Math 4324 Continuous maps

Hand in Friday, March 15.

1. Suppose that X and Y are topological spaces, that $A \subset X$, and that f and g are continuous maps from X to Y that satisfy, for all $x \in A$, $f(x) = g(x)$. If Y is Hausdorff, show that, for all z in the closure of A, $f(z) = g(z)$.

Proof. Assume X and Y are topological spaces, that $A \subset X$, and that f and g are continuous maps from X to Y that satisfy, for all $x \in A$, $f(x) = g(x)$. Assuming Y is Hausdorff and that there there exists some point z in the closure of A where $f(z) \neq g(z)$. Then as Y is Hausdorff we have two neighborhoods U_f of $f(z)$ and U_g of $g(z)$ where $U_f \cap U_g = \emptyset$. As f, g are continuous we have that $f^{-1}(U_f)$ and $g^{-1}(U_g)$ are both open. We have that $z \in f^{-1}(U_f) \cap g^{-1}(U_g)$ but we have that z is a limit point as if $z \in A$ then that would be an immediate contraction on $f(z) \neq g(z)$ hence we have $A \cap f^{-1}(U_f) \cap g^{-1}(U_g) \setminus \{z\} \neq \emptyset$ hence we have for some $a \in A \cap f^{-1}(U_f) \cap g^{-1}(U_g) \setminus \{z\}$ then we have a neighborhood U_a of a with $U_a \subset f^{-1}(U_f) \cap g^{-1}(U_g)$ hence we have $f(a) \in U_f \cap U_g$ which contradicts U_f and U_g being disjoint. □

2. Let $f : X_1 \to Y_1$ and $g : X_2 \to Y_2$ be continuous maps between topological spaces. Give the products $X_1 \times X_2$ and $Y_1 \times Y_2$ their product topologies. Show that the map $H : X_1 \times X_2 \to Y_1 \times Y_2$ defined by $H((x_1, x_2)) = (f(x_1), g(x_2))$ is continuous.

Proof. Assume that $f : X_1 \to Y_1$ and $g : X_2 \to Y_2$ are continuous maps between topological spaces. Assume we have $X_1 \times X_2$ and $Y_1 \times Y_2$ with their product topologies with the map $H : X_1 \times X_2 \to Y_1 \times Y_2$ where $H((x_1, x_2)) = (f(x_1), g(x_2))$. Then consider an arbitrary basis element $U_1 \times U_2 \subset Y_1 \times Y_2$ then we have by the definition of product topology that U_1 is open in Y_1 and U_2 is open in Y_2 and as f, g are continuous we have $f^{-1}(U_1)$ and $g^{-1}(U_2)$ are open hence $f^{-1}(U_1) \times g^{-1}(U_2)$ is open in $X_1 \times X_2$ this implies $H^{-1}(U_1 \times U_2) = f^{-1}(U_1) \times g^{-1}(U_2)$ is open which shows that H is continuous.

□

3. An injective (one-to-one) continuous map $f : X \to Y$ between topological spaces is a bijection from X to the image $f(X)$. Give $f(X)$ the subspace topology induced by Y's topology. We call such an f an imbedding if it is a homeomorphism from X to $f(X)$.

In this context, let Y be $X \times X$ with the product topology. Let x_0 be an arbitrary element of X.

a. Show that $f : X \to X \times X$ defined by $f(x) = (x, x_0)$ is an imbedding.

Proof. Assume that $f : X \to X \times X$ is a map defined by $f(x) = (x, x_0)$ where x_0 is an arbitrary element of X. Suppose $x, y \in X$ with $f(x) = f(y)$ then we have $(x, x_0) = (y, x_0)$ hence $x = y$ which shows that f is injective. Let $U_1 \times U_2 \subset f(X)$ be an arbitrary basis element. Then as $f(X) = X \times \{x_0\}$ we get that $U_2 = \{x_0\}$ additionally from the definition of subspace topology we get $U_1 = X \cap U_3$ where U_3 is some open set in X hence U_1 is also open. As $f^{-1}(U_1 \times U_2) = U_1$ we have $f^{-1}(U_1 \times U_2)$ is open. Hence f is continuous.

We have $f^{-1} : f(X) \to X$ where $f^{-1}(x, y) = x$ is a bijection as we have shown that f is injective and it is surjective with its own range. So we just need to show that f^{-1} is continuous. Given an arbitrary basis element U of X. We have that $(f^{-1})^{-1} = f$ hence $f(U) = U \times \{x_0\}$ and as $U \times \{x_0\} = f(X) \cap (U \times X)$ and $U \times X$ is open in the product topology we get $U \times \{x_0\}$ is open in the subspace topology hence which shows that f^{-1} is continuous which implies f is a homeomorphism which shows that f is an imbedding.

□

b. Show that $g : X \to X \times X$ defined by $g(x) = (x, x)$ is an imbedding.

Proof. Let $g : X \to X \times X$ be defined by $g(x) = (x, x)$. Let $x, y \in X$ be two elements with $g(x) = g(y)$ then $(x, x) = (y, y)$ which implies $x = y$ hence g is injective. Now let $U_1 \times U_2 \subset g(X)$ be an open set. Then we have $U_1 \times U_2 = \{(x, x) : x \in X\} \cap A \times B$ for some open sets A, B in X which implies $U_1 \times U_2 = \{(x, x) : x \in A \cap B\}$. Then $g^{-1}(U_1 \times U_2) = A \cap B$ and as A, B are open we get $A \cap B$ is open which shows that g is continuous.

Let the map $g^{-1} : g(X) \to X$ be defined by $g^{-1}(x, x) = x$ we have already shown g is injective and as g is surjective with its own range we have that g^{-1} is a bijection. Let $U \subset X$ be an open set then we have $(g^{-1})^{-1} = g$ hence $(g^{-1})^{-1}(U) = g(U) = \{(x, x) : x \in U\}$. As $\{(x, x) : x \in U\} = (U \times U) \cap g(X)$ we get $\{(x, x) : x \in U\}$ is open in the subspace topology hence g^{-1} is continuous which implies that g is an imbedding.

$\square$

4. Suppose that $h : X \to Y$ is a homeomorphism of topological spaces. If Z is any other topological space and if $g : Y \to Z$ is a continuous map, we know that the composition $g \circ h$ is a continuous map from X to Z. Show that every continuous map $f : X \to Z$ arises this way, i.e. for any continuous $f : X \to Z$, there exists a continuous $G : Y \to Z$ for which $f = G \circ h$.

5. a. Show that a linearly ordered set with the order topology is Hausdorff.

b. Suppose that X is a topological space. Show that X is Hausdorff if and only if the diagonal subset $\{(x, x) : x \in X\}$ of the product $X \times X$ is a closed subset of the product. Assume here that the topology on $X \times X$ is the product topology.

6. text p. 111-112, problem 8.