AATM 500 – Atmospheric Dynamics

Crizzia Mielle De Castro
Nov. 10, 2022
Homework 5

Problem 1 (a) Perform a scale analysis of the Boussinesq vorticity equation for large-scale flows in the midlatitude ocean based on the following characteristic scales:

$$\frac{D\zeta}{Dt} = (\zeta + f)\frac{\partial w}{\partial z} + \left(\frac{\partial w}{\partial y}\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}\frac{\partial v}{\partial z}\right) - v\frac{df}{dy} \tag{1.1}$$

$$\begin{aligned} f &\approx 10^{-4}\ s^{-1} \\ U, V &\approx 1\ ms^{-1} \\ W &\approx 10^{-3}\ ms^{-1} \\ L &\approx 10^{6}\ m \\ H &\approx 10^{3}\ m \end{aligned} \tag{1.2}$$

What does each term scale as? What is the order of magnitude of each term? Write down a simplified Boussinesq vorticity equation based on the leading-order balance, and describe the balance.

(b) Integrate the simplified Boussinesq vorticity equation with respect to height. Assume the vertical velocity vanishes at the bottom. Express your solution as $\int v dz = \cdots$

(c) In Figure 1, where the wind stress curl is negative (anticyclonic), there is Ekman pumping and downwelling in the upper ocean. Where the wind stress curl is positive (cyclonic), there is Ekman suction and upwelling in the upper ocean. The magnitude of the wind stress curl is proportional to the magnitude of the vertical motion. Based on this information, and your answer in part b), sketch what you expect the column-mean meridional motion to conceptually look like as a function of latitude in the North Pacific.

Solution.

(a) Relative vorticity is given by

$$\zeta = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} \tag{1.3}$$

which we can scale as

$$\zeta \approx \frac{V}{L} - \frac{U}{L} = 10^{-6} s^{-1} \tag{1.4}$$

The time scale is also given by $T \approx L/U = 10^6 s$. The table below shows the summary of magnitude scaling for each term.

	$\frac{D\zeta}{Dt}$	$\zeta\frac{\partial w}{\partial z}$	$f\frac{\partial w}{\partial z}$	$\frac{\partial w}{\partial y}\frac{\partial u}{\partial z}$	$\frac{\partial w}{\partial x}\frac{\partial v}{\partial z}$	$v\frac{df}{dy}$
scaling for each term	$\frac{\zeta}{T}$	$\zeta\frac{W}{H}$	$f\frac{W}{H}$	$\frac{W}{L}\frac{U}{H}$	$\frac{W}{L}\frac{V}{H}$	$V\frac{f}{L}$
magnitude	10^{-12}	10^{-12}	10^{-10}	10^{-12}	10^{-12}	10^{-10}

If we ignore non-leading order terms, we get

$$f\frac{\partial w}{\partial z} = v\frac{df}{dy} \tag{1.5}$$

Here, we get a balance between the vertical stretching of the planetary vorticity (LHS) and the meridional advection of the planetary vorticity (RHS).

(b) By integrating our simplified equation with respect to height, we have

$$\int f\frac{\partial w}{\partial z}dz = \int v\frac{df}{dy}dz \tag{1.6}$$

We can take out expressions with f outside the integral since it is only dependent on y. Rewriting our equation, we get

$$f\int_{w_0}^{w} dw = \frac{df}{dy}\int_{z_0}^{z} vdz \tag{1.7}$$

where w_0 is the vertical velocity at the bottom of the ocean, and z_0 is its depth. We were given that $w_0 = 0$, so

$$\int_{z_0}^{z} vdz = \frac{fw}{df/dy} \tag{1.8}$$

(c) Since $f = 2\Omega\sin\theta$, then

$$\frac{df}{dy} = 2\Omega\cos\theta\frac{d\theta}{dy} \tag{1.9}$$

From our first HW, we can estimate dy as $\Delta y = a\Delta\theta$ (let θ be in radians), so $\frac{\Delta\theta}{\Delta y} = 1/a$. From here, we can say that

$$\frac{df}{dy} \approx \frac{2\Omega}{a}\cos\theta \tag{1.10}$$

Using this, we can approximate our integral as

$$\int_{z_0}^{z} vdz \approx \frac{2\Omega\sin\theta}{\frac{2\Omega}{a}\cos\theta}w = (a\tan\theta)w \tag{1.11}$$

Thus, as latitude increases, the magnitude of the mean meridional motion (LHS) exponentially increases. Besides this, we know that a positive wind stress curl means upwelling, and vice versa. Thus, when wind stress curl is positive, the ocean vertical motion w is positive, and when wind stress curl is negative, w is negative. The sign of the mean meridional motion is dependent on the sign of w. Lastly, the stronger the magnitude of the wind stress curl, the stronger the vertical motion, and consequently, the stronger the mean meridional motion. In the plot I made, the amplitude of the wave increases, because $\tan\theta$ increases with increasing latitude θ, and the peaks and trough designate sections with higher wind stress curl (and w) magnitude.

□

Problem 2

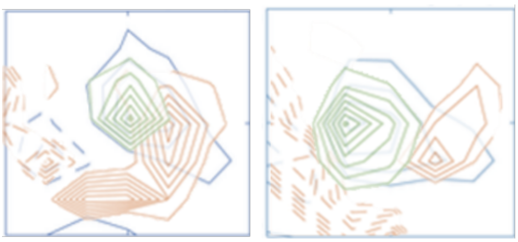
The scientific community does not fully agree on the dynamics of how tropical cyclones form. Figure 2 shows an illustration of typical cloud features in a developing tropical cyclone. One prominent feature that often exists is one or more mesoscale convective system(s) (MCSs), which is the mature/decaying stage of a cluster of convection. Within an MCS, a mesoscale convective vortex (MCV) often exists, with a maximum, positive relative vorticity/circulation in the middle troposphere (around 5 km). Another prominent feature that often exists are vortical hot towers (VHTs), which are deep turrets of intense convection.

One hypothesis is that these vortical hot towers are key to the dynamics of tropical cyclone formation. Figure 3 shows an analysis of the 2.5-km absolute vorticity, following a VHT that erupts within an MCV, in a numerical simulation of tropical cyclone formation. The first column shows the updraft core of the VHT. Over 20 min, the absolute vorticity quadruples in magnitude! Based on Figure 3, which you can treat as a Lagrangian frame of reference, describe the processes forming and amplifying the positive absolute vorticity. In your description, explicitly link to the mathematics. It may be helpful to draw some sketches to explain the patterns you are seeing. Assume solenoidal effects are negligible.

Solution. Our full equation for the change in absolute vorticity following a parcel without the solenoidal terms is given by

$$\frac{D\eta}{Dt} = -\eta\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) + \left(\frac{\partial w}{\partial y}\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}\frac{\partial v}{\partial z}\right) \tag{2.1}$$

The first term describes the divergence or vortex stretching, and the second term describes the tilting of the horizontal vorticity into the vertical. If $\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}$ is negative, then we have horizontal wind convergence, which results to an increase in η. We see in the fourth column that the increasing stretching term accompanies a rapid increase in η on the second column. Initially, the absolute vorticity is primarily dependent on the positive tilting, but as the wind starts to converge towards the VHT, the stretching term increases. The rapid increase of the stretching term begin to dominate the effects of tilting, so absolute vorticity starts to behave more similarly with the stretching term. If we try to overlap the plots of the absolute vorticity, tilting, and stretching, we observe that the maximum absolute vorticity coincides with the maximum stretching.



(a) At $t = 40$ min (b) At $t = 50$ min

Figure 1: Green: stretching, orange: tilting, blue: absolute vorticity

Next, we observe that when the tilting term is negative, the absolute vorticity is also negative, and vice versa. We learned in class that the stretching term can't change the sign of η, but here, we notice that the tilting term can. At $t = 30$ min, the tilting term can be related to w, but over time, the wind shear becomes more chaotic, so the relationship between w and the tilting becomes a little bit messier. We can observe this in the disorganized structure of the tilting term plots over time. Since the effects of the stretching becomes more relevant at later times, I will just relate the tilting term with w at $t = 30$ min, which is when tilting is the most relevant.

From the plot for w and the tilting term, we can infer the following:

Thus, when tilting is positive,

$$\left[\frac{(-)\partial w}{\partial y}\frac{(-)\partial u}{\partial z} - \frac{(-)\partial w}{\partial x}\frac{(+)\partial v}{\partial z}\right] > 0 \tag{2.2}$$

and when tilting is negative,

$$\left[\frac{(+)\partial w}{\partial y}\frac{(-)\partial u}{\partial z} - \frac{(+)\partial w}{\partial x}\frac{(+)\partial v}{\partial z}\right] < 0 \tag{2.3}$$

which shows a clear dipole between positive and negative tilting.

$\square$

Problem 3 (a) Figure 4 shows a two-layer, shallow-water system. Derive an expression for the pressure **in the bottom layer** at some height z.

(b) Show that the horizontal pressure gradient force may be expressed as

$$\mathrm{PGF} = -\frac{\rho_1 - \rho_2}{\rho_1} g \nabla_z h_1 = -g_r \nabla_z h_1 \tag{3.1}$$

where $gr = g(\rho_1 \rho_2)/\rho_1$ is called the reduced gravity.

(c) The zonal momentum equation, ignoring Coriolis and viscous forces, and the conservation of mass for the bottom layer are

$$\begin{aligned} \frac{Du_1}{Dt} &= -g_r \frac{\partial h_1}{\partial x} \\ \frac{Dh_1}{Dt} &= -h_1 \nabla \cdot \vec{u}_1 \end{aligned} \tag{3.2}$$

Expand the total derivative, and linearize the two equations about zero mean flow.

$$\begin{aligned} u_1 &= u'(x,t) \\ v_1 &= v'(x,t) \\ h_1 &= H_1 + \eta'(x,t) \end{aligned} \tag{3.3}$$

Assume $|\eta'| \ll H_1$ As you did in the previous assignment, throw out terms that have two perturbation quantities multiplied together, which we will assume to be small.

(d) From these two linearized equations, form a single second-order PDE with η' as the independent variable.

(e) Assume a wavy solution for η' that has an amplitude A, zonal wavenumber k, and frequency ω:

$$\eta' = Ae^{i(kx - \omega t)} \tag{3.4}$$

Substitute in this wavy solution into your PDE. Solve for ω, the dispersion relationship for an internal wave in this system.

(f) Your solution should look very similar to ω for non-rotating shallow water waves, with one important difference. Explain why the frequency depends on reduced gravity, i.e., the density difference between the two layers.

(g) The background density can be expressed as

$$\tilde{\rho} = \rho_1 - (\rho_1 - \rho_2)\,\mathcal{H}\left[z - h_1\right] \tag{3.5}$$

where $\mathcal{H}\left[z - h_1\right]$ is the Heaviside step function, which is 0 for $z < h_1$ and is 1 for $z > h_1$. Using this expression for $\tilde{\rho}$, relate the buoyancy frequency N^2 to g_r.

Solution.

(a) For a shallow water system, we assume hydrostatic balance and layers with constant density.

$$\frac{\partial p}{\partial z} = -\rho g \tag{3.6}$$

To obtain the pressure at the interface, we apply hydrostatic balance to the top layer and integrate with respect to z from height h_1 to height H.

$$\begin{aligned} \int_{h_1}^{H} dp &= -\int_{h_1}^{H} \rho g dz \\ p\Big|_{H} - p\Big|_{h_1} &= -\rho_2 g(H - h_1) \end{aligned} \tag{3.7}$$

The pressure and g are both constants, so we can take them out of the integral. We assume that $p\Big|_{H}$ is negligible, and since $h_2 = H - h_1$, the pressure at the interface becomes

$$p\Big|_{h_1} = \rho_2 g h_2 \tag{3.8}$$

To obtain the pressure at the bottom layer, we apply hydrostatic balance again and integrate with respect to z from height z to height h_1.

$$\begin{aligned} \int_{z}^{h_1} dp &= -\int_{z}^{h_1} \rho g dz \\ p\Big|_{h_1} - p\Big|_{z} &= -\rho_1 g(h_1 - z) \end{aligned} \tag{3.9}$$

Using our expression for the pressure at the interface, we get the pressure at the bottom layer at some height z.

$$p_1 = \rho_2 g h_2 + \rho_1 g(h_1 - z) \tag{3.10}$$

(b) The pressure at the bottom layer can also be written as

$$p_1 = \rho_2 g(H - h_1) + \rho_1 g(h_1 - z) \tag{3.11}$$

The horizontal gradient of pressure is given by

$$\nabla_h p_1 = \rho_2 g(\nabla_h H - \nabla_h h_1) + \rho_1 g(\nabla_h h_1 - \nabla_h z) \tag{3.12}$$

H is just a constant so $\nabla_h H = 0$, and $\nabla_h z = 0$, because we're only looking at the horizontal gradient. Thus, this simplifies to

$$\nabla_h p_1 = -\rho_2 g \nabla_h h_1 + \rho_1 g \nabla_h h_1 = g\nabla_h h_1(\rho_1 - \rho_2) \tag{3.13}$$

The pressure gradient force at the bottom layer is given by

$$-\frac{1}{\rho_1}\nabla_h p_1 = -\frac{\rho_1 - \rho_2}{\rho_1} g \nabla_h h_1 = -g_r \nabla_h h_1 \tag{3.14}$$

where $g_r = g\frac{\rho_1 - \rho_2}{\rho_1}$

(c) Expanding the total derivative for u_1, we get

$$\begin{aligned} \frac{Du_1}{Dt} &= \frac{\partial u_1}{\partial t} + \vec{v} \cdot \nabla u_1 \\ &= \frac{\partial u'}{\partial t} + u' \frac{\partial u'}{\partial x} \\ \frac{Du_1}{Dt} &= \frac{\partial u'}{\partial t} \end{aligned} \tag{3.15}$$

Expanding the total derivative for h_1, we get no H_1 terms since its just a constant.

$$\begin{aligned}\frac{Dh_1}{Dt} &= \frac{\partial h_1}{\partial t} + \vec{v}\cdot\nabla h_1 \\ &= \frac{\partial \eta'}{\partial t} + u'\frac{\partial \eta'}{\partial x} \\ \frac{Dh_1}{Dt} &= \frac{\partial \eta'}{\partial t}\end{aligned} \tag{3.16}$$

Expanding the horizontal velocity divergence, we get

$$\nabla \cdot \vec{u_1} = \frac{\partial u_1}{\partial x} + \frac{\partial v_1}{\partial y} = \frac{\partial u'}{\partial x} \tag{3.17}$$

Lastly, $\frac{\partial h_1}{\partial x} = \frac{\partial \eta'}{\partial x}$, since H is just a constant. Plugging all these approximations back in our momentum equations, and assuming that H_1 is much larger in magnitude than η', we get

$$\begin{aligned}\frac{\partial u'}{\partial t} &= -g_r\frac{\partial \eta'}{\partial x} \\ \frac{\partial \eta'}{\partial t} &= -H_1\frac{\partial u'}{\partial x}\end{aligned} \tag{3.18}$$

(d) By getting the partial derivative with respect to t of the second equation, we have

$$\frac{\partial^2 \eta'}{\partial t^2} = -H_1\frac{\partial}{\partial x}\left(\frac{\partial u'}{\partial t}\right) \tag{3.19}$$

Plugging in the first equation, we get a second order ODE.

$$\frac{\partial^2 \eta'}{\partial t^2} = H_1 g_r\frac{\partial^2 \eta'}{\partial x^2} \tag{3.20}$$

(e) For simplicity, I let $a = i(kx - \omega t)$. Substituting the solution for η' into the partial derivative with respect to t, we get

$$\frac{\partial^2 \eta'}{\partial t^2} = -\omega^2 A e^a \tag{3.21}$$

Doing the same for the partial derivative with respect to x,

$$\frac{\partial^2 \eta'}{\partial x^2} = -k^2 A e^a \tag{3.22}$$

Plugging these derivatives into our ODE, we get

$$\begin{aligned}-\omega^2 A e^a &= -H_1 g_r k^2 A e^a \\ \omega^2 &= H_1 g_r k^2 \\ \omega &= \pm k\sqrt{H_1 g_r}\end{aligned} \tag{3.23}$$

(f) The buoyant force along the interface acts as the restoring force, since by nature, the fluids want to return to equilibrium after any displacements. The buoyant force points from the interface displacement towards equilibrium. Parcel displacement in the interface can be described through Archimedes's principle, which states that the upward buoyant force is equal to the weight of the displaced fluid.

For example, as the top layer pushes down on the bottom layer, fluid along the interface of the bottom layer gets displaced away. It's weight is equal to the buoyant force. The ratio $\frac{\rho_1-\rho_2}{\rho_1}$ describes this fluid displacement along the interface. In the two-layer model, We now consider a reduced gravity, because there is an added weight on top of the bottom layer, as opposed to the single layer we considered in class where $\rho_2 = 0$.

(g) The buoyancy frequency is given by

$$N^2 = -\frac{g}{\tilde{\rho}_\theta}\frac{\partial \tilde{\rho}_\theta}{\partial z} \tag{3.24}$$

For small displacements, we can assume that $\tilde{\rho} = \tilde{\rho}_\theta$. Getting the partial derivative with respect to z of our expression of $\tilde{\rho}$, we get

$$\frac{\partial \tilde{\rho}}{\partial z} = -\left(\rho_1 - \rho_2\right)\frac{\partial \mathcal{H}\left[z - h_1\right]}{\partial z} = -\left(\rho_1 - \rho_2\right)\delta(z - h_1) \tag{3.25}$$

Note that the derivative of the Heaviside function centered at h_1 is the Dirac-delta function centered at h_1. The buoyancy frequency is then

$$N^2 = g\frac{(\rho_1 - \rho_2)}{\rho_1 - \left(\rho_1 - \rho_2\right)\mathcal{H}\left[z - h_1\right]}\delta(z - h_1) \tag{3.26}$$

The Heaviside function has two cases: when $z < h_1$ or when $z > h_1$. At $z < h_1$, this becomes

$$N^2 = g\frac{(\rho_1 - \rho_2)}{\rho_1}\delta(z - h_1) = g_r\delta(z - h_1) \quad \text{at } z < h_1 \tag{3.27}$$

At $z > h_1$, this becomes

$$\begin{aligned} N^2 &= g\frac{(\rho_1 - \rho_2)}{\rho_1 - (\rho_1 - \rho_2)}\delta(z - h_1) \\ &= g\frac{(\rho_1 - \rho_2)}{\rho_2}\delta(z - h_1) \\ N^2 &= \frac{\rho_1}{\rho_2}g_r\delta(z - h_1) \quad \text{at } z > h_1 \end{aligned} \tag{3.28}$$

Here, we obtain a discrete relationship between the buoyancy frequency N^2 and the reduced gravity g_r.

□