1. In each of the following topological spaces, give an example of an intersection of infinitely many open sets that is not itself an open set.

(1) $\mathbb{R}$ with it's standard topology. Consider the intersection

$$\bigcap_{n\in\mathbb{N}}(-1/n, 1/n)$$

We have $\bigcap_{n\in\mathbb{N}}(-1/n, 1/n) = \{0\}$. This is shown as $-1/n < 0 < 1/n$ for all $n \in \mathbb{N}$. Any element $j \in (0, 1)$ is not in the intersection as there exists $n \in \mathbb{N}$ such that $1/n < j$ by the Archimedean property. Likewise for any $-j \in (-1, 0)$ there exists a $n \in \mathbb{N}$ such that $-j < -1/n$ again by the Archimedean property. Now $\{0\}$ is not open as there exists no basis element in the standard topology that is a subset of $\{0\}$. This is shown as all basis elements are of the form (a, b) where $a, b \in \mathbb{R}$ where $a < b$ but $|\{0\}| = 1$ but $|(a, b)|$ is uncountable.

(2) $\mathbb{R}$ with its lower limit topology. Consider

$$\bigcap_{n\in\mathbb{N}}[0, 1/n)$$

. We have that $\bigcap_{n\in\mathbb{N}}[0, 1/n) = \{0\}$ using the same reasoning as above. Now $|\{0\}| = 1$ but any basis element $[a, b)$ where $a, b \in \mathbb{R}$ with $a < b$ is uncountable hence no basis element is a subset of $\{0\}$ which implies it is not open.

(3) $\mathbb{R}$ with the finite complement topology. Consider

$$\bigcap_{n\in\mathbb{N}}\mathbb{R} \setminus \{1/n\}$$

. We have each $n \in \mathbb{N}$ that $\mathbb{R} \setminus (\mathbb{R} \setminus \{1/n\}) = \{1/n\}$ hence $\mathbb{R} \setminus \{1/n\}$ is open. But $\bigcap_{n\in\mathbb{N}}\mathbb{R} \setminus \{1/n\} = \mathbb{R} \setminus \{1, 1/2, 1/3, ...\}$ the complement of this set is not finite hence not open.

2. Let $\mathbb{R}$ have the lower limit topology is $(0, 1)$ open? Yes

Proof. Consider the union $\bigcup_{n\in\mathbb{N}}[1/n, 1)$ as each of the sets is open and this is a union we have that it is open in the lower limit topology so just need to demonstrate double containment. Let $j \in \bigcup_{n\in\mathbb{N}}[1/n, 1)$ then for some $n \in \mathbb{N}$ we have $j \in [1/n, 1)$ but as $0 < 1/n$ for all $n \in \mathbb{N}$ we get the inequality $0 < j < 1$ hence $j \in (0, 1)$. Now let $j \in (0, 1)$ then by the Archimedean property for some $n \in \mathbb{N}$ we have $1/n < j < 1$ hence $j \in \bigcup[1/n, 1)$. Which shows double containment hence $\bigcup_{n\in\mathbb{N}}[1/n, 1) = (0, 1)$ which completes the proof. □

3. In the set $\mathbb{R}$, consider the collection of subsets consisting of $\mathbb{R}, \emptyset$, and all sets whose complements are finite sets of irrational numbers. Is this collection a topology on $\mathbb{R}$?

Yes

Proof. As $\emptyset, \mathbb{R}$ are in this collection $\mathcal{C}$ we just need to demonstrate finite intersections and arbitrary unions are in $\mathcal{C}$.

Consider the intersection of two elements $A, B \in \mathcal{C}$ then we have $\mathbb{R} \setminus A \cap B = (\mathbb{R} \setminus A) \cup (\mathbb{R} \setminus B)$ as the union of two finite sets is finite that completes the base case. Now assume for some $n \in \mathbb{N}$ where $n \geq 2$ we have that the intersection of n elements of $\mathcal{C}$ is in $\mathcal{C}$. Then given $n + 1$ elements $A_1, ..., A_{n+1}$ consider the intersection $A_1 \cap ... \cap A_{n+1}$ then we have $\mathbb{R} \setminus (A_1 \cap ... \cap A_2) = (\mathbb{R} \setminus A_1 \cup ... \cup \mathbb{R} \setminus A_n) \cup \mathbb{R} \setminus A_{n+1}$ using the induction hypothesis we have $\mathbb{R} \setminus A_1 \cup ... \cup \mathbb{R} \setminus A_n \in \mathcal{C}$ by the base case the intersection of two elements of $\mathcal{C}$ is also in $\mathcal{C}$ hence that completes finite intersections.

Let $\mathcal{B} \subset \mathcal{C}$ consider the arbitrary union of elements $\bigcup_{b\in\mathcal{B}} U_b$ where $U_b \in \mathcal{B}$. Then $\mathbb{R} \setminus \bigcup_{b\in\mathcal{B}} U_b \subset \mathbb{R} \setminus U_b$ where U_b is any $U_b \in \mathcal{B}$ as subsets of finite sets are finite this shows that arbitrary unions are in $\mathcal{C}$ hence it is a topology. □

4. Suppose that Y is a Hausdorff topological space. Let a, b distinct elements of Y. Suppose that (a_n) is a sequence in Y that converges to a and (b_n) is a sequence in Y that converges to b. Show that there exists an N such that, for all $n > N$, $a_n \neq b_n$.

Proof. As Y is a Hausdorff space and a, b are distinct elements then there exists two neighborhoods U_a, U_b for a, b respectively where $U_a \cap U_b = \emptyset$. But as (a_n) is convergent we have for some $N_1 \in \mathbb{N}$ that for all $n \geq N_1$ that $a_n \in U_a$. Likewise for (b_n) for some $N_2 \in \mathbb{N}$ we have for all $n \geq N_2$ that $b_n \in U_b$. Let $N = \max(N_1, N_2)$ then for all $n \geq N$ we have $a_n \in U_a$ and $b_n \in U_b$ but as these sets are disjoint we have $a_n \neq b_n$. □

5.

(1) Show that, in any metric space (X, d) and for any $x_0 \in X$, the closure of $B_d(x_0, 1)$ is contained in $D_d(x_0, 1)$.

Proof. Let (X, d) be an arbitrary metric space and $x_0 \in X$. We have $\overline{B_d(x_0, 1)} = B_d(x_0, 1) \cup B_d(x_0, 1)'$. We have $B_d(x_0, 1) \subset D_d(x_0, 1)$ which follows from the strict inequality on $B_d(x_0, 1)$. Now let $y \in B_d(x_0, 1)'$ then for every ϵ-neighborhood we have for some distinct $x \in B_d(x_0, 1)$ that $0 < d(x, y) < \epsilon$ from the definition of the open ball we have $d(x_0, x) < 1$ applying the triangle inequality we get $d(x_0, y) \leq d(x_0, x) + d(x_0, y) < 1 + \epsilon$ as this is true for all $\epsilon > 0$ we get the strict inequality $d(x_0, y) \leq 1$ hence $y \in D_d(x_0, 1)$ which shows $B_d(x_0, 1)' \subset D_d(x_0, 1)$ as both the set and its limit points are subsets of $D_d(x_0, 1)$ this completes the proof.

□

(2) Is it the case that, in every metric space (X, d) and for every $x_0 \in X, D_d(x_0, 1)$ is contained in the closure of $B_d(x_0, 1)$?

Proof. No this is not true consider the metric $(\mathbb{R}, d)$ where $d(x, y) = \begin{cases} 1 & \text{if } x \neq y \\ 0 & \text{if } x = y \end{cases}$. Consider the open ball $B_d(0, 1) = \{0\}$ which follows because of the strict inequality but $D_d(0, 1) = \mathbb{R}$. The closure of $B_d(0, 1) = B_d(0, 1) \cup B_d(0, 1)'$ but for any $x \in \mathbb{R}$ with $x \neq 0$ we have that any epsilon neighborhood of x with $0 < \epsilon < 1$ that $B_d(x, \epsilon) = \{x\}$ hence not every neighborhoods even intersects $B_d(0, 1)$ so the limit point is the emptyset. Therefore $\overline{B_d(0, 1)} = \{0\} \not\supset D_d(0, 1) = \mathbb{R}$. □

6. For all natural numbers j let X_j be $\mathbb{R}$ with the standard topology. Let $X = \prod_{j \in \mathbb{N}} X_j$ define the elements of X by $\vec{x} = (x_1, x_2, ...)$

(1) Show that the set of $5-$bounded elements of X is closed in both the product topology and the box topology.

Proof. Denote the set of 5 bounded elements by B_5 we have that a set is closed if and only if it contains it's limit points. Let $\vec{x} \in B_5$ then we have for any neighborhood U of $\vec{x}$ that $U_i \neq \mathbb{R}_i$ for a finite number of i. As this is the product topology we have for each $U_i \neq \mathbb{R}_i$ as U_i is open a basis element $u_i = (a_i, b_i) \subset U_i$ where $a_i, b_i \in \mathbb{R}$ with $-5 \leq a_i < \pi_i(\vec{x}) < b_i \leq 5$ then we have the set $A = \mathbb{R}_1 \times u_2 \times ... \times u_i \times ... \times \mathbb{R}_n \times ...$ is open and $A \subset U$ and $A \cap B_5 \setminus \{\vec{x}\} \neq \emptyset$ as there exists some $\vec{y} \in B_5 \setminus \{\vec{x}\}$ with $a_i < \pi_i(\vec{y}) < b_i$ where the a_i, b_i come from the bases element u_i when $U_i \neq \mathbb{R}_i$ if $U_i = \mathbb{R}_i$ then $\pi_i(\vec{y}) \in [-5, 5]$ hence this shows $A \cap B_5 \setminus \{\vec{x}\} \neq \emptyset$. Which implies every element of B_5 is a limit point of B_5.

If $\vec{x} \notin B_5$ then there exists some $i \in \mathbb{N}$ we have $|\pi_i(\vec{x})| > 5$. If $\pi_i(\vec{x}) > 5$ then consider the open set $\pi_i^{-1}((a, b))$ where $a, b \in \mathbb{R}$ with $5 < a < \pi_i(\vec{x}) < b$. We have $\pi_i^{-1}((a, b)) \cap B_5 = \emptyset$ as every element of B_5 is $5-$bounded. If $\pi_i(\vec{x}) < -5$ then we we have the open set $\pi_i^{-1}(a, b)$ with $a < \pi_i(\vec{x}) < b < -5$ again this open set $\pi_i^{-1}(a, b) \cap B_5 = \emptyset$ as B_5 is 5 bounded. This shows that B_5 contains its limit points hence it is closed in the product topology.

We have that the box topology is finer than the product topology. Which implies that B_5 if it is closed in the product topology then it is also closed in the box topology.

□

(2) Show that the set of bounded elements of X is closed in the box topology.

Proof. Denote the set of bounded elements by B. We have that B is closed if and only if it contains it's limit points. Let $\vec{x} \in B$ then consider the open set $U = \prod_{i\in\mathbb{N}}(\pi_i(\vec{x}) - 1, \pi(\vec{x}) + 1)$ we have that $U \cap B = \emptyset$ because if it did not then there would exist $\vec{y} \in B$ with $|\pi_i(\vec{x}) - \pi_i(\vec{y})| < 1$ for all $i \in \mathbb{N}$ but as $\vec{y}$ is bounded we have for some $M \in \mathbb{R}$ with $M > 0$ that $-M \leq \pi_i(\vec{y}) \leq M$ for all $i \in \mathbb{N}$. Then we have for all $i \in \mathbb{N}$ that $|\pi_i(\vec{x}) - \pi_i(\vec{y})| < 1$ which by the triangle inequality yields $|\pi_i(\vec{x})| - |\pi_i(\vec{y})| < 1$ which implies $|\pi_i(\vec{x})| < 1 + M$ which is a contradiction on $\vec{x}$ being bounded. Hence $U \cap B = \emptyset$. Now suppose that $\vec{x}$ is bounded then for any neighborhood U of $\vec{x}$ we have a basis element where for each $i \in \mathbb{N}$ we have $u_i = (a_i, b_i) \subset U_i$ where $a_i, b_i \in \mathbb{R}$ and $a_i < \pi_i(\vec{x}) < b_i$. Then $\vec{x} \in \prod_{i\in\mathbb{N}} u_i \subset U$ then we have $\prod_{i\in\mathbb{N}} u_i \cap B \setminus \{\vec{x}\} \neq \emptyset$ as consider the element $\vec{y} \in X$ where $\pi_1(\vec{y}) = (\pi_1(\vec{x}) + b_1)/2$ where b_1 is the upper bound on u_1 and $\pi_i(\vec{y}) = \pi_i(\vec{x})$ for all $i \geq 2$ we have $\vec{y} \neq \vec{x}$ and it is bounded hence $\prod_{i\in\mathbb{N}} u_i \cap B \setminus \{\vec{x}\} \neq \emptyset$ which shows that B contains its limit points hence it is closed. □

(3) Is the set of bounded elements of X closed in the product topology?

Proof. No. Denote the set of bounded elements by B. Then consider an element $\vec{x} \notin B$. Then we have that any open set containing $\vec{x}$ only a finite amount of times $U_i \neq \mathbb{R}_i$ for each of the $U_i \neq \mathbb{R}_i$ we have a basis element $u_i = (a_i, b_i) \subset U_i$ with $a_i, b_i \in \mathbb{R}$ and $a_i < \pi_i(\vec{x}) < b_i$. Then we have construct the open set $A = \mathbb{R} \times u_1 \times ... \times u_i \times R \times ...$ we have that $A \cap B \setminus \{\vec{x}\} \neq \emptyset$ as consider the element $\vec{y} \in B$ where when $U_i \neq \mathbb{R}_i$ we have $\pi_i(\vec{y}) = (\pi(\vec{x}) + b_i)/2$ where b_i is the upper bound on u_i and when $U_i = \mathbb{R}_i$ we have $\pi_i(\vec{y}) = 1$. We have $\vec{y}$ is bounded and an arbitrary $\vec{x} \notin B$ was a limit point of B hence B does not contain its limit points so it is not bounded. □