcup_{\beta\in A_1 \text{ and } \gamma\in A_2} (U_\beta \bigcap U_\gamma).$$

Proof. This will be a proof by double containment. Assume A_1, A_2 are both subsets of A. Assume that $x \in \left(\bigcup_{\beta\in A_1}\right) \bigcap \left(\bigcup_{\gamma\in A_2} U_\gamma\right)$. Then for some $\beta \in A_1$ and some $\gamma \in A_2$ we have $x \in U_\beta \cap U_\gamma$ which implies $\bigcup_{\beta\in A_1 \text{and} \gamma\in A_2} (U_\beta \cap U_\gamma)$. This implies

$$\left(\bigcup_{\beta\in A_1} U_\beta\right) \bigcap \left(\bigcup_{\gamma\in A_2} U_\gamma\right) \subseteq \bigcup_{\beta\in A_1 \text{ and } \gamma\in A_2} (U_\beta \bigcap U_\gamma)$$

Now let $x \in \bigcup_{\beta\in A_1 \text{ and } \gamma\in A_2} (U_\beta \bigcap U_\gamma)$ then $x \in U_\beta \cap U_\gamma$ for some $\beta \in A_1$, $\gamma \in A_2$. This implies $x \in \left(\bigcup_{\beta\in A_1} U_\beta\right) \cap (U_{\gamma\in A_2} U_\gamma)$ which implies

$$\left(\bigcup_{\beta\in A_1} U_\beta\right) \bigcap \left(\bigcup_{\gamma\in A_2} U_\gamma\right) \supseteq \bigcup_{\beta\in A_1 \text{ and } \gamma\in A_2} (U_\beta \bigcap U_\gamma)$$

By double containment we have that the sets are equal.

□

2. Let X be an infinite set and let $\mathcal{T}_c$ be the set of subsets U of X satisfying: $U = X$ or $U = \emptyset$ or $X \setminus U$ is countable. Show that $\mathcal{T}_c$ is a topology on X. This is often called the countable-complement topology.

Proof. Assume X is an infinite set and $\mathcal{T}_c$ is as stated. We have $\emptyset, X \in \mathcal{T}_c$ by the definition of $\mathcal{T}_c$. Consider the intersection of $U_1 \cap U_2$ where $U_1, \in U_2 \in \mathcal{T}_c$. If either $U_1 = \emptyset$ or $U_2 = \emptyset$ we have $U_1 \cap U_2 = \emptyset$ and if $U_1 = X$ then $U_1 \cap U_2 = U_2$. In either case the intersection is in $\mathcal{T}_c$. Now given $U_1, U_2 \in \mathcal{T}_c$ assuming neither is the empty set or X. Then $X \setminus (U_1 \cap U_2) = X \cap \overline{(U_1 \cap U_2)} = X \cap (\overline{U_1} \cup \overline{U_2}) = X \cap \overline{U}_1 \cup X \cap \overline{U_2}$. (note the 'overline' is set complement, I used De Morgan's laws and also distributivity of sets over intersection, union). We have that the union of two countable sets is countable following the equality that implies $X \setminus (U_1 \cap U_2)$ is countable which implies $U_1 \cap U_2 \in \mathcal{T}_c$. That covers the base case so now assume for some $k \in \mathbb{N}$ with $k \geq 2$ we have that the intersection of k elements of $\mathcal{T}_c$ is in $\mathcal{T}_c$. Then given the intersection of $k+1$ elements of $\mathcal{T}_c$ we have $U_1 \cap ... \cap U_{k+1}$ then we have the intersection of k of the elements is in $\mathcal{T}_c$. WLOG $U_{k+1} \neq \emptyset$ or $U_{k+1} \neq X$.

- If $U_1 \cap ... \cap U_k = \emptyset$ then $U_1 \cap ... \cap U_{k+1} = \emptyset$
- If $U_1 \cap ... \cap U_k = X$ then $U_1 \cap ... U_{k+1} = U_{k+1}$
- If $X \setminus U_1 \cap ... \cap U_k$ is countable then using the same reasoning as above we have $X \setminus U_1 \cap ... \cap U_{k+1}$ is countable as well.

In all cases we have that $\mathcal{T}_c$ is closed under finite intersections.

Now given a subset of $S \subset \mathcal{T}_c \setminus \{\emptyset\}$ we have $X \setminus \bigcup_{U \in S} U \subseteq X \setminus U$ where $U \in S$. As subsets of countable sets are countable then we have arbitrary unions are closed in $\mathcal{T}_c$ as well. In the case that the empty set is in a union the resulting set is equal to not having it be in the union hence the reason for the removal in S. □

Munkres introduces several topologies on the real line $\mathbb{R}$. The standard topology $\mathcal{T}$ is generated by $\{U_{a,b} : a \in \mathbb{R},\ b \in \mathbb{R},\ \text{and } a < b\}$. Here $U_{a,b} = \{x \in \mathbb{R} : a < x < b\}$. The lower limit topology $\mathcal{T}_l$ is generated by $\{I_{a,b} : a \in \mathbb{R},\ b \in \mathbb{R},\ \text{and } a < b\}$, where $I_{a,b} = \{x \in \mathbb{R} : a \leq x < b\}$.

In doing problem 3. you may **assume** without proof that the sets I have chosen to generate the topologies are bases for the topologies, so that Munkres Lemma 13.3 applies.

3. Let $\mathcal{T}_Q$ be the topology on $\mathbb{R}$ generated by $\{U_{a,b} : a \in \mathbb{Q} \text{ and } b \in \mathbb{Q}\}$. Let $\mathcal{T}_{Q,l}$ be the topology on $\mathbb{R}$ generated by $\{I_{a,b} : a \in \mathbb{Q} \text{ and } b \in \mathbb{Q}\}$. Show that $\mathcal{T}_Q = \mathcal{T}$. Show that $\mathcal{T}_{Q,l} \neq \mathcal{T}_l$.

Proof. $\mathcal{T}_Q = \mathcal{T}$

Let $x \in \mathbb{R}$ and $x \in (a,b)$ where $a, b \in \mathbb{Q}$ then as the rationals are a subset of the real numbers the same open interval is a basis for $\mathcal{T}$. Which implies by Lemma 13.3 $\mathcal{T}_Q \subset \mathcal{T}$. Now let $x \in \mathbb{R}$ and $x \in (a,b)$ where $a, b \in \mathbb{R}$. Then as the rationals are dense in the reals we have two rational numbers c, d such that $a < c < x < d < b$. This implies $x \in (c,d) \subset (a,b)$ by Lemma 13.3 we have $\mathcal{T} \subset \mathcal{T}_c$. By double containment $\mathcal{T}_c = \mathcal{T}$ □

The next proof using contrapositive of Lemma 13.3 which says let $\mathcal{B}$ and $\mathcal{B}'$ be bases for the topologies $\mathcal{T}$ and $\mathcal{T}'$ on X. Then the following are equivalent:

(1) $\mathcal{T}'$ is not finer than $\mathcal{T}$
(2) There exists $x \in X$ and there exists a basis element $B \in \mathcal{B}$ with $x \in B$ such that for all $B' \in \mathcal{B}'$ with $x \in B'$ we have $B' \not\subset B$

Proof. $\mathcal{T}_{Q,l} \neq \mathcal{T}_l$

□

Definitions. A **sequence** in a topological space X is a function $f : \mathbb{N} \to X$. Using the notation x_n for $f(n)$, we also can think of a sequence as a list, indexed by the natural numbers, of elements of X. Thinking this way, we often use the notation (x_n) for the sequence. We say that (x_n) **converges** to $y \in X$ if, for every open set U containing y, there is an $N \in \mathbb{N}$ such that for all $n > N$, $x_n \in U$. When this happens, we call y a **limit** of the sequence (x_n). We say that (x_n) **converges** if one or more elements of X is a limit or are limits of (x_n). The elements x_n in X are called the **values** of the sequence. Note that the list of values in the sequence is always infinitely long but that some sequences have only finitely many different values. For example, in $\mathbb{R}$, the sequence in which $x_n = 0$ when n is even and $x_n = 1$ when n is odd has only two different values, and a constant sequence has only one value.

4. a. Suppose that (x_n) is a sequence in an infinite space X that has the finite-complement topology. Suppose that for all distinct k and m, $x_k \neq x_m$. Must (x_n) converge? If so, what element or elements does it converge to? As always, prove your assertions.

b. Suppose that (x_n) is a sequence in an infinite space X that has the finite-complement topology. Suppose that for every even k, x_k equals the same value $z \in X$, and assume that for all distinct odd m and l, $x_m \neq x_l$. Does (x_n) converge? If so, state what element(s) it converges to and prove that that convergence does happen. If not, prove that it does not converge to any element in X.

5. a. Suppose that (x_n) is a sequence in an infinite space X with the discrete topology. Show that (x_n) converges if and only if there exists $c \in X$ and there exists $N \in \mathbb{N}$ such that for all $n > N$, $x_n = c$.

b. Suppose that (x_n) is a sequence in an infinite space X with the countable-complement topology. Show that (x_n) converges if and only if there exists $c \in X$ and there exists $N \in \mathbb{N}$ such that for all $n > N$, $x_n = c$.