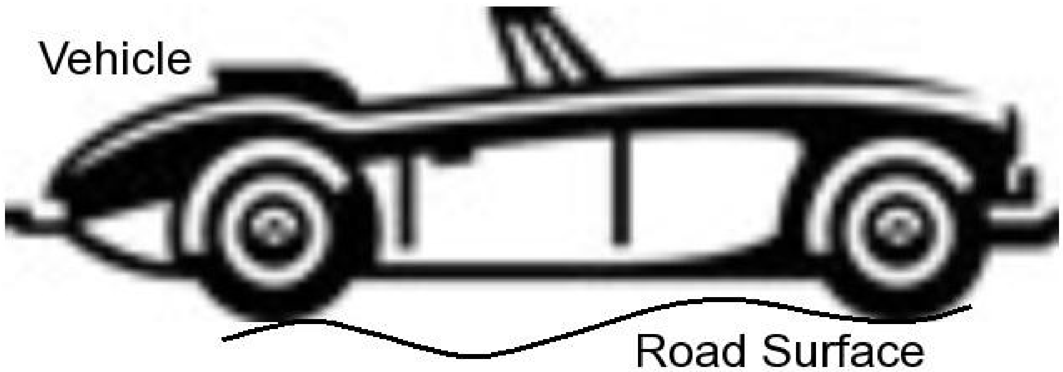
Vehicle Suspension Modeling

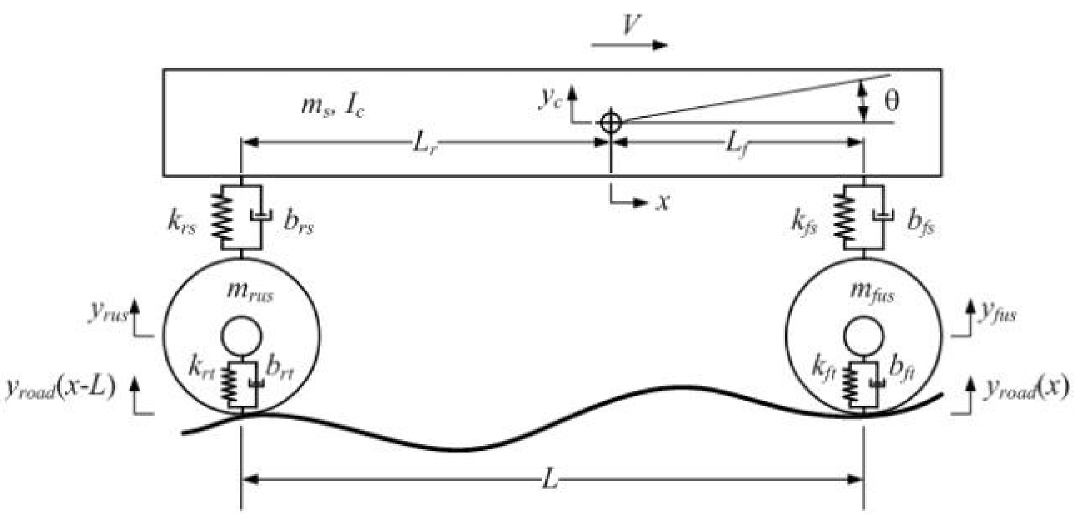
Dr. Javier A. Kypuros
Vehicle Systems Modeling and Control
Department of Mechanical Engineering
The University of Texas – Pan American

Reasons for Modeling Vehicle Suspensions

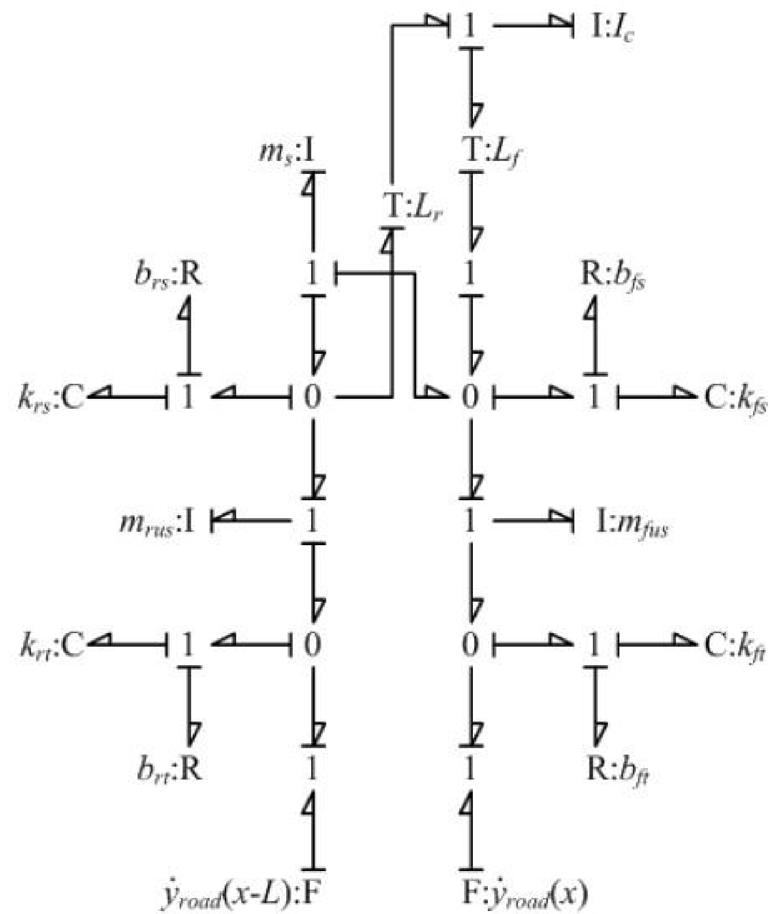
- Virtual suspension tuning
  - Spring rate
  - Shock absorber damping constant
  - Tire wall stiffness and air pressure
  - Mass of unsprung weight (wheel, rotor, brakes, etc.)
- Design: suspension travel and geometry
- Frequency response analysis

Half-Car Suspension
Component Modeling



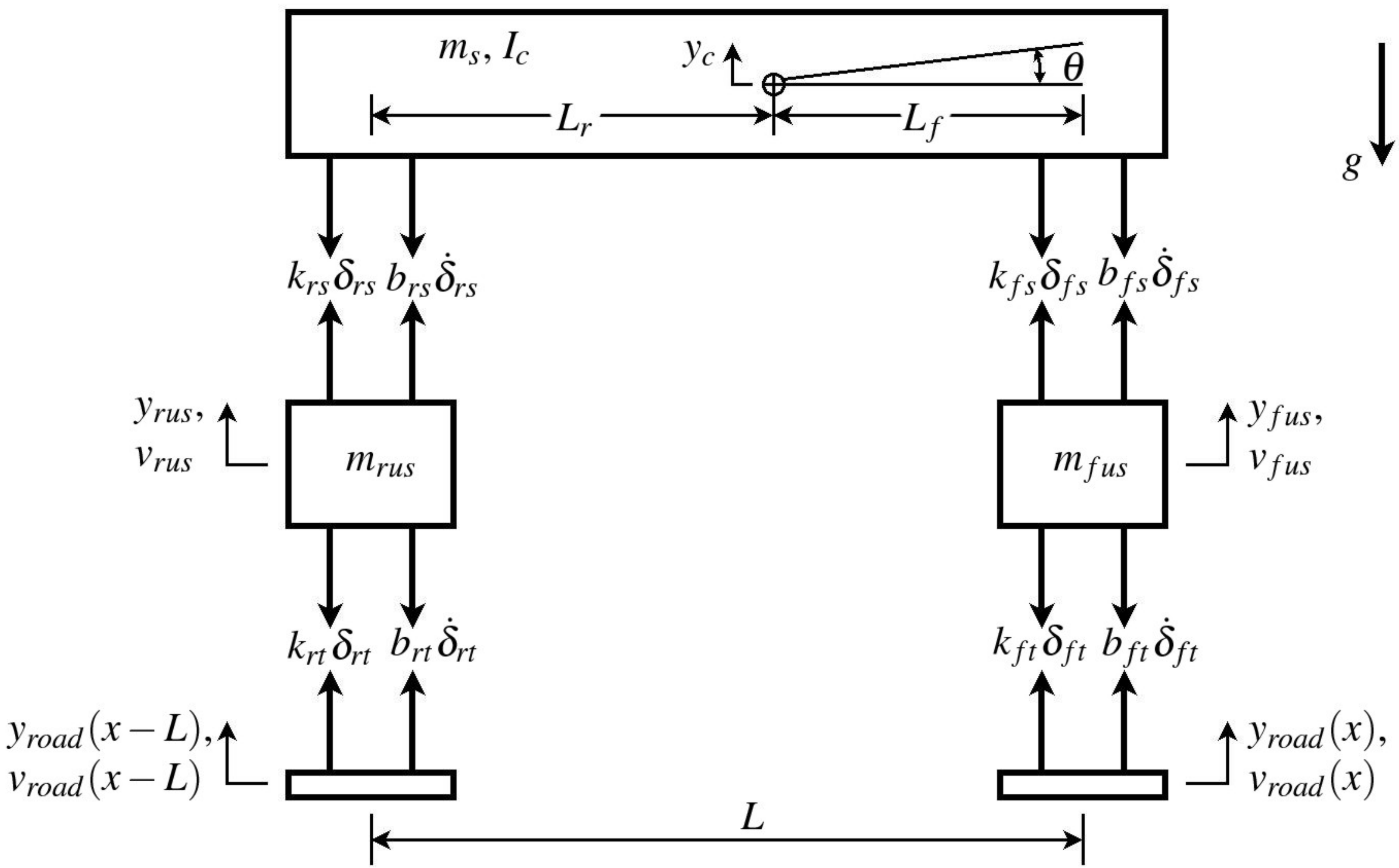


2D Vehicle Suspension Schematic

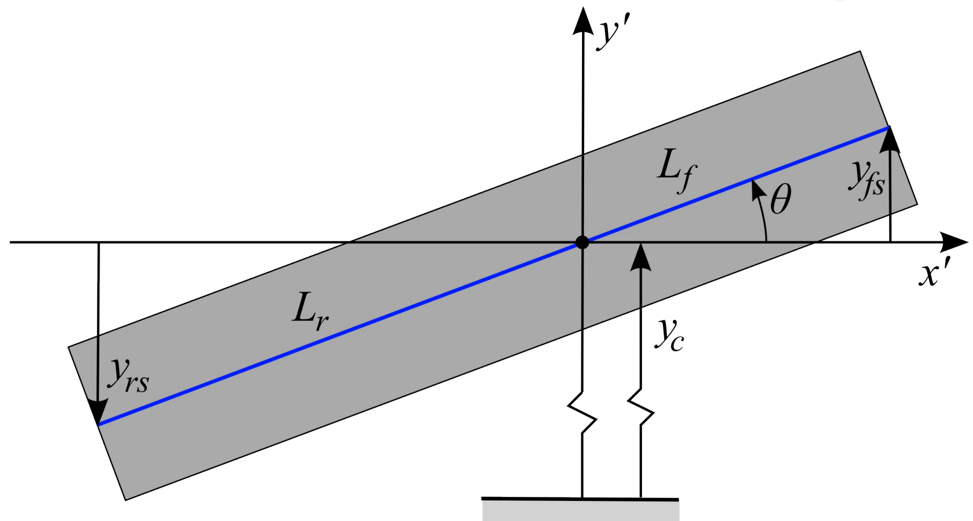


Bond Graph for 2D Vehicle Suspension Model

Half-Car Suspension
FBD



Half-Car Suspension
Kinematic Constraint Equations



From the kinematic constraints, the diplacements from equilibrium of the front and rear of the vehicle are

$$y_{fs} = y_c + L_f \sin\theta \;\text{ and }\; y_{rs} = y_c - L_r \sin\theta.$$

The transverse velocities of the front and rear of the vehicle are derived by differentiating the above equations :

$$v_{fs} = \dot{y}_{fs} = v_c + L_f \cos\theta\,\omega \text{ and } v_{rs} = \dot{y}_{rs} = v_c - L_r \cos\theta\,\omega.$$

Apply a small angle approximation the approximate velocities are

$$v_{fs} \approx v_c + L_f\omega \;\text{ and }\; v_{rs} \approx v_c - L_r\omega.$$

Half-Car Suspension
Mathematical Modeling

Assuming small amplitude pitch about the center of mass the state-space representation is:

$$
\begin{bmatrix} \dot{\delta}_{ft} \\ \dot{p}_{fus} \\ \dot{\delta}_{fs} \\ \dot{h}_c \\ \dot{p}_c \\ \dot{\delta}_{rs} \\ \dot{p}_{rus} \\ \dot{\delta}_{rt} \end{bmatrix}
=
\begin{bmatrix}
0 & \frac{1}{m_{fus}} & 0 & 0 & 0 & 0 & 0 & 0 \\
-k_{ft} & -\frac{b_{ft}+b_{fs}}{m_{fus}} & k_{fs} & \frac{b_{fs}L_f}{I_c} & \frac{b_{fs}}{m_s} & 0 & 0 & 0 \\
0 & -\frac{1}{m_{fus}} & 0 & \frac{L_f}{I_c} & \frac{1}{m_s} & 0 & 0 & 0 \\
0 & \frac{b_{fs}L_f}{m_{fus}} & -k_{fs}L_f & -\frac{b_{rs}L_r^2+b_{fs}L_f^2}{I_c} & \frac{b_{rs}L_r-b_{fs}L_f}{m_s} & k_{rs}L_r & -\frac{b_{rs}L_r}{m_{rus}} & 0 \\
0 & \frac{b_{fs}}{m_{fus}} & -k_{fs} & \frac{b_{rs}L_r-b_{fs}L_f}{I_c} & -\frac{b_{rs}+b_{fs}}{m_s} & -k_{rs} & \frac{b_{rs}}{m_{rus}} & 0 \\
0 & 0 & 0 & -\frac{L_r}{I_c} & \frac{1}{m_s} & 0 & -\frac{1}{m_{rus}} & 0 \\
0 & 0 & 0 & -\frac{b_{rs}L_r}{I_c} & \frac{b_{rs}}{m_s} & k_{rs} & -\frac{b_{rs}+brt}{m_{rus}} & -k_{rt} \\
0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{m_{rus}} & 0
\end{bmatrix}
\begin{bmatrix} \delta_{ft} \\ p_{fus} \\ \delta_{fs} \\ h_c \\ p_c \\ \delta_{rs} \\ p_{rus} \\ \delta_{rt} \end{bmatrix}
+
\begin{bmatrix} -1 & 0 \\ b_{ft} & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & b_{rt} \\ 0 & -1 \end{bmatrix}
\begin{bmatrix} v_{road}(x) \\ v_{road}(x-L) \end{bmatrix}
$$

$$
\mathbf{y} =
\begin{bmatrix} \delta_{ft} \\ \delta_{fs} \\ \dot{\theta} \\ \dot{y}_c \\ \delta_{rs} \\ \delta_{rt} \end{bmatrix}
=
\begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & \frac{1}{I_c} & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & \frac{1}{m_s} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1
\end{bmatrix}
\begin{bmatrix} \delta_{ft} \\ p_{fus} \\ \delta_{fs} \\ h_c \\ p_c \\ \delta_{rs} \\ p_{rus} \\ \delta_{rt} \end{bmatrix}
$$

From the outputs, the velocity of the center of mass and the pitch velocity, can be integrated to get the vertical motion and pitch (i.e. θ and y_c).

Time Domain Responses

First-Order Systems

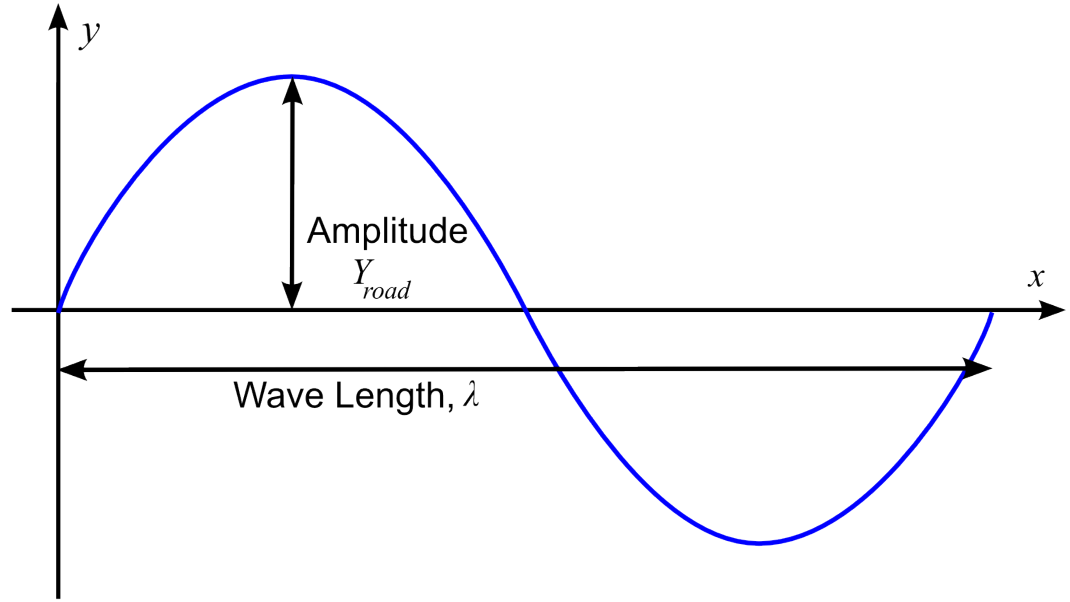
Time Domain Responses
Second-Order Systems

Time Domain Responses
Higher-Order Systems

Half-Car Suspension
Time-Domain Responses

Road Surface Inputs

- When conducting frequency response analysis, it is useful to expose the suspension model to a bandwidth of frequency inputs
- This can be simulated by exciting the vehicle suspension model with fixed amplitude oscillations of different frequencies
- The frequency of the oscillations can be varied 1 of 2 ways:
  - Constant wave length and variable vehicle speed
  - Constant vehicle speed and variable wave length
- This procedure is similar to a sine-sweep test in electronic circuits



The wave number is $k = 2\pi/\lambda$. The road profiles at the front and rear axles in terms of the wave number are

$$y_{road}(x) = Y_{road}\sin(kx) \quad \text{and} \quad y_{road}(x-L) = Y_{road}\sin\left(k(x-L)\right)$$

where

$$x = \int V\,dt = Vt$$

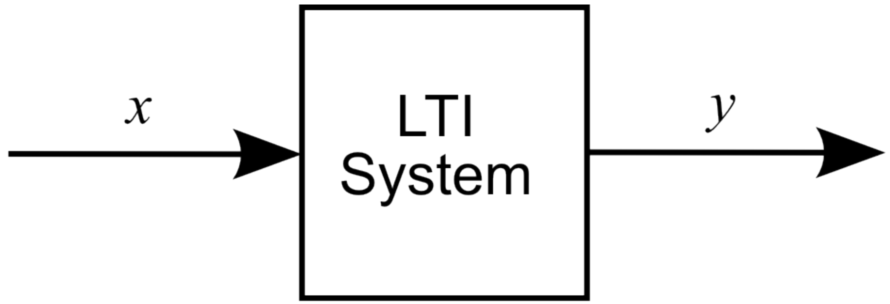
because the vehicle velocity, V, is assumed to be constant. Thus the vertical velocity inputs at the front and rear axle are

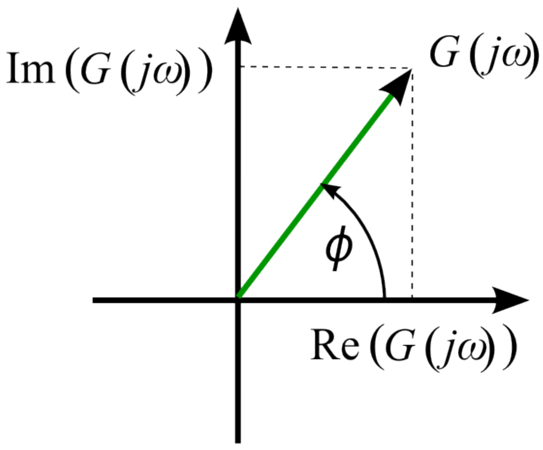
$$v_{road}(x) = \dot{y}_{road}(x) = Y_{road}kV\cos(kVt) = Y_{road}kV\cos(kVt)$$

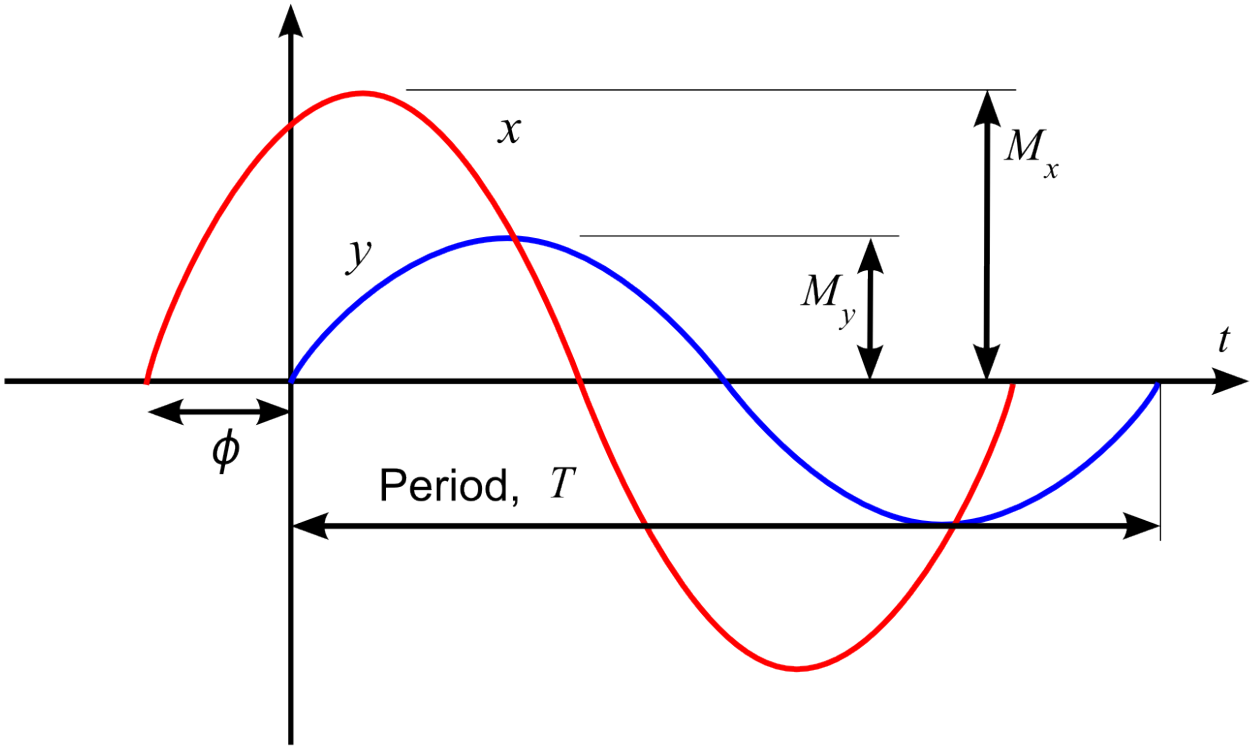
and

$$v_{road}(x-L) = \dot{y}_{road}(x-L) = Y_{road}kV\cos\left(k(Vt-L)\right) = Y_{road}kV\cos\left(k(Vt-L)\right).$$

Sinusoidal Frequency Response







Magnitude and Phase Angle

$$M(\omega) = |G(j\omega)| = \frac{M_y(\omega)}{M_x(\omega)} = \sqrt{\mathrm{Re}(G(j\omega))^2 + \mathrm{Im}(G(j\omega))^2}$$

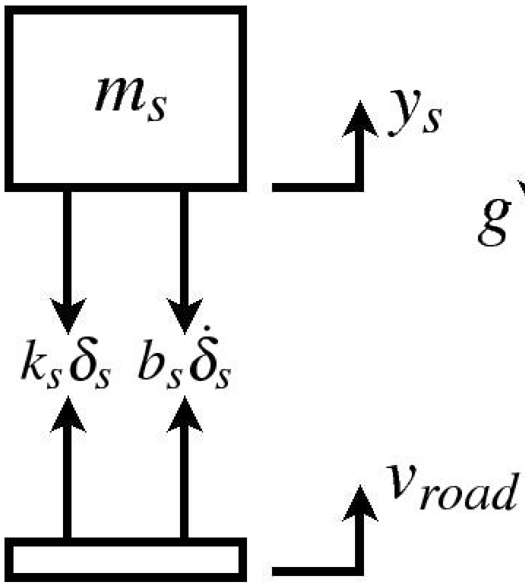
$$\phi(\omega) = \angle G(j\omega) = \phi_y(\omega) - \phi_x(\omega) = \tan^{-1}\left[\frac{\mathrm{Im}(G(j\omega))}{\mathrm{Re}(G(j\omega))}\right]$$

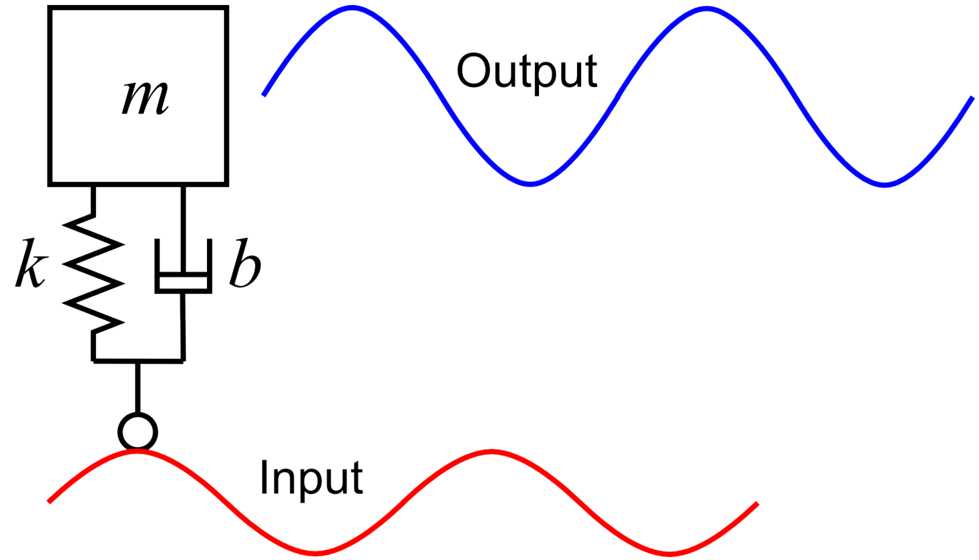
- The ***Magnitude*** is the ratio, in decibels ($\mathrm{dB} = 20 \log M$), of the output signal amplitude to the input signal amplitude.
- The ***Phase Angle,*** relative to the output signal, is the fraction of a cycle, in degrees, that has gone by since the input signal passed through a given value.

A Sine Sweep Test

- Basic procedure is to input a sinusoid of known frequency and amplitude to excite the system
- Measure the output amplitude and phase shift
- If the system is linear, the output sinusoid frequency should be the same as the input
- Repeat the procedure for a range of frequency inputs
- Results plotted on a semi-logarithmic plot
- Magnitude (amplitude) ratio, $M(\omega)$, converted to dB and phase angle, $\varphi(\omega)$, in degrees
- Maintain the amplitude constant
- Refer to Bode Plot Notes at class Web site for more details

Motorcycle Suspension
Sine Sweep Test (1 of 3)



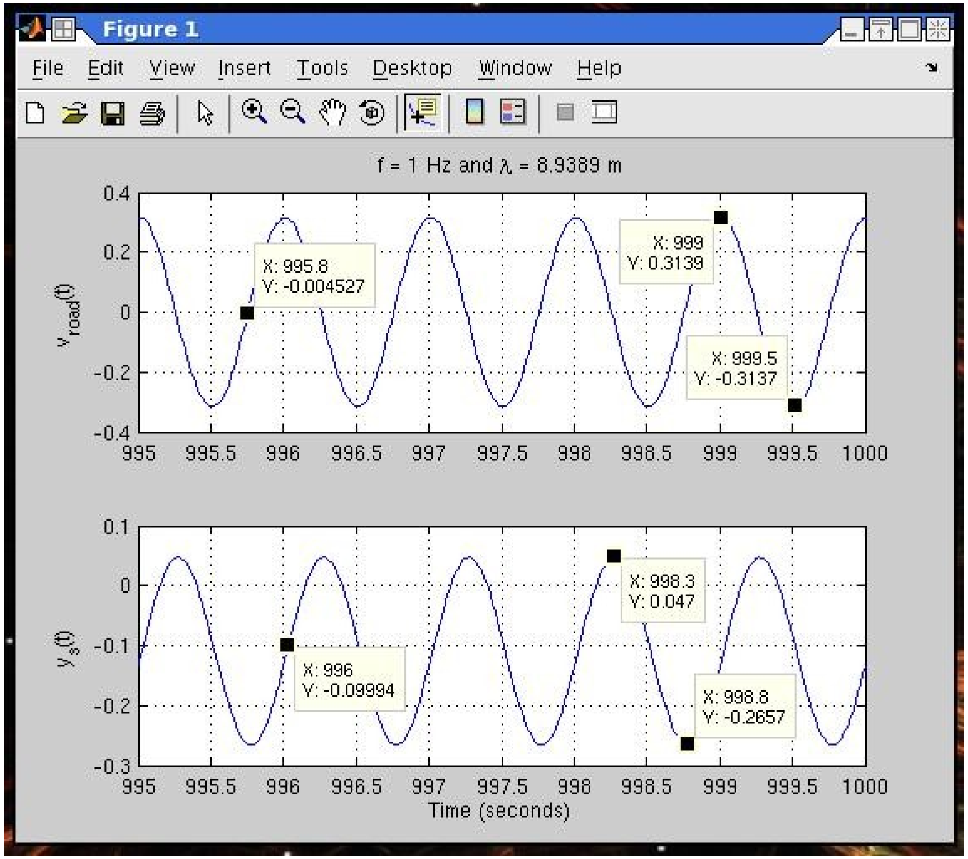


$$\frac{Y_s(s)}{Y_{road}(s)} = \frac{s\,V_s(s)}{s\,V_{road}(s)} = \frac{V_s(s)}{V_{road}(s)} = \frac{bs+k}{ms^2+bs+k}$$

Note that though the transfer function is in terms of the velocities, the output/input relation in terms of displacements is the same because the velocities are related to the displacements through derivatives.

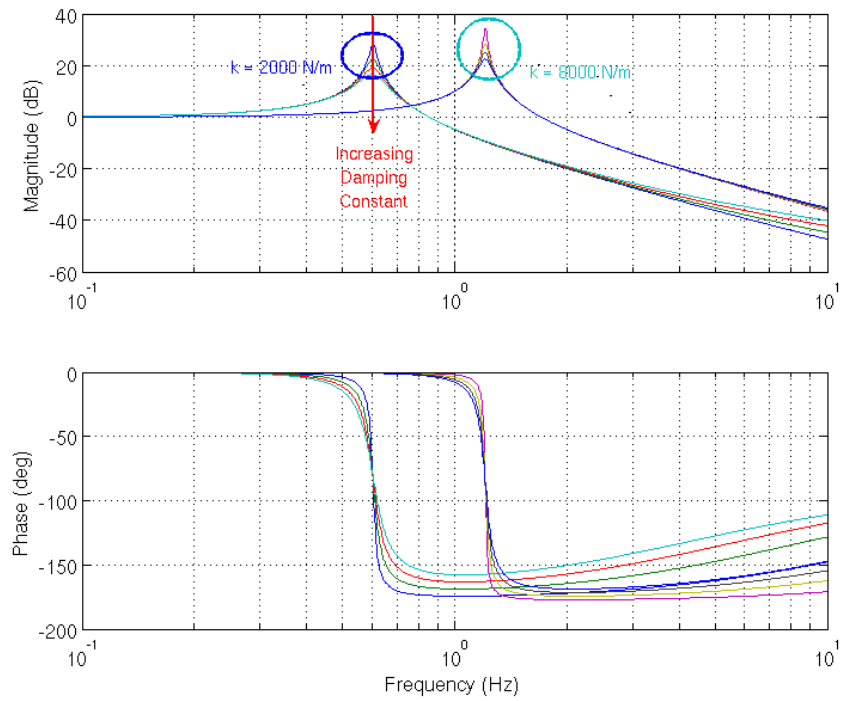
Motorcycle Suspension
Sine Sweep Test (2 of 3)

- Assuming constant velocity, the frequency of oscillation is $\omega = kV$ (rad/sec) and $f = kV/2\pi$
- Thus the frequency can be set by setting V or k
  - $V = 2\pi f / k$
  - $k = 2\pi f / V$
- Magnitude and phase angle can be measured from simulated results



Motorcycle Suspension
Sine Sweep Test (3 of 3)

- Increasing the damping constant, b, decreases the magnitude amplification at the natural frequency
- Increasing the spring stiffness, k, increases the frequency at which the natural frequency occurs
- Video of response at resonant frequency
- Video of response at higher frequency

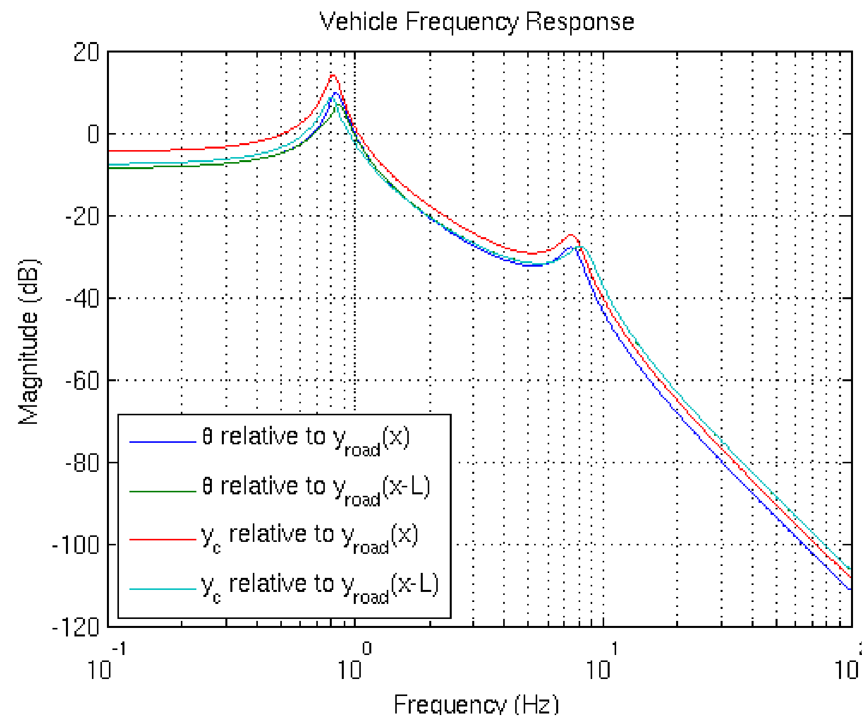


Notes on Sinusoidal Frequency Response

- Recall that M is the ratio of the output amplitude to the input amplitude
- Also recall that in Bode plots magnitude is in dB; because $20 \log_{10}(1) = 0$,
  - $\mathrm{dB} > 0$ implies $M > 1$
  - $\mathrm{dB} < 0$ implies $M < 1$
- Spikes in magnitude will occur in and around the natural frequencies of the system
- You must be concerned with frequency ranges where the magnitude spikes above $0\ \mathrm{dB}$

Half Car Suspension
Vehicle Sinusoidal Frequency Response

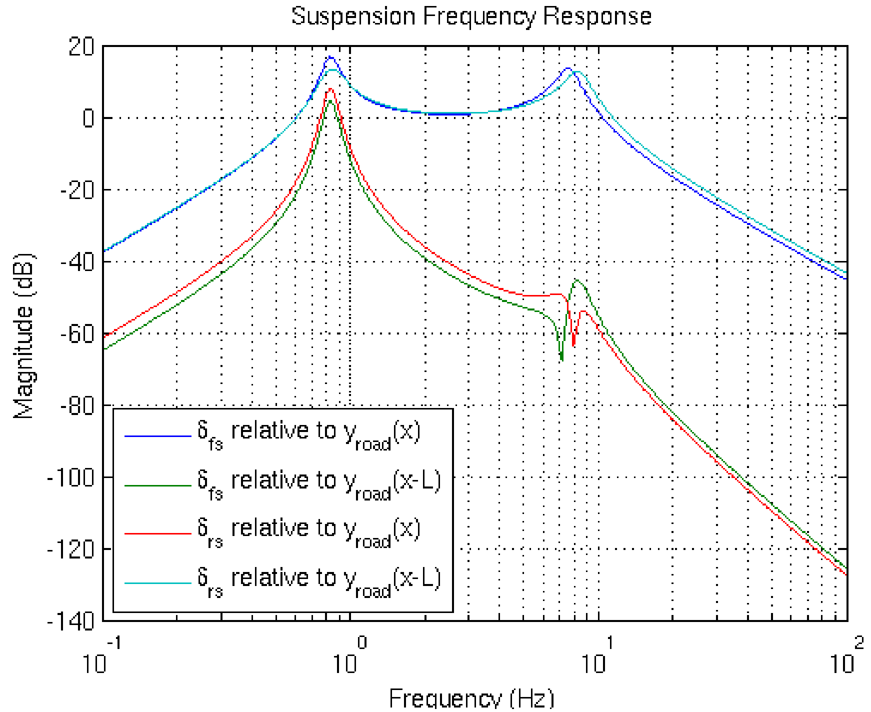
- At around $0.8\ \mathrm{Hz}$ the vehicle frequency response peaks above $0\ \mathrm{dB}$
- The vehicle translational motion, y_c, is the most magnified



$$\frac{d\theta}{dt}=\omega=\frac{h_c}{I_c} \quad \text{and} \quad \frac{d\,y_c}{dt}=v_c=\frac{p_c}{m_s}$$

Half Car Suspension
Suspension Sinusoidal Frequency Response

- Natural frequencies at around 0.8 and 8 Hz
- Response spikes above 0 dB at both frequencies, however, we the suspension should absorb the energy and not the vehicle so magnitude amplification is not necessarily unreasonable as long as the motion does not exceed the allowable suspension travel
- Note that the front suspension deflection, δ_{fs}, is most affected by the front input, $y_{road}(x)$, and the rear suspension deflection, δ_{fs}, is affected most by the rear input, $y_{road}(x)$

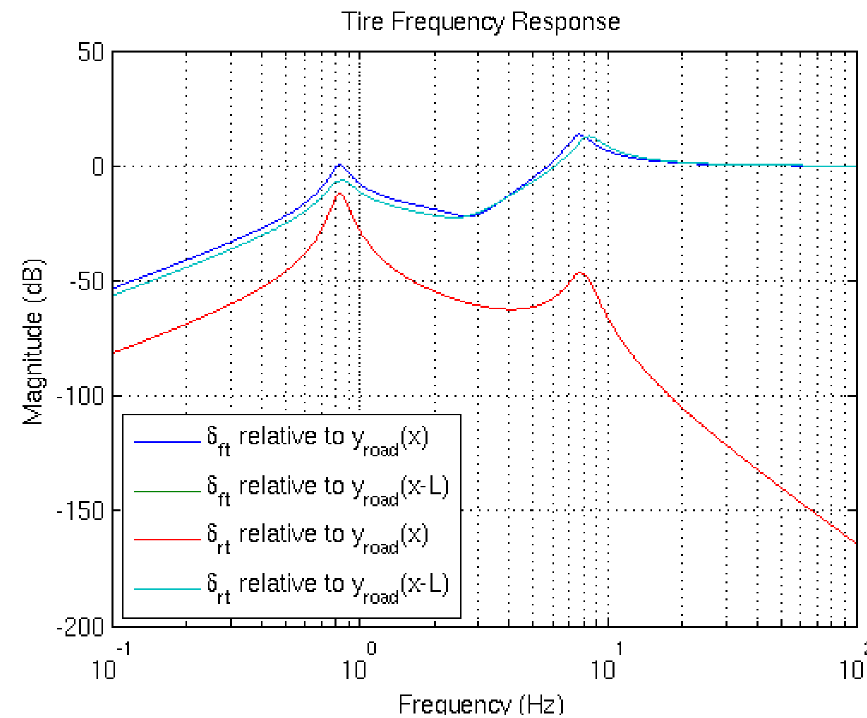


$$\frac{d\delta_{fs}}{dt} = v_{fs} - v_{fus} = (v_c + L_f\omega) - v_{fus} = \left(\frac{p_c}{m_s} + L_f\frac{h_c}{I_c}\right) - \frac{p_{fus}}{m_{fus}}$$

$$\frac{d\delta_{rs}}{dt} = v_{rs} - v_{rus} = (v_c - L_r\omega) - v_{rus} = \left(\frac{p_c}{m_s} - L_r\frac{h_c}{I_c}\right) - \frac{p_{rus}}{m_{rus}}$$

Half Car Suspension
Tire Sinusoidal Frequency Response

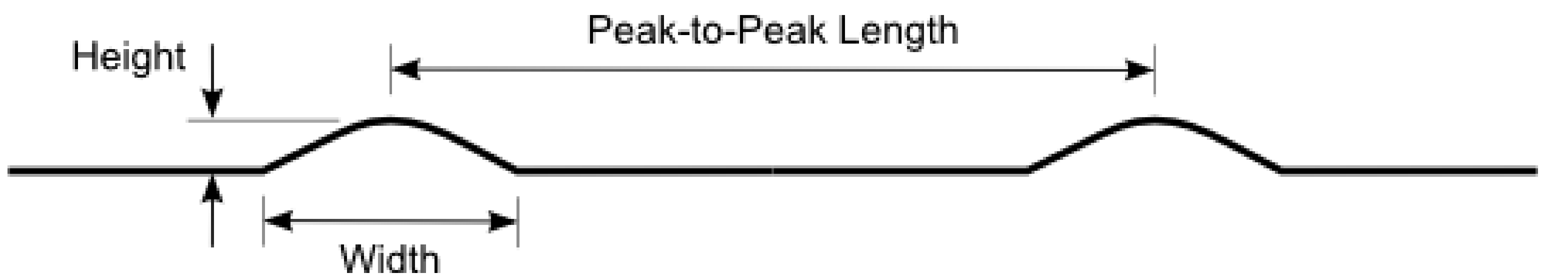
- Again natural frequencies occur at around 0.8 and 8 Hz
- Magnitudes spike above 0 dB at the latter
- Large amplitude deflections of the tire wall can result in damage to the rim



$$\frac{d\,\delta_{ft}}{dt} = v_{fus} - v_{road}(x) = \frac{p_{fus}}{m_{fus}} - v_{road}(x)$$

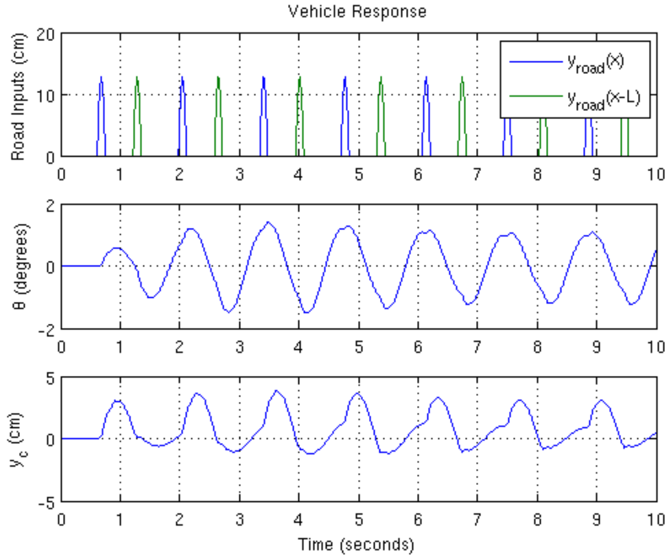
$$\frac{d\,\delta_{rs}}{dt} = v_{fus} - v_{road}(x-L) = \frac{p_{rus}}{m_{rus}} - v_{road}(x-L)$$

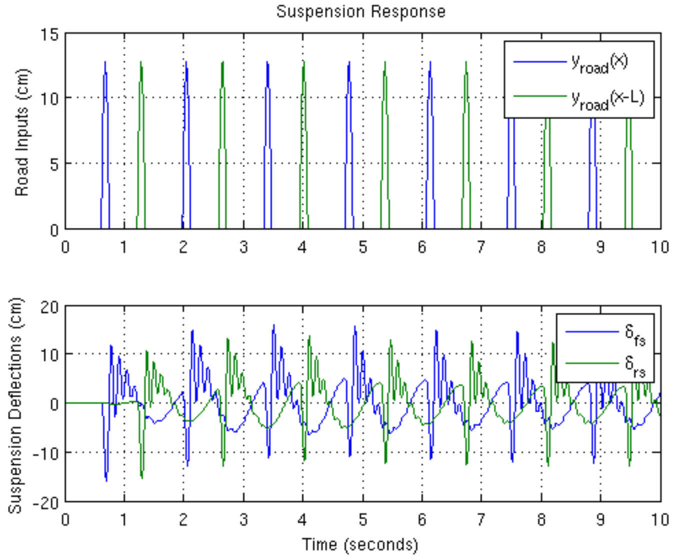
Half Car Suspension
Speed Bumps

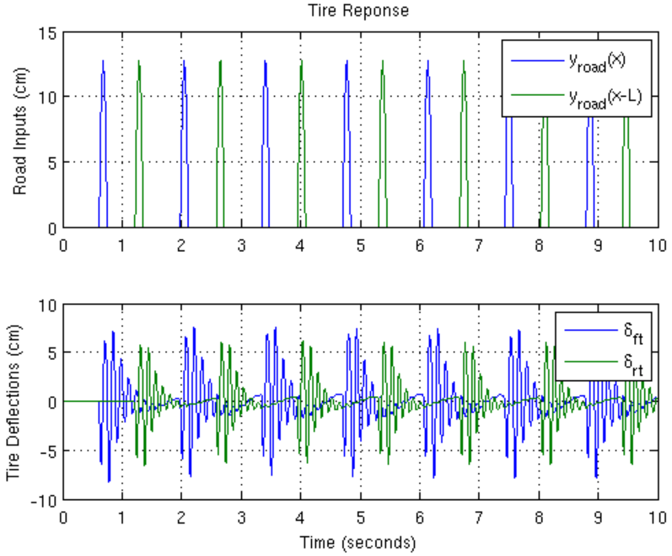


- Speed bumps in a parking lot
  - 10 MPH (~ 4.7 m/s)
  - 12 inches wide (width ~ 0.3 m)
  - 5 inches tall (height ~ 0.13 m)
  - Separated by 20 ft (peak-to-peak ~ 6m)

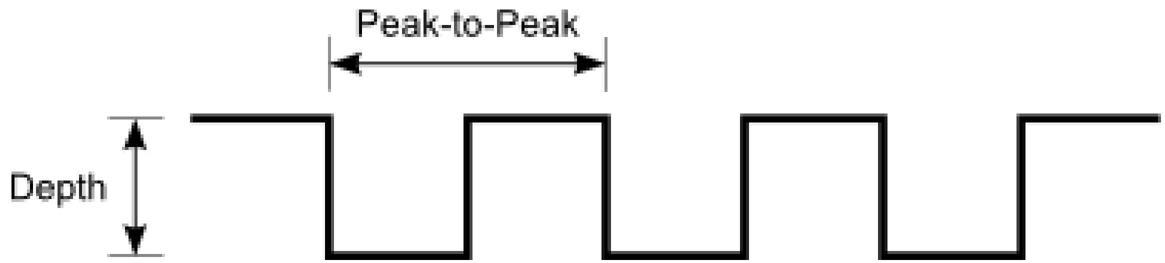
Half Car Suspension
Speed Bumps Results







Half Car Suspension
Highway Warning Track



- Warning track is a series of ruts stamped into the asphalt on the shoulder
- Rut geometry
  - ~ 2 inches deep (depth ~ 0.05 m)
  - ~ 3 inches wide or ~ 6 inches peak-to-peak (peak-to-peak ~ 0.015 m)
- Vehicle traveling on the highway at about 65-70 MPH (~ 30 m/s)

Half Car Suspension

Highway Warning Track Results