Taylor Series or Taylor Polynomial

Given $f(x)$, we approximate $f(x)$ around point $x=a$, such that

$$P_n(a) = f(a)$$
$$P_n'(a) = f'(a)$$
$$\vdots$$
$$P_n^{(n)}(a) = f^{(n)}(a)$$

Then, the polynomial P (of degree n) is given by

$$\boxed{P_n(x) = f(a) + (x-a)f'(a) + \frac{(x-a)^2}{2!}f^{(2)}(a) + \frac{(x-a)^3}{3!}f^{(3)}(a) + \cdots + \frac{(x-a)^n}{n!}f^{(n)}(a)}$$

The error can be bounded by

$$R_n(x) = \frac{(x-a)^{n+1}}{(n+1)!}f^{(n+1)}(c_x)$$

when c_x is between a and x

Thus

$$f(x) = P_n(x) + R_n(x)$$

- This can be used to find (estimate) how many terms are needed to achieve desired accuracy

Bisection Method

Let $f(x)$ be continuous on $[a, b]$ and $f(a).f(b) < 0$

Then, ~~we take~~ ~~$a = (a+b)/2$ (the midpoint)~~

For required precision ϵ we repeat the fllg

B1 : Let $c = (a+b)/2$ (midpoint)

B2 : If $b - c \leq \epsilon$ then c is root. STOP!

B3 : Else If ~~sign~~ $f(b).f(c) \leq 0$ then $a = c$

else $b = c$, return to B1

Every iteration the interval is halved.

Number of iterations to achieve ϵ is given by

$$\boxed{n = \log\left(\frac{b_0 - a_0}{\epsilon}\right) / \log 2}$$

Newton's Method

$$\boxed{x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}}$$

Converges quadratically (order = 2) if $f'(\alpha) \neq 0$ — if $f'(\alpha) = 0$ then linear!

$$\boxed{(\alpha - x_{n+1}) = (\alpha - x_n)^2 \left[-\frac{f''(\alpha)}{2f'(\alpha)} \right]}$$

when x_n close to α

let $M = -\dfrac{f''(\alpha)}{2f'(\alpha)}$

then

$$M(\alpha - x_n) \approx \left[M(\alpha - x_0) \right]^{2^n}$$

Interval of convergence is given by

$$|\alpha - x_0| < \frac{1}{|M|} = \left| \frac{2f'(\alpha)}{f''(\alpha)} \right|$$

number of iterations required to achieve ϵ,

$$\boxed{n \propto \log \log \frac{1}{\epsilon}}$$

Secant Method

$$\boxed{x_{n+1} = x_n - f(x_n) \frac{x_n - x_{n-1}}{f(x_n) - f(x_{n-1})}}$$

Converges with order = 1.62

$$\alpha - x_{n+1} = (\alpha - x_n)(\alpha - x_{n-1}) \left[-\frac{f''(\alpha)}{2f'(\alpha)} \right]$$

$$|\alpha - x_{n+1}| \approx c\,|\alpha - x_n|^{1.62} \qquad \text{where } c = \left| \frac{f''(\alpha)}{2f'(\alpha)} \right|^{0.62}$$

Fixed Point Iteration

- Iterations ~~[illegible]~~ of the form

$$x_{n+1} = g(x_n)$$

converges to fixed point α such that $\alpha = g(\alpha)$

- Newton's Method is a particular case of this.

- Converges if $|g'(\alpha)| < 1$

  Diverges if $|g'(\alpha)| > 1$

  if $|g'(\alpha)| = 1$ then it could either converge or diverge, <u>very slowly</u>

- If $g'(\alpha) \neq 0$ then convergence is <u>linear</u> given by

$$\boxed{(\alpha - x_{n+1}) \approx g'(\alpha)(\alpha - x_n)}$$

$\lambda = g'(\alpha)$ is rate of convergence

- If $g'(\alpha) = 0$ then higher order convergence is possible.

  If $g''(\alpha) \neq 0$ it is quadratic

  If $g''(\alpha) = 0$ and $g'''(\alpha) \neq 0$ then cubic

  and so on ....

Order of convergence

$$|\alpha - x_{n+1}| \le c\,|\alpha - x_n|^p$$

then p is order of convergence.

$$g(x_n) = g(\alpha) + (x_n - \alpha)\, g'(\alpha) + \frac{(x_n-\alpha)^2}{2}\, g''(\alpha)$$

$$+ \cdots$$

So whenever we find first p such that

$g^{(p)}(\alpha) \neq 0$ then

$$g(x_n) - g(\alpha) = x_{n+1} - \alpha = \frac{(x_n - \alpha)^p}{p!} \cdot g^{(p)}(\alpha)$$

hence p is the order of convergence.

Multiplicity of Roots (for Newton's Method)

$f(x)$ has multiple root at α

if $f(x) = (x-\alpha)^m h(x)$, $h(\alpha) \neq 0$

then m = multiplicity , ~~$h(\alpha) \neq 0$~~

Equivalently,

$f(\alpha) = f'(\alpha) = \cdots f^{(m-1)}(\alpha) = 0$ and $f^{(m)}(\alpha) \neq 0$

then m = multiplicity

When $m \geq 2$ convergence of Newton's Method is linear ~~rate~~

Rate of convergence $\lambda = \frac{m-1}{m}$

Thus
$$(\alpha - x_n) \simeq \lambda (\alpha - x_{n-1})$$

Stability of Roots

Let $F_\epsilon(x) = f(x) + \epsilon g(x)$

where $\epsilon g(x)$ is perturbation

let α be root of $f(x)$ ~~the~~ $\alpha = \alpha(0)$

~~the~~ and $\alpha(\epsilon)$ be root of $F_\epsilon(x)$

then
$$\boxed{\alpha(\epsilon) \simeq \alpha(0) - \epsilon \frac{g(\alpha(0))}{f'(\alpha(0))}}$$

Interpolation

Given $(x_0, y_0) \dots (x_n, y_n)$

Lagrange's Formula

$$y = P_n(x) = y_0 L_0(x) + y_1 L_1(x) + \dots y_n L_n(x)$$

where
$$L_i(x) = \frac{(x-x_0)(x-x_1)\dots(x-x_{i-1})(x-x_{i+1})\dots(x-x_n)}{(x_i-x_0)\dots(x_i-x_{i-1})(x_i-x_{i+1})\dots(x_i-x_n)}$$

Newton's Interpolation

$$P_n(x) = f(x_0) + (x-x_0)\, f[x_0, x_1] + (x-x_0)(x-x_1)\, f[x_0\, x_1\, x_2] + \ldots\ldots (x-x_0)(x-x_1)\ldots(x-x_{n-1})\, f[x_0\, x_1 \ldots x_n]$$

where <u>divided difference</u>

$$f[x_0\, x_1] = \frac{f(x_1) - f(x_0)}{x_1 - x_0}$$

$$f[x_0 \ldots x_n] = \frac{f[x_1 \ldots x_n] - f[x_0 \ldots x_{n-1}]}{x_n - x_0}$$

Divided differences are calculated using the pyramid structure.

$$f[x_0\, x_0] = f'(x_0)$$

$$f[x_0\, x_0\, x_0] = \frac{f''(x_0)}{2!}$$

$$f[x_0\, x_1\, x_2\, x_3] = f[x_1\, x_0\, x_3\, x_2] = f[x_0\, x_2\, x_1\, x_3]$$

(the order doesn't matter)