AATM 529 – Air Sea Interactions

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Homework 4

Problem 1

Let C be the concentration of hockipuculis bacteria in the air. This contagious bacteria, which sweeps across the northern states each winter, is known to increase as ice forms on the lakes. Researchers at the Institute for Sieve Studies have discovered the following conservation equation for hockipuculis in the air:

$$\frac{\mathrm{dC}}{\mathrm{dt}} = \frac{\mathrm{a}C}{\overline{\theta}} \tag{1.1}$$

where a is a constant. Find the conservation equation for C in a turbulent atmosphere. Assume horizontal homogeneity and no subsidence.

Solution.

Expanding the total derivative, we have

$$\frac{\partial C}{\partial t} + U_j \frac{\partial C}{\partial x_j} = \frac{\mathrm{a}}{\overline{\theta}}(\overline{C} - C') \tag{1.2}$$

Expanding C and U_j into its mean and turbulent parts, we get

$$\frac{\partial \overline{C}}{\partial t} + \frac{\partial C'}{\partial t} + \overline{U}_j \frac{\partial \overline{C}}{\partial x_j} + U'_j \frac{\partial \overline{C}}{\partial x_j} + \overline{U}_j \frac{\partial C'}{\partial x_j} + U'_j \frac{\partial C'}{\partial x_j} = \frac{\mathrm{a}}{\overline{\theta}}(\overline{C} - C') \tag{1.3}$$

Expanding this further for all components of U_j, we have

$$\begin{aligned} &\frac{\partial \overline{C}}{\partial t} + \frac{\partial C'}{\partial t} + \overline{U}\frac{\partial \overline{C}}{\partial x} + \overline{V}\frac{\partial \overline{C}}{\partial y} + \overline{W}\frac{\partial \overline{C}}{\partial z} + U'\frac{\partial \overline{C}}{\partial x} + V'\frac{\partial \overline{C}}{\partial y} + W'\frac{\partial \overline{C}}{\partial z} \\ &+ \overline{U}\frac{\partial C'}{\partial x} + \overline{V}\frac{\partial C'}{\partial y} + \overline{W}\frac{\partial C'}{\partial z} + U'\frac{\partial C'}{\partial x} + V'\frac{\partial C'}{\partial y} + W'\frac{\partial C'}{\partial z} = \frac{\mathrm{a}}{\overline{\theta}}(\overline{C} - C') \end{aligned} \tag{1.4}$$

Assuming horizontal homogeneity ($\partial\overline{\alpha}/\partial x = \partial\overline{\alpha}/\partial y = 0$) and negligible subsidence ($\overline{W} = 0$), the 3rd, 4th, 5th, 6th, 7th, and 11th terms all disappear. We get

$$\frac{\partial \overline{C}}{\partial t} + \frac{\partial C'}{\partial t} + \overline{U}\frac{\partial C'}{\partial x} + \overline{V}\frac{\partial C'}{\partial y} + U'\frac{\partial C'}{\partial x} + V'\frac{\partial C'}{\partial y} + W'\frac{\partial C'}{\partial z} = \frac{\mathrm{a}}{\overline{\theta}}(\overline{C} - C') \tag{1.5}$$

After averaging both sides and applying the Reynolds averaging rules, the 2nd, 3rd, 4th, and last terms disappear, leaving us with

$$\frac{\partial \overline{C}}{\partial t} + \overline{U'\frac{\partial C'}{\partial x}} + \overline{V'\frac{\partial C'}{\partial y}} + \overline{W'\frac{\partial C'}{\partial z}} = \frac{\mathrm{a}\overline{C}}{\overline{\theta}} \tag{1.6}$$

Using the flux form of the turbulent advection terms and rewriting this into summation notation, we can simplify it into

$$\frac{\partial \overline{C}}{\partial t} = \frac{\mathrm{a}\overline{C}}{\overline{\theta}} - \frac{\partial\left(\overline{U'_j C'}\right)}{\partial x_j} \tag{1.7}$$

□

Problem 2

Given a kinematic heat flux of 0.2 $K\ m/s$ at the ground, and a flux of -0.1 $K\ m/s$ at the top of a 1 km thick mixed layer, calculate the average warming rate of the mixed layer.

Solution.

Here, we're given that $\overline{w'\theta'}\big|_{0\ m} = 0.2\ K\ m/s$, $\overline{w'\theta'}\big|_{100\ m} = -0.1\ K\ m/s$, and $\Delta z = 1\ km = 100\ m$. We're asked to find the average warming rate $\frac{\partial\bar{\theta}}{\partial t}$ of the mixed layer. We start with the equation for the conservation of heat in turbulent flow. We also only consider turbulence along the z-axis.

$$\frac{\partial\bar{\theta}}{\partial t} + \bar{U}_j\frac{\partial\bar{\theta}}{\partial x_j} = -\frac{1}{\bar{\rho}C_p}\left[L_vE + \frac{\partial\bar{Q}_j^*}{\partial x_j}\right] - \frac{\partial\left(\overline{w'\theta'}\right)}{\partial z} \tag{2.1}$$

Since no other information is given, we assume horizontal homogeneity, and neglect radiation divergence, latent heat exchange, and subsidence. Thus, only the first and last terms are left.

$$\begin{aligned}
\frac{\partial\bar{\theta}}{\partial t} &= -\frac{\partial\overline{w'\theta'}}{\partial z}\\
\frac{\partial\bar{\theta}}{\partial t} &= -\frac{\Delta\overline{w'\theta'}}{\Delta z} = -\frac{\overline{w'\theta'}\big|_{100\ m} - \overline{w'\theta'}\big|_{0\ m}}{\Delta z}\\
\frac{\partial\bar{\theta}}{\partial t} &= -\frac{-0.1\ K\ m/s - 0.2\ K\ m/s}{100\ m} = \mathbf{0.003\ K/s}
\end{aligned} \tag{2.2}$$

□

Problem 3

If a volume of boundary layer air initially contains 2 g/kg of liquid water droplets, and these droplets completely evaporate during 15 minutes, then find $\partial\bar{q}/\partial t$ and $\partial\bar{\theta}/\partial t$ associated with this evaporation. What is the value (with its units) of E, in (3.2.4b) and (3.2.5)?

Solution. Here, we're given that $q\big|_{t=0\ s} = 2\ g/kg$, $q\big|_{t=900\ s} = 0\ g/kg$, and $\Delta t = 900\ s$. We're asked to find $\partial\bar{q}/\partial t$, $\partial\bar{\theta}/\partial t$, and the latent heat E. We start with the equation for the conservation of water vapor (general form).

$$\frac{\partial q}{\partial t} + U_j\frac{\partial q}{\partial x_j} = v_q\frac{\partial^2 q}{\partial x_j^2} + \frac{S_q}{\rho_{\text{air}}} + \frac{E}{\rho_{\text{air}}} \tag{3.1}$$

Since no other information is given, we assume horizontal homogeneity, no body source for water vapor, no viscosity, and no subsidence. Thus, we are left with the first and last terms. The density of air is approximately $\rho_{\text{air}} = 1.225\ kg/m^3$.

$$\begin{aligned}
\frac{\partial q}{\partial t} &= \frac{E}{\rho_{\text{air}}}\\
E &= \rho_{\text{air}}\frac{\Delta q}{\Delta t} = \rho_{\text{air}}\frac{q\big|_{t=900\ s} - q\big|_{t=0\ s}}{\Delta t}\\
\mathbf{E} &= \left(1.225\ kg/m^3\right)\frac{0\ g/kg - 2\ g/kg}{900\ s} = \mathbf{-2.72\times10^{-6}\ kg\ m^{-3}s^{-1}}
\end{aligned} \tag{3.2}$$

From this equation, assuming $q = \overline{q}$, $\partial\overline{\mathbf{q}}/\partial\mathbf{t} = \mathbf{-2.2\times10^{-3}\ g/kg\ s^{-1}}$. To get $\partial\bar{\theta}/\partial t$, we start with the equation for the conservation of heat.

$$\frac{\partial\theta}{\partial t} + U_j\frac{\partial\theta}{\partial x_j} = v_\theta\frac{\partial^2\theta}{\partial x_j^2} - \frac{1}{\rho C_p}\left(\frac{\partial Q_j^*}{\partial x_j}\right) - \frac{L_pE}{\rho C_p} \tag{3.3}$$

Since no other information is given, we assume horizontal homogeneity, no subsidence, no viscosity, and no radiation. Thus, just like with the conservation of moisture, we are left with the first and last terms.

$$\frac{\partial\theta}{\partial t} = -\frac{L_p E}{\rho C_p} \tag{3.4}$$

The latent heat of vaporization of water at $0^o\ C$ is $L_v = 2.5 \times 10^6\ J/kg$. The specific heat of moist air is given by $C_p = C_{pd}(1 + 0.84q)$, where $C_{pd} = 1004.67\ J\ kg^{-1}\ K^{-1}$. Lastly, $q = 2\ g/kg$ is just from the given initial moisture content. Thus, $C_p = C_{pd}(1 + 0.84q) = 1.002C_{pd}$. Plugging in all our constants and the previously solved for E, we get

$$\frac{\partial\theta}{\partial t} = -\frac{\left(2.5 \times 10^6\ J/kg\right)\left(-2.72 \times 10^{-6}\ kg\ m^{-3}s^{-1}\right)}{\left(1.225\ kg/m^3\right)\left(1.002 \cdot 1004.67\ J\ kg^{-1}\ K^{-1}\right)} = 5.5 \times 10^{-3}\ K/s \tag{3.5}$$

Assuming again that $\frac{\partial\theta}{\partial t} = \partial\overline{\theta}/\partial t$, $\partial\overline{\theta}/\partial t = \mathbf{5.5 \times 10^{-3}\ K/s}$

□

Problem 4 (a) Determine the warming rate in the mixed layer, given the soundings of Fig 3.5.

(b) Using the heat flux data of Fig 3.2, what percentage of the warming rate from part (a) can be explained by the turbulent flux divergence term?

(c) Suggest physical mechanisms to explain the remaining percentages of warming.

Solution.

(a) For simplicity, I looked at one pressure level within the mixing layer and determined from Fig. 3.5 in Stull the potential temperature θ for each leg of the flight. The top of the mixing layer is clearly defined by the temperature inversion at around $90 - 87.5\ kPa$. Looking at $95\ kPa$, $\theta_{leg\ 1} = 320\ K$, $\theta_{leg\ 12} = 325\ K$, and $\theta_{leg\ 22} = 329\ K$. We are also given that leg 1 was done at 1438 CDT, leg 12 at 1604 CDT, and leg 22 at 1731 CDT. Looking at the time difference between the first two legs, last two legs, and first and last legs, we have $\Delta t_1 = 5160\ s$ and $\Delta t_2 = 5220\ s$, $\Delta t_3 = 10380\ s$, respectively. The warming rates between the first two legs, last two legs, and first and last legs are given by

$$\begin{aligned}
\left(\frac{\partial\theta}{\partial t}\right)_1 &= \left(\frac{\Delta\theta}{\Delta t}\right)_1 = \frac{325\ K - 320\ K}{5160\ s} = 9.69 \times 10^{-4}\ K/s \\
\left(\frac{\partial\theta}{\partial t}\right)_2 &= \left(\frac{\Delta\theta}{\Delta t}\right)_2 = \frac{329\ K - 325\ K}{5220\ s} = 7.66 \times 10^{-4}\ K/s \\
\left(\frac{\partial\theta}{\partial t}\right)_3 &= \left(\frac{\Delta\theta}{\Delta t}\right)_3 = \frac{329\ K - 320\ K}{10380\ s} = 8.67 \times 10^{-4}\ K/s
\end{aligned} \tag{4.1}$$

On average, the warming rate in the mixing layer is about $\mathbf{8.67 \times 10^{-4}\ K/s}$.

(b) Based on Fig. 3.2 in Stull, we look at the slope of the line plot for heat flux. At $z = 450\ m$, $\overline{w'\theta'} = 0\ K\ m/s$, and at $z = 300\ m$, $\overline{w'\theta'} = 0.02\ K\ m/s$. Thus,

$$\frac{\partial\left(\overline{w'\theta'}\right)}{\partial z} = \frac{\Delta\left(\overline{w'\theta'}\right)}{\Delta z} = \frac{0\ K\ m/s - 0.02\ K\ m/s}{450\ m - 300\ m} = \mathbf{-1.33 \times 10^{-4}\ K/s} \tag{4.2}$$

This represents the final term in the equation for the conservation of heat in turbulent motion.

$$\frac{\partial\overline{\theta}}{\partial t} + \overline{U}_j\frac{\partial\overline{\theta}}{\partial x_j} = -\frac{1}{\overline{\rho}C_p}\left[L_v E + \frac{\partial\overline{Q^*_j}}{\partial x_j}\right] - \frac{\partial\left(\overline{u'_j\theta'}\right)}{\partial x_j} \tag{4.3}$$

Calculating what percent of the answer in (a) turbulent heat flux is, we get

$$100 \cdot \frac{1.33 \times 10^{-4}\ K/s}{8.67 \times 10^{-4}\ K/s} = \mathbf{15.34}\ \% \tag{4.4}$$

(c) Looking at our equation for heat conservation, the remaining contributions to the warming rate comes from advection, radiation, and latent heat exchange. Since latent heat has a negative contribution to the temperature change, we expect E to be negative to achieve warming over time. Thus, within the mixing layer water droplets get evaporated into water vapor over time. Similarly, advection must also be negative to achieve warming temperatures over time. The vertical wind could be advecting the warmer temperatures above the mixing layer downwards to the mixing layer causing it to warm up. Lastly, radiation divergence must also be negative to achieve warming over time. In other words, radiation Q^* must be converging towards the mixing layer causing it to warm up.

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