Problem 1. Let U be the subset of $\mathcal{P}_3(\mathbb{R})$ given by

$$U = \{p \in \mathcal{P}_3(\mathbb{R}) \ : \ p(0) = p(1)\}.$$

Define the function

$$\alpha : U \times U \to \mathbb{R}$$
$$\alpha(p,q) = \int_0^1 p(x)q'(x)\,dx$$

1. Prove that α is an alternating bilinear form.
2. Find the matrix $\mathcal{M}(\alpha, \mathcal{B})$ for α with respect to the following basis

$$\mathcal{B} = \left\{x^3 - x,\ x^2 - x,\ 1\right\}$$

Proof. First I will show that α (as defined above) is a bilinear form.

First fix some $p \in U$ then for any $q_1, q_2 \in U$ and $\lambda \in \mathbb{R}$ we have

$$\begin{aligned}
\alpha(p, \lambda q_1 + q_2) &= \int_0^1 p(x)(\lambda q_1 + q_2)'(x)dx \\
&= \int_0^1 p(x)\lambda q_1'(x) + p(x)q_2'(x)dx \\
&= \lambda \int_0^1 p(x)q_1'(x)dx + \int_0^1 p(x)q_2'(x)dx \\
&= \lambda\alpha(p,q_1) + \alpha(p,q_2)
\end{aligned}$$

Now fix some $q \in U$ then for any $p_1, p_2 \in U$ and $\lambda \in \mathbb{R}$ we have

$$\alpha(\lambda p_1 + p_2, q) = \int_0^1 (\lambda p_1 + p_2)(x)q'(x)dx$$

Using the same logic as above we get $\alpha(\lambda p_1 + p_2, q) = \lambda\alpha(p_1, q) + \alpha(p_2, q)$. Hence we have that α is a bilinear form.

Now let $p \in U$ then we have

$$\begin{aligned}
\alpha(p,p) &= \int_0^1 p(x)p'(x)dx \\
&= \frac{1}{2}p(x)^2\Big|_0^1 \\
&= \frac{1}{2}\left(p(1)^2 - p(0)^2\right) \\
&= 0
\end{aligned}$$

Hence we get that it is an alternating bilinear form.

$\square$

Now finding the matrix $\mathcal{M}(\alpha, \mathcal{B})$ I will say that $b_1 = x^3 - x, \ \ b_2 = x^2 - x, \ \ b_3 = 1$. We have immediately that $\alpha(b_i, b_3) = 0$ for $i \in \{1,2,3\}$ additionally as this is an alternating form we get $\alpha(b_i, b_i) = 0$ and we also get $\alpha(b_i, b_j) = -\alpha(b_j, b_i)$ by Theorem 9.16. Hence we only need to examine the values of $\alpha(b_1, b_2)$

$$\begin{aligned}
\alpha(b_1, b_2) &= \int_0^1 (x^3 - x)(x^2 - x)' dx \\
&= \int_0^1 (x^3 - x)(2x - 1) dx \\
&= \int_0^1 2x^4 - x^3 - 2x^2 + x dx \\
&= \frac{2}{5}x^5 - \frac{1}{4}x^4 - \frac{2}{3}x^3 + \frac{1}{2}x^2 \Big|_0^1 \\
&= \frac{2}{5} - \frac{1}{4} - \frac{2}{3} + \frac{1}{2} \\
&= -\frac{1}{60}
\end{aligned}$$

Hence we get the matrix

$$\mathcal{M}(\alpha, \mathcal{B}) = \begin{pmatrix} 0 & -\frac{1}{60} & 0 \\ \frac{1}{60} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Problem 2. If V and W are $\mathbb{K}$-vector spaces, observe that the Cartesian $V \times W$ is a $\mathbb{K}$-vector space with the following addition and scalar multiplication operations:

$$(\mathbf{v_1}, \mathbf{w_1}) + (\mathbf{v_2}, \mathbf{w_2}) = (\mathbf{v_1} + \mathbf{v_2}, \mathbf{w_1} + \mathbf{w_2}) \qquad \text{and} \qquad k(\mathbf{v}, \mathbf{w}) = (k\mathbf{v}, k\mathbf{w}).$$

Show that, in general, a bilinear form $\beta \in V^{(2)}$ is <u>not</u> a linear functional, $\mathcal{L}(V \times V, \mathbb{K})$.

Let $\beta \in V^{(2)}$ and $(a_1, b_1), (a_2, b_2) \in V \times V$ then if β where linear we would have

$$\begin{aligned}\beta((a_1, b_1) + (a_2, b_2)) &= \beta(a_1 + a_2, b_1 + b_2) \\ &= \beta(a_1 + a_2, b_1) + \beta(a_1 + a_2, b_2) \\ &= \beta(a_1, b_1) + \beta(a_2, b_1) + \beta(a_1, b_2) + \beta(a_2, b_2)\end{aligned}$$

We only get the equality $\beta((a_1, b_1)+(a_2, b_2)) = \beta((a_1, b_1))+\beta((a_2, b_2))$ if and only if $\beta(a_2, b_1)+\beta(a_1, b_2) = 0$.

Now let $(a, b) \in V \times V$ and $\lambda \in \mathbb{K}$ then we have

$$\beta(\lambda(a, b)) = \beta(\lambda a, \lambda b) = \lambda\beta(a, \lambda b) = \lambda^2\beta(a, b)$$

Hence we only get the equality $\beta(\lambda(a, b)) = \lambda\beta((a, b))$ if and only if $\lambda = \lambda^2$ this doesn't hold for all the scalars hence it is not linear.

Problem 3. The notion of a bilinear form can be extended to a **bilinear map** in the following way: Let U, V, W be $\mathbb{K}$-vector spaces. The function $\Gamma : V \times W \to U$ is a bilinear map if it satisfies the following: for all scalars k and vectors $\mathbf{v}, \mathbf{w}$:

$$\Gamma(\mathbf{v_1} + \mathbf{v_2}, \mathbf{w}) = \Gamma(\mathbf{v_1}, \mathbf{w}) + \Gamma(\mathbf{v_2}, \mathbf{w}) \qquad \text{and} \qquad \Gamma(k\mathbf{v}, \mathbf{w}) = k\Gamma(\mathbf{v}, \mathbf{w}),$$

$$\Gamma(\mathbf{v}, \mathbf{w_1} + \mathbf{w_2}) = \Gamma(\mathbf{v}, \mathbf{w_1}) + \Gamma(\mathbf{v}, \mathbf{w_2}) \qquad \text{and} \qquad \Gamma(\mathbf{v}, k\mathbf{w}) = k\Gamma(\mathbf{v}, \mathbf{w}).$$

1. Go find your old multivariable calculus textbook and look up the definition of the cross product on $\mathbb{R}^3$.
2. Prove that $\Gamma : \mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{R}^3$ given by
$$\Gamma(\mathbf{v}, \mathbf{w}) = \underbrace{\mathbf{v} \times \mathbf{w}}_{\text{cross product}}$$
is a bilinear map.
3. A bilinear map $\Gamma : V \times V \to U$ is said to be **alternating** if $\Gamma(\mathbf{v}, \mathbf{v}) = \mathbf{0}$ for all $\mathbf{v}$. Prove that the cross product map above is alternating.

(1) Stewart's Calculus book defines it as

If $\mathbf{a} = \langle a_1, a_2, a_3\rangle$ and $\mathbf{b} = \langle b_1, b_2, b_3\rangle$, then the **cross product** of $\mathbf{a}$ and $\mathbf{b}$ is the vector

$$\mathbf{a} \times \mathbf{b} = \langle a_2b_3 - a_3b_2, a_3b_1 - a_1b_3, a_1b_2 - a_2b_1\rangle$$

(2) Let $a = (a_1, a_2, a_3), b = (b_1, b_2, b_3), c = (c_1, c_2, c_3) \in \mathbb{R}^3$ and Γ be the cross product map. Then we have

$$\Gamma((a_1, a_2, a_3) + (b_1, b_2, b_3), (c_1, c_2, c_3)) =$$

$$((a_2 + b_2)c_3 - (a_3 + b_3)c_2, (a_3 + b_3)c_1 - (a_1 + b_1)c_3, (a_1 + b_1)c_2 - (a_2 + b_2)c_1)$$

distributing the c_i's we get the equality

$$(a_2c_3 - a_3c_2, a_3c_1 - a_1c_3, a_1c_2 - a_2c_1) + (b_2c_3 - b_3c_2, b_3c_1 - b_1c_3, b_1c_2 - b_2c_1)$$

which is equal to $\Gamma(a, c) + \Gamma(b, c)$ hence we get the first condition.

Now

$$\Gamma((a_1, a_2, a_3), (b_1, b_2, b_3) + (c_1, c_2, c_3)) =$$

$$(a_2(b_3 + c_3) - a_3(b_2 + c_2), a_3(b_1 + c_1) - a_1(b_3 + c_3), a_1(b_2 + c_2) - a_2(b_1 + c_1))$$

distributing each a_i we get the equality

$$(a_2b_3 - a_3b_2, a_3b_1 - a_1b_3, a_1b_2 - a_2b_1) + (a_2c_3 - a_3c_2, a_3c_1 - a_1c_3, a_1c_2 - a_2c_1)$$

which is equal to $\Gamma(a,b)+\Gamma(a,c)$ hence we have the second condition satisfied.

Now let $k \in K$ then we have

$$\begin{aligned}\Gamma(k(a_1,a_2,a_3),(b_1,b_2,b_3)) &= (ka_2b_3 - ka_3b_2, ka_3b_1 - ka_1b_3, ka_1b_2 - ka_2b_1) && (1)\\ &= k(a_2b_3 - a_3b_2, a_3b_1 - a_1b_3, a_1b_2 - a_2b_1) && (2)\\ &= k\Gamma((a_1,a_2,a_3),(b_1,b_2,b_3)) && (3)\end{aligned}$$

Now as $\Gamma(a,kb) = (ka_2b_3 - ka_3b_2, ka_3b_1 - ka_1b_3, ka_1b_2 - ka_2b_1)$ we get both $\Gamma(ka,b) = k\Gamma(a,b) = \Gamma(a,kb)$ hence we have completed all the conditions and so Γ is a bilinear map.

(3)

Proof. Let $(v_1,v_2,v_3) \in \mathbb{R}^3$ then we get

$$\begin{aligned}\Gamma((v_1,v_2,v_3),(v_1,v_2,v_3)) &= (v_2v_3 - v_3v_2, v_3v_1 - v_1v_3, v_1v_2 - v_2v_1)\\ &= (0,0,0)\end{aligned}$$

□