

# Matching Theory Notes

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**Lemma 1.0.1.** For any graph,  $G$ ,  $\alpha(G) + \tau(G) = |V(G)|$ .

*Proof.* Let  $G$  be an arbitrary graph and let  $M$  be an arbitrary point cover where  $|M| = \tau(G)$ . Then  $V(G) - M$  is an independent set of points. This is true because if  $V(G) - M$  was not an independent set of points then there are at least two points  $u, v \in V(G) - M$  where  $u$  and  $v$  are adjacent. If they are adjacent the line incident with  $u$  and  $v$  is not covered by  $M$  which is a contradiction. As  $V(G) - M$  is an independent set of points we get the inequality

$$\alpha(G) \geq |V(G) - M| = |V(G)| - \tau(G)$$

Now assume that  $L$  is an arbitrary independent set of points where  $|L| = \alpha(G)$ . Then  $V(G) - L$  is a point cover of  $G$ . If  $V(G) - L$  is not a point cover of  $G$  then there exists a line  $l$  such that  $l$  is not covered by  $V(G) - L$ . However this would imply that  $l$  is incident with two points in  $L$  this is a contradiction on the fact  $L$  is independent points hence  $V(G) - L$  is a point cover of  $G$ . As  $V(G) - L$  is a point cover of  $G$  we get the inequality

$$\tau(G) \leq |V(G) - L| = |V(G)| - \alpha(G)$$

Combining both inequalities we get  $\alpha(G) \geq |V(G)| - \tau(G)$  and  $\tau(G) \leq |V(G)| - \alpha(G)$ . This implies that

$$\alpha(G) + \tau(G) = |V(G)|$$

□

**Lemma 1.0.2.** For any graph  $G$  with no isolated points,  $v(G) + \rho(G) = |V(G)|$ .

*Proof.* Let  $G$  be an arbitrary graph with no isolated points and let  $C$  be a line cover of  $G$  where  $|C| = \rho(G)$ . Let  $\langle C \rangle$  be the graph formed from lines the set of lines  $C$  and the set of points  $V(C)$ . We have that  $\langle C \rangle$  is a union of stars. This is because if  $\langle C \rangle$  was not a union of stars. Then there would be two points in the graph that that are adjacent two one or more point in  $\langle C \rangle$ . If they are adjacent to a single point then removing either of the lines incident would create a smaller minimal line cover. If they are adjacent to two or more points then removing any of the incident lines would create a smaller minimal line cover. Now if we let  $n$  be the number of components in  $\langle C \rangle$  we get that  $n = |V(G)| - \rho(G)$  which arises because in each star there is one more point than there are lines. If we take one line from each star we get a matching hence

$$v(G) \geq |V(G)| - \rho(G)$$

Now let  $M$  be an arbitrary matching of  $G$  where  $|M| = v(G)$ , and  $U$  be the set of points that are not covered by  $M$ . We get that  $U$  is an independent set of points because if  $U$  was not an independent set of points then there would be two points  $u, v \in U$  such that  $u$  and  $v$  are adjacent. If  $u$  and  $v$  are adjacent then there is a line incident with  $u$  and  $v$  that is not covered by  $M$  which is a contradiction. As  $U$  is an independent set of points and  $G$  has no isolated points we get that  $|U| = |V(G)| - 2v(G)$ . Let  $S$  be a line covering of  $U$  we get that  $M \cup S$  is a line covering. Then we get the inequality

$$\rho(G) \leq |M \cup S| = v(G) + |V(G)| - 2v(G) = |V(G)| - v(G)$$

Combining both inequalities we get

$$v(G) + \rho(G) = |V(G)|$$

□

### Exercise 1.0.3.

1. A minimal line cover is minimum if and only if it contains a maximum matching.
2. A maximal matching is maximum if and only if it is contained in a minimum line cover.

*Proof.* (1)

Let  $L$  be a line cover where  $|L| = \rho(G)$ . As  $L$  is a minimum line cover we have that it forms the lines from a union of stars on  $G$ . We have that the number of stars is  $v(G)$  which comes from selecting one matching from each star. Hence  $L$  contains a maximum matching.

Let  $L$  be a minimal line cover that contains a maximum matching. Then as  $L$  is a union of stars we get a maximum matching by selecting a line from each star. Hence  $|L| = |V(G)| - v(G) = \rho(G)$  hence it is a minimum line cover. □

*Proof.* (2) Let  $M$  be a maximum matching. Let  $L$  be an line cover obtained by taking an arbitrary line covering of the edges incident with  $V(G) - V(M)$  and  $M$ . We have that  $|L| = |V(G)| - 2v(G) + v(G) = \rho(G)$  hence  $L$  contains a minimum line cover.

Let  $M$  be a maximal matching that is contained in a minimum line cover  $L$ . Then we have that the matchings in  $M$  are formed from individual lines of the stars formed by  $L$ . Hence  $|M| = V(G) - \rho(G) = v(G)$  hence  $M$  is a maximum matching. □

### Exercise 1.0.4. For any graph $G$ , $v(G) \leq \tau(G)$

*Proof.* Let  $G$  be an arbitrary graph and let  $M$  be a maximum matching in  $G$ . As  $M$  is a maximum matching we get that it is contained in a minimum line cover  $L$ . As  $L$  is a minimum line cover we have that it is the union of stars where each matching in  $M$  comes from a single line in each star. We get that the number of stars is  $\tau(G)$  hence  $v(G) \leq \tau(G)$  □

**Theorem 1.1.1.** König's Minimax Theorem: If  $G$  is bipartite, then  $v(G) = \tau(G)$

*Proof.* Remove lines from  $G$  as long as possible while keeping  $\tau(G)$  the same. Denote the resulting minimal graph by  $G'$ . Hence  $\tau(G') = \tau(G)$ , but for any line  $l \in E(G)$  we have  $\tau(G' - l) < \tau(G)$ . Suppose that  $G'$  does not consist of independent lines. Then there are two lines  $l_1, l_2 \in E(G')$  that are both incident with a point  $a \in V(G')$ . As  $G'$  is minimal we have that there exists a point cover of  $G' - a$  we denote this point cover by  $S_x$  where  $|S_x| = \tau(G') - 1$ . We also have the point cover  $S_y$  which is covering  $G' - a$  with  $|S_x| = |S_y|$ . Form the induced subgraph  $G''$  of  $G'$ ,  $G'' = G'[\{a\} \cup S_x \cup S_y \cup \overline{S_x \cap S_y}]$  and let  $t = |S_x \cap S_y|$ . Then we have  $|V(G'')| = 2(\tau(G') - 1 - t) + 1$ . As  $G''$  is a subgraph of a bipartite graph we have that it is bipartite. Hence we have a set  $T$  which is a point cover of  $G''$  where  $|T| = \tau(G') - 1 - t$ . We claim that the set of points  $T' = T \cup (S_x \cap S_y)$  covers  $G'$ . Let  $z$  be a line in  $G'$  if  $z \neq x$  or  $z \neq y$  then we have that  $z$  is covered by  $S_x$  and  $S_y$  hence it is covered by  $S_x \cap S_y$  or it connects  $S_x - S_y$  to  $S_y - S_x$ . If  $z = x$  or  $z = y$  then we have it is covered by  $T$ . So  $\tau(G') \leq |T'| = |T \cup (S_x \cap S_y)| = |T| + |S_x \cap S_y| \leq \tau(G') - 1 - t + t = \tau(G') - 1$  a contradiction. Hence  $G'$  is an independent set of lines. Hence

$$\tau(G) = \tau(G') = v(G') \leq v(G)$$

□

**Theorem 1.1.2.** In any (possibly infinite) bipartite graph there exists a matching  $M$  and a point cover  $P$  such that every line in  $M$  contains exactly one point in  $P$  and every point in  $P$  is contained in exactly one line of  $M$ .

*Proof.* Let  $G$  be an arbitrary bipartite graph and let  $L$  be a minimum point cover of  $G$ . Create the new graph  $G'$  by removing lines from  $G$  as long as  $\tau(G') = \tau(G)$ . We have that  $G'$  consists of an independent set of lines from each of the points in the point cover  $\tau(G')$  select one line. As  $G'$  consists of an independent set of lines we have a maximum matching by König's Minimax Theorem which satisfies the implication. □

If  $X$  is any set of points in a graph  $G$ , we denote the set of all points which are adjacent to at least one point of  $X$  as  $\Gamma(X)$ .

**Theorem 1.1.3.** (P. Hall's Theorem). Let  $G = (A, B)$  be a bipartite graph. Then  $G$  has a matching of  $A$  into  $B$  if and only if  $|\Gamma(X)| \geq |X|$  for all  $X \subseteq A$ .

*Proof.* Let  $G = (A, B)$  be a bipartite graph and suppose that there is a matching  $A$  to  $B$  and suppose that for some subgraph  $X \subseteq A$  we have  $|\Gamma(X)| < |X|$ . If  $X$  is any subgraph of  $A$  and there exists a matching from  $A$  to  $B$  we have that there exists a matching from  $X$  to  $B$  as well. As  $|\Gamma(X)| < |X|$  we have that there is at least 1 point in  $X$  that is not adjacent to any point in  $B$  hence it can not be a complete matching.

Suppose  $|\Gamma(X)| \geq |X|$  for all  $X \subseteq A$ . Then using mathematical induction on  $|A|$ . If  $|A| = 0$  then we would have a complete matching. If  $|A| = 1$  then by our assumption we have that  $|\Gamma(A)| \geq |A|$  hence there is a matching.

Case 1:

Suppose that for all  $X \subset A$ ,  $X \neq \emptyset$ ,  $|X| < |\Gamma(X)|$  holds. Let  $a \in A, b \in B$  be two adjacent points. Let  $G' = G - a - b$  and let  $X$  be any subset of  $A - a$ . If  $X = \emptyset$ , then

$|X| = |\Gamma_{G'}(X)|$  so assume  $X \neq \emptyset$ . Since  $X \neq A$ ,  $|X| < |\Gamma_{G'}(X)|$  based on the assumption then we have  $|\Gamma_{G'}(X)| \geq |\Gamma_G(X)| - 1 \geq |X|$ . Therefore by the induction hypothesis, there is a matching  $M'$  of  $G'$  which covers all points of  $A - a$ . But then  $M = M' \cup \{ab\}$  matches  $A$  and  $B$  as desired.

Case 2:

Suppose there is a set  $A' \subset A$ ,  $A' \neq \emptyset$  with  $|\Gamma_G(A')| = |A'|$ . We proceed to split  $G$  into two smaller subgraphs by letting  $G_1$  be the subgraph induced by  $A' \cup \Gamma(A')$  and  $G_2 = G - A' - \Gamma(A')$ . Suppose  $X \subseteq A'$ . Then  $\Gamma_G(X) \subseteq \Gamma_G(A')$ , so  $\Gamma_{G_1}(X) = \Gamma_G(X)$  and hence  $|\Gamma_{G_1}(X)| = |\Gamma_G(X)| \geq |X|$ . Now in  $G_2$  assume  $X \subseteq A - A'$ . Then  $\Gamma_G(X \cup A') = \Gamma_{G_2}(X) \cup \Gamma_G(A')$  and therefore  $|\Gamma_{G_2}(X)| = |\Gamma_G(X \cup A')| - |\Gamma_G(A')| \geq |X \cup A'| - |\Gamma_G(A')| = |X \cup A'| - |A'| = |X|$ .

By applying the induction hypothesis to both  $G_1$  and  $G_2$ , we see that there must exist matchings  $M_1$  of  $A'$  into  $\Gamma_G(A')$  and  $M_2$  of  $A - A'$  into  $B - \Gamma_G(A')$ . The union of  $M = M_1 \cup M_2$  is the desired matching. □

**Corollary 1.1.4.** (The Marriage Theorem). A bipartite graph  $G = (A, B)$  has a prefect matching if and only if  $|A| = |B|$  and for each  $X \subseteq A$ ,  $|X| \leq |\Gamma(X)|$ .

*Proof.* Let  $G$  be a bipartite graph with  $G = (A, B)$  and assume that  $G$  has a prefect matching  $M$ . Then we have  $|A| = |B|$  otherwise it would not be a prefect matching. By Hall's theorem we have for each  $X \subseteq A$ ,  $|X| \leq |\Gamma(X)|$ .

Assume  $G$  is a bipartite graph  $G = (A, B)$  where  $|A| = |B|$  and for each  $X \subseteq A$ ,  $|X| \leq |\Gamma(X)|$ . Then by Hall's Theorem we have that there is a matching  $M$  of  $A$  into  $B$ . However as we have  $|A| = |B|$  we get that this is a prefect matching. □

**Corollary 1.1.7.** If  $G$  is bipartite,  $\rho(G) = \alpha(G)$ .

*Proof.* Suppose  $G$  is a bipartite graph then by König's Minimax Theorem we have  $v(G) = \tau(G)$  by the Gallai Identities we have

$$v(G) = |V(G)| - \rho(G)$$

and

$$\tau(G) = |V(G)| - \alpha(G)$$

combining with König's Minimax Theorem we get

$$|V(G)| - \rho(G) = |V(G)| - \alpha(G)$$

$$\rho(G) = \alpha(G)$$

□

**Theorem 1.2.1.** Let  $M$  be a matching in a graph  $G$ . Then  $M$  is a maximum matching if and only if there exists no augmenting path in  $G$  relative to  $M$ .

*Proof.* Let  $G$  be a graph and  $M$  be a maximum matching in  $G$  and assume that there exists a augmenting path  $P$  in  $G$  relative to  $M$ . Then consider the matching  $M' = (M \setminus M \cap P) \cup (P \setminus M)$  this would imply  $|M| < |M'|$  which is a clear contradiction.

Let  $G$  be a graph and  $M$  be a matching that is not maximum and suppose there are no augmenting paths in  $G$  relative to  $M$ . As  $M$  is not a maximum matching there exists a larger matching  $M'$ . Create the graph  $G' = G[M \setminus M' \cup M' \setminus M]$  as this creates an alternating path and there are at least two points in  $V(G')$  that are not in  $V(M)$  we have an augmenting path.  $\square$