Problem 1. Show that if f and $\bar{f}$ are analytic on a domain D, then f is constant.

Proof. Assume that $f = u + iv$ and $\bar{f} = u - iv$ are both analytic on domain D. Then we have that both satisfy the Cauchy-Riemann equations: That is

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \qquad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x},$$
$$\frac{\partial u}{\partial x} = -\frac{\partial v}{\partial y}, \qquad \frac{\partial u}{\partial x} = \frac{\partial v}{\partial x}.$$

Then we have $\frac{\partial u}{\partial x} = -\frac{\partial u}{\partial x} \implies \frac{\partial u}{\partial x} = 0$ using the same reasoning for the rest we get $\frac{\partial v}{\partial x} = \frac{\partial u}{\partial y} = \frac{\partial v}{\partial y} = 0$. This implies that u and v are both constant functions therefore f is constant. □

Problem 2. Let a be a complex number, $a \neq 0$, and $f(z)$ be an analytic branch of z^a on $\mathbb{C} \setminus (-\infty, 0]$. Show that $f'(z) = af(z)/z$.

Proof. Let $f(z) = z^a$ where $a \neq 0$, and $f(z)$ be an analytic branch of z^a on $\mathbb{C} \setminus (-\infty, 0]$. Then we have $f(z) = e^{a\mathrm{Log} z}$. Using the chain rule we get $f'(z) = ae^{a\mathrm{Log} z} \cdot \frac{1}{z}$. As $f(z) = e^{a\mathrm{Log}}$ is the same branch we can do the substitution $f'(z) = ae^{a\mathrm{Log} z}/z = af(z)/z$ which completes the proof. □

In accordance with the Hokie Honor Code, I affirm that I have neither given nor received unauthorized assistance on this assignment.