CS 4124
Solutions to Homework Assignment 4
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[30] 1. Let $\Sigma = \{0, 1\}$, the binary alphabet. Let

$$F \quad = \quad \{h \mid h : \Sigma^\star \to \Sigma^\star\},$$

the set of functions from strings to strings.

Prove that F is uncountable.

HINT: **Use the fact that $\Sigma^\star$ is countably infinite and employ a proof by diagonalization.**

My proof strategy:

I will prove that the set of functions $F' = \{h : h : \Sigma^\star \to \{0, 1\}\}$ is uncountable and as this set is a subset of F that would imply that F is uncountable. I will do this by constructing a bijection between $H : F' \to \mathcal{P}(\Sigma^\star)$ (the power set of $\Sigma^\star$) in the last homework we proved that no set can have a bijection with its power set and as $\Sigma^\star$ is countable we get that $\mathcal{P}(\Sigma^\star)$ is uncountable hence if a bijection H exists we would have that F' is uncountable.

Let $H : F' \to \mathcal{P}(\Sigma^\star)$ where $H(f) = \{w \in \Sigma^\star : f(w) = 1\}$ (informally this is the set of elements from $\Sigma^\star$ who get mapped to 1).

Now to show that this is a bijection I will come up with an inverse. We create the function

$$H^{-1} : \mathcal{P}(\Sigma^\star) \to F'$$

where for $A \in \mathcal{P}(\Sigma^\star)$ we have $H^{-1}(A) = f$ where for all $w \in \Sigma^\star$ we have $f(w) = 1$ if and only if $w \in A$. The fact that H^{-1} is not a multivalued function can be proven by assuming that for some $A \in \mathcal{P}(\Sigma^\star)$ that there exists two $f_1, f_2 \in F'$ such that $H^{-1}(A) = f_1$ and $H^{-1}(A) = f_2$ then we have that for all $w \in \Sigma^\star$ that $f_1(w) = 1$ and $f_2(w) = 1$ if and only if $w \in A$ which directly implies that for any $\omega \in \Sigma^\star$ where $\omega \notin A$ that $f_1(\omega) = 0$ and $f_2(\omega) = 0$. As f_1, f_2 are equal at every input we have that $f_1 = f_2$.

Now to show that let $f \in F'$ then we have $H(f) = \{w \in \Sigma^\star : f(w) = 1\}$. Then take $H^{-1}(\{w \in \Sigma^\star : f(w) = 1\}) = f_0$ for some $f_0 \in F'$. But from the definition of H^{-1} we get that $f_0(\omega) = 1$ if and only if $\omega \in \{w \in \Sigma^\star : f(w) = 1\}$ which is the same as $f(\omega) = 1$ so we have that $f_0 = f$. This implies that $H^{-1} \circ H = \mathrm{Id}_{F'}$. Now let $A \in \mathcal{P}(\Sigma^\star)$ then we have that $H^{-1}(A) = f$ for some $f \in F'$. Then taking $H(f) = \{w \in \Sigma^\star : f(w) = 1\}$ but from the definition of f we have that for all $\omega \in \Sigma^\star$ that $f(\omega) = 1$ if and only if $\omega \in A$. This implies that $H(f) = A$ so we have $H \circ H^{-1}(A) = A$ so we have that H is the inverse of H^{-1} so we have that H is a bijection.

Then as a bijection between $F' \subset F$ to $\mathcal{P}(\Sigma^\star)$ exists and $\mathcal{P}(\Sigma^\star)$ is uncountable we get F' is uncountable and so F is uncountable as well.

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[30] 2. Let

$$E \quad = \quad \{w \in \{0, 1\}^\star \mid \ell(w) \text{ is even}\},$$

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Program P_x
if x halts on input y
 then run program x on input y
   if if y is a has even length
    then accept
    else reject
else loop forever
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the language of binary strings of even length. See Figure 2 for a sample program that decides E, just for LaTeX formatting purposes. Let

$$L \quad = \quad \{x \mid \text{program } x \text{ decides } E\}$$

be the language of programs that decide E.

Prove that L is undecidable by using m-reducibility.

I wil use the M reduction $HP \leq_M L$. I will do this by creating the function sending $\langle x, y \rangle \rightarrow P_x$ where P_x is the program defined above.

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[40] 3. Let

$$\text{DOUBLE} \quad = \quad \{ww \mid w \in \{a, b\}^\star\}.$$

Let

$$W \quad = \quad \{x \mid \text{program } x \text{ decides DOUBLE}\}$$

be the language of programs that decide DOUBLE.

A. Is DOUBLE **decidable? Justify your answer.**

B. Is W decidable? Prove your answer.

A: Yes DOUBLE is decidable. We have created a turing machine in class that accepts this language hence such a program exists that decides it.
