Hand in Friday, March 15.

1. Suppose that X and Y are topological spaces, that $A \subset X$, and that f and g are continuous maps from X to Y that satisfy, for all $x \in A$, $f(x) = g(x)$. If Y is Hausdorff, show that, for all z in the closure of A, $f(z) = g(z)$.

Proof. Assume X and Y are topological spaces, that $A \subset X$, and that f and g are continuous maps from X to Y that satisfy, for all $x \in A$, $f(x) = g(x)$. Assuming Y is Hausdorff and that there there exists some point z in the closure of A where $f(z) \neq g(z)$. Then as Y is Hausdorff we have two neighborhoods U_f of $f(z)$ and U_g of $g(z)$ where $U_f \cap U_g = \emptyset$. As f, g are continuous we have that $f^{-1}(U_f)$ and $g^{-1}(U_g)$ are both open. We have that $z \in f^{-1}(U_f) \cap g^{-1}(U_g)$ but we have that z is a limit point as if $z \in A$ then that would be an immediate contraction on $f(z) \neq g(z)$ hence we have $A \cap f^{-1}(U_f) \cap g^{-1}(U_g) \setminus \{z\} \neq \emptyset$ hence we have for some $a \in A \cap f^{-1}(U_f) \cap g^{-1}(U_g) \setminus \{z\}$ then we have a neighborhood U_a of a with $U_a \subset f^{-1}(U_f) \cap g^{-1}(U_g)$ hence we have $f(a) \in U_f \cap U_g$ which contradicts U_f and U_g being disjoint. □

2. Let $f : X_1 \to Y_1$ and $g : X_2 \to Y_2$ be continuous maps between topological spaces. Give the products $X_1 \times X_2$ and $Y_1 \times Y_2$ their product topologies. Show that the map $H : X_1 \times X_2 \to Y_1 \times Y_2$ defined by $H((x_1, x_2)) = (f(x_1), g(x_2))$ is continuous.

Proof. Assume that $f : X_1 \to Y_1$ and $g : X_2 \to Y_2$ are continuous maps between topological spaces. Assume we have $X_1 \times X_2$ and $Y_1 \times Y_2$ with their product topologies with the map $H : X_1 \times X_2 \to Y_1 \times Y_2$ where $H((x_1, x_2)) = (f(x_1), g(x_2))$. Then consider an arbitrary basis element $U_1 \times U_2 \subset Y_1 \times Y_2$ then we have by the definition of product topology that U_1 is open in Y_1 and U_2 is open in Y_2 and as f, g are continuous we have $f^{-1}(U_1)$ and $g^{-1}(U_2)$ are open hence $f^{-1}(U_1) \times g^{-1}(U_2)$ is open in $X_1 \times X_2$ this implies $H^{-1}(U_1 \times U_2) = f^{-1}(U_1) \times g^{-1}(U_2)$ is open which shows that H is continuous.

□

3. An injective (one-to-one) continuous map $f : X \to Y$ between topological spaces is a bijection from X to the image $f(X)$. Give $f(X)$ the subspace topology induced by Y's topology. We call such an f an imbedding if it is a homeomorphism from X to $f(X)$.

In this context, let Y be $X \times X$ with the product topology. Let x_0 be an arbitrary element of X.

a. Show that $f : X \to X \times X$ defined by $f(x) = (x, x_0)$ is an imbedding.

Proof. Assume that $f : X \to X \times X$ is a map defined by $f(x) = (x, x_0)$ where x_0 is an arbitrary element of X. Suppose $x, y \in X$ with $f(x) = f(y)$ then we have $(x, x_0) = (y, x_0)$ hence $x = y$ which shows that f is injective. Let $U_1 \times U_2 \subset f(X)$ be an arbitrary basis element. Then as $f(X) = X \times \{x_0\}$ we get that $U_2 = \{x_0\}$ additionally from the definition of subspace topology we get $U_1 = X \cap U_3$ where U_3 is some open set in X hence U_1 is also open. As $f^{-1}(U_1 \times U_2) = U_1$ we have $f^{-1}(U_1 \times U_2)$ is open. Hence f is continuous.

We have $f^{-1} : f(X) \to X$ where $f^{-1}(x, y) = x$ is a bijection as we have shown that f is injective and it is surjective with its own range. So we just need to show that f^{-1} is continuous. Given an arbitrary basis element U of X. We have that $(f^{-1})^{-1} = f$ hence $f(U) = U \times \{x_0\}$ and as $U \times \{x_0\} = f(X) \cap (U \times X)$ and $U \times X$ is open in the product topology we get $U \times \{x_0\}$ is open in the subspace topology hence which shows that f^{-1} is continuous which implies f is a homeomorphism which shows that f is an imbedding.

□

b. Show that $g : X \to X \times X$ defined by $g(x) = (x, x)$ is an imbedding.

Proof. Let $g : X \to X \times X$ be defined by $g(x) = (x, x)$. Let $x, y \in X$ be two elements with $g(x) = g(y)$ then $(x, x) = (y, y)$ which implies $x = y$ hence g is injective. Now let $U_1 \times U_2 \subset g(X)$ be an open set. Then we have $U_1 \times U_2 = \{(x, x) : x \in X\} \cap A \times B$ for some open sets A, B in X which implies $U_1 \times U_2 = \{(x, x) : x \in A \cap B\}$. Then $g^{-1}(U_1 \times U_2) = A \cap B$ and as A, B are open we get $A \cap B$ is open which shows that g is continuous.

Let the map $g^{-1} : g(X) \to X$ be defined by $g^{-1}(x, x) = x$ we have already shown g is injective and as g is surjective with its own range we have that g^{-1} is a bijection. Let $U \subset X$ be an open set then we have $(g^{-1})^{-1} = g$ hence $(g^{-1})^{-1}(U) = g(U) = \{(x, x) : x \in U\}$. As $\{(x, x) : x \in U\} = (U \times U) \cap g(X)$ we get $\{(x, x) : x \in U\}$ is open in the subspace topology hence g^{-1} is continuous which implies that g is an imbedding.

□

4. Suppose that $h : X \to Y$ is a homeomorphism of topological spaces. If Z is any other topological space and if $g : Y \to Z$ is a continuous map, we know that the composition $g \circ h$ is a continuous map from X to Z. Show that every continuous map $f : X \to Z$ arises this way, i.e. for any continuous $f : X \to Z$, there exists a continuous $G : Y \to Z$ for which $f = G \circ h$.

Proof. Assume $h : X \mapsto Y$ is a homeomorphism of topological spaces and Z is some other topological space. Then given an arbitrary continuous function $f : X \to Z$ we need to create a function $G : Y \to Z$ such that $G \circ h = f$. First I will prove the existence and then that it is continuous. Define $G : Y \to Z$ where $G(y) = f(h^{-1}(y))$.

Now to prove that G is continuous let $U \subset Z$ be an open set. Then as $f^{-1}(U)$ is open in X we have that $h(f^{-1}(U))$ is open in Y as homeomorphisms preserve open sets (this is immediate based off both directions of homeomorphisms being continuous) we just need to show $G^{-1}(U) = h(f^{-1}(U))$.

Let $y \in G^{-1}(U)$ as h is a homeomorphism we have for some unique $x \in X$ that $h(x) = y$ and as $G(y) = f(x) \in U$ we get $x \in f^{-1}(U)$ which implies $y = h(x) \in h(f^{-1}(U))$ hence $y \in h(f^{-1}(U))$ which gives $G^{-1}(U) \subset h(f^{-1}(U))$

Now let $y \in h(f^{-1}U)$. Then as h is a homeomorphism we have for some $x \in f^{-1}(U)$ that $y = h(x)$ but then $G(y) = f(h^{-1}(y)) = f(x) \in U$ which implies $y \in G^{-1}(U)$ hence $h(f^{-1}U) \subset G^{-1}(U)$ then $h(f^{-1}U) = G^{-1}(U)$ which implies G is continuous. As for any $x \in X$ we have $G \circ h(x) = f(h^{-1}(h(x))) = f(x)$ we get $f = G \circ h$. □

5. a. Show that a linearly ordered set with the order topology is Hausdorff.

Proof. Suppose X is a linearly ordered set with the order topology. Let $a, b \in X$ with $a \neq b$ and without loss of generality assume $a < b$. I will proceed by cases.

(1) If there is no element $c \in X$ with $a < c < b$ and a is the minimum element of X and b is the maximum element of X consider the open sets $a \in [a, b)$ and $b \in (a, b]$ we have each of the sets is open and $[a, b) \cap (a, b] = \emptyset$.
(2) If there exists an element $c \in X$ with $a < c < b$ and a is the minimum element of X and b is the maximum. Then consider the open sets $U_a \cap [a, c)$ and $U_b \cap (c, b]$ then we have $a \in [a, c)$ and $b \in (c, b]$ but $[a, c) \cap (c, b] = \emptyset$.
(3) If there is no element $c \in X$ with $a < c < b$ and a is not a minimum element of X and b is not a maximum of X then we have the sets (d, b) where $d \in X$ with $d < a$ is open and the set (a, l) where $l \in X$ with $b < l$ is open. We have $a \in (d, b)$ and $b \in (a, l)$ but $(d, b) \cap (a, l) = \emptyset$
(4) If there exists some element $c \in X$ with $a < c < b$ and a is not a minimum of X and b is not a maximum of X then we have the sets (d, c) where $d \in X$ and $d < a$ is open and the set (c, l) where $l \in X$ with $b < l$ is open. We have $a \in (d, c)$ and $b \in (c, l)$ where $l \in X$ with $b < l$ but $(d, c) \cap (c, l) = \emptyset$
(5) If there exists no element $c \in X$ with $a < c < b$ and without loss of generality a is a minimum of X and b is not the maximum (the case where a is not min of X and b is max of X will follow by almost the same exact reasoning) then consider the open sets $a \in [a, b)$ and $b \in (a, d)$ where $d \in X$ with $b < d$ then we have $[a, b) \cap (a, d) = \emptyset$.
(6) If there exists some element $c \in X$ with $a < c < b$ and without loss of generality a is a minimum of X and b is not the maximum. Then $a \in [a, c)$ and $b \in (c, d)$ where $d \in X$ with $b < d$ then $[a, c) \cap (c, d) = \emptyset$

This completes all the cases hence we get X is a Hausdorff space. □

b. Suppose that X is a topological space. Show that X is Hausdorff if and only if the diagonal subset $\{(x,x) : x \in X\}$ of the product $X \times X$ is a closed subset of the product. Assume here that the topology on $X \times X$ is the product topology.

Proof.
($\Rightarrow$)

Suppose X is a topological space and X is Hausdorff. Now assume that $X \times X$ has the product topology. Assume that $(a,b) \in X \times X$ with $a \neq b$ is a limit point of $\{(x,x) : x \in X\}$. Then as $a \neq b$ and X is Hausdorff we have two neighborhoods U_a, U_b of a, b respectively with $U_a \cap U_b = \emptyset$ then we have $(a,b) \in U_a \times U_b$ but as $U_a \cap U_b = \emptyset$ we get $(U_a \times U_b) \cap \{(x,x) : x \in X\} = \emptyset$ hence (a,b) is not a limit point of $\{(x,x) : x \in X\}$.

This implies either there are no limit points of $\{(x,x) : x \in X\}$ or $\{(x,x) : x \in X\}' \subset \{(x,x) : x \in X\}$ in either case we get $\overline{\{(x,x) : x \in X\}} = \{(x,x) : x \in X\} \cup \{(x,x) : x \in X\}' = \{(x,x) : x \in X\}$ hence $\{(x,x) : x \in X\}$ contains it's limit points so its closed.

($\Leftarrow$)

Assume that X is a topological space and that $X \times X$ has the product topology and $\{(x,x) : x \in X\}$ is closed in $X \times X$. Then for any $a, b \in X$ with $a \neq b$ we have $(a,b) \in \{(x,x) : x \in X\}^c$ as there exists a basis element of the form $(a,b) \in U_a \times U_b \subset \{(x,x) : x \in X\}^c$ but as we have $U_a \times U_b \cap \{(x,x) : x \in X\} = \emptyset$ we get that there exists no $x \in X$ such that $(x,x) \in U_a \times U_b$ which implies $U_a \cap U_b = \emptyset$ this implies that X is Hausdorff.

□

6. Let Y be an ordered set in the order topology. Let $f, g : X \to Y$ be continuous.

(1) Show that the set $\{x : f(x) \leq g(x)\}$ is closed in X.

Proof. Assume that X is a topological space and Y is an ordered set in the order topology and $f, g : X \to Y$ are continuous. We just need to show that $\{x : f(x) \leq g(x)\}^c = \{x : f(x) > g(x)\}$ is open. Let $x \in \{x : f(x) > g(x)\}$ then $f(x) \neq g(x)$. I will proceed by cases.

(a) If there exists no $y \in Y$ such that $f(x) > y > g(x)$ and $f(x) = \max(Y)$ and $g(x) = \min(Y)$ we have the neighborhoods $f(x) \in (g(x), f(x)]$ and $g(x) \in [g(x), g(x))$ as f, g are continuous we have $f^{-1}((g(x), f(x)])$ and $g^{-1}([g(x), f(x)))$ are open in X with for all $x_f \in f^{-1}((g(x), f(x)])$ for all $x_g \in g^{-1}([g(x), f(x)))$ we have $f(x_f) > g(x_g)$ hence we get
$$f^{-1}((g(x), f(x)]) \cup g^{-1}([g(x), f(x))) \subset \{x : f(x) > g(x)\}$$

(b) If there exists some $y \in Y$ such that $f(x) > y > g(x)$ and $f(x) = \max(Y)$ and $g(x) = \min(Y)$ then we have the neighborhoods $f(x) \in (y, f(x)]$ and $g(x) \in [g(x), f(x))$ and $f^{-1}((y, f(x)])$ is open and $g^{-1}([g(x), f(x)))$ is open. Using the same reasoning as above we have
$$f^{-1}((y, f(x)]) \cup g^{-1}([g(x), y)) \subset \{x : f(x) > g(x)\}$$

(c) If there exists no $y \in Y$ with $f(x) > y > g(x)$ and $f(x) \neq \max(Y)$ and $g(x) \neq \min(Y)$ then we have the neighborhoods $f(x) \in (g(x), a)$ where $a \in Y$ with $a > f(x)$ and $g(x) \in (b, f(x))$ where $b \in Y$ with $b < g(x)$ then using same reasoning
$$f^{-1}((g(x), a)) \cup g^{-1}((b, f(x))) \subset \{x : f(x) > g(x)\}$$

(d) If there exists some $y \in Y$ where $f(x) > y > g(x)$ and $f(x) \neq \max(Y)$ and $g(x) \neq \min(Y)$ then we have the the neighborhoods $f(x) \in (y, b)$ where $b \in Y$ with $b > f(x)$ and $g(x) \in (c, y)$ where $x \in Y$ with $c < g(x)$. Then
$$f^{-1}((y, b)) \cup g^{-1}((c, y)) \subset \{x : f(x) > g(x)\}$$

(e) If there exists no $y \in Y$ where $f(x) > y > g(x)$ and without loss of generality $f(x) = \max(Y)$ and $g(x) \neq \min(Y)$ then we have the neighborhoods $f(x) \in (g(x), f(x)]$ and $g(x) \in (c, f(x))$ where $c \in Y$ with $c < g(x)$. Then
$$f^{-1}((g(x), f(x)]) \cup g^{-1}((c, f(x))) \subset \{x : f(x) > g(x)\}$$

(f) If there is some $y \in Y$ where $f(x) > y > g(x)$ and without loss of generality $f(x) = \max(Y)$ and $g(x) \neq \min(Y)$ then we have the neighborhoods $f(x) \in (y, f(x)]$ and $g(x) \in (b, y)$ where $b \in Y$ with $b < g(x)$ then

$$f^{-1}((y, f(x)]) \cup g^{-1}((b, y)) \subset \{x : f(x) > g(x)\}$$

Hence for all $x \in \{x : f(x) > g(x)\}$ we have the existence of two open sets U, V in Y with $x \in f^{-1}(U) \cup g^{-1}(V) \subset \{x : f(x) > g(x)\}$ then we get

$$\bigcup_{x \in \{x : f(x) > g(x)\}} f^{-1}(U_x) \cup g^{-1}(V_x) = \{x : f(x) > g(x)\}$$

This implies that $\{x : f(x) > g(x)\}$ is open which implies that $\{x : f(x) \leq g(x)\}$ is closed. □

(2) Let $h : X \to Y$ be the function

$$h(x) = \min\{f(x), g(x)\}$$

. Show that h is continuous.

Proof. Assume that X is a topology and Y is an ordered set with the order topology. Let $g, f : X \to Y$ be continuous maps. Let $h : X \to Y$ be the function where $h(x) = \min\{f(x), g(x)\}$. We have $X = \{x : f(x) \leq g(x)\} \cup \{x : g(x) \leq f(x)\}$ both $\{x : g(x) \leq f(x)\}, \{x : f(x) \leq g(x)\}$ are closed by part (a). As f, g are continuous then $f' : \{x : f(x) \leq g(x)\} \to Y$ where $f'(x) = f(x)$ is continuous and $g' : \{x : g(x) \leq f(x)\} \to Y$ where $g'(x) = g(x)$ is continuous.

We also have $\{x : f(x) \leq g(x)\} \cap \{x : g(x) \leq f(x)\} = \{x : f(x) = g(x)\}$ hence $f'(x) = g'(x)$ for all $x \in \{x : f(x) \leq g(x)\} \cap \{x : g(x) \leq f(x)\} = \{x : f(x) = g(x)\}$ then by the Munkres Theorem 18.3 we have the continuous function $h' : X \to Y$ where $h'(x) = g'(x)$ if $x \in \{x : g(x) \leq f(x)\}$ and $h'(x) = f'(x)$ if $x \in \{x : f(x) \leq g(x)\}$.

To show $h = h'$ let $x \in X$ then $h(x) = \min\{f(x), g(x)\}$ and $h'(x) = g(x)$ if $g(x) \leq f(x)$ or $h'(x) = f(x)$ if $f(x) \leq g(x)$ which shows $h'(x) = \min\{g(x), f(x)\}$. □