Matching Theory Notes

Collin McDevitt

Lemma 1.0.1. For any graph, G, $\alpha(G) + \tau(G) = |V(G)|$.

Proof. Let G be an arbitrary graph and let M be an arbitrary point cover where $|M| = \tau(G)$. Then $V(G) - M$ is an independent set of points. This is true because if $V(G) - M$ was not an independent set of points then there are at least two points $u, v \in V(G) - M$ where u and v are adjacent. If they are adjacent the line incident with u and v is not covered by M which is a contradiction. As $V(G) - M$ is an independent set of points we get the inequality

$$\alpha(G) \geq |V(G) - M| = |V(G)| - \tau(G)$$

Now assume that L is an arbitrary independent set of points where $|L| = \alpha(G)$. Then $V(G) - L$ is a point cover of G. If $V(G) - L$ is not a point cover of G then there exists a line l such that l is not covered by $V(G) - L$. However this would imply that l is incident with two points in L this is a contradiction on the fact L is independent points hence $V(G) - L$ is a point cover of G. As $V(G) - L$ is a point cover of G we get the inequality

$$\tau(G) \leq |V(G) - L| = |V(G)| - \alpha(G)$$

Combining both inequalities we get $\alpha(G) \geq |V(G)| - \tau(G)$ and $\tau(G) \leq |V(G)| - \alpha(G)$. This implies that

$$\alpha(G) + \tau(G) = |V(G)|$$

. □

Lemma 1.0.2. For any graph G with no isolated points, $v(G) + \rho(G) = |V(G)|$.

Proof. Let G be an arbitrary graph with no isolated points and let C be a line cover of G where $|C| = \rho(G)$. Let $\langle C \rangle$ be the graph formed from lines the set of lines C and the set of points $V(C)$. We have that $\langle C \rangle$ is a union of stars. This is because if $\langle C \rangle$ was not a union of stars. Then there would be two points in the graph that that are adjacent two one or more point in $\langle C \rangle$. If they are adjacent to a single point then removing either of the lines incident would create a smaller minimal line cover. If they are adjacent to two or more points then removing any of the incident lines would create a smaller minimal line cover. Now if we let n be the number of components in $\langle C \rangle$ we get that $n = |V(G)| - \rho(G)$ which arises because in each star there is one more point than there are lines. If we take one line from each star we get a matching hence

$$v(G) \geq |V(G)| - \rho(G)$$

Now let M be an arbitrary matching of G where $|M| = v(G)$, and U be the set of points that are not covered by M. We get that U is an independent set of points because if U was not an independent set of points then there would be two points $u, v \in U$ such that u and v are adjacent. If u and v are adjacent then there is a line incident with u and v that is not covered by M which is a contradiction. As U is an independent set of points and G has no isolated points we get that $|U| = |V(G)| - 2v(G)$. Let S be a line covering of U we get that $M \cup S$ is a line covering. Then we get the inequality

$$\rho(G) \leq |M \cup S| = v(G) + |V(G)| - 2v(G) = |V(G)| - v(G)$$

Combining both inequalities we get

$$v(G) + \rho(G) = |V(G)|$$

□

Exercise 1.0.3.

1. A minimal line cover is minimum if and only if it contains a maximum matching.

2. A maximal matching is maximum if and only if it is contained in a minimum line cover.

Proof. (1)

Let L be a line cover where $|L| = \rho(G)$. As L is a minimum line cover we have that it forms the lines from a union of stars on G. We have that the number of stars is $v(G)$ which comes from selecting one matching from each star. Hence L contains a maximum matching.

Let L be a minimal line cover that contains a maximum matching. Then as L is a union of stars we get a maximum matching by selecting a line from each star. Hence $|L| = |V(G)| - v(G) = \rho(G)$ hence it is a minimum line cover. □

Proof. (2) Let M be a maximum matching. Let L be an line cover obtained by taking an arbitrary line covering of the edges incident with $V(G) - V(M)$ and M. We have that $|L| = |V(G)| - 2v(G) + v(G) = \rho(G)$ hence L contains a minimum line cover.

Let M be a maximal matching that is contained in a minimum line cover L. Then we have that the matchings in M are formed from individual lines of the stars formed by L. Hence $|M| = V(G) - \rho(G) = v(G)$ hence M is a maximum matching. □

Exercise 1.0.4. For any graph G, $v(G) \leq \tau(G)$

Proof. Let G be an arbitrary graph and let M be a maximum matching in G. As M is a maximum matching we get that it is contained in a minimum line cover L. As L is a minimum line cover we have that it is the union of stars where each matching in M comes from a single line in each star. We get that the number of stars is $\tau(G)$ hence $v(G) \leq \tau(G)$ □

Theorem 1.1.1. König's Minimax Theorem: If G is bipartite, then $v(G) = \tau(G)$

Proof. Remove lines from G as long as possible while keeping $\tau(G)$ the same. Denote the resulting minimal graph by G'. Hence $\tau(G') = \tau(G)$, but for any line $l \in E(G)$ we have $\tau(G' - l) < \tau(G)$. Suppose that G' does not consist of independent lines. Then there are two lines $l_1, l_2 \in E(G')$ that are both incident with a point $a \in V(G')$. As G' is minimal we have that there exists a point cover of $G' - x$ we denote this point cover by S_x where $|S_x| = \tau(G') - 1$. We also have the point cover S_y which is covering $G' - y$ with $|S_x| = |S_y|$. Form the induced subgraph G'' of G', $G'' = G'[\{a\} \cup S_x \cup S_y \cap \overline{S_x \cap S_y}]$ and let $t = |S_x \cap S_y|$. Then we have $|V(G'')| = 2(\tau(G') - 1 - t) + 1$. As G'' is a subgraph of a bipartite graph we have that it is bipartite. Hence we have a set T which is a point cover of G'' where $|T| = \tau(G') - 1 - t$. We claim that the set of points $T' = T \cup (S_x \cap S_y)$ covers G'. Let z be a line in G' if $z \neq x$ or $z \neq y$ then we have that z is covered by S_x and S_y hence it is covered by $S_x \cap S_y$ or it connects $S_x - S_y$ to $S_y - S_x$. If $z = x$ or $z = y$ then we have it is covered by T. So $\tau(G') \leq |T'| = |T \cup (S_x \cap S_y)| = |T| + |S_x \cap S_y| \leq \tau(G') - 1 - t + t = \tau(G') - 1$ a contradiction. Hence G' is an independent set of lines. Hence

$$\tau(G) = \tau(G') = v(G') \leq v(G)$$

□

Theorem 1.1.2. In any (possibly infinite) bipartite graph there exists a matching M and a point cover P such that every line in M contains exactly one point in P and every point in P is contained in exactly one line of M.

Proof. Let G be an arbitrary bipartite graph and let L be a minimum point cover of G. From each point in the point cover select one line that is incident with it. Assume that this is not a matching then there are two points that are incident □