AATM 500 – Atmospheric Dynamics

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Homework 3

Problem 1 (a) Prove that the Coriolis force, acting alone, cannot change the kinetic energy of a parcel.

(b) Based on this fact, explain why the method used to solve the problem in Worksheet 3 is problematic for a very long time step.

Solution.

(a) We have determined in class that the change in kinetic energy is given by

$$\frac{DK}{Dt} = \mathbf{v} \cdot \frac{D\mathbf{v}}{Dt} \tag{1.1}$$

We know that the fundamental momentum equation in a rotating frame is just

$$\frac{D\mathbf{v}}{Dt} = \mathbf{g} - \frac{1}{\rho}\nabla p + \nu\nabla^2\mathbf{v} - (2\boldsymbol{\Omega} \times \mathbf{v}) \tag{1.2}$$

If the Coriolis force is acting alone then

$$\frac{D\mathbf{v}}{Dt} = -(2\boldsymbol{\Omega} \times \mathbf{v}) \tag{1.3}$$

Plugging this into our equation for the change in kinetic energy, we get

$$\frac{DK}{Dt} = -2\mathbf{v} \cdot (\boldsymbol{\Omega} \times \mathbf{v}) \tag{1.4}$$

We can rearrange this to become

$$\frac{DK}{Dt} = -2\boldsymbol{\Omega} \cdot (\mathbf{v} \times \mathbf{v}) \tag{1.5}$$

The self cross product of a vector is just the zero vector: $\mathbf{v} \times \mathbf{v} = \mathbf{0}$. Thus,

$$\frac{DK}{Dt} = -2\boldsymbol{\Omega} \cdot \mathbf{0} = 0 \tag{1.6}$$

We have shown that the Coriolis force acting alone produces zero change in the kinetic energy.

(b) The Coriolis force does not do work on the parcel, so it cannot change its kinetic energy. This means that the Coriolis force can only change the direction of the parcel and not its speed. In worksheet 3, at much longer time steps, the parcel would never slow down (or speed up); it's speed would remain the same. The parcel will just keep moving and spiralling, a very unrealistic phenomena. Models that use this approximation could blow up at longer time scales.

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Problem 2 (a) The figure below shows an observed parcel path in the Baltic Sea. The parcel makes a clockwise looping motion as it moves northward. Such motions are commonly observed by drifting buoys the wake of storms. The problem will explore this curious behavior.

Consider only horizontal flow (u, v) on an f-plane. Assume that the Coriolis force is the only force acting to accelerate the parcel. In this case, the horizontal momentum equations are

$$\begin{aligned}\frac{Du}{Dt} &= f_0 v \\ \frac{Dv}{Dt} &= -f_0 u\end{aligned} \tag{1.7}$$

Using these equations, form a homogenous 2nd order ordinary differential equation with constant coefficients.

(b) Solve the 2nd order ordinary differential equations to arrive at expressions for both u and v, given an initial horizontal velocity at $t = 0$ of (u_0, v_0) which are constants.

(c) Presume that $u_0 > 0$ and $v_0 = 0$. The initial horizontal position at $t = 0$ is at the origin $(0, 0)$. Solve for the position (x, y) as a function of time, and draw a sketch of the trajectory.

(d) Based on your solution to part (c), what is the theoretical period of oscillation? Given an average latitude of the Baltic Sea of $58^o\ N$, and approximate 11 oscillations made by the parcel over 6.5 days, is the observed period of oscillation constant with theory?

(e) The figure shows that the amplitude of the oscillation gradually decreases with time. Hypothesize what might be causing this amplitude decrease, and explain how your hypothesis is consistent with your solution.

Solution.

(a) Here, we isolate v from the first equation as

$$v = \frac{1}{f_0}\frac{du}{dt} \tag{1.8}$$

Plugging this into the second equation, we get our first 2nd order differential equation.

$$\frac{d^2u}{dt^2} + f_0^2 u = 0 \tag{1.9}$$

Repeating the same process for u in the second equation, we get

$$\frac{d^2v}{dt^2} + f_0^2 v = 0 \tag{1.10}$$

(b) Second order differential equations are generally of the form

$$\frac{d^2y}{dx^2} + P(x)\frac{dy}{dx} + Q(x)y = f(x) \tag{1.11}$$

In a homogeneous differential equation, $f(x) = 0$, which applies to our equations. In a second order differential equation with constant coefficients, $P(x)$ and $Q(x)$ are both constants, which also applies to our equations. In a homogenous second order differential equation with constant coefficients, we assume a general solution

$$u = e^{rt} \tag{1.12}$$

where r is just some arbitrary value. Thus,

$$\frac{d^2u}{dt^2} = r^2 e^{rt} \tag{1.13}$$

Plugging these into our differential equation, we get

$$\begin{aligned} r^2 e^{rt} + f_0^2 e^{rt} &= 0 \\ r^2 + f_0^2 &= 0 \\ r &= \pm f_0 i \end{aligned} \tag{1.14}$$

Here, we get two imaginary solutions for r, which means the solution for our differential equation should have the form

$$u(t) = e^{at}\left(A\sin(bt) + B\cos(bt)\right) \tag{1.15}$$

where a and b are the real and imaginary parts, respectively, of our solution for r. There is no real part, so $a = 0$. Lastly, A and B are constants we need to solve for given the boundary conditions. We get

$$u(t) = A\sin(f_0 t) + B\cos(f_0 t) \tag{1.16}$$

At $t = 0$, $u = u_0$. Plugging this in, we determine that $B = u_0$, so

$$u(t) = A\sin(f_0 t) + u_0\cos(f_0 t) \tag{1.17}$$

We also know that $v = \frac{1}{f_0}\frac{du}{dt}$ and

$$\frac{du}{dt} = Af_0\cos(f_0 t) - u_0 f_0 \sin(f_0 t) \tag{1.18}$$

From here, we can get an expression for v as

$$v(t) = A\cos(f_0 t) - u_0\sin(f_0 t) \tag{1.19}$$

Similar to u, we are given that at $t = 0$, $v = v_0$. Plugging this in, we determine that $A = v_0$. We have our final expressions for u and v as

$$\begin{aligned} u(t) &= v_0\sin(f_0 t) + u_0\cos(f_0 t) \\ v(t) &= v_0\cos(f_0 t) - u_0\sin(f_0 t) \end{aligned} \tag{1.20}$$

(c) If $v_0 = 0$, then our velocity equations become

$$\begin{aligned} u(t) &= u_0\cos(f_0 t) \\ v(t) &= -u_0\sin(f_0 t) \end{aligned} \tag{1.21}$$

We can also write our velocities in terms of derivatives of the displacement.

$$\begin{aligned} u(t) &= \frac{dx}{dt} = u_0\cos(f_0 t) \\ v(t) &= \frac{dy}{dt} = -u_0\sin(f_0 t) \end{aligned} \tag{1.22}$$

Integrating both sides, we get something that looks similar to what we did in Worksheet 3.

$$\begin{aligned}\int_0^x dx &= u_0 \int_0^t \cos(f_0 t)\ dt \\ \int_0^y dy &= -u_0 \int_0^t \sin(f_0 t)\ dt\end{aligned} \tag{1.23}$$

Here, the 0 values in the integral limits denote that the initial horizontal position at $t = 0$ is at the origin. Solving these integrals, we get

$$\begin{aligned}x(t) &= \frac{u_0}{f_0} \sin(f_0 t) \\ y(t) &= \frac{u_0}{f_0} \left(\cos(f_0 t) - 1\right)\end{aligned} \tag{1.24}$$

The trajectory of the parcel should follow a clockwise circle as shown below.

(d) In our solution from part (c), the theoretical period of oscillation is just $\frac{2\pi}{f_0}$. The sine and cosine functions complete one period every 2π, which we can get at $t = \frac{2\pi}{f_0}$. Recall that $f_0 = 2\Omega \sin\theta_0$, where $\Omega = 2\pi$ rad/day is Earth's angular speed. At $\theta_0 = 58^o\ N$,

$$\frac{2\pi}{f_0} = \frac{2\pi}{2(2\pi \text{ rad/day}) \sin(58^o)} = 0.59 \text{ days} \tag{1.25}$$

The observed period of oscillation is

$$T = \frac{6.5 \text{ days}}{11 \text{ oscillations}} = 0.59 \text{ days} \tag{1.26}$$

The theoretical period of oscillations matches the observed period of oscillations.

(e) In our displacement equations for $x(t)$ and $y(t)$, the amplitude of oscillations are dependent on the coefficient $\frac{u_0}{f_0}$. We know that u_0 is a constant, so we look at f_0 instead. As f_0 increases, the amplitude of the oscillations decrease. The Coriolis factor f_0 is given by $f_0 = 2\Omega \sin\theta_0$, which increases as the latitude θ_0 increases. We observe that the parcel loops northward, meaning its latitude gradually increases. Thus, the amplitude of the parcel's oscillation gradually decreases because of the gradually increasing Coriolis effects as the parcel moves northward (thereby increasing its latitude).

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Problem 3 (a) We performed a synoptic scale analysis of the momentum equations. Redo the scale analysis for a thunderstorm updraft at 45^o N with the following characteristic scales:

$$\begin{aligned} U, V &\approx 10\ ms^{-1} \\ W &\approx 10\ ms^{-1} \\ L &\approx 10^3\ m \\ H &\approx 10^4\ m \\ \delta P/\rho &\approx 10^2\ m^2\ s^{-2}\ (\textbf{horizontal direction}) \\ \delta P/\rho &\approx 10^5 m^2\ s^{-2}\ (\textbf{vertical direction}) \end{aligned} \tag{1.27}$$

(b) Compare this new scaling with the synoptic scaling done in class. Which terms can we now not ignore? What does this imply about the predominate balance of force and acceleration in a thunderstorm (i.e., at the mesoscale)?

Solution.

(a) Besides what we are given, we also need the following characteristic scales. We have already determined the characteristic scale of f_0 at 45^o N in class. We also note that $\cos\theta_o = \sin\theta_o$ at 45^o N.

$$\begin{aligned} T = L/U &\approx 10^2\ s^{-1} \\ f_0 &\approx 10^{-4}\ s^{-1} \\ a &\approx 10^7\ m \\ \tan\theta &= 1 \end{aligned} \tag{1.28}$$

The table below shows the synoptic scale analysis for the zonal momentum equation.

	$\frac{Du}{Dt}$	$-2\Omega v\sin\theta$	$2\Omega w\cos\theta$	$-\frac{uv\tan\theta}{a}$	$\frac{uw}{a}$	$-\frac{1}{\rho a\cos\theta}\frac{\partial p}{\partial\lambda}$
scaling for each term	$\frac{U}{T}$	f_0V	f_0W	$\frac{UV}{a}$	$\frac{UW}{a}$	$\frac{\delta p}{\rho L}$
magnitude	10^{-1}	10^{-3}	10^{-3}	10^{-5}	10^{-5}	10^{-1}

The leading order terms are highlighted in pink, while the terms within 2 orders of magnitude from the leading terms are highlighted in orange. This also applies for the remaining tables. The table below shows the synoptic scale analysis for the meridional momentum equation.

	$\frac{Dv}{Dt}$	$2\Omega u\sin\theta$	$\frac{vw}{a}$	$\frac{u^2\tan\theta}{a}$	$-\frac{1}{\rho a}\frac{\partial p}{\partial\theta}$
scaling for each term	$\frac{V}{T}$	f_0U	$\frac{VW}{a}$	$\frac{U^2}{a}$	$\frac{\delta p}{\rho L}$
magnitude	10^{-1}	10^{-3}	10^{-5}	10^{-5}	10^{-1}

Lastly, the table below shows the synoptic scale analysis for the vertical momentum equation.

	$\frac{Dw}{Dt}$	$-2\Omega u\cos\theta$	$\frac{u^2+v^2}{a}$	$-\frac{1}{\rho}\frac{\partial p}{\partial z}$	$-g$
scaling for each term	$\frac{W}{T}$	f_0U	$\frac{U^2+V^2}{a}$	$\frac{\delta p}{\rho H}$	g
magnitude	10^{-1}	10^{-3}	10^{-5}	10	10

After only considering the highlighted terms we get

$$\begin{aligned}
\frac{Du}{Dt} &= 2\Omega v \sin\theta - 2\Omega w \cos\theta - \frac{1}{\rho a \cos\theta}\frac{\partial p}{\partial \lambda} \\
\frac{Dv}{Dt} &= -2\Omega u \sin\theta - \frac{1}{\rho a}\frac{\partial p}{\partial \theta} \\
\frac{Dw}{Dt} &= -\frac{1}{\rho}\frac{\partial p}{\partial z} - g
\end{aligned} \tag{1.29}$$

(b) Comparing the mesoscale with the synoptic scale characteristics, we notice that along the horizontal, we only consider the pressure gradient force and the Coriolis force as the dominant forces. On the other hand, along the vertical, the acceleration now becomes comparable with the pressure gradient force and gravity. Under mesoscale characteristics, the curvature terms now become negligible.

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