

**CS 4124**  
**Solutions to Homework Assignment 1**  
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[25] 1. Textbook Problem 5 in B.1 on Page 520.

Prove Lemma A.4: For all alphabets  $\Sigma$  and all languages  $L \subseteq \Sigma^*$ , the equivalence relation  $\equiv_L$  is right-invariant.

The lemma and the definition of the equivalence relation are on Page 505.

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*Proof.* Let  $\Sigma$  be an alphabet and  $L \subseteq \Sigma^*$  and  $\equiv_L$  be an equivalence relation as defined by (A.3). Let  $x, y, z, w \in \Sigma^*$  and  $x \equiv_L y$  then we have  $xz, yz \in L$  as  $zw \in \Sigma^*$  we have  $xzw, yzw \in L$  which implies  $xz \equiv_L yz$ .  $\square$

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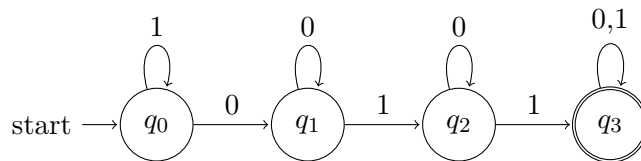
[25] 2. Textbook Problem 3(c) in B.2 on Page 522.

Design an FA  $M$ , with alphabet  $\Sigma = \{0, 1\}$ , that recognizes (c) the set of all strings that contain the string 011, in that order, but not necessarily consecutively.

Be certain you understand the set (language) of  $M$  before you start designing. Constructing some examples of strings both in  $L(M)$  and not in  $L(M)$  can be helpful.

You may use an algebraic specification or a transition diagram (labeled directed graph) specification to present your design for  $M$ . Be certain to explain why your design works.

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This design for a **FA** works as reading through the input string we have it only enters state  $q_1$  if it has read 0 and will loop otherwise. Then for each of the next states  $q_1, q_2$  they will loop if they receive a 0 otherwise they will go to  $q_2$  and  $q_3$  respectively.  $q_3$  is an accepting state and will loop for any input which ensures that the string only need to contain 011 not end in that.

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[25] 3. Consider the OA  $M_4$  in Figure 3.4 on Page 60. Give a complete and careful algebraic specification

$$M_4 = (Q, \Sigma, \delta, q_0, F)$$

for  $M_4$ .

$Q = \{C, A_1, B_1, A_2, B_2, \dots\}$  is the set of states for  $M_4$ .

$\Sigma = \{a, b\}$  is  $M_4$ 's input alphabet.

$\delta : Q \mapsto Q$  is  $M_4$ 's state transition function. Where  $\delta(A_k, a) = A_{k+1}$  and  $\delta(A_k, b) = B_k$  for  $k \geq 2$  and  $\delta(A_0, b) = \delta(B_1, b) = \delta(B_1, a) = C$  and  $\delta(B_k, b) = B_{k-1}$  for  $k \geq 2$  and  $\delta(B_K, a) = C$  for  $k \geq 1$ .

$q_0 = A_0$  the starting state.

$F = \{A_0, B_1\}$  is the final/accepting states. These are only reached if the input are from the set  $\{a^n b^n : n \in \mathbb{N}\}$ .

**[25] 4. Textbook Problem 7(c) in B.2 on Pages 522 and 523.**

Use a (fooling set)-plus-(Continuation Lemma) argument to prove that the following language is *not* Regular:

$$L_5 = \{a^i b^j c^k \mid i = j \text{ OR } j = k\}.$$

**Your argument will be a proof by contradiction.**

The Continuation Lemma is Lemma 3.2 on Page 57, while the fooling set kind of argument is first utilized on Pages 76 and 77.

*Proof.* Assume that  $L_5$  is regular then there is a finite automata

$$M = \{Q, \{a, b, c\}, \delta, q_0, F\}$$

such that  $L(M) = L_5$ . As  $Q$  is finite and  $\{a^k : k \in \mathbb{N}\}$  has infinite cardinality then the function  $\delta : J \subseteq Q \times \{a^k : k \in \mathbb{N}\} \mapsto Q$  cannot be injective by the pigeonhole principle. Then for some  $k, j \in \mathbb{N}$  with  $k \neq j$  we have  $\delta(q_0, a^k) = \delta(q_0, a^j) = q$  where  $q \in Q$ . Then we have by the continuation lemma  $\delta(q, b^i) = \delta(q_0, a^k b^i) = \delta(q_0, a^j b^i) = q_1$  where  $i \in \mathbb{N}$ . Again by the pigeonhole principle we have that there exists  $t, s \in \mathbb{N}$  where  $t \neq s$  and  $\delta(q_1, c^s) = \delta(q_1, c^t)$ . Then this would imply  $\delta(q_0, a^k b^i c^s) = \delta(q_0, a^j b^i c^t)$  now if they are accepted then we have  $i = k$  or  $i = s$  however as  $k \neq j$  and  $t \neq s$  then  $M$  would accept strings not belonging to  $L_5$ . If  $M$  accepts neither string that would then there would be some strings in  $L_5$  that are not accepted by  $M$ . Such a string can be shown to exist by letting  $i = k$  then we have  $\delta(q_0, a^k b^i c^s) = \delta(q_0, a^k b^k c^s)$  a contradiction on the universal quantifier of  $i$ . In both cases  $M$  does not accept the language of  $L_5$   $\square$