CS 4124
Solutions to Homework Assignment 3
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March 15, 2024

[25] 1. Let

$$M_1 \quad = \quad (Q^{(\text{poll,acc})}, Q^{(\text{poll,rej})}, Q^{(\text{aut})}, \Sigma, \Gamma, \delta, q_0)$$

be the OTM with the following algebraic specification

$$\begin{aligned} Q^{(\text{poll,acc})} &= \{q_0\} \\ Q^{(\text{poll,rej})} &= \{q_1\} \\ Q^{(\text{aut})} &= \emptyset \\ \Sigma &= \{0,1\} \\ \Gamma &= \{0,1,\boxed{\text{B}}\}, \end{aligned}$$

where the specification of δ is given in a transition table; see Table 1.

A. On input $w = 01100$, give the computation (sequence of configurations) that M_1 goes through.

B. Let L_1 be the language accepted by M_1. What is L_1? Explain.

The sequence of configurations is

$$\begin{aligned}
C_0^{M_1}(01100) &= \langle \varepsilon, \boxed{\text{B}} q_0 \boxed{\text{B}}\boxed{\text{B}} \rangle \\
C_1^{M_1}(01100) &= \langle 0, \boxed{\text{B}} 0 q_1 \boxed{\text{B}} \rangle \\
C_2^{M_1}(01100) &= \langle 01, \boxed{\text{B}} 01 q_0 \boxed{\text{B}} \rangle \\
C_3^{M_1}(01100) &= \langle 011, \boxed{\text{B}} 011 q_1 \boxed{\text{B}} \rangle \\
C_4^{M_1}(01100) &= \langle 0110, \boxed{\text{B}} 0110 q_0 \boxed{\text{B}} \rangle \\
C_5^{M_1}(01100) &= \langle 01100, \boxed{\text{B}} 01100 q_1 \boxed{\text{B}} \rangle
\end{aligned}$$

The language that M_1 accepts is given by

$$L_1 = \{w \in \{0,1\}^\star : w \text{ has even length}\}$$

q	$\delta((q,0),0)$	$\delta((q,1),0)$	$\delta((q,0),1)$	$\delta((q,1),1)$	$\delta((q,0),\boxed{\text{B}})$	$\delta((q,1),\boxed{\text{B}})$
q_0	$(q_1,0,R)$	$(q_1,1,R)$	$(q_1,0,R)$	$(q_1,1,R)$	$(q_1,0,R)$	$(q_1,1,R)$
q_1	$(q_0,0,R)$	$(q_0,1,R)$	$(q_0,0,R)$	$(q_0,1,R)$	$(q_0,0,R)$	$(q_0,1,R)$

Table 1: Transition table for Problem 1.

. We have no autonomous states and all the state transition functions only ever move right hence we only have to examine when the final state is q_0. Looking at the transition table we see that on any input and any tape symbol the state goes from q_0 to q_1 or from q_1 to q_0 informally the state flips at each step in the configuration. As the initial state is q_0 then for us to end up in q_0 again we need to have flipped an even number of times which implies that the word was even length.

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[25] 2. Let $L_2 \subseteq \{a,b\}^\star$ be the language

$$L_2 \;=\; \{(aab)^i \mid i \geq 0\}.$$

Three example strings in L_2 are ϵ, $aabaab$, and $aabaabaabaabaab$. Three example strings not in L_2 are baa, $aaaab$, and $aabaa$.

Design an OTM M_2 that accepts L_2. Give the complete algebraic specification for M_2:

$$M_2 \;=\; (Q^{(\text{poll,acc})}, Q^{(\text{poll,rej})}, Q^{(\text{aut})}, \Sigma, \Gamma, \delta, q_0).$$

You may want to put δ in a transition table; otherwise, you can specify all the transitions as equations in an `eqnarray*` or a `displaymath` environment.

HINT: **You will need a dead state that is in $Q^{(\text{poll,rej})}$.**

Let $M_2 = (Q^{(\text{poll,acc})}, Q^{(\text{poll,rej})}, Q^{(\text{aut})}, \Sigma, \Gamma, \delta, q_0)$ be an OTM with the following algebraic specification.

$$Q^{(\text{pull,acc})} = \{q_0, q_3\}$$
$$Q^{(\text{pull,rej})} = \{q_1, q_2, q_4\}$$
$$Q^{(\text{aut})} = \emptyset$$
$$\Sigma = \{a, b\}$$
$$\Gamma = \{a, b, \boxed{\text{B}}\}$$

Where δ is given by the state transition Table 2. Note each of the transition functions will be doing the same thing regardless of what is on the tape (for same input and same state) so for the sake of readability I will construct the table with an arbitrary tape element $d \in \Gamma$.

Explanation of why M_2 works. If M_2 reads in an a from the initial configuration, then we have it enter state q_1 (non-accepting state). If we get a b in state q_1, then we would have had a string of the form ab, hence we go to the dead state q_4. If instead we get a a in q_1, we go on to state q_2. If we get another a in q_2 (non-accepting state), then we would have read three continues a's, hence we enter the dead state q_4. If q_2 reads a b, then we enter the accepting state q_3. If q_3 reads a b, then we would have read two b's in a row, hence we enter q_4. If we read another a in q_3, we enter state q_1, and the process continues. If we enter state q_4 we just write what ever our input is to the tape and move right while staying in state q_4 (non accepting).

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q	$\delta((q,a),d)$	$\delta((q,b),d)$
q_0	(q_1, a, R)	(q_4, b, R)
q_1	(q_2, a, R)	(q_4, b, R)
q_2	(q_4, a, R)	(q_3, b, R)
q_3	(q_1, a, R)	(q_4, b, R)
q_4	(q_4, a, R)	(q_4, b, R)

Table 2: Transition table for Problem 2 with $d \in \Gamma$

[20] 3. Let D be a set of cards from a standard deck, 52 cards in all. Let C be a set of 50 pennies, each with a different year so they are easily distinguished.

Prove by contradiction that there is no injection

$$h : D \to C.$$

Hint: **Assume that such a injection h exists and derive a contradiction. Think about the Pigeonhole Principle.**

Assume that D, C are sets as described above. Also assume that there exists an injective function $h : D \to C$. Then from the definition of injective we get for all $x, y \in D$ with $x \neq y$ we have $h(x) \neq h(y)$ but as $|D| = 52$ and $|C| = 50$ we have by the Pigeonhole Principle that there exists two distinct elements $a, b \in D$ where $h(a) = h(b)$ this contradicts the assumption that h is injective hence we get that no such injective function exists.

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[30] 4. Let A be any nonempty set. Let $\mathcal{P}(A)$ be the power set of A, the set of all subsets of A; see Page 492.

Prove by contradiction (diagonalization) that there is no bijection

$$h : A \to \mathcal{P}(A).$$

Hint: **Assume that such a bijection h exists and derive a contradiction. Be careful with your argument.**

Let A be an arbitrary set and assume that such a bijection $h : A \to \mathcal{P}(A)$ exists. Then we create the set $\{x : x \notin h(x)\} \in \mathcal{P}(A)$. As h is a bijection we have that for some $a \in A$ that $h(a) = \{x : x \notin h(x)\}$. We have two cases $a \in h(a)$ or $a \notin h(a)$ if $a \in h(a)$ then $a \in \{x : x \notin h(x)\}$ but from the definition we get $a \notin h(a)$ this is a contradiction. Now in the other case if $a \notin h(a)$ then from the definition of the set we get $a \in h(a)$ again a contradiction. This implies that the assumption that there exists some $a \in A$ such that $h(a) = \{x : x \notin h(x)\}$ is not correct. Hence we have that h can not be surjective which is sufficient to prove it is not a bijection this contradicts the assumption that h is a bijection which implies there exists no bijection.
