

1. In each of the following topological spaces, give an example of an intersection of infinitely many open sets that is not itself an open set.

- (1) \mathbb{R} with its standard topology. Consider the intersection

$$\bigcap_{n \in \mathbb{N}} (-1/n, 1/n)$$

We have $\bigcap_{n \in \mathbb{N}} (-1/n, 1/n) = \{0\}$. This is shown as $-1/n < 0 < 1/n$ for all $n \in \mathbb{N}$. Any element $j \in (0, 1)$ is not in the intersection as there exists $n \in \mathbb{N}$ such that $1/n < j$ by the Archimedean property. Likewise for any $-j \in (-1, 0)$ there exists a $n \in \mathbb{N}$ such that $-j < -1/n$ again by the Archimedean property. Now $\{0\}$ is not open as there exists no basis element in the standard topology that is a subset of $\{0\}$. This is shown as all basis elements are of the form (a, b) where $a, b \in \mathbb{R}$ where $a < b$ but $|\{0\}| = 1$ but $|(a, b)|$ is uncountable.

- (2) \mathbb{R} with its lower limit topology. Consider

$$\bigcap_{n \in \mathbb{N}} [0, 1/n)$$

. We have that $\bigcap_{n \in \mathbb{N}} [0, 1/n) = \{0\}$ using the same reasoning as above. Now $|\{0\}| = 1$ but any basis element $[a, b)$ where $a, b \in \mathbb{R}$ with $a < b$ is uncountable hence no basis element is a subset of $\{0\}$ which implies it is not open.

- (3) \mathbb{R} with the finite complement topology. Consider

$$\bigcap_{n \in \mathbb{N}} \mathbb{R} \setminus \{1/n\}$$

. We have each $n \in \mathbb{N}$ that $\mathbb{R} \setminus (\mathbb{R} \setminus \{1/n\}) = \{1/n\}$ hence $\mathbb{R} \setminus \{1/n\}$ is open. But $\bigcap_{n \in \mathbb{N}} \mathbb{R} \setminus \{1/n\} = \mathbb{R} \setminus \{1, 1/2, 1/3, \dots\}$ the complement of this set is not finite hence not open.

2. Let \mathbb{R} have the lower limit topology is $(0, 1)$ open? Yes

Proof. Consider the union $\bigcup_{n \in \mathbb{N}} [1/n, 1)$ as each of the sets is open and this is a union we have that it is open in the lower limit topology so just need to demonstrate double containment. Let $j \in \bigcup_{n \in \mathbb{N}} [1/n, 1)$ then for some $n \in \mathbb{N}$ we have $j \in [1/n, 1)$ but as $0 < 1/n$ for all $n \in \mathbb{N}$ we get the inequality $0 < j < 1$ hence $j \in (0, 1)$. Now let $j \in (0, 1)$ then by the Archimedean property for some $n \in \mathbb{N}$ we have $1/n < j < 1$ hence $j \in [1/n, 1)$. Which shows double containment hence $\bigcup_{n \in \mathbb{N}} [1/n, 1) = (0, 1)$ which completes the proof. \square

3. In the set \mathbb{R} , consider the collection of subsets consisting of \mathbb{R}, \emptyset , and all sets whose complements are finite sets of irrational numbers. Is this collection a topology on \mathbb{R} ?

Yes

Proof. As \emptyset, \mathbb{R} are in this collection \mathcal{C} we just need to demonstrate finite intersections and arbitrary unions are in \mathcal{C} .

Consider the intersection of two elements $A, B \in \mathcal{C}$ then we have $\mathbb{R} \setminus A \cap B = (\mathbb{R} \setminus A) \cup (\mathbb{R} \setminus B)$ as the union of two finite sets is finite that completes the base case. Now assume for some $n \in \mathbb{N}$ where $n \geq 2$ we have that the intersection of n elements of \mathcal{C} is in \mathcal{C} . Then given $n + 1$ elements A_1, \dots, A_{n+1} consider the intersection $A_1 \cap \dots \cap A_{n+1}$ then we have $\mathbb{R} \setminus (A_1 \cap \dots \cap A_{n+1}) = (\mathbb{R} \setminus A_1 \cup \dots \cup \mathbb{R} \setminus A_{n+1}) \cup \mathbb{R} \setminus A_{n+1}$ using the induction hypothesis we have $\mathbb{R} \setminus A_1 \cup \dots \cup \mathbb{R} \setminus A_n \in \mathcal{C}$ by the base case the intersection of two elements of \mathcal{C} is also in \mathcal{C} hence that completes finite intersections.

Let $\mathcal{B} \subset \mathcal{C}$ consider the arbitrary union of elements $\bigcup_{b \in \mathcal{B}} U_b$ where $U_b \in \mathcal{B}$. Then $\mathbb{R} \setminus \bigcup_{b \in \mathcal{B}} U_b \subset \mathbb{R} \setminus U_b$ where U_b is any $U_b \in \mathcal{B}$ as subsets of finite sets are finite this shows that arbitrary unions are in \mathcal{C} hence it is a topology. \square

4. Suppose that Y is a Hausdorff topological space. Let a, b distinct elements of Y . Suppose that (a_n) is a sequence in Y that converges to a and (b_n) is a sequence in Y that converges to b . Show that there exists an N such that, for all $n > N$, $a_n \neq b_n$.

Proof. As Y is a Hausdorff space and a, b are distinct elements then there exists two neighborhoods U_a, U_b for a, b respectively where $U_a \cap U_b = \emptyset$. But as (a_n) is convergent we have for some $N_1 \in \mathbb{N}$ that for all $n \geq N_1$ that $a_n \in U_a$. Likewise for (b_n) for some $N_2 \in \mathbb{N}$ we have for all $n \geq N_2$ that $b_n \in U_b$. Let $N = \max(N_1, N_2)$ then for all $n \geq N$ we have $a_n \in U_a$ and $b_n \in U_b$ but as these sets are disjoint we have $a_n \neq b_n$. \square

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