

1. In each of the following topological spaces, give an example of an intersection of infinitely many open sets that is not itself an open set.

- (1)  $\mathbb{R}$  with its standard topology. Consider the intersection

$$\bigcap_{n \in \mathbb{N}} (-1/n, 1/n)$$

We have  $\bigcap_{n \in \mathbb{N}} (-1/n, 1/n) = \{0\}$ . This is shown as  $-1/n < 0 < 1/n$  for all  $n \in \mathbb{N}$ . Any element  $j \in (0, 1)$  is not in the intersection as there exists  $n \in \mathbb{N}$  such that  $1/n < j$  by the Archimedean property. Likewise for any  $-j \in (-1, 0)$  there exists a  $n \in \mathbb{N}$  such that  $-j < -1/n$  again by the Archimedean property. Now  $\{0\}$  is not open as there exists no basis element in the standard topology that is a subset of  $\{0\}$ . This is shown as all basis elements are of the form  $(a, b)$  where  $a, b \in \mathbb{R}$  where  $a < b$  but  $|\{0\}| = 1$  but  $|(a, b)|$  is uncountable.

- (2)  $\mathbb{R}$  with its lower limit topology. Consider

$$\bigcap_{n \in \mathbb{N}} [0, 1/n)$$

. We have that  $\bigcap_{n \in \mathbb{N}} [0, 1/n) = \{0\}$  using the same reasoning as above. Now  $|\{0\}| = 1$  but any basis element  $[a, b)$  where  $a, b \in \mathbb{R}$  with  $a < b$  is uncountable hence no basis element is a subset of  $\{0\}$  which implies it is not open.

- (3)  $\mathbb{R}$  with the finite complement topology. Consider

$$\bigcap_{n \in \mathbb{N}} \mathbb{R} \setminus \{1/n\}$$

. We have each  $n \in \mathbb{N}$  that  $\mathbb{R} \setminus (\mathbb{R} \setminus \{1/n\}) = \{1/n\}$  hence  $\mathbb{R} \setminus \{1/n\}$  is open. But  $\bigcap_{n \in \mathbb{N}} \mathbb{R} \setminus \{1/n\} = \mathbb{R} \setminus \{1, 1/2, 1/3, \dots\}$  the complement of this set is not finite hence not open.

2. Let  $\mathbb{R}$  have the lower limit topology is  $(0, 1)$  open? Yes

*Proof.* Consider the union  $\bigcup_{n \in \mathbb{N}} [1/n, 1)$  as each of the sets is open and this is a union we have that it is open in the lower limit topology so just need to demonstrate double containment. Let  $j \in \bigcup_{n \in \mathbb{N}} [1/n, 1)$  then for some  $n \in \mathbb{N}$  we have  $j \in [1/n, 1)$  but as  $0 < 1/n$  for all  $n \in \mathbb{N}$  we get the inequality  $0 < j < 1$  hence  $j \in (0, 1)$ . Now let  $j \in (0, 1)$  then by the Archimedean property for some  $n \in \mathbb{N}$  we have  $1/n < j < 1$  hence  $j \in [1/n, 1)$ . Which shows double containment hence  $\bigcup_{n \in \mathbb{N}} [1/n, 1) = (0, 1)$  which completes the proof.  $\square$

3. In the set  $\mathbb{R}$ , consider the collection of subsets consisting of  $\mathbb{R}, \emptyset$ , and all sets whose complements are finite sets of irrational numbers. Is this collection a topology on  $\mathbb{R}$ ?

Yes

*Proof.* As  $\emptyset, \mathbb{R}$  are in this collection  $\mathcal{C}$  we just need to demonstrate finite intersections and arbitrary unions are in  $\mathcal{C}$ .

Consider the intersection of two elements  $A, B \in \mathcal{C}$  then we have  $\mathbb{R} \setminus A \cap B = (\mathbb{R} \setminus A) \cup (\mathbb{R} \setminus B)$  as the union of two finite sets is finite that completes the base case. Now assume for some  $n \in \mathbb{N}$  where  $n \geq 2$  we have that the intersection of  $n$  elements of  $\mathcal{C}$  is in  $\mathcal{C}$ . Then given  $n + 1$  elements  $A_1, \dots, A_{n+1}$  consider the intersection  $A_1 \cap \dots \cap A_{n+1}$  then we have  $\mathbb{R} \setminus (A_1 \cap \dots \cap A_{n+1}) = (\mathbb{R} \setminus A_1 \cup \dots \cup \mathbb{R} \setminus A_{n+1}) \cup \mathbb{R} \setminus A_{n+1}$  using the induction hypothesis we have  $\mathbb{R} \setminus A_1 \cup \dots \cup \mathbb{R} \setminus A_n \in \mathcal{C}$  by the base case the intersection of two elements of  $\mathcal{C}$  is also in  $\mathcal{C}$  hence that completes finite intersections.

Let  $\mathcal{B} \subset \mathcal{C}$  consider the arbitrary union of elements  $\bigcup_{b \in \mathcal{B}} U_b$  where  $U_b \in \mathcal{B}$ . Then  $\mathbb{R} \setminus \bigcup_{b \in \mathcal{B}} U_b \subset \mathbb{R} \setminus U_b$  where  $U_b$  is any  $U_b \in \mathcal{B}$  as subsets of finite sets are finite this shows that arbitrary unions are in  $\mathcal{C}$  hence it is a topology.  $\square$

4. Suppose that  $Y$  is a Hausdorff topological space. Let  $a, b$  distinct elements of  $Y$ . Suppose that  $(a_n)$  is a sequence in  $Y$  that converges to  $a$  and  $(b_n)$  is a sequence in  $Y$  that converges to  $b$ . Show that there exists an  $N$  such that, for all  $n > N$ ,  $a_n \neq b_n$ .

*Proof.* As  $Y$  is a Hausdorff space and  $a, b$  are distinct elements then there exists two neighborhoods  $U_a, U_b$  for  $a, b$  respectively where  $U_a \cap U_b = \emptyset$ . But as  $(a_n)$  is convergent we have for some  $N_1 \in \mathbb{N}$  that for all  $n \geq N_1$  that  $a_n \in U_a$ . Likewise for  $(b_n)$  for some  $N_2 \in \mathbb{N}$  we have for all  $n \geq N_2$  that  $b_n \in U_b$ . Let  $N = \max(N_1, N_2)$  then for all  $n \geq N$  we have  $a_n \in U_a$  and  $b_n \in U_b$  but as these sets are disjoint we have  $a_n \neq b_n$ .  $\square$

5.

- (1) Show that, in any metric space  $(X, d)$  and for any  $x_0 \in X$ , the closure of  $B_d(x_0, 1)$  is contained in  $D_d(x_0, 1)$ .

*Proof.* Let  $(X, d)$  be an arbitrary metric space and  $x_0 \in X$ . We have  $\overline{B_d(x_0, 1)} = B_d(x_0, 1) \cup B_d(x_0, 1)'$ . We have  $B_d(x_0, 1) \subset D_d(x_0, 1)$  which follows from the strict inequality on  $B_d(x_0, 1)$ . Now let  $y \in B_d(x_0, 1)'$  then for every  $\epsilon$ -neighborhood we have for some distinct  $x \in B_d(x_0, 1)$  that  $0 < d(x, y) < \epsilon$  from the definition of the open ball we have  $d(x_0, x) < 1$  applying the triangle inequality we get  $d(x_0, y) \leq d(x_0, x) + d(x, y) < 1 + \epsilon$  as this is true for all  $\epsilon > 0$  we get the strict inequality  $d(x_0, y) \leq 1$  hence  $y \in D_d(x_0, 1)$  which shows  $B_d(x_0, 1)' \subset D_d(x_0, 1)$  as both the set and its limit points are subsets of  $D_d(x_0, 1)$  this completes the proof.  $\square$

- (2) Is it the case that, in every metric space  $(X, d)$  and for every  $x_0 \in X$ ,  $D_d(x_0, 1)$  is contained in the closure of  $B_d(x_0, 1)$ ?

*Proof.* No this is not true consider the metric  $(\mathbb{R}, d)$  where  $d(x, y) = \begin{cases} 1 & \text{if } x \neq y \\ 0 & \text{if } x = y \end{cases}$ . Consider the open ball  $B_d(0, 1) = \{0\}$  which follows because of the strict inequality but  $D_d(0, 1) = \mathbb{R}$ . The closure of  $B_d(0, 1) = B_d(0, 1) \cup B_d(0, 1)'$  but for any  $x \in \mathbb{R}$  with  $x \neq 0$  we have that any epsilon neighborhood of  $x$  with  $0 < \epsilon < 1$  that  $B_d(x, \epsilon) = \{x\}$  hence not every neighborhoods even intersects  $B_d(0, 1)$  so the limit point is the emptyset. Therefore  $\overline{B_d(0, 1)} = \{0\} \not\supset D_d(0, 1) = \mathbb{R}$ .  $\square$

6. For all natural numbers  $j$  let  $X_j$  be  $\mathbb{R}$  with the standard topology. Let  $X = \prod_{j \in \mathbb{N}} X_j$  define the elements of  $X$  by  $\vec{x} = (x_1, x_2, \dots)$

- (1) Show that the set of 5-bounded elements of  $X$  is closed in both the product topology and the box topology.

*Proof.* Denote the set of 5 bounded elements by  $B_5$  we have that a set is closed if and only if it contains its limit points. Let  $\vec{x} \in B_5$  then we have for any neighborhood  $U$  of  $\vec{x}$  that  $U_i \neq \mathbb{R}_i$  for a finite number of  $i$ . As this is the product topology we have for each  $U_i \neq \mathbb{R}_i$  as  $U_i$  is open a basis element  $u_i = (a_i, b_i) \subset U_i$  where  $a_i, b_i \in \mathbb{R}$  with  $-5 \leq a_i < \pi_i(\vec{x}) < b_i \leq 5$  then we have the set  $A = \mathbb{R}_1 \times u_2 \times \dots \times u_i \times \dots \times \mathbb{R}_n \times \dots$  is open and  $A \subset U$  and  $A \cap B_5 \setminus \{\vec{x}\} \neq \emptyset$  as there exists some  $\vec{y} \in B_5 \setminus \{\vec{x}\}$  with  $a_i < \pi_i(\vec{y}) < b_i$  where the  $a_i, b_i$  come from the bases element  $u_i$  when  $U_i \neq \mathbb{R}_i$  if  $U_i = \mathbb{R}_i$  then  $\pi_i(\vec{y}) \in [-5, 5]$  hence this shows  $A \cap B_5 \setminus \{\vec{x}\} \neq \emptyset$ . Which implies every element of  $B_5$  is a limit point of  $B_5$ .

If  $\vec{x} \notin B_5$  then there exists some  $i \in \mathbb{N}$  we have  $|\pi_i(\vec{x})| > 5$ . If  $\pi_i(\vec{x}) > 5$  then consider the open set  $\pi_i^{-1}((a, b))$  where  $a, b \in \mathbb{R}$  with  $5 < a < \pi_i(\vec{x}) < b$ . We have  $\pi_i^{-1}((a, b)) \cap B_5 = \emptyset$  as every element of  $B_5$  is 5-bounded. If  $\pi_i(\vec{x}) < -5$  then we have the open set  $\pi_i^{-1}(a, b)$  with  $a < \pi_i(\vec{x}) < b < -5$  again this open set  $\pi_i^{-1}(a, b) \cap B_5 = \emptyset$  as  $B_5$  is 5 bounded. This shows that  $B_5$  contains its limit points hence it is closed in the product topology.

We have that the box topology is finer than the product topology. Which implies that  $B_5$  if it is closed in the product topology then it is also closed in the box topology.  $\square$

- (2) Show that the set of bounded elements of  $X$  is closed in the box topology.

*Proof.* Denote the set of bounded elements by  $B$ . We have that  $B$  is closed if and only if it contains its limit points. Let  $\vec{x} \in B$  then consider the open set  $U = \prod_{i \in \mathbb{N}} (\pi_i(\vec{x}) - 1, \pi_i(\vec{x}) + 1)$  we have that  $U \cap B = \emptyset$  because if it did not then there would exist  $\vec{y} \in B$  with  $|\pi_i(\vec{x}) - \pi_i(\vec{y})| < 1$  for all  $i \in \mathbb{N}$  but as  $\vec{y}$  is bounded we have for some  $M \in \mathbb{R}$  with  $M > 0$  that  $-M \leq \pi_i(\vec{y}) \leq M$  for all  $i \in \mathbb{N}$ . Then we have for all  $i \in \mathbb{N}$  that  $|\pi_i(\vec{x}) - \pi_i(\vec{y})| < 1$  which by the triangle inequality yields  $|\pi_i(\vec{x})| - |\pi_i(\vec{y})| < 1$  which implies  $|\pi_i(\vec{x})| < 1 + M$  which is a contradiction on  $\vec{x}$  being bounded. Hence  $U \cap B = \emptyset$ . Now suppose that  $\vec{x}$  is bounded then for any neighborhood  $U$  of  $\vec{x}$  we have a basis element where for each  $i \in \mathbb{N}$  we have  $u_i = (a_i, b_i) \subset U_i$  where  $a_i, b_i \in \mathbb{R}$  and  $a_i < \pi_i(\vec{x}) < b_i$ . Then  $\vec{x} \in \prod_{i \in \mathbb{N}} u_i \subset U$  then

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