

Problem 1. Show that if f and \bar{f} are analytic on a domain D , then f is constant.

Proof. Assume that $f = u + iv$ and $\bar{f} = u - iv$ are both analytic on domain D . Then we have that both satisfy the Cauchy-Riemann equations: That is

$$\begin{aligned} \frac{\partial u}{\partial x} &= \frac{\partial v}{\partial y}, & \frac{\partial u}{\partial y} &= -\frac{\partial v}{\partial x}, \\ \frac{\partial u}{\partial x} &= -\frac{\partial v}{\partial y}, & \frac{\partial u}{\partial y} &= \frac{\partial v}{\partial x}. \end{aligned}$$

Then we have $\frac{\partial u}{\partial x} = -\frac{\partial u}{\partial x} \implies \frac{\partial u}{\partial x} = 0$ using the same reasoning for the rest we get $\frac{\partial v}{\partial x} = \frac{\partial u}{\partial y} = \frac{\partial v}{\partial y} = 0$. This implies that u and v are both constant functions therefore f is constant. \square

Problem 2. Let a be a complex number, $a \neq 0$, and $f(z)$ be an analytic branch of z^a on $\mathbb{C} \setminus (-\infty, 0]$. Show that $f'(z) = af(z)/z$.

Proof. Let $f(z) = z^a$ where $a \neq 0$, and $f(z)$ be an analytic branch of z^a on $\mathbb{C} \setminus (-\infty, 0]$. Then we have $f(z) = e^{a \operatorname{Log} z}$. Using the chain rule we get $f'(z) = ae^{a \operatorname{Log} z} \cdot \frac{1}{z}$. As $f(z) = e^{a \operatorname{Log} z}$ is the same branch we can do the substitution $f'(z) = ae^{a \operatorname{Log} z}/z = af(z)/z$ which completes the proof. \square

Problem 3. Show that if $h(z)$ is a complex valued harmonic function such that $zh(z)$ is also harmonic, then $h(z)$ is analytic.

Proof. Assume that $h(z)$ is a complex valued harmonic function such that $zh(z)$ is also harmonic. Then $h(z) = u(z) + iv(z)$ we get

$$\begin{aligned} \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} &= 0 \\ \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} &= 0 \end{aligned}$$

Additionally as we have $zh(z)$ is harmonic we get

$$(x + iy)(u(z) + iv(z)) = xu(z) - yv(z) + i(xv(z) + yu(z))$$

We get

$$\frac{\partial^2(xu(z) - yv(z))}{\partial x^2} + \frac{\partial^2(xu(z) - yv(z))}{\partial y^2} = 0 \tag{1}$$

$$\frac{\partial^2(xv(z) + yu(z))}{\partial x^2} + \frac{\partial^2(xv(z) + yu(z))}{\partial y^2} = 0 \tag{2}$$

Applying the partial derivatives to (1) we get

$$\begin{aligned} x \frac{\partial^2 u}{\partial x^2} + \frac{\partial u}{\partial x} - y \frac{\partial^2 v}{\partial x^2} + x \frac{\partial^2 u}{\partial y^2} - y \frac{\partial^2 v}{\partial y^2} - \frac{\partial v}{\partial y} &= 0 \\ x \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right) - y \left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} \right) + \frac{\partial u}{\partial x} - \frac{\partial v}{\partial y} &= 0 \\ \frac{\partial u}{\partial x} &= \frac{\partial v}{\partial y} \end{aligned}$$

Applying the partial derivatives to (2) we get.

$$\begin{aligned} x \frac{\partial^2 v}{\partial x^2} + \frac{\partial v}{\partial x} + y \frac{\partial^2 u}{\partial x^2} + x \frac{\partial^2 v}{\partial y^2} + y \frac{\partial^2 u}{\partial y^2} + \frac{\partial u}{\partial y} &= 0 \\ x \left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} \right) + y \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right) + \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} &= 0 \\ \frac{\partial u}{\partial y} &= -\frac{\partial v}{\partial x} \end{aligned}$$

Therefore we have that the Cauchy Riemann equations are satisfied which implies that $h(z)$ is analytic. \square

Problem 4. Let $f = u + iv$ be a continuously differentiable complex valued function on a domain D such that the Jacobian matrix of f does not vanish at any point of D . Show that if f maps orthogonal curves to orthogonal curves, then either f or \bar{f} is analytic, with a nonvanishing derivative.

Proof. Assume that f maps orthogonal curves to orthogonal. We have that derivative of the tangent curves. Let $x_0 + iy_0 \in D$ and consider the four curves $c_1(t) = (x_0 + t, y_0 + t)$, $c_2(t) = (x_0 + t, y_0 - t)$, $c_3(t) = (x_0 + t, y_0)$, $c_4(t) = (x_0, y_0 + t)$. We have that tangent vectors for c_1 and c_2 are orthogonal and the tangent vectors for c_3 and c_4 are orthogonal. Now for some curve $c(t) = (x(t), y(t))$ the following equation

$$\frac{f(c(t))}{dt} = \left(\frac{\partial u}{\partial x} \frac{dx}{dt} + \frac{\partial u}{\partial y} \frac{dy}{dt}, \frac{\partial v}{\partial x} \frac{dx}{dt} + \frac{\partial v}{\partial y} \frac{dy}{dt} \right)$$

which shows that the image of the tangent vector for a curve under a function is given by the Jacobian.

Applying the equation for the curves c_1, c_2 we get

$$\begin{aligned} f(c_1(t)) &= \left(\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y}, \frac{\partial v}{\partial x} + \frac{\partial v}{\partial y} \right) \\ f(c_2(t)) &= \left(\frac{\partial u}{\partial x} - \frac{\partial u}{\partial y}, \frac{\partial v}{\partial x} - \frac{\partial v}{\partial y} \right) \\ f(c_3(t)) &= \left(\frac{\partial u}{\partial x}, \frac{\partial v}{\partial x} \right) \\ f(c_4(t)) &= \left(\frac{\partial u}{\partial y}, \frac{\partial v}{\partial y} \right) \end{aligned}$$

As f maps orthogonal curves to orthogonal curves we have get by the dot product of $f(c_1(t))$ and $f(c_2(t))$

$$\left(\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y}\right)\left(\frac{\partial u}{\partial x} - \frac{\partial u}{\partial y}\right) + \left(\frac{\partial v}{\partial x} + \frac{\partial v}{\partial y}\right)\left(\frac{\partial v}{\partial x} - \frac{\partial v}{\partial y}\right) = 0 \quad (3)$$

By the dot product of $f(c_3(t))$ and $f(c_4(t))$ we get

$$\frac{\partial u}{\partial x} \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \frac{\partial v}{\partial y} = 0 \quad (4)$$

Expanding (3) we get

$$\begin{aligned} \left(\frac{\partial u}{\partial x}\right)^2 - \left(\frac{\partial u}{\partial y}\right)^2 + \left(\frac{\partial v}{\partial x}\right)^2 - \left(\frac{\partial v}{\partial y}\right)^2 &= 0 \\ \left(\frac{\partial u}{\partial x}\right)^2 \left(\left(\frac{\partial u}{\partial x}\right)^2 - \left(\frac{\partial u}{\partial y}\right)^2 + \left(\frac{\partial v}{\partial x}\right)^2 - \left(\frac{\partial v}{\partial y}\right)^2\right) &= 0 \\ \left(\frac{\partial u}{\partial x}\right)^4 - \left(\frac{\partial u}{\partial x}\right)^2 \left(\frac{\partial u}{\partial y}\right)^2 + \left(\frac{\partial u}{\partial x}\right)^2 \left(\frac{\partial v}{\partial x}\right)^2 - \left(\frac{\partial u}{\partial x}\right)^2 \left(\frac{\partial v}{\partial y}\right)^2 &= 0 \end{aligned}$$

Applying the substitution $\left(\frac{\partial u}{\partial x} \frac{\partial u}{\partial y}\right)^2 = \left(\frac{\partial v}{\partial x} \frac{\partial v}{\partial y}\right)^2$ which comes from equation (4).

$$\begin{aligned} \left(\frac{\partial u}{\partial x}\right)^4 - \left(\frac{\partial v}{\partial x}\right)^2 \left(\frac{\partial v}{\partial y}\right)^2 + \left(\frac{\partial u}{\partial x}\right)^2 \left(\frac{\partial v}{\partial x}\right)^2 - \left(\frac{\partial u}{\partial x}\right)^2 \left(\frac{\partial v}{\partial y}\right)^2 &= 0 \\ \left(\frac{\partial u}{\partial x}\right)^2 \left(\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial x}\right)^2\right) - \left(\frac{\partial v}{\partial y}\right)^2 \left(\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial x}\right)^2\right) &= 0 \\ \left(\left(\frac{\partial u}{\partial x}\right)^2 - \left(\frac{\partial v}{\partial y}\right)^2\right) \left(\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial x}\right)^2\right) &= 0 \end{aligned}$$

This implies either $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ or $-\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$. Substituting $\frac{\partial v}{\partial y}$ for $\frac{\partial u}{\partial x}$ in equation (4) we have $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$ hence this gives us f is analytic. Substituting $-\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ in (3) we get $\frac{\partial u}{\partial y} = \frac{\partial v}{\partial x}$ which corresponds to \bar{f} being analytic. As the Jacobian is nonvanishing we get that that the derivatives for f and \bar{f} are non vanishing.

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