AATM 500 – Atmospheric Dynamics

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Homework 1

Problem 1

The table below gives data from four New York State Mesonet stations at 10 UTC 18 February 2022. Note that the provided locations have been slightly adjusted from their actual locations in order to make the calculations more straightforward.

Station	Longitude	Latitude	u (m/s)	v (m/s)	T (C)
Oppenheim	-74.6	43.0	6.9	-1.2	-0.5
Edinburg	-74.1	43.2	4.9	0.9	9.4
Voorheesville	-74.1	42.8	3.8	1.9	12.8
Schuylerville	-73.6	43.0	1.5	4.2	14.4

(a) Using this data, calculate the following quantities at the midpoint between these stations:
- Horizontal divergence
- Relative vorticity (vertical component)
- Horizontal temperature advection

(b) Assume the density is constant. Would you expect there to be rising or sinking motion just above the surface at the midpoint? Justify your answer.

(c) Are your answers from parts (a) and (b) consistent with a passing low-pressure area (as we saw in Worksheet 1) over the Capital Region? Explain.

(d) The temperature drops from $11.7^o\ C$ at 09 UTC to $2.2^o\ C$ at 11 UTC at Ballston Spa, which is near the midpoint location. Determine how much heating/cooling has occurred over this 2-h period for a parcel being advected by the horizontal flow through this location.

Solution.

(a) The divergence of a velocity vector field is given by $\nabla \cdot \vec{v} = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}$. However, we're only interested in the horizontal divergence, so we use

$$\nabla \cdot \vec{v_h} = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \tag{1.1}$$

From here, we follow the same steps as worksheet 1. Using our estimation of derivatives for discrete data points, we have

$$\frac{\partial u}{\partial x} = \frac{u_S - u_O}{\Delta x} \quad \text{and} \quad \frac{\partial v}{\partial y} = \frac{v_E - v_V}{\Delta y} \tag{1.2}$$

We determined in class how to solve for Δx and Δy.

$$\Delta x = \Delta\lambda(a\cos\theta) \quad \text{and} \quad \Delta y = a\Delta\theta \tag{1.3}$$

where $\Delta\lambda$ is the change in longitude along x, $\Delta\theta$ is the change in latitude along y, and $a = 6.38 \times 10^6\ m$ is the Earth's radius. We also have to convert degrees into radians, so

$$\Delta x = \Delta\lambda(a\cos\theta)\frac{\pi}{180} \qquad \text{and} \qquad \Delta y = a\Delta\theta\frac{\pi}{180} \tag{1.4}$$

We can then rewrite our partial derivatives as

$$\frac{\partial u}{\partial x} = \frac{u_S - u_O}{\Delta\lambda(a\cos\theta)\frac{\pi}{180}} \qquad \text{and} \qquad \frac{\partial v}{\partial y} = \frac{v_E - v_V}{a\Delta\theta\frac{\pi}{180}} \tag{1.5}$$

Finally, our horizontal divergence becomes

$$\nabla \cdot \vec{v_h} = \frac{u_S - u_O}{\Delta\lambda(a\cos\theta)\frac{\pi}{180}} + \frac{v_E - v_V}{a\Delta\theta\frac{\pi}{180}} \tag{1.6}$$

We can now plug in our given values.

$$\nabla \cdot \vec{v_h} = \frac{(1.5 - 6.9)m/s}{(6.38 \times 10^6\ m \cdot \cos 43^o)\left(1^o \cdot \frac{\pi}{180}\right)} + \frac{(0.9 - 1.9)m/s}{(6.38 \times 10^6\ m)\left(0.4^o \cdot \frac{\pi}{180}\right)} = -8.88 \times 10^{-5}\frac{m/s}{m} \tag{1.7}$$

The relative vorticity is given by $\hat{k} \cdot (\nabla \times \vec{v})$. In other words, it's the vertical component of the curl of the velocity vector field. We have determined in class that

$$\vec{\zeta_v} = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} \tag{1.8}$$

Similar to the horizontal divergence, we have

$$\frac{\partial v}{\partial x} = \frac{v_S - v_O}{\Delta x} \qquad \text{and} \qquad \frac{\partial v}{\partial y} = \frac{u_E - u_V}{\Delta y} \tag{1.9}$$

Using what we have solved for previously, we have the final expression for the relative vorticity.

$$\vec{\zeta_v} = \frac{v_S - v_O}{\Delta\lambda(a\cos\theta)\frac{\pi}{180}} - \frac{u_E - u_V}{a\Delta\theta\frac{\pi}{180}} \tag{1.10}$$

Plugging in our given values, we get

$$\vec{\zeta_v} = \frac{(4.2 + 1.2)m/s}{(6.38 \times 10^6\ m \cdot \cos 43^o)\left(1^o \cdot \frac{\pi}{180}\right)} - \frac{(4.9 - 3.8)m/s}{(6.38 \times 10^6\ m)\left(0.4^o \cdot \frac{\pi}{180}\right)} = 4.16 \times 10^{-5}\frac{m/s}{m} \tag{1.11}$$

The horizontal temperature advection is given by

$$A = -\vec{v_h} \cdot \nabla T_h \tag{1.12}$$

where $\vec{v_h}$ is the horizontal velocity vector at the midpoint. Since we are dealing with horizontal temperature advection, we have

$$A_h = -(u\hat{\imath} + v\hat{\jmath}) \cdot \left(\frac{\partial T}{\partial x}\hat{\imath} + \frac{\partial T}{\partial y}\hat{\jmath}\right) \tag{1.13}$$

Simplifying this, we get

$$A_h = -\left(u_m\frac{\partial T}{\partial x} + v_m\frac{\partial T}{\partial y}\right) \tag{1.14}$$

Similar to the previous problems, we can estimate the partial derivatives as

$$\frac{\partial T}{\partial x} = \frac{T_S - T_O}{\Delta x} \qquad \text{and} \qquad \frac{\partial T}{\partial y} = \frac{T_E - T_V}{\Delta y} \tag{1.15}$$

Again, after rewriting the denominator, we get

$$\frac{\partial T}{\partial x} = \frac{T_S - T_O}{\Delta\lambda(a\cos\theta)\frac{\pi}{180}} \qquad \text{and} \qquad \frac{\partial T}{\partial y} = \frac{T_E - T_V}{a\Delta\theta\frac{\pi}{180}} \tag{1.16}$$

To get the velocity vector at the midpoint, we use the horizontal velocity gradient, $\nabla v_h = \frac{\partial u}{\partial x}\hat{i} + \frac{\partial v}{\partial y}\hat{j}$. The velocity gradient tells us how much the wind velocity changes with respect to distance traveled. At the midpoint, the wind has only traveled $\frac{\Delta x}{2}$ along x, and $\frac{\Delta y}{2}$ along y. Thus, using our estimates for partial derivatives, the horizontal velocity at midpoint must be

$$u_m = u_O + \frac{\partial u}{\partial x}\frac{\Delta x}{2} \qquad \text{and} \qquad v_m = v_V + \frac{\partial v}{\partial y}\frac{\Delta y}{2} \tag{1.17}$$

or

$$u_m = u_O + \frac{u_S - u_O}{\Delta x}\frac{\Delta x}{2} \qquad \text{and} \qquad v_m = v_V + \frac{v_E - v_V}{\Delta y}\frac{\Delta y}{2} \tag{1.18}$$

Simplifying this, we get

$$u_m = u_O + \frac{u_S - u_O}{2} \qquad \text{and} \qquad v_m = v_V + \frac{v_E - v_V}{2} \tag{1.19}$$

which is essentially just the average of the given velocities from the surrounding stations, or

$$u_m = \frac{u_S + u_O}{2} \qquad \text{and} \qquad v_m = \frac{v_E + v_V}{2} \tag{1.20}$$

Our final equation for the horizontal temperature advection becomes

$$A_h = -\left[\left(\frac{u_S + u_O}{2}\right)\left(\frac{T_S - T_O}{\Delta\lambda(a\cos\theta)\frac{\pi}{180}}\right) + \left(\frac{v_E + v_V}{2}\right)\left(\frac{T_E - T_V}{a\Delta\theta\frac{\pi}{180}}\right)\right] \tag{1.21}$$

Plugging in all our given values, we get

$$\begin{aligned} A_h &= -\left[\left(\frac{(1.5 + 6.9)m/s}{2}\right)\left(\frac{(14.4 + 0.5)^oC}{(6.38 \times 10^6\ m \cdot \cos 43^o)\left(1^o \cdot \frac{\pi}{180}\right)}\right)\right. \\ &\quad \left. + \left(\frac{(0.9 + 1.9)m/s}{2}\right)\left(\frac{(9.4 - 12.8)^oC}{(6.38 \times 10^6\ m)\left(0.4^o \cdot \frac{\pi}{180}\right)}\right)\right] \\ A_h &= -6.62 \times 10^{-4}\ {}^oC/s \end{aligned} \tag{1.22}$$

(b) We expect there to be a rising motion just above the surface at the midpoint. We calculated a negative horizontal divergence, so wind converges at the midpoint of these four stations. The positive vorticity also indicates a cyclonic rotation of wind (since we're in the northern hemisphere). Thus, as air accumulates from the convergence of winds, it begins to rise in a cyclonic motion.

(c) A passing low pressure area is consistent with the convergent winds near the surface, and the cyclonic upward rotation of wind. As the winds near the surface converge, it begins to lift upward (within the troposphere) until they begin to diverge aloft.

(d) We have previously obtained our horizontal temperature advection as about $-6.62 \times 10^{-4}\ ^oC/s$. In two hours, we have 7200 seconds. Using our horizontal temperature advection,

$$\frac{-6.62 \times 10^{-4}\ ^oC}{1\ s} \cdot \frac{7200\ s}{2\ hrs} = \frac{-4.76^oC}{2\ hrs} \tag{1.23}$$

Thus, a parcel being advected through this location cooled by about 4.76^oC within this 2-hour period.

$\square$

Problem 2 (a) Derive a conservation equation for the water vapor density, ρ_w. Begin with the conservation equation for the water vapor mixing ratio, q (kg of water vapor per kg of dry air):

$$\frac{Dq}{Dt} = E - C \tag{1.24}$$

where E is the evaporation rate and C is the condensation rate. Combine this equation with the conservation of mass equation (for dry air) to arrive at a conservation equation for $\rho_w = \rho q$. Simplify to the extent possible, and express your answer as

$$\frac{\partial(\rho q)}{\partial t} = \cdots \tag{1.25}$$

(b) Vertically integrate your answer from part (a) from $z = 0$ to ∞. Assume that the vertical velocity vanishes at $z = 0$ and ∞. Also, use

$$\int_0^\infty = \rho(E - C)dz = F_s - P \tag{1.26}$$

where F_s is evaporation rate from the surface and P is the precipitation rate at the surface. Describe qualitatively what each term represents. *This equation is central to understanding the water vapor budget of atmospheric systems, such as atmospheric rivers.*

(c) Instead of integrating with respect to height, we can integrate with respect to pressure by applying a transformation via the hydrostatic relationship: $dp = \rho g dz$. Transform the integral on the left-hand side of your answer to part (b) to integrate with respect to pressure. What advantage does this transformation offer?

(d) The following table gives radiosonde moisture data at KALB on 12 UTC 4 August 2022 and 00 UTC 5 August 2022. The data are given at mandatory pressure levels. Estimate the vertical integral, without the time derivative, you derived in part (c) (i.e., the "precipitable water") at these two times. In doing so, you may assume that the limits of integration are 1000 and 200 hPa. *Remember that the SI unit of pressure is a Pa.*

Pressure (hPa)	**q at 12 UTC (kg/kg)**	**q at 00 UTC (kg/kg)**
1000	11.87×10^{-3}	15.53×10^{-3}
925	12.03×10^{-3}	9.35×10^{-3}
850	8.31×10^{-3}	7.14×10^{-3}
700	3.51×10^{-3}	6.18×10^{-3}
500	0.15×10^{-3}	3.86×10^{-3}
400	0.18×10^{-3}	1.92×10^{-3}
300	0.07×10^{-3}	0.57×10^{-3}
200	0.02×10^{-3}	0.05×10^{-3}

(e) Make a hypothesis for what caused the 12-h change in precipitable water. Your hypothesis does not have to be correct, but should be reasonable. How would you test your hypothesis (i.e., what data would you need and what calculations would you make to validate or refute your hypothesis)?

Solution.

(a) Here, we first expand the total derivative as

$$\frac{Dq}{Dt} = \frac{\partial q}{\partial t} + \vec{v} \cdot \nabla q = E - C \tag{1.27}$$

The water vapor mixing ratio is given by $q = \frac{\rho_w}{\rho}$, so our local derivative becomes

$$\frac{\partial q}{\partial t} = \frac{\partial \rho_w \rho^{-1}}{\partial t} \tag{1.28}$$

Using chain rule, we have

$$\begin{aligned} \frac{\partial \rho_w \rho^{-1}}{\partial t} &= \frac{1}{\rho}\frac{\partial \rho_w}{\partial t} - \frac{\rho_w}{\rho^2}\frac{\partial \rho}{\partial t} \\ &= \frac{1}{\rho}\left(\frac{\partial \rho_w}{\partial t} - \frac{\rho_w}{\rho}\frac{\partial \rho}{\partial t}\right) \\ \frac{\partial q}{\partial t} &= \frac{1}{\rho}\left(\frac{\partial \rho_w}{\partial t} - q\frac{\partial \rho}{\partial t}\right) \end{aligned} \tag{1.29}$$

We now have a $\frac{\partial \rho}{\partial t}$ term, so we can use the Eulerian form of the mass conservation equation for dry air: $\frac{\partial \rho}{\partial t} = -\nabla \cdot (\rho\vec{v})$. Thus, we get

$$\frac{\partial q}{\partial t} = \frac{1}{\rho}\left(\frac{\partial \rho_w}{\partial t} + q\nabla \cdot (\rho\vec{v})\right) \tag{1.30}$$

By plugging this into our equation for conservation of water vapor mixing ratio, we get

$$\frac{1}{\rho}\left(\frac{\partial \rho_w}{\partial t} + q\nabla \cdot (\rho\vec{v})\right) + \vec{v} \cdot \nabla q = E - C \tag{1.31}$$

Simplifying this further by multiplying both sides by ρ, we have

$$\frac{\partial \rho_w}{\partial t} + q\nabla \cdot (\rho\vec{v}) + (\rho\vec{v}) \cdot \nabla q = \rho(E - C) \tag{1.32}$$

Note that we have the identity $\nabla \cdot (b\vec{a}) = \vec{a} \cdot \nabla b + b\nabla \cdot \vec{a}$. Thus, we can say that $\nabla \cdot (q\rho\vec{v}) = (\rho\vec{v}) \cdot \nabla q + q\nabla \cdot (\rho\vec{v})$. Our final equation becomes

$$\frac{\partial \rho_w}{\partial t} = \frac{\partial(\rho q)}{\partial t} = \rho(E - C) - \nabla \cdot (q\rho\vec{v}) \tag{1.33}$$

(b) To integrate by height z, we have

$$\int_0^\infty \frac{\partial(\rho q)}{\partial t} dz = \int_0^\infty \rho(E - C) dz - \int_0^\infty \nabla \cdot (q\rho\vec{v})\, dz \tag{1.34}$$

We already know that $\int_0^\infty \rho(E - C)dz = F_s - P$. We expand the integral of the divergence as

$$\int_0^\infty \nabla \cdot (q\rho\vec{v})\, dz = \int_0^\infty \frac{\partial(\rho q u)}{\partial x} dz + \int_0^\infty \frac{\partial(\rho q v)}{\partial y} dz + \int_0^\infty \frac{\partial(\rho q w)}{\partial z} dz \tag{1.35}$$

The first two terms is just the horizontal divergence and can be combined as

$$\int_0^\infty \frac{\partial(\rho q u)}{\partial x} dz + \int_0^\infty \frac{\partial(\rho q v)}{\partial y} dz = \int_0^\infty \nabla_h \cdot (\rho q \vec{v}_h) dz \tag{1.36}$$

The last term is simply

$$\int_0^\infty \frac{\partial(\rho q w)}{\partial z} dz = (\rho q w)\Big|_0^\infty = 0 \tag{1.37}$$

This cancels to zero since we were given the boundary conditions that the vertical velocity w vanishes at 0 and ∞. Thus, the integral of our divergence is just

$$\int_0^\infty \nabla \cdot (q\rho\vec{v})\, dz = \int_0^\infty \nabla_h \cdot (\rho q \vec{v}_h) dz \tag{1.38}$$

Our final equation is then

$$\int_0^\infty \frac{\partial(\rho q)}{\partial t} dz = F_s - P - \int_0^\infty \nabla_h \cdot (\rho q \vec{v}_h) dz \tag{1.39}$$

The left-hand side is just the total change in water vapor density (and consequently, the water vapor) in the atmosphere. F_s dictates how much water vapor gets evaporated from the surface, and P dictates how much gets precipitated. Lastly, the final term dictates the total convergence of water vapor density flux.

(c) We are given the hydrostatic equilibrium equation

$$dz = -\frac{1}{\rho g} dp \tag{1.40}$$

By plugging this into the left hand side of our final equation in (b), we get

$$\int_{p_0}^{p_\infty} \frac{\partial(\rho q)}{\partial t} \left(-\frac{1}{\rho g}\frac{dp}{dz}\right) dz = -\frac{1}{g}\int_{p_0}^{p_\infty} \frac{\partial q}{\partial t} dp \tag{1.41}$$

Here, we would need to change the limits of our integration from the surface pressure p_s to 0.

$$-\frac{1}{g}\int_{p_s}^{0} \frac{\partial q}{\partial t} dp \tag{1.42}$$

This equation removes the need to measure the density of dry air; only the water vapor mixing ratio is needed.

(d) Without the time derivative in our integral at (c), we would get the precipitable water PW

$$PW = -\frac{1}{g}\int_{p_s}^{0} q dp \tag{1.43}$$

We change the limits of integration to the given values

$$PW = -\frac{1}{g}\int_{1000\ hPa}^{200\ hPa} q dp \tag{1.44}$$

To estimate this integral, we use the summation approximation introduced in class.

$$\int f(t) dt \approx \sum_{i=1}^{N} f(t_i) \cdot \Delta t_i \tag{1.45}$$

For our integral, this would become

$$PW = -\frac{1}{g}\sum_{i=1}^{N} q_i \cdot \Delta p_i \tag{1.46}$$

where g is gravity (constant), q_i is the midpoint (or average) between consecutive q values, and Δp_i is the change in consecutive pressures. At 12 UTC, we expand this to get

$$
\begin{aligned}
PW &= \frac{1}{2g}\big[(11.87\times10^{-3}+12.03\times10^{-3})(1000-925)\\
&\quad+(12.03\times10^{-3}+8.31\times10^{-3})(925-850)+(8.31\times10^{-3}+3.51\times10^{-3})(850-700)\\
&\quad+(3.51\times10^{-3}+0.15\times10^{-3})(700-500)+(0.15\times10^{-3}+0.18\times10^{-3})(500-400)\\
&\quad+(0.18\times10^{-3}+0.07\times10^{-3})(400-300)+(0.07\times10^{-3}+0.02\times10^{-3})(300-200)\big]\\
&= \frac{1}{2\cdot 9.81\ m/s^2}\left(5.89\frac{kg\cdot hPa}{kg}\right)\\
&= 30.1\ \frac{kg\cdot Pa}{kg\cdot m/s^2} = 30.1\ \frac{kg^2/(ms^2)}{kg\cdot m/s^2}\\
PW &= 30.1\ kg/m^2
\end{aligned}
\tag{1.47}
$$

The 2 in the denominator accounts for the average of each q pair. Pascals can also be written as $kg/(ms^2)$. (Note that I wrote the units only at the end just to save space.) We repeat the same at 00 UTC to get

$$
\begin{aligned}
PW &= \frac{1}{2g}\big[(15.53\times10^{-3}+9.35\times10^{-3})(1000-925)\\
&\quad+(9.35\times10^{-3}+7.14\times10^{-3})(925-850)+(7.14\times10^{-3}+6.18\times10^{-3})(850-700)\\
&\quad+(6.18\times10^{-3}+3.86\times10^{-3})(700-500)+(3.86\times10^{-3}+1.92\times10^{-3})(500-400)\\
&\quad+(1.92\times10^{-3}+0.57\times10^{-3})(400-300)+(0.57\times10^{-3}+0.05\times10^{-3})(300-200)\big]\\
&= \frac{1}{2\cdot 9.81\ m/s^2}\left(7.99775\frac{kg\cdot hPa}{kg}\right)\\
&= 40.8\ \frac{kg\cdot Pa}{kg\cdot m/s^2} = 40.8\ \frac{kg^2/(ms^2)}{kg\cdot m/s^2}\\
PW &= 40.8\ kg/m^2
\end{aligned}
\tag{1.48}
$$

(e) One component that can contribute to an increase in precipitable water is an increase in evaporation rate. Possibly, the temperature increased between August 4 12 UTC and August 5 00 UTC. Another possible component to consider is the horizontal divergence of the wind. Since the amount of precipitable water increased, then maybe the horizontal wind was diverging. If we have horizontal wind measurements between August 4 12 UTC and August 5 00 UTC, we can calculate the changes in their divergence.

□