

Problem 1. Consider $V = \mathbb{C}$, the complex numbers, as a \mathbb{C} -vector space. Define a function $\Re : \mathbb{C} \rightarrow \mathbb{C}$ by

$$\Re(x + iy) = x$$

Is \Re a linear map? If so, prove it. If not, explain why not.

No it is not

Proof. No as it does not satisfy homogeneity. Consider $(1 + i) \cdot \Re(1 + i) = 1 + i$ but $\Re((1 + i) \cdot (1 + i)) = \Re(2i) = 0$ so it does not satisfy homogeneity.

□

Problem 2. Extending a linear map. Let V be a finite-dimensional \mathbb{K} -vector space, W a \mathbb{K} -vector space, and U a subspace of V . Furthermore, let

$$\begin{aligned} \{\mathbf{u}_1, \dots, \mathbf{u}_m\} &\text{ be a basis for } U \text{ and} \\ \{\mathbf{u}_1, \dots, \mathbf{u}_m, \mathbf{v}_{m+1}, \dots, \mathbf{v}_n\} &\text{ be the (extended) basis for } V. \end{aligned}$$

Taking $T : U \rightarrow W$ to be any linear map, define a function $f : V \rightarrow W$ by

$$f(a_1 \mathbf{u}_1 + \dots + a_m \mathbf{u}_m + a_{m+1} \mathbf{v}_{m+1} + \dots + a_n \mathbf{v}_n) = T(a_1 \mathbf{u}_1 + \dots + a_m \mathbf{u}_m).$$

- (a) Is f a linear map? If so, prove it. If not, explain why not.
 (b) What happens if the definition of f is changed to the following?

$$f(\mathbf{x}) = \begin{cases} T(\mathbf{x}) & \text{if } \mathbf{x} \in U \\ \mathbf{0} & \text{otherwise} \end{cases}$$

- (a) *Proof.* Yes it is a linear map

Let $\lambda \in \mathbb{K}$ and let $\vec{v} = \mathbf{a}_1 \mathbf{u}_1 + \dots + \mathbf{a}_m \mathbf{u}_m + \mathbf{a}_{m+1} \mathbf{v}_{m+1} + \dots + \mathbf{a}_n \mathbf{v}_n \in V$ and $u = \mathbf{b}_1 \mathbf{v}_1 + \dots + \mathbf{b}_m \mathbf{v}_m + \mathbf{v}_{m+1} \mathbf{v}_{m+1} + \dots + \mathbf{b}_m \mathbf{v}_n \in V$. Where $u_i, w_i \in \text{Span}(U)$ and $a_i \in \mathbb{K}$

$$\begin{aligned} f(\lambda \vec{v} + \vec{u}) &= T(\lambda(a_1 u_1 + \dots + a_m u_m) + b_1 w_1 + \dots + b_m w_m) \\ &= T(\lambda(a_1 u_1 + \dots + a_m u_m)) + T(b_1 w_1 + \dots + b_m w_m) \\ &= \lambda T(a_1 u_1 + \dots + a_m u_m) + T(b_1 w_1 + \dots + b_m w_m) \\ &= \lambda f(\vec{v}) + f(\vec{u}) \end{aligned}$$

□

- (b) *Proof.* Then it is no longer a linear map. Let $x \in U$ and $y \in V$ where $y \notin U$. Then $f(x + y) = 0$ but $f(x) + f(y) = T(x) + 0$ hence it is not linear. □

Problem 3. Prove that the following is a subspace of $\mathcal{L}(\mathbb{K}^2)$:

$$U = \left\{ f \in \mathcal{L}(\mathbb{K}^2) : \begin{array}{l} a, b, c \in \mathbb{K} \text{ and} \\ f(x, y) = (ax + by, bx + cy) \end{array} \right\}$$

Proof.

□

Problem 4. For any linear map $T \in \mathcal{L}(V)$, we say that a subspace $U \subseteq V$ is an **invariant subspace of T** if and only if $T(U) \subseteq U$.

Let $T : \mathbb{R}^3 \rightarrow \mathbb{R}^3$ be the linear map given by

$$T\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} 7x - 3y + 5z \\ 12x - 4y + 12z \\ -x + y + z \end{bmatrix}.$$

Show that each of the following subspaces of \mathbb{R}^3 are invariant subspaces of T .

$$U_1 = \text{Span}\left(\begin{bmatrix} -2 \\ -3 \\ 1 \end{bmatrix}\right) \quad \text{and} \quad U_2 = \text{Span}\left(\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}\right)$$

Proof.

□

Problem 5. Let U_1, U_2 be the subspaces from Problem 4. Prove that

$$\mathbb{R}^3 = U_1 \oplus U_2.$$

Proof.

□