Hand in Friday, February 16.

1. In $\mathbb{R}$ let $A = \{1/k : k \in \mathbb{N}\}$.

a. If $\mathbb{R}$ has the standard topology, what is the boundary of A, and what is the set of limit points of A. Show that your answers are correct.

The boundary points of A is the set $A \cup \{0\}$. For an arbitrary $1/n \in A$ any neighborhood O of $1/n$ we have $1/n \in \mathbb{R} \cap O$. Now as O is open then there exists a basis element B such that $x \in B$ as this is the standard topology we have $B = (a, b)$ for some $a, b \in \mathbb{R}$ with $a < x < b$. Then $(a + x)/2 \in O \cap A^c$ therefore $A \subset \partial A$. For any neighborhood O of 0 there exists a basis element (a, b) with $a, b \in \mathbb{R}$ and $a < 0 < b$ and for some $n \in \mathbb{N}$ we have $1/n < b$ hence $1/n \in O \cap A \neq \emptyset$ additionally we have for some $c \in \mathbb{R}$ where $c \notin A$ with $0 < c < b$ that $c \in O \cap A^c \neq \emptyset$. Now assume $x \in \mathbb{R}$ with $x \notin A \cup \{0\}$. Then if $x < 0$ we immediately have a neighborhood $(x - 1, 0)$ with $(x - 1, 0) \cap A = \emptyset$ like wise if $x > 0$ then we have the neighborhood $(1, x + 1)$ with $(1, x + 1) \cap A = \emptyset$. If $0 < x < 1$ we have for some $n \in \mathbb{N}$ that $1/(n + 1) < x < 1/n$ we have the neighborhood $(1/(n + 1), 1/n)$ of x with $A \cap (1/(n + 1), 1/n) = \emptyset$. This shows that $A \cup \{0\} = \partial A$.

We have that the limit point of A is $\{0\}$. This is shown as let O be a neighborhood of 0 . Then as it is open we have a basis element $(a, b) \subset O$ where $a, b \in \mathbb{R}$ with $a < 0 < b$. We have that there exists $n \in \mathbb{N}$ such that $0 < 1/n < b$ therefore $1/n \in O$ which shows that 0 is a limit point. We have no element of A is a limit point as for all $1/n \in A$ we have a neighborhood (a, b) with $a, b \in \mathbb{R}$ where $1/n \in (a, b)$ but $1/(n + 1), 1/(n - 1) \notin (a, b)$ with the condition $n > 1$. This shows that no element of A is a limit point. Lastly assume $x \in \mathbb{R} \setminus (\{0\} \cup A)$ then if $x < 0$ we have the neighborhood $(x - 1, 0)$ with $(x - 1, 0) \cap A = \emptyset$ and for $x > 0$ we have the neighborhood $(1, x + 1)$ with $(1, x + 1) \cap A = \emptyset$. If $0 < x < 1$ then for some $n \in \mathbb{N}$ we have $(1/(n + 1), 1/n)$ with $x \in (1/(n + 1), 1/n)$ but $(1/(n + 1), 1/n) \cap A = \emptyset$. This implies that the only limit point of A is $\{0\}$.

b. If $\mathbb{R}$ has the upper limit topology, what is the boundary of A, and what is the set of limit points of A. Show that your answers are correct.

The boundary for $\mathbb{R}$ is A.

Let $1/n \in A$ then for any neighborhood O of $1/n$ we have $1/n \in O \cap A$ therefore $O \cap A \neq \emptyset$. Now as O is open we have a basis element with $(a, 1/n] \subset O$ with $a \in \mathbb{R}$ and $a < 1/n$. Then consider the intersection $O \cap A^c$ we have as $(a + \frac{1}{n})/2 \in (a, 1/n) \subset O$ and $(a + \frac{1}{n})/2 \in A^c$ that $O \cap A^c \neq \emptyset$. This implies that $A \subset \partial A$. Now let $x \in \mathbb{R}$ if $x \leq 0$ then we have the neighborhood $(x - 1, 0]$ and $(x - 1, 0] \cap A = \emptyset$. If $x > 1$ then we have the neighborhood $(1, x + 1]$ with $(1, x + 1] \cap A = \emptyset$. If $0 < x < 1$ then we have the neighborhood $(a, x]$ where $a \in \mathbb{R}$ where $(a, x] \cap A = \emptyset$ this a exists as for some $n \in \mathbb{N}$ we have $1/(n + 1) < a < x < 1/n$. This shows $A = \partial A$.

We have that there are no limit points of A.

Let $1/n \in A$ then we have the neighborhood $(1/(n+1), 1/n]$ then as $(1/(n+1), 1/n) \cap A = \{1/n\}$ we have no element of A is a limit point. Now suppose $x \in \mathbb{R}$ if $x \leq 0$ then we have the neighborhood $(x - 1, 0]$ with $(x - 1, 0] \cap A = \emptyset$ if $x > 1$ then $(1, x]$ is a neighborhood of x and $(1, x] \cap A = \emptyset$. If $0 < x < 1$ then we have the neighborhood $(a, x]$ where $a \in \mathbb{R}$ and $(a, x] \cap A = \emptyset$ such an a exists as for some $n \in \mathbb{N}$ the inequality is true $1/(n + 1) < a < x < 1/n$. This implies that there are no limit points of A with the upper limit topology.

2. Let A and B be subsets of a topological space X.

a. If $A \subset B$, show that $\overline{A} \subset \overline{B}$.

Proof. Assume A and B are subsets of a topological space X with $A \subset B$. Then we have $\bar{A} = A \cup A' \subset B \cup A'$. Let $x \in A'$ then for all neighborhoods O of x we have $O \cap A \setminus \{x\} \neq \emptyset$ as $A \subset B$ then $O \cap B \setminus \{x\} \neq \emptyset$ hence $x \in B'$. This implies $A' \subset B'$ so we have $\bar{A} = A \cup A' \subset B \cup B' = \bar{B}$. □

b. Show that $\overline{A \bigcup B} = \overline{A} \bigcup \overline{B}$.

Proof. As $\overline{A \bigcup B} = A \cup B \cup (A \cup B)'$ we just need to show $(A \cup B)' = A' \cup B'$. Let $x \in (A \cup B)'$ then for all neighborhoods O of x we have $O \cap (A \cup B) \setminus \{x\} \neq \emptyset$. Then

$$O \cap (A \cup B) \setminus \{x\} = (O \cap A) \cap \{x\}^c \cup (O \cap B) \cap \{x\}^c = (O \cap A \setminus \{x\}) \cup (O \cap B \setminus \{x\}) \tag{1}$$

this shows that $x \in A' \cup B'$ hence $(A \cup B)' \subset A' \cup B'$. Then for the other inclusion the equation (1) also holds hence $A' \cup B' \subset (A \cup B)'$ which implies $(A \cup B)' = A' \cup B'$. Then we have $\overline{A \bigcup B} = (A \cup B) \cup (A \cup B)' = A \cup B \cup A' \cup B' = (A \cup A') \cup (B \cup B') = \overline{A} \bigcup \overline{B}$ □

3. a. Give an example of a topological space X and subsets A and B for which $\overline{A \bigcap B} \neq \overline{A} \bigcap \overline{B}$. Show that your example has the asserted property.

Let $\mathbb{R}$ be a topological space with the standard topology. Let the two sets be the open interval $(0, 1)$ and $(1, 2)$ then we have $(0, 1)' = [0, 1]$ as for any $x \in (0, 1)$ any neighborhood O of x we have a basis element (a, b) where $a, b \in \mathbb{R}$ with $x \in (a, b) \subset U$ and as $(a, b) \cap (0, 1) \setminus \{x\} \neq \emptyset$ then we have x is a limit point. Now to show that $0 \in (0, 1)'$ we have for any neighborhood O of 0 that there exists a basis element (a, b) with $a, b \in \mathbb{R}$ and $a < 0 < b$ such that $(a, b) \subset U$ then $(a, b) \cap (0, 1) \setminus \{0\} \neq \emptyset$ as $b/2 \in (0, 1)$. To show $1 \in (0, 1)'$ for any neighborhood O of 1 we have a element of the basis (a, b) where $a, b \in \mathbb{R}$ with $a < 1 < b$ and $(a, b) \subset O$. Then we have $(a, b) \cap (0, 1) \neq \emptyset$ as there exists a real number c with $a < c < 1$. This shows that $1 \in (0, 1)'$. For any real number $r \in \mathbb{R} \setminus [0, 1]$ if $r < 0$ we have the neighborhood $(r - 1, 0)$ and $(r - 1, 0) \cap (0, 1) = \emptyset$ which shows that it is not a limit point. If $r > 1$ then we have the neighborhood $(1, r + 1)$ and $(1, r + 1) \cap (0, 1) = \emptyset$. This shows that the limit points of $(0, 1)$ are $[0, 1]$. From the definition of closure we have $\overline{(0, 1)} = \overline{(0, 1) \cup [0, 1]} = [0, 1]$. A similar argument gives that $(1, 2)' = [1, 2]$. Now as $(0, 1) \cap (1, 2) = \emptyset$. We have $\overline{(0, 1) \bigcap (1, 2)} = \emptyset$ and $\overline{(0, 1)} \cap \overline{(1, 2)} = 1$ hence they are not equal. n

b. Give an example of a topological space X and a collection of subsets $\{A_j : j \in \mathbb{N}\}$ for which

$$\bigcup_{j \in \mathbb{N}} \overline{A_j}$$

is not equal to

$$\overline{\bigcup_{j \in \mathbb{N}} A_j}.$$

Show that your example has the asserted property.

Claim that $A_j = \{1/j\}$ has this property in the Real numbers with the standard topology.

Proof. From problem (1a) we have that the limit point of $\bigcup_{j \in \mathbb{N}} A_j$ is 0. By the definition of complement $\overline{\bigcup_{j \in \mathbb{N}} A_j} = (\bigcup_{j \in \mathbb{N}} A_j) \cup (\bigcup_{j \in \mathbb{N}} A_j)' = \{1/n : n \in \mathbb{N}\} \cup \{0\}$. But for each $\overline{A_j} = A_j \cup A_j'$ we have $A' = \emptyset$ which follows from (1a) therefore we have $\overline{A_j} = A_j$ which implies $\bigcup_{j \in \mathbb{N}} \overline{A_j} = \{1/n : n \in \mathbb{N}\}$ but this set does not contain 0 hence $\overline{\bigcup_{j \in \mathbb{N}} A_j} \neq \bigcup_{j \in \mathbb{N}} \overline{A_j}$. □

4. Show that if X is a Hausdorff topological space and Y is a finite subset of X, then Y has no limit points.

Proof. Suppose X is a Hausdorff topological space and $Y = \{y_1, ..., y_n\}$ is a finite subset of X. Then for any $x \in X$ construct the set $\bigcap_{j=1}^{n} X_j$ where $x \in X_j$ but if $y_j \neq x$ then $y_j \notin X_j$ such a set exists based off the assumption X is Hausdorff topological space. Then we have $\bigcap_{j=1}^{n} X_j$ is a neighborhood of x as it contains x and finite sets are open. As $\bigcap_{j=1}^{n} X_j \cap Y \subset \{x\}$ this shows that no element of X is a limit point which completes the proof.

□

5. a. If X is an infinite set with the finite complement topology and if Y is a finite subset of X, what are the limit points of Y? Show that your answer is correct.

Claim that the limit points of Y is the empty set.

Proof. Assume that X is an infinite set with the finite complement topology and Y is a finite subset of X. Now let $x \in X$ then we have that $(X \setminus Y) \cup \{x\}$ is a neighborhood as it it's complement is finite and contains x but as $((X \setminus Y) \cup \{x\}) \cap Y \subset \{x\}$ we have x is not a limit point. As it was chosen arbitrarily we have $Y' = \emptyset$ $\square$

b. If X is an infinite set with the finite complement topology and if Y is an infinite subset of X, what are the limit points of Y? Show that your answer is correct.

Claim it is all of X

Proof. Let X,Y be as definined. Then let $x \in X$ we have that any neighborhood O of x contains an infinite number of elements of Y as if it only contained a finite number of elements of Y then $X \setminus O$ would not be finite because it would at least contain an infinite number of elements of Y. Therefore we have $O \cap Y \neq \emptyset$. Hence x is a limit point and as x was arbitrarily chosen from X we have that $Y' = X$. $\square$

6. a. Let X be the product of countably many copies of $\mathbb{R}$, i.e. $X = \prod_{j \in \mathbb{N}} X_j$, where each X_j is $\mathbb{R}$ with the standard topology. Let Y be the subset of X of elements $\vec{x} = (x_1, x_2, ...)$ for which at most finitely many of the entries x_j are nonzero. If we give X the box topology, what is the closure of Y? Show that your answer is correct.

Claim is that $\bar{Y} = Y$

Proof. Let X, Y be as definition. From the definition of closure if we prove that $Y' \subset Y$ then that proves the claim. Let $\vec{x} \in X \backslash Y$ then consider the neighborhood $O = \prod_{j \in \mathbb{N}} u_j$ where $u_j = \begin{cases} (0, \pi_j(\vec{x}) + 1) & \text{if } \pi_j(\vec{x}) \neq 0 \\ (-1, 1) & \text{if } \pi_j(\vec{x}) = 0 \end{cases}$ we have O contains nonzero entires an infinite number of times based on $\vec{x} \notin Y$. But as each element of Y contains finitely many nonzero terms we have $U \cap Y = \emptyset$. This implies that $Y' \neq X \setminus Y$ which implies that $Y' \subset Y$ so $\bar{Y} = Y \cup Y' = Y$ which completes the proof.

$\square$

b. Let X be the product of countably many copies of $\mathbb{R}$, i.e. $X = \prod_{j \in \mathbb{N}} X_j$, where each X_j is $\mathbb{R}$ with the standard topology. Let Y be the subset of X of elements $\vec{x} = (x_1, x_2, ...)$ for which at most finitely many of the entries x_j are nonzero. If we give X the product topology, what is the closure of Y? Show that your answer is correct.

Claim that $Y' = X$

Proof. Assume that X, Y are defined as above. Then let $\vec{x} \in X$ then for any neighborhood U of x as this is the product topology for some $N \in \mathbb{N}$ we have for all $n \in \mathbb{N}$ with $n > N$ that $U_n = \mathbb{R}$. Then take the element $\vec{y} \in Y$ where $\pi_j(\vec{y}) = \pi_j(\vec{x})$ for $j \in \mathbb{N}$ with $j \leq N$ and zero for the rest. Then we have $\vec{y} \in U \cap Y \neq \emptyset$. This shows that $Y' = X$ hence $\bar{Y} = Y \cup Y' = X$. $\square$