Problem 1. Compute the fractional linear transformation determined by the correspondence:

$$(0, 1, \infty) \mapsto (1, 1 + i, 2)$$

$$f(z) = \frac{z(2 - 2i) - 2}{z(1 - i) - 2}$$

Then $f(0) = 1$, $f(1) = \frac{2i}{1+i} = \frac{2i(1-i)}{|1+i|} = i(1 - i) = i + 1$, and $f(\infty) = 2$.

Problem 2. Show that the differential

$$\frac{-ydx + xdy}{x^2 + y^2}, \quad (x, y) \neq (0, 0)$$

is closed. Show that it is not independent of path on any annulus centered at 0.

First to show that the differential is closed.
Computing $\frac{\partial P}{\partial y}$ we get

$$\frac{\partial \left(\frac{-y}{x^2+y^2}\right)}{\partial y} = \frac{2y^2}{(x^2 + y^2)} + \frac{-1}{x^2 + y^2} = \frac{-x^2 + y^2}{(x^2 + y^2)^2}$$

Computing $\frac{\partial Q}{\partial x}$

$$\frac{\partial \left(\frac{x}{x^2+y^2}\right)}{\partial x} = \frac{-2x^2}{(x^2 + y^2)^2} + \frac{1}{x^2 + y^2} = \frac{-x^2 + y^2}{(x^2 + y^2)^2}$$

As $\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$ we have that the differential is closed.
Now to show that it is not independent of path on any annulus centered at 0.
Let $r > 0$

$$\oint_{|z|=r} \frac{-ydx + xdy}{x^2 + y^2}$$

Using the parametrization $\gamma(t) = (r \cos t, r \sin t)$ for $t \in [0, 2\pi)$

$$\begin{aligned}
\oint_{|z|=r} \frac{-ydx + xdy}{x^2 + y^2} &= \int_0^{2\pi} \frac{-r \sin t \cdot -r \sin t + r \cos t \cdot rc \cos t}{r^2 \cos^2 t + r^2 \sin^2 t} dt \\
&= \int_0^{2\pi} dt \\
&= 2\pi
\end{aligned}$$

Therefore as this is a closed curve it is not independent of path.

Problem 3. Show that a complex valued function $h(z)$ on a simply connected domain is harmonic if and only if $h(z) = f(z) + \overline{g(z)}$, where $f(z), g(z)$ are analytic on D.

Proof. Assume that $h(z)$ is harmonic on a simply connected domain D. Then we have $h(z) = u(z) + iv(z)$ where $u(z), v(z)$ are both harmonic on D as well. Then for $u(z)$ as this is a simply connected domain there exists a harmonic conjugate $\mu(z)$. Likewise for $v(z)$ there exists a harmonic conjugate $\phi(z)$. From this we get the two analytic equations $a(z) = u(z) + i\mu(z)$ and $b(z) = v(z) + i\phi(z)$. Solving for $u(z)$ and $v(z)$ we get $u(z) = \frac{a(z)+\overline{a(z)}}{2}$ and $v(z) = \frac{b(z)+\overline{b(z)}}{2}$. Then we have

$$\begin{aligned} h(z) &= u(z) + v(z) \\ &= \frac{a(z) + \overline{a(z)}}{2} + i\frac{b(z) + \overline{b(z)}}{2} \\ &= \frac{a(z) + ib(z)}{2} + \frac{\overline{a(z) - ib(z)}}{2} \end{aligned}$$

Letting $f(z) = \frac{a(z)+ib(z)}{2}$ and $\overline{g(z)} = \frac{\overline{a(z)-ib(z)}}{2}$. Both f, g are analytic as a, b are and the sum of two differentiable functions is differentiable with their derivatives still being continuos as well. This completes the forward direction.

For the backwards direction assume that $h(z)$ is a complex valued function on the simply connected domain D and $f(z), g(z)$ are analytic on D. With $h(z) = f(z) + \overline{g(z)}$

Then we have $h = u + iv$ with $u = \text{Re}f + \text{Re}g$ and $v = \text{Im}f - \text{Im}g$. Then as

$$\frac{\partial^2 \text{Re}f + \text{Re}g}{\partial x^2} = \frac{\partial}{\partial x}\frac{\partial \text{Im}f + \text{Im}g}{\partial y} = \frac{\partial}{\partial y}\frac{\partial \text{Im}f + \text{Im}g}{\partial x} = \frac{\partial^2 \text{Re}f + \text{Re}g}{\partial y^2}$$

Which shows that u is harmonic.

Similarly with v.

$$\frac{\partial^2 \text{Im}f - \text{Im}g}{\partial x^2} = \frac{\partial}{\partial x}\frac{\partial \text{Re}f - \text{Re}g}{\partial y} = \frac{\partial}{\partial y}\frac{\partial \text{Re}f - \text{Re}g}{\partial x} = \frac{\partial \text{Im}f - \text{Im}g}{\partial y^2}$$

Hence v is harmonic as well which implies that $h = u + iv$ is harmonic.

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