

**Hand in Friday, March 15.**

1. Suppose that  $X$  and  $Y$  are topological spaces, that  $A \subset X$ , and that  $f$  and  $g$  are continuous maps from  $X$  to  $Y$  that satisfy, for all  $x \in A$ ,  $f(x) = g(x)$ . If  $Y$  is Hausdorff, show that, for all  $z$  in the closure of  $A$ ,  $f(z) = g(z)$ .

*Proof.* Assume  $X$  and  $Y$  are topological spaces, that  $A \subset X$ , and that  $f$  and  $g$  are continuous maps from  $X$  to  $Y$  that satisfy, for all  $x \in A$ ,  $f(x) = g(x)$ . Assuming  $Y$  is Hausdorff and that there exists some point  $z$  in the closure of  $A$  where  $f(z) \neq g(z)$ . Then as  $Y$  is Hausdorff we have two neighborhoods  $U_f$  of  $f(z)$  and  $U_g$  of  $g(z)$  where  $U_f \cap U_g = \emptyset$ . As  $f, g$  are continuous we have that  $f^{-1}(U_f)$  and  $g^{-1}(U_g)$  are both open. We have that  $z \in f^{-1}(U_f) \cap g^{-1}(U_g)$  but we have that  $z$  is a limit point as if  $z \in A$  then that would be an immediate contradiction on  $f(z) \neq g(z)$  hence we have  $A \cap f^{-1}(U_f) \cap g^{-1}(U_g) \setminus \{z\} \neq \emptyset$  hence we have for some  $a \in A \cap f^{-1}(U_f) \cap g^{-1}(U_g) \setminus \{z\}$  then we have a neighborhood  $U_a$  of  $a$  with  $U_a \subset f^{-1}(U_f) \cap g^{-1}(U_g)$  hence we have  $f(a) \in U_f \cap U_g$  which contradicts  $U_f$  and  $U_g$  being disjoint.  $\square$

2. Let  $f : X_1 \rightarrow Y_1$  and  $g : X_2 \rightarrow Y_2$  be continuous maps between topological spaces. Give the products  $X_1 \times X_2$  and  $Y_1 \times Y_2$  their product topologies. Show that the map  $H : X_1 \times X_2 \rightarrow Y_1 \times Y_2$  defined by  $H((x_1, x_2)) = (f(x_1), g(x_2))$  is continuous.

3. An injective (one-to-one) continuous map  $f : X \rightarrow Y$  between topological spaces is a bijection from  $X$  to the image  $f(X)$ . Give  $f(X)$  the subspace topology induced by  $Y$ 's topology. We call such an  $f$  an imbedding if it is a homeomorphism from  $X$  to  $f(X)$ .

In this context, let  $Y$  be  $X \times X$  with the product topology. Let  $x_0$  be an arbitrary element of  $X$ .

a. Show that  $f : X \rightarrow X \times X$  defined by  $f(x) = (x, x_0)$  is an imbedding.

b. Show that  $g : X \rightarrow X \times X$  defined by  $g(x) = (x, x)$  is an imbedding.

4. Suppose that  $h : X \rightarrow Y$  is a homeomorphism of topological spaces. If  $Z$  is any other topological space and if  $g : Y \rightarrow Z$  is a continuous map, we know that the composition  $g \circ h$  is a continuous map from  $X$  to  $Z$ . Show that every continuous map  $f : X \rightarrow Z$  arises this way, i.e. for any continuous  $f : X \rightarrow Z$ , there exists a continuous  $G : Y \rightarrow Z$  for which  $f = G \circ h$ .

5. a. Show that a linearly ordered set with the order topology is Hausdorff.

b. Suppose that  $X$  is a topological space. Show that  $X$  is Hausdorff if and only if the diagonal subset  $\{(x, x) : x \in X\}$  of the product  $X \times X$  is a closed subset of the product. Assume here that the topology on  $X \times X$  is the product topology.

6. text p. 111-112, problem 8.