AATM 500 – Atmospheric Dynamics

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Homework 6

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Problem 1 (a) Consider the shallow-water equations on a beta-plane centered at the Equator. In this case, $f = \beta y$, and the shallow water equations become:

$$\begin{aligned} \frac{\partial u'}{\partial t} - \beta y v' &= -g\frac{\partial \eta'}{\partial x} \\ \frac{\partial v'}{\partial t} + \beta y u' &= -g\frac{\partial \eta'}{\partial y} \\ \frac{\partial \eta'}{\partial t} + H\left(\frac{\partial u'}{\partial x} + \frac{\partial v'}{\partial y}\right) &= 0 \end{aligned} \tag{1.1}$$

where we have linearized the equations about a background state of rest. Notice the differences between these equations with those we used to investigate Kelvin waves on an f-plane. In this problem, we will investigate equatorial Kelvin waves on a β-plane instead. Let $v' = 0$. Then (1) and (3) end up being identical to the f-plane versions, so $c = \omega/k = \pm\sqrt{gH}$ is the same. Assume a wavelike solution for η':

$$\eta' = \tilde{\eta}(y)e^{i(kx-\omega t)} \tag{1.2}$$

where $\tilde{\eta}(y)$ is some function that varies with y. Combine equations and substitute this expression in to form an ODE for $\tilde{\eta}(y)$. Solve and get a solution for η'.

(b) Does c have to be positive or negative (i.e., do equatorial Kelvin waves propagate eastward or westward)? Why?

(c) Take the **real** part of η', using Euler's formula for complex numbers (i.e., $e^{ia} = \cdots$). Using the real part of η', get an expression for u'.

(d) Plot or sketch the pattern of the real part of η' (using contours) and u' (using vectors) for $k = 1$ at $t = 0$. Use $H = 25$ m, and let the unknown coefficient have a value of c/g to simplify things.

(e) Figure 1 shows a composite average of Kelvin waves over the equatorial Pacific. Compare your plot/sketch with the observed structure of Kelvin waves. What similarities and differences do you notice? Why does the anomalous convection (dark area) occur where it does in the Kelvin wave composite?

Solution.

(a) If we let $v' = 0$, then

$$\begin{aligned} \frac{\partial u'}{\partial t} &= -g\frac{\partial \eta'}{\partial x} \\ \beta y u' &= -g\frac{\partial \eta'}{\partial y} \\ \frac{\partial \eta'}{\partial t} &= -H\frac{\partial u'}{\partial x} \end{aligned} \tag{1.3}$$

By taking the partial derivative with respect to x of the second equation, we get

$$\beta y \frac{\partial u'}{\partial x} = -g \frac{\partial}{\partial x}\left(\frac{\partial \eta'}{\partial y}\right) \tag{1.4}$$

and combining this with the third equation, we arrive at

$$\frac{-\beta y}{H}\frac{\partial \eta'}{\partial t} = -g \frac{\partial}{\partial x}\left(\frac{\partial \eta'}{\partial y}\right) \tag{1.5}$$

Plugging in our solution for η' and simplifying to get an ODE for $\tilde{\eta}$, we get

$$\begin{aligned}\frac{-\beta y}{H}\tilde{\eta}(-i\omega)e^{i(kx-\omega t)} &= -g\frac{d\tilde{\eta}}{dy}(ik)e^{i(kx-\omega t)}\\ -\beta y \frac{\omega/k}{gH}\tilde{\eta} &= \frac{d\tilde{\eta}}{dy}\\ -\frac{\beta y}{c}\tilde{\eta} &= \frac{d\tilde{\eta}}{dy}\end{aligned} \tag{1.6}$$

This is a separable ODE, so we can solve it by rearranging variables and integrating both sides.

$$\begin{aligned}-\frac{\beta}{c}\int y dy &= \int \frac{d\tilde{\eta}}{\tilde{\eta}}\\ -\frac{\beta y^2}{2c} &= ln\left(\frac{\tilde{\eta}}{\eta_0}\right)\end{aligned} \tag{1.7}$$

Isolating $\tilde{\eta}$, we get

$$\tilde{\eta} = \eta_0 e^{-\frac{\beta y^2}{2c}} \tag{1.8}$$

Our final solution for η' is then

$$\eta' = \eta_0 e^{-\frac{\beta y^2}{2c}} e^{i(kx-\omega t)} \tag{1.9}$$

(b) For Kelvin waves, we want $e^{-\frac{\beta y^2}{2c}}$ to decay to satisfy our meridional boundary conditions. In the exponent, $y^2 = (a\theta)^2$ (centered at equator) is always positive no matter where we are. On the other hand, $\beta = \frac{2\Omega}{a}$ (centered at equator) is just a constant. Since we want our exponent to be negative, **c must be positive** no matter our y. Kelvin waves propagate eastward.

(c) Euler's formula is given by $e^{ia} = \cos a + i \sin a$. Thus, the real part of η' is

$$\eta'_R = \tilde{\eta}\cos(kx - \omega t) \tag{1.10}$$

Plugging this into our second shallow water equation, we get

$$\beta y u' = -g\frac{d\tilde{\eta}}{dy}\cos(kx-\omega t) \tag{1.11}$$

From Eqn. 1.6, we see that $\frac{d\tilde{\eta}}{dy} = -\frac{\beta y}{c}\tilde{\eta}$. Isolating u', we have

$$u' = \frac{g}{c}\tilde{\eta}\cos(kx-\omega t) \tag{1.12}$$

where $\tilde{\eta}$ is what we have solved for previously in (a).

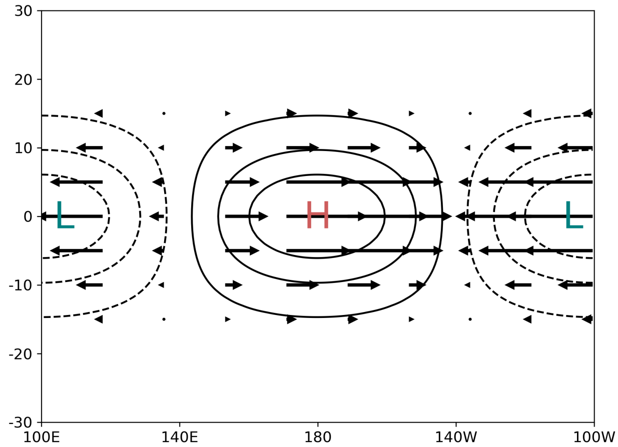
(d) At $k = 1$, $t = 0$, and $\eta_0 = c/g$, η'_R becomes

$$\eta' = \frac{c}{g} e^{-\frac{\beta y^2}{2c}} \cos(x) \tag{1.13}$$

and u' becomes

$$u' = e^{-\frac{\beta y^2}{2c}} \cos(x) \tag{1.14}$$

By plotting these we get the following. Here, the solid lines indicate positive contours, while dashed lines indicate negative contours. The "H" and "L" indicate regions of higher or lower η contours. Note also that the length of the arrows are proportional to the magnitude of u'.



(e) Similar to my plot above, Figure 1 shows stronger winds near the equator. Where Figure 1 has geopotential height contours, my sketch has η contours. Thus, we can say that in Figure 1, geopotential height is analogous to η. Figure 1 also shows eastward winds over the positive geopotential height, and westward winds over the negative geopotential height. The locations of the positive versus negative geopotential height in Figure 1, and the positive and negative η in my sketch also match. Note that my sketch is only accurate up to around 15^o N and S, and east of $140^o E$; Figure 1 gets a little more complicated outside these latitudes and longitudes. The anomalous convection (dark shading) occurs where there is convergence of eastward and westward winds, which acts as a precursor to upward wind convection.

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Problem 2 (a) In lecture, we looked at geostrophic adjustment starting from an unbalanced mass (η) state. One could also start with an unbalanced velocity state, which we will do in this problem. Let the initial meridional velocity be

$$v = \begin{cases} v_0 \sin\left(\frac{x}{L}\right) & \text{if } -L \leq x \leq L \\ 0 & \text{if } x < -L \text{ or } x > L \end{cases} \tag{2.1}$$

The shallow-water layer has a constant depth H at the initial time. Determine the initial potential vorticity q, using the linear form with constant Coriolis parameter that we previously used for geostrophic adjustment.

(b) Using conservation of potential vorticity, expressed in terms of the streamfunction, find a solution for ψ. To do so, you will need to consider three separate regions: $x < -L$, $-L \leq x \leq L$, and $x > L$. For the outer two regions, ψ must be bounded as $x \to \pm\infty$. You should find that the middle region is characterized by a nonhomogeneous ODE, which requires you to find a particular solution and add it to the general solution. To find the particular solution, use the method of undetermined coefficients by assuming a solution of the form

$$\psi_p = A \cos\left(\frac{x}{L}\right) \tag{2.2}$$

and then solve for A. There will be other unknown coefficients in your full solution, which you can leave as unknowns.

(c) For the remainder of this problem, we will focus **only on the particular solution in the middle region**, assuming that the particular solution dominates here. Using ψ_p, find η'_p. For $v_0 > 0$ and $f_0 > 0$, does the free surface bulge upward or downward around $x = 0$? Why does your answer make sense from a balanced perspective?

(d) Show that the maximum response in η'_p occurs when $L = L_d$. That is, scale matching between the initial, unbalanced disturbance and the deformation radius produces the biggest bang for your buck.

(e) Determine the final v' from ψ_p, and show that the $|v'_{\text{final}}| < |v'_{\text{initial}}|$ for $L > 0$. Explain why the meridional velocity must decrease in magnitude in order to conserve the potential vorticity.

Solution. (a) The potential vorticity is given by

$$q = \zeta - \frac{f_0 \eta'}{H} = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} - \frac{f_0 \eta'}{H} \tag{2.3}$$

During the initial state, the shallow water layer has constant height H ($\eta' = 0$) and no u, so q simplifies into

$$q = \frac{\partial v}{\partial x} \tag{2.4}$$

Given our initial v, we get

$$q(x, y) = \begin{cases} \frac{v_0}{L} \cos\left(\frac{x}{L}\right) & \text{if } -L \leq x \leq L \\ 0 & \text{if } x < -L \text{ or } x > L \end{cases} \tag{2.5}$$

(b) The conservation of potential vorticity states that $\frac{\partial q}{\partial t} = 0$, and

$$q(x, y) = \left(\nabla^2 - \frac{1}{L_d^2}\right)\psi \tag{2.6}$$

where ψ is the stream function defined by $\psi = g\eta/f_0$. Given our equation for $q(x, y)$ from (a), and assuming there are no variations along y, we get

$$\frac{d^2\psi}{dx^2} - \frac{1}{L_d^2}\psi = \begin{cases} \frac{v_0}{L}\cos\left(\frac{x}{L}\right) & \text{if } -L \leq x \leq L \\ 0 & \text{if } x < -L \text{ or } x > L \end{cases} \tag{2.7}$$

First, we look at $x < -L$ or $x > L$. The homogenous ODE looks like $\frac{d^2\psi}{dx^2} - \frac{1}{L_d^2}\psi = 0$. It's characteristic equation $r^2 - \frac{1}{L_d^2} = 0$ has two real roots $r = \pm\frac{1}{L_d}$. Thus, the homogenous solution is

$$\psi = c_1 e^{x/L_d} + c_2 e^{-x/L_d} \tag{2.8}$$

We want ψ to be bounded such that $\psi \to 0$ as $x \to \pm\infty$. For $x < -L$ or $x \to -\infty$,

$$\psi(-\infty) = 0 = c_1 e^{-\infty/L_d} + c_2 e^{\infty/L_d} \tag{2.9}$$

so we want $c_2 = 0$. On the other hand, for $x > L$ or $x \to \infty$,

$$\psi(\infty) = 0 = c_1 e^{\infty/L_d} + c_2 e^{-\infty/L_d} \tag{2.10}$$

so we want $c_1 = 0$. In summary,

$$\psi(x) = \begin{cases} c_1 e^{x/L_d} & \text{if } x < -L \\ c_2 e^{-x/L_d} & \text{if } x > L \end{cases} \tag{2.11}$$

Next, we look at $-L \leq x \leq L$. We already know that the homogenous solution looks like $\psi = Be^{x/L_d} + Ce^{-x/L_d}$. The solution to the inhomogenous ODE is just the sum of the homogenous solution and the particular solution. Thus, for $-L \leq x \leq L$,

$$\psi = A\cos\left(\frac{x}{L}\right) + Be^{x/L_d} + Ce^{-x/L_d} \tag{2.12}$$

Taking the derivatives with respect x of our solution, we get

$$\begin{aligned} \frac{d\psi}{dx} &= -\frac{A}{L}\sin\left(\frac{x}{L}\right) + \frac{B}{L_d}e^{x/L_d} - \frac{C}{L_d}e^{-x/L_d} \\ \frac{d^2\psi}{dx^2} &= -\frac{A}{L^2}\cos\left(\frac{x}{L}\right) + \frac{B}{L_d^2}e^{x/L_d} + \frac{C}{L_d^2}e^{-x/L_d} \end{aligned} \tag{2.13}$$

Plugging these into our inhomogenous ODE, we get,

$$\begin{aligned} -\frac{A}{L^2}\cos\left(\frac{x}{L}\right) + \frac{B}{L_d^2}e^{x/L_d} + \frac{C}{L_d^2}e^{-x/L_d} - \frac{1}{L_d^2}\left(A\cos\left(\frac{x}{L}\right) + Be^{x/L_d} + Ce^{-x/L_d}\right) &= \frac{v_0}{L}\cos\left(\frac{x}{L}\right) \\ -A\left(\frac{1}{L^2} + \frac{1}{L_d^2}\right) &= \frac{v_0}{L} \\ A &= -\frac{LL_d^2}{L^2 + L_d^2}v_0 \end{aligned} \tag{2.14}$$

Thus, our complete solution for ψ for all three domains is given by

$$\psi(x) = \begin{cases} c_1 e^{x/L_d} & \text{if } x < -L \\ c_2 e^{-x/L_d} & \text{if } x > L \\ -\frac{LL_d^2}{L^2+L_d^2}v_0\cos\left(\frac{x}{L}\right) + Be^{x/L_d} + Ce^{-x/L_d} & \text{if } -L \leq x \leq L \end{cases} \tag{2.15}$$

Here, c_1, c_2, B, and C are just unknown coefficients.

(c) Since $\psi_p = g\eta_p/f_0$, and $\psi_p = -\frac{LL_d^2}{L^2+L_d^2} v_0 \cos\left(\frac{x}{L}\right)$, then

$$\eta_p(x) = -\frac{LL_d^2}{L^2+L_d^2}\frac{f_0 v_0}{g}\cos\left(\frac{x}{L}\right) \tag{2.16}$$

At $x = 0$, this becomes

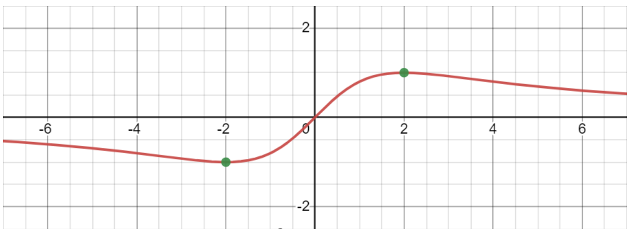
$$\eta_p(x=0) = -\frac{LL_d^2}{L^2+L_d^2}\frac{f_0 v_0}{g} \tag{2.17}$$

If $f_0 > 0$ and $v_0 > 0$, then $\eta_p < 0$, which means the **free surface is bulging down** around $x = 0$. Since our meriodional velocity follows a sine function, it points downward at $x < 0$, and it points upward at $x > 0$. The Coriolis force should point to the right of the flow, so the Coriolis force points to the left at $x < 0$, and to the right at $x > 0$. To maintain geostrophic balance, the PGF should point opposite the Coriolis force, so the PGF points to the right at $x < 0$, and to the left at $x > 0$. The PGF should point from higher η (higher pressure) to lower η (lower pressure), so η should be higher at $x < 0$ and $x > 0$ than at $x = 0$.

(d) Here, we are concerned about the amplitude of η_p, more specifically $\frac{LL_d^2}{L^2+L_d^2}$. We want to show that this is maximum when $L = L_d$. To do that, we take its derivative with respect to L and equate it to zero to solve for L.

$$\begin{aligned}\frac{d}{dL}\left(\frac{LL_d^2}{L^2+L_d^2}\right) = 0 &= \frac{(L^2+L_d^2)L_d^2 - LL_d^2(2L)}{(L^2+L_d^2)^2} \\ 0 &= L_d^4 + L^2L_d^2 - 2L^2L_d^2 \\ L_d^4 &= L^2L_d^2 \\ L &= L_d\end{aligned} \tag{2.18}$$

To check, say $L_d = 2$, then the plot below shows that $\frac{LL_d^2}{L^2+L_d^2}$ is maximum at $L = 2 = L_d$.



(e) The geostrophic meridional flow is given by $v_g = \frac{\partial \psi}{\partial x}$, and consequently, $v' = \frac{\partial \psi_p}{\partial x}$. Thus,

$$v' = \frac{L_d^2}{L^2 + L_d^2} v_0 \sin\left(\frac{x}{L}\right) \tag{2.19}$$

In the middle region, our initial v' is given by $v_0 \sin\left(\frac{x}{L}\right)$. We want to show that $|v'_{\text{final}}| < |v'_{\text{initial}}|$.

$$\begin{aligned} \frac{L_d^2}{L^2 + L_d^2} \left| v_0 \sin\left(\frac{x}{L}\right) \right| &< \left| v_0 \sin\left(\frac{x}{L}\right) \right| \\ \frac{L_d^2}{L^2 + L_d^2} &< 1 \\ L_d^2 &< L^2 + L_d^2 \end{aligned} \tag{2.20}$$

which should always be true for all real values of L that are not zero. Given conservation of potential vorticity, we can say that $\frac{\partial q}{\partial t} = 0$, and

$$\frac{\partial q}{\partial t} = \frac{\partial}{\partial t}\left(\frac{\partial v}{\partial x}\right) - \frac{\partial}{\partial t}\left(\frac{\partial u}{\partial y}\right) - \frac{f_0}{H}\frac{\partial \eta}{\partial t} = 0 \tag{2.21}$$

Here, we assume there are no variations along y, so

$$\frac{\partial}{\partial x}\left(\frac{\partial v}{\partial t}\right) = \frac{f_0}{H}\frac{\partial \eta}{\partial t} \tag{2.22}$$

The initial η is zero, and the final η is given by Eqn. 2.16, so $\frac{\partial \eta}{\partial t}$ becomes more negative over time. Thus, the left hand side must also become more negative over time, which can only occur if the magnitude of v decreases with time.

We can also look at this through an energy conservation perspective. Initially, our system has high potential energy, but as the system gets disturbed, this potential energy is converted to kinetic energy. Some parts of the system's kinetic energy gets transported by the gravity wave to the left and right (beyond the deformation radius), so the middle region's kinetic energy decreases over time. However, the gravity wave cannot transport the potential vorticity so all of it remains within the middle region. Thus, the magnitude of v decreases over time (kinetic energy $\propto |v|$), while keeping the potential vorticity within our middle region constant (or conserved).

□

Problem 3 (a) The dispersion relationship for quasi-geostrophic Rossby waves in a stratified atmosphere with background zonal flow U is

$$\omega = Uk - \frac{\beta k}{k^2 + l^2 + m^2 f_0^2/N^2} \tag{3.1}$$

Show that

$$m^2 = \frac{N^2}{f_0^2}\left[\frac{\beta}{U - c_{p,x}} - \left(k^2 + l^2\right)\right] \tag{3.2}$$

where $c_{p,x}$ is the zonal phase speed.

(b) Assume stationary waves, so that $c_{p,x} = 0$. Also assume $l = 0$ for simplicity. Show that for m to be real (i.e., for Rossby waves to propagate vertically), the following condition must be satisfied:

$$0 < U < \frac{\beta}{k^2} \tag{3.3}$$

(c) Use this condition to explain the differences in planetary Rossby wave patterns between 500 hPa (in the troposphere) and 50 hPa (in the stratosphere) seen in Figure 2.

Solution.

(a) The zonal phase speed is given by $c_{p,x} = \omega/k$. Dividing both sides of our equation by k, we get

$$c_{p,x} = U - \frac{\beta}{k^2 + l^2 + m^2 f_0^2/N^2} \tag{3.4}$$

Removing the denominator on both sides, we get

$$c_{p,x}(k^2 + l^2 + m^2 f_0^2/N^2) = U(k^2 + l^2 + m^2 f_0^2/N^2) - \beta \tag{3.5}$$

Through simplification and rearranging variables, we have

$$\begin{aligned} m^2\left(c_{p,x} f_0^2/N^2\right) + c_{p,x}(k^2 + l^2) &= m^2\left(U f_0^2/N^2\right) + U(k^2 + l^2) - \beta \\ -m^2\left(U - c_{p,x}\right) f_0^2/N^2 &= \left(U - c_{p,x}\right)(k^2 + l^2) - \beta \\ m^2 \frac{f_0^2}{N^2} &= \frac{\beta}{U - c_{p,x}} - (k^2 + l^2) \\ m^2 &= \frac{N^2}{f_0^2}\left[\frac{\beta}{U - c_{p,x}} - \left(k^2 + l^2\right)\right] \end{aligned} \tag{3.6}$$

(b) If $c_{p,x} = 0$ and $l = 0$, our equation simplifies into

$$\begin{aligned} m^2 &= \frac{N^2}{f_0^2}\left[\frac{\beta}{U} - k^2\right] \\ m &= \pm \frac{N}{f_0}\sqrt{\frac{\beta}{U} - k^2} \end{aligned} \tag{3.7}$$

For m to be real, $\frac{\beta}{U} - k^2$ must be positive, i.e. $\frac{\beta}{U} - k^2 > 0$.

$$
\begin{aligned}
k^2 &< \frac{\beta}{U} \\
U &< \frac{\beta}{k^2}
\end{aligned}
\tag{3.8}
$$

The second step is only valid if both U and k^2 have the same sign. Assuming k is real, k^2 is always positive. Thus, $U > 0$. In summary, $0 < U < \frac{\beta}{k^2}$.

(c) Figure 2 shows that in the troposphere, the Rossby waves are a combination of high and low wave numbers, while in the stratosphere, the Rossby waves are composed primarily of just low wave numbers. Based on our condition ($0 < U < \frac{\beta}{k^2}$), the allowable range of U to maintain vertical propagation of Rossby waves is smaller in the troposphere than in the stratosphere. In other words, motions in the stratosphere can extend to larger scales than in the troposphere. In the troposphere, Rossby waves tend to get more excited, but shorter waves or higher wave numbers are trapped and only long waves or lower wave numbers reach the stratosphere. Note that somewhere between the troposphere and the stratosphere is the jet stream, a strong eastward flow, which is one of the things that can "filter out" shorter waves, and only let long waves propagate vertically to the stratosphere. Rossby waves can only propagate in the presence of eastward winds that are not too strong.

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