Problem 1. Recall the following useful technique for computing the determinant of a matrix.

> **Theorem. (Cofactor Expansion, Laplace).** Let A be an $n \times n$ matrix and let $M_{i,j}$ denote the $(n-1) \times (n-1)$ submatrix obtained by deleting Row i and Column j from A. The determinant of an $n \times n$ matrix A can be computed *along the i^{th} row* as the sum
>
> $$\det A = \sum_{\text{col. } j} (-1)^{i+j} A_{i,j} \det(M_{i,j})$$
>
> or *along the j^{th} column* as the sum
>
> $$\det A = \sum_{\text{row. } i} (-1)^{i+j} A_{i,j} \det(M_{i,j})$$

Let $T \in \mathcal{L}(\mathbb{C}^4)$ be an operator with matrix (in the standard basis) given by

$$A = \begin{pmatrix} 1 & -1 & 1 & -2 \\ 0 & 0 & 0 & -1 \\ -1 & 1 & -1 & 2 \\ 0 & 1 & 0 & 2 \end{pmatrix}.$$

1. Find the characteristic polynomial for A.
2. Find a eigenvalues for A.
3. Find basis $\mathcal{G}$ for $\mathbb{C}^4$ so that $\mathcal{M}(T, \mathcal{G})$ is upper triangular with eigenvalues along the diagonal.
4. Is A diagonalizable? Why or why not?

1. Expanding along row 2 we get

$$\det(A - \lambda \mathrm{Id}) = -\lambda \det \begin{pmatrix} 1-\lambda & 1 & -2 \\ -1 & -1-\lambda & 2 \\ 0 & 0 & 2-\lambda \end{pmatrix} - 1 \det \begin{pmatrix} 1-\lambda & -1 & 1 \\ -1 & 1 & -1-\lambda \\ 0 & 1 & 0 \end{pmatrix}$$

Choosing row 3 for each of these determinants we get.

$$= -\lambda(2-\lambda)((1-\lambda)(-1-\lambda)+1)) + ((1-\lambda)(-1-\lambda)+1)$$

$$= \big((1-\lambda)(-1-\lambda)+1\big)\big(-\lambda(2-\lambda)+1\big)$$

The left is a difference of squares hence we get

$$= \lambda^2(\lambda^2 - 2\lambda + 1) = \lambda^2(\lambda-1)^2$$

2. The eigenvalues are the roots of the characteristic polynomial. Hence $\lambda = 0, 1$.
3. First to find the eigenvectors for $\lambda = 0$

Solving for $A(a, b, c, d) = (0, 0, 0, 0)$ we get the system of equations

$$\begin{cases} a - b + c - 2d = 0 \\ -d = 0 \\ -a + b - c + 2d = 0 \\ b + 2d = 0 \end{cases} \tag{1}$$

This immediately shows that $d = 0$ from which it follows that $b = 0$. This implies that $a = c$ hence $\operatorname{Span}((1, 0, -1, 0)^t) = E(0, A)$.

Now for $\lambda = 1$ we get the system of equations

$$\begin{cases} -b + c - 2 = 0 \\ -b - d = 0 \\ -a + b - 2c + 2d = 0 \\ b + d = 0 \end{cases} \tag{2}$$

This implies that $b = -d$ and $c = 2 + b$ subbing these into the 3rd equation we get $-a + b - 2(2 + b) - 2b = 0$ which implies $a = -3b - 4$. Hence $\operatorname{Span}((-1, -1, 1, 1)^t) = E(1, A)$.

Now finding a basis to span $\operatorname{Null} A^2$. We have $A^2(a, b, c, d) = (0, 0, 0, 0)$ gives the system of equations

$$\begin{cases} 2b + 9d = 0 \\ -1b - 2d = 0 \\ 2b - d = 0 \\ 2b + 3d = 0 \end{cases} \tag{3}$$

Hence it is spanned by $\{(1, 0, 0, 0)^t, (0, 0, 1, 0)^t\}$

Doing the same for $\operatorname{Null}(A - \operatorname{Id})^2$ we have that $(A - \operatorname{Id})^2(a, b, c, d) = (0, 0, 0, 0)$ gives the system of equations

$$\begin{cases} -a - 2c + d = 0 \\ 2a + 3c - d = 0 \end{cases} \tag{4}$$

Which gives $a = -c$ and $a = -d$ hence we have that $\{(1, 0, -1, -1)^t, (0, 1, 0, 0)^t\}$ spans $\operatorname{Null}(A - \operatorname{Id})^2$. Then we choose the basis

$$\mathcal{B} = \{b_1 = ((1, 0, -1, 0)^t, b_2 = (1, 0, 0, 0)^t, b_3 = (-1, -1, 1, 1)^t, b_4 = (0, 1, 0, 0)^t)\}$$

solving $A(1, 0, 0, 0)^t = (1, 0, -1, 0)^t = 1 \cdot b_1$ doing the same for $A(0, 1, 0, 0)^t = (-1, 0, 1, 1)^t = b_4 + b_3$ hence we get the matrix

$$\mathcal{M}(A, \mathcal{B}) = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

4. This is not diagonalizable as the geometric multiplicity of both the eigenvalues values is 1 while the algebraic is 2.

Problem 2. Let $T \in \mathcal{L}(\mathbb{R}^4)$ be an operator whose matrix (in the standard basis) is given by

$$\begin{pmatrix} -2 & 1 & 0 & 3 \\ -2 & 0 & 1 & 1 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 1 & 1 \end{pmatrix}$$

Find a basis $\mathcal{B}$ for $\mathbb{R}^4$ so that $\mathcal{M}(T, \mathcal{B})$ is block-diagonal. That is,

$$\mathcal{M}(T, \mathcal{B}) = \begin{pmatrix} * & * & 0 & 0 \\ * & * & 0 & 0 \\ 0 & 0 & * & * \\ 0 & 0 & * & * \end{pmatrix}$$

Hint: Given $S \in \mathcal{L}(\mathbb{C}^2)$ with

$$\mathcal{M}(S, \{\mathbf{b_1}, \mathbf{b_2}\}) = \begin{pmatrix} ke^{i\theta} & 0 \\ 0 & ke^{-i\theta} \end{pmatrix}$$

then

$$\mathcal{M}(S, \{\mathrm{Re}(\mathbf{b_1}), \mathrm{Im}(\mathbf{b_1})\}) = \begin{pmatrix} k\cos\theta & -k\sin\theta \\ k\sin\theta & k\cos\theta \end{pmatrix}.$$

First to find the eigenvalues.

$$\det \begin{pmatrix} -2-\lambda & 1 & 0 & 3 \\ -2 & -\lambda & 1 & 1 \\ 0 & 0 & 1-\lambda & -1 \\ 0 & 0 & 1 & 1-\lambda \end{pmatrix}$$

Using cofactor expansion on row 4. We get

$$= \det \begin{pmatrix} -2-\lambda & 1 & 3 \\ -2 & -\lambda & 1 \\ 0 & 0 & -1 \end{pmatrix} - (-1-\lambda) \det \begin{pmatrix} -2-\lambda & 1 & 0 \\ -2 & -\lambda & 1 \\ 0 & 0 & 1-\lambda \end{pmatrix}$$

$$= -((-2-\lambda)(-\lambda)+2) + (1-\lambda)(1-\lambda)((-2-\lambda)-\lambda+2)$$

$$= ((2+\lambda)\lambda+2) + (1-\lambda)(1-\lambda)((2+\lambda)\lambda+2)$$
$$= ((2+\lambda)\lambda+2)((1-\lambda)^2+1)$$
$$= (\lambda^2+2\lambda+2)(\lambda^2-2\lambda+2)$$

So $\lambda = 1 \pm i, -1 \pm i$. The eigenvectors are $(1, i, i, 1), (1, -i, -i, 1), (1-i, 2, 0, 0), (1+i, 2, 0, 0)$. Then taking the basis to be

$$\mathcal{B} = \{b_1 = (1, 0, 0, 1), b_2(0, 1, 1, 0), b_3 = (1, 2, 0, 0), b_4 = (1, 0, 0, 0)\}$$

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We have

$$\begin{aligned} T(1,0,0,1) &= (1,-1,-1,1) = b_1 - b_2, \\ T(0,1,1,0) &= (1,1,1,1) = b_1 + b_2 \\ T(1,2,0,0) &= (0,-2,0,0) = -b_3 + b_4 \\ T(1,0,0,0) &= (-2,-2,0,0) = -b_3 - b_4 \end{aligned}$$

From that we get the matrix

$$\mathcal{M}(T, \mathcal{B}) = \begin{pmatrix} 1 & -1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & -1 & -1 \end{pmatrix}$$