Define explicitly a continuous branch of $\log z$ in the complex plane slit along the negative imaginary axis, $\mathbb{C} \setminus [0, -i\infty)$.

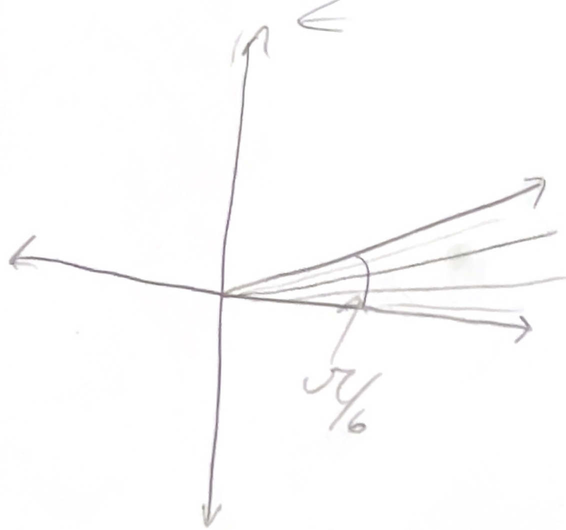
This branch is given by $f(re^i\theta) = \log r + i\theta$ where $\theta \in (-\frac{\pi}{2}, \frac{3\pi}{2})$.

Sketch the image of the sector $\{0 < \arg z < \frac{\pi}{6}\}$ under the map $w = z^a$.

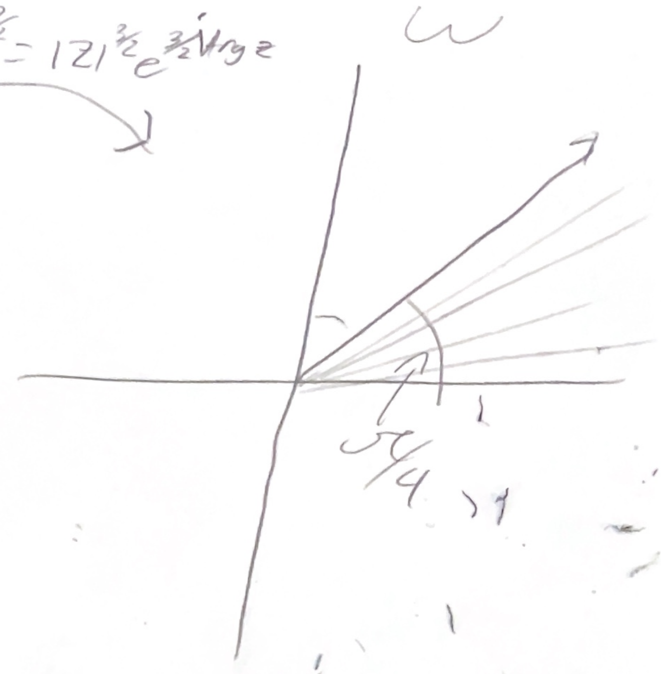
- $a = \frac{3}{2}$
- $a = i$

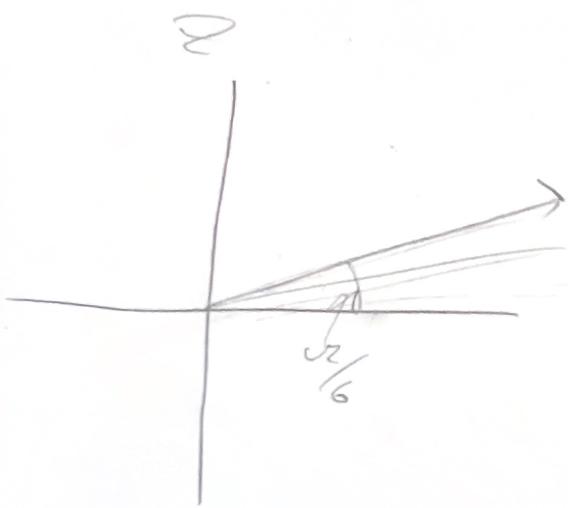
$$w = z^{\frac{3}{2}} = e^{\frac{3}{2}(\log|z| + i\mathrm{Arg}z)} = |z|^{\frac{3}{2}} e^{\frac{3}{2}i\mathrm{Arg}z}$$

$$w = z^i = e^{i\log z} = e^{i(\log|z| + i\mathrm{Arg}z)} = e^{-\mathrm{Arg}z + i\log|z|}$$

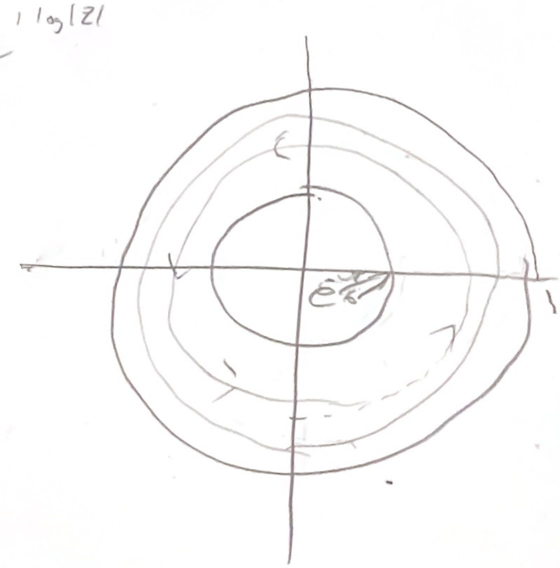


$$w = z^{3/2} = |z|^{3/2} e^{\frac{3}{2} i \operatorname{Arg} z}$$





$$w = e^{-\operatorname{Arg} z} e^{i \log|z|}$$



Determine the phase factors of the function $z^a(1-z)^b$ at the branch points $z = 0$ and $z = 1$. What conditions on a and b are necessary for the function to be single-valued on $\mathbb{C} \setminus [0, 1]$?

The phase factor at the branch point $z = 0$ is given by $e^{2\pi ia}$ and the phase factor at the branch point $z = 1$ is given by $e^{2\pi ib}$. The function is single-valued on $\mathbb{C} \setminus [0, 1]$ if we have $e^{i\pi a}e^{i\pi b} = 1$ which happens when $a + b$ is an integer.

Show that if f is analytic on D then $g(z) = \overline{f(\overline{z})}$ is analytic on the reflected domain $D^\star = \{\overline{z} : z \in D\}$, and $g'(z) = \overline{f'(\overline{z})}$.

Proof. Suppose f is analytic on D. Then for any $z_0 \in D^\star$ we have

$$\begin{aligned}
\lim_{\Delta z \mapsto 0} \frac{g(z_0 + \Delta z) - g(z_0)}{\Delta z} &= \lim_{\Delta z \mapsto 0} \frac{\overline{f(\overline{z_0 + \Delta z}) - \overline{f(\overline{z_0})}}}{\Delta z} \\
&= \lim_{\Delta z \mapsto 0} \overline{\frac{f(\overline{z}_0 + \overline{\Delta z}) - f(\overline{z}_0)}{\overline{\Delta z}}} \\
&= \overline{\lim_{\Delta z \mapsto 0} \frac{f(\overline{z}_0 + \overline{\Delta z}) - f(\overline{z}_0)}{\overline{\Delta z}}} \\
&= \overline{f'(\overline{z}_0)}
\end{aligned}$$

This shows that $g'(z_0)$ and $g'(z_0) = \overline{f'(\overline{z}_0)}$. Now to show that g is analytic on $D^\star$. As f is analytic on D we have for any $\overline{z} \in D^\star$ for any $\epsilon > 0$ there exists $\delta > 0$ such that for all $\overline{z_0} \in D^\star$ with $|\overline{z} - \overline{z}_0| < \delta$ implies $|g'(\overline{z}) - g'(\overline{z}_0)| = |\overline{f'(z)} - \overline{f'(z_0)}| = |f'(z) - f'(z_0)| < \epsilon$. $\square$

Let $h(t)$ be a continuous function on $[0, 1]$ and define

$$H(z) = \int_0^1 \frac{h(t)}{t - z} dt, \quad z \in \mathbb{C} \setminus [0, 1]$$

Show that $H(z)$ is analytic and compute its derivative.

Proof. Let $H(z)$ be as defined let $z \in \mathbb{C} \setminus [0, 1]$ then

$$\begin{aligned}
\lim_{\Delta z \mapsto 0} \frac{H(z+\Delta z) - H(z)}{\Delta z} &= \lim_{\Delta z \mapsto 0} \frac{1}{\Delta z} \left(\int_0^1 \frac{h(t)}{t-(z+\Delta z)} dt - \int_0^1 \frac{h(t)}{t-z} dt \right) \\
&= \lim_{\Delta z \mapsto 0} \frac{1}{\Delta z} \int_0^1 \frac{h(t)}{t-z-\Delta z} - \frac{h(t)}{t-z} dt \\
&= \lim_{\Delta z \mapsto 0} \int_0^1 \frac{h(t)}{(t-z-\Delta z)(t-z)} dt \\
&= \int_0^1 \lim_{\Delta z \mapsto 0} \frac{h(t)}{(t-z-\Delta z)(t-z)} dt \\
&= \int_0^1 \frac{h(t)}{(t-z)^2} dt
\end{aligned}$$

$\square$