AATM 504 – Radiation Laws and TOA Irradiances

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Homework 2

Problem 1

Most terrestrial surfaces on Earth have surface temperatures (T_s) around 300 K, ranging from hot tropical ($+30^oC$) to cold polar regions (-40^oC). However, there are special cases such as forest fires and volcanoes with much higher T_s. **A plot of spectral exitance ($\mathbf{Wm^{-2}\mu m^{-1}}$) of a forest fire peaks at 3.0 μm.**

(a) For regular terrestrial surfaces with Ts ranging from -40^oC to $+30^oC$, what is the wavelength range of peak spectral exitance?

(b) For regular terrestrial surfaces with T_s ranging from -40^oC to $+30^oC$, what is the range of total exitance Wm^{-2}?

(c) Assume the fire radiates as a blackbody, calculate its T_s (K).

(d) Assume the fire radiates as a blackbody, calculate the total exitance (Wm^{-2}) from the burning surface.

(e) Assume the fire radiates as a blackbody, what is the brightness temperature of the burning surface (K)?

(f) If the fire radiates as a graybody with an emissivity of ϵ and a surface temperature of T_s, use integrals to express its total exitance (Wm^{-2}) as a function of T_s and ϵ. Assume that the fire surface is a Lambertian.

(g) If the fire radiates as a graybody with an emissivity of ϵ and its total exitance (Wm^{-2}) measured is equal to (d), what is its actual T_s (K)? Explain if the answer is different from (c). Assume that the fire surface is a Lambertian.

(h) If the fire radiates as a colored body with a spectral emissivity of ϵ_λ and a surface temperature of T_s, use integrals to express its total exitance (Wm^{-2}) as a function of T_s, ϵ_λ, and wavelength (λ). Assume that the fire surface is a Lambertian.

Solution.

(a) First, we need to convert the given temperatures to Kelvin.

$$-40^oC \equiv -40^oC + 273.15 = 233.15\ K \qquad \text{and} \qquad 30^oC \equiv 30^oC + 273.15 = 303.15\ K \tag{1.1}$$

Next, we use Wien's displacement law, $\lambda_m = \frac{2897\ \mu m\cdot K}{T}$.

$$\lambda_{m,p} = \frac{2897\ \mu m \cdot K}{233.15\ K} = 12.4\ \mu m \qquad \text{and} \qquad \lambda_{m,t} = \frac{2897\ \mu m \cdot K}{303.15\ K} = 9.56\ \mu m \tag{1.2}$$

Thus, wavlength range of peak spectral exitance is from 9.56 μm to 12.4 μm.

(b) Here, we use the Stefan-Boltzmann law $F(T) = \sigma T^4$, where $\sigma = 5.67 \times 10^{-8} \frac{W}{m^2} K^{-4}$. Thus,

$$F_p(233.15\ K) = (5.67 \times 10^{-8} \frac{W}{m^2} K^{-4})(233.15\ K)^4 = 168\ \frac{W}{m^2} \tag{1.3}$$

and

$$F_t(303.15\ K) = (5.67 \times 10^{-8} \frac{W}{m^2} K^{-4})(303.15\ K)^4 = 479\ \frac{W}{m^2} \tag{1.4}$$

The range of total exitance is from 168 $\frac{W}{m^2}$ to 479 $\frac{W}{m^2}$.

(c) Here, we use Wien's displacement law again: $T_s = \frac{2897\ \mu m \cdot K}{\lambda_m}$.

$$T_s = \frac{2897 \mu m \cdot K}{3.0 \mu m} = 966\ K \text{ or } 692.52^o C \tag{1.5}$$

(d) Here, we use the Stefan-Boltzmann law again.

$$F(966\ K) = (5.67 \times 10^{-8} \frac{W}{m^2} K^{-4}) \left(\frac{2897 \mu m \cdot K}{3.0 \mu m} \right)^4 = 4.93 \times 10^4 \frac{W}{m^2} \tag{1.6}$$

(e) According to Wallace and Hobbs, the measured radiance of a radiating body at any given wave number (or wavelength) has an equivalent blackbody temperature at that wavelength. These are known as the brightness temperatures, which are dependent on the wavelength. Since we assume the fire acts as a blackbody, then the brightness temperature (at all wavelengths) is just 966 K, which we have solved from (c).

(f) The emissivity ϵ_λ for non-blackbodies with surface temperature T_s is the ratio between the monochromatic intensity of the radiation emitted by the body and the corresponding blackbody radiation. In other words,

$$\epsilon_\lambda = \frac{I_\lambda(emitted)}{B_\lambda(T_s)} \tag{1.7}$$

The total exitance for any radiating body is

$$F_\lambda = \int_0^\infty \pi I_\lambda d\lambda \tag{1.8}$$

Note that the π coefficient takes into account the cosine law we established in class. We use π since we assume a Lambertian surface. The integral limits 0 to ∞ just means we're integrating over all wavelengths to get the total exitance. Combining our two equations, we get

$$F_\lambda = \int_0^\infty \pi \epsilon_\lambda B_\lambda(T_s) d\lambda \tag{1.9}$$

For a graybody, emissivity is constant for all wavelengths, so we can take it out of the integral.

$$F_\lambda = \epsilon \int_0^\infty \pi B_\lambda(T_s) d\lambda \tag{1.10}$$

We know that this integral is just σT_s^4 from the Stefan-Boltzmann law. Finally, the total exitance for a graybody is simply

$$F(T_s) = \epsilon \sigma T_s^4 \tag{1.11}$$

(g) We obtained the total exitance value in (d) as

$$F(T_s) = \sigma \left(\frac{2897 \mu m \cdot K}{3.0\ \mu m} \right)^4 \tag{1.12}$$

From (f), we know that the total exitance of the fire as a graybody is

$$\sigma \left(\frac{2897 \mu m \cdot K}{3.0\ \mu m} \right)^4 = \epsilon \sigma T_s^4 \tag{1.13}$$

By isolating T_s, we get

$$T_s = \epsilon^{-1/4} \left(\frac{2897 \mu m \cdot K}{3.0 \ \mu m} \right) = \epsilon^{-1/4} (966 \ K) \tag{1.14}$$

We get a much different value from (c), since we are now considering the fire as a graybody rather than a blackbody.

(h) We determined in (f) that the total exitance of any non-blackbody is just

$$F_\lambda(T_s) = \int_0^\infty \pi \epsilon_\lambda B_\lambda(T_s) d\lambda = \pi \int_0^\infty \epsilon_\lambda B_\lambda(T_s) d\lambda \tag{1.15}$$

Unlike in graybodies, we can't take ϵ_λ out of the integral, since the emissivity of colored bodies varies depending on what wavelength we're looking at. Thus, this is already our final equation for the total exitance of colored bodies.

$\square$

Problem 2

Derivations

(a) Use calculus to derive Wien's Displacement Law from the Planck equation.

(b) Verify that the constants in the end of the derivations are indeed giving the constant $2897\mu m \cdot K$ that is seen in the Wien's Displacement Law.

(c) For a particular day, the total integrated irradiance over the entire spectrum (S_0, W/m^2) of Sun on the top of atmosphere (TOA) of Earth can be easily calculated based on the inverse square law we learn in class. However, solar constant $\bar{S_0}$ is often used in some equations to calculate S_0. Derive the relationship between TOA solar irradiance and solar constant (Lecture 5) as shown below:

$$S_0 = \bar{S_0}(d_0/d)^2 = \bar{S_0}(d_m)^2 \tag{2.1}$$

Solution.

(a) The Planck equation is given by

$$B_\lambda(T) = \frac{c_1 \cdot \lambda^{-5}}{\pi(e^{c_2/\lambda T} - 1)} \tag{2.2}$$

To derive Wien's displacement law, we get the derivative of Planck's equation with respect to λ and equate it to zero. We first divide the equation into manageable chunks.

$$\text{Let } x = \frac{1}{e^{c_2/\lambda T} - 1} \text{ and } y = \frac{1}{\lambda^5} \tag{2.3}$$

Thus, the derivative becomes

$$\frac{dB_\lambda(T)}{d\lambda} = \frac{c_1}{\pi}\left(x\frac{dy}{d\lambda} + y\frac{dx}{d\lambda}\right) \tag{2.4}$$

Next, we get the derivatives of x and y using the chain rule.

$$\frac{dx}{d\lambda} = \left(\frac{-1}{\left(e^{c_2/\lambda T} - 1\right)^2}\right)\left(\frac{c_2}{T}e^{\lambda/c_2 T}\right)\left(\frac{-1}{\lambda^2}\right) = \frac{\frac{c_2}{T}e^{c_2/\lambda T}}{\lambda^2\left(e^{c_2/\lambda T} - 1\right)^2} \tag{2.5}$$

$$\frac{dy}{d\lambda} = \frac{-5}{\lambda^6} \tag{2.6}$$

Thus,

$$\frac{dB_\lambda(T)}{d\lambda} = \frac{c_1}{\pi}\left[\frac{-5}{\lambda^6\left(e^{c_2/\lambda T} - 1\right)} + \frac{\frac{c_2}{T}e^{c_2/\lambda T}}{\lambda^7\left(e^{c_2/\lambda T} - 1\right)^2}\right] \tag{2.7}$$

We now equate this to zero.

$$\begin{aligned} 0 &= \frac{c_1}{\pi}\left[\frac{-5}{\lambda^6\left(e^{c_2/\lambda T} - 1\right)} + \frac{\frac{c_2}{T}e^{c_2/\lambda T}}{\lambda^7\left(e^{c_2/\lambda T} - 1\right)^2}\right] \\ &= -5 + \frac{c_2}{\lambda T}\frac{e^{c_2/\lambda T}}{\left(e^{c_2/\lambda T} - 1\right)} = \frac{c_2}{\lambda T}e^{c_2/\lambda T} - 5\left(e^{c_2/\lambda T} - 1\right) \end{aligned} \tag{2.8}$$

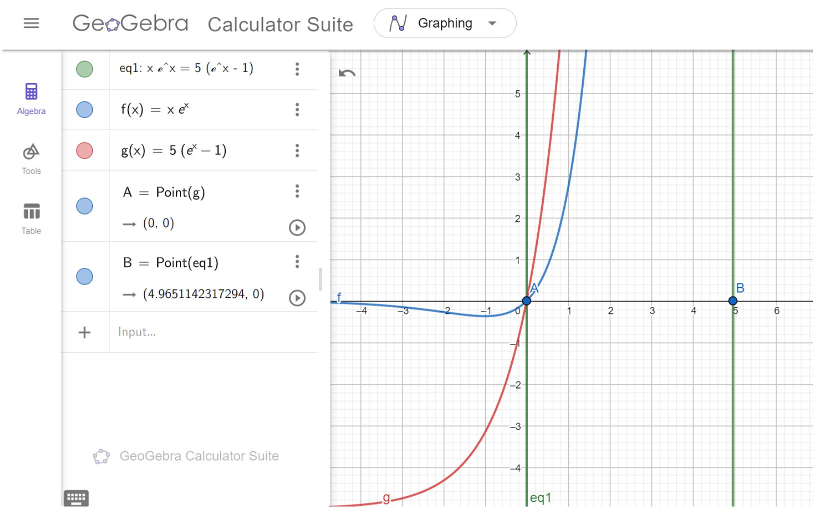
For simplicity, we let $X = c_2/\lambda T$, so

$$0 = Xe^X - 5\left(e^X - 1\right) \tag{2.9}$$

Wallace and Hobbs provides a hint that the exponential term e^X is much greater than 1 in the wavelength range we're interested in. Thus, we can estimate that $e^X - 1 \approx e^X$. Solving for X, we get $X = 5$, so

$$\lambda_m = \frac{c_2}{5T} = \frac{1.45 \times 10^4\ \mu m \cdot K}{5T} = \frac{2900\ \mu m \cdot K}{T} \tag{2.10}$$

To get a more exact answer, I used a graphing calculator. We see in the figure below that the two roots of this equation are at point A $X = 0$ (trivial) and point B $X = 4.965\ldots$.



Using the non-zero root, we get Wien's displacement law

$$\lambda_m = \frac{c_2}{(4.965\ldots)T} = \frac{2897\ \mu m \cdot K}{T} \tag{2.11}$$

I show in the next problem how $\frac{c_2}{4.965\ldots}$ equates to $2897\ \mu m \cdot K$.

(b) The final constant $2897\ \mu m \cdot K$ is just $\frac{c_2}{x}$, which is

$$\frac{c_2}{x} = \frac{(6.626 \times 10^{-34}\ J \cdot s)(3 \times 10^8\ m/s)}{(4.9651\ldots)(1.381 \times 10^{-23}\ J/K)} = 2.879 \times 10^{-3}\ m \cdot K \equiv 2897\ \mu m \cdot K \tag{2.12}$$

Note that x is dimentionless. Explicitly, the units are just

$$\left[\frac{c_2}{x}\right] = \frac{(J \cdot s)(m/s}{J/K} \tag{2.13}$$

(c) The inverse square law gives us the total irradiance S_o (W/m^2) of the Sun on the top of Earth's atmosphere.

$$S_o = F_{\text{Sun}} \frac{R_{\text{Sun}}}{d^2} \tag{2.14}$$

where F_{Sun} is the total irradiance or flux density of the Sun (given by the Stefan-Boltzmann law), R_{Sun} is the Sun's radius, and d is the actual distance between the Sun and Earth at any point in its orbit. The inverse square law also gives us the solar constant $\bar{S}_o$.

$$\bar{S}_o = F_{\text{Sun}} \frac{R_{\text{Sun}}}{d_o^2} \tag{2.15}$$

where d_o is the average distance between the Sun and Earth. By isolating, F_{Sun}, we get

$$F_{\text{Sun}} = \bar{S}_o \frac{d_o^2}{R_{\text{Sun}}} \tag{2.16}$$

Plugging this into our first equation for S_o, we get

$$S_o = \bar{S}_o \frac{d_o^2}{R_{\text{Sun}}} \frac{R_{\text{Sun}}}{d^2} \tag{2.17}$$

We rewrite the ratio $\frac{d_o^2}{d^2}$ as d_m^2.

$$S_o = \bar{S}_o \frac{d_o^2}{d^2} = \bar{S}_o d_m^2 \tag{2.18}$$

□

Problem 3

We showed in class that the integral of the Planck Function $I_{BB\lambda}(T)$ over all zenith angles $0-\pi/2$ and all azimuth angles $0-2\pi$ is $F_{BB\lambda}(T) = \pi \cdot I_{BB\lambda}(T)$, the spectral irradiance of a blackbody at temperature T (in Kelvin). Integration over all wavelengths $F(T) = \int_0^\infty F_{BB}(T) \cdot d\lambda$ yields

$$F(T) = \text{constant} \cdot T^4 \tag{3.1}$$

with constant $= \frac{\pi^4}{15}\frac{c_1}{c_2^4}$.

$$c_1 = 2\pi hc^2 \qquad \text{and} \qquad c_2 = hck^{-1} \tag{3.2}$$

Show that the constant is consistent with the Stefan-Boltzmann constant σ in both, the numerical value and the physical units!

(a) Look up the numerical values and the units of the physical constants and list them (speed of light in vacuum (c), the Boltzmann constant (k), the Planck constant (h), and the Stefan-Boltzmann constant (σ).

(b) Calculate the numerical values of c_1, c_2, and the constant.

(c) Show how you get the physical units for c_1 c_2, and the constant (step by step).

(d) Finally compare the calculated results with the Stefan-Boltzmann constant (σ) and conclude with a short sentence what you found out.

Solution.

(a) The speed of light in a vacuum c is about 3×10^8 m/s. The Boltzmann constant k is about 1.381×10^{-23} J/K. The Planck constant h is about $6.626 \times 10^{-34} J \cdot s$. Lastly, the Stefan-Boltzmann constant σ is about $5.67 \times 10^{-8} \frac{W}{m^2} K^{-4}$.

(b) For the first constant, we use Planck's constant and the speed of light in a vacuum.

$$c_1 = 2\pi(6.626 \times 10^{-34})(3 \times 10^8)^2 = 3.75 \times 10^{-16} \tag{3.3}$$

For the second constant, we use Planck's constant, the speed of light, and the Boltzmann constant.

$$c_2 = (6.626 \times 10^{-34})(3 \times 10^8)(1.381 \times 10^{-23})^{-1} = 1.44 \times 10^{-2} \tag{3.4}$$

Lastly, the constant is given by

$$\begin{aligned} \text{constant} &= \frac{\pi^4}{15}\frac{2\pi hc^2}{(hck^{-1})^4} \\ &= \frac{\pi^4}{15}\frac{2\pi(6.626 \times 10^{-34})(3 \times 10^8)^2}{[(6.626 \times 10^{-34})(3 \times 10^8)(1.381 \times 10^{-23})^{-1}]^4} \\ \text{constant} &= 5.67 \times 10^{-8} \end{aligned} \tag{3.5}$$

(c) For the first constant, we use the SI units for the speed of light and Planck's constant.

$$[c_1] \equiv (J \cdot s)(m^2/s^2) \equiv (J/s)m^2 \equiv W \cdot m^2 \tag{3.6}$$

Note that $W = \frac{J}{s}$ and 2π is dimensionless. Thus, in the end, we get $c_1 = 3.75 \times 10^{-16}$ $W \cdot m^2$. For the second constant, we use the SI units for Planck's constant, the speed of light, and the Boltzmann constant.

$$[c_2] \equiv (J \cdot s)(m/s)(K/J) \equiv m \cdot K \tag{3.7}$$

Thus, in the end, we get $c_2 = 1.44 \times 10^{-2}\ m \cdot K$. For the constant, we have

$$[\text{constant}] \equiv \frac{W \cdot m^2}{(m \cdot K)^4} \equiv \frac{W}{m^2} \cdot K^{-4} \tag{3.8}$$

Thus, in the end, we get constant $= 5.67 \times 10^{-8}\ \frac{W}{m^2} \cdot K^{-4}$

(d) The Stefan-Boltzmann constant is simply composed of other fundamental constants (c, k, and h) such that

$$\sigma = \frac{\pi^4}{15} \frac{2\pi hc^2}{hck^{-1}} = 5.67 \times 10^{-8}\ \frac{W}{m^2} \cdot K^{-4} \tag{3.9}$$

$\square$

Problem 4

Earth's Atmosphere. Venus, Earth, Mars are sibling planets in our solar system. They all have atmospheres, weather surfaces, massive volcanoes, and chemically and thermally evolved interiors. Comparison of them can provide a context of how unique or not our planet Earth is. It also helps to have a basic knowledge of the properties of our two nearest neighboring planets. Please answer the following questions:

(a) Use the given planetary albedo values, calculate the reflected and absorbed values of the TOA solar irradiance (in W/m^2).

Planet	**Mean Sun-Planet Distance (A.U.)**	**Planetary Albedo**
Venus	0.723	0.75
Earth	1.000	0.30
Mars	1.524	0.16

(b) If there was no atmosphere, the global mean surface temperature (often called equilibrium temperature, T_e) of a planet could be estimated based on the energy balance between the absorbed solar radiation and the loss of longwave radiation at the TOA as given next (we will learn and derive this equation later):

$$\bar{S}_0 \frac{1}{4}(1-\alpha_p) = \sigma T_\epsilon^4 \tag{4.1}$$

where $\bar{S}_0$ is the annual mean total solar irradiance (W/m^2), α_p is the planetary albedo, and σ is Boltzmann's constant. Using the planetary albedo given above, estimate the global mean surface temperature each of Venus, Earth and Mars would have without an atmosphere. Also compare your estimates with their actual temperatures listed in Lecture 2 here. What do the differences tell us?

Solution.

(a) The planetary albedo dictates how much incoming irradiance each planet reflects back. So, 1.0 minus each value is how much irradiance is absorbed. We recall from the first homework that the TOA incoming solar irradiance of Earth is 1361 W/m^2. Thus, for earth

$$\begin{aligned} R_E(\text{reflected}) &= 0.30(1361\ W/m^2) = 408.3\ W/m^2 \\ A_E(\text{absorbed}) &= 0.70(1361\ W/m^2) = 952.7\ W/m^2 \end{aligned} \tag{4.2}$$

I previously showed in the first HW and in (2c) that the TOA incoming solar irradiance of other planets with respect to Earth is just

$$\text{TOA}_{\text{planet}} = \text{TOA}_{\text{Earth}} \cdot \left(\frac{r_{\text{Earth}}}{r_{\text{planet}}}\right)^2 \tag{4.3}$$

where r denotes the distance of the planet from the Sun. Thus, the TOA incoming solar irradiances for Venus and Mars are

$$\begin{aligned} \text{TOA}_{\text{Venus}} &= (1361\ W/m^2) \cdot \left(\frac{1.000\ \text{AU}}{0.723\ \text{AU}}\right)^2 \\ \text{TOA}_{\text{Mars}} &= (1361\ W/m^2) \cdot \left(\frac{1.000\ \text{AU}}{1.524\ \text{AU}}\right)^2 \end{aligned} \tag{4.4}$$

For Venus,

$$
\begin{aligned}
R_V &= 0.75(1361\ W/m^2) \cdot \left(\frac{1.000\ \text{AU}}{0.723\ \text{AU}}\right)^2 = 1953\ W/m^2 \\
A_V &= 0.25(1361\ W/m^2) \cdot \left(\frac{1.000\ \text{AU}}{0.723\ \text{AU}}\right)^2 = 650.9\ W/m^2
\end{aligned}
\tag{4.5}
$$

For Mars,

$$
\begin{aligned}
R_M &= 0.16(1361\ W/m^2) \cdot \left(\frac{1.000\ \text{AU}}{1.524\ \text{AU}}\right)^2 = 93.8\ W/m^2 \\
A_M &= 0.84(1361\ W/m^2) \cdot \left(\frac{1.000\ \text{AU}}{1.524\ \text{AU}}\right)^2 = 492.2\ W/m^2
\end{aligned}
\tag{4.6}
$$

(b) Using our given equation, we isolate T_ϵ^4.

$$
T_\epsilon = \left(\frac{\bar{S}_0(1-\alpha_p)}{4\sigma}\right)^{1/4} \tag{4.7}
$$

We already solved for the solar irradiance from (a). For Earth,

$$
T_\epsilon(E) = \left(\frac{(1361\ W/m^2)(1-0.30)}{4(5.67\times10^{-8}\ W/m^2\cdot K^{-4})}\right)^{1/4} = 254.6\ K \equiv -18.6^oC \tag{4.8}
$$

For Venus,

$$
T_\epsilon(V) = \left(\frac{(1361\ W/m^2)\cdot\left(\frac{1.000\ \text{AU}}{0.723\ \text{AU}}\right)^2\cdot(1-0.75)}{4(5.67\times10^{-8}\ W/m^2\cdot K^{-4})}\right)^{1/4} = 231.5\ K \equiv -41.7^oC \tag{4.9}
$$

For Mars,

$$
T_\epsilon(M) = \left(\frac{(1361\ W/m^2)\cdot\left(\frac{1.000\ \text{AU}}{1.524\ \text{AU}}\right)^2\cdot(1-0.16)}{4(5.67\times10^{-8}\ W/m^2\cdot K^{-4})}\right)^{1/4} = 215.8\ K \equiv -57.3^oC \tag{4.10}
$$

The actual mean surface temperature of Earth is 14^oC, 467^oC for Venus, and -46^oC for Mars. We see that without an atmosphere, Earth and Venus would have a much colder surface. The difference in Venus's surface temperature is more prominent since it has the thickest atmosphere of the three. On the other hand, the actual surface temperature of Mars is not very far from its surface temperature without an atmosphere. Mars has a much thinner atmosphere than the other two planets, so taking its atmosphere into account does not greatly impact its surface temperature.

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Problem 5

Please give us feedback on how the homework assignments have so far been in terms of:

(a) How much time you spend on doing the homework (1 and this one 2)?

(b) How clear are the instructions written? Is it confusing or easy to understand what the question is and what you are asked to do?

(c) Do you feel mathematically prepared to deal with the equations or is there something missing that deserves some review?

(d) Is the lecture material and are the book sections helping you finding the answers?

(e) Any other comments?

Solution.

(a) I spent roughly five days to finish HW1 and a little longer at 6 days to finish HW2. A week to finish the HWs is not so bad as long as the due dates never coincide with our ATM 500 (Dynamics) HWs.

(b) Most of the instructions are clearly written. However, sometimes they're a little confusing, so we have to either consult Prof. Oliver or Tyler for clarifications. Office hours are a big help since they can explain to use what the questions mean and what they're actually asking for.

(c) I've already encountered a lot of the math discussed so far in class, so they're not really a problem for me yet.

(d) The lecture material and book sections are somewhat helpful, but I still need to Google things I'm not sure of or to get through the first few steps of the problem at least. The lecture material is way more helpful than the book, since a lot of topics are better covered in class versus the book.

(e) We also have problem sets in our other class in Dynamics. We get two weeks to submit those, since they're longer and more complicated. However, I would greatly appreciate it if the due dates for our Physics and Dynamics class don't intersect. They haven't so far, since this HW2's due date was moved.

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