Problem 1. Consider $V = \mathbb{C}$, the complex numbers, as a $\mathbb{C}$-vector space. Define a function $\Re : \mathbb{C} \to \mathbb{C}$ by

$$\Re(x + iy) = x$$

Is $\Re$ a linear map? If so, prove it. If not, explain why not.

No it is not

Proof. No as it does not satisfy homogeneity. Consider $(1 + i) \cdot \Re(1 + i) = 1 + i$ but $\Re((1 + i) \cdot (1 + i)) = \Re(2i) = 0$ so it does not satisfy homogeneity. $\square$

Problem 2. Extending a linear map. Let V be a finite-dimensional $\mathbb{K}$-vector space, W a $\mathbb{K}$-vector space, and U asubspace of V. Furthermore, let

$$\{\mathbf{u_1}, \ldots, \mathbf{u_m}\} \text{ be a basis for } U \text{ and}$$
$$\{\mathbf{u_1}, \ldots, \mathbf{u_m}, \mathbf{v_{m+1}}, \ldots, \mathbf{v_n}\} \text{ be the (extended) basis for } V.$$

Taking $T : U \to W$ to be any linear map, define a function $f : V \to W$ by

$$f\big(a_1\mathbf{u_1} + \cdots + a_m\mathbf{u_m} + a_{m+1}\mathbf{v_{m+1}} + \cdots + a_n\mathbf{v_n}\big) = T\big(a_1\mathbf{u_1} + \cdots + a_m\mathbf{u_m}\big).$$

(a) Is f a linear map? If so, prove it. If not, explain why not.

(b) What happens if the definition of f is changed to the following?

$$f(\mathbf{x}) = \begin{cases} T(\mathbf{x}) & \text{if } \mathbf{x} \in U \\ \mathbf{0} & \text{otherwise} \end{cases}$$

(a) *Proof.* Yes it is a linear map

Let $\lambda \in \mathbb{K}$ and let $\vec{v} = \mathbf{a_1 u_1} + ... + \mathbf{a_1 u_m} + \mathbf{a_{m+1} v_{m+1}} + ... + \mathbf{a_n v_n} \in V$ and $u = \mathbf{b_1 w_1} + ... + \mathbf{b_m w_m} + \mathbf{v_{m+1} v_{m+1}} + ... + \mathbf{b_m v_n} \in V$. Where $u_i, w_i \in \text{Span}(U)$ and $a_i \in \mathbb{K}$

$$\begin{aligned} f(\lambda\vec{v} + \vec{u}) &= T(\lambda(a_1u_1 + ... + a_1u_m) + b_1w_1 + ... + b_mw_m) \\ &= T(\lambda(a_1u_1 + ... + a_1u_m)) + T(b_1w_1 + ... + b_mw_m) \\ &= \lambda T(a_1u_1 + ... + a_1u_m) + T(b_1w_1 + ... + b_mw_m) \\ &= \lambda f(\vec{v}) + f(\vec{u}) \end{aligned}$$

□

(b) *Proof.* Then it is no longer a linear map. Let $x \in U$ and $y \in V$ where $y \notin U$. Then $f(x + y) = 0$ but $f(x) + f(y) = T(x) + 0$ hence it is not linear. □

Problem 3. Prove that the following is a subspace of $\mathcal{L}(\mathbb{K}^2)$:

$$U = \left\{ f \in \mathcal{L}(\mathbb{K}^2) \ : \ \begin{array}{l} a, b, c \in \mathbb{K} \text{ and} \\ f(x,y) = (ax + by, bx + cy) \end{array} \right\}$$

Proof. We have the function $f(x,y) = (0,0) \in U$ as we can just let $a = b = c = 0$ then $f(x,y) = (0x + 0y, 0x + 0y) = (0,0)$. Now suppose that we have two different vectors f_1, f_2 of U where $f_1(x,y) = (a_1x + b_1y, b_1x + c_1y)$ and $f_2(x,y) = (a_2x + b_2y, b_2x + c_2y)$ where $a_i, b_i \in \mathbb{K}$. Then $f_1(x,y) + f_2(x,y) = (a_1x + b_1y, b_1x + c_1y) + (a_2x + b_2y, b_2x + c_2y) = ((a_1 + a_2)x + (b_1 + b_2)y, (b_1 + b_2)x + (c_1 + c_2)y)$ now as $(a_1 + a_2), (b_1 + b_2), (c_1 + c_2) \in \mathbb{K}$ we get $f_1 + f_2 \in U$. Now let $\lambda \in \mathbb{K}$ then $\lambda f_1(x,y) = \lambda(a_1x + b_1y, b_1x + c_1y) = (\lambda a_1x + \lambda b_1y, \lambda b_1x + \lambda c_1y)$ and as $\lambda a_1, \lambda b_1, \lambda c_1 \in \mathbb{K}$ we have $\lambda f_1 \in U$. Therefore it satisfies the three step subspace test hence it is a subspace.

$\square$

Problem 4. For any linear map $T \in \mathcal{L}(V)$, we say that a subspace $U \subseteq V$ is an **invariant subspace of $\boldsymbol{T}$** if and only if $T(U) \subseteq U$.

Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be the linear map given by

$$T\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} 7x - 3y + 5z \\ 12x - 4y + 12z \\ -x + y + z \end{bmatrix}.$$

Show that each of the following subspaces of $\mathbb{R}^3$ are invariant subspaces of T.

$$U_1 = \operatorname{Span}\left(\begin{bmatrix} -2 \\ -3 \\ 1 \end{bmatrix}\right) \qquad \text{and} \qquad U_2 = \operatorname{Span}\left(\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}\right)$$

Proof. Assume T, U_1, U_2 are as defined above. Let $\vec{v} \in U_1$ then by the definition of span we have for some scalar λ that $\vec{v} = \lambda \begin{bmatrix} -2 \\ -3 \\ 1 \end{bmatrix}$ likewise as for any vector $\vec{w} \in U_2$ there exists two scalars α, β such that $\vec{w} = \alpha \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} + \beta \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}$. Then

$$T(\vec{v}) = T\left(\lambda \begin{bmatrix} -2 \\ -3 \\ 1 \end{bmatrix}\right) = \lambda T\left(\begin{bmatrix} -2 \\ -3 \\ 1 \end{bmatrix}\right) = \lambda \begin{bmatrix} -14 + 9 + 5 \\ -24 + 12 + 12 \\ 2 - 3 + 1 \end{bmatrix} = \lambda \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

As this is just the subspace containing the zero vector we have $T(U_1) \subseteq U_2$.

Now computing $T(\vec{w})$.

$$\alpha T\left(\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}\right) + \beta T\left(\begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}\right) = \alpha \begin{bmatrix} -7 + 5 \\ -12 + 12 \\ 1 + 1 \end{bmatrix} + \beta \begin{bmatrix} 7 - 6 \\ 12 - 8 \\ -1 + 2 \end{bmatrix} = 2\alpha \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} + 2\beta \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} + \beta \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

As this is just equal to a linear combination of the basis elements of U_2 we have $T(U_2) \subseteq U_2$. $\square$

Problem 5. Let U_1, U_2 be the subspaces from Problem 4. Prove that

$$\mathbb{R}^3 = U_1 \oplus U_2.$$

Proof. Let $(a, b, c) \in \mathbb{R}^3$ then we have the linear combination

$$(-b + 2a + 2c)\langle -2, -3, 1\rangle + (b - 2a + 2c)\langle -1, 0, 1\rangle + (3a - b + 3c)\langle 1, 2, 0\rangle = \langle a, b, c\rangle$$

Therefore we have $\mathbb{R}^3 \subseteq U_1 + U_2$. Now the other direction we have have the sum of two subspaces of $\mathbb{R}^3$ therefore $U_1 + U_2 \subseteq \mathbb{R}^3$ which implies $R^3 = U_1 + U_2$. Now to show the direct sum. Let $(a, b, c) \in U_1 \cap U_2$ then we have for some scalars α, β, λ that

$$\alpha(-2, -3, 1) = \beta(-1, 0, 1) + \lambda(1, 2, 0)$$

$$\begin{cases} -\alpha = -\beta + \lambda \\ -3\alpha = 2\lambda \\ \alpha = \beta \end{cases}$$

Substituting β for α in the first equation we get $\lambda = 0$ which implies $\alpha = 0$ which finally implies $\beta = 0$.

Therefore the only vector that is in the intersection is the zero vector. This implies that it is a direct sum.

$\bar{A} \cap \overline{X \setminus A}$

$\square$