Define explicitly a continuous branch of $\log z$ in the complex plane slit along the negative imaginary axis, $\mathbb{C} \setminus [0, -i\infty)$.

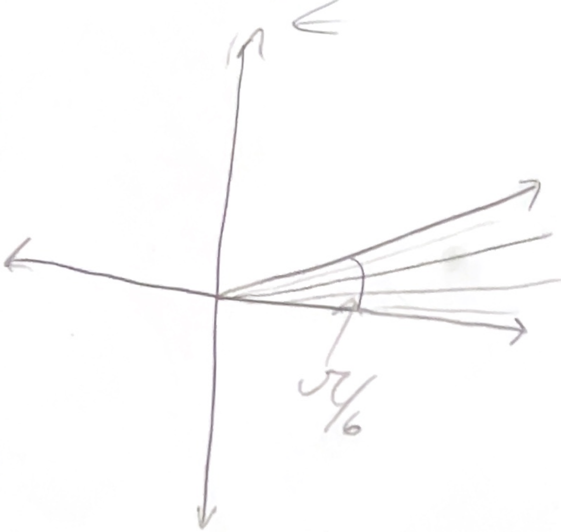
This branch is given by $f(re^i\theta) = \log r + i\theta$ where $\theta \in (-\frac{\pi}{2}, \frac{3\pi}{2})$.

Sketch the image of the sector $\{0 < \arg z < \frac{\pi}{6}\}$ under the map $w = z^a$.

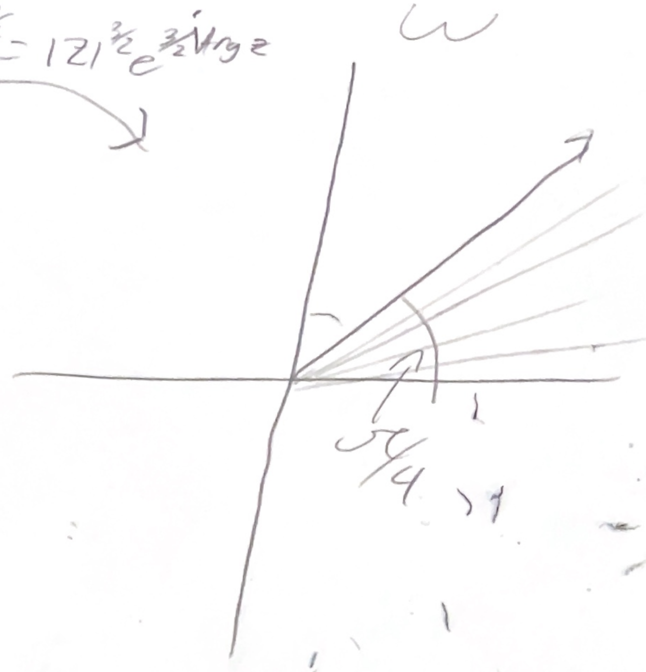
- $a = \frac{3}{2}$
- $a = i$

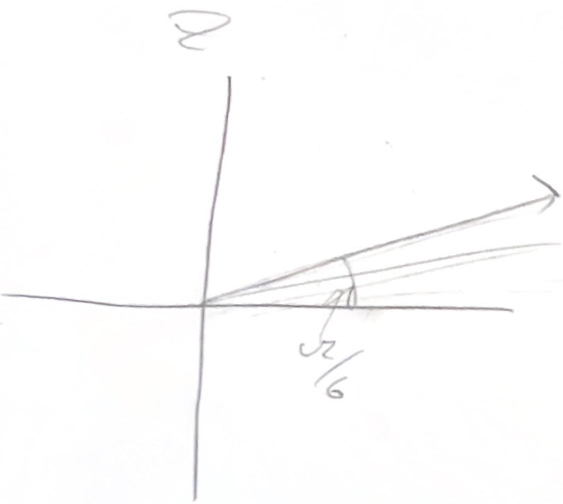
$$w = z^{\frac{3}{2}} = e^{\frac{3}{2}(\log|z| + i\mathrm{Arg}z)} = |z|^{\frac{3}{2}} e^{\frac{3}{2}i\mathrm{Arg}z}$$

$$w = z^i = e^{i\log z} = e^{i(\log|z| + i\mathrm{Arg}z)} = e^{-\mathrm{Arg}z + i\log|z|}$$

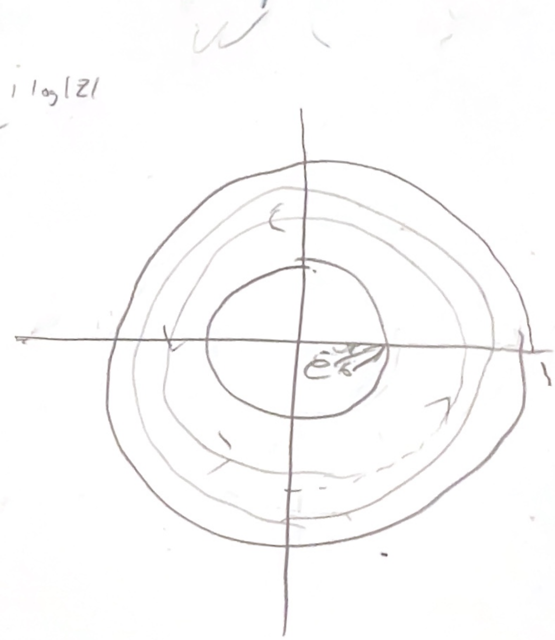


$w = z^{3/2} = |z|^{3/2} e^{\frac{3}{2} i \operatorname{Arg} z}$





$w = e^{-\operatorname{Arg} z} e^{i \log|z|}$



Determine the phase factors of the function $z^a(1-z)^b$ at the branch points $z=0$ and $z=1$. What conditions on a and b are necessary for the function to be single-valued on $\mathbb{C}\setminus[0,1]$?

The phase factor at the branch point $z=0$ is given by $e^{2\pi ia}$ and the phase factor at the branch point $z=1$ is given by $e^{2\pi ib}$. The function is single-valued on $\mathbb{C}\setminus[0,1]$ if we have $e^{i\pi a}e^{i\pi b}=1$ which happens when $a+b$ is an integer.

Show that if f is analytic on D then $g(z)=\overline{f(\bar{z})}$ is analytic on the reflected domain $D^\star=\{\bar{z}:z\in D\}$, and $g'(z)=\overline{f'(\bar{z})}$.

Proof. Suppose f is analytic on D. Then for any $z_0\in D^\star$ we have

$$\begin{aligned}
\lim_{\Delta z\mapsto 0}\frac{g(z_0+\Delta z)-g(z_0)}{\Delta z} &= \lim_{\Delta z\mapsto 0}\frac{\overline{f(\overline{z_0+\Delta z})}-\overline{f(\overline{z_0})}}{\Delta z} \\
&= \lim_{\Delta z\mapsto 0}\overline{\frac{f(\bar{z}_0+\overline{\Delta z})-f(\overline{z}_0)}{\overline{\Delta z}}} \\
&= \overline{\lim_{\Delta z\mapsto 0}\frac{f(\bar{z}_0+\overline{\Delta z})-f(\overline{z}_0)}{\overline{\Delta z}}} \\
&= \overline{f'(\overline{z}_0)}
\end{aligned}$$

This shows that $g'(z_0)$ and $g'(z_0)=\overline{f'(\bar{z}_0)}$. Now to show that g is analytic on $D^\star$. As f is analytic on D we have for any $z\in D$ that for any $\epsilon>0$ there exists $\delta>0$ such that for all $z_0\in D$ with $0<|z-z_0|<\delta$ implies □