

# Matching Theory Notes

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**Lemma 1.0.1.** For any graph,  $G$ ,  $\alpha(G) + \tau(G) = |V(G)|$ .

*Proof.* Let  $G$  be an arbitrary graph and let  $M$  be an arbitrary point cover where  $|M| = \tau(G)$ . Then  $V(G) - M$  is an independent set of points. This is true because if  $V(G) - M$  was not an independent set of points then there are at least two points  $u, v \in V(G) - M$  where  $u$  and  $v$  are adjacent. If they are adjacent the line incident with  $u$  and  $v$  is not covered by  $M$  which is a contradiction. As  $V(G) - M$  is an independent set of points we get the inequality

$$\alpha(G) \geq |V(G) - M| = |V(G)| - \tau(G)$$

Now assume that  $L$  is an arbitrary independent set of points where  $|L| = \alpha(G)$ . Then  $V(G) - L$  is a point cover of  $G$ . If  $V(G) - L$  is not a point cover of  $G$  then there exists a line  $l$  such that  $l$  is not covered by  $V(G) - L$ . However this would imply that  $l$  is incident with two points in  $L$  this is a contradiction on the fact  $L$  is independent points hence  $V(G) - L$  is a point cover of  $G$ . As  $V(G) - L$  is a point cover of  $G$  we get the inequality

$$\tau(G) \leq |V(G) - L| = |V(G)| - \alpha(G)$$

Combining both inequalities we get  $\alpha(G) \geq |V(G)| - \tau(G)$  and  $\tau(G) \leq |V(G)| - \alpha(G)$ . This implies that

$$\alpha(G) + \tau(G) = |V(G)|$$

□

**Lemma 1.0.2.** For any graph  $G$  with no isolated points,  $v(G) + \rho(G) = |V(G)|$ .

*Proof.* Let  $G$  be an arbitrary graph with no isolated points and let  $C$  be a line cover of  $G$  where  $|C| = \rho(G)$ . Let  $\langle C \rangle$  be the graph formed from lines the set of lines  $C$  and the set of points  $V(C)$ . We have that  $\langle C \rangle$  is a union of stars. This is because if  $\langle C \rangle$  was not a union of stars. Then there would be two points in the graph that that are adjacent two one or more point in  $\langle C \rangle$ . If they are adjacent to a single point then removing either of the lines incident would create a smaller minimal line cover. If they are adjacent to two or more points then removing any of the incident lines would create a smaller minimal line cover. Now if we let  $n$  be the number of components in  $\langle C \rangle$  we get that  $n = |V(G)| - \rho(G)$  which arises because in each star there is one more point than there are lines. If we take one line from each star we get a matching hence

$$v(G) \geq |V(G)| - \rho(G)$$

Now let  $M$  be an arbitrary matching of  $G$  where  $|M| = v(G)$ , and  $U$  be the set of points that are not covered by  $M$ . We get that  $U$  is an independent set of points because if  $U$  was not an independent set of points then there would be two points  $u, v \in U$  such that  $u$  and  $v$  are adjacent. If  $u$  and  $v$  are adjacent then there is a line incident with  $u$  and  $v$  that is not covered by  $M$  which is a contradiction. As  $U$  is an independent set of points and  $G$  has no isolated points we get that  $|U| = |V(G)| - 2v(G)$ . Let  $S$  be a line covering of  $U$  we get that  $M \cup S$  is a line covering. Then we get the inequality

$$\rho(G) \leq |M \cup S| = v(G) + |V(G)| - 2v(G) = |V(G)| - v(G)$$

Combining both inequalities we get

$$v(G) + \rho(G) = |V(G)|$$

□

### Exercise 1.0.3.

1. A minimal line cover is minimum if and only if it contains a maximum matching.
2. A maximal matching is maximum if and only if it is contained in a minimum line cover.

*Proof.* (1)

Let  $L$  be a line cover where  $|L| = \rho(G)$ . As  $L$  is a minimum line cover we have that it forms the lines from a union of stars on  $G$ . We have that the number of stars is  $v(G)$  which comes from selecting one matching from each star. Hence  $L$  contains a maximum matching.

Let  $L$  be a minimal line cover that contains a maximum matching. Then as  $L$  is a union of stars we get a maximum matching by selecting a line from each star. Hence  $|L| = |V(G)| - v(G) = \rho(G)$  hence it is a minimum line cover. □

*Proof.* (2) Let  $M$  be a maximum matching. Let  $L$  be an line cover obtained by taking an arbitrary line covering of the edges incident with  $V(G) - V(M)$  and  $M$ . We have that  $|L| = |V(G)| - 2v(G) + v(G) = \rho(G)$  hence  $L$  contains a minimum line cover.

Let  $M$  be a maximal matching that is contained in a minimum line cover  $L$ . Then we have that the matchings in  $M$  are formed from individual lines of the stars formed by  $L$ . Hence  $|M| = V(G) - \rho(G) = v(G)$  hence  $M$  is a maximum matching. □

### Exercise 1.0.4. For any graph $G$ , $v(G) \leq \tau(G)$

*Proof.* Let  $G$  be an arbitrary graph and let  $M$  be a maximum matching in  $G$ . As  $M$  is a maximum matching we get that it is contained in a minimum line cover  $L$ . As  $L$  is a minimum line cover we have that it is the union of stars where each matching in  $M$  comes from a single line in each star. We get that the number of stars is  $\tau(G)$  hence  $v(G) \leq \tau(G)$  □

### Theorem 1.1.1. König's Minimax Theorem: If $G$ is bipartite, then $v(G) = \tau(G)$

*Proof.* Remove lines from  $G$  as long as possible while keeping  $\tau(G)$  the same. Denote the resulting minimal graph by  $G'$ . Hence  $\tau(G') = \tau(G)$ , but for any line  $l \in E(G)$  we have  $\tau(G' - l) < \tau(G)$ . Suppose that  $G'$  does not consist of independent lines. Then there are two lines  $l_1, l_2 \in E(G')$  that are both incident with a point  $c \in V(G')$  □