Hand in Monday, April 26.

1. Let D be a disk with boundary circle C, and let $f : D \to \mathbb{R}^2$ be a continuous map. Suppose P is a point in $\mathbb{R}^2$ with $P \notin f(C)$, and the winding number of the restriction $f|_C$ of f to C around P is not zero. Show that there is some point Q in D such that $f(Q) = P$.

Proof. Suppose that $f : D \to \mathbb{R}^2$ where D has boundary circle C. Suppose that $P \in \mathbb{R}^2$ with $P \notin f(D)$. Choose any point $Q \in C$ and and create the function $g : C \times [0, 1] \to D$ by the equation $g(\vec{v}, t) = \vec{v}(1-t)+t\vec{c}$ this is a continuous function as it is a weighted sum of continuous functions. Now we have that $h : C \times [0, 1] \to \mathbb{R}^2$ defined by the equation $h(\vec{v}, t) = f(g(\vec{v}, t))$ is a homotopy between $f|_C$ and the constant curve at $\vec{c}$. This is shown to be a homotopy as $h(\vec{v}, 0) = f(g(\vec{v}, 0)) = f(\vec{v})$ and $h(\vec{v}, 1) = f(g(\vec{v}, 1)) = f(\vec{c})$. We have that h is continuous as it is the composition of two continuous functions. As the constant curve at $\vec{c}$ has winding number 0 we get that $W(f|_C, P) = 0$. This proves the contrapositive of the statement. □

2. If $f : C \to C$ is a continuous mapping with no fixed point, show that degree of f must be 1.

Proof. It suffices to assume that C is centered at the origin as we could create a homeomorphism that shifts it. Now assume that $f : C \to C$ is a continuous map with no fixed points. Then consider the homotopy $h : [0, 1] \times C \to C$ defined by the equation $h(t, \vec{c}) = (1-t)f(\vec{c}) + t(-\vec{c})$ this is continuous as it is two weighted continuous functions. As no point is fixed we get the line that intersects $f(\vec{c})$ and $g(\vec{c}) = -\vec{c}$ does not intersect $\vec{0}$. Lastly as $h(0, \vec{c}) = f(\vec{c})$ and $h(1, \vec{c}) = -\vec{c}$ we get that h is a homotopy.

On problem 2b on last weeks homework we showed that $\gamma(x, y) = (-x, -y)$ on the unit circle to the unit circle has degree 1. We have that g is homotopic to γ (changing radius of circle) so we get that g has degree 1. As f is homotopic to g we get that f has degree 1. □

3. If $f : D^2 \to \mathbb{R}^2$ is continuous and $f(P) \cdot P \neq 0$ for all P in S^1, show that there is some $Q \in D^2$ with $f(Q) = 0$.

Proof. Assume that $f : D^2 \to \mathbb{R}^2$ is continuous and $f(P) \cdot P \neq 0$ for all $P \in S^1$. Then as for all $\vec{c} \in C$ we can not have the line that intersects $f(\vec{c})$ and $g(\vec{c})$ intersects $\vec{0}$ which follows from the fact that $f(\vec{v}) \cdot \vec{v} \neq 0$ and $g(\vec{v}) \cdot \vec{0} = 0$ hence we get by the Dog-on-a-Leash theorem that $W(f|_{S^1}, \vec{0}) = W(g, \vec{0})$. As $W(g, \vec{0})$ has a non zero winding number then we get by problem 1 that there exists a point $Q \in D^2$ such that $f(Q) = 0$. □

4. Show that if $f : C \to C'$ is a map between circles such that $f(P^\star) = f(P)$ for all P, then the degree of f is even.

Proof. We have C, C' are both homeomorphic to S^1. So let $f : S^1 \to S^1$ such that $f(P^\star) = f(P)$ for all P.

Let $\gamma_1(\theta) = f(\cos\theta, \sin\theta)$ for $0 \leq \theta \leq \pi$ and $\gamma_2(\theta) = f(\cos\theta + \pi, \sin\theta + \pi)$ for $\pi \leq \theta \leq 2\pi$.

As $f(\cos\theta, \sin\theta) = f(\cos\theta+\pi, \sin\theta+\pi)$ we get both γ_1, γ_2 are closed path hence $W(\gamma_1, \vec{0}), W(\gamma_2, \vec{0})$ both have integer winding numbers. As $\gamma_1 = \gamma_2$ we get $W(\gamma_1, \vec{0}) = W(\gamma_2, \vec{0}) = n$ for some $n \in \mathbb{Z}$. Hence we have $W(f, \vec{0}) = W(\gamma_1, \vec{0}) + W(\gamma_2, \vec{0}) = 2n$. Hence the degree of f is even. □

5. If f and g are continuous real-valued functions on a sphere S such that that $f(P^\star) = -f(P)$ and $g(P^\star) = -g(P)$ for all P, show that f and g must have a common zero on the sphere.

Proof. Create the function $h : S \to \mathbb{R}^2$ where $h(\vec{x}) = (f(\vec{x}), g(\vec{x}))$ this is continuous by Munkres Theorem 18.4. Then we have by Borsuk-Ulam theorem that there exists a point such that $h(\vec{x}) = h(\vec{x}^\star)$ this implies $(f(\vec{x}), g(\vec{x})) = (-f(\vec{x}), -g(\vec{x}))$ hence we get $f(\vec{x}) = 0$ and $g(\vec{x}) = 0$. $\square$