

**Hand in Friday, April 19.**

**Definition.** Let  $f : S \rightarrow C$  be a continuous map from a circle in the plane to a circle in the plane. Define the **degree** of  $f$  to be the winding number of this map around the point  $\vec{c}$  at the center of  $C$ . If you prefer to think of winding numbers in terms of continuous maps from intervals, give the name  $\theta$  to a variable running through the interval  $[0, 2\pi]$ , let  $\gamma : [0, 2\pi] \rightarrow S$  parametrize  $S$  by  $\gamma(\theta) = (x_0 + r \cos \theta, y_0 + r \sin \theta)$  for appropriate  $x_0, y_0$ , and  $r$ , and define the degree of  $f$  to be the winding number of  $f \circ \gamma$  around  $\vec{c}$ . [from the textbook by Fulton]

1. Show that, for an  $f$  as in the above definition, if  $f$  is not surjective, then the degree of  $f$  equals zero.

*Proof.* Using the parametrization  $\gamma : [0, 2\pi] \rightarrow S$  by  $\gamma(\theta) = (x_0 + r \cos \theta, y_0 + r \sin \theta)$  where  $(x_0, y_0)$  is the center of  $S$  and  $r$  is the radius of  $S$ . Then we have that  $f$  is a closed curve as

$$f \circ \gamma(0) = f(x_0 + r \cos 0, y_0 + r \sin 0) = f(x_0 + r \cos 2\pi, y_0 + r \sin 2\pi) = f \circ \gamma(2\pi)$$

Now let  $\vec{p}_1 \in C$  be a point that is not in the image of  $f$ . Then we have a point  $\vec{p}_2 \in C$  that is colinear with the line intersecting  $(x_0, y_0)$  and  $\vec{p}_1$ .

We have the constant curve  $g : S \rightarrow C$  given by the equation  $g(\vec{x}) = \vec{p}_2$  for all  $\vec{x} \in S$ .

Then we create the homotopy  $H : [0, 2\pi] \times [0, 1] \rightarrow \mathbb{R}^2 \setminus \{(x_0, y_0)\}$  given by

$$H(\theta, s) = f(\gamma(\theta))(1 - s) + s \cdot \vec{p}_2$$

for all  $\theta \in [0, 2\pi]$  and  $s \in [0, 1]$ .

We have  $H(\theta, 0) = f(\gamma(\theta)) + 0 \cdot \vec{p}_2 = f(\gamma(\theta))$ , and  $H(\theta, 1) = f(\gamma(\theta)) \cdot 0 + 1 \cdot \vec{p}_2 = g(\gamma(\theta))$ .

We have that  $H$  is continuous as it is the sum of two weighted continuous functions.

Additionally we have that the image of  $H$  is contained in  $\mathbb{R}^2 \setminus \{(x_0, y_0)\}$  as for any  $\theta \in [0, 2\pi]$  we have for all  $s \in [0, 1]$  that  $H(\theta, s) \neq (x_0, y_0)$  as the only point colinear with  $(x_0, y_0)$  and  $\vec{p}_2$  is  $\vec{p}_1$  and by our assumption that  $\vec{p}_1$  is not in the image of  $f$ . Then we have that  $H$  is a homotopy between  $f$  and  $g$ .

Then we have that  $f$  and  $g$  are homotopic and thus have the same winding number. We have that the winding number of  $g$  is zero as it is a constant curve. Then we have that the winding number of  $f$  is zero.  $\square$

2. Calculate the degree of each of the following maps from the unit circle centered at the origin to the unit circle centered at the origin.

a.  $f(x, y) = (x, y)$

Using the parametrization  $\gamma(\theta) = (\cos \theta, \sin \theta)$ , for  $\theta \in [0, 2\pi]$ . With the four sectors

$$U_1 = \{(x, y) : x, y \in \mathbb{R} \text{ with } x > 0\}$$

$$U_2 = \{(x, y) : x, y \in \mathbb{R} \text{ with } y > 0\}$$

$$U_3 = \{(x, y) : x, y \in \mathbb{R} \text{ with } x < 0\}$$

$$U_4 = \{(x, y) : x, y \in \mathbb{R} \text{ with } y < 0\}$$

Then we have the six subdivisions  $t_0 = 0, t_1 = \pi/4, t_2 = 3\pi/4, t_3 = 5\pi/4, t_4 = 7\pi/4, t_5 = 2\pi$ . With the angle function being for  $\theta_1$  being its angle in the range  $(-\pi/2, \pi/2)$  for  $\theta_2$  being its angle in the range  $(0, \pi)$  and for  $\theta_3$  being its angle in the range  $(\pi/2, 3\pi/2)$  and  $\theta_4$  being its angle in the range  $(\pi, 2\pi)$ .

Then

$$W(f \circ \gamma, \vec{0}) = \frac{1}{2\pi} (\theta_1(f(\gamma(t_1))) - \theta_1(f(\gamma(t_0))) + \theta_2(f(\gamma(t_2))) - \theta_2(f(\gamma(t_1))) + \theta_3(f(\gamma(t_3))) - \theta_3(f(\gamma(t_2))) + \theta_4(f(\gamma(t_4))) - \theta_4(f(\gamma(t_3))) + \theta_1(f(\gamma(t_5))) - \theta_1(f(\gamma(t_4))))$$

$$\text{Then } W(f \circ \gamma, \vec{0}) = \frac{1}{2\pi} \left( \frac{\pi}{4} - 0 + \frac{3\pi}{4} - \frac{\pi}{4} + \frac{5\pi}{4} - \frac{3\pi}{4} + \frac{7\pi}{4} - \frac{5\pi}{4} + 0 - \frac{\pi}{4} \right) = \frac{1}{2\pi} \left( \frac{7\pi}{4} + \frac{\pi}{4} \right) = 1$$

b.  $g(x, y) = (-x, -y)$

Using the same parametrization with  $\gamma$  as in a. With the four sectors

$$U_1 = \{(x, y) : x, y \in \mathbb{R} \text{ with } x < 0\}$$

$$U_2 = \{(x, y) : x, y \in \mathbb{R} \text{ with } y < 0\}$$

$$U_3 = \{(x, y) : x, y \in \mathbb{R} \text{ with } x > 0\}$$

$$U_4 = \{(x, y) : x, y \in \mathbb{R} \text{ with } y > 0\}$$

With the same subdivisions as in **a.** and the angle functions  $\theta_1$  being its angle in the range  $(\frac{\pi}{2}, \frac{3\pi}{2})$  for  $\theta_2$  being its angle in the range  $(\pi, 2\pi)$  and for  $\theta_3$  being its angle in the range  $(\frac{3\pi}{2}, \frac{5\pi}{2})$  and  $\theta_4$  being its angle in the range  $(2\pi, 3\pi)$ .

Then  $W(g \circ \gamma, \vec{0}) = \frac{1}{2\pi} (\theta_1(g(\gamma(t_1))) - \theta_1(g(\gamma(t_0))) + \theta_2(g(\gamma(t_2))) - \theta_2(g(\gamma(t_3))) + \theta_3(g(\gamma(t_3))) - \theta_3(g(\gamma(t_2))) + \theta_4(g(\gamma(t_4))) - \theta_4(g(\gamma(t_3))) + \theta_1(g(\gamma(t_5))) - \theta_1(g(\gamma(t_4))))$

$$\text{Then } W(g \circ \gamma, \vec{0}) = \frac{1}{2\pi} \left( \frac{5\pi}{4} - \pi + \frac{7\pi}{4} - \frac{5\pi}{4} + \frac{9\pi}{4} - \frac{7\pi}{4} + 3\pi - \frac{9\pi}{4} \right) = \frac{1}{2\pi} (3\pi - \pi) = 1$$

**c.**  $h(x, y) = (x, -y)$

Using the same parametrization with  $\gamma$ , and the following four sectors.

$$U_1 = \{(x, y) : x, y \in \mathbb{R} \text{ with } x > 0\}$$

$$U_2 = \{(x, y) : x, y \in \mathbb{R} \text{ with } y < 0\}$$

$$U_3 = \{(x, y) : x, y \in \mathbb{R} \text{ with } x < 0\}$$

$$U_4 = \{(x, y) : x, y \in \mathbb{R} \text{ with } y > 0\}$$

With the same subdivisions as in **a.** With the following angle functions  $\theta_1$  being its angle in the range  $(-\pi/2, \pi/2)$   $\theta_2$  being its angle in the range  $(-\pi, 0)$   $\theta_3$  being its angle in the range  $(-3\pi/2, -\pi/2)$  and  $\theta_4$  being its angle in the range  $(-2\pi, -\pi)$ .

Then  $W(h \circ \gamma, \vec{0}) = \frac{1}{2\pi} (\theta_1(h(\gamma(t_1))) - \theta_1(h(\gamma(t_0))) + \theta_2(h(\gamma(t_2))) - \theta_2(h(\gamma(t_1))) + \theta_3(h(\gamma(t_3))) - \theta_3(h(\gamma(t_2))) + \theta_4(h(\gamma(t_4))) - \theta_4(h(\gamma(t_3))) + \theta_1(h(\gamma(t_5))) - \theta_1(h(\gamma(t_4))))$

$$\text{Then } W(h \circ \gamma, \vec{0}) = \frac{1}{2\pi} \left( -\frac{\pi}{4} - 0 + -\frac{3\pi}{4} + \frac{\pi}{4} - \frac{5\pi}{4} + \frac{3\pi}{4} - \frac{7\pi}{4} + \frac{5\pi}{4} - 2\pi + \frac{7\pi}{4} \right) = -1$$

**d.**  $k(\cos(\theta), \sin(\theta)) = (\cos(n\theta), \sin(n\theta))$ , where  $n$  is an arbitrary integer. Using the same parametrization with  $\gamma$ . With the four sectors

$$U_1 = \{(x, y) : x, y \in \mathbb{R} \text{ with } x > 0\}$$

$$U_2 = \{(x, y) : x, y \in \mathbb{R} \text{ with } y > 0\}$$

$$U_3 = \{(x, y) : x, y \in \mathbb{R} \text{ with } x < 0\}$$

$$U_4 = \{(x, y) : x, y \in \mathbb{R} \text{ with } y < 0\}$$

With the subdivisions being  $t_0 = 0, t_n = 2\pi, t_i =$  and for each  $i \in \{1, \dots, n-1\}$

**Definition.** If  $Y$  is a topological subspace of a topological space  $X$ , a **retraction** from  $X$  to  $Y$  is a continuous map  $r : X \rightarrow Y$  that satisfies, for all  $y \in Y$ ,  $r(y) = y$ . When such a retraction exists, we call  $Y$  a **retract** of  $X$ . [from the textbook by Fulton]

**3.** Show that, if  $Y$  is a retract of  $X$  and if every continuous map from  $X$  to  $X$  has a fixed point, then every continuous map from  $Y$  to  $Y$  has a fixed point. **Hint.** Start with an arbitrary continuous map  $f : Y \rightarrow Y$ . How can you make a continuous map  $g : X \rightarrow X$  whose behavior has the needed implications for  $f$ 's behavior?

*Proof.* Let  $Y$  be a retraction of  $X$  given by the map  $r : X \rightarrow Y$  and  $f : Y \rightarrow Y$  be an arbitrary continuous map. Then consider the function  $g : X \rightarrow X$  where  $g = j \circ f \circ r$  where  $j$  is the inclusion map given by (Munkres Theorem 18.2). We have as  $g$  is a continuous and  $g : X \rightarrow X$  that there exists a fixed point in  $x_0 \in X$ . Then we have that  $r(x_0) = y_0$  for some  $y_0 \in Y$  and hence we get  $g(x_0) = j(f(r(x_0))) = j(f(y_0)) = f(y_0) = x_0$  as  $j$  is the inclusion map this implies that  $r \circ g(x_0) = y_0$  but hence we get  $r(j(f(y_0))) = y_0$  which implies that  $f(y_0) = y_0$  as the codomain of  $f$  is  $Y$  and  $r$  is a retraction and  $j$  is the inclusion map. □

4. Let  $B$  be the open unit disk in  $\mathbb{R}^2$  and let  $D$  be the closed unit disk in  $\mathbb{R}^2$ . Show that, for any  $\vec{p} \in B$ , the unit circle  $C$  in  $\mathbb{R}^2$  is a retract of  $D \setminus \{\vec{p}\}$ . **Hint.** When  $\vec{p}$  is the origin, the map  $\vec{x} \mapsto \frac{\vec{x}}{|\vec{x}|}$  is the retraction. When  $\vec{p}$  is more general, consider solving  $|\vec{p} + t(\vec{x} - \vec{p})| = 1$  for  $t$ .

5. Let  $S$  and  $C$  be circles in the plane, and let  $f : S \rightarrow C$  be a continuous map. Show that, for every  $\vec{p}$  in the open disk bounded by  $C$ , the winding number of  $f$  around  $\vec{p}$  equals the degree of  $f$ . (In particular the winding number is the same, regardless of which  $\vec{p}$  in the open disk is used.)

*Proof.*

Let  $S, C$  be circles in the plane and  $f : S \rightarrow C$  be a continuous map. Let  $\gamma : [0, 2\pi] \rightarrow S$  be a parametrization of  $S$  by  $\gamma(\theta) = (x_0 + r \cos \theta, y_0 + r \sin \theta)$  for appropriate  $x_0, y_0, r$ .

Let  $U_1, \dots, U_n$  be the sectors used in calculating the degree of  $f$  and  $\theta_1, \dots, \theta_n$  be the corresponding angle functions.

Now let  $\vec{p}$  be a point in the open disk bounded by  $C$ . Then consider the sectors given by  $U_1 + \vec{p} = \{\vec{x} + \vec{p} : \vec{x} \in U_1\}, \dots, U_n + \vec{p}$  and the corresponding angle functions □