

**Hand in Friday, March 15.**

1. Suppose that  $X$  and  $Y$  are topological spaces, that  $A \subset X$ , and that  $f$  and  $g$  are continuous maps from  $X$  to  $Y$  that satisfy, for all  $x \in A$ ,  $f(x) = g(x)$ . If  $Y$  is Hausdorff, show that, for all  $z$  in the closure of  $A$ ,  $f(z) = g(z)$ .

*Proof.* Assume  $X$  and  $Y$  are topological spaces, that  $A \subset X$ , and that  $f$  and  $g$  are continuous maps from  $X$  to  $Y$  that satisfy, for all  $x \in A$ ,  $f(x) = g(x)$ . Assuming  $Y$  is Hausdorff and that there exists some point  $z$  in the closure of  $A$  where  $f(z) \neq g(z)$ . Then as  $Y$  is Hausdorff we have two neighborhoods  $U_f$  of  $f(z)$  and  $U_g$  of  $g(z)$  where  $U_f \cap U_g = \emptyset$ . As  $f, g$  are continuous we have that  $f^{-1}(U_f)$  and  $g^{-1}(U_g)$  are both open. We have that  $z \in f^{-1}(U_f) \cap g^{-1}(U_g)$  but we have that  $z$  is a limit point as if  $z \in A$  then that would be an immediate contradiction on  $f(z) \neq g(z)$  hence we have  $A \cap f^{-1}(U_f) \cap g^{-1}(U_g) \setminus \{z\} \neq \emptyset$  hence we have for some  $a \in A \cap f^{-1}(U_f) \cap g^{-1}(U_g) \setminus \{z\}$  then we have a neighborhood  $U_a$  of  $a$  with  $U_a \subset f^{-1}(U_f) \cap g^{-1}(U_g)$  hence we have  $f(a) \in U_f \cap U_g$  which contradicts  $U_f$  and  $U_g$  being disjoint.  $\square$

2. Let  $f : X_1 \rightarrow Y_1$  and  $g : X_2 \rightarrow Y_2$  be continuous maps between topological spaces. Give the products  $X_1 \times X_2$  and  $Y_1 \times Y_2$  their product topologies. Show that the map  $H : X_1 \times X_2 \rightarrow Y_1 \times Y_2$  defined by  $H((x_1, x_2)) = (f(x_1), g(x_2))$  is continuous.

*Proof.* Assume that  $f : X_1 \rightarrow Y_1$  and  $g : X_2 \rightarrow Y_2$  are continuous maps between topological spaces. Assume we have  $X_1 \times X_2$  and  $Y_1 \times Y_2$  with their product topologies with the map  $H : X_1 \times X_2 \rightarrow Y_1 \times Y_2$  where  $H((x_1, x_2)) = (f(x_1), g(x_2))$ . Then consider an arbitrary basis element  $U_1 \times U_2 \subset Y_1 \times Y_2$  then we have by the definition of product topology that  $U_1$  is open in  $Y_1$  and  $U_2$  is open in  $Y_2$  and as  $f, g$  are continuous we have  $f^{-1}(U_1)$  and  $g^{-1}(U_2)$  are open hence  $f^{-1}(U_1) \times g^{-1}(U_2)$  is open in  $X_1 \times X_2$  this implies  $H^{-1}(U_1 \times U_2) = f^{-1}(U_1) \times g^{-1}(U_2)$  is open which shows that  $H$  is continuous.  $\square$

3. An injective (one-to-one) continuous map  $f : X \rightarrow Y$  between topological spaces is a bijection from  $X$  to the image  $f(X)$ . Give  $f(X)$  the subspace topology induced by  $Y$ 's topology. We call such an  $f$  an imbedding if it is a homeomorphism from  $X$  to  $f(X)$ .

In this context, let  $Y$  be  $X \times X$  with the product topology. Let  $x_0$  be an arbitrary element of  $X$ .

a. Show that  $f : X \rightarrow X \times X$  defined by  $f(x) = (x, x_0)$  is an imbedding.

*Proof.* Assume that  $f : X \rightarrow X \times X$  is a map defined by  $f(x) = (x, x_0)$  where  $x_0$  is an arbitrary element of  $X$ . Suppose  $x, y \in X$  with  $f(x) = f(y)$  then we have  $(x, x_0) = (y, x_0)$  hence  $x = y$  which shows that  $f$  is injective. Let  $U_1 \times U_2 \subset f(X)$  be an arbitrary basis element. Then as  $f(X) = X \times \{x_0\}$  we get that  $U_2 = \{x_0\}$  additionally from the definition of subspace topology we get  $U_1 = X \cap U_3$  where  $U_3$  is some open set in  $X$  hence  $U_1$  is also open. As  $f^{-1}(U_1 \times U_2) = U_1$  we have  $f^{-1}(U_1 \times U_2)$  is open. Hence  $f$  is continuous.

We have  $f^{-1} : f(X) \rightarrow X$  where  $f^{-1}(x, y) = x$  is a bijection as we have shown that  $f$  is injective and it is surjective with its own range. So we just need to show that  $f^{-1}$  is continuous. Given an arbitrary basis element  $U$  of  $X$ . We have that  $(f^{-1})^{-1} = f$  hence  $f(U) = U \times \{x_0\}$  and as  $U \times \{x_0\} = f(X) \cap (U \times X)$  and  $U \times X$  is open in the product topology we get  $U \times \{x_0\}$  is open in the subspace topology hence which shows that  $f^{-1}$  is continuous which implies  $f$  is a homeomorphism which shows that  $f$  is an imbedding.  $\square$

b. Show that  $g : X \rightarrow X \times X$  defined by  $g(x) = (x, x)$  is an imbedding.

*Proof.* Let  $g : X \rightarrow X \times X$  be defined by  $g(x) = (x, x)$ . Let  $x, y \in X$  be two elements with  $g(x) = g(y)$  then  $(x, x) = (y, y)$  which implies  $x = y$  hence  $g$  is injective. Now let  $U_1 \times U_2 \subset g(X)$  be an open set. Then we have  $U_1 \times U_2 = \{(x, x) : x \in X\} \cap A \times B$  for some open sets  $A, B$  in  $X$  which implies  $U_1 \times U_2 = \{(x, x) : x \in A \cap B\}$ . Then  $g^{-1}(U_1 \times U_2) = A \cap B$  and as  $A, B$  are open we get  $A \cap B$  is open which shows that  $g$  is continuous.

Let the map  $g^{-1} : g(X) \rightarrow X$  be defined by  $g^{-1}(x, x) = x$  we have already shown  $g$  is injective and as  $g$  is surjective with its own range we have that  $g^{-1}$  is a bijection. Let  $U \subset X$  be an open set then we have  $(g^{-1})^{-1} = g$  hence  $(g^{-1})^{-1}(U) = g(U) = \{(x, x) : x \in U\}$ . As  $\{(x, x) : x \in U\} = (U \times U) \cap g(X)$  we get  $\{(x, x) : x \in U\}$  is open in the subspace topology hence  $g^{-1}$  is continuous which implies that  $g$  is an imbedding.  $\square$

4. Suppose that  $h : X \rightarrow Y$  is a homeomorphism of topological spaces. If  $Z$  is any other topological space and if  $g : Y \rightarrow Z$  is a continuous map, we know that the composition  $g \circ h$  is a continuous map from  $X$  to  $Z$ . Show that every continuous map  $f : X \rightarrow Z$  arises this way, i.e. for any continuous  $f : X \rightarrow Z$ , there exists a continuous  $G : Y \rightarrow Z$  for which  $f = G \circ h$ .

*Proof.* Assume  $h : X \rightarrow Y$  is a homeomorphism of topological spaces and  $Z$  is some other topological space. Then given an arbitrary continuous function  $f : X \rightarrow Z$  we need to create a function  $G : Y \rightarrow Z$  such that  $G \circ h = f$ . First I will prove the existence and then that it is continuous. Define  $G : Y \rightarrow Z$  where  $G(y) = f(h^{-1}(y))$ .

Now to prove that  $G$  is continuous let  $U \subset Z$  be an open set. Then as  $f^{-1}(U)$  is open in  $X$  we have that  $h(f^{-1}(U))$  is open in  $Y$  as homeomorphisms preserve open sets (this is immediate based off both directions of homeomorphisms being continuous) we just need to show  $G^{-1}(U) = h(f^{-1}(U))$ .

Let  $y \in G^{-1}(U)$  as  $h$  is a homeomorphism we have for some unique  $x \in X$  that  $h(x) = y$  and as  $G(y) = f(x) \in U$  we get  $x \in f^{-1}(U)$  which implies  $y = h(x) \in h(f^{-1}(U))$  hence  $y \in h(f^{-1}(U))$  which gives  $G^{-1}(U) \subset h(f^{-1}(U))$

Now let  $y \in h(f^{-1}(U))$ . Then as  $h$  is a homeomorphism we have for some  $x \in f^{-1}(U)$  that  $y = h(x)$  but then  $G(y) = f(h^{-1}(y)) = f(x) \in U$  which implies  $y \in G^{-1}(U)$  hence  $h(f^{-1}(U)) \subset G^{-1}(U)$  then  $h(f^{-1}(U)) = G^{-1}(U)$  which implies  $G$  is continuous. As for any  $x \in X$  we have  $G \circ h(x) = f(h^{-1}(h(x))) = f(x)$  we get  $f = G \circ h$ .  $\square$

5. a. Show that a linearly ordered set with the order topology is Hausdorff.

*Proof.* Suppose  $X$  is a linearly ordered set with the order topology. Let  $a, b \in X$  with  $a \neq b$  and without loss of generality assume  $a < b$ . Then let  $U_a$  and  $U_b$  be neighborhoods of  $a, b$  respectively. I will proceed by cases.

- (1) If there is no element  $c \in X$  with  $a < c < b$  and  $a$  is the minimum element of  $X$  and  $b$  is the maximum element of  $X$  consider the open sets  $a \in U_a \cap [a, b)$  and  $b \in U_b \cap (a, b]$  we have each of the sets is open and  $U_a \cap [a, b) \cap U_b \cap (a, b] = \emptyset$ .
- (2) If there exists an element  $c \in X$  with  $a < c < b$  and  $a$  is the minimum element of  $X$  and  $b$  is the maximum. Then consider the open sets  $U_a \cap [a, c)$  and  $U_b \cap (c, b]$  then we have  $a \in U_a \cap [a, c)$  and  $U_b \cap (c, b]$  but  $U_a \cap [a, c) \cap U_b \cap (c, b] = \emptyset$ .
- (3) If there is no element  $c \in X$  with  $a < c < b$  and  $a$  is not a minimum element of  $X$  and  $b$  is not a maximum of  $X$  then we have the sets  $U_a \cap (d, b)$  where  $d \in X$  with  $d < a$  is open and the set  $U_b \cap (a, l)$  where  $l \in X$  with  $b < l$  is open. We have  $a \in U_a \cap (d, b)$  and  $b \in U_b \cap (a, l)$  but  $U_a \cap (d, b) \cap U_b \cap (a, l) = \emptyset$ .
- (4) If there exists some element  $c \in X$  with  $a < c < b$  and  $a$  is not a minimum of  $X$  and  $b$  is not a maximum of  $X$  then we have the sets  $U_a \cap (d, c)$  where  $d \in X$  and  $d < a$  is open and the set  $U_b \cap (c, l)$  where  $l \in X$  with  $b < l$  is open. We have  $a \in U_a \cap (d, c)$  and  $b \in U_b \cap (c, l)$  where  $l \in X$  with  $b < l$  but  $U_a \cap (d, c) \cap U_b \cap (c, l) = \emptyset$ .
- (5) If there exists no element  $c \in X$  with  $a < c < b$  and without loss of generality  $a$  is a minimum of  $X$  and  $b$  is not the maximum (the case where  $a$  is not min of  $X$  and  $b$  is max of  $X$  will follow by almost the same exact reasoning) then consider the open sets  $a \in U_a \cap [a, b)$  and  $b \in U_b \cap (a, d)$  where  $d \in X$  with  $b < d$  then we have  $U_a \cap [a, b) \cap U_b \cap (a, d) = \emptyset$ .
- (6) If there exists some element  $c \in X$  with  $a < c < b$  and without loss of generality  $a$  is a minimum of  $X$  and  $b$  is not the maximum. Then  $a \in U_a \cap [a, c)$  and  $b \in U_b \cap (c, d)$  where  $d \in X$  with  $b < d$  then  $U_a \cap [a, c) \cap U_b \cap (c, d) = \emptyset$ .

This completes all the cases hence we get  $X$  is a Hausdorff space.  $\square$

**b.** Suppose that  $X$  is a topological space. Show that  $X$  is Hausdorff if and only if the diagonal subset  $\{(x, x) : x \in X\}$  of the product  $X \times X$  is a closed subset of the product. Assume here that the topology on  $X \times X$  is the product topology.

*Proof.*

( $\Rightarrow$ )

Suppose  $X$  is a topological space and  $X$  is Hausdorff. Now assume that  $X \times X$  has the product topology. Assume that  $(a, b) \in X \times X$  with  $a \neq b$  is a limit point of  $\{(x, x) : x \in X\}$ . Then as  $a \neq b$  and  $X$  is Hausdorff we have two neighborhoods  $U_a, U_b$  of  $a, b$  respectively with  $U_a \cap U_b = \emptyset$  then we have  $(a, b) \in U_a \times U_b$  but as  $U_a \cap U_b = \emptyset$  we get  $(U_a \times U_b) \cap \{(x, x) : x \in X\} = \emptyset$  hence  $(a, b)$  is not a limit point of  $\{(x, x) : x \in X\}$ .

This implies either there are no limit points of  $\{(x, x) : x \in X\}$  or  $\{(x, x) : x \in X\}' \subset \{(x, x) : x \in X\}$  in either case we get  $\overline{\{(x, x) : x \in X\}} = \{(x, x) : x \in X\} \cup \{(x, x) : x \in X\}' = \{(x, x) : x \in X\}$  hence  $\{(x, x) : x \in X\}$  contains its limit points so its closed.

( $\Leftarrow$ )

Assume that  $X$  is a topological space and that  $X \times X$  has the product topology and  $\{(x, x) : x \in X\}$  is closed in  $X \times X$ . Then for any  $a, b \in X$  with  $a \neq b$  we have  $(a, b) \in \{(x, x) : x \in X\}^c$  as there exists a basis element of the form  $(a, b) \in U_a \times U_b \subset \{(x, x) : x \in X\}^c$  but as we have  $U_a \times U_b \cap \{(x, x) : x \in X\} = \emptyset$  we get that there exists no  $x \in X$  such that  $(x, x) \in U_a \times U_b$  which implies  $U_a \cap U_b = \emptyset$  this implies that  $X$  is Hausdorff. □

**6.** Let  $Y$  be an ordered set in the order topology. Let  $f, g : X \rightarrow Y$  be continuous.

(1) Show that the set  $\{x : f(x) \leq g(x)\}$  is closed in  $X$ .

*Proof.* Assume that  $X$  is a topological space and  $Y$  is an ordered set in the order topology and  $f, g : X \rightarrow Y$  are continuous. We have by Munkres Theorem 17.11 that simply ordered sets are Hausdorff in the order topology. So I just need to show  $\{x : f(x) \leq g(x)\}^c$  is open. We have  $\{x : f(x) \leq g(x)\}^c = \{x : f(x) > g(x)\}$  let  $x \in \{x : f(x) > g(x)\}$  then as  $f(x) \neq g(x)$  and  $Y$  is Hausdorff we have two disjoint neighborhoods  $U_f$  and  $U_g$  of  $f(x), g(x)$  respectively. □

(2) Let  $h : X \rightarrow Y$  be the function

$$h(x) = \min\{f(x), g(x)\}$$

. Show that  $h$  is continuous.