

**Problem 1.** Let  $V$  be a finite-dimensional  $\mathbb{K}$ -vector space with basis  $\{\mathbf{v}_1, \dots, \mathbf{v}_n\}$ . The vector space  $\mathcal{L}(V, \mathbb{K})$  is called the **dual space** of  $V$ , and is denoted  $V'$ . Last time you proved that  $\{\varphi_1, \dots, \varphi_n\}$  was a basis for  $V'$  (this basis is called the **dual basis**).

Let  $W$  be another finite-dimensional  $\mathbb{K}$ -vector space with basis  $\{\mathbf{w}_1, \dots, \mathbf{w}_m\}$ , and dual basis  $\{\omega_1, \dots, \omega_m\}$ . Given a map  $T \in \mathcal{L}(V, W)$ , there is another map  $T' \in \mathcal{L}(W', V')$  defined by  $T'(\psi) = \psi \circ T$ . (The notation is a bit strange, but  $T'(\psi)$  is a function in  $\mathcal{L}(V, \mathbb{K})$ , and for every  $\mathbf{v} \in V$ , we define  $T'(\psi)(\mathbf{v}) = \psi(T(\mathbf{v}))$ .)

Show that  $\mathcal{M}(T') = (\mathcal{M}(T))^t$ .

*Proof.* Assume that  $V, W$  are both  $\mathbb{K}$  vector spaces with basis  $\{\mathbf{v}_1, \dots, \mathbf{v}_n\}$  and  $\{\mathbf{w}_1, \dots, \mathbf{w}_m\}$  respectively. Assume that the basis for  $V'$  is  $\{\varphi_1, \dots, \varphi_n\}$  and the basis for  $W'$  is  $\{\omega_1, \dots, \omega_m\}$ . Now consider two arbitrary linear transformations  $T \in \mathcal{L}(V, W)$  and  $T' \in \mathcal{L}(W', V')$ . Then we have the entries of  $B$  are given by

Then by the definition of  $T'$  we get

$$T'\omega_k(\mathbf{v}_j) = \omega_k \circ T(\mathbf{v}_j)$$

where  $1 \leq k \leq m$  and  $1 \leq j \leq n$ .

We also get

$$T'\omega_k = \sum_{i=1}^n B_{i,k} \varphi_i$$

substituting this into the equation above we get

$$\sum_{i=1}^n B_{i,k} \varphi_i(\mathbf{v}_j) = \omega_k \circ \sum_{c=1}^m A_{c,j} \mathbf{w}_c$$

which follows from the definition of matrix of a linear map.

Then we have  $B_{k,j} = \sum_{i=1}^n B_{i,k} \varphi_i(\mathbf{v}_j)$  by the definition of matrix of a linear map. We also get  $\omega_k \circ \sum_{c=1}^m A_{c,j} \mathbf{w}_c = \sum_{c=1}^m A_{c,j} \omega_k(\mathbf{w}_c)$  which follows due to  $\omega_k$  being linear. Then by the definition of dual basis we get  $\sum_{c=1}^m A_{c,j} \omega_k(\mathbf{w}_c) = A_{k,j}$  this implies  $\mathcal{M}(T') = (\mathcal{M}(T))^t$  □

**Problem 2.** Let  $D : \mathcal{P}_4(\mathbb{R}) \rightarrow \mathcal{P}_3(\mathbb{R})$  be the derivative map  $D(p(x)) = \frac{dp}{dx}$ . When using the standard polynomial bases  $\{1, x, x^2, x^3, x^4\}$  and  $\{1, x, x^2, x^3\}$ , the matrix  $\mathcal{M}(D)$  is

$$\mathcal{M}(D) = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 0 & 4 \end{pmatrix}.$$

Find bases  $\mathcal{B}$  for  $\mathcal{P}_4(\mathbb{R})$  and  $\mathcal{C}$  for  $\mathcal{P}_3(\mathbb{R})$  so that

$$\mathcal{M}(D, \mathcal{B}, \mathcal{C}) = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix}.$$

We have that the basis  $\mathcal{B} = \{\mathbf{v}_1 = x^4, \mathbf{v}_2 = x^3, \mathbf{v}_3 = x^2, \mathbf{v}_4 = x, \mathbf{v}_5 = 1\}$  and  $\mathcal{C} = \{\mathbf{w}_1 = 4x^3, \mathbf{w}_2 = 3x^2, \mathbf{w}_3 = 2x, \mathbf{w}_4 = 1\}$  work. First to show that these are basis  $\mathcal{B}$  is the standard basis for  $\mathcal{P}_4(\mathbb{R})$  hence it is a basis. Now for  $\mathcal{C}$  consider  $\alpha_1 4x^3 + \alpha_2 3x^2 + \alpha_3 2x + \alpha_4 1 = 0$  where each  $\alpha_i$  is an arbitrary scalar as each  $\alpha_1, \alpha_2, \alpha_3, \alpha_4$  is a coefficient for a unique degree polynomial we get  $\alpha_1 = \alpha_2 = \alpha_3 = \alpha_4 = 0$  hence  $\mathcal{C}$  is linear independent and as  $\dim \mathcal{P}_3(\mathbb{R}) = 4$  we get it is a basis.

Now computing  $\mathcal{M}(D, \mathcal{B}, \mathcal{C})$  we have

$$D(x^4) = 1 \cdot 4x^3 + 0 \cdot 3x^2 + 0 \cdot 2x + 0 \cdot 1$$

$$D(x^3) = 0 \cdot 4x^3 + 1 \cdot 3x^2 + 0 \cdot 2x + 0 \cdot 1$$

$$D(x^2) = 0 \cdot 4x^3 + 0 \cdot 3x^2 + 1 \cdot 2x + 0 \cdot 1$$

$$D(x^1) = 0 \cdot 4x^3 + 0 \cdot 3x^2 + 0 \cdot 2x + 1 \cdot 1$$

$$D(x^0) = 0 \cdot 4x^3 + 0 \cdot 3x^2 + 0 \cdot 2x + 0 \cdot 1$$

From this we get the matrix above.

**Problem 3.** Let  $\mathcal{B} = \{\mathbf{b}_1 = (1, -1, 0), \mathbf{b}_2 = (1, 0, 2), \mathbf{b}_3 = (0, 2, -1)\}$  be a basis for  $\mathbb{K}^3$  and let  $\mathcal{E}$  denote the standard basis for  $\mathbb{K}^3$ .

- (a) Find scalars  $k_1, k_2, k_3$  satisfying  $k_1\mathbf{b}_1 + k_2\mathbf{b}_2 + k_3\mathbf{b}_3 = (3, 5, 1)$ .
- (b) Find the change of basis matrix  $\mathcal{M}(\text{Id}, \mathcal{B}, \mathcal{E})$ .
- (c) Compute the following matrix product. How does this relate to your work in part (a)?

$$\mathcal{M}(\text{Id}, \mathcal{B}, \mathcal{E}) \begin{pmatrix} 1 & 1 & 0 & 3 \\ -1 & 0 & 2 & 5 \\ 0 & 2 & -1 & 1 \end{pmatrix}$$

- (a) The scalars  $k_1 = 1, k_2 = 2, k_3 = 3$  work this is shown by computing  $1(1, -1, 0) + 2(1, 0, 2) + 3(0, 2, -1) = (3, -1 + 6, 4 - 3) = (3, 5, 1)$
- (b) In this case it is easier to find the change of basis matrix  $(\text{Id}, \mathcal{E}, \mathcal{B})$  and then compute it's inverse. We have

$$\mathcal{M}(\text{Id}, \mathcal{E}, \mathcal{B}) = \begin{pmatrix} 1 & 1 & 0 \\ -1 & 0 & 2 \\ 0 & 2 & -1 \end{pmatrix}$$

Then

$$\mathcal{M}(\text{Id}, \mathcal{E}, \mathcal{B})^{-1} = \begin{pmatrix} \frac{4}{5} & \frac{-1}{5} & \frac{-2}{5} \\ \frac{1}{5} & \frac{1}{5} & \frac{2}{5} \\ \frac{2}{5} & \frac{2}{5} & \frac{-1}{5} \end{pmatrix}$$

Computing the transformations of each of the basis of  $\mathcal{B}$  we get  $\mathcal{M}(\text{Id}, \mathcal{E}, \mathcal{B})^{-1} \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} =$

$$\begin{pmatrix} 4/5 + 1/5 \\ 1/5 - 1/5 \\ 0 \end{pmatrix} = \mathbf{b}_1.$$

Doing the same calculation for  $\mathbf{b}_2$  we get  $\mathcal{M}(\text{Id}, \mathcal{E}, \mathcal{B})^{-1} \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix} = \begin{pmatrix} 4/5 - 4/5 \\ 1/5 + 4/5 \\ 2/5 - 2/5 \end{pmatrix} = \mathbf{b}_2$

Lastly with  $\mathbf{b}_3$  we get  $\mathcal{M}(\text{Id}, \mathcal{E}, \mathcal{B})^{-1} \begin{pmatrix} 0 \\ 2 \\ -1 \end{pmatrix} = \begin{pmatrix} -2/5 + 2/5 \\ 2/5 - 2/5 \\ 4/5 + 1/5 \end{pmatrix} = \mathbf{b}_3$ . Therefore we have found the basis transformation.

(c)

$$\begin{pmatrix} \frac{4}{5} & \frac{-1}{5} & \frac{-2}{5} \\ \frac{1}{5} & \frac{1}{5} & \frac{2}{5} \\ \frac{2}{5} & \frac{2}{5} & \frac{-1}{5} \end{pmatrix} \begin{pmatrix} 1 & 1 & 0 & 3 \\ -1 & 0 & 2 & 5 \\ 0 & 2 & -1 & 1 \end{pmatrix} = \begin{pmatrix} 4/5 + 1/5 & 4/5 - 4/5 \\ 1/5 - 1/5 & 1/5 + 4/5 \\ 2/5 - 2/5 & 2/5 - 1/5 \end{pmatrix}$$

**Problem 4.** Have a lovely Spring Break!