Problem 1. For a fixed $a \in \mathbb{C}$, show that $\frac{|z-a|}{|1-\bar{a}z|} = 1$ if $|z| = 1$ and $1 - \bar{a}z \neq 0$.

Proof. Assume $a \in \mathbb{C}$ and $z \in \mathbb{Z}$ with $|z| = 1$ and $1 - \bar{a}z \neq 0$. Then for some $x, y, c, d \in \mathbb{R}$ we have $a = x + iy$ and $z = c + id$.

Calculating $|1 - \bar{a}z|$ yields

$$\begin{aligned}
|1-\bar{a}z| &= |1-(x-iy)(c+id)| && (1)\\
&= \sqrt{(1-xc-yd)^2+(yc-xd)^2} && (2)\\
&= \sqrt{1-2xc-2yd+2xcyd+x^2c^2+y^2d^2+y^2c^2-2ycxd+x^2d^2} && (3)\\
&= \sqrt{1-2xc-2yd+x^2c^2+x^2d^2+y^2d^2+y^2c^2} && (4)\\
&= \sqrt{1-2xc-2yd+x^2(c^2+d^2)+y^2(c^2+d^2)} && (5)\\
&= \sqrt{1-2xc-2yd+x^2+y^2} && (6)\\
&= \sqrt{c^2+d^2+x^2+y^2-2xc-2yd} && (7)\\
&= \sqrt{(c-x)^2+(d-y)^2} && (8)\\
&= |z-a| && (9)
\end{aligned}$$

Based on the assumption of $|1 - \bar{a}z| \neq 0$ we have

$$\frac{|z-a|}{|1-\bar{a}z|} = 1$$

□