Problem 1.

(a) Considering $\mathbb{C}$ as a $\mathbb{R}$-vector space, find a basis for $\mathbb{C}$.

(b) Considering $\mathbb{C}$ as a $\mathbb{C}$-vector space, find a basis for $\mathbb{C}$.

(a) *Proof.* A basis for $\mathbb{C}$ as a $\mathbb{R}$-vector space is $B = \{1, i\}$. This is shown to be a basis by the following: Let $a + bi \in \mathbb{C}$ then consider the linear combination $a(1) + b(i) = a + bi$ as an arbitrary element of $\mathbb{C}$ is a linear combination of the elements of B we have $\mathrm{Span}(B) = \mathbb{C}$. Lastly to show linear independence of B consider the linear combination $a(1) + b(i) = 0 = 0 + 0i$ as two complex numbers are equal if and only if their real parts are equal and imaginary parts are equal we get $a = 0$ and $b = 0$. Thus B is a basis for $\mathbb{C}$ as a $\mathbb{R}$-vector space. $\square$

(b) *Proof.* A basis for $\mathbb{C}$ as a $\mathbb{C}$-vector space is given by $B = \{1 + i\}$. Let $a + bi \in \mathbb{C}$ then consider the linear combination $(\frac{a+b}{2} + \frac{b-a}{2}i)(1 + i) = \frac{a+b}{2} + \frac{a+b}{2}i + \frac{b-a}{2}i - \frac{b-a}{2} = a + bi$ as an arbitrary element of $\mathbb{C}$ is a linear combination of the element of B we have $\mathrm{Span}(B) = \mathbb{C}$. *Note that $\frac{a+b}{2} + \frac{b-a}{2}i \in C$ is a scalar as this is a $\mathbb{C}$-vector space.* Lastly to show linear independence of B consider the linear combination $a + bi \in \mathbb{C}$ and $1 + i \in B$ we have $(a + bi)(1 + i) = 0$ if and only if $a + bi = 0$ as $\mathbb{C}$ is a field hence no zero divisors. $\square$

Problem 2. Let V be a $\mathbb{K}$-vector space, and suppose that S_1, S_2 are subsets of V satisfying the following conditions:

- S_1 and S_2 are both finite.
- $S_1 \cap S_2 = \emptyset$.
- $S_1 \cup S_2$ is a linearly independent set.

(a) Prove that $\text{Span}(S_1 \cup S_2) = \text{Span}(S_1) \oplus \text{Span}(S_2)$.

(b) What would change about the claim in (a) if $S_1 \cup S_2$ was not assumed to be linearly independent?

(a) *Proof.* Assume that V is a $\mathbb{K}$-vector space, and that S_1, S_2 are as described. First I will show that $\text{Span}(S_1) + \text{Span}(S_2)$ is a direct sum. As $\text{Span}(S_1), \text{Span}(S_2)$ are both vector spaces we have that $\vec{0} \in \text{Span}(S_1) + \text{Span}(S_2) \neq \emptyset$. Now assume that $\vec{v} \in \text{Span}(S_1) \cap \text{Span}(S_2)$ then $\vec{v} \in \text{Span}(S_1)$ and $\vec{v} \in \text{Span}(S_2)$ so $v = k_1\vec{s_1} + ... + k_n\vec{s_n}$ where $k_i \in \mathbb{K}$ and $\vec{s} \in S_1$ and $v = c_1\vec{u_1} + ... + c_n\vec{u_n}$ where $c_i \in \mathbb{K}$ and $\vec{u_i} \in S_2$. Then $c_1\vec{u_1} + ... + c_n\vec{u_n} + -(k_1\vec{s_1} + ... + k_n\vec{s_n}) = c_1\vec{u_1} + ... + c_n\vec{u_n} + (-k_1\vec{s_1}) + ... + (-k_n\vec{s_n}) = \vec{0}$ but as $S_1 \cup S_2$ is linearly independent and the previous equation is a linear combination of $S_1 \cup S_2$ we have that for each $c_i, k_i \in \mathbb{K}$ that $c_i = k_i = 0$ hence only $\vec{0} \in \text{Span}(S_1) \cap \text{Span}(S_2)$ which implies $\text{Span}(S_1) + \text{Span}(S_2)$ is a direct sum.

Now suppose $\vec{v} \in \text{Span}(S_1 \cup S_2)$ then $\vec{v} = k_1\vec{s_1} + ... + k_n\vec{s_n} + c_1\vec{u_1} + ... + c_n\vec{c_n}$ where $k_i, c_i \in \mathbb{K}$ and $\vec{s_i} \in S_1$ and $\vec{u_i} \in S_2$. As $k_1\vec{s_1} + ... + k_n\vec{s_n} \in \text{Span}(S_1)$ and $c_1\vec{u_1} + ... + c_n\vec{u_n} \in \text{Span}(S_2)$ we have that $\vec{v} \in \text{Span}(S_1) \oplus \text{Span}(S_2)$. Thus $\text{Span}(S_1 \cup S_2) \subseteq \text{Span}(S_1) \oplus \text{Span}(S_2)$.

Let $\vec{v} \in \text{Span}(S_1) \oplus \text{Span}(S_2)$ then $\vec{v} = k_1\vec{s_1} + ... + k_n\vec{s_n} + c_1\vec{u_1} + ... + c_n\vec{c_n}$ where $k_i, c_i \in \mathbb{K}$ and $\vec{s_i} \in S_1$ and $\vec{u_i} \in S_2$. Then we have as this is just a linear combination of the elements of $S_1 \cup S_2$ we have that $\vec{v} \in \text{Span}(S_1 \cup S_2)$ therefore $\text{Span}(S_1) \oplus \text{Span}(S_2) \subseteq \text{Span}(S_1 \cup S_2)$ which implies $\text{Span}(S_1 \cup S_2) = \text{Span}(S_1) \oplus \text{Span}(S_2)$ ☐

(b) Then $\text{Span}(S_1) + \text{Span}(S_2)$ would no longer be a direct sum. But the equation would still be true if you replaced '$\oplus$' with '$+$' i.e. $\text{Span}(S_1 \cup S_2) = \text{Span}(S_1) + \text{Span}(S_2)$.

Problem 3. Let $V = \mathbb{Q}^4$, considered as a $\mathbb{Q}$-vector space, and let U be the subspace

$$U = \operatorname{Span}\big(\mathbf{u_1} = (1, -1, 2, 1),\ \mathbf{u_2} = (2, -3, 6, 3)\big).$$

Extend the set $\{\mathbf{u_1}, \mathbf{u_2}\}$ into a basis for V. That is, find two vectors $\mathbf{v_1}, \mathbf{v_2} \in V$ so that

$$\{\mathbf{u_1}, \mathbf{u_2}, \mathbf{v_1}, \mathbf{v_2}\}$$

is a basis for V.

Proof. Adding the two vectors $(0, 0, 1, 0)$ and $(0, 0, 0, 1)$ will make a basis.

Let $\langle a, b, c, d\rangle \in V$ and consider the linear combination

$$(3a + 2b)\langle 1, -1, 2, 1\rangle + (-a - b)\langle 2, -3, 6, 3\rangle + (c + 2b)\langle 0, 0, 1, 0\rangle + (d + b)\langle 0, 0, 0, 1\rangle =$$

$$\langle (3a+2b)+(-a-b)2, -(3a+2b)-3(-a-b), 2(3a+2b)+6(-a-b)+(c+2b), (3a+2b)+3(-a-b)+(d+b)\rangle =$$

$$\langle 3a + 2b - 2a - 2b, -3a - 2b + 3a + 3b, 6a + 4b - 6a - 6b + c + 2b, 3a + 2b - 3a - 3b + d + b\rangle =$$

$$\langle a, b, c, d\rangle$$

Therefore we have an arbitrary element of V as a linear combination of the vectors $\mathbf{u_1}, \mathbf{u_2}, \mathbf{v_1}, \mathbf{v_2}$. Lastly to show linear independence. Consider the linear combination $a, b, c, d \in \mathbb{Q}$

$$a(1, -1, 2, 1) + b(2, -3, 6, 3) + c(0, 0, 1, 0) + d(0, 0, 0, 1) = 0$$

$$(a + 2b, -a - 3b, 2a + 6b + c, a + 3b + d) = (0, 0, 0, 0)$$

From this we get the system of equations

$$\begin{cases} a + 2b = 0 \\ -a - 3b = 0 \\ 2a + 6b + c = 0 \\ a + 3b + d = 0 \end{cases}$$

From the first two equations we get $a + 2b - a - 3b = 0$ which implies $b = 0$ substituting in 0 for b we get $a + 2 \cdot 0 = 0$ which implies $a = 0$. Replacing a, b in the bottom equations we get that $c = 0$ and $d = 0$ as well. As the scalars where chosen arbitrarily we have that the set is linear independent hence it is a basis for V.

□

Problem 4. Let $V = \mathcal{P}_3(\mathbb{R})$ be the $\mathbb{R}$-vector space of polynomials of degree 3 or less. Let U be the subspace (you can take this for granted)

$$U = \{p(x) \in \mathcal{P}_3(\mathbb{R}) \,:\, p'(7) = 0\}$$

where $p'(x)$ is the derivative of $p(x)$ and $p'(7)$ is the derivative evaluated at $x = 7$. Find a basis for U.

Proof. Consider the set $B = \left\{\frac{x^3}{147} + \frac{x^2}{14} - 2x,\ \frac{x^2}{14} - x,\ 1\right\}$. First I will show the linear independent of B. Consider the linear combination $a\left(\frac{x^3}{147} + \frac{x^2}{14} - 2x\right) + b\left(\frac{x^2}{14} - x\right) + c(1) = 0$ then as the right of the equation has no x^3 and the coefficient on x^3 is $\frac{a}{147}$ we have that $a = 0$. Likewise we have that $b = 0$ as the right of the equation has no x^2 lastly we have that $c = 0$. Now we have that $\dim(U) \leq \mathcal{P}_3(\mathbb{R}) = 4$. If $\dim(U) = 4$ then we would be able to add another vector to B and have B still be linearly independent denote this vector by $ax^3 + bx^2 + cx + d$ where $a, b, c, d \in \mathbb{R}$ however consider the linear combination

$$ax^3 + bx^2 + cx + d - a147\left(\frac{x^3}{147} + \frac{x^2}{14} - 2x\right) + -(14b - a147)(\frac{x^2}{14} - x) + d \cdot 1 =$$

$$ax^3 + bx^2 + cx + d - ax^3 - \frac{a147x^2}{14} + 294ax - 14bx^2 + 14bx + \frac{147ax^2}{14} - a147x - d =$$
$$cx + 294ax + 14bx = x(c + 294a + 14b)$$

If $c + 294a + 14b = 0$ then the set would not be linearly independent. If $c + 294a + 14b \neq 0$ then taking the derivative and evaluating at $x = 7$ would give a non-zero value. This implies that no such vector exists and that $\dim(U) = 3$. Thus B is a basis for U by **Theorem 2.39**. $\square$

Problem 5. The classical "Inclusion-Exclusion Principle" states that, for two finite sets A_1, A_2, the cardinality of the union satisfies:

$$|A_1 \cup A_2| = |A_1| + |A_2| - |A_1 \cap A_2|.$$

Notice that we have a similar formula for vector spaces V_1, V_2:

$$\dim(V_1 + V_2) = \dim(V_1) + \dim(V_2) - \dim(V_1 \cap V_2).$$

For three sets, A_1, A_2, A_3, the Inclusion-Exclusion Principle says

$$\begin{aligned} |A_1 \cup A_2 \cup A_3| = &|A_1| + |A_2| + |A_3| \\ &- |A_1 \cap A_2| - |A_1 \cap A_3| - |A_2 \cap A_3| \\ &+ |A_1 \cap A_2 \cap A_3|. \end{aligned}$$

Give an example showing that, sadly, the following analogous formula does not hold for vector spaces V_1, V_2, V_3:

$$\begin{aligned} \dim(V_1 + V_2 + V_3) = &\dim(V_1) + \dim(V_2) + \dim(V_3) \\ &- \dim(V_1 \cap V_2) - \dim(V_1 \cap V_3) - \dim(V_2 \cap V_3) \\ &+ \dim(V_1 \cap V_2 \cap V_3). \end{aligned}$$

Hint: consider subspaces of a familiar low-dimensional vector space.

Consider the three $\mathbb{R}$-vector spaces $V_1 = \{(a, 0) : a \in \mathbb{R}\}, V_2 = \{(0, b) : b \in \mathbb{R}\}, V_3 = \{(0, 0)\}$ We have $\dim(V_1 + V_2 + V_3) = 2$. However $\dim(V_1) + \dim(V_2) + \dim(V_3) - \dim(V_1 \cap V_2) - \dim(V_1 \cap V_3) - \dim(V_2 \cap V_3) + \dim(V_1 \cap V_2 \cap V_3) = 1 + 1 + 0 - 1 - 0 - 0 + 0 = 1$. Thus the formula does not hold.