Define explicitly a continuous branch of $\log z$ in the complex plane slit along the negative imaginary axis, $\mathbb{C} \setminus [0, -i\infty)$.

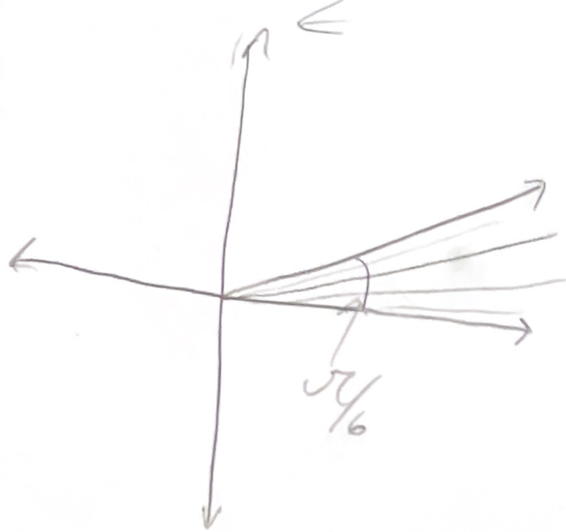
This branch is given by $f(re^i\theta) = \log r + i\theta$ where $\theta \in (-\frac{\pi}{2}, \frac{3\pi}{2})$.

Sketch the image of the sector $\{0 < \arg z < \frac{\pi}{6}\}$ under the map $w = z^a$.

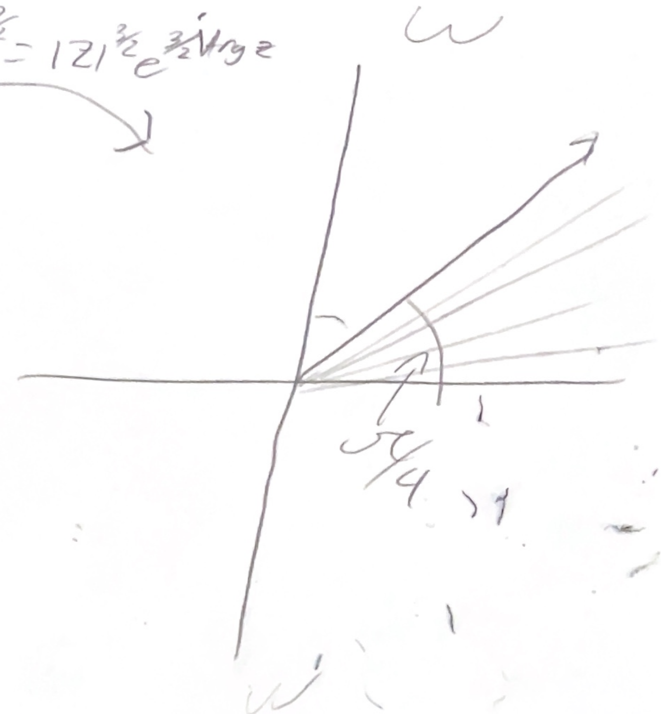
- $a = \frac{3}{2}$
- $a = i$

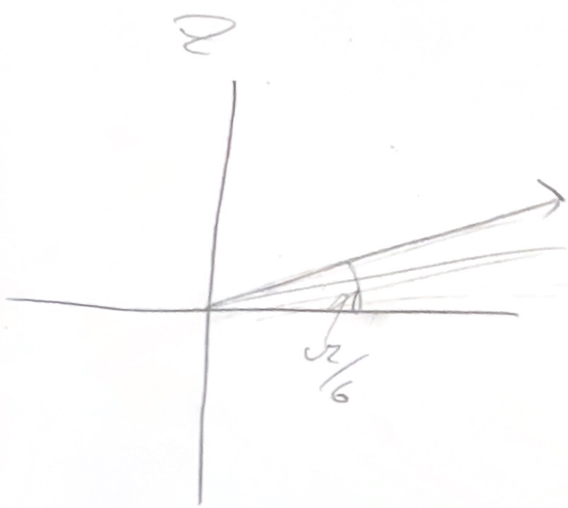
$$w = z^{\frac{3}{2}} = e^{\frac{3}{2}(\log|z| + i\mathrm{Arg} z)} = |z|^{\frac{3}{2}} e^{\frac{3}{2} i \mathrm{Arg} z}$$

$$w = z^i = e^{i \log z} = e^{i(\log|z| + i\mathrm{Arg} z)} = e^{-\mathrm{Arg} z + i\log|z|}$$

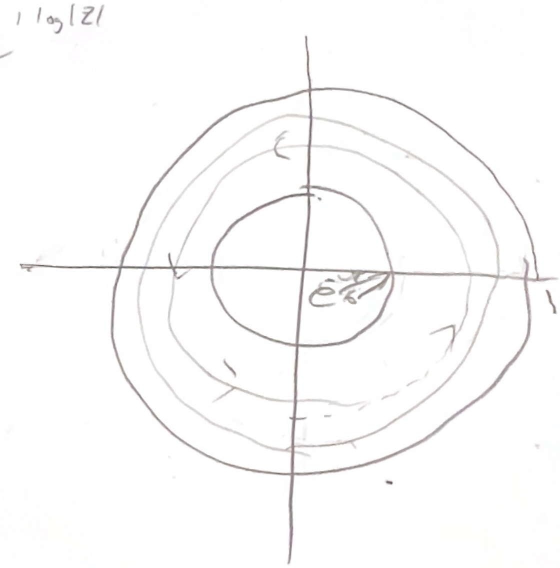


$$w = z^{3/2} = |z|^{3/2} e^{\frac{3}{2} i \operatorname{Arg} z}$$





$$w = e^{-\operatorname{Arg} z} e^{i \log |z|}$$



Determine the phase factors of the function $z^a(1-z)^b$ at the branch points $z=0$ and $z=1$. What conditions on a and b are necessary for the function to be single-valued on $\mathbb{C}\setminus[0,1]$?

The phase factor at the branch point $z=0$ is given by $e^{2\pi ia}$ and the phase factor at the branch point $z=1$ is given by $e^{2\pi ib}$. The function is single-valued on $\mathbb{C}\setminus[0,1]$ if we have $e^{i\pi a}e^{i\pi b}=1$ which happens when $a+b$ is an integer.

Show that if f is analytic on D then $g(z)=\overline{f(\bar{z})}$ is analytic on the reflected domain $D^\star=\{\bar{z} : z\in D\}$, and $g'(z)=\overline{f'(\bar{z})}$.

Proof. Suppose f is analytic on D. Then for any $z_0\in D^\star$ we have

$$\begin{aligned}
\lim_{\Delta z\mapsto 0}\frac{g(z_0+\Delta z)-g(z_0)}{\Delta z} &= \lim_{\Delta z\mapsto 0}\frac{\overline{f(\overline{z_0+\Delta z})-\overline{f(\overline{z_0})}}}{\Delta z}\\
&= \lim_{\Delta z\mapsto 0}\overline{\frac{f(\overline{z}_0+\overline{\Delta z})-f(\overline{z}_0)}{\overline{\Delta z}}}\\
&= \overline{\lim_{\Delta z\mapsto 0}\frac{f(\overline{z}_0+\overline{\Delta z})-f(\overline{z}_0)}{\overline{\Delta z}}}\\
&= \overline{f'(\overline{z}_0)}
\end{aligned}$$

This shows that $g'(z_0)$ and $g'(z_0)=\overline{f'(\overline{z}_0)}$. Now to show that g is analytic on $D^\star$. As f is analytic on D we have for any $\overline{z}\in D^\star$ for any $\epsilon>0$ there exists $\delta>0$ such that for all $\overline{z_0}\in D^\star$ with $|\overline{z}-\overline{z}_0|<\delta$ implies $|g'(\overline{z})-g'(\overline{z}_0)|=|\overline{f'(z)}-\overline{f'(z_0)}|=|f'(z)-f'(z_0)|<\epsilon$. ☐

Let $h(t)$ be a continuous function on $[0,1]$ and define

$$H(z)=\int_0^1\frac{h(t)}{t-z}dt,\quad z\in\mathbb{C}\setminus[0,1]$$

Show that $H(z)$ is analytic and compute its derivative.

Proof. Let $H(z)$ be as defined let $z\in\mathbb{C}\setminus[0,1]$ then

$$\begin{aligned}
\lim_{\Delta z \mapsto 0} \frac{H(z+\Delta z) - H(z)}{\Delta z} &= \lim_{\Delta z \mapsto 0} \frac{1}{\Delta z} \left(\int_0^1 \frac{h(t)}{t-(z+\Delta z)} dt - \int_0^1 \frac{h(t)}{t-z} dt \right) \\
&= \lim_{\Delta z \mapsto 0} \frac{1}{\Delta z} \int_0^1 \frac{h(t)}{t-z-\Delta z} - \frac{h(t)}{t-z} dt \\
&= \lim_{\Delta z \mapsto 0} \int_0^1 \frac{h(t)}{(t-z-\Delta z)(t-z)} dt \\
&= \int_0^1 \lim_{\Delta z \mapsto 0} \frac{h(t)}{(t-z-\Delta z)(t-z)} dt \\
&= \int_0^1 \frac{h(t)}{(t-z)^2} dt
\end{aligned}$$

This shows that $H'(z)$ exists on $\mathbb{C} \setminus (0,1)$.

$\square$