# Matching Theory Notes

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**Lemma 1.0.1.** For any graph, $G$, $\alpha(G) + \tau(G) = |V(G)|$.

*Proof.* Let $G$ be an arbitrary graph and let $M$ be an arbitrary point cover where $|M| = \tau(G)$. Then $V(G) - M$ is an independent set of points. This is true because if $V(G) - M$ was not an independent set of points then there are at least two points $u, v \in V(G) - M$ where $u$ and $v$ are adjacent. If they are adjacent the line incident with $u$ and $v$ is not covered by $M$ which is a contradiction. As $V(G) - M$ is an independent set of points we get the inequality

$$\alpha(G) \geq |V(G) - M| = |V(G)| - \tau(G)$$

Now assume that $L$ is an arbitrary independent set of points where $|L| = \alpha(G)$. Then $V(G) - L$ is a point cover of $G$. If $V(G) - L$ is not a point cover of $G$ then there exists a line $l$ such that $l$ is not covered by $V(G) - L$. However this would imply that $l$ is incident with two points in $L$ this is a contradiction on the fact $L$ is independent points hence $V(G) - L$ is a point cover of $G$. As $V(G) - L$ is a point cover of $G$ we get the inequality

$$\tau(G) \leq |V(G) - L| = |V(G)| - \alpha(G)$$

Combining both inequalities we get $\alpha(G) \geq |V(G)| - \tau(G)$ and $\tau(G) \leq |V(G)| - \alpha(G)$. This implies that
$$\alpha(G) + \tau(G) = |V(G)|$$
. $\square$

**Lemma 1.0.2.** For any graph $G$ with no isolated points, $v(G) + \rho(G) = |V(G)|$.

*Proof.* Let $G$ be an arbitrary graph with no isolated points and let $C$ be a line cover of $G$ where $|C| = \rho(G)$. Let $\langle C \rangle$ be the graph formed from lines the set of lines $C$ and the set of points $V(C)$. We have that $\langle C \rangle$ is a union of stars. This is because if $\langle C \rangle$ was not a union of stars. Then there would be two points in the graph that that are adjacent two one or more point in $\langle C \rangle$. If they are adjacent to a single point then removing either of the lines incident would create a smaller minimal line cover. If they are adjacent to two or more points then removing any of the incident lines would create a smaller minimal line cover. Now if we let $n$ be the number of components in $\langle C \rangle$ we get that $n = |V(G)| - \rho(G)$ which arises because in each star there is one more point than there are lines. If we take one line from each star we get a matching hence
$$v(G) \geq |V(G)| - \rho(G)$$

Now let $M$ be an arbitrary matching of $G$ where $|M| = v(G)$, and $U$ be the set of points that are not covered by $M$. We get that $U$ is an independent set of points because if $U$ was not an independent set of points then there would be two points $u, v \in U$ such that $u$ and $v$ are adjacent. If $u$ and $v$ are adjacent then there is a line incident with $u$ and $v$ that is not covered by $M$ which is a contradiction. As $U$ is an independent set of points and $G$ has no isolated points we get that $|U| = |V(G)| - 2v(G)$. Let $S$ be a line covering of $U$ we get that $M \cup S$ is a line covering. Then we get the inequality

$$\rho(G) \le |M \cup S| = v(G) + |V(G)| - 2v(G) = |V(G)| - v(G)$$

Combining both inequalities we get

$$v(G) + \rho(G) = |V(G)|$$

$\square$

**Exercise 1.0.3.**

1. A minimal line cover is minimum if and only if it contains a maximum matching.

2. A maximal matching is maximum if and only if it is contained in a minimum line cover.

*Proof.* (1)
Let $L$ be a line cover where $|L| = \rho(G)$. As $L$ is a minimum line cover we have that it forms the lines from a union of stars on $G$. We have that the number of stars is $v(G)$ which comes from selecting one matching from each star. Hence $L$ contains a maximum matching.

Let $L$ be a minimal line cover that contains a maximum matching. Then as $L$ is a union of stars we get a maximum matching by selecting a line from each star. Hence $|L| = |V(G)| - v(G) = \rho(G)$ hence it is a minimum line cover. $\square$

*Proof.* (2) Let $M$ be a maximum matching. Let $L$ be an line cover obtained by taking an arbitrary line covering of the edges incident with $V(G) - V(M)$ and $M$. We have that $|L| = |V(G)| - 2v(G) + v(G) = \rho(G)$ hence $L$ contains a minimum line cover.

Let $M$ be a maximal matching that is contained in a minimum line cover $L$. Then we have that the matchings in $M$ are formed from individual lines of the stars formed by $L$. Hence $|M| = V(G) - \rho(G) = v(G)$ hence $M$ is a maximum matching. $\square$

**Exercise 1.0.4.** For any graph $G$, $v(G) \le \tau(G)$

*Proof.* Let $G$ be an arbitrary graph and let $M$ be a maximum matching in $G$. As $M$ is a maximum matching we get that it is contained in a minimum line cover $L$. As $L$ is a minimum line cover we have that it is the union of stars where each matching in $M$ comes from a single line in each star. We get that the number of stars is $\tau(G)$ hence $v(G) \le \tau(G)$ $\square$

**Theorem 1.1.1.** König's Minimax Theorem: If $G$ is bipartite, then $v(G) = \tau(G)$

*Proof.* Remove lines from $G$ as long as possible while keeping $\tau(G)$ the same. Denote the resulting minimal graph by $G'$. Hence $\tau(G') = \tau(G)$, but for any line $l \in E(G)$ we have $\tau(G' - l) < \tau(G)$. Suppose that $G'$ does not consist of independent lines. Then there are two lines $l_1, l_2 \in E(G')$ that are both incident with a point $a \in V(G')$. As $G'$ is minimal we have that there exists a point cover of $G' - x$ we denote this point cover by $S_x$ where $|S_x| = \tau(G') - 1$. We also have the point cover $S_y$ which is covering $G' - y$ with $|S_x| = |S_y|$. Form the induced subgraph $G''$ of $G'$, $G'' = G'[\{a\} \cup S_x \cup S_y \cap \overline{S_x \cap S_y}]$ and let $t = |S_x \cap S_y|$. Then we have $|V(G'')| = 2(\tau(G') - 1 - t) + 1$. As $G''$ is a subgraph of a bipartite graph we have that it is bipartite. Hence we have a set $T$ which is a point cover of $G''$ where $|T| = \tau(G') - 1 - t$. We claim that the set of points $T' = T \cup (S_x \cap S_y)$ covers $G'$. Let $z$ be a line in $G'$ if $z \neq x$ or $z \neq y$ then we have that $z$ is covered by $S_x$ and $S_y$ hence it is covered by $S_x \cap S_y$ or it connects $S_x - S_y$ to $S_y - S_x$. If $z = x$ or $z = y$ then we have it is covered by $T$. So $\tau(G') \leq |T'| = |T \cup (S_x \cap S_y)| = |T| + |S_x \cap S_y| \leq \tau(G') - 1 - t + t = \tau(G') - 1$ a contradiction. Hence $G'$ is an independent set of lines. Hence

$$\tau(G) = \tau(G') = v(G') \leq v(G)$$

$\square$

**Theorem 1.1.2.** In any (possibly infinite) bipartite graph there exists a matching $M$ and a point cover $P$ such that every line in $M$ contains exactly one point in $P$ and every point in $P$ is contained in exactly one line of $M$.

*Proof.* Let $G$ be an arbitrary bipartite graph and let $L$ be a minimum point cover of $G$. Create the new graph $G'$ by removing lines from $G$ as long as $\tau(G') = \tau(G)$. We have that $G'$ consists of an independent set of lines from each of the points in the point cover $\tau(G')$ select one line. As $G'$ consists of an independent set of lines we have a maximum matching by König's Minimax Theorem which satisfies the implication. $\square$

If $X$ is any set of points in a graph $G$, we denote the set of all points which are adjacent to at least one point of $X$ as $\Gamma(X)$.

**Theorem 1.1.3..** (P. Hall's THeorem). Let $G = (A, B)$ be a bipartite graph. Then $G$ has a matching of $A$ into $B$ if and only if $|\Gamma(X)| \geq |X|$ for all $X \subseteq A$.

*Proof.* Let $G = (A, B)$ be a bipartite graph and suppose that there is a matching $A$ to $B$ and suppose that for some subgraph $X \subseteq A$ we have $|\Gamma(X)| < |X|$. If $X$ is any subgraph of $A$ and there exists a matching from $A$ to $B$ we have that there exists a matching from $X$ to $B$ as well. As $|\Gamma(X)| < |X|$ we have that there is at least 1 point in $X$ that is not adjacent to any point in $B$ hence it can not be a complete matching.

Suppose $|\Gamma(X)| \geq |X|$ for all $X \subseteq A$. Then using mathematical induction on $|A|$. If $|A| = 0$ then we would have a complete matching. If $|A| = 1$ then by our assumption we have that $|\Gamma(A)| \geq |A|$ hence there is a matching. Now suppose that for all $X \subset A$ $\square$