Problem 1. Consider $V = \mathbb{C}$, the complex numbers, as a $\mathbb{C}$-vector space. Define a function $\Re : \mathbb{C} \to \mathbb{C}$ by

$$\Re(x + iy) = x$$

Is $\Re$ a linear map? If so, prove it. If not, explain why not.

No it is not

Proof. No as it does not satisfy homogeneity. Consider $(1 + i) \cdot \Re(1 + i) = 1 + i$ but $\Re((1 + i) \cdot (1 + i)) = \Re(2i) = 0$ so it does not satisfy homogeneity. $\square$

Problem 2. Extending a linear map. Let V be a finite-dimensional $\mathbb{K}$-vector space, W a $\mathbb{K}$-vector space, and U asubspace of V. Furthermore, let

$$\{\mathbf{u_1}, \ldots, \mathbf{u_m}\} \text{ be a basis for } U \text{ and}$$
$$\{\mathbf{u_1}, \ldots, \mathbf{u_m}, \mathbf{v_{m+1}}, \ldots, \mathbf{v_n}\} \text{ be the (extended) basis for } V.$$

Taking $T : U \to W$ to be any linear map, define a function $f : V \to W$ by

$$f\big(a_1\mathbf{u_1} + \cdots + a_m\mathbf{u_m} + a_{m+1}\mathbf{v_{m+1}} + \cdots + a_n\mathbf{v_n}\big) = T\big(a_1\mathbf{u_1} + \cdots + a_m\mathbf{u_m}\big).$$

(a) Is f a linear map? If so, prove it. If not, explain why not.

(b) What happens if the definition of f is changed to the following?

$$f(\mathbf{x}) = \begin{cases} T(\mathbf{x}) & \text{if } \mathbf{x} \in U \\ \mathbf{0} & \text{otherwise} \end{cases}$$

(a) *Proof.* Let $\lambda \in \mathbb{K}$ and $a_1\mathbf{u_1} + a_2\mathbf{u_2} + ... + a_m\mathbf{u_m}$ □

(b)

Problem 3. Prove that the following is a subspace of $\mathcal{L}(\mathbb{K}^2)$:

$$U = \left\{ f \in \mathcal{L}(\mathbb{K}^2) \; : \; \begin{array}{l} a, b, c \in \mathbb{K} \text{ and} \\ f(x,y) = (ax+by, bx+cy) \end{array} \right\}$$

Proof. $\square$

Problem 4. For any linear map $T \in \mathcal{L}(V)$, we say that a subspace $U \subseteq V$ is an **invariant subspace of $\boldsymbol{T}$** if and only if $T(U) \subseteq U$.

Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be the linear map given by

$$T\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} 7x - 3y + 5z \\ 12x - 4y + 12z \\ -x + y + z \end{bmatrix}.$$

Show that each of the following subspaces of $\mathbb{R}^3$ are invariant subspaces of T.

$$U_1 = \operatorname{Span}\left(\begin{bmatrix} -2 \\ -3 \\ 1 \end{bmatrix}\right) \qquad \text{and} \qquad U_2 = \operatorname{Span}\left(\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}\right)$$

Proof. $\square$

Problem 5. Let U_1, U_2 be the subspaces from Problem 4. Prove that

$$\mathbb{R}^3 = U_1 \oplus U_2.$$

Proof. $\square$