

**Problem 1.** Recall the following useful technique for computing the determinant of a matrix.

**Theorem. (Cofactor Expansion, Laplace).** Let  $A$  be an  $n \times n$  matrix and let  $M_{i,j}$  denote the  $(n-1) \times (n-1)$  submatrix obtained by deleting Row  $i$  and Column  $j$  from  $A$ . The determinant of an  $n \times n$  matrix  $A$  can be computed *along the  $i^{\text{th}}$  row* as the sum

$$\det A = \sum_{\text{col. } j} (-1)^{i+j} A_{i,j} \det(M_{i,j})$$

or *along the  $j^{\text{th}}$  column* as the sum

$$\det A = \sum_{\text{row. } i} (-1)^{i+j} A_{i,j} \det(M_{i,j})$$

Let  $T \in \mathcal{L}(\mathbb{C}^4)$  be an operator with matrix (in the standard basis) given by

$$A = \begin{pmatrix} 1 & -1 & 1 & -2 \\ 0 & 0 & 0 & -1 \\ -1 & 1 & -1 & 2 \\ 0 & 1 & 0 & 2 \end{pmatrix}.$$

1. Find the characteristic polynomial for  $A$ .
2. Find a eigenvalues for  $A$ .
3. Find basis  $\mathcal{G}$  for  $\mathbb{C}^4$  so that  $\mathcal{M}(T, \mathcal{G})$  is upper triangular with eigenvalues along the diagonal.
4. Is  $A$  diagonalizable? Why or why not?

1. Expanding along row 2 we get

$$\det(A - \lambda \text{Id}) = -\lambda \det \begin{pmatrix} 1-\lambda & 1 & -2 \\ -1 & -1-\lambda & 2 \\ 0 & 0 & 2-\lambda \end{pmatrix} - 1 \det \begin{pmatrix} 1-\lambda & -1 & 1 \\ -1 & 1 & -1-\lambda \\ 0 & 1 & 0 \end{pmatrix}$$

Choosing row 3 for each of these determinants we get.

$$= -\lambda(2-\lambda)((1-\lambda)(-1-\lambda)+1)) + ((1-\lambda)(-1-\lambda)+1)$$

$$= ((1-\lambda)(-1-\lambda)+1)(-\lambda(2-\lambda)+1)$$

The left is a difference of squares hence we get

$$= \lambda^2(\lambda^2 - 2\lambda + 1) = \lambda^2(\lambda - 1)^2$$

2. The eigenvalues are the roots of the characteristic polynomial. Hence  $\lambda = 0, 1$ .

3. First to find the eigenvectors for  $\lambda = 0$

Solving for  $A(a, b, c, d) = (0, 0, 0, 0)$  we get the system of equations

$$\begin{cases} a - b + c - 2d = 0 \\ -d = 0 \\ -a + b - c + 2d = 0 \\ b + 2d = 0 \end{cases} \quad (1)$$

This immediately shows that  $d = 0$  from which it follows that  $b = 0$ . This implies that  $a = c$  hence  $\text{Span}((1, 0, -1, 0)^t) = E(0, A)$ .

Now for  $\lambda = 1$  we get the system of equations

$$\begin{cases} -b + c - 2 = 0 \\ -b - d = 0 \\ -a + b - 2c + 2d = 0 \\ b + d = 0 \end{cases} \quad (2)$$

This implies that  $b = -d$  and  $c = 2 + b$  subbing these into the 3rd equation we get  $-a + b - 2(2 + b) - 2b = 0$  which implies  $a = -3b - 4$ . Hence  $\text{Span}((-1, -1, 1, 1)^t) = E(1, A)$ .

Now finding a basis to span  $\text{Null } A^2$ . We have  $A^2(a, b, c, d) = (0, 0, 0, 0)$  gives the system of equations

$$\begin{cases} 2b + 9d = 0 \\ -1b - 2d = 0 \\ 2b - d = 0 \\ 2b + 3d = 0 \end{cases} \quad (3)$$

Hence it is spanned by  $\{(1, 0, 0, 0)^t, (0, 0, 1, 0)^t\}$

Doing the same for  $\text{Null}(A - \text{Id})^2$  we have that  $(A - \text{Id})^2(a, b, c, d) = (0, 0, 0, 0)$  gives the system of equations

$$\begin{cases} -a - 2c + d = 0 \\ 2a + 3c - d = 0 \end{cases} \quad (4)$$

Which gives  $a = -c$  and  $a = -d$  hence we have that  $\{(1, 0, -1, -1)^t, (0, 1, 0, 0)^t\}$  spans  $\text{Null}(A - \text{Id})^2$ . Then we choose the basis

$$\mathcal{B} = \{b_1 = ((1, 0, -1, 0)^t, b_2 = (1, 0, 0, 0)^t, b_3 = (-1, -1, 1, 1)^t, b_4 = (0, 1, 0, 0)^t)\}$$

solving  $A(1, 0, 0, 0)^t = (1, 0, -1, 0)^t = 1 \cdot b_1$  doing the same for  $A(0, 1, 0, 0)^t = (-1, 0, 1, 1)^t = b_4 + b_3$  hence we get the matrix

$$\mathcal{M}(A, \mathcal{B}) = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

4. This is not diagonalizable as the geometric multiplicity of both the eigenvalues values is 1 while the algebraic is 2.

**Problem 2.** Let  $T \in \mathcal{L}(\mathbb{R}^4)$  be an operator whose matrix (in the standard basis) is given by

$$\begin{pmatrix} -2 & 1 & 0 & 3 \\ -2 & 0 & 1 & 1 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 1 & 1 \end{pmatrix}$$

Find a basis  $\mathcal{B}$  for  $\mathbb{R}^4$  so that  $\mathcal{M}(T, \mathcal{B})$  is block-diagonal. That is,

$$\mathcal{M}(T, \mathcal{B}) = \begin{pmatrix} * & * & 0 & 0 \\ * & * & 0 & 0 \\ 0 & 0 & * & * \\ 0 & 0 & * & * \end{pmatrix}$$

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HINT: Given  $S \in \mathcal{L}(\mathbb{C}^2)$  with

$$\mathcal{M}(S, \{\mathbf{b}_1, \mathbf{b}_2\}) = \begin{pmatrix} ke^{i\theta} & 0 \\ 0 & ke^{-i\theta} \end{pmatrix}$$

then

$$\mathcal{M}(S, \{\operatorname{Re}(\mathbf{b}_1), \operatorname{Im}(\mathbf{b}_1)\}) = \begin{pmatrix} k \cos \theta & -k \sin \theta \\ k \sin \theta & k \cos \theta \end{pmatrix}.$$

First to find the eigenvalues.

$$\det \begin{pmatrix} -2 - \lambda & 1 & 0 & 3 \\ -2 & -\lambda & 1 & 1 \\ 0 & 0 & 1 - \lambda & -1 \\ 0 & 0 & 1 & 1 - \lambda \end{pmatrix}$$

Using cofactor expansion on row 4. We get

$$\begin{aligned} &= \det \begin{pmatrix} -2 - \lambda & 1 & 3 \\ -2 & -\lambda & 1 \\ 0 & 0 & -1 \end{pmatrix} - (-1 - \lambda) \det \begin{pmatrix} -2 - \lambda & 1 & 0 \\ -2 & -\lambda & 1 \\ 0 & 0 & 1 - \lambda \end{pmatrix} \\ &= -((-2 - \lambda)(-\lambda) + 2) + (1 - \lambda)(1 - \lambda)((-2 - \lambda) - \lambda + 2) \\ &= ((2 + \lambda)\lambda + 2) + (1 - \lambda)(1 - \lambda)((2 + \lambda)\lambda + 2) \\ &= ((2 + \lambda)\lambda + 2)((1 - \lambda)^2 + 1) \\ &= (\lambda^2 + 2\lambda + 2)(\lambda^2 - 2\lambda + 2) \end{aligned}$$

So  $\lambda = 1 \pm i, -1 \pm i$ . The eigenvectors are  $(1, i, i, 1), (1, -i, -i, 1), (1 - i, 2, 0, 0), (1 + i, 2, 0, 0)$ . Then taking the basis to be

$$\mathcal{B} = \{b_1 = (1, 0, 0, 1), b_2 = (0, 1, 1, 0), b_3 = (1, 2, 0, 0), b_4 = (1, 0, 0, 0)\}$$

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