Concise Description of the Sequences of Conformal Iterates of the Unit Disk

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1 Prerequisite Tools

Theorem 1 (Iterates of $\mathbb{D}$)**.** *Let $f : \mathbb{D} \to \mathbb{D}$ be a biholomorUwU, that is bijective and holomorphic (hence conformal) self map of $\mathbb{D}$, then f takes the form*

$$f(z) = e^{i\theta}\frac{z-a}{1-\overline{a}z}$$

Where $a \in \mathbb{D}$ and $\theta \in \mathbb{R}$

We wish to fully explain the condition for convergence of the sequence of iterates of functions of this form. The reason for doing so, is due to the Riemann mapping theorem if we can solve this problem on this domain, we will be able to fully understand the iterates for any domain of $\mathbb{C}$.

Theorem 2 (Reimann Mapping Theorem)**.** *Let $\Omega \subset \mathbb{C}$ be a domain of $\mathbb{C}$, then for all $z_0 \in \Omega$ there exists some biholomorUwU $F : \Omega \to \mathbb{D}$ such that $F(z_0) = 0$ and $F'(z_0) > 0$*

Corollary 3. *Any two domains of $\mathbb{C}$, say Ω_1 Ω_2 are conformally equivalent, meaning there exists some biholomorUwU between them.*

To see why, let F_i be the biholomorUwUs from Ω_i to $\mathbb{D}$, then clearly $F_2^{-1} \circ F_1 : \Omega_1 \to \Omega_2$ and clearly a biholomorUwU

Theorem 4 (The Hyperbolic Plane)**.** *Consider the metric space $\mathcal{D} = (\mathbb{D}, d_{\mathbb{D}})$ where $d_{\mathbb{D}}$ is defined by*

$$d_{\mathbb{D}}(z,w) = 2\tanh^{-1}\left|\frac{z-w}{1-z\overline{w}}\right|$$

We call this this Hyperbolic Plane.

The key to this space is that the biholomorUwUs of $\mathbb{D}$ are the isometries of $\mathcal{D}$, this fact we will exploit to study the sequences of iterates of the biholomorUwUs of $\mathbb{D}$

2 Iterates on $\mathbb{D}$

First, setting $f(z) = z$ and solving we will find a quadratic equation, hence these biholomorUwUs have at most two unique fixed points.

Theorem 5. *Let f be a biholomorUwU of $\mathbb{D}$, if we have a fixed point $z^* \in \mathbb{D}$ then the sequence defined by $z_n = f(z_{n-1})$ with $z_0 \in \mathbb{D} \setminus \{z^*\}$ will not converge.*

Proof. We noted before that f must be a hyperbolic isometry, we know that these sequences of iterates must only converge to fixed points, however

$$d_{\mathbb{D}}(f(z_n), f(z^*)) = d_{\mathbb{D}}(z_n, z^*) = d_{\mathbb{D}}(z_0, z^*) = K$$

From this one may see, $d_{\mathbb{D}}(z_n, z^*) = 2\tanh^{-1}\left|\frac{z_n - z^*}{1-\overline{z^*}z_n}\right| = K$ which we then see

$$d_E(z_n, z^*) = K'|1 - \overline{z^*}z_n|$$

We cannot have convergence as the RHS of this equation does not tend to zero. For $w, z \in \mathbb{D}$ clearly $|w||z| < 1$ so the term $|1 - \overline{z^*}z_n|$ is bounded below by some constant $c \in \mathbb{R}^+$ $\square$

Theorem 6. *Let $f : \mathbb{D} \to \mathbb{D}$ be a biholomorUwU of $\mathbb{D}$ in the usual form, then the sequence of iterates converges if and only if $|a| > \sin(\theta/2)$*

Proof. $\square$