Homework 1 MTH5210

March 12, 2022

1. Let Y_1, Y_2, Y_3, Y_4 be i.i.d. $\mathcal{N}(0,1)$ random variables. Let $X_1 = (Y_1 + Y_2)/\sqrt{2}$, $X_2 = (Y_1 + Y_2 + Y_3 + Y_4)/2$, and $X_3 = (Y_1 - Y_2)/\sqrt{2}$. State with reason if the following is true.

(a) X_1, X_2 and X_3 are identically distributed $\mathcal{N}(0,1)$

This is true because $\mathcal{N}(\mu_X, \sigma_X^2) \pm \mathcal{N}(\mu_Y, \sigma_Y^2) \sim \mathcal{N}(\mu_X \pm \mu_Y, \sigma_X^2 + \sigma_Y^2)$

(b) (X_1, X_2, X_3) is a Gaussian vector

Yes

(c) X_1 and X_2 are independent

False (d) X_1 and X_3 are independent

True

7. Let B_t and W_t be two independent Brownian. Show that $X_t = (B_t + W_t)/\sqrt{2}$ is also a Brownian motion. Find the covariance between B_t and X_t
For a process to be a brownian motion it must, be a continuous function of t, have normal increments and independent increments.

First it is clear that X_t is a continuous function of t as it is the sum of two continuous functions in t. It can also be shown that X_t has normal increments for $t > s$

$$X_t - X_s = (W_t + B_t)/\sqrt{2} - (W_s + B_s)/\sqrt{2} = (W_t - W_s + B_t - B_s)/\sqrt{2}$$

Since W_t and B_t are independent brownian motions we have $X_t - X_s \sim \mathcal{N}(0, (t-s)) + \mathcal{N}(0, t-s))/\sqrt{2}$ which then we find

$$X_t - X_s \sim \frac{1}{\sqrt{2}} \mathcal{N}(0, 2(t-s)) \sim \mathcal{N}(0, t-s)$$

Then finally for independent increments, consider $X_{t+s} - X_t$ we must show that this is independent of $u \leq t$

$$X_{t+s} - X_t = \frac{1}{\sqrt{2}}(W_{t+s} - W_t) + \frac{1}{\sqrt{2}}(B_{t+s} - B_t)$$

Both $W_{t+s} - W_t$ and $B_{t+s} - B_t$ are independent of W_u and B_u and since W_t and B_t are independent of each other $\frac{1}{\sqrt{2}}(W_{t+s} - W_t) + \frac{1}{\sqrt{2}}(B_{t+s} - B_t)$ is independent of $\frac{1}{\sqrt{2}}(W_u + B_u)$ for $u \leq t$ hecne X_t is independent of all X_u for $u \leq t$
Now calculating the covariance of B_t and X_t

$$cov(X_t, B_t) = \mathbb{E}[X_t B_t] - \mathbb{E}[X_t]\mathbb{E}[B_t]$$

Then using the definition of X_t

$$cov(X_t, B_t) = \frac{1}{\sqrt{2}}\mathbb{E}[B_t^2 + B_t W_t] - \frac{1}{\sqrt{2}}\mathbb{E}[W_t + B_t]\mathbb{E}[B_t]$$

By linearity of the expectation

$$cov(X_t, B_t) = \frac{1}{\sqrt{2}}(\mathbb{E}[B_t^2] - \mathbb{E}[B_t]^2 + \mathbb{E}[B_t W_t] - \mathbb{E}[W_t]\mathbb{E}[B_t])$$

Then finally we have

$$cov(X_t, B_t) = \frac{1}{\sqrt{2}}(Var(B_t) + cov(W_t, B_t)) = \sqrt{2}t$$