Problem 1. For a fixed $a \in \mathbb{C}$, show that $\frac{|z-a|}{|1-\bar{a}z|} = 1$ if $|z| = 1$ and $1 - \bar{a}z \neq 0$.

Proof. Assume $a \in \mathbb{C}$ and $z \in \mathbb{Z}$ with $|z| = 1$ and $1 - \bar{a}z \neq 0$. Then for some $x, y, c, d \in \mathbb{R}$ we have $a = x + iy$ and $z = c + id$.

Calculating $|1 - \bar{a}z|$ yields

$$\begin{align}
|1-\bar{a}z| &= |1-(x-iy)(c+id)| \tag{1}\\
|1-\bar{a}z| &= \sqrt{(1-xc-yd)^2+(yc-xd)^2} \tag{2}\\
|1-\bar{a}z| &= \sqrt{1-2xc-2yd+2xcyd+x^2c^2+y^2d^2+y^2c^2-2ycxd+x^2d^2} \tag{3}\\
|1-\bar{a}z| &= \sqrt{1-2xc-2yd+x^2c^2+x^2d^2+y^2d^2+y^2c^2} \tag{4}\\
|1-\bar{a}z| &= \sqrt{1-2xc-2yd+x^2(c^2+d^2)+y^2(c^2+d^2)} \tag{5}\\
|1-\bar{a}z| &= \sqrt{1-2xc-2yd+x^2+y^2} \tag{6}\\
|1-\bar{a}z| &= \sqrt{c^2+d^2+x^2+y^2-2xc-2yd} \tag{7}\\
|1-\bar{a}z| &= \sqrt{(c-x)^2+(d-y)^2} \tag{8}\\
|1-\bar{a}z| &= |z-a| \tag{9}
\end{align}$$

Based on the assumption of $|1 - \bar{a}z| \neq 0$ we have

$$\frac{|z-a|}{|1-\bar{a}z|} = 1$$

□

Problem 2. For which n is i an nth root of unity?

For any positive integer n where $n \equiv 0 \bmod 4$.

Problem 3. If the point P on the sphere corresponds to z under stereographic projection, show that the antipodal point $-P$ on the sphere corresponds to $-\frac{1}{\bar{z}}$.

Proof. Assume that the point P corresponds to $z_0 = x_0 + iy_0$ under stereographic projection. Then solving for the point z from the projection $-P$ by using the equations on page 12

$$\begin{cases} X = \frac{2x_0}{|z_0|^2+1} \\ \\ Y = \frac{2y_0}{|z_0|^2+1} \\ \\ Z = \frac{|z_0|^2-1}{|z_0|^2+1} \end{cases}$$

substituting in for $z = -\frac{X}{1+Z} - i\frac{Y}{1+Z}$ we get

$$z = -\frac{\frac{2x_0}{|z_0|^2+1}}{1+\frac{|z_0|^2-1}{|z_0|^2+1}} - i\frac{\frac{2y_0}{|z_0|^2+1}}{1+\frac{|z_0|^2-1}{|z_0|^2+1}}$$

$$z = -\frac{\frac{2x_0+i2y_0}{|z_0|^2+1}}{1+\frac{|z_0|^2-1}{|z_0|^2+1}}$$

$$z = -\frac{\frac{2x_0+i2y_0}{|z_0|^2+1}}{\frac{2|z_0|^2}{|z_0|^2+1}}$$

$$z = -\frac{x_0+iy_0}{|z_0|^2}$$

$$z = -\frac{x_0+iy_0}{|z_0|^2} \cdot \frac{\frac{1}{z_0}}{\frac{1}{z_0}}$$

$$z = -\frac{1}{\bar{z}_0}$$

□