Problem 1. Prove or provide a counterexample: For any $\alpha \in V_{alt}^{(4)}$, the set

$$U = \left\{(\mathbf{v_1}, \mathbf{v_2}, \mathbf{v_3}, \mathbf{v_4}) \in V^4 \ : \ \alpha(\mathbf{v_1}, \mathbf{v_2}, \mathbf{v_3}, \mathbf{v_4}) = 0\right\}$$

is a subspace of V^4.

This is not true.

Proof. Let $V = \mathbb{R}^4$ be an $\mathbb{R}$ vector space. Define $\alpha \in \mathbb{R}_{\text{alt}}^{(4)}$ by

$$\alpha(v_1, v_2, v_3, v_4) = \det \begin{pmatrix} v_{1,1} & v_{1,2} & v_{1,3} & v_{1,4} \\ v_{2,1} & v_{2,2} & v_{2,3} & v_{2,4} \\ v_{3,1} & v_{3,2} & v_{3,3} & v_{3,4} \\ v_{4,1} & v_{4,2} & v_{4,3} & v_{4,4} \end{pmatrix}$$

Which is an alternating 4-multilinear form by Theorem 9.4.5.

Now take the two vectors of V^4 (I will put them in the form of a matrix for clarity).

$$v_1 = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad \text{and} \quad v_2 = \begin{pmatrix} 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ -1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

The fact that each of these vectors have a zero determinant is based off the fact that each contain a column of all 0s hence any permutation will have a 0 in the product. But

$$\alpha(v_1 + v_2) = \det \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = 1$$

We have this is a diagonal matrix hence by problem 3 we have it's determinant is 1.

□

Problem 2. Let $A = \begin{pmatrix} A_{1,1} & A_{1,2} & A_{1,3} \\ A_{2,1} & A_{2,2} & A_{2,3} \\ A_{3,1} & A_{3,2} & A_{3,3} \end{pmatrix}$. Let's compute the determinant from the definition, regarding it as a map $\det : \mathbb{K}^3 \to \mathbb{K}$ given by

$$\det(\mathbf{v_1}, \mathbf{v_2}, \mathbf{v_3}) = \sum_{\substack{\text{perm.} \\ \sigma}} \operatorname{sgn}(\sigma) A_{\sigma(1),1} \cdots A_{\sigma(3),3}.$$

1. Fill out the table to describe all permutations of the triple $(1, 2, 3)$, and find their signs.
2. Compute $\det(A)$.

Using the equations

1. Fill out the table below.

Permutation $\boldsymbol{\sigma}$	**Explicit Description**	**sgn($\boldsymbol{\sigma}$)**
$\sigma_1 = \mathrm{Id}$	$(1,2,3) \mapsto (1,2,3) = (1)(2)(3)$	1
σ_2	$(1,2,3) \mapsto (2,1,3) = (12)(3)$	-1
σ_3	$(1,2,3) \mapsto (3,2,1) = (13)(2)$	-1
σ_4	$(1,2,3) \mapsto (1,3,2) = (23)(1)$	-1
σ_5	$(1,2,3) \mapsto (3,1,2) = (132)$	1
σ_6	$(1,2,3) \mapsto (2,3,1) = (123)$	1

2.

$$\det(\mathbf{v_1}, \mathbf{v_2}, \mathbf{v_3}) = \sum_{\mathbf{i=1}}^{\mathbf{6}} \operatorname{sgn}(\sigma_{\mathbf{i}}) \mathbf{A}_{\sigma_{\mathbf{i}}(\mathbf{1}),\mathbf{1}} \mathbf{A}_{\sigma_{\mathbf{i}}(\mathbf{2}),\mathbf{2}} \mathbf{A}_{\sigma_{\mathbf{i}}(\mathbf{3}),\mathbf{3}}$$

$$= A_{1,1}A_{1,2}A_{1,3} - A_{2,1}A_{1,2}A_{3,3} - A_{3,1}A_{2,2}A_{1,3} - A_{1,1}A_{3,2}A_{2,3} + A_{3,1}A_{1,2}A_{2,3} + A_{2,1}A_{3,2}A_{1,3}$$

Problem 3. Suppose $A = \begin{pmatrix} A_{1,1} & * & \cdots & * \\ 0 & A_{2,2} & \cdots & * \\ 0 & 0 & \ddots & \vdots \\ 0 & 0 & 0 & A_{n,n} \end{pmatrix}$. Then

$$\det(A) = A_{1,1}A_{2,2}\cdots A_{n,n}.$$

It may be helpful to acknowledge that A can be described by the fact that $A_{i,j} = 0$ whenever $j < i$.

Proof. Using the fact that $A_{i,j} = 0$ whenever $j < i$. It suffices to show for any permutation other then the identity we have some $\sigma(i) < i$.

Assume that there exists a permutation of n elements $\sigma \neq \mathrm{Id}$ such that $\sigma(i) \geq i$ for all i. Then $\sigma(n) = \sigma(n)$ (because n is the max of the n elements) based on the fact that this is a bijection we get that $\sigma(n-1) = n-1$. Continuing this process we get that $\sigma(i) = i$ for all $i \in \{1, ..., n\}$ which contradicts the fact that $\sigma \neq \mathrm{Id}$. Hence we have that $\sigma(i) < i$ for some $i \in \{1, ..., n\}$.

Then all the terms in the sum are 0 except for the identity permutation. Hence we get $\det(A) = A_{\mathrm{Id}(1),1}, ..., A_{\mathrm{Id}(n),n} = A_{1,1}, ..., A_{n,n}$

□