CS 4124
Homework Assignment 6

Given: April 10, 2024 **Due:** April 26, 2024

General directions. The point value of each problem is shown in [ ]. Each solution must include all details and an explanation of why the given solution is correct. **In particular, write complete sentences. A correct answer without an explanation is worth no credit.** The completed assignment must be submitted on Canvas as a PDF by 5:00 PM on April 26, 2024. **No late homework will be accepted.**

Digital preparation of your solutions is mandatory. This includes digital preparation of any drawings; see syllabus concerning neat drawings included in LaTeX solutions. Use of LaTeX is required. Also, please include your name.

Use of LaTeX (required).

- Retrieve this LaTeX source file, named `homework6.tex`, from the course web site.
- Rename the file <*Your VT PID*>`_solvehw6.tex`, For example, for the instructor, the file name would be `heath_solvehw6.tex`.
- Use a **text editor** (such as `vi`, `emacs`, or `pico`) to accomplish the next three steps. Alternately, use Overleaf as your LaTeX platform.
- Uncomment the line

  `% \setboolean{solutions}{True}`

  in the document preamble by deleting the `%`.
- Find the line

  `\renewcommand{\author}{Lenwood S. Heath}`

  and replace the instructor's name with your name.
- Enter your solutions where you find the LaTeX comments

  `% PUT YOUR SOLUTION HERE`
- Generate a PDF and turn it in on Canvas by 5:00 PM on April 26, 2024.

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[20] 1. Textbook Problem 5(a) in B.15 on Page 538.
Prove that the following set (language) belongs to the class NP:
a. The set of binary representations of composite (i.e., nonprime) integers.

This belongs to **P** as given an integer n we can just check if it is divisible by any number greater then 1 and less then n. This can be done in polynomial time by checking $\{(i,j) : 2 \leq i,j \leq n-1\}$ just multiplying each of the entries and checking if it equals n. Will tell weather it is composite or not.

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[30] 2. Textbook Problem 12(a) in B.15 on Page 538.
Provide a rigorously analyzed answer to the following question:
a. What is the "size" (in the complexity-theoretic sense of the term) of a 3-CNF formula that has m clauses and n variables?

HINT: **How would you design a TM that could process *any* 3-CNF formula? The only "tough" issue is how to represent "names" (in this case, of the variables). You should use the most compact (to within constant factors) fixed-length encoding of "names". Use the binary alphabet $\Sigma = \{0, 1\}$ for the encoding.**

We need to a fixed-length encoding of the names. This can done by if we know that there is n variables then we can use $\lceil \log_2(n) \rceil$ bits to encode the name of the variable by using the binary representation of the variable number. As there is m clauses and each clause has 3 variables we can encode the entire formula in $3m\lceil \log_2(n) \rceil$ bits. Hence we can say that the size of the 3-CNF formula is $O(m \log(n))$.

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[50] 3. For each integer $n \geq 1$, let $S_n = \{1, 2, 3, \ldots, n\}$. A *family* of S_n is a set F of nonempty subsets of S_n; note that the empty set is not allowed to be an element of F. If $f \in F$ and $i \in f$, then i is a *representative* of f. A *representative set* R for F is a subset $R \subseteq S_n$ such that R contains at least one representative of each $f \in F$. The MINIMUM FAMILY REPRESENTATION problem is to find a minimum-size representative set R for F.

A formal statement of the MINIMUM FAMILY REPRESENTATION problem follows:

MINIMUM FAMILY REPRESENTATION

INSTANCE: Integer $n \geq 1$; F, a family of S_n.

SOLUTION: A subset $R \subseteq S_n$ that is a representative set of F and such that $|R|$ is minimum.

To obtain a decision problem, we add the parameter K to the instance and formulate the maximization problem in terms of a question involving K, as follows:

DECISION FAMILY REPRESENTATIVE

INSTANCE: Integer $n \geq 1$; F, a family of S_n; integer $K \geq 0$.

QUESTION: Is there representative set R of F such that $|R| \leq K$?

The resulting language is just

DFR $=$ $\{\rho(n, F, K) \mid F$ is a family of S_n with a representative set of size $\leq K\}$.

Prove that DFR **is NP-complete. Follow the NP-completeness Proof Paradigm from class. Hint: Use a polynomial-time reduction from** VERTEX-COVER**.**

For a graph $G = (V, E)$ denote the neighbors of a vertex $v \in V$ by the set

$$\mathcal{N}(v) := \{a \in V : a \text{ is neighbors with } v\} \cup \{v\}$$

. Then we have the following reduction from VERTEX-COVER $\leq_{poly}$ DFR which is given by the function $f : \Sigma^\star \to \Sigma^\star$. Given a graph $G = (V, E)$ with $|V| = n$ create an enumeration of the vertexes from $1, ..., n$. Then create the set of subsets of vertexes

$$U := \bigcup_{i=1}^{n} \{\mathcal{N}(v_i)\}$$

. Then let f be given by the equation $f(G, K) \to (|V|, U, K)$.

Now to show that this function can be computed in polynomial time. We can enumerate the neighbors of each vertex in $O(n^2)$ time hence we can create U in $O(n^2)$ time.

Now there are two cases if $(G, K) \in$ VERTEX-COVER then there exists a vertex cover of size K. Find one such vertex cover of size K (this can be done in nondeterministic polynomial time) and look at of the edges in the vertex cover. For each edge in the vertex cover choose a single vertex from the edge and add it to the set R. We have that $|R| = K$ and by the construction of U we have that R is a representative set of U. Hence we have that $f(G, K) \in$ DFR.
