1.

Proof. Suppose X is a compact topological space and $f : X \to \mathbb{R}$ is as described. As X is compact we have the open cover $\{U_x\}_{x\in X}$ where for each $U_x \in \{U_x\}_{x\in X}$ we have that there exists a constant M_x such that for all $z \in U_x$ that $|f(z)| \leq M_x$.

Then as X is compact we have that the collection has a finite subcover $\{U_{x_1}, ..., U_{x_n}\}$. Each of these in the finite subcover has a corresponding constant $\{M_{x_1}, ..., M_{x_n}\}$ with the property that for all $z \in U_{x_i}$ that $|f(z)| \leq M_{x_i}$ where $i = 1, ..., n$. Let $M = \max\{M_{x_1}, ..., M_{x_n}\}$ then for any $z \in X$ we have that $z \in U_{x_i}$ for some $i = 1, ..., n$. Hence $|f(z)| \leq M_{x_i} \leq M$ so this choice of M works for all $z \in X$. □

2.

Proof. Suppose that X is a topological space C is a connected subset of X and C_α is a connected subset of X. With for all αA that $C_\alpha \bigcap C \neq \emptyset$. Suppose that $C \bigcup \big(\cup_{\alpha\in A} C_\alpha\big)$ is not connected. Then there exists a separation $U \cup V$ of $C \bigcup \big(\cup_{\alpha\in A} C_\alpha\big)$.

Then as C is connected we have either $C \subset U$ with $C \cap V = \emptyset$ or $C \subset V$ with $C \cap U = \emptyset$. If C wasn't fully contained in only one then $(C \cap U)$, $(C \cap V)$ would be a separation of C but C is connected. For each $\alpha \in A$ we get that C_α is contained in exactly one U or V.

WLOG suppose $C \cup U$ then as $C \cap C_\alpha \neq \emptyset$ we get that $C_\alpha \subset U$. Hence we get $C \bigcup \big(\cup_{\alpha\in A} C_\alpha\big) \subset U$ and $\big(C \bigcup \big(U_{\alpha\in A} C_\alpha\big)\big) \bigcap V = \emptyset$ hence $V = \emptyset$ so this separation can not exist. So we have $C \bigcup \big(\cup_{\alpha\in A} C_\alpha\big)$ is connected. □

3.

(a) We have that a set is closed if and only if it contains it's limit points. As any singleton in $\{1/n : n \in \mathbb{N}\}$ is ope. Which is shown as we have for any $1/n \in \{1/n : n \in \mathbb{N}\}$ we have that $(1/n - \epsilon, 1/n + \epsilon)$ for $\epsilon > 0$ is open in $\mathbb{R}$ choosing a sufficiently small $\epsilon > 0$ we get $(1/n-\epsilon, 1/n+\epsilon) \cap \{1/n : n \in \mathbb{N}\} = \{1/n\}$. As the choice of $1/n$ was arbitrary we get that all singletons are open.

Hence any $B \subset \{1/n : n \in \mathbb{N}\}$ we get that $B' = \emptyset$ as for any $b \in \{1/n : n \in \mathbb{N}\}$ we have the neighborhood $\{b\}$ that doesn't intersect B at any place other then possibly itself. Hence B is closed as B is arbitrary we have that all subsets are closed.

(b) We have that all singletons other then $\{0\}$ are open using the same reasoning as above. Hence $\{1/n : n \in \mathbb{N}\}$ is open so we get $\{1/n : n \in \mathbb{N}\}^c = \{0\}$ is closed. We have that all singletons are closed as any singleton other than $\{0\}$ don't have any limit points. As finite unions of closed sets are closed then all finite subsets are closed.

Lastly any infinite set containing 0 is closed. This is shown as let $B \subset \{0\} \cup \{1/n : n \in \mathbb{N}\}$ be an infinite set containing 0. Then we have B^c is a union of singletons each of which are open hence B^c is open so B is closed.

Any infinite set not containing 0 is not closed as 0 is a limit point of said set.

4.

Proof.
$(\to)$

Assume that f is continuous and that there exists a $m, n \in \mathbb{N}$ with $m < n$ where $f(m) > f(n)$. Choose the neighborhood $U_{f(n)} = \{1, ..., f(n)\}$ of $f(n)$. Then we have that there exists a neighborhood U_n of n where $f(U_n) \subset U_{f(n)}$. From the definition of the open sets we get that $m \in U_n$ but as $f(m) > f(n)$ we have $f(m) \in f(U_n)$ but $f(m) \notin U_{f(n)}$. This is a contradiction on f being continuous hence $f(m) \leq f(n)$

$(\leftarrow)$

Let $n \in \mathbb{N}$ then take an arbitrary neighborhood $U_{f(n)}$ of $f(n)$. Then we have $U_n = \{1, ..., n\}$ that $f(U_n) \subset U_{f(n)}$ this follows as $f(n) \in f(U_n)$ and $f(n) \in U_{f(n)}$ and for any $m \in U_n$ with $m < n$ we have that $f(m) \leq f(n)$ hence $f(m) \in U_{f(n)}$. This is one of the equivalent definitions of continuity. Hence f is continuous. □

5.

Proof. Let X be a first countable topological space. Let $A \subset X$ and $c \in \overline{A}$. If $c \in A$ a constant sequence works hence assume that $c \in A'$ but $c \notin A$. We have the collection of neighborhoods $\{U_{c_i}\}_{i\in\mathbb{N}}$ of c where any neighborhood V of c that for some $i \in \mathbb{N}$ that $U_{c_i} \subset V$.

Create the sequence where the n'th element is taken from $A \bigcap \big(\cap_{i=1}^{n} U_{c_i} \big)$. First to show that $A \bigcap \big(\cap_{i=1}^{n} U_{c_i} \big) \neq \emptyset$ for any $n \in \mathbb{N}$. As $\bigcap_{i=1}^{n} U_{c_i}$ is a finite intersection of nonempty open sets (c is in each) we have it is nonempty and open. As $c \in A'$ then it intersects A at some place other then itself. Hence not empty.

Create the sequence (a_n) where $a_n \in A \bigcap \big(\cap_{i=1}^{n} U_{c_i} \big)$.

Now to show that $a_n \to c$.

Let U_c be an arbitrary neighborhood of c then for some $i \in \mathbb{N}$ we have $U_{c_i} \subset U$ from the definition of (a_n) we get that for all $j \geq i$ where $j \in \mathbb{N}$ that $a_j \in U_{c_i} \subset U$ which follows from the intersection in the construction of (a_n). Hence $a_n \to c$. $\square$