Matching Theory Notes

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Lemma 1.0.1. For any graph, G $\alpha(G)$, $\alpha(G) + \tau(G) = |V(G)|$.

Proof. Let G be an arbitrary graph and let M be an arbitrary point cover where $|M| = \tau(G)$. Then $V(G) - M$ is an independent set of points. This is true because if $V(G) - M$ was not an independent set of points then there are at least two points $u, v \in V(G) - M$ where u and v are adjacent. If they are adjacent the line incident with u and v is not covered by M which is a contradiction. As $V(G) - M$ is an independent set of points we get the inequality

$$\alpha(G) \geq |V(G) - M| = |V(G)| - \tau(G)$$

Now assume that L is an arbitrary independent set of points where $|L| = \alpha(G)$. Then $V(G) - L$ is a point cover of G. If $V(G) - L$ is not a point cover of G then there exists a line l such that l is not covered by $V(G) - L$. However this would imply that l is incident with two points in L this is a contradiction on the fact L is independent points hence $V(G) - L$ is a point cover of G. As $V(G) - L$ is a point cover of G we get the inequality

$$\tau(G) \leq |V(G) - L| = |V(G)| - \alpha(G)$$

Combining both inequalities we get $\alpha(G) \geq |V(G)| - \tau(G)$ and $\tau(G) \leq |V(G)| - \alpha(G)$. This implies that

$$\alpha(G) + \tau(G) = |V(G)|$$

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Lemma 1.0.2. For any graph G with no isolated points, $v(G) + \rho(G) = |V(G)|$.

Proof. Let G be an arbitrary graph with no isolated points and let C be a line cover of G where $|C| = \rho(G)$. Let $\langle C \rangle$ be the graph formed from lines the set of lines C and the set of points $V(C)$. We have that $\langle C \rangle$ is a union of stars. This is because if $\langle C \rangle$ was not a union of stars. Then there would be two points in the graph that that are adjacent two one or more point in $\langle C \rangle$. If they are adjacent to a single point then removing either of the lines incident would create a smaller minimal line cover. If they are adjacent to two or more points then removing any of the incident lines would create a smaller minimal line cover. Now if we let n be the number of components in $\langle C \rangle$ we get that $n = |V(G)| - \rho(G)$ which arises because in each star there is one more point than there are lines. If we take one line from each star we get a matching hence

$$v(G) \geq |V(G)| - \rho(G)$$

Now let M be an arbitrary matching of G where $|M| = v(G)$, and U be the set of points that are not covered by M. We get that U is an independent set of points because if U was not an independent set of points then there would be two points $u, v \in U$ such that u and v are adjacent. If u and v are adjacent then there is a line incident with u and v that is not covered by M which is a contradiction. As U is an independent set of points and G has no isolated points we get that $|U| = |V(G)| - 2v(G)$. Let S be a line covering of U we get that $M \cup S$ is a line covering. Then we get the inequality

$$\rho(G) \leq |M \cup S| = v(G) + |V(G)| - 2v(G) = |V(G)| - v(G)$$

Combining both inequalities we get

$$v(G) + \rho(G) = |V(G)|$$

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