

**Problem 1.** Consider  $V = \mathbb{C}$ , the complex numbers, as a  $\mathbb{C}$ -vector space. Define a function  $\Re : \mathbb{C} \rightarrow \mathbb{C}$  by

$$\Re(x + iy) = x$$

Is  $\Re$  a linear map? If so, prove it. If not, explain why not.

No it is not

*Proof.* No as it does not satisfy homogeneity. Consider  $(1 + i) \cdot \Re(1 + i) = 1 + i$  but  $\Re((1 + i) \cdot (1 + i)) = \Re(2i) = 0$  so it does not satisfy homogeneity.

□

**Problem 2. Extending a linear map.** Let  $V$  be a finite-dimensional  $\mathbb{K}$ -vector space,  $W$  a  $\mathbb{K}$ -vector space, and  $U$  a subspace of  $V$ . Furthermore, let

$$\{\mathbf{u}_1, \dots, \mathbf{u}_m\} \text{ be a basis for } U \text{ and} \\ \{\mathbf{u}_1, \dots, \mathbf{u}_m, \mathbf{v}_{m+1}, \dots, \mathbf{v}_n\} \text{ be the (extended) basis for } V.$$

Taking  $T : U \rightarrow W$  to be any linear map, define a function  $f : V \rightarrow W$  by

$$f(a_1 \mathbf{u}_1 + \dots + a_m \mathbf{u}_m + a_{m+1} \mathbf{v}_{m+1} + \dots + a_n \mathbf{v}_n) = T(a_1 \mathbf{u}_1 + \dots + a_m \mathbf{u}_m).$$

- (a) Is  $f$  a linear map? If so, prove it. If not, explain why not.  
 (b) What happens if the definition of  $f$  is changed to the following?

$$f(\mathbf{x}) = \begin{cases} T(\mathbf{x}) & \text{if } \mathbf{x} \in U \\ \mathbf{0} & \text{otherwise} \end{cases}$$

- (a) *Proof.* Yes it is a linear map

Let  $\lambda \in \mathbb{K}$  and let  $\vec{v} = \mathbf{a}_1 \mathbf{u}_1 + \dots + \mathbf{a}_m \mathbf{u}_m + \mathbf{a}_{m+1} \mathbf{v}_{m+1} + \dots + \mathbf{a}_n \mathbf{v}_n \in V$  and  $u = \mathbf{b}_1 \mathbf{u}_1 + \dots + \mathbf{b}_m \mathbf{u}_m + \mathbf{b}_{m+1} \mathbf{v}_{m+1} + \dots + \mathbf{b}_n \mathbf{v}_n \in V$ . Where  $u_i, w_i \in \text{Span}(U)$  and  $a_i \in \mathbb{K}$

$$\begin{aligned} f(\lambda \vec{v} + \vec{u}) &= T(\lambda(a_1 u_1 + \dots + a_m u_m) + b_1 w_1 + \dots + b_m w_m) \\ &= T(\lambda(a_1 u_1 + \dots + a_m u_m)) + T(b_1 w_1 + \dots + b_m w_m) \\ &= \lambda T(a_1 u_1 + \dots + a_m u_m) + T(b_1 w_1 + \dots + b_m w_m) \\ &= \lambda f(\vec{v}) + f(\vec{u}) \end{aligned}$$

□

- (b) *Proof.* Then it is no longer a linear map. Let  $x \in U$  and  $y \in V$  where  $y \notin U$ . Then  $f(x + y) = 0$  but  $f(x) + f(y) = T(x) + 0$  hence it is not linear. □

**Problem 3.** Prove that the following is a subspace of  $\mathcal{L}(\mathbb{K}^2)$ :

$$U = \left\{ f \in \mathcal{L}(\mathbb{K}^2) : \begin{array}{l} a, b, c \in \mathbb{K} \text{ and} \\ f(x, y) = (ax + by, bx + cy) \end{array} \right\}$$

*Proof.* We have the function  $f(x, y) = (0, 0) \in U$  as we can just let  $a = b = c = 0$  then  $f(x, y) = (0x + 0y, 0x + 0y) = (0, 0)$ . Now suppose that we have two different vectors  $f_1, f_2$  of  $U$  where  $f_1(x, y) = (a_1x + b_1y, b_1x + c_1y)$  and  $f_2(x, y) = (a_2x + b_2y, b_2x + c_2y)$  where  $a_i, b_i \in \mathbb{K}$ . Then  $f_1(x, y) + f_2(x, y) = (a_1x + b_1y, b_1x + c_1y) + (a_2x + b_2y, b_2x + c_2y) = ((a_1 + a_2)x + (b_1 + b_2)y, (b_1 + b_2)x + (c_1 + c_2)y)$  now as  $(a_1 + a_2), (b_1 + b_2), (c_1 + c_2) \in \mathbb{K}$  we get  $f_1 + f_2 \in U$ . Now let  $\lambda \in \mathbb{K}$  then  $\lambda f_1(x, y) = \lambda(a_1x + b_1y, b_1x + c_1y) = (\lambda a_1x + \lambda b_1y, \lambda b_1x + \lambda c_1y)$  and as  $\lambda a_1, \lambda b_1, \lambda c_1 \in \mathbb{K}$  we have  $\lambda f_1 \in U$ . Therefore it satisfies the three step subspace test hence it is a subspace. □

**Problem 4.** For any linear map  $T \in \mathcal{L}(V)$ , we say that a subspace  $U \subseteq V$  is an **invariant subspace of  $T$**  if and only if  $T(U) \subseteq U$ .

Let  $T : \mathbb{R}^3 \rightarrow \mathbb{R}^3$  be the linear map given by

$$T\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} 7x - 3y + 5z \\ 12x - 4y + 12z \\ -x + y + z \end{bmatrix}.$$

Show that each of the following subspaces of  $\mathbb{R}^3$  are invariant subspaces of  $T$ .

$$U_1 = \text{Span}\left(\begin{bmatrix} -2 \\ -3 \\ 1 \end{bmatrix}\right) \quad \text{and} \quad U_2 = \text{Span}\left(\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}\right)$$

*Proof.*

□

**Problem 5.** Let  $U_1, U_2$  be the subspaces from Problem 4. Prove that

$$\mathbb{R}^3 = U_1 \oplus U_2.$$

*Proof.*

□