AATM 500 – Atmospheric Dynamics

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Homework 2

Problem 1

This problem will explore equatorial currents using dynamics and observations. Assume that sea surface height has deviations (η) from mean sea level, as illustrated in the diagram. Assume that the ocean is in hydrostatic balance.

(a) Derive an expression for the zonal (in the x-direction) pressure gradient force at some constant level $z = D$ at or below the ocean surface. In doing so, assume that the pressure at the ocean surface is constant.

(b) Write down an equation describing the acceleration for an ocean parcel in the x-direction (i.e., $du/dt = \cdots$), with the following forces acting on the parcel

- pressure gradient force from part (a)
- viscous force
- wind stress $= \epsilon\frac{2\rho_a C_d |u_s| u_s}{\rho_w h}$, where ρ_a is the density of air, ρ_w is the density of sea water, C_d is the drag coefficient, u_s is the surface zonal wind, and h is the height of the wind measurement.

The wind stress is the momentum imparted to the ocean by winds blowing along the ocean surface. The given formula for the wind stress is a **parametrization**. The constant ϵ represents a fraction of the stress that drives ocean currents. Below the surface, $\epsilon = 0$.

(c) Figure 1 shows TAO array data in the tropical Pacific. At $(150^o W, 0^o)$, estimate the i) zonal pressure gradient force and ii) wind stress at the ocean surface.

(d) Then assuming there is a force balance between the pressure gradient force and wind stress at the ocean surface, estimate ϵ. What does your answer imply about the efficiency at which winds drive ocean currents?

(e) Figure 2 shows a cross section ($x - z$ plane) along the Equator of the zonal ocean currents. At about 150 m depth, there is a belt of positive (west tio east) ocean currents. Explain why this "equatorial countercurrent" exists using dynamical reasoning.

(f) During an El Niño, easterly surface winds typically weaken (become less negative). Describe what must happen to the zonal structure of η and the strength of the equatorial countercurrent during El Niño.

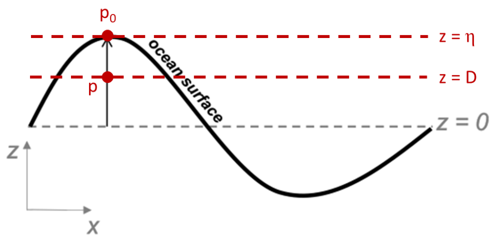
Solution.

(a) The pressure gradient force is given by

$$\frac{\vec{F}_p}{m} = -\frac{1}{\rho_w}\nabla p \tag{1.1}$$

where ρ_w is the ocean water density. However, since we're only interested in the horizontal pressure gradient along x, we use

$$\frac{F_{px}}{m} \underset{1}{=} -\frac{1}{\rho_w}\frac{\partial p}{\partial x} \tag{1.2}$$



Here, we need to find an expression for the pressure p with an x dependence. We use the hydrostatic balance equation to do this.

$$dp = -\rho_w g dz \tag{1.3}$$

where we can integrate both sides.

$$\int_p^{p_0} dp = -\int_D^{\eta} \rho_w g dz \tag{1.4}$$

Here, we integrate both sides from our height of interest to the ocean surface. The limits of the integrals on both sides of the equation should correspond to each other. At the ocean surface, the surface deviates by η, and its pressure is a constant p_0. At some constant height $z = D$, the pressure p is unknown. Evaluating this integral, we get

$$p_0 - p = -\rho_w g(\eta - D) \tag{1.5}$$

Isolating pressure, we have

$$p = p_0 + \rho_w g(\eta - D) \tag{1.6}$$

In this equation, p_0, ρ_w, g, and D are all constants, and only η has an x dependence such that

$$\frac{\partial p}{\partial x} = \rho_w g \frac{\partial \eta}{\partial x} \tag{1.7}$$

Plugging this into our equation for the horizontal pressure gradient, we get

$$\frac{F_{px}}{m} = -g\frac{\partial \eta}{\partial x} \tag{1.8}$$

(b) From Newton's 2nd law, we know the acceleration along x to be

$$\frac{du}{dt} = \frac{F_x}{m} \tag{1.9}$$

We are given that the total force along x is composed of the pressure gradient force, the viscous force, and the wind stress: $\frac{F_x}{m} = \frac{F_{px}}{m} + \frac{F_{vx}}{m} + \frac{F_{\text{stress}}}{m}$. The viscous force is given by

$$\frac{\vec{F_v}}{m} = \nu \nabla^2 \vec{v} \tag{1.10}$$

We solved in the 2nd worksheet the x component of the viscous force as

$$\frac{F_{vx}}{m} = \nu \nabla^2 u \tag{1.11}$$

We leave the equation for wind stress as is, so our horizontal acceleration becomes

$$\frac{du}{dt} = -g\frac{\partial \eta}{\partial x} + \nu \nabla^2 u + \epsilon \frac{2\rho_a C_d |u_s| u_s}{\rho_w h} \tag{1.12}$$

(c) Similar to our previous homework we can estimate: $\frac{\partial \eta}{\partial x} = \frac{\Delta \eta}{\Delta x}$. Thus, we can solve for the zonal pressure gradient force as

$$\frac{F_{px}}{m} = -g\frac{\Delta \eta}{\Delta x} \tag{1.13}$$

In Figure 1, the change in sea surface height deviations along x at 0^o latitude is about $\Delta\eta = -5\ cm = -0.05\ m$ for every $\Delta\lambda = 10^o$ longitude eastward. From worksheet 1, we determined how to convert a change in longitude to meters: $\Delta x = \Delta\lambda(a\cos\theta)\frac{\pi}{180}$. We're looking at latitude $\theta = 0^o$, so this just becomes

$$\Delta x = a\Delta\lambda\frac{\pi}{180} \tag{1.14}$$

where $a = 6.38 \times 10^6\ m$ is the Earth's radius. Plugging all our given values to our equation for zonal pressure gradient force, we get

$$\frac{F_{px}}{m} = -(9.8\ m/s^2)\frac{-0.05\ m}{(6.38 \times 10^6\ m)(10^o)\frac{\pi}{180}} = 4.4 \times 10^{-7}\ m/s^2 \tag{1.15}$$

From Figure 1, the zonal wind on the ocean's surface at $(150^oW, 0^o)$ is about $-8\ m/s$. Plugging this and the given constants into our equation for wind stress, we get

$$\begin{aligned} \text{wind stress} &= \epsilon\frac{2(1\ kg/m^3)(1.5 \times 10^{-3})|-8\ m/s|(-8\ m/s)}{(1000\ kg/m^3)(4\ m)} \\ \text{wind stress} &= \epsilon(-4.8 \times 10^{-5}\ m/s^2) \end{aligned} \tag{1.16}$$

(d) We assume a force balance between the pressure gradient force and wind stress at the ocean surface, so

$$0 = -g\frac{\Delta\eta}{\Delta x} + \nu\nabla^2 u + \epsilon\frac{2\rho_a C_d|u_s|u_s}{\rho_w h} \tag{1.17}$$

Here, we ignore the viscosity term, so

$$g\frac{\Delta\eta}{\Delta x} = \epsilon\frac{2\rho_a C_d|u_s|u_s}{\rho_w h} \tag{1.18}$$

Isolating ϵ, we get

$$\epsilon = g\frac{\Delta\eta}{\Delta x}\frac{\rho_w h}{2\rho_a C_d|u_s|u_s} = g\frac{\Delta\eta}{a\Delta\lambda\frac{\pi}{180}}\frac{\rho_w h}{2\rho_a C_d|u_s|u_s} \tag{1.19}$$

Plugging in all our constants and estimated values, we get

$$\begin{aligned} \epsilon &= (9.8\ m/s^2)\frac{-0.05\ m}{(6.38 \times 10^6\ m)(10^o)\frac{\pi}{180}}\frac{(1000\ kg/m^3)(4\ m)}{2(1\ kg/m^3)(1.5 \times 10^{-3})|-8\ m/s|(-8\ m/s)} \\ \epsilon &= 9.2 \times 10^{-3} \end{aligned} \tag{1.20}$$

We notice that the force due to the non-parametrized wind stress is much greater than the pressure gradient force. In fact, only a very small portion of the wind stress is needed to drive ocean currents, as indicated by the very small ϵ. Thus, we can say that wind stress is a very efficient driver of ocean currents.

(e) Here, we return to our equation for horizontal acceleration Eqn. 1.12. Below the ocean surface, the wind stress disappears since $\epsilon = 0$. We only consider the pressure gradient force and the viscosity, where the PGF is the primary driver of currents below the ocean surface. Since sea surface deviations are higher to the west, the pressure is higher to the west at 150 km depth and lower to the east. Thus, the pressure gradient force, which is pointing from high to low, is pointing from west to east, causing the currents to follow the direction of the PGF.

(f) In Eq. 1.17, if u_s becomes less negative, then the change of η should decrease along the eastward direction. Since the change of η decreases, the ocean surface becomes a bit more uniform throughout the Pacific. Consequently, the pressure gradient force also decreases at deeper levels. Thus, since the equatorial countercurrents are driven by the pressure gradient force, it weakens during El Niño.

$\square$

Problem 2 (a) This problem will explore the energetics of a thermally direct circulation. The diagram below shows a fluid circulating in a closed box. Heat is added into the fluid at the bottom left, and heat is removed out of the fluid at the top right. The resulting circulation is thermally direct because it rises where it is warm and sinks where it is cold, so is driven by buoyancy differences across the fluid.

Beginning with this form of the first law of thermodynamics

$$d(c_pT) = dQ + \alpha dp \tag{1.21}$$

Assuming a reversible system, use the second law of thermodynamics (in differential form) tio eliminate dQ.

(b) We will also assume a steady-state system. As a result, Bernoulli's equation states that along streamlines,

$$d\left(\frac{1}{2}|\vec{v}|^2\right) + d(gz) + \alpha dp + Fdl = 0 \tag{1.22}$$

This form looks a bit different from the Bernoulli function we derived in class. Additionally, we have included the effects of friction, where F is friction and dl is an incremental distance along a streamline. Use your answer in part (a) to eliminate the αdp term in Bernoulli's equation.

(c) Line integrate your answer in part (b) around the closed loop in the diagram, which we will take to be a streamline. The loop consists of four branches: (i) an isothermal branch at constant temperature T_1, (ii) a rising adiabatic branch at constant entropy η_1, (iii) an isothermal branch at constant temperature T_2, (iv) a sinking adiabaticbranch at constant entropy η_2.

(d) Assume that friction can be parametrized as the drag coefficient times the wind speed:

$$\oint Fdl = C_d|\vec{v}|^2 \tag{1.23}$$

and that the heat input, by the second law of thermodynamics is

$$(\eta_1 - \eta_2) = \frac{Q_{in}}{T_1} \tag{1.24}$$

Derive an expression for the wind speed $|\vec{v}|^2$, and the describe what would cause the wind speed to increase.

(e) A prominent example of a thermally direct circulation occurs in a hurricane, where can you imagine the left hand side of the diagram is the eyewall and the right side of the diagram is the far environment. How does this simple model help us understand analogous processes that control hurricane intensity? What might be some major limitations of this simple model?

Solution.

(a) The second law of thermodynamics (assuming a reversible system) in differential form is given by

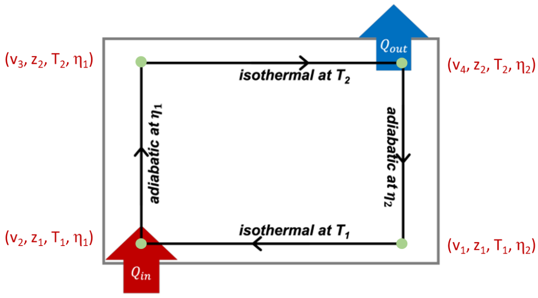
$$Td\eta = dQ \tag{1.25}$$

where T is the temperature and η is the entropy. Plugging this into the first law of thermodynamics, we get

$$d(c_pT) = Td\eta + \alpha dp \tag{1.26}$$

(b) From our answer in (a), we see that $\alpha dp = d(c_pT) - Td\eta$. Plugging this into Bernoulli's equation, we get

$$d\left(\frac{1}{2}|\vec{v}|^2\right) + d(gz) + d(c_pT) - Td\eta + Fdl = 0 \tag{1.27}$$



(c) We perform the close loop integral for each term in our equation.

$$\oint d\left(\frac{1}{2}|\vec{v}|^2\right) + \oint d(gz) + \oint d(c_pT) - \oint Td\eta + \oint Fdl = 0 \tag{1.28}$$

We separate the close loop integral into integrals per branch such that

$$\oint = \int_{\text{branch}_1} + \int_{\text{branch}_2} + \int_{\text{branch}_3} + \int_{\text{branch}_4} \tag{1.29}$$

We evaluate the close loop integrals per term. For the first term, we let $|\vec{v_1}|^2$, $|\vec{v_2}|^2$, $|\vec{v_3}|^2$, and $|\vec{v_4}|^2$ be the wind speeds at the corners starting from the lower right going clockwise. We can also ignore the $\frac{1}{2}$ for now since it's just a constant. Thus, the close loop integral becomes

$$\begin{aligned} \oint d|\vec{v}|^2 &= |\vec{v}|^2\Big|_{\vec{v_1}}^{\vec{v_2}} + |\vec{v}|^2\Big|_{\vec{v_2}}^{\vec{v_3}} + |\vec{v}|^2\Big|_{\vec{v_3}}^{\vec{v_4}} + |\vec{v}|^2\Big|_{\vec{v_4}}^{\vec{v_1}} \\ &= (|\vec{v_2}|^2 - |\vec{v_1}|^2) + (|\vec{v_3}|^2 - |\vec{v_2}|^2) + (|\vec{v_4}|^2 - |\vec{v_3}|^2) + (|\vec{v_1}|^2 - |\vec{v_4}|^2) \\ \frac{1}{2}\oint d|\vec{v}|^2 &= 0 \end{aligned} \tag{1.30}$$

For the second term, we let z_1 be the altitude at the lower segment and z_2 the altitude at the upper segment. We can take g out of the integral, since it's constant with respect to z.

$$\begin{aligned} g\oint dz &= gz\Big|_{z_1}^{z_1} + gz\Big|_{z_1}^{z_2} + gz\Big|_{z_2}^{z_2} + gz\Big|_{z_2}^{z_1} \\ &= g(z_1 - z_1) + g(z_2 - z_1) + g(z_2 - z_2) + g(z_1 - z_2) \\ g\oint dz &= 0 \end{aligned} \tag{1.31}$$

We repeat a similar process with the remaining terms. We can take c_p out of the third integral, since it's constant with respect to T. Note that T is constant along the first and third branch, since they're both isotherms.

$$
\begin{aligned}
c_p \oint dT &= gT\Big|_{T_1}^{T_1} + c_pT\Big|_{T_1}^{T_2} + gT\Big|_{T_2}^{T_2} + c_pT\Big|_{T_2}^{T_1} \\
&= c_p(T_1 - T_1) + c_p(T_2 - T_1) + c_p(T_2 - T_2) + c_p(T_1 - T_2) \\
c_p \oint dT &= 0
\end{aligned} \tag{1.32}
$$

Finally, we evaluate the fourth term. Here, we also take T out of the integral since we're integrating with respect to η.

$$
\begin{aligned}
T \oint d\eta &= T\eta\Big|_{\eta_2}^{\eta_1} + T\eta\Big|_{\eta_1}^{\eta_1} + T\eta\Big|_{\eta_1}^{\eta_2} + T\eta\Big|_{\eta_2}^{\eta_2} \\
&= T_{\text{branch}_1}(\eta_1 - \eta_2) + T_{\text{branch}_2}(\eta_1 - \eta_1) + T_{\text{branch}_3}(\eta_2 - \eta_1) + T_{\text{branch}_4}(\eta_2 - \eta_2) \\
-\oint T d\eta &= -T_1(\eta_1 - \eta_2) - T_2(\eta_2 - \eta_1)
\end{aligned} \tag{1.33}
$$

Note that the integrals along the second and fourth branches disappear since they have constant entropy, and the temperatures at branch 1 and 3 are just constant values T_1 and T_2, respectively. For now, we leave the final integral as is. Thus, the close loop integral of our Bernoulli equation becomes

$$
-T_1(\eta_1 - \eta_2) - T_2(\eta_2 - \eta_1) + \oint F dl = 0 \tag{1.34}
$$

(d) Using the second law of thermodynamics, we can also say that the heat output is given by

$$
(\eta_2 - \eta_1) = -\frac{Q_{out}}{T_2} \tag{1.35}
$$

Here, the right side is negative since the system is losing heat. Plugging all our given equations to the close loop integral of the Bernoulli equation, we get

$$
\begin{aligned}
-T_1\frac{Q_{in}}{T_1} + T_2\frac{Q_{out}}{T_2} + C_d|\vec{v}|^2 &= 0 \\
Q_{out} - Q_{in} + C_d|\vec{v}|^2 &= 0
\end{aligned} \tag{1.36}
$$

By isolating the wind speed, we get

$$
|\vec{v}|^2 = \frac{Q_{in} - Q_{out}}{C_d} \tag{1.37}
$$

Based on this equation, we can increase the system's wind speed by either increasing the heat input or decreasing the heat output of the system.

(e) Hurricanes (and tropical cyclones) typically form over warm oceans (e.g. the west Pacific warm pool). Warm oceans provide a huge amount of heat and energy to fuel and strengthen a hurricane. However, as the hurricane makes landfall, it gets less heat and energy from land. In equation 1.36, the warm oceans increase Q_{in}, while land decreases Q_{in}. As the hurricane convects warm air upwards, the warm air begins to cool, disperse, and sink, causing the hurricane system to lose heat and energy (increasing Q_{out}). While traversing warm oceans, the heat input Q_{in} greatly compensates for the slowly dissipating Q_{out}. However, once the hurricane makes landfall, the land cannot provide enough Q_{in} to offset Q_{out}. Thus, the hurricanes weaken. Other major factors that affect hurricanes (and tropical cyclones) intensity are land terrain (lowlands vs mountains) and wind shear, which our simple model doesn't necessarily consider.

□