

**CS 4124**  
**Solutions to Homework Assignment 4**  
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[30] 1. Let  $\Sigma = \{0, 1\}$ , the binary alphabet. Let

$$F = \{h \mid h : \Sigma^* \rightarrow \Sigma^*\},$$

the set of functions from strings to strings.

**Prove that  $F$  is uncountable.**

**HINT: Use the fact that  $\Sigma^*$  is countably infinite and employ a proof by diagonalization.**

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My proof strategy:

I will prove that the set of functions  $F' = \{h : \Sigma^* \rightarrow \{0, 1\}\}$  is uncountable and as this set is a subset of  $F$  that would imply that  $F$  is uncountable. I will do this by constructing a bijection between  $H : F' \rightarrow \mathcal{P}(\Sigma^*)$  (the power set of  $\Sigma^*$ ) in the last homework we proved that no set can have a bijection with its power set and as  $\Sigma^*$  is countable we get that  $\mathcal{P}(\Sigma^*)$  is uncountable hence if a bijection exists  $H$  exists we would have that  $F'$  is uncountable.

Let  $H : F' \rightarrow \mathcal{P}(\Sigma^*)$  where  $H(f) = \{w \in \Sigma^* : f(w) = 1\}$  (informally this is the set of elements from  $\Sigma^*$  who get mapped to 1).

Now to show that this is a bijection I will come up with an inverse. We create the function

$$H^{-1} : \mathcal{P}(\Sigma^*) \rightarrow F'$$

where for  $A \in \mathcal{P}(\Sigma^*)$  we have  $H^{-1}(A) = f$  where for all  $w \in \Sigma^*$  we have  $f(w) = 1$  if and only if  $w \in A$ . The fact that  $H^{-1}$  is not a multivalued function can be proven by assuming that for some  $A \in \mathcal{P}(\Sigma^*)$  that there exists two  $f_1, f_2 \in F'$  such that  $H^{-1}(A) = f_1$  and  $H^{-1}(A) = f_2$  then we have that for all  $w \in \Sigma^*$  that  $f_1(w) = 1$  and  $f_2(w) = 1$  if and only if  $w \in A$  which directly implies that for any  $\omega \in \Sigma^*$  where  $\omega \notin A$  that  $f_1(\omega) = 0$  and  $f_2(\omega) = 0$ . As  $f_1, f_2$  are equal at every input we have that  $f_1 = f_2$ .

Now to show that let  $f \in F'$  then we have  $H(f) = \{w \in \Sigma^* : f(w) = 1\}$ . Then take  $H^{-1}(\{w \in \Sigma^* : f(w) = 1\}) = f_0$  for some  $f_0 \in F'$ . But from the definition of  $H^{-1}$  we get that  $f_0(w) = 1$  if and only if  $\omega \in \{w \in \Sigma^* : f(w) = 1\}$  which is the same as  $f(w) = 1$  so we have that  $f_0 = f$ . This implies that  $H^{-1} \circ H = \text{Id}_{F'}$ . Now let  $A \in \mathcal{P}(\Sigma^*)$  then we have that  $H^{-1}(A) = f$  for some  $f \in F'$ . Then taking  $H(f) = \{w \in \Sigma^* : f(w) = 1\}$  but from the definition of  $f$  we have that for all  $\omega \in \Sigma^*$  that  $f(\omega) = 1$  if and only if  $\omega \in A$ . This implies that  $H(f) = A$  so we have  $H \circ H^{-1}(A) = A$  so we have that  $H$  is the inverse of  $H^{-1}$  so we have that  $H$  is a bijection.

Then as a bijection between  $F' \subset F$  to  $\mathcal{P}(\Sigma^*)$  exists and  $\mathcal{P}(\Sigma^*)$  is uncountable we get  $F'$  is uncountable and so  $F$  is uncountable as well.

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[30] 2. Let

$$E = \{w \in \{0, 1\}^* \mid \ell(w) \text{ is even}\},$$

**Program  $P$**   
**Input**  $w$   
**if**  $\ell(w)$  is even  
**then** accept  
**else** reject

Figure 1: A sample program to decide  $E$ , just to show how to format a program in L<sup>A</sup>T<sub>E</sub>X.

the language of binary strings of even length. See Figure 1 for a sample program that decides  $E$ , just for L<sup>A</sup>T<sub>E</sub>X formatting purposes. Let

$$L = \{x \mid \text{program } x \text{ decides } E\}$$

be the language of programs that decide  $E$ .

**Prove that  $L$  is undecidable by using m-reducibility.**

[40] **3.** Let

$$\text{DOUBLE} = \{ww \mid w \in \{a, b\}^*\}.$$

Let

$$W = \{x \mid \text{program } x \text{ decides DOUBLE}\}$$

be the language of programs that decide DOUBLE.

**A. Is DOUBLE decidable? Justify your answer.**

**B. Is  $W$  decidable? Prove your answer.**