Problem 1. Show that if f and $\bar{f}$ are analytic on a domain D, then f is constant.

Proof. Assume that $f = u + iv$ and $\bar{f} = u - iv$ are both analytic on domain D. Then we have that both satisfy the Cauchy-Riemann equations: That is

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \qquad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x},$$
$$\frac{\partial u}{\partial x} = -\frac{\partial v}{\partial y}, \qquad \frac{\partial u}{\partial x} = \frac{\partial v}{\partial x}.$$

Then we have $\frac{\partial u}{\partial x} = -\frac{\partial u}{\partial x} \implies \frac{\partial u}{\partial x} = 0$ using the same reasoning for the rest we get $\frac{\partial v}{\partial x} = \frac{\partial u}{\partial y} = \frac{\partial v}{\partial y} = 0$. This implies that u and v are both constant functions therefore f is constant. □

Problem 2. Let a be a complex number, $a \neq 0$, and $f(z)$ be an analytic branch of z^a on $\mathbb{C} \setminus (-\infty, 0]$. Show that $f'(z) = af(z)/z$.

Proof. Let $f(z) = z^a$ where $a \neq 0$, and $f(z)$ be an analytic branch of z^a on $\mathbb{C} \setminus (-\infty, 0]$. Then we have $f(z) = e^{a\text{Log}z}$. Using the chain rule we get $f'(z) = ae^{a\text{Log}z} \cdot \frac{1}{z}$. As $f(z) = e^{a\text{Log}}$ is the same branch we can do the substitution $f'(z) = ae^{a\text{Log}z}/z = af(z)/z$ which completes the proof. □

Problem 3. Show that if $h(z)$ is a complex valued harmonic function such that $zh(z)$ is also harmonic, then $h(z)$ is analytic.

Proof. Assume that $h(z)$ is a complex valued harmonic function such that $zh(z)$ is also harmonic. Then $h(z) = u(z) + iv(z)$ we get

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$
$$\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0$$

Additionally as we have $zh(z)$ is harmonic we get

$$(x + iy)(u(z) + iv(z)) = xu(z) - yv(z) + i(xv(z) + yu(z))$$

We get

$$\frac{\partial^2 (xu(z) - yv(z))}{\partial x^2} + \frac{\partial^2 (xu(z) - yv(z))}{\partial y^2} = 0 \tag{1}$$
$$\frac{\partial^2 (xv(z) + yu(z))}{\partial x^2} + \frac{\partial^2 (xv(z) + yu(z))}{\partial y^2} = 0 \tag{2}$$

Applying the partial derivatives to (1) we get

$$x\frac{\partial^2 u}{\partial x^2} + \frac{\partial u}{\partial x} - y\frac{\partial^2 v}{\partial x^2} + x\frac{\partial^2 u}{\partial y^2} - y\frac{\partial^2 v}{\partial y^2} - \frac{\partial v}{\partial y} = 0$$

$$x(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}) - y(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}) + \frac{\partial u}{\partial x} - \frac{\partial v}{\partial y} = 0$$

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$$

Applying the partial derivatives to (2) we get.

$$x\frac{\partial^2 v}{\partial x^2} + \frac{\partial v}{\partial x} + y\frac{\partial^2 u}{\partial x^2} + x\frac{\partial^2 v}{\partial y^2} + y\frac{\partial^2 u}{\partial y^2} + \frac{\partial u}{\partial y} = 0$$

$$x(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}) + y(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}) + \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} = 0$$

$$\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

Therefore we have that the Cauchy Riemann equations are satisfied which implies that $h(z)$ is analytic.

$\square$