

**Hand in Friday, February 16.**

1. In  $\mathbb{R}$  let  $A = \{1/k : k \in \mathbb{N}\}$ .

a. If  $\mathbb{R}$  has the standard topology, what is the boundary of  $A$ , and what is the set of limit points of  $A$ . Show that your answers are correct.

The boundary points of  $A$  are the set  $A \cup \{0\}$ . This is shown as for all points of  $A$  are in  $\partial A$  as for any  $1/n \in A$  any neighborhood  $O$  of  $1/n$  we have  $1/n \in \mathbb{R} \cap O$ . Now as  $O$  is open then there exists a basis element  $B$  such that  $x \in B$  and as this is the standard topology we have  $B = (a, b)$  for some  $a, b \in \mathbb{R}$  with  $a < x < b$ . We have  $(a + x)/2 \in O \cap A^c$  therefore  $A \subset \partial A$ . For any neighborhood  $O$  containing 0 that again as  $O$  is open there exists a basis element  $(a, b)$  with  $a, b \in \mathbb{R}$  and  $a < 0 < b$  and for some  $n \in \mathbb{N}$  we have  $1/n < b$  hence  $1/n \in O \cap A \neq \emptyset$  additionally we have for some  $c \in \mathbb{R}$  where  $c \notin A$  with  $0 < c < b$  that  $c \in O \cap A^c \neq \emptyset$ . Now assume  $x \in \mathbb{R}$  with  $x \notin A \cup \{0\}$ . Then if  $x < 0$  we immediately have a neighborhood  $(x - 1, 0)$  with  $(x - 1, 0) \cap A = \emptyset$  like wise if  $x > 0$  then we have the neighborhood  $(1, x + 1)$  with  $(1, x + 1) \cap A = \emptyset$ . If  $0 < x < 1$  we have for some  $n \in \mathbb{N}$  that  $1/(n + 1) < x < 1/n$  we have the neighborhood of  $x$   $(1/(n + 1), 1/n)$  with  $A \cap (1/(n + 1), 1/n) = \emptyset$ . This shows that  $A \cup \{0\} = \partial A$ .

We have that the limit point of  $A$  is  $\{0\}$ . This is shown as let  $O$  be a neighborhood of 0. Then as it is open we have a basis element  $(a, b) \subset O$  where  $a, b \in \mathbb{R}$  with  $a < 0 < b$ . We have that there exists  $n \in \mathbb{N}$  such that  $0 < 1/n < b$  therefore  $1/n \in O$  which shows that 0 is a limit point. We have no element of  $A$  is a limit point as for all  $1/n \in A$  we have a neighborhood  $(a, b)$  with  $a, b \in \mathbb{R}$  where  $1/n \in (a, b)$  but  $1/(n + 1), 1/(n - 1) \notin (a, b)$  with the condition  $n > 1$ . This shows that no element of  $A$  is a limit point. Lastly assume  $x \in \mathbb{R} \setminus (\{0\} \cup A)$  then if  $x < 0$  we have the neighborhood  $(x - 1, 0)$  with  $(x - 1, 0) \cap A = \emptyset$  and for  $x > 0$  we have the neighborhood  $(1, x + 1)$  with  $(1, x + 1) \cap A = \emptyset$ . If  $0 < x < 1$  then for some  $n \in \mathbb{N}$  we have  $(1/(n + 1), 1/n)$  with  $x \in (1/(n + 1), 1/n)$  but  $(1/(n + 1), 1/n) \cap A = \emptyset$ . This implies that the only limit point of  $A$  is  $\{0\}$ .

b. If  $\mathbb{R}$  has the upper limit topology, what is the boundary of  $A$ , and what is the set of limit points of  $A$ . Show that your answers are correct.

The boundary for  $\mathbb{R}$  is  $A$ .

Let  $1/n \in A$  then for any neighborhood  $O$  of  $1/n$  we have  $1/n \in O \cap A$  therefore  $O \cap A \neq \emptyset$ . Now as  $O$  is open we have a basis element with  $(a, 1/n] \subset O$  with  $a \in \mathbb{R}$  and  $a < 1/n$ . Then consider the intersection  $O \cap A^c$  as  $(a + \frac{1}{n})/2 \in (a, 1/n) \subset O$  and  $(a + \frac{1}{n})/2 \in A^c$  that  $O \cap A^c \neq \emptyset$ . This implies that  $A \subset \partial A$ . Now let  $x \in \mathbb{R}$  if  $x \leq 0$  then we have the neighborhood  $(x - 1, 0]$  and  $(x - 1, 0] \cap A = \emptyset$ . If  $x > 1$  then we have the neighborhood  $(1, x + 1]$  with  $(1, x + 1] \cap A = \emptyset$ . If  $0 < x < 1$  then we have the neighborhood  $(a, x]$  where  $a \in \mathbb{R}$  where  $(a, x] \cap A = \emptyset$  this  $a$  exists as for some  $n \in \mathbb{N}$  we have  $1/(n + 1) < a < x < 1/n$ . This shows  $A = \partial A$ .

We have that there are no limit points of  $A$ .

Let  $1/n \in A$  then we have the neighborhood  $(1/(n + 1), 1/n]$  then as  $(1/(n + 1), 1/n) \cap A = \{1/n\}$  we have no element of  $A$  is a limit point. Now suppose  $x \in \mathbb{R}$  if  $x \leq 0$  then we have the neighborhood  $(x - 1, 0]$  with  $(x - 1, 0] \cap A = \emptyset$  if  $x > 1$  then  $(1, x]$  is a neighborhood of  $x$  and  $(1, x] \cap A = \emptyset$ . If  $0 < x < 1$  then we have the neighborhood  $(a, x]$  where  $a \in \mathbb{R}$  and  $(a, x] \cap A = \emptyset$  such an  $a$  exists as for some  $n \in \mathbb{N}$  the inequality is true  $1/(n + 1) < a < x < 1/n$ . This implies that there are no limit points of  $A$  with the upper limit topology.

2. Let  $A$  and  $B$  be subsets of a topological space  $X$ .

a. If  $A \subset B$ , show that  $\overline{A} \subset \overline{B}$ .

*Proof.* Assume  $A$  and  $B$  are subsets of a topological space  $X$  with  $A \subset B$ . Then we have  $\overline{A} = A \cup A' \subset B \cup A'$ . Let  $x \in A'$  then for all neighborhoods  $O$  of  $x$  we have  $O \cap A \setminus \{x\} \neq \emptyset$  as  $A \subset B$  then  $O \cap B \setminus \{x\} \neq \emptyset$  hence  $x \in B'$ . This implies  $A' \subset B'$  so we have  $\overline{A} = A \cup A' \subset B \cup B' = \overline{B}$ .  $\square$

b. Show that  $\overline{A \cup B} = \overline{A} \cup \overline{B}$ .

*Proof.* As  $\overline{A \cup B} = A \cup B \cup (A \cup B)'$  we just need to show  $(A \cup B)' = A' \cup B'$ . Let  $x \in (A \cup B)'$  then for all neighborhoods  $O$  of  $x$  we have  $O \cap (A \cup B) \setminus \{x\} \neq \emptyset$ . Then

$$(1) \quad O \cap (A \cup B) \setminus \{x\} = (O \cap A) \cap \{x\}^c \cup (O \cap B) \cap \{x\}^c = (O \cap A \setminus \{x\}) \cup (O \cap B \setminus \{x\})$$

this shows that  $x \in A' \cup B'$  hence  $(A \cup B)' \subset A' \cup B'$ . Then for the other inclusion the equation (1) also holds hence  $A' \cup B' \subset (A \cup B)'$  which implies  $(A \cup B)' = A' \cup B'$ . Then we have  $\overline{A \cup B} = (A \cup B) \cup (A \cup B)' = A \cup B \cup A' \cup B' = (A \cup A') \cup (B \cup B') = \overline{A} \cup \overline{B}$   $\square$

**3. a.** Give an example of a topological space  $X$  and subsets  $A$  and  $B$  for which  $\overline{A \cap B} \neq \overline{A} \cap \overline{B}$ . Show that your example has the asserted property.

Let  $\mathbb{R}$  be a topological space with the standard topology. Let the two sets be the open interval  $(0, 1)$  and  $(1, 2)$  then we have  $(0, 1)' = [0, 1]$  as for any  $x \in (0, 1)$  any neighborhood  $O$  of  $x$  we have a basis element  $(a, b)$  where  $a, b \in \mathbb{R}$  with  $x \in (a, b) \subset U$  and as  $(a, b) \cap (0, 1) \setminus \{x\} \neq \emptyset$  then we have  $x$  is a limit point. Now to show that  $0 \in (0, 1)'$  we have for any neighborhood  $O$  of  $0$  that there exists a basis element  $(a, b)$  with  $a, b \in \mathbb{R}$  and  $a < 0 < b$  such that  $(a, b) \subset U$  then  $(a, b) \cap (0, 1) \setminus \{0\} \neq \emptyset$  as  $b/2 \in (0, 1)$ . To show  $1 \in (0, 1)'$  for any neighborhood  $O$  of  $1$  we have a element of the basis  $(a, b)$  where  $a, b \in \mathbb{R}$  with  $a < 1 < b$  and  $(a, b) \subset O$ . Then we have  $(a, b) \cap (0, 1) \neq \emptyset$  as there exists a real number  $c$  with  $a < c < 1$ . This shows that  $1 \in (0, 1)'$ . For any real number  $r \in \mathbb{R} \setminus [0, 1]$  if  $r < 0$  we have the neighborhood  $(r - 1, 0)$  and  $(r - 1, 0) \cap (0, 1) = \emptyset$  which shows that it is not a limit point. If  $r > 1$  then we have the neighborhood  $(1, r + 1)$  and  $(1, r + 1) \cap (0, 1) = \emptyset$ . This shows that the limit points of  $(0, 1)$  are  $[0, 1]$ . From the definition of closure we have  $\overline{(0, 1)} = (0, 1) \cup [0, 1] = [0, 1]$ . A similar argument gives that  $(1, 2)' = [1, 2]$ . Now as  $(0, 1) \cap (1, 2) = \emptyset$ . We have  $\overline{(0, 1) \cap (1, 2)} = \emptyset$  and  $\overline{(0, 1)} \cap \overline{(1, 2)} = [0, 1] \cap [1, 2] = \{1\}$  hence they are not equal.  $\square$

**b.** Give an example of a topological space  $X$  and a collection of subsets  $\{A_j : j \in \mathbb{N}\}$  for which

$$\bigcup_{j \in \mathbb{N}} \overline{A_j}$$

is not equal to

$$\overline{\bigcup_{j \in \mathbb{N}} A_j}.$$

Show that your example has the asserted property.

Claim that  $A_j = \{1/j\}$  has this property in the Real numbers with the standard topology.

*Proof.* From problem (1a) we have that the limit point of  $\bigcup_{j \in \mathbb{N}} A_j$  is  $0$ . By the definition of complement  $\overline{\bigcup_{j \in \mathbb{N}} A_j} = (\bigcup_{j \in \mathbb{N}} A_j) \cup (\bigcup_{j \in \mathbb{N}} A_j)' = \{1/n : n \in \mathbb{N}\} \cup \{0\}$ . But for each  $\overline{A_j} = A_j \cup A_j'$  we have  $A_j' = \emptyset$  which follows from (1a) therefore we have  $\overline{A_j} = A_j$  which implies  $\bigcup_{j \in \mathbb{N}} \overline{A_j} = \{1/n : n \in \mathbb{N}\}$  but this set does not contain  $0$  hence  $\overline{\bigcup_{j \in \mathbb{N}} A_j} \neq \bigcup_{j \in \mathbb{N}} \overline{A_j}$ .  $\square$

**4.** Show that if  $X$  is a Hausdorff topological space and  $Y$  is a finite subset of  $X$ , then  $Y$  has no limit points.

**5. a.** If  $X$  is an infinite set with the finite complement topology and if  $Y$  is a finite subset of  $X$ , what are the limit points of  $Y$ ? Show that your answer is correct.

Claim that the limit points of  $Y$  is the emptyset.

*Proof.* Assume that  $X$  is an infinite set with the finite complement topology and  $Y$  is a finite subset of  $X$ . Now let  $x \in X$  then we have that  $(X \setminus Y) \cup \{x\}$  is a neighborhood as it's complement is finite and contains  $x$  but as  $((X \setminus Y) \cup \{x\}) \cap Y = \{x\}$  we have  $x$  is not a limit point. As it was chosen arbitrarily we have  $Y' = \emptyset$   $\square$

**b.** If  $X$  is an infinite set with the finite complement topology and if  $Y$  is an infinite subset of  $X$ , what are the limit points of  $Y$ ? Show that your answer is correct.

Claim it is all of  $X$

*Proof.* Let  $X, Y$  be as defined. Then let  $x \in X$  we have that any neighborhood  $O$  of  $x$  contains an infinite number of elements of  $Y$  as if it only contained a finite number of elements of  $Y$  then  $X \setminus O$  would not be countable because it would at least contain an infinite number of elements of  $Y$ . Therefore we have  $O \cap Y \neq \emptyset$ . Hence  $x$  is a limit point and as  $x$  was arbitrarily chosen from  $X$  we have that  $Y' = X$ .  $\square$

**6. a.** Let  $X$  be the product of countably many copies of  $\mathbb{R}$ , i.e.  $X = \prod_{j \in \mathbb{N}} X_j$ , where each  $X_j$  is  $\mathbb{R}$  with the standard topology. Let  $Y$  be the subset of  $X$  of elements  $\vec{x} = (x_1, x_2, \dots)$  for which at most finitely many of the entries  $x_j$  are nonzero. If we give  $X$  the box topology, what is the closure of  $Y$ ? Show that your answer is correct.

Claim is that  $\bar{Y} = Y$

*Proof.* Let  $X, Y$  be as definition. From the definition of closure if we prove that  $Y' \subset Y$  then that proves the claim. Let  $\vec{x} \in X \setminus Y$  then consider the neighborhood  $O = \prod_{j \in \mathbb{N}} u_j$  where  $u_j = \begin{cases} (0, \pi_j(\vec{x}) + 1) & \text{if } \pi_j(\vec{x}) \neq 0 \\ (-1, 1) & \text{if } \pi_j(\vec{x}) = 0 \end{cases}$  we have  $O$  does not contain zero an infinite number of times based on  $\vec{x} \notin Y$ . But as each element of  $Y$  contains zero an infinite number of times we have  $O \cap Y = \emptyset$ . This implies that  $Y' \neq X \setminus Y$  which implies that  $Y' \subset Y$  so  $\bar{Y} = Y \cup Y' = Y$  which completes the proof.  $\square$

**b.** Let  $X$  be the product of countably many copies of  $\mathbb{R}$ , i.e.  $X = \prod_{j \in \mathbb{N}} X_j$ , where each  $X_j$  is  $\mathbb{R}$  with the standard topology. Let  $Y$  be the subset of  $X$  of elements  $\vec{x} = (x_1, x_2, \dots)$  for which at most finitely many of the entries  $x_j$  are nonzero. If we give  $X$  the product topology, what is the closure of  $Y$ ? Show that your answer is correct.

Claim that  $Y' = X$

*Proof.* Assume that  $X, Y$  are defined as above. Then let  $\vec{x} \in X$  then for any neighborhood  $U$  of  $x$  as this is the product topology for some  $N \in \mathbb{N}$  we have for all  $n \in \mathbb{N}$  with  $n > N$  that  $U_n = \mathbb{R}$ . Then we have the element  $\vec{y} \in Y$  where  $\pi_j(\vec{y}) = \pi_j(\vec{x})$  for  $j \in \mathbb{N}$  with  $j \leq N$  and zero for the rest. Then we have  $\vec{y} \in U \cap Y \neq \emptyset$ . This shows that  $Y' = X$  hence  $\bar{Y} = Y \cup Y' = X$ .  $\square$