Math 4324 Applications of the winding number

Hand in Monday, April 26.

The **final exam** will be on Friday, May 3, 10:05 am – 12:05 pm, in the classroom. No books, no notes, no calculators. Bring paper and something to write with. The exam will cover important points from throughout the semester. The concepts needed to do homework assignments will be the concepts needed to do final exam problems.

1. Let D be a disk with boundary circle C, and let $f : D \to \mathbb{R}^2$ be a continuous map. Suppose P is a point in $\mathbb{R}^2$ with $P \notin f(C)$, and the winding number of the restriction $f|_C$ of f to C around P is not zero. Show that there is some point Q in D such that $f(Q) = P$.

Proof. Suppose that $P \notin f(C)$ and that there is some $Q \in D$ such that $f(Q) = P$. Create the function $g : D \to \mathbb{R}^2$ defined by the equation $g(Q) = f(Q) - Q$ this is continuous as it is adding two continuous functions. $\square$

2. Fulton textbook, page 52, exercise 4.15. **Hint.** Is f homotopic to the map $g(P) = P^*$, where P^* refers to the point on the circle at the other end of the diameter segment that hits P? When the circle has center the origin, $P^* = -P$.

If $f : C \to C$ is a continuous mapping with no fixed point, show that degree of f must be 1. In particular, if f has no fixed point, show that f must be surjective.

Proof. It suffices to assume that C is centered at the origin as we could create a homeomorphism that shifts it. Now assume that $f : C \to C$ is a continuous map with no fixed points. Then consider the homeomorphism $h : [0,1] \times C \to C$ defined by the equation $h(t, \vec{c}) = (1-t)f(\vec{c}) + t(-\vec{c})$ this is continuous as it is two weighted continuous functions. As no point is fixed we get the line that intersects $f(\vec{c})$ and $g(\vec{c}) = -\vec{c}$ does not intersect $\vec{0}$. Lastly as $h(0, \vec{c}) = f(\vec{c})$ and $h(1, \vec{c}) = -\vec{c}$ we get that h is a homotopy.

On problem 2b on last weeks homework we showed that $\gamma(x, y) = (-x, -y)$ on the unit circle to the unit circle has degree 1. We have that g is homotopic to γ (changing radius of circle) so we get that g has degree 1. As f is homotopic to g we get that f has degree 1. $\square$

3. Fulton textbook, page 53, exercise 4.17. **Hint.** Is f homotopic to the map $g(P) = \widetilde{P}$, where $\widetilde{P}$ refers to the point that is a counterclockwise rotation of $\pi/2$ from P.

If $f : D^2 \to \mathbb{R}^2$ is continuous and $f(P) \cdot P \neq 0$ for all P in S^1, show that there is some $Q \in D^2$ with $f(Q) = 0$.

Proof. Suppose that $f : D^2 \to \mathbb{R}^2$ and $f(P) \cdot P \neq 0$ for all $P \in S^1$ and for all $Q \in D^2$ that $f(Q) \neq \vec{0}$. Then create the homotopy $h : [0,1] \times D^2 \to D^2$ by the equation $h(t, \vec{c}) = f(\vec{c})(1-t) + t \cdot g(\vec{c})$ $\square$

4. Show that if $f : C \to C'$ is a map between circles such that $f(P^\star) = f(P)$ for all P, then the degree of f is even.

Proof. We have C, C' are both homeomorphic to S^1. So let $f : S^1 \to S^1$ such that $f(P^\star) = f(P)$ for all P.

Let $\gamma_1(\theta) = f(\cos\theta, \sin\theta)$ for $0 \leq \theta \leq \pi$ and $\gamma_2(\theta) = f(\cos\theta + \pi, \sin\theta + \pi)$ for $\pi \leq \theta \leq 2\pi$.

As $f(\cos\theta, \sin\theta) = f(\cos\theta + \pi, \sin\theta + \pi)$ we get both γ_1, γ_2 are closed path hence $W(\gamma_1, \vec{0}), W(\gamma_2, \vec{0})$ both have integer winding numbers. As $\gamma_1 = \gamma_2$ we get $W(\gamma_1, \vec{0}) = W(\gamma_2, \vec{0}) = n$ for some $n \in \mathbb{Z}$. Hence we have $W(f, \vec{0}) = W(\gamma_1, \vec{0}) + W(\gamma_2, \vec{0}) = 2n$. Hence the degree of f is even. $\square$

5. Fulton textbook, page 55, exercise 4.27.

If f and g are continuous real-valued functions on a sphere S such that that $f(P^\star) = -f(P)$ and $g(P^\star) = -g(P)$ for all P, show that f and g must have a common zero on the sphere.

Proof. Create the function $h : S \to \mathbb{R}^2$ where $h(\vec{x}) = (f(\vec{x}), g(\vec{x}))$ this is continuous by Munkres Theorem 18.4. Then we have by Borsuk-Ulam theorem that there exists a point such that $h(\vec{x}) = h(\vec{x}^\star)$ this implies $(f(\vec{x}), g(\vec{x})) = (-f(\vec{x}), -g(\vec{x}))$ hence we get $f(\vec{x}) = 0$ and $g(\vec{x}) = 0$. $\square$