Matching Theory Notes

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Lemma 1.0.1. For any graph, G, $\alpha(G) + \tau(G) = |V(G)|$.

Proof. Let G be an arbitrary graph and let M be an arbitrary point cover where $|M| = \tau(G)$. Then $V(G) - M$ is an independent set of points. This is true because if $V(G) - M$ was not an independent set of points then there are at least two points $u, v \in V(G) - M$ where u and v are adjacent. If they are adjacent the line incident with u and v is not covered by M which is a contradiction. As $V(G) - M$ is an independent set of points we get the inequality

$$\alpha(G) \geq |V(G) - M| = |V(G)| - \tau(G)$$

Now assume that L is an arbitrary independent set of points where $|L| = \alpha(G)$. Then $V(G) - L$ is a point cover of G. If $V(G) - L$ is not a point cover of G then there exists a line l such that l is not covered by $V(G) - L$. However this would imply that l is incident with two points in L this is a contradiction on the fact L is independent points hence $V(G) - L$ is a point cover of G. As $V(G) - L$ is a point cover of G we get the inequality

$$\tau(G) \leq |V(G) - L| = |V(G)| - \alpha(G)$$

Combining both inequalities we get $\alpha(G) \geq |V(G)| - \tau(G)$ and $\tau(G) \leq |V(G)| - \alpha(G)$. This implies that

$$\alpha(G) + \tau(G) = |V(G)|$$

. □

Lemma 1.0.2. For any graph G with no isolated points, $v(G) + \rho(G) = |V(G)|$.

Proof. Let G be an arbitrary graph with no isolated points and let C be a line cover of G where $|C| = \rho(G)$. Let $\langle C \rangle$ be the graph formed from lines the set of lines C and the set of points $V(C)$. We have that $\langle C \rangle$ is a union of stars. This is because if $\langle C \rangle$ was not a union of stars. Then there would be two points in the graph that that are adjacent two one or more point in $\langle C \rangle$. If they are adjacent to a single point then removing either of the lines incident would create a smaller minimal line cover. If they are adjacent to two or more points then removing any of the incident lines would create a smaller minimal line cover. Now if we let n be the number of components in $\langle C \rangle$ we get that $n = |V(G)| - \rho(G)$ which arises because in each star there is one more point than there are lines. If we take one line from each star we get a matching hence

$$v(G) \geq |V(G)| - \rho(G)$$

Now let M be an arbitrary matching of G where $|M| = v(G)$, and U be the set of points that are not covered by M. We get that U is an independent set of points because if U was not an independent set of points then there would be two points $u, v \in U$ such that u and v are adjacent. If u and v are adjacent then there is a line incident with u and v that is not covered by M which is a contradiction. As U is an independent set of points and G has no isolated points we get that $|U| = |V(G)| - 2v(G)$. Let S be a line covering of U we get that $M \cup S$ is a line covering. Then we get the inequality

$$\rho(G) \leq |M \cup S| = v(G) + |V(G)| - 2v(G) = |V(G)| - v(G)$$

Combining both inequalities we get

$$v(G) + \rho(G) = |V(G)|$$

□

Exercise 1.0.3.

1. A minimal line cover is minimum if and only if it contains a maximum matching.

2. A maximal matching is maximum if and only if it is contained in a minimum line cover.

Proof. (1)
Let L be a line cover where $|L| = \rho(G)$. As L is a minimum line cover we have that it forms the lines from a union of stars on G. We have that the number of stars is $v(G)$ which comes from selecting one matching from each star. Hence L contains a maximum matching.
Let L be a minimal line cover that contains a maximum matching. Then as L is a union of stars we get a maximum matching by selecting a line from each star. Hence $|L| = |V(G)| - v(G) = \rho(G)$ hence it is a minimum line cover. □

Proof. (2) Let M be a maximum matching. Let L be an line cover obtained by taking an arbitrary line covering of the edges incident with $V(G) - V(M)$ and M. We have that $|L| = |V(G)| - 2v(G) + v(G) = \rho(G)$ hence L contains a minimum line cover.
Let M be a maximal matching that is contained in a minimum line cover L. Then we have that the matchings in M are formed from individual lines of the stars formed by L. Hence $|M| = V(G) - \rho(G) = v(G)$ hence M is a maximum matching. □

Exercise 1.0.4. For any graph G, $v(G) \leq \tau(G)$

Proof. Let G be an arbitrary graph and let M be a maximum matching in G. As M is a maximum matching we get that it is contained in a minimum line cover L. As L is a minimum line cover we have that it is the union of stars where each matching in M comes from a single line in each star. We get that the number of stars is $\tau(G)$ hence $v(G) \leq \tau(G)$ □

Theorem 1.1.1. König's Minimax Theorem: If G is bipartite, then $v(G) = \tau(G)$

Proof. Remove lines from G as long as possible while keeping $\tau(G)$ the same. Denote the resulting minimal graph by G'. Hence $\tau(G') = \tau(G)$, but for any line $l \in E(G)$ we have $\tau(G' - l) < \tau(G)$. Suppose that G' does not consist of independent lines. Then there are two lines $l_1, l_2 \in E(G')$ that are both incident with a point $a \in V(G')$. As G' is minimal we have that there exists a point cover of $G' - x$ we denote this point cover by S_x where $|S_x| = \tau(G') - 1$. We also have the point cover S_y which is covering $G' - y$ with $|S_x| = |S_y|$. Form the induced subgraph G'' of G', $G'' = G'[\{a\} \cup S_x \cup S_y \cap \overline{S_x \cap S_y}]$ and let $t = |S_x \cap S_y|$. Then we have $|V(G'')| = 2(\tau(G') - 1 - t) + 1$. As G'' is a subgraph of a bipartite graph we have that it is bipartite. Hence we have a set T which is a point cover of G'' where $|T| = \tau(G') - 1 - t$. We claim that the set of points $T' = T \cup (S_x \cap S_y)$ covers G'. Let z be a line in G' if $z \neq x$ or $z \neq y$ then we have that z is covered by S_x and S_y hence it is covered by $S_x \cap S_y$ or it connects $S_x - S_y$ to $S_y - S_x$. If $z = x$ or $z = y$ then we have it is covered by T. So $\tau(G') \leq |T'| = |T \cup (S_x \cap S_y)| = |T| + |S_x \cap S_y| \leq \tau(G') - 1 - t + t = \tau(G') - 1$ a contradiction. Hence G' is an independent set of lines. Hence

$$\tau(G) = \tau(G') = v(G') \leq v(G)$$

□

Theorem 1.1.2. In any (possibly infinite) bipartite graph there exists a matching M and a point cover P such that every line in M contains exactly one point in P and every point in P is contained in exactly one line of M.

Proof. Let G be an arbitrary bipartite graph and let L be a minimum point cover of G. Create the new graph G' by removing lines from G as long as $\tau(G') = \tau(G)$. We have that G' consists of an independent set of lines from each of the points in the point cover $\tau(G')$ select one line. As G' consists of an independent set of lines we have a maximum matching by König's Minimax Theorem which satisfies the implication. □

If X is any set of points in a graph G, we denote the set of all points which are adjacent to at least one point of X as $\Gamma(X)$.

Theorem 1.1.3. (P. Hall's Theorem). Let $G = (A, B)$ be a bipartite graph. Then G has a matching of A into B if and only if $|\Gamma(X)| \geq |X|$ for all $X \subseteq A$.

Proof. Let $G = (A, B)$ be a bipartite graph and suppose that there is a matching A to B and suppose that for some subgraph $X \subseteq A$ we have $|\Gamma(X)| < |X|$. If X is any subgraph of A and there exists a matching from A to B we have that there exists a matching from X to B as well. As $|\Gamma(X)| < |X|$ we have that there is at least 1 point in X that is not adjacent to any point in B hence it can not be a complete matching.

Suppose $|\Gamma(X)| \geq |X|$ for all $X \subseteq A$. Then using mathematical induction on $|A|$. If $|A| = 0$ then we would have a complete matching. If $|A| = 1$ then by our assumption we have that $|\Gamma(A)| \geq |A|$ hence there is a matching.

Case 1:

Suppose that for all $X \subset A$, $X \neq \emptyset, |X| < |\Gamma(X)|$ holds. Let $a \in A, b \in B$ be two adjacent points. Let $G' = G - a - b$ and let X be any subset of $A - a$. If $X = \emptyset$, then

$|X| = |\Gamma_{G'}(X)|$ so assume $X \neq \emptyset$. Since $X \neq A$, $|X| < |\Gamma_{G'}(X)|$ based on the assumption then we have $|\Gamma_{G'}(X)| \geq |\Gamma_G(X)| - 1 \geq |X|$. Therefore by the induction hypothesis, there is a matching M' of G' which covers all points of $A - a$. But then $M = M' \cup \{ab\}$ matches A and B as desired.

Case 2:
Suppose there is a set $A' \subset A$, $A' \neq \emptyset$ with $|\Gamma_G(A')| = |A'|$. We proceed to split G into two smaller subgraphs by letting G_1 be the subgraph induced by $A' \cup \Gamma(A')$ and $G_2 = G - A' - \Gamma(A')$. Suppose $X \subseteq A'$. Then $\Gamma_G(X) \subseteq \Gamma_G(A')$, so $\Gamma_{G_1}(X) = \Gamma_G(X)$ and hence $|\Gamma_{G_1}(X)| = |\Gamma_G(X)| \geq |X|$. Now in G_2 assume $X \subseteq A - A'$. Then $\Gamma_G(X \cup A') = \Gamma_{G_2}(X) \cup \Gamma_G(A')$ and therefore $|\Gamma_{G_2}(X)| = |\Gamma_G(X \cup A')| - |\Gamma_G(A')| \geq |X \cup A'| - |\Gamma_G(A')| = |X \cup A'| - |A'| = |X|$.

By applying the induction hypothesis to both G_1 and G_2, we see that there must exist matchings M_1 of A' into $\Gamma_G(A')$ and M_2 of $A - A'$ into $B - \Gamma_G(A')$. The union of $M = M_1 \cup M_2$ is the desired matching.

□

Corollary 1.1.4. (The Marriage Theorem). A bipartite graph $G = (A, B)$ has a prefect matching if and only if $|A| = |B|$ and for each $X \subseteq A$, $|X| \leq |\Gamma(X)|$.

Proof. Let G be a bipartite graph with $G = (A, B)$ and assume that G has a prefect matching M. Then we have $|A| = |B|$ otherwise it would not be a prefect matching. By Hall's theorem we have for each $X \subseteq A$, $|X| \leq |\Gamma(X)|$.

Assume G is a bipartite graph $G = (A, B)$ where $|A| = |B|$ and for each $X \subseteq A$, $|X| \leq |\Gamma(X)|$. Then by Hall's Theorem we have that there is a matching M of A into B. However as we have $|A| = |B|$ we get that this is a prefect matching. □

Corollary 1.1.7. If G is bipartite, $\rho(G) = \alpha(G)$.

Proof. Suppose G is a bipartite graph then by König's Minimax Theorem we have $v(G) = \tau(G)$ by the Gallai Identities we have

$$v(G) = |V(G)| - \rho(G)$$

and

$$\tau(G) = |V(G)| - \alpha(G)$$

combining with König's Minimax Theorem we get

$$|V(G)| - \rho(G) = |V(G)| - \alpha(G)$$

$$\rho(G) = \alpha(G)$$

□

Theorem 1.2.1. Let M be a matching in a graph G. Then M is a maximum matching if and only if there exists no augmenting path in G relative to M.

Proof. Let G be a graph and M be a maximum matching in G and assume that there exists a augmenting path P in G relative to M. Then consider the matching $M' = (M \setminus M \cap P) \cup (P \setminus M)$ this would imply $|M| < |M'|$ which is a clear contradiction.

Let G be a graph and M be a matching that is not maximum and suppose there are no augmenting paths in G relative to M. As M is not a maximum matching there exists a larger matching M. Create the graph $G' = G[M \setminus M' \cup M' \setminus M]$ as this creates an alternating path and there are at least two points in $V(G')$ that are not in $V(M)$ we have an augmenting path. □

Lemma 1.2.2. If $G = (A, B)$ is a bipartite graph and M is a matching on G where $A_1 \subseteq A, B_1 \subseteq B$ are the sets of exposed points where F is the maximal forest in G with the properties:

- For each point b of F has degree 2 and of the incident lines of b is in M.
- Each component of F contains a point in A_1.

,then M is a maximum matching if and only if no point of B_1 is adjacent to any point of F.

Proof. Assume M is a maximum matching and there exists a point b of B_1 that is adjacent to a point in F. Then we have an $M-$*augmenting* path which by Theorem 1.2.1 implies M is not a maximum matching.

Suppose that no point of B_1 is adjacent to any point of F. We have M covers $A \setminus V(F) \cup (B \cap V(F))$ which is immediate as M covers A and B. We have that no line of M joins a point from $A \setminus (F)$ to a point of $B \cap V(F)$. As if there was then we would have both points would have to be in $V(F)$ however that would contradict $A \setminus V(F)$. Now as M covers $A \setminus V(F) \cup (B \cap V(F))$ and each line of M contains exactly one single point of $A \setminus V(F) \cup (B \cap V(F))$. We have $|A \setminus V(F) \cup (B \cap V(F))| = |M|$.

Now to show that $A \setminus V(F) \cup (B \cap V(F))$ is a point cover. Suppose that it is not and there is a line ab that is not covered by $A \setminus V(F) \cup (B \cap V(F))$ where $a \in A$ and $b \in B$ then we have $a \in V(F)$ and $b \notin V(F)$. By the hypothesis we have $b \notin B_1$ therefore the matching covers b by a line $a'b$. We have $a \neq a'$

□