Math 4324 Second Compactness Assignment

Hand in Friday, March 29. In this assignment I'll hyphenate "ω-accumulation-point-compact" to make it clearer that the full expression is one adjective describing a space.

1. Show that, if the topological space X is compact, then every infinite subset of X has an ω-accumulation point in X, i.e. that every compact space is ω-accumulation-point-compact. **Hint.** If Y is a subset that has no ω-accumulation point, choose a neighborhood of each point of X that helps you show that Y must be finite.

Proof. Let X be a compact topological space and $Y \subset X$ be a set with no $\omega-$accumulation point. Then for each $y \in Y$ we have a neighborhood $y \in U_y$ but $\mathcal{U}_y = Y \cap U_y$ is finite. Then we create the open cover $\{\mathcal{U}_y\}_{y \in Y}$ but as X is compact we have a finite subcover $\{\mathcal{U}_{y_1}, \ldots, \mathcal{U}_{y_n}\}$. But then $Y = \bigcup_{i=1}^{n} \mathcal{U}_{y_i}$ and so Y is finite. This shows that any set that does not have an $\omega-$accumulation point is finite hence any infinite set must have an $\omega-$accumulation point. $\square$

2. Show that, if a topological space X is sequentially compact (every sequence in X has a subsequence converging to a point in X), then X is ω-accumulation-point-compact.

3. Show that, if a topological space X is ω-accumulation-point-compact and first countable, then X is sequentially compact. **Hint.** For an arbitrary sequence, show that, if no single value appears with infinitely many indices, then the sequence must have infinitely many distinct values. Use first countable to show that an ω-accumulation point is a limit of a subsequence.

Definition 0.1. A topological space X is called *second countable* if X's topology has a countable basis.

4. a. Show that, if a topological space X is second countable, then every open cover of X has a countable subcover.

b. Show that every second countable, ω-accumulation-point-compact space X is compact. **Hint.** For a countable open cover $\{U_j\}$ with no finite subcover, construct an infinite subset Y of X with the property that each U_j contains at most finitely many elements of Y.

Definition 0.2. In a metric space (X, d), the *diameter* of a nonempty bounded subset Y is the supremum (least upper bound) of $\{d(y_1, y_2) : y_1, y_2 \in Y\}$.

Definition 0.3. Let $\{U_\alpha\}_{\alpha \in A}$ be an open cover of a metric space X. A *Lebesgue number* for this cover is a number δ satisfying: for every subset Y of X with diameter less than δ, there is a U_α for which $Y \subset U_\alpha$.

Problem 5 is an **extra credit** problem, again with no penalty for students who do not do problem 5, as a way of making sure the assignment is of reasonable length.

5. The results in this problem apply to any metrizable topological space, that is to any space whose topology comes from a metric, regardless of which metric is used, as long as the topology the metric generates is the space's given topology. So that no one will worry about the implications of a choice of metric, I'll phrase the problem in terms of a metric space, i.e. I'll choose at the beginning a specific metric that defines the topology.

a. Show that, if X is an ω-accumulation-point-compact metric space, then every open cover of X has a Lebesgue number. **Hint.** Consider the implications of having a sequence of subsets (Y_n), satisfying, for each n, the diameter of Y_n is less than $1/n$ and, for each n, no open set from the cover contains Y_n.

b. Show that, if X is an ω-accumulation-point-compact metric space, then, for each $\epsilon > 0$, there is a finite collection of open balls of radius ϵ that covers X. **Hint.** If there is some $\epsilon > 0$ for which no finite collection of open balls of radius ϵ that covers X, exhibit an infinite set with no ω-accumulation point.

c. Show that, if X is an ω-accumulation-point-compact metric space, then X is compact. **Hint.** Use parts a. and b.