1. In each of the following topological spaces, give an example of an intersection of infinitely many open sets that is not itself an open set.

(1) $\mathbb{R}$ with it's standard topology. Consider the intersection

$$\bigcap_{n\in\mathbb{N}}(-1/n,1/n)$$

We have $\bigcap_{n\in\mathbb{N}}(-1/n,1/n)=\{0\}$. This is shown as $-1/n<0<1/n$ for all $n\in\mathbb{N}$. Any element $j\in(0,1)$ is not in the intersection as there exists $n\in\mathbb{N}$ such that $1/n<j$ by the Archimedean property. Likewise for any $-j\in(-1,0)$ there exists a $n\in\mathbb{N}$ such that $-j<-1/n$ again by the Archimedean property. Now $\{0\}$ is not open as there exists no basis element in the standard topology that is a subset of $\{0\}$. This is shown as all basis elements are of the form (a,b) where $a,b\in\mathbb{R}$ where $a<b$ but $|\{0\}|=1$ but $|(a,b)|$ is uncountable.

(2) $\mathbb{R}$ with its lower limit topology. Consider

$$\bigcap_{n\in\mathbb{N}}[0,1/n)$$

. We have that $\bigcap_{n\in\mathbb{N}}[0,1/n)=\{0\}$ using the same reasoning as above. Now $|\{0\}|=1$ but any basis element $[a,b)$ where $a,b\in\mathbb{R}$ with $a<b$ is uncountable hence no basis element is a subset of $\{0\}$ which implies it is not open.

(3) $\mathbb{R}$ with the finite complement topology. Consider

$$\bigcap_{n\in\mathbb{N}}\mathbb{R}\setminus\{1/n\}$$

. We have each $n\in\mathbb{N}$ that $\mathbb{R}\setminus(\mathbb{R}\setminus\{1/n\})=\{1/n\}$ hence $\mathbb{R}\setminus\{1/n\}$ is open. But $\bigcap_{n\in\mathbb{N}}\mathbb{R}\setminus\{1/n\}=\mathbb{R}\setminus\{1,1/2,1/3,...\}$ the complement of this set is not finite hence not open.

2. Let $\mathbb{R}$ have the lower limit topology is $(0,1)$ open? Yes

Proof. Consider the union $\bigcup_{n\in\mathbb{N}}[1/n,1)$ as each of the sets is open and this is a union we have that it is open in the lower limit topology so just need to demonstrate double containment. Let $j\in\bigcup_{n\in\mathbb{N}}[1/n,1)$ then for some $n\in\mathbb{N}$ we have $j\in[1/n,1)$ but as $0<1/n$ for all $n\in\mathbb{N}$ we get the inequality $0<j<1$ hence $j\in(0,1)$. Now let $j\in(0,1)$ then by the Archimedean property for some $n\in\mathbb{N}$ we have $1/n<j<1$ hence $j\in\bigcup[1/n,1)$. Which shows double containment hence $\bigcup_{n\in\mathbb{N}}[1/n,1)=(0,1)$ which completes the proof. $\square$

3. In the set $\mathbb{R}$, consider the collection of subsets consisting of $\mathbb{R},\emptyset$, and all sets whose complements are finite sets of irrational numbers. Is this collection a topology on $\mathbb{R}$?

Yes

Proof. As $\emptyset,\mathbb{R}$ are in this collection $\mathcal{C}$ we just need to demonstrate finite intersections and arbitrary unions are in $\mathcal{C}$.

Consider the intersection of two elements $A,B\in\mathcal{C}$ then we have $\mathbb{R}\setminus A\cap B=(\mathbb{R}\setminus A)\cup(\mathbb{R}\setminus B)$ as the union of two finite sets is finite that completes the base case. Now assume for some $n\in\mathbb{N}$ where $n\geq 2$ we have that the intersection of n elements of $\mathcal{C}$ is in $\mathcal{C}$. Then given $n+1$ elements $A_1,...,A_{n+1}$ consider the intersection $A_1\cap...\cap A_{n+1}$ then we have $\mathbb{R}\setminus(A_1\cap...\cap A_2)=(\mathbb{R}\setminus A_1\cup...\cup\mathbb{R}\setminus A_n)\cup\mathbb{R}\setminus A_{n+1}$ using the induction hypothesis we have $\mathbb{R}\setminus A_1\cup...\cup\mathbb{R}\setminus A_n\in\mathcal{C}$ by the base case the intersection of two elements of $\mathcal{C}$ is also in $\mathcal{C}$ hence that completes finite intersections.

Let $\mathcal{B}\subset\mathcal{C}$ consider the arbitrary union of elements $\bigcup_{b\in\mathcal{B}}U_b$ where $U_b\in\mathcal{B}$. Then $\mathbb{R}\setminus\bigcup_{b\in\mathcal{B}}U_b\subset\mathbb{R}\setminus U_b$ where U_b is any $U_b\in\mathcal{B}$ as subsets of finite sets are finite this shows that arbitrary unions are in $\mathcal{C}$ hence it is a topology. $\square$

4. Suppose that Y is a Hausdorff topological space. Let a, b distinct elements of Y. Suppose that (a_n) is a sequence in Y that converges to a and (b_n) is a sequence in Y that converges to b. Show that there exists an N such that, for all $n > N$, $a_n \neq b_n$.

Proof. As Y is a Hausdorff space and a, b are distinct elements then there exists two neighborhoods U_a, U_b for a, b respectively where $U_a \cap U_b = \emptyset$. But as (a_n) is convergent we have for some $N_1 \in \mathbb{N}$ that for all $n \geq N_1$ that $a_n \in U_a$. Likewise for (b_n) for some $N_2 \in \mathbb{N}$ we have for all $n \geq N_2$ that $b_n \in U_b$. Let $N = \max(N_1, N_2)$ then for all $n \geq N$ we have $a_n \in U_a$ and $b_n \in U_b$ but as these sets are disjoint we have $a_n \neq b_n$. $\square$