Problem 1. For a fixed $a \in \mathbb{C}$, show that $\frac{|z-a|}{|1-\bar{a}z|} = 1$ if $|z| = 1$ and $1 - \bar{a}z \neq 0$.

Proof. Assume $a \in \mathbb{C}$ and $z \in \mathbb{Z}$ with $|z| = 1$ and $1 - \bar{a}z \neq 0$. Then for some $x, y, c, d \in \mathbb{R}$ we have $a = x + iy$ and $z = c + id$. We also have $|z| = \sqrt{c^2 + d^2} = 1 = c^2 + d^2$

Calculating $|1 - \bar{a}z|$ yields

$$\begin{aligned}
|1-\bar{a}z| &= |1-(x-iy)(c+id)| \\
|1-\bar{a}z| &= \sqrt{(1-xc-yd)^2 + (yc-xd)^2} \\
|1-\bar{a}z| &= \sqrt{1-2xc-2yd+2xcyd+x^2c^2+y^2d^2+y^2c^2-2ycxd+x^2d^2} \\
|1-\bar{a}z| &= \sqrt{1-2xc-2yd+x^2c^2+x^2d^2+y^2d^2+y^2c^2} \\
|1-\bar{a}z| &= \sqrt{1-2xc-2yd+x^2(c^2+d^2)+y^2(c^2+d^2)} \\
|1-\bar{a}z| &= \sqrt{1-2xc-2yd+x^2+y^2} \\
|1-\bar{a}z| &= \sqrt{c^2+d^2+x^2+y^2-2xc-2yd} \\
|1-\bar{a}z| &= \sqrt{(c-x)^2+(d-y)^2} \\
|1-\bar{a}z| &= |z-a|
\end{aligned}$$

Based on the assumption of $|1 - \bar{a}z| \neq 0$ we have

$$\frac{|z-a|}{|1-\bar{a}z|} = 1$$

□

Problem 2. For which n is i an nth root of unity?

For any positive integer n where $n \equiv 0 \bmod 4$.

Problem 3. If the point P on the sphere corresponds to z under stereographic projection, show that the antipodal point $-P$ on the sphere corresponds to $-\frac{1}{\bar{z}}$.

Proof. Assume that the point P corresponds to $z_0 = x_0 + iy_0$ under stereographic projection. Then solving for the point z from the projection $-P$ by using the equations on page 12

$$\begin{cases}
X = \frac{2x_0}{|z_0|^2+1} \\
\\
Y = \frac{2y_0}{|z_0|^2+1} \\
\\
Z = \frac{|z_0|^2-1}{|z_0|^2+1}
\end{cases}$$

substituting in for $z = -\frac{X}{1+Z} - i\frac{Y}{1+Z}$ we get

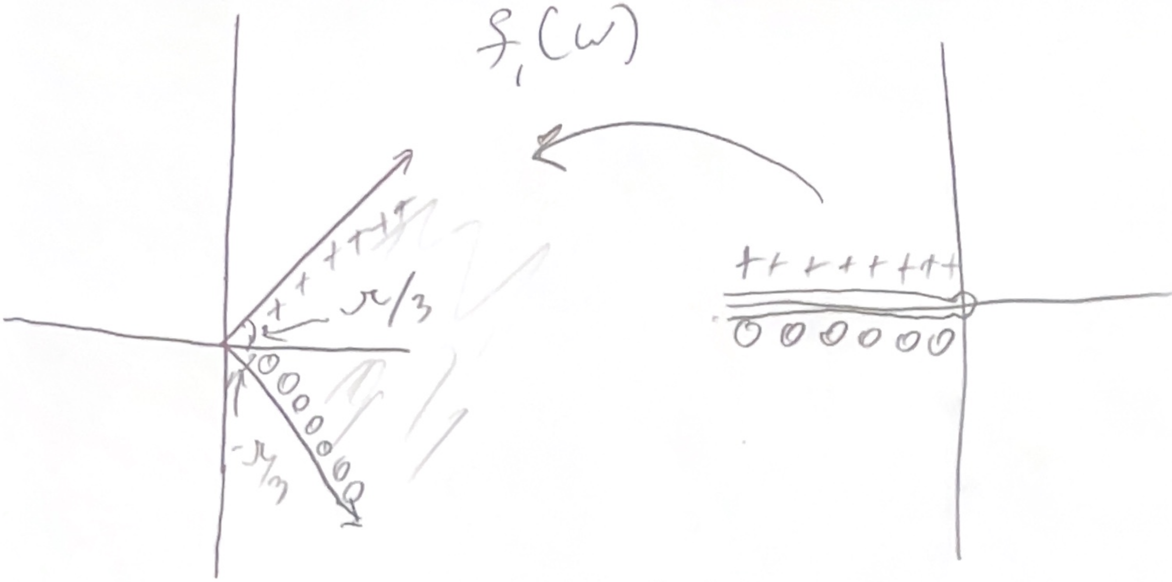
$$
\begin{aligned}
z &= -\frac{\frac{2x_0}{|z_0|^2+1}}{1+\frac{|z_0|^2-1}{|z_0|^2+1}} - i\frac{\frac{2y_0}{|z_0|^2+1}}{1+\frac{|z_0|^2-1}{|z_0|^2+1}} \\
z &= -\frac{\frac{2x_0+i2y_0}{|z_0|^2+1}}{1+\frac{|z_0|^2-1}{|z_0|^2+1}} \\
z &= -\frac{\frac{2x_0+i2y_0}{|z_0|^2+1}}{\frac{2|z_0|^2}{|z_0|^2+1}} \\
z &= -\frac{x_0+iy_0}{|z_0|^2} \\
z &= -\frac{x_0+iy_0}{|z_0|^2}\cdot\frac{\frac{1}{z_0}}{\frac{1}{z_0}} \\
z &= -\frac{1}{\bar{z}_0}
\end{aligned}
$$

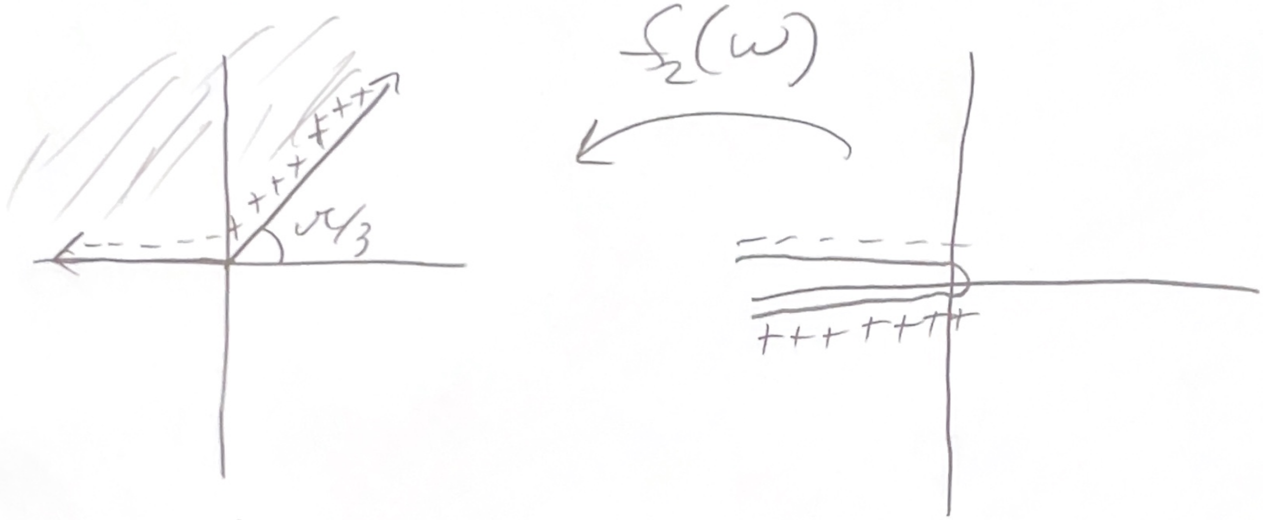
$\square$

Problem 4. (a) Give a brief description of the function $z \mapsto w = z^3$, considered as a mapping from the z-plane to the w-plane. (b) Make branch cuts and define explicitly three branches of the inverse mapping.

(a) As $z = \rho e^{i\theta_0}$ traces a ray from the origin then $w = \rho^3 e^{i3\theta}$ hence the angle is three times the angle of z while $|w| = |z|^3$. As z traverses a circle centred at the origin we have for every single loop z completes w completes 3 in the same direction and the radius of the circle in the w-plane is $|w| = |z|^3$.

(b) The branch cut is $\mathbb{C} \setminus (-\infty, 0]$ and the three branches of the inverse mapping are
$f_1(w) = |w|^{\frac{1}{3}} e^{i\frac{\text{Arg} w}{3}}, w \in \mathbb{C} \setminus (-\infty, 0]$
$f_2(w) = |w|^{\frac{1}{3}} e^{i(\frac{\text{Arg} w}{3}+\frac{2\pi}{3})}, w \in \mathbb{C} \setminus (-\infty, 0]$
$f_3(w) = |w|^{\frac{1}{3}} e^{i(\frac{\text{Arg} w}{3}+\frac{4\pi}{3})}, w \in \mathbb{C} \setminus (-\infty, 0]$.


$f_1(w)$
$\pi/3$
$-\pi/3$


$f_2(w)$
$\pi/3$


$f_3(w)$
$-\pi/3$