AATM 500 – Atmospheric Dynamics

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Homework 4

Problem 1 (a) For the thunderstorm scaling, recall that the predominate horizontal balance is between the acceleration and pressure gradient forces. Based on this result, we will ignore the Coriolis force. We will also use the Boussinesq approximation to conceptually explore what causes mesoscale pressure perturbations in convection. The momentum equations are

$$\begin{aligned} \frac{Du}{Dt} &= -\frac{\partial \phi}{\partial x} \\ \frac{Dv}{Dt} &= -\frac{\partial \phi}{\partial y} \\ \frac{Dw}{Dt} &= -\frac{\partial \phi}{\partial z} + b \end{aligned} \tag{1.1}$$

Form an equation for the local change in the divergence, i.e., $\partial/\partial t(\nabla \cdot \vec{v}) = \cdots$, and simplify using the Boussinesq conservation of mass.

(b) Linearize this equation about a background state, similar to what we did in deriving the Boussinesq equations, but for the horizontal flow instead of the density and pressure. The overbars represent the background flow that is only a function of height, which you can think of as an area average, and the primes represent flow perturbations.

$$\begin{aligned} u &= \bar{u}(z) + u'(x, y, z, t) \\ v &= \bar{v}(z) + v'(x, y, z, t) \end{aligned} \tag{1.2}$$

Assume $|u'| \ll |\bar{u}|$, $|v'| \ll |\bar{v}|$, and $|w| \ll |u|$ & $|v|$. Expand the total derivative, and substitute in the expressions for u and v. Determine which terms containing partial derivatives of flow perturbations are relatively small and can be neglected. After neglecting terms, group the remaining flow terms and express in vector form.

(c) Assume a wavelike solution to ϕ: $\phi = ce^{i(kx+ly+mz)}$, where c is a constant, $i = \sqrt{-1}$, and k, l, and m are constant wavenumbers. Show that $\nabla^2\phi$ is proportional to $-\phi$. This result tells us that ϕ should be proportional to the negative of the sum of the terms on the right-hand side of your result from part b).

(d) Let's first consider the buoyancy term in isolation, ignoring the flow terms. Consider a thermal, with a maximum in b in the center of the thermal. What would the sign of ϕ be above and below the thermal? Would this "dynamic" pressure perturbation distribution promote or inhibit the upward acceleration of the thermal?

(e) Now consider the flow terms in isolation, ignoring the buoyancy term. Consider a background flow that is veering with height (turning clockwise), such that the mean flow is toward the northeast, but the background shear vector is toward the east. A thunderstorm updraft has a maximum in w at midlevels, as pictured. What would the sign of ϕ be upshear and downshear of the updraft? Relative to the mean flow, where would this midlevel, dynamic pressure perturbation distribution promote and inhibit the updraft, and thus cause deviate motion of the thunderstorm from the mean flow?

Solution.

(a) Expanding the divergence, we get

$$
\begin{aligned}
\frac{\partial}{\partial t}(\nabla \cdot \vec{v}) &= \frac{\partial}{\partial t}\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right) \\
&= \frac{\partial^2 u}{\partial x \partial t} + \frac{\partial^2 v}{\partial y \partial t} + \frac{\partial^2 w}{\partial z \partial t}
\end{aligned} \tag{1.3}
$$

Using the definition of the total derivative and getting their partial derivatives with respect to each Cartesian coordinate, we can show that

$$
\begin{aligned}
\frac{\partial}{\partial x}\left(\frac{Du}{Dt}\right) &= \frac{\partial^2 u}{\partial x \partial t} + \frac{\partial}{\partial x}\left(\vec{v} \cdot \nabla u\right) \\
\frac{\partial}{\partial y}\left(\frac{Dv}{Dt}\right) &= \frac{\partial^2 v}{\partial y \partial t} + \frac{\partial}{\partial y}\left(\vec{v} \cdot \nabla v\right) \\
\frac{\partial}{\partial z}\left(\frac{Dw}{Dt}\right) &= \frac{\partial^2 w}{\partial z \partial t} + \frac{\partial}{\partial z}\left(\vec{v} \cdot \nabla w\right)
\end{aligned} \tag{1.4}
$$

By adding all our total derivatives, we get

$$
\frac{\partial}{\partial x}\left(\frac{Du}{Dt}\right)+\frac{\partial}{\partial y}\left(\frac{Dv}{Dt}\right)+\frac{\partial}{\partial z}\left(\frac{Dw}{Dt}\right) = \left(\frac{\partial^2 u}{\partial x \partial t} + \frac{\partial^2 v}{\partial y \partial t} + \frac{\partial^2 w}{\partial z \partial t}\right)+\frac{\partial}{\partial x}(\vec{v}\cdot\nabla u)+\frac{\partial}{\partial y}(\vec{v}\cdot\nabla v)+\frac{\partial}{\partial z}(\vec{v}\cdot\nabla w) \tag{1.5}
$$

The first term in the right hand side is just $\frac{\partial}{\partial t}(\nabla \cdot \vec{v})$. Plugging in the approximated momentum equations, we get

$$
-\frac{\partial^2 \phi}{\partial x^2} - \frac{\partial^2 \phi}{\partial y^2} - \frac{\partial^2 \phi}{\partial z^2} + \frac{\partial b}{\partial z} = \frac{\partial}{\partial t}(\nabla \cdot \vec{v}) + \frac{\partial}{\partial x}(\vec{v} \cdot \nabla u) + \frac{\partial}{\partial y}(\vec{v} \cdot \nabla v) + \frac{\partial}{\partial z}(\vec{v} \cdot \nabla w) \tag{1.6}
$$

The left hand side can be simplified as $-\nabla^2 \phi$, and the final term can be simplified as $\nabla \cdot (\vec{v} \cdot \nabla \vec{v})$. From the Boussinesq approximation of the conservation of mass, $\frac{\partial}{\partial t}(\nabla \cdot \vec{v}) = 0$. By isolating ϕ, we get

$$
\nabla^2 \phi = \frac{\partial b}{\partial z} - \nabla \cdot (\vec{v} \cdot \nabla \vec{v}) \tag{1.7}
$$

Note that we are still going to use the expanded flow terms for 1b).

(b) Since w is very small, we let $w = w'$ to denote this. Here, whenever we see two prime terms multiplied together, we can just neglect it. Again, the expanded form of the flow terms is given by

$$
\frac{\partial}{\partial x}(\vec{v} \cdot \nabla u) + \frac{\partial}{\partial y}(\vec{v} \cdot \nabla v) + \frac{\partial}{\partial z}(\vec{v} \cdot \nabla w) \tag{1.8}
$$

By expanding the very first term, we have

$$
\frac{\partial}{\partial x}\left(u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z}\right) \tag{1.9}
$$

Since $\bar{u}$ is just in terms of z, then $\frac{\partial(\bar{u}+u')}{\partial x} = \frac{\partial u'}{\partial x}$ and $\frac{\partial(\bar{u}+u')}{\partial y} = \frac{\partial u'}{\partial y}$. This simplifies our equation into

$$
\frac{\partial}{\partial x}\left(u\frac{\partial u'}{\partial x} + v\frac{\partial u'}{\partial y} + w\frac{\partial u}{\partial z}\right) \tag{1.10}
$$

Expanding the velocities into the background state and perturbations, we have

$$\frac{\partial}{\partial x}\left((\bar{u}+u')\frac{\partial u'}{\partial x}+(\bar{v}+v')\frac{\partial u'}{\partial y}+w'\frac{\partial(\bar{u}+u')}{\partial z}\right) \tag{1.11}$$

Here, we notice some two prime terms multiplied together. After neglecting these terms, we get

$$\frac{\partial}{\partial x}\left(\bar{u}\frac{\partial u'}{\partial x}+\bar{v}\frac{\partial u'}{\partial y}+w'\frac{\partial \bar{u}}{\partial z}\right) \tag{1.12}$$

Doing the same for the second flow term (along y), we get

$$\frac{\partial}{\partial y}\left(\bar{u}\frac{\partial v'}{\partial x}+\bar{v}\frac{\partial v'}{\partial y}+w'\frac{\partial \bar{v}}{\partial z}\right) \tag{1.13}$$

We have to do something a little different for the final term (along z). Expanding it, we get

$$\frac{\partial}{\partial z}\left(u\frac{\partial w}{\partial x}+v\frac{\partial w}{\partial y}+w\frac{\partial w}{\partial z}\right) \tag{1.14}$$

Expanding the velocities again, we have

$$\frac{\partial}{\partial z}\left((\bar{u}+u')\frac{\partial w'}{\partial x}+(\bar{v}+v')\frac{\partial w'}{\partial y}+w'\frac{\partial w'}{\partial z}\right) \tag{1.15}$$

By neglecting all two prime terms multiplied together, we get

$$\frac{\partial}{\partial z}\left(\bar{u}\frac{\partial w'}{\partial x}+\bar{v}\frac{\partial w'}{\partial y}\right) \tag{1.16}$$

By bringing together all flow terms, we get

$$\frac{\partial}{\partial x}\left(\bar{u}\frac{\partial u'}{\partial x}+\bar{v}\frac{\partial u'}{\partial y}+w'\frac{\partial \bar{u}}{\partial z}\right)+\frac{\partial}{\partial y}\left(\bar{u}\frac{\partial v'}{\partial x}+\bar{v}\frac{\partial v'}{\partial y}+w'\frac{\partial \bar{v}}{\partial z}\right)+\frac{\partial}{\partial z}\left(\bar{u}\frac{\partial w'}{\partial x}+\bar{v}\frac{\partial w'}{\partial y}\right) \tag{1.17}$$

Grouping like terms, we get

$$\begin{split}\frac{\partial}{\partial x}\left(\bar{u}\frac{\partial u'}{\partial x}+\bar{v}\frac{\partial u'}{\partial y}\right)+\frac{\partial}{\partial y}\left(\bar{u}\frac{\partial v'}{\partial x}+\bar{v}\frac{\partial v'}{\partial y}\right)+\frac{\partial}{\partial z}\left(\bar{u}\frac{\partial w'}{\partial x}+\bar{v}\frac{\partial w'}{\partial y}\right)\\+\frac{\partial}{\partial x}\left(w'\frac{\partial \bar{u}}{\partial z}\right)+\frac{\partial}{\partial y}\left(w'\frac{\partial \bar{v}}{\partial z}\right)\end{split} \tag{1.18}$$

Simplifying this into a vector form, we get

$$\nabla\cdot(\vec{\bar{v}}\cdot\nabla_h\vec{v'})+\nabla_h\cdot\left(w'\frac{\partial\vec{\bar{v}}}{\partial z}\right) \tag{1.19}$$

Thus, our final linearized equation for the Laplacian of ϕ is

$$\nabla^2\phi=\frac{\partial b}{\partial z}-\nabla\cdot(\vec{\bar{v}}\cdot\nabla_h\vec{v'})-\nabla_h\cdot\left(w'\frac{\partial\vec{\bar{v}}}{\partial z}\right) \tag{1.20}$$

where $\vec{\bar{v}}$ denotes the background flow vector and $\vec{v'}$ denotes the perturbation flow vector. The h subscript denotes the horizontal gradient or divergence.

(c) We expand the Laplacian as $\nabla^2\phi = \frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} + \frac{\partial^2\phi}{\partial z^2}$. Plugging in our solution for ϕ, we get

$$\begin{aligned}\nabla^2\phi &= c\left(i^2k^2e^{i(kx+ly+mz)} + i^2l^2e^{i(kx+ly+mz)} + i^2m^2e^{i(kx+ly+mz)}\right)\\ &= -ce^{i(kx+ly+mz)}\left(k^2+l^2+m^2\right)\\ \nabla^2\phi &= -\phi\left(k^2+l^2+m^2\right)\end{aligned} \tag{1.21}$$

Note that $i^2 = -1$. Thus, $\nabla^2\phi \propto -\phi$.

(d) As a consequence of our answers in a) and c), we see that $-\frac{\partial b}{\partial z} \propto \phi$ when we ignore the flow terms. If we go above the thermal (higher altitude), then

$$-\frac{\partial b}{\partial z}\bigg|_{\text{above}} = -\frac{b_{\text{lower}} - b_{\text{max}}}{z_{\text{higher}} - z_{\text{lower}}} \to -\frac{(-)}{(+)} \to (+) \tag{1.22}$$

If we go below the thermal (lower altitude), then

$$-\frac{\partial b}{\partial z}\bigg|_{\text{below}} = -\frac{b_{\text{lower}} - b_{\text{max}}}{z_{\text{lower}} - z_{\text{higher}}} \to -\frac{(-)}{(-)} \to (-) \tag{1.23}$$

Thus, above the thermal, ϕ is positive, while below the thermal, ϕ is negative. Our equation for upward acceleration is $\frac{Dw}{Dt} = -\frac{\partial\phi}{\partial z} + b$. Given our signs for ϕ, we see that as we go up,

$$-\frac{\partial\phi}{\partial z} = -\frac{\phi_{+} - \phi_{-}}{z_{\text{higher}} - z_{\text{lower}}} \to -\frac{(+)-(-)}{(+)} \to (-) \tag{1.24}$$

The change in ϕ with respect to height z has a negative effect on the upward acceleration. Thus, this pressure perturbation inhibits the upward acceleration of the thermal.

(e) The $\bar{u}$ and $\bar{v}$ describe the mean flow. Since the mean flow is pointing towards the northeast, this means that $\bar{u}$ and $\bar{v}$ are both positive. We also need to look at the sign of $\frac{\partial w'}{\partial x}$ upshear and downshear. At upshear, $\frac{\partial w}{\partial x}$ is positive.

$$\frac{\partial w}{\partial x}\bigg|_{\text{upshear}} = \frac{w_{\text{lower}} - w_{\text{max}}}{x_{\text{left}} - x_{\text{right}}} \to \frac{(-)}{(-)} \to (+) \tag{1.25}$$

At downshear, $\frac{\partial w}{\partial x}$ is negative.

$$\frac{\partial w}{\partial x}\bigg|_{\text{downshear}} = \frac{w_{\text{lower}} - w_{\text{max}}}{x_{\text{right}} - x_{\text{left}}} \to \frac{(-)}{(+)} \to (-) \tag{1.26}$$

Ignoring the buoyancy for now, we see that the expanded flow terms are proportional to ϕ.

$$\left[\frac{\partial}{\partial x}\left(\bar{u}\frac{\partial u'}{\partial x} + \bar{v}\frac{\partial u'}{\partial y} + w\frac{\partial \bar{u}}{\partial z}\right) + \frac{\partial}{\partial y}\left(\bar{u}\frac{\partial v'}{\partial x} + \bar{v}\frac{\partial v'}{\partial y} + w\frac{\partial \bar{v}}{\partial z}\right) + \frac{\partial}{\partial z}\left(\bar{u}\frac{\partial w}{\partial x} + \bar{v}\frac{\partial w}{\partial y}\right)\right] \propto \phi \tag{1.27}$$

Here, we only look at terms with a $\frac{\partial w}{\partial x}$ for now, so

$$\frac{\partial w}{\partial x}\left(\frac{\partial \bar{u}}{\partial z}\right) + \frac{\partial}{\partial z}\left(\bar{u}\frac{\partial w'}{\partial x}\right) = \left[\frac{\partial w}{\partial x}\left(\frac{\partial \bar{u}}{\partial z}\right) + \frac{\partial \bar{u}}{\partial z}\left(\frac{\partial w}{\partial x}\right) + \bar{u}\frac{\partial^2 w}{\partial z\partial x}\right] \propto \phi \tag{1.28}$$

Using sign analysis for upshear, we see that ϕ is positive.

$$[(+)(+) + (+)(+) + (+)(+)] \propto \phi \tag{1.29}$$

Using sign analysis for downshear, we see that ϕ is negative.

$$[(-)(+) + (+)(-) + (+)(-)] \propto \phi \tag{1.30}$$

Thus, the pressure is relatively lower to the right of the mean flow, and relatively higher to the left of the mean flow. Similar to 1d), higher pressure aloft inhibits vertical motion, while lower pressure aloft promotes vertical motion. Thus, the pressure difference inhibits the updraft to the left of the mean flow, and promotes it to the right of the mean flow.

□

Problem 2 (a) Cylindrical coordinates are useful for the study of vortex dynamics, such as tropical cyclones. The horizontal momentum equation in cylindrical coordinates consists of radial and tangential components. Using the f -plane approximation and pressure coordinates, the radial component is

$$\frac{Du}{Dt} = f_0 v - \frac{\partial \Phi}{\partial r} + \frac{v^2}{r} \tag{2.1}$$

where u is the radial wind, v is the tangential wind, Φ is the geopotential, and r is the radius. The last term is analogous to the curvature terms in spherical coordinates. Explain how this term can be interpreted as the local centrifugal force of a parcel about the vortex center.

(b) Assume $Du/Dt = 0$. Then there is a balance between the Coriolis, pressure gradient, and local centrifugal forces. Form a Rossby number (R_o) equal to the ratio of the local centrifugal and Coriolis forces. $R_o \ll 1$ implies geostrophic balance, $R_o \approx 1$ implies gradient wind balance, and $R_o \gg 1$ implies cyclostrophic balance. Using reconnaissance aircraft data taken in Hurricane Ian, located at 25.6^oN, calculate the Rossby number at the radii in the table below. Characterize the balance. Illustrate and describe the predominate forces at each radii.

(c) Form a thermal wind equation using gradient wind balance and hydrostatic balance:

$$\begin{aligned} \frac{\partial \Phi}{\partial r} &= f_0 v + \frac{v^2}{r} \\ \frac{\partial \Phi}{\partial p} &= -\alpha \end{aligned} \tag{2.2}$$

(d) The figure on the next page shows an example of an azimuthally averaged Advanced Microwave Sounding Unit temperature retrieval through a hurricane, displayed as a temperature anomaly relative to a background state. Develop a procedure for how such satellite temperature data, combined with the thermal wind equation, can be used to estimate the tropical cyclone gradient wind structure. Include any assumptions. This technique could be especially useful to gauge intensity changes in ocean basins where there are not routine reconnaissance observations.

Solution.

(a) The centrifugal force for an air parcel with respect to Earth's rotation is given by

$$\frac{\vec{F}_{\text{centrifugal}}}{m} = \Omega^2 \vec{r}_{\perp} \tag{2.3}$$

where Ω is the Earth's angular speed and $\vec{r}_{\perp}$ is the perpendicular distance of the parcel from the axis of rotation. It is pointing radially outward. Locally, v is interpreted as the tangential speed of a parcel about the vortex center. It is analogous to Ω. In terms of the angular speed ω of the parcel about the vortex center, it can be written as $v = \omega r$. On the other hand, r is the distance of the parcel from the vortex center. It is analogous to $\vec{r}_{\perp}$. Thus,

$$\frac{F_{\text{local centrifugal}}}{m} = \frac{v^2}{r} = \frac{(\omega r)^2}{r} = \omega^2 r \tag{2.4}$$

which looks very similar to our original equation.

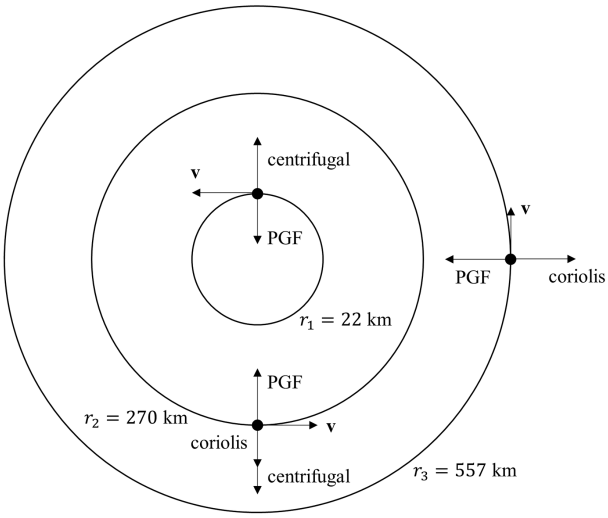
(b) If $Du/Dt = 0$, then $0 = f_0 v - \frac{\partial \Phi}{\partial r} + \frac{v^2}{r}$. We let our Rossby number R_o be the ratio between the local centrifugal force $\frac{v^2}{r}$ and Coriolis force $f_0 v$.

$$R_0 = \frac{\frac{v^2}{r}}{f_0 v} = \frac{v}{f_0 r} \tag{2.5}$$

Here, $f_0 = 2\Omega \sin\theta$. At 25.6^{o}N, this becomes $f_0 = 2(7.292 \times 10^{-5} s^{-1}) \sin(25.6^o)$. Using our equation for the Rossby number and plugging in the given values, we get the following table

r (km)	**v (m/s)**	$\mathbf{R_0}$	**balance**
557	10	0.285	geostrophic
270	18	1.06	gradient wind
22	70	50.5	cyclostrophic

At $r = 557\ km$, the Coriolis force is much greater than the local centrifugal force, so there is geostrophic balance (balance between Coriolis force and the PGF): $f_0 v = \frac{\partial \Phi}{\partial r}$. At $r = 270\ km$, the local centrifugal force is almost equal to the Coriolis force, so there is gradient wind balance (balance between the Coriolis & centrifugal force and the PGF): $f_0 v + \frac{v^2}{r} = \frac{\partial \Phi}{\partial r}$. At $r = 22\ km$, the local centrifugal force is much greater than the Coriolis force, so there is cyclostrophic balance (balance between centrifugal force and the PGF): $\frac{v^2}{r} = \frac{\partial \Phi}{\partial r}$.



At higher r (farther from the hurricane's eye), the PGF and the Coriolis force maintain the cyclonic motion of a parcel. At lower r (closer to the eye), the PGF and the centrifugal force maintain the cyclonic motion of the parcel. At middle r values, all three forces balanced together maintain the cyclonic motion of a parcel. This is all assuming $Du/Dt = 0$.

(c) By differentiating the gradient wind balance equation with respect to p, we get

$$\frac{\partial^2 \Phi}{\partial p \partial r} = \frac{\partial^2 \Phi}{\partial r \partial p} = \frac{\partial (f_0 v)}{\partial p} + \frac{\partial}{\partial p}\left(\frac{v^2}{r}\right) \tag{2.6}$$

Combining this with hydrostatic balance, we get

$$-\frac{\partial \alpha}{\partial r} = \frac{\partial (f_0 v)}{\partial p} + \frac{\partial}{\partial p}\left(\frac{v^2}{r}\right) \tag{2.7}$$

Using the ideal gas law, $\alpha = \frac{RT}{p}$, we get

$$-\frac{R}{p}\frac{\partial T}{\partial r} = f_0\frac{\partial v}{\partial p} + \frac{1}{r}\frac{\partial v^2}{\partial p} \tag{2.8}$$

We can take out p, the Coriolis coefficient, and the radius outside of the derivative since we're considering an isobaric surface with radius r from the vortex center and with central latitude θ_0. By simplifying the right hand side, we get

$$-\frac{R}{p}\frac{\partial T}{\partial r} = \left(f_0 + \frac{2v}{r}\right)\frac{\partial v}{\partial p} \tag{2.9}$$

Isolating the gradient wind, we get

$$\frac{\partial v}{\partial p} = -\frac{R}{p}\frac{r}{f_0 r + 2v}\frac{\partial T}{\partial r} \tag{2.10}$$

(d) In our equation for the thermal wind, the satellite data gives us information on $\frac{\partial T}{\partial r}$. We want to find a way to estimate v. Under the f-plane approximation, f_0 is constant about a central latitude θ_0. Here, we choose θ_0 depending on the latitude where the satellite data was taken. Besides f_0, R is also a universal constant. For simplicity, we can choose the hurricane center as the reference radius r_0 such that

$$\frac{\partial T}{\partial r} = \frac{\Delta T}{\Delta r} = \frac{T - T_0}{r - r_0} = \frac{T - T_0}{r} \tag{2.11}$$

where T and T_0 are the temperature at r and $r_0 = 0$, respectively, for a specific height, and $r_0 = 0$ is just the hurricane's center. Even if we are given the temperature anomalies, this equation should still be applicable. To solve for v, we apply a numerical solution. We set an initial solution for v to be equal to 0. Thus,

$$\frac{\partial v}{\partial p} = -\frac{R}{pf_0}\frac{\partial T}{\partial r} \qquad \text{or} \qquad dv = -\frac{R}{pf_0}\frac{\partial T}{\partial r}dp \tag{2.12}$$

We can integrate both sides to get an expression for v. To code this, one can use the summation estimation we use for integrals introduced during the first weeks of class.

$$\int f(t)dt \approx \sum_{i=1}^{N} f(t_i)\Delta t_i \tag{2.13}$$

For each pressure level p, the change in temperature with radius $\frac{\partial T}{\partial r}$ changes, so we need to keep that term inside the summation. After getting a new expression for v, we can plug this back in

$$\frac{\partial v}{\partial p} = -\frac{R}{p}\frac{r}{f_0 r + 2v}\frac{\partial T}{\partial r} \tag{2.14}$$

We keep integrating and solving for v until the difference between the new and old equations for v are small enough (set a threshold difference when coding). The final equation for v describes the gradient wind structure of the hurricane.

□