**1.** Suppose that $X$ is a compact topological space and $f : X \to \mathbb{R}$ is a function, not necessarily continuous, with the following property: for each $x \in X$, there is an open neighborhood $U_x$ of $x$ and a constant $M_x$ such that, for all $z \in U_x$, $|f(z)| \leq M_x$. Show that there is a constant $M$ such that, for all $y \in X$, $|f(y)| \leq M$.

*Proof.* Suppose $X$ is a compact topological space and $f : X \to \mathbb{R}$ is as described. As $X$ is compact we have the open cover $\{U_x\}_{x\in X}$ where for each $U_x \in \{U_x\}_{x\in X}$ we have that there exists a constant $M_x$ such that for all $z \in U_x$ that $|f(z)| \leq M_x$.

Then as $X$ is compact we have that the collection has a finite subcover $\{U_{x_1}, ..., U_{x_n}\}$. Each of these in the finite subcover has a corresponding constant $\{M_{x_1}, ..., M_{x_n}\}$ with the property that for all $z \in U_{x_i}$ that $|f(z)| \leq M_{x_i}$ where $i = 1, ..., n$. Let $M = \max\{M_{x_1}, ..., M_{x_n}\}$ then for any $z \in X$ we have that $z \in U_{x_i}$ for some $i = 1, ..., n$. Hence $|f(z)| \leq M_{x_i} \leq M$ so this choice of $M$ works for all $z \in X$. □

**2.** Suppose that $X$ is a topological space, that $C$ is a connected subset of $X$ and that, for each $\alpha$ in some index set $A$, $C_\alpha$ is a connected subset of $X$. Suppose also that, for every $\alpha \in A, C_\alpha \bigcap C \neq \emptyset$. Show that $C \bigcup \left(\bigcup_{\alpha\in A} C_\alpha\right)$ is connected.

*Proof.* Suppose that $X$ is a topological space $C$ is a connected subset of $X$ and $C_\alpha$ is a connected subset of $X$. With for all $\alpha A$ that $C_\alpha \bigcap C \neq \emptyset$. Suppose that $C \bigcup \left( \cup_{\alpha\in A} C_\alpha\right)$ is not connected. Then there exists a separation $U \cup V$ of $C \bigcup \left( \cup_{\alpha\in A} C_\alpha\right)$.

Then as $C$ is connected we have either $C \subset U$ with $C \cap V = \emptyset$ or $C \subset V$ with $C \cap U = \emptyset$. If $C$ wasn't fully contained in only one then $(C \cap U)$, $(C \cap V)$ would be a separation of $C$ but $C$ is connected. For each $\alpha \in A$ we get that $C_\alpha$ is contained in exactly one $U$ or $V$.

WLOG suppose $C \cup U$ then as $C \cap C_\alpha \neq \emptyset$ we get that $C_\alpha \subset U$. Hence we get $C \bigcup \left( \cup_{\alpha\in A} C_\alpha\right) \subset U$ and $\left(C \bigcup \left(U_{\alpha\in A} C_\alpha\right)\right) \bigcap V = \emptyset$ hence $V = \emptyset$ so this separation can not exist. So we have $C \bigcup \left( \cup_{\alpha\in A} C_\alpha\right)$ is connected. □

**3.** Give $\mathbb{R}$ its standard topology and give its subsets their subspace topologies.

(a) Let $A$ be $\mathbb{R}$'s topological subspace $\{1/n : n \in \mathbb{N}\}$. Describe as explicitly as possible which subsets of $A$ are closed subsets of $A$.

We have that a set is closed if and only if it contains it's limit points. As any singleton in $\{1/n : n \in \mathbb{N}\}$ is ope. Which is shown as we have for any $1/n \in \{1/n : n \in \mathbb{N}\}$ we have that $(1/n - \epsilon, 1/n + \epsilon)$ for $\epsilon > 0$ is open in $\mathbb{R}$ choosing a sufficiently small $\epsilon > 0$ we get $(1/n - \epsilon, 1/n + \epsilon) \cap \{1/n : n \in \mathbb{N}\} = \{1/n\}$. As the choice of $1/n$ was arbitrary we get that all singletons are open.

Hence any $B \subset \{1/n : n \in \mathbb{N}\}$ we get that $B' = \emptyset$ as for any $b \in \{1/n : n \in \mathbb{N}\}$ we have the neighborhood $\{b\}$ that doesn't intersect $B$ at any place other then possibly itself. Hence $B$ is closed as $B$ is arbitrary we have that all subsets are closed.

(b) Let $B$ be $\mathbb{R}$'s topological subspace $\{0\} \bigcup \{1/n : n \in \mathbb{N}\}$. Describe as explicitly as possible which infinite subsets of $B$ are closed subsets of $B$.

We have that all singletons other then $\{0\}$ are open using the same reasoning as above. Hence $\{1/n : n \in \mathbb{N}\}$ is open so we get $\{1/n : n \in \mathbb{N}\}^c = \{0\}$ is closed. We have that all singletons are closed as any singleton other than $\{0\}$ don't have any limit points. As finite unions of closed sets are closed then all finite subsets are closed.

Lastly any infinite set containing 0 is closed. This is shown as let $B \subset \{0\} \cup \{1/n : n \in \mathbb{N}\}$ be an infinite set containing 0. Then we have $B^c$ is a union of singletons each of which are open hence $B^c$ is open so $B$ is closed.

Any infinite set not containing 0 is not closed as 0 is a limit point of said set.

**4.** Give the natural numbers $\mathbb{N}$ the topology in which the open sets are: the emptyset; all of $\mathbb{N}$; and, for each $k \in \mathbb{N}$, the set $U_k = \{n \in \mathbb{N} : n \leq k\}$. Using this topology on both the domain and range space of $f : \mathbb{N} \to \mathbb{N}$, show that $f$ is continuous if and only if, for all $m, n \in \mathbb{N}$ with $m < n$, $f(m) \leq f(n)$

*Proof.*
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Assume that $f$ is continuous and that there exists a $m, n \in \mathbb{N}$ with $m < n$ where $f(m) > f(n)$. Choose the neighborhood $U_{f(n)} = \{1, ..., f(n)\}$ of $f(n)$. Then we have that there exists a neighborhood $U_n$ of $n$ where $f(U_n) \subset U_{f(n)}$. From the definition of the open sets we get that $m \in U_n$ but as $f(m) > f(n)$ we have $f(m) \in f(U_n)$ but $f(m) \notin U_{f(n)}$. This is a contradiction on $f$ being continuous hence $f(m) \leq f(n)$

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Let $n \in \mathbb{N}$ then take an arbitrary neighborhood $U_{f(n)}$ of $f(n)$. Then we have $U_n = \{1, ..., n\}$ that $f(U_n) \subset U_{f(n)}$ this follows as $f(n) \in f(U_n)$ and $f(n) \in U_{f(n)}$ and for any $m \in U_n$ with $m < n$ we have that $f(m) \leq f(n)$ hence $f(m) \in U_{f(n)}$. This is one of the equivalent definitions of continuity. Hence $f$ is continuous. $\square$

**5.** Let $X$ be a first countable topological space, let $A$ be a subset of $X$, and let $c$ be in $A$'s closure. SHow that there exists a sequence $(a_n)$ of elements of $A$ that converge to $c$.

*Proof.* Let $X$ be a first countable topological space. Let $A \subset X$ and $c \in \overline{A}$. If $c \in A$ a constant sequence works hence assume that $c \in A'$ but $c \notin A$. We have the collection of neighborhoods $\{U_{c_i}\}_{i \in \mathbb{N}}$ of $c$ where any neighborhood $V$ of $c$ that for some $i \in \mathbb{N}$ that $U_{c_i} \subset V$.

Create the sequence where the n'th element is taken from $A \bigcap \big( \cap_{i=1}^{n} U_{c_i} \big)$. First to show that $A \bigcap \big( \cap_{i=1}^{n} U_{c_i} \big) \neq \emptyset$ for any $n \in \mathbb{N}$. As $\bigcap_{i=1}^{n} U_{c_i}$ is a finite intersection of nonempty open sets ($c$ is in each) we have it is nonempty and open. As $c \in A'$ then it intersects $A$ at some place other then itself. Hence not empty.

Create the sequence $(a_n)$ where $a_n \in A \bigcap \big( \cap_{i=1}^{n} U_{c_i} \big)$.

Now to show that $a_n \to c$.

Let $U_c$ be an arbitrary neighborhood of $c$ then for some $i \in \mathbb{N}$ we have $U_{c_i} \subset U$ from the definition of $(a_n)$ we get that for all $j \geq i$ where $j \in \mathbb{N}$ that $a_j \in U_{c_i} \subset U$ which follows from the intersection in the construction of $(a_n)$. Hence $a_n \to c$. $\square$