1.

Proof. Suppose X is a compact topological space and $f : X \to \mathbb{R}$ is as described. As X is compact we have the open cover $\{U_x\}_{x\in X}$ where for each $U_x \in \{U_x\}_{x\in X}$ we have that there exists a constant M_x such that for all $z \in U_x$ that $|f(z)| \leq M_x$.

Then as X is compact we have that the collection has a finite subcover $\{U_{x_1}, ..., U_{x_n}\}$. Each of these in the finite subcover has a corresponding constant $\{M_{x_1}, ..., M_{x_n}\}$ with the property that for all $z \in U_{x_i}$ that $|f(z)| \leq M_{x_i}$ where $i = 1, ..., n$. Let $M = \max\{M_{x_1}, ..., M_{x_n}\}$ then for any $z \in X$ we have that $z \in U_{x_i}$ for some $i = 1, ..., n$. Hence $|f(z)| \leq M_{x_i} \leq M$ so this choice of M works for all $z \in X$. $\square$

2.

Proof. Suppose that X is a topological space C is a connected subset of X and C_α is a connected subset of X. With for all αA that $C_\alpha \bigcap C \neq \emptyset$. Suppose that $C \bigcup \big(\cup_{\alpha\in A} C_\alpha \big)$ is not connected. Then there exists a separation $U \cup V$ of $C \bigcup \big(\cup_{\alpha\in A} C_\alpha \big)$.

Then as C is connected we have either $C \subset U$ with $C \cap V = \emptyset$ or $C \subset V$ with $C \cap U = \emptyset$. If C wasn't fully contained in only one then $(C \cap U)$, $(C \cap V)$ would be a separation of C but C is connected. For each $\alpha \in A$ we get that C_α is contained in exactly one U or V.

WLOG suppose $C \cup U$ then as $C \cap C_\alpha \neq \emptyset$ we get that $C_\alpha \subset U$. Hence we get $C \bigcup \big(\cup_{\alpha\in A} C_\alpha \big) \subset U$ and $\big(C \bigcup \big(U_{\alpha\in A} C_\alpha\big)\big) \bigcap V = \emptyset$ hence $V = \emptyset$ so this separation can not exist. So we have $C \bigcup \big(\cup_{\alpha\in A} C_\alpha \big)$ is connected. $\square$