Math 4324 Connected Spaces and Compact Spaces

Hand in Friday, April 5.

Convention. Assume that each subset of Euclidean space that you encounter on this assignment has the subspace topology it gets from the standard topology on Euclidean space. You may use without proof results such as: closed, bounded subsets of Euclidean space are compact; polynomial functions on Euclidean space are continuous; and sums, products, etc. of continuous functions are continuous; but be explicit about the dependence of your arguments on compactness and continuity.

1. a. Show that no one of the three intervals $[0,1]$, $(0,1]$, and $(0,1)$ is homeomorphic to any other one of them. **Hint.** Look for a homeomorphism-invariant property one has that the others don't have. Where that's not enough, consider an argument that the space resulting from removing some point from one of these cannot be homeomorphic to the space resulting from removing an arbitrary point from another of these. If you think of such an argument, be sure to express it clearly and justify its steps.

Proof. That $[0,1]$ not homeomorphic to $(0,1]$

Suppose that a homeomorphism $h : [0,1] \to (0,1]$ exists. Then consider the two subspaces $(0,1]$ of $[0,1]$ and $(0,1] \setminus \{a_0\}$ of $(0,1]$ where $a_0 \in (0,1]$. We have that $h\mid_{(0,1]}: (0,1] \to (0,1] \setminus \{a_0\}$ is a homeomorphism for some $a_0 \in (0,1]$.

Now I claim that $(0,1]$ is connected. Suppose that U, V are a separation of $(0,1]$ then we would have that two intervals of the form $(a,b] \subset U$ and $(b,c) \subset V$ for some $a,b,c \in (0,1)$. But then $b \in V'$ and $b \in U$ hence this can not be a separation by Munkres Lemma 23.1.

By Munkres Theorem 23.5 we have that $(0,1] \setminus \{a_0\}$ is connected as well. Now if $a_0 \in (0,1)$ then we would have the separation $(0,a_0)$ and $(a_0,1]$ this implies that $a_0 = 1$. By the assumption we would have that $h\mid_{(0,1]}: (0,1] \to (0,1)$ is a homeomorphism. But then we would have $h|_{(0,1)} : (0,1) \to (0,1) \setminus \{a_1\}$ is a homeomorphism for some $a_1 \in (0,1)$. But using similar reasoning to above we get that $(0,1)$ is connected while $(0,1) \setminus \{a_1\}$ is not. This is a contradiction and so no such homeomorphism exists.

Now suppose that a homeomorphism $h : (0,1] \to (0,1)$ exists. Then consider the two subspaces $(0,1)$ of $(0,1]$ and $(0,1) \setminus \{a_0\}$ of $(0,1)$ where $a_0 \in (0,1)$. We have that $h\mid_{(0,1)}: (0,1) \to (0,1) \setminus \{a_0\}$ is a homeomorphism for some $a_0 \in (0,1)$. But as $(0,1)$ is connected (using same argument as above) we have that $(0,1) \setminus \{a_0\}$ is connected. But we have the separation $(0,a_0), (a_0,1)$ hence $(0,1) \setminus \{a_0\}$ is not connected. This is a contradiction and so no such homeomorphism exists.

The argument for $[0,1]$ and $(0,1)$ is essentially the same as the argument for $(0,1]$ and $(0,1)$.

□

b. Show that $\mathbb{R}$ is not homeomorphic to $\mathbb{R}^2$.

Proof. Suppose that a homeomorphism $h : \mathbb{R}^2 \to \mathbb{R}$ exists. Then we have that

$$h|_{\mathbb{R}^2 \setminus \{\vec{0}\}} : \mathbb{R}^2 \setminus \{\vec{0}\} \to \mathbb{R} \setminus \{a\}$$

is a homeomorphism for some $a \in \mathbb{R}$.

I will prove that $\mathbb{R}^2 \setminus \{\vec{0}\}$ is path connected which implies that $\mathbb{R} \setminus \{a\}$ is connected. Let $\vec{x}, \vec{y} \in \mathbb{R}^2 \setminus \{\vec{0}\}$.

Then consider the function $f : [0,1] \to \mathbb{R}^2 \setminus \{\vec{0}\}$ where if for all $t \in [0,1]$ we have $\vec{x}(1-t) + t \cdot \vec{y} \neq \vec{0}$ then $f(t) = \vec{x}(1-t) + t(\vec{y})$ otherwise

$$f(t) = \begin{cases} \vec{a}(1-2t) + 2t(\vec{b} - (0,1)) & \text{if } t \in [0,1/2] \\ (2-2t)(\vec{b} - (0,1)) + (2t-1)\vec{b} & \text{if } t \in (1/2,1] \end{cases} \tag{1}$$

The fact that f does not intersect the origin is clear for the first case in the definition of the function. For the second we would have that the line that intersects the two points $\vec{a}, \vec{b}$ would also intersect $\vec{0}$ hence we get that the line that intersects $\vec{a}, (\vec{b} - (0,1))$ would not intersect $\vec{0}$. Using similar reasoning we get the line that intersects $(\vec{b} - (0,1)), \vec{b}$ does not intersect the origin.

We have that each of the piecewise definitions of f are continuous on their respective domains hence by the pasting lemma we get f is continuous. This implies that $\mathbb{R}^2 \setminus \{\vec{0}\}$ is path connected which implies that it is connected. But we have $\mathbb{R} \setminus \{a\}$ is not connected as we have the separation $(-\infty, a), (a, \infty)$ as homeomorphisms preserve connectedness we get that no such homeomorphism exists.

$\square$

2. a. Show that any continuous map $f : [0,1] \to [0,1]$ has a fixed point, by which I mean a point x for which $f(x) = x$. **Hint.** Consider the function $F(x) = f(x) - x$. **Something to think about.** Is the same result true for continuous maps $[0,1] \times [0,1] \to [0,1] \times [0,1]$? I am asking you to think briefly about, not write an answer to, this question about the square. It may be instructive to explore briefly the challenges of trying to extend the interval result to the product.

Proof. Assume that there exists a continuous function $f : [0,1] \to [0,1]$ that has no fixed points and consider the function $F(x) = f(x) - x$. We have that F is continuous and as $F(0) = f(0) - 0 > 0$ and $F(1) = f(1) - 1 < 0$ we have by the intermediate value theorem that there exists $r \in [0,1]$ such that $F(r) = 0 = f(r) - r$ which implies $f(r) = r$ which is a contradiction. So we have all continuous functions $f : [0,1] \to [0,1]$ have a fixed point. $\square$

b. Is it true that any continuous $g : [0,1) \to [0,1)$ must have a fixed point? Show that your answer is correct. **Aside.** Consider (but you need not write): if you provide an example to show that the answer is no, why does your example not extend to contradict what you proved in part a?

No the function $f : [0,1) \to [0,1)$ where $f(x) = \frac{1}{2}x + \frac{1}{10}$ is a counterexample.

3. a. Suppose that X is homeomorphic to $[0,1]$. Must any continuous $\phi : X \to X$ have a fixed point?

Yes this is true.

Proof. Assume that X is homeomorphic to $[0,1]$ with homeomorphism $h : [0,1] \to X$ and suppose that $\phi : X \to X$ is an arbitrary continuous function. Then we have the function $h^{-1} \circ \phi \circ h : [0,1] \to [0,1]$ is a continuous function hence it has a fixed point $x_0 \in [0,1]$. This implies that $h(x_0)$ is a fixed point of ϕ which follows from $h \circ h^{-1} \circ \phi \circ h(x_0) = h(x_0) \implies \phi \circ h(x_0) = h(x_0)$. This shows that all continuous functions $\phi : X \to X$ have a fixed point. $\square$

b. Show that there is a continuous map $f : [-2,-1] \bigcup [1,2] \to [-2,-1] \bigcup [1,2]$ that has no fixed point.

Proof. Consider the function $f : [-2,-1] \bigcup [1,2] \to [-2,-1] \bigcup [1,2]$ defined by the equation $f(x) = -x$.

Now to show that f is continuous. Let U be an arbitrary basis element of $[-2,-1] \bigcup [1,2]$ then $f^{-1}(U) = \{-u : u \in U\}$ which is open in $[-2,-1] \bigcup [1,2]$ hence f is continuous. The fact that f does not have any fixed points is due to the domain not including 0 as all other points switch their signs. $\square$

4. Show that for any continuous $f : S^1 \to \mathbb{R}$, there is a point $\vec{p}$ for which $f(\vec{p}) = f(-\vec{p})$. Here S^1 is the unit circle, $\{(x,y) : x^2 + y^2 = 1\}$ in $\mathbb{R}^2$. **Hint.** Consider the function $F(\vec{p}) = f(\vec{p}) - f(-\vec{p})$.

Proof. Suppose that there exists $f : S^1 \to \mathbb{R}$ where $f(\vec{p}) \neq f(-\vec{p})$ for some $\vec{p} \in S^1$. Then consider the function $F : S^1 \to \mathbb{R}$ where $F(x) = f(x) - f(-x)$. We have that F is continuos and by the assumption that $f(\vec{p}) \neq f(-\vec{p})$ we get either $F(\vec{p}) = f(\vec{p}) - f(-\vec{p}) > 0$ or $F(\vec{p}) = f(\vec{p}) - f(-\vec{p}) < 0$. Without loss of generality assume that $F(\vec{p}) = f(\vec{p}) - f(-\vec{p}) > 0$ then we also have $F(-\vec{p}) = f(-\vec{p}) - f(\vec{p}) < 0$ hence by Munkres Theorem 24.3 (intermediate value theorem) we get that there exists $\vec{q} \in S^1$ such that $F(\vec{q}) = 0 = f(\vec{q}) - f(-\vec{q})$ which implies $f(\vec{q}) = f(-\vec{q})$ which is a contradiction. Hence we have that for all continuous functions $f : S^1 \to \mathbb{R}$ there exists a point $\vec{p}$ such that $f(\vec{p}) = f(-\vec{p})$. $\square$

5. Suppose that X is a Hausdorff topological space and that C and K are compact subspaces of X that satisfy $C \bigcap K = \emptyset$. Show that there exist open subsets of X, U and V, that satisfy $C \subset U$, $K \subset V$, and $U \bigcap V = \emptyset$.

Proof. Suppose that X is a Hausdorff topological space and C, K are compact subspaces of X.

For $c_0 \in C$ we have that that for each $k \in K$ that there are two disjoint neighborhoods $k \in V_k$ and $c_0 \in U_k$ we have that $\{V_k : k \in K\}$ is a cover of K (this cover is created from the neighborhoods of elements of K which are disjoint with some neighborhood of a fixed element of C).

Now as K is compact we have that $\{V_k : k \in K\}$ has a finite subcover $K \subset \mathcal{V}_{c_0} := V_{k_1} \cup ... \cup V_{k_n}$ and $c_0 \in \mathcal{U}_{c_o} := U_{k_1} \cap ... \cap U_{k_n}$ where each $c_0 \in U_{k_i}$ is the corresponding disjoint set of V_{k_i}.

We have that $\mathcal{V}_{c_0} \cap \mathcal{U}_{c_0} = \emptyset$ which follows because each V_{k_i} is disjoint from its corresponding U_{k_i}.

Then take the cover $\{\mathcal{V}_c : c \in C\}$ and $\{\mathcal{U}_c : c \in K\}$ of C and K respectively. We have that C is compact hence there exists a finite subcover $C \subset \mathcal{V}_{c_1} \cup ... \cup \mathcal{V}_{c_n}$ and we have from the definition of $\mathcal{U}_c$ that $K \subset \mathcal{U}_c$ which implies $K \subset \mathcal{U}_{c_1} \cap ... \cap \mathcal{U}_{c_n}$. Then as each $\mathcal{V}_{c_i}$ is disjoint from the corresponding $\mathcal{U}_{c_i}$ we get that $\left(\mathcal{V}_{c_1} \cup ... \cup \mathcal{V}_{c_n}\right) \bigcap \left(\mathcal{U}_{c_1} \cap ... \cap \mathcal{U}_{c_n}\right) = \emptyset$

$\square$