
AATM 504 – Moisture in the atmosphere

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Homework 7

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Problem 1

Lapse rate of dew point temperature T_d

Consider an unsaturated moist air parcel. When the moist unsaturated air parcel rises adiabatically then the temperature of the air decreases and the water vapor pressure e decreases with altitude as the environmental temperature T_e and pressure p_e decrease with height. We assume that the air parcel and environmental air temperature have the same pressure and temperature (T). One can show that for adiabatic lifting of the air parcel, the dew point temperature T_d will decrease as well. We skip the derivation and present an equation for the lapse rate of T_d (before condensation happens):

$$\Gamma_{\text{dew}} = \frac{dT_d}{dz} = -\frac{gR_v}{L_v R_d} * \frac{T_d^2}{(1 + 0.61q)T} \approx -6.31 * 10^{-6} * \frac{T_d^2}{T} \quad (1.1)$$

with dewpoint temperature T_d and air temperature T in kelvin.

- (a) Calculate the dew point temperature lapse rate for a tropical air mass with $T = 300 \text{ K}$ and dew point temperature $T_d = 284 \text{ K}$, and for air with $T = 255 \text{ K}$ and $T_d = 250 \text{ K}$.
- (b) Prove that Γ_d has units Kelvin per meter by analyzing all involved constants and variables and their units.
- (c) A student argues they have found a simpler method for estimating the height of the Lifting Condensation Level (LCL) for an air parcel with an initial state given by the temperature T , dew point T_d temperature and pressure p : *Simply follow the dry adiabat until the dry adiabat intersects the isotherm that equals T_d .* Explain why this method is too simple and why it leads to a systematic over- or under-estimation of the LCL.
- (d) You are given an air parcel's initial state at $p = 980 \text{ hPa}$, $T = 20^\circ\text{C}$, and a wet bulb temperature of $T_w = 12^\circ\text{C}$. Estimate the LCL with help of the skew-T diagram with the graph methods that were introduced in the lectures and described in Wallace and Hobbs (Section 3.5):
 - (i) with Normand's rule. Explain the individual steps so that it is 100% clear which lines we have to follow to get to the LCL.
 - (ii) For comparison show the student's simple method in skew-T plot, too (dew point temperature assumed constant).
 - (iii) the 1/3 2/3 rule for relationship between wet-bulb and dry bulb temperature and dew point temperature and the equation:

$$z_{LCL} \approx \left[\frac{8K}{1000m} \right]^{-1} \cdot (T - T_d) \quad (1.2)$$

Solution.

- (a) At $T_d = 284 \text{ K}$ and $T = 300 \text{ K}$, the lapse rate is given by

$$\Gamma_{\text{dew}} = -6.31 \times 10^{-6} m^{-1} \left[\frac{(284 \text{ K})^2}{(300 \text{ K})} \right] = -1.70 \times 10^{-3} \text{ K/m} \quad (1.3)$$

At $T_d = 250 \text{ K}$ and $T = 255 \text{ K}$, the lapse rate is given by

$$\Gamma_{\text{dew}} = -6.31 \times 10^{-6} m^{-1} \left[\frac{(250 \text{ K})^2}{(255 \text{ K})} \right] = -1.55 \times 10^{-3} \text{ K/m} \quad (1.4)$$

- (b) Here, we note the following constants: (1) $g = 9.8 m/s^2$, (2) water vapor gas constant $R_v = 461.5 \text{ Jkg}^{-1}K^{-1}$, (3) dry air gas constant $R_d = 287.05 \text{ Jkg}^{-1}K^{-1}$, and (4) specific latent heat of water vapor $L_v = 2.50 \times 10^6 \text{ Jkg}^{-1}$. Note also that the moist air gas constant is given by $R_m = R_d(1 + 0.61q)$, which we obtained in the previous HW. Thus, we can rewrite our equation for the lapse rate as

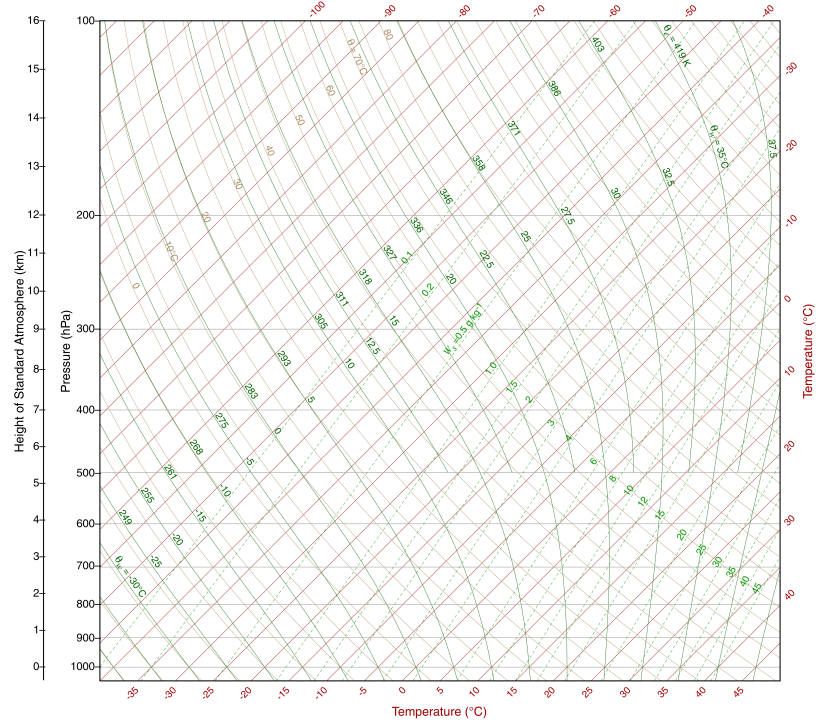
$$\Gamma_{\text{dew}} = -\frac{gR_v}{L_v R_m} * \frac{T_d^2}{T} \quad (1.5)$$

In terms of the units, we have

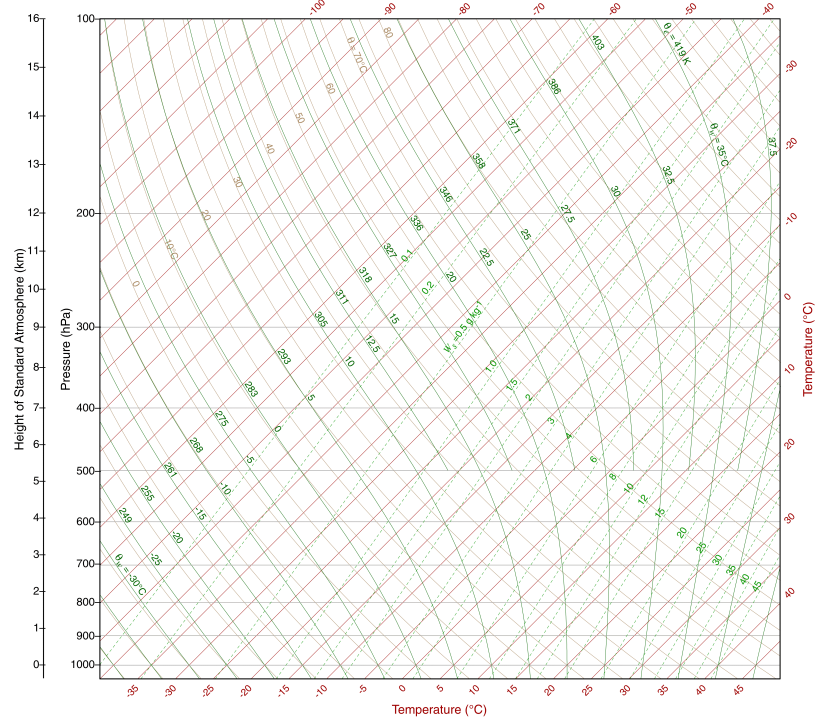
$$\begin{aligned} [\Gamma_{\text{dew}}] &= -\frac{[m/s^2][Jkg^{-1}K^{-1}][K^2]}{[Jkg^{-1}][Jkg^{-1}K^{-1}][K]} \\ &= \frac{[m/s^2]}{[m^2/s^2]}[K] \\ [\Gamma_{\text{dew}}] &= K/m \end{aligned} \quad (1.6)$$

Note that in SI units, $[J] \equiv [Nm] \equiv [kg \text{ m}^2/s^2]$.

- (c) The student argues to follow the dew point temperature isotherm. However, based on our skew-T diagram, following the isotherm with height means the saturation mixing ratio would increase with height. The saturation mixing ratio is given by $w_s = \frac{\text{mass of water vapor}}{\text{mass of dry air}}$. Thus, as w_s increases, the air parcel obtains more water vapor, which *breaks our rule on parcel theory that the mixing ratio should be conserved until it reaches saturation*. In the same line, dew point temperature is not necessarily conserved. The student is correct to follow the dry adiabat, but we need to follow the saturation mixing ratio that equals to T_d instead.
- (d) (i) According to Normand's rule, at the LCL, the following lines should intersect: (1) the dry adiabat passing through the initial temperature $T = 20^\circ C$, and (2) the moist adiabat passing through the initial wet bulb temperature $T_w = 12^\circ C$, (3) and the line of constant mixing ratio passing through the initial dew point temperature T_d . For this problem, we only consider (1) and (2), since we don't know T_d . First, at the pressure level $p = 980 \text{ hPa}$, we look for the two isotherms corresponding to T and T_w . Next, T follows a line parallel to the dry adiabats, while T_w follows a curve parallel to the moist adiabats. Lastly, their intersection is the LCL.



(ii) By following the student's suggested method, we obtain the following skew-T diagram.



(iii) The 1/3, 2/3 rule states that $T_w = \frac{2}{3}T + \frac{1}{3}T_d$. Thus, the initial dew point temperature is given by

$$T_d = 3T_w - 2T = 3(12^\circ C) - 2(20^\circ C) = -4^\circ C \quad (1.7)$$

Plugging this into our estimated equation for the LCL height, we get

$$z_{LCL} \approx \left[\frac{8K}{1000m} \right]^{-1} \cdot [(293.15 K) - (269.15 K)] \approx 3000 m \quad (1.8)$$

□

Problem 2**Dewpoint and relative humidity**

Show mathematically that the two definitions of relative humidity

$$RH \equiv 100 \cdot \frac{e}{e_s} \quad (2.1)$$

$$RH \equiv 100 \cdot \frac{w}{w_s} \quad (2.2)$$

are numerically **not** identical! Start from the basic definition of mixing ratio and the ideal gas law and derive expressions for w and w_s as ratios of the vapor pressure e and pressure of the moist air p (and e_s , p_s).

- (a) Derive the following expressions:

$$\begin{aligned} w &= \epsilon \cdot \frac{e}{p - e} \\ \text{and} \\ w_s &= \epsilon \cdot \frac{e_s}{p - e_s} \end{aligned} \quad (2.3)$$

- (b) Explain with the help of these equations the differences between the two RH definitions and why it practically can be considered the same.
- (c) Take a look at Figure 1a in the article of Lawrence (2005). The shown graphs provide support for the simple rule that RH changes by +5% per 1 K increase in dewpoint temperature (or per 1 K dry bulb temperature decrease). Justify with the curvature of the graphs that a single number 5%/1 K is applicable over a wide range of temperatures and relative humidity levels.

Solution.

- (a) From the ideal gas law and Dalton's law of partial pressures, the pressure exerted by a constituent of a mixture of gases is proportional to the number of moles that constituent in the mixture. Thus, $\frac{e}{p} = \frac{n_v}{n}$, where e is the vapor pressure, p is the moist air pressure, n_v is the number of vapor moles, and n is the total number of moist air moles. By conservation of mass, we can rewrite this as

$$\frac{e}{p} = \frac{n_v}{n_v + n_d} = \frac{m_v/M_v}{m_v/M_v + m_d/M_d} \quad (2.4)$$

where m denotes mass, M denotes the molecular mass, and the subscripts denote water vapor v and dry air d . Multiplying the right hand side by $\frac{M_v/m_d}{M_v/m_d}$, we get

$$\frac{e}{p} = \frac{m_v/m_d}{m_v/m_d + M_v/M_d} = \frac{w}{w + \epsilon} \quad (2.5)$$

Here, we let $\epsilon = M_v/M_d$. By definition, the mixing ratio is given by $w = m_v/m_d$. Isolating w ,

we have

$$\begin{aligned}
(w + \epsilon) \frac{e}{p} &= w \\
\epsilon \frac{e}{p} &= w \left(1 - \frac{e}{p} \right) \\
w &= \epsilon \frac{e/p}{1 - \frac{e}{p}} \cdot \frac{p}{p} \\
w &= \epsilon \frac{e}{p - e}
\end{aligned} \tag{2.6}$$

Similarly, we can follow the same logic for the saturation mixing ratio w_s , except that we use m_s (saturation vapor mass) instead of m_v , and e_s (saturation vapor pressure) instead of e . They mean the same, but here, the air is *saturated* with water vapor. The meaning of the variables p and ϵ remain unchanged. Thus,

$$w_s = \epsilon \frac{e_s}{p - e_s} \tag{2.7}$$

(b) Using these two equations, we see that

$$\frac{w}{w_s} = \frac{e}{e_s} \frac{p - e_s}{p - e} \tag{2.8}$$

According to the article, for many applications e and e_s are very small compared to p . Thus, we can approximate that $p - e \approx p \approx p - e_s$.

$$\frac{w}{w_s} \approx \frac{e}{e_s} \frac{p}{p} \approx \frac{e}{e_s} \tag{2.9}$$

From here, we observe that numerically, $\frac{w}{w_s} \neq \frac{e}{e_s}$, but for practical applications, we can assume they are equivalent.

- (c) According to the article, the 5% increase in RH per $1^\circ C$ increase in dew point temperature (and vice versa) is most applicable at $RH > 50\%$, where the curves follow a more linear trend. By looking at the *slope of the linear trend*, we can determine how much RH increases when T_d increases. For example, at $T = 30^\circ C$, when $RH = 80\%$, $T_d \approx 24^\circ C$, and when $RH = 60\%$, $T_d \approx 20^\circ C$. Thus, the slope is approximately $5\%/1^\circ C$. If we read further in the article, they also showed that this approximation is applicable to a $\pm 15\%$ deviation for temperatures T between $0^\circ C$ to $30^\circ C$.

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Problem 3

Skew-T diagram: Estimating relative humidity with help of the right lines

Explain how you can use the dry bulb temperature (air temperature) and dew point temperature and the saturation mixing ratio lines in the skew-T diagram to estimate relative humidity. Write up a “user guide” with example values depicted in the skew-T diagram with illustrating lines drawn into or highlighted in the diagram.

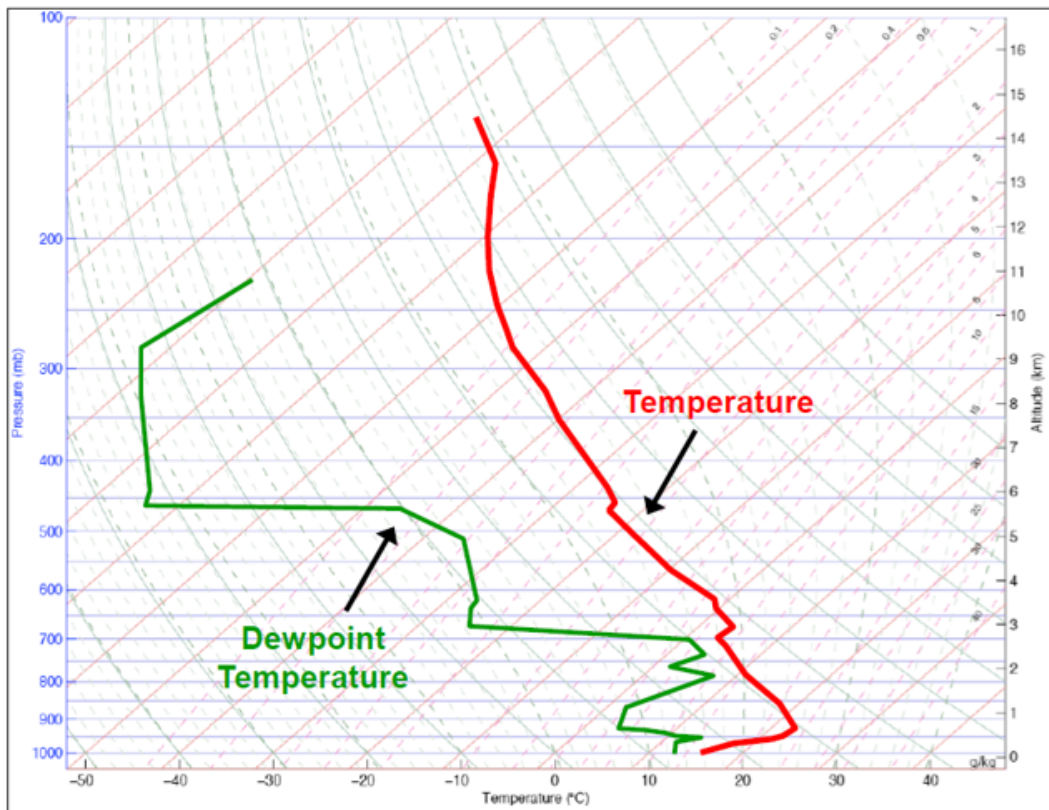
Solution.

Here, we use the following equation presented in the article

$$RH \equiv 100 \cdot \frac{w}{w_s} \quad (3.1)$$

1. Determine what pressure level you are looking at. In our example below, $p = 900 \text{ hPa}$.
2. Determine where the isobar intersects with the plots of the dew point temperature and ambient temperature. Following the isotherms at the intersections, we have our dew point temperature T_d and ambient temperature T at pressure level p .
3. We draw two lines: (1) the saturation mixing ratio passing through T_d , and (2) the saturation mixing ratio passing through T . Line (1) denotes the saturation mixing ratio w_s , while line (2) denotes the mixing ratio w .
4. Lastly, we plug in our values for w_s and w in Eqn. 3.1, to solve for the RH at pressure level p .

Reference: National Weather Service. (2022). Skew-T Parameters. Retrieved 11/20/2022, from www.weather.gov/source/zhu/ZHU_Training_Page/convective.parameters/skewt/skewtinfo.html



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Problem 4**Thermodynamics of global warming**

It is often stated that in a warming climate, for a 1% increase in the earth's surface temperature, we get a $\sim 18\%$ increase in saturation vapor pressure and specific humidity, or $\sim 6\%$ per degree C or K . Please do the following:

- (a) Start from the differential form of the Clausius-Clapeyron equation, eliminate (that is, replace with other state variables) the specific volume from the equation and obtain an expression for the derivative of saturation vapor pressure e_s with respect to temperature:

$$\Delta e_s = \frac{de_s}{dT} \cdot \Delta T \quad (4.1)$$

Then, prove that above statement ($\sim 6\%$ per degree C or K) is a correct statement.

- (b) Explain with this relationship why the % increase in saturation vapor pressure per degree warming in T is the largest in the coldest polar regions.

Solution.

- (a) The differential form of the Clausius Clapeyron equation for phase change from phase 1 (liquid) to phase 2 (vapor) is given by

$$\frac{dp}{dT} = \frac{L_v}{T(\alpha_2 - \alpha_1)} \approx \frac{L_{12}}{T\alpha_2} \quad (4.2)$$

where L_v denotes the latent heat of vaporization, and α denotes the specific volume of the two phases. Here, we assume that the specific volume of liquid water is much smaller than the specific volume of water vapor. From the ideal gas law, we can say that $e_s\alpha_2 = R_v T$, where e_s is the saturation vapor pressure and R_v is the vapor gas constant. Thus, in terms of the saturation vapor pressure, the Clausius-Clapeyron equation becomes

$$\frac{de_s}{dT} = \frac{L_v e_s}{R_v T^2} \quad (4.3)$$

That specific latent heat of vaporization is about $L_v = 2.50 \times 10^6 \text{ J kg}^{-1}$ (at $0^\circ C$) and the vapor gas constant is $R_v = 461.5 \text{ J kg}^{-1} \text{ K}^{-1}$. By letting $T = 300 \text{ K}$ and $\Delta T = 1 \text{ K}$, and plugging in all our constants to our working equations, we get

$$\Delta e_s = \frac{L_v e_s}{R_v T^2} \cdot \Delta T = \frac{2.50 \times 10^6 \text{ J kg}^{-1}}{(461.5 \text{ J kg}^{-1} \text{ K}^{-1})(300 \text{ K})^2} (1 \text{ K}) e_s = 0.06 e_s \equiv (6\%) e_s \quad (4.4)$$

Thus, for every 1 K increase in temperature, the saturation vapor pressure increases by 6% its initial value.

- (b) In the colder polar regions, we expect T to be lower, and L_v to be slightly higher. The magnitude of the specific latent heat L increases slightly with decreasing T , and vice versa. In other words, L is a weak function of T . In our equation for $\frac{de_s}{dT}$, the numerator slightly increases, while the denominator decreases, resulting to a higher value. The polar regions have the lowest temperatures T , so the effect there must be the greatest.

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