Math 4324 Winding Number and Degree

Hand in Friday, April 19.

Definition. Let $f : S \to C$ be a continuous map from a circle in the plane to a circle in the plane. Define the **degree** of f to be the winding number of this map around the point $\vec{c}$ at the center of C. If you prefer to think of winding numbers in terms of continuous maps from intervals, give the name θ to a variable running through the interval $[0, 2\pi]$, let $\gamma : [0, 2\pi] \to S$ parametrize S by $\gamma(\theta) = (x_0 + r\cos\theta, y_0 + r\sin\theta)$ for appropriate x_0, y_0, and r, and define the degree of f to be the winding number of $f \circ \gamma$ around $\vec{c}$. [from the textbook by Fulton]

1. Show that, for an f as in the above definition, if f is not surjective, then the degree of f equals zero.

Proof. Using the parametrization $\gamma : [0, 2\pi] \to S$ by $\gamma(\theta) = (x_0 + r\cos\theta, y_0 + r\sin\theta)$ where (x_0, y_0) is the center of S and r is the radius of S. Then we have that f is a closed curve as

$$f \circ \gamma(0) = f(x_0 + r\cos 0, y_0 + r\sin 0) = f(x_0 + r\cos 2\pi, y_0 + r\sin 2\pi) = f \circ \gamma(2\pi)$$

Now let $\vec{p_1} \in C$ be a point that is not in the image of f. Then we have a point $\vec{p_2} \in C$ that is colinear with the line intersecting (x_0, y_0) and $\vec{p_1}$.

We have the constant curve $g : S \to C$ given by the equation $g(\vec{x}) = \vec{p_2}$ for all $\vec{x} \in S$.

Then we create the homotopy $H : [0, 2\pi] \times [0, 1] \to \mathbb{R}^2 \setminus \{(x_0, y_0)\}$ given by

$$H(\theta, s) = f(\gamma(\theta))(1 - s) + s \cdot \vec{p_2}$$

for all $\theta \in [0, 2\pi]$ and $s \in [0, 1]$.

We have $H(\theta, 0) = f(\gamma(\theta)) + 0 \cdot \vec{p_2} = f(\gamma(\theta))$, and $H(\theta, 1) = f(\gamma(\theta)) \cdot 0 + 1 \cdot \vec{p_2} = g(\gamma(\theta))$.

We have that H is continuous as it is the sum of two weighted continuous functions.

Additionally we have that the image of H is contained in $\mathbb{R}^2 \setminus \{(x_0, y_0)\}$ as for any $\theta \in [0, 2\pi]$ we have for all $s \in [0, 1]$ that $H(\theta, s) \neq (x_0, y_0)$ as the only point colinear with (x_0, y_0) and $\vec{p_2}$ is $\vec{p_1}$ and by our assumption that $\vec{p_1}$ is not in the image of f. Then we have that H is a homotopy between f and g.

Then we have that f and g are homotopic and thus have the same winding number. We have that the winding number of g is zero as it is a constant curve. Then we have that the winding number of f is zero. □

2. Calculate the degree of each of the following maps from the unit circle centered at the origin to the unit circle centered at the origin.

a. $f(x, y) = (x, y)$ Using the parametrization $\gamma(\theta) = (\cos\theta, \sin\theta)$ for $\theta \in [0, 2\pi]$.
I will be using the three sectors

$$U_1 = \{(x, y) : 0 < \text{angle in polar}(x, y) < 3\pi/2\}$$
$$U_2 = \{(x, y) : \pi/2 < \text{angle in polar}(x, y) < 2\pi\}$$
$$U_3 = \{(x, y) : \pi < \text{angle in polar}(x, y) < 5\pi/2\}$$

With the following four subdivisions $t_0 = 0, t_1 = \pi/2, t_2 = \pi, t_3 = 2\pi$.
Each angle function θ_i just gives the angle in polar coordinates.
Then

$$W(f, \vec{0}) = \frac{1}{2\pi}\big(\theta_1(\gamma(t_1)) - \theta_1(\gamma(t_0)) + \theta_2(\gamma(t_2)) - \theta_2(\gamma(t_1)) + \theta_3(\gamma(t_3)) - \theta_3(\gamma(t_2))\big)$$

We have that for each angle function θ_i that $\theta_i(f(\gamma(t_i))) = \theta_i(\gamma(t_i)) = t_i$.
After canceling terms in the equation we get $W(f, \vec{0}) = \frac{1}{2\pi}\big(2\pi\big) = 1$

b. $g(x, y) = (-x, -y)$
Using the same parametrization as before. With the sectors

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$$U_2 = \{(x, y) : \pi/2 < \text{angle in polar}(x, y) < 2\pi\}$$

$$U_3 = \{(x, y) : \pi < \text{angle in polar}(x, y) < 5\pi/2\}$$

$$W(g, \vec{0}) = \frac{1}{2\pi}\big(\theta_3(g(\gamma(t_3)) - \theta_1(g(\gamma(t_0))))\big)$$

c. $h(x, y) = (x, -y)$

d. $k(\cos(\theta), \sin(\theta)) = (\cos(n\theta), \sin(n\theta))$, where n is an arbitrary integer

Definition. If Y is a topological subspace of a topological space X, a **retraction** from X to Y is a continuous map $r : X \to Y$ that satisfies, for all $y \in Y$, $r(y) = y$. When such a retraction exists, we call Y a **retract** of X. [from the textbook by Fulton]

3. Show that, if Y is a retract of X and if every continuous map from X to X has a fixed point, then every continuous map from Y to Y has a fixed point. **Hint.** Start with an arbitrary continuous map $f : Y \to Y$. How can you make a continuous map $g : X \to X$ whose behavior has the needed implications for f's behavior?

4. Let B be the open unit disk in $\mathbb{R}^2$ and let D be the closed unit disk in $\mathbb{R}^2$. Show that, for any $\vec{p} \in B$, the unit circle C in $\mathbb{R}^2$ is a retract of $D \setminus \{\vec{p}\}$. **Hint.** When $\vec{p}$ is the origin, the map $\vec{x} \mapsto \frac{\vec{x}}{|\vec{x}|}$ is the retraction. When $\vec{p}$ is more general, consider solving $|\vec{p} + t(\vec{x} - \vec{p})| = 1$ for t.

5. Let S and C be circles in the plane, and let $f : S \to C$ be a continuous map. Show that, for every $\vec{p}$ in the open disk bounded by C, the winding number of f around $\vec{p}$ equals the degree of f. (In particular the winding number is the same, regardless of which $\vec{p}$ in the open disk is used.)