MATH 4324 ORDER TOPOLOGY AND INTRODUCTION TO CONTINUITY

Hand in Friday, March 1.

1. Recall that a map $f : X \to Y$ between topological spaces is said to be continuous at a point $x_0 \in X$ if and only if, for each open set V that contains $f(x_0)$, there is an open U satisfying $x_0 \in U \subset f^{-1}(V)$. Let $\mathcal{B}_X$ be a basis for the topology of X and $\mathcal{B}_Y$ a basis for the topology of Y. Show that f is continuous at $x_0 \in X$ if and only if, for each $W \in \mathcal{B}_Y$ that contains $f(x_0)$, there is a $\widetilde{W} \in \mathcal{B}_X$ satisfying $x_0 \in \widetilde{W} \subset f^{-1}(W)$.

Proof. Suppose that $f : X \mapsto Y$ is continuous at $x_0 \in X$ then we have for all neighborhoods $V_{f(x_0)}$ of $f(x_0)$ that there exists an open neighborhood U_{x_0} of x_0 with $x_0 \in U_{x_0} \subset f^{-1}(V_{f(x_0)})$ but as U_{x_0} is open then it is the union of bases elements hence there exists a basis element U_b with $x_0 \in U_b \subset f^{-1}(V_{f(x_0)})$ as $V_{f(x_0)}$ is an arbitrary neighborhood of $f(x_0)$ then this implies that the above is true for all neighborhoods $V_b \in \mathcal{B}_Y$ of $f(x_0)$ which completes the forward direction.

Assume that for $f : X \mapsto Y$ we have for each $x_0 \in X$ that for each $W \in \mathcal{B}_Y$ that contains $f(x_0)$, there is a $\widetilde{W} \in \mathcal{B}_X$ with $x_0 \in \widetilde{W} \subset f^{-1}(W)$. Now for any $x_0 \in X$ consider an arbitrary neighborhood V of $f(x_0)$. Then we have $V = \bigcup U_\alpha$ where $U_\alpha \in \mathcal{B}_Y$ which implies for some $U'_\alpha \in \mathcal{B}_y$ that $f(x_0) \in U'_\alpha \subset V$ this implies (by the assumption) that there exists $\widetilde{W} \in \mathcal{B}_X$ with $x_0 \in \widetilde{W} \subset f^{-1}(U'_\alpha) \subset f^{-1}(\bigcup U_\alpha) = f^{-1}(V)$ which shows that f is continuous at $x_0 \in X$.

□

2. When X is a metric space, we know that the set of open balls, $\{B(x,r) : x \in X \text{ and } r > 0\}$ is a basis for X's metric topology $\mathcal{T}$.

a. Show that $\{B(x,r) : x \in X \text{ and } 0 < r < 1\}$ is a basis for a topology on X. Call this topology $\mathcal{T}_1$.

Proof. Let $x \in X$ then we have $x \in B(x, 0.5)$ hence we have the first condition for being a basis satisfied. Now suppose that we have $x \in B(y_1,r_1) \cap B(y_2,r_2)$ where $y_1, y_2 \in X$ and $0 < r_1 < 1$ and $0 < r_2 < 1$. Then let $r_3 = \min(r_1 - d(x,y_1), r_2 - d(x,y_2))$ then consider the set $B(x,r_3)$. Then for any $x_0 \in B(x,r_3)$ we have $d(x_0,y_1) \le d(x_0,x) + d(x,y_1) < r_3 + d(x,y_1) \le r_1 - d(x,y_1) + d(x,y_1) \le r_1$ note that the strict inequality follows from $d(x_0,x) < r_3$ and I used the triangle inequality for the first inequality and the third inequality follows due to the choice of r_3. Then for any $x_0 \in B(x,r_3)$ we have $d(x_0,y_2) \le d(x_0,y_2) + d(x,y_2) < r_3 + d(x,y_2) \le r_2 - d(x,y_2) + d(x,y_2) = r_2$ which shows $x_0 \in B(y_2,r_2)$ hence we get $B(x,r_3) \subset B(y_1,r_1)$ and $B(x,r_3) \subset B(y_2,r_2)$ which implies $x \in B(x,r_3) \subset B(y_1,r_1) \cap B(y_2,r_2)$. Which shows that it is a basis. □

b. Show that $\mathcal{T} = \mathcal{T}_1$.

Proof. Suppose $B(x_0,r_0) \in \{B(x,r) : x \in X \text{ and } 0 < r < 1\}$ then as $x \in X$ and $0 < r_0$ we get $B(x_0,r_0) \in \{B(x,r) : x \in X \text{ and } r > 0\}$ hence $\{B(x,r) : x \in X \text{ and } 0 < r < 1\} \subset \{B(x,r) : x \in X \text{ and } r > 0\}$ which implies $\mathcal{T}_1 \subset \mathcal{T}$.

Now using Munkres Lemma 13.3 for an arbitrary $x \in X$ and an arbitrary $B(x_0,r_0) \in \{B(x,r) : x \in X \text{ and } 0 < r < 1\}$ with $x \in B(x_0,r_0)$ then we have $B(x_0,r_0) \in \{B(x,r) : x \in X \text{ and } 0 < r\}$ then as $x \in B(x_0,r_0) \subset B(x_0,r_0)$ we get $\mathcal{T} \subset \mathcal{T}_1$ as we have double inclusion we get $\mathcal{T}_1 = \mathcal{T}$ □

3. Let X be $\{0\} \bigcup \{1/k : k \in \mathbb{N}\}$ be ordered by the usual "less than" $<$, i.e. the order it gets from "less than" when X is regarded as a subset of $[0,1]$. Give X the associated order topology. In the associated product topology on $X \times X$, show that every open set that contains $(0,1)$ contains infinitely many points with second coordinate 1 and that there is an open set that contains $(0,1)$ and in which every point has second coordinate 1.

Proof. Let $X = \{0\} \bigcup \{1/k : k \in \mathbb{N}\}$ with the order topology. Now let $A \times B$ be an arbitrary open set in the product topology with $(0,1) \in A \times B$. Then as this is the product topology we have A, B are open sets of X with $0 \in A$ and $1 \in B$. Then we have two basis elements of the form $[0,b) \subset A$ where $b = 1/n$ for some

$n \in \mathbb{N}$ and $(a, 1]$ where $a = 1/k$ for some $k \in \mathbb{K}$. Then we have $(1/j, 1) \in [0, b) \times (a, 1] \subset A \times B$ where $j \in \mathbb{N}$ and $j \geq n$ this shows that there is an infinite number of points where the second coordinate is 1.

We have $(1/2, 1] = \{1\}$ and $[0, 1/2)$ are open in X and as this is the product topology we have that $[0, 1/2) \times \{1\}$ is open in $X \times X$ but $[0, 1/2) \times \{1\} = \{(a, 1) : a = 0 \text{ or } a = 1/n \text{ where } n > 2\}$ hence every element has 2nd coordinate 1. $\square$

If we start with the same order on X and give $X \times X$ the associated dictionary order, then in the topology $X \times X$ gets from the dictionary order, show that, for every $y \in X$, every open set that contains $(0, 1)$ contains infinitely many points with second coordinate y.

Proof. Assume that $X \times X$ has the dictionary order and the order topology. Let $A \times B$ be an open set with $(0, 1) \in A \times B$. Then we have a basis element of the form $\big((a, b), (c, d)\big) = \{(x, z) \in X \times X : (a, b) < (x, z) \text{ and } (x, z) < (c, d)\}$ where $a, b, c, d \in X$ with $(0, 1) \in \big((a, b), (c, d)\big) \subset A \times B$. Then as $(0, 1) \in \big((a, b), (c, d)\big)$ we have $(a, b) < (0, 1) < (c, d)$. This implies that $a = 0$ and $b < 1$ as this is the order topology we get that $0 < c$ as if it where not then we would have $c = 0$ and $d = 1$ which would imply that the basis is $\big((0, 1), (0, 1)\big)$ which is not an element of the order topology. Hence we get the strict inequality $0 < c$ then we have for all $n \in \mathbb{N}$ with $1/n < c$ that for all $y \in X$ the inequality $(a, b) < (1/n, y) < (c, d)$ as there is an infinite number of natural numbers $k \in \mathbb{N}$ such that $0 = a < 1/k < c$ this completes the proof. $\square$