AATM 529 – Air Sea Interactions

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Homework 3

Problem 1

Why is an understanding of turbulence necessary for studying and modeling the boundary layer?

Solution. Turbulence in the boundary layer induce motions that can transfer momentum, moisture, heat and other scalars such as pollutants or aerosols. If we want to model this quantities (and we most definitely do), we need to consider turbulence in our forecasts specially since majority of the weather we experience are within the boundary layer. Turbulence plays an important role in forecasting and modeling mean quantities. This is evident in the governing equations of mean variables in turbulent flow characteristic of the very turbulent boundary layer. □

Problem 2

Expand the following term, and describe its physical meaning.

$$\delta_{ij}\overline{U}_k \frac{\partial(\overline{u_i'u_j'})}{\partial x_k} \tag{2.1}$$

Solution. The index k is repeating so we sum over $k = 1, 2, 3$ to get

$$\delta_{ij}\overline{U}\frac{\partial(\overline{u_i'u_j'})}{\partial x} + \delta_{ij}\overline{V}\frac{\partial(\overline{u_i'u_j'})}{\partial y} + \delta_{ij}\overline{W}\frac{\partial(\overline{u_i'u_j'})}{\partial z} \tag{2.2}$$

The Kronecker-delta ensures that only terms with $i = j$ are non zero, so we further expand this using $i = j = 1$, $i = j = 2$, and $i = j = 3$ to get

$$\overline{U}\frac{\partial(\overline{u'^2})}{\partial x}+\overline{V}\frac{\partial(\overline{u'^2})}{\partial y}+\overline{W}\frac{\partial(\overline{u'^2})}{\partial z}+\overline{U}\frac{\partial(\overline{v'^2})}{\partial x}+\overline{V}\frac{\partial(\overline{v'^2})}{\partial y}+\overline{W}\frac{\partial(\overline{v'^2})}{\partial z}+\overline{U}\frac{\partial(\overline{w'^2})}{\partial x}+\overline{V}\frac{\partial(\overline{w'^2})}{\partial y}+\overline{W}\frac{\partial(\overline{w'^2})}{\partial z} \tag{2.3}$$

This term can be thought of as the mean advection of the variance of wind, since $\overline{u'^2}, \overline{v'^2}, \overline{w'^2}$ are interpreted as variances. This can also be rearranged and simplified into

$$\overline{U}\frac{\partial}{\partial x}\left(\overline{u'^2}+\overline{v'^2}+\overline{w'^2}\right)+\overline{V}\frac{\partial}{\partial x}\left(\overline{u'^2}+\overline{v'^2}+\overline{w'^2}\right)+\overline{W}\frac{\partial}{\partial x}\left(\overline{u'^2}+\overline{v'^2}+\overline{w'^2}\right) \tag{2.4}$$

Thus, this term can also be thought of as the mean advection of turbulent kinetic energy.

□

Problem 3

List the steps, assumptions, simplifications and substitutions (in their proper order) used to get the following equation from (3.2.3b). Do NOT do the whole derivation, just list the steps.

$$\frac{\partial\overline{U}}{\partial t} = -f_c\left(\overline{V}_g - \overline{V}\right) - \frac{\partial\overline{u'w'}}{\partial z} \tag{3.1}$$

Solution.

1. Make the Boussinesq approximation in Eqn. 3.2.3b.
2. Expand variables into their mean and turbulent parts. Multiply and separate terms.
3. Average the whole equation, and apply Reynolds averaging rules to remove terms that become zero.
4. Rewrite the turbulent advection term into flux form and move it to the right hand side of the equation.
5. Neglect molecular diffusion and viscosity.
6. Let $i = 1$ to get the x-component of the momentum equation.
7. Use the definition of the averaged geostrophic wind, $f_c\overline{V_g} = \frac{1}{\bar{\rho}}\frac{\partial \bar{p}}{\partial x}$.
8. Let $j = 3$.
9. Neglect subsidence such that $\overline{W} = 0$, and assume horizontal homogeneity.

□

Problem 4

Very briefly define the following, and comment or give examples of their use in micrometeorology:

(a) kinematic heat flux

(b) Reynolds stress

(c) horizontal homogeneity

(d) Boussinesq approximation

Solution.

(a) The kinematic heat flux is defined by $Q_H = \frac{\tilde{Q_H}}{\rho_{air}C_{p\ air}}$. It is the transfer of heat per unit area per unit time then divided by the density of moist air ρ_{air} for convenience. It's units $[Km/s]$ let it be expressed in terms of temperature and wind speed for easier measurement.

(b) Reynolds stress is turbulent momentum flux and is only present in turbulent motion. Say we have a cube of fluid in Cartesian coordinates. For each of the three Cartesian directions, we need to consider three further directions of deformations across the face. Thus, the Reynolds stress tensor has nine components. Reynolds stress can be used to describe eddies mixing surrounding airflow towards our fluid of interest. Reynolds stress is symmetric, so it can be written as

$$\begin{bmatrix} \overline{u'u'} & \overline{u'v'} & \overline{u'w'} \\ \overline{v'u'} & \overline{v'v'} & \overline{v'w'} \\ \overline{w'u'} & \overline{w'v'} & \overline{w'w'} \end{bmatrix} = \begin{bmatrix} \overline{u'u'} & \overline{u'v'} & \overline{u'w'} \\ \overline{u'v'} & \overline{v'v'} & \overline{v'w'} \\ \overline{u'w'} & \overline{v'w'} & \overline{w'w'} \end{bmatrix} \tag{4.1}$$

(c) In assuming horizontal homogeneity, the mean of any variable $\bar{\xi}$ does not change along the x or y direction such that $\frac{\partial\bar{\xi}}{\partial x} = \frac{\partial\bar{\xi}}{\partial y} = 0$. Sometimes in micrometeorology, we want to focus on turbulent effects alone while neglecting mean advections to simplify derivations. However, these are rarely valid in the actual atmosphere.

(d) In the Boussinesq approximation, we neglect the density variations in the inertia term, but keep it in the buoyancy/gravity term. The Boussinesq approximation must satisfy the shallow convection conditions. The Boussinesq approximation can be used to describe the dynamics behind updrafts and downdrafts caused by changes in buoyancy due to temperature perturbations.

□

Problem 5

The forecast equation for mean wind in a turbulent flow is:

$$\underset{A}{\frac{\partial \overline{U_i}}{\partial t}} + \underset{B}{\overline{U_j}\frac{\partial \overline{U_i}}{\partial x_j}} = \underset{C}{-\delta_{i3}g} + \underset{D}{f_c\varepsilon_{ij3}\overline{U_j}} - \underset{E}{\frac{1}{\overline{\rho}}\frac{\partial \overline{P}}{\partial x_i}} + \underset{F}{\nu\frac{\partial^2 \overline{U_i}}{\partial {x_j}^2}} - \underset{G}{\frac{\partial \overline{u_i' u_j'}}{\partial x_j}} \tag{5.1}$$

(a) Name each term, and give its physical interpretation.

(b) Starting with the equation above, derive the equation for $\partial\overline{V}/\partial t$, assuming $\overline{U} = 0$.

Solution.

(a)
- A = inertia, storage of mean momentum
- B = advection of mean momentum by the mean wind
- C = gravity only acts along the vertical direction
- D = Coriolis term, effects of the earth's rotation on turbulent flow
- E = pressure gradient force points from a region of high pressure to low pressure
- F = viscous stress, effects of the viscosity of the medium on the diffusion of momentum around the flow
- G = divergence of turbulent momentum flux, effects of Reynolds' stress on the mean motions

(b) First, let $i = 2$ and use the definition of averaged geostrophic wind $f_c\overline{U_g} = \frac{1}{\overline{\rho}}\frac{\partial \overline{p}}{\partial y}$ to get

$$\frac{\partial \overline{V}}{\partial t} = -\overline{U_j}\frac{\partial \overline{V}}{\partial x_j} + f_c\varepsilon_{2j3}\overline{U_j} - f_c\overline{U_g} + \nu\frac{\partial^2 \overline{V}}{\partial {x_j}^2} - \frac{\partial \overline{v' u_j'}}{\partial x_j} \tag{5.2}$$

Simplifying the Coriolis term and expanding the advection term, we get

$$\frac{\partial \overline{V}}{\partial t} = -\overline{U}\frac{\partial \overline{V}}{\partial x} - \overline{V}\frac{\partial \overline{V}}{\partial y} - \overline{W}\frac{\partial \overline{V}}{\partial z} + f_c(\overline{U_g} - \overline{U}) + \nu\frac{\partial^2 \overline{V}}{\partial {x_j}^2} - \frac{\partial \overline{v' u_j'}}{\partial x_j} \tag{5.3}$$

Assuming $\overline{U} = 0$, we can simplify this further into

$$\frac{\partial \overline{V}}{\partial t} = -\left(\overline{V}\frac{\partial \overline{V}}{\partial y} + \overline{W}\frac{\partial \overline{V}}{\partial z}\right) + f_c\overline{U_g} + \nu\frac{\partial^2 \overline{V}}{\partial {x_j}^2} - \frac{\partial \overline{v' u_j'}}{\partial x_j} \tag{5.4}$$

□

Problem 6

Given the nighttime data of Fig 3.10, estimate the vertical profile of temperature change associated with the advective contribution between 2100 and 2230 CDT.

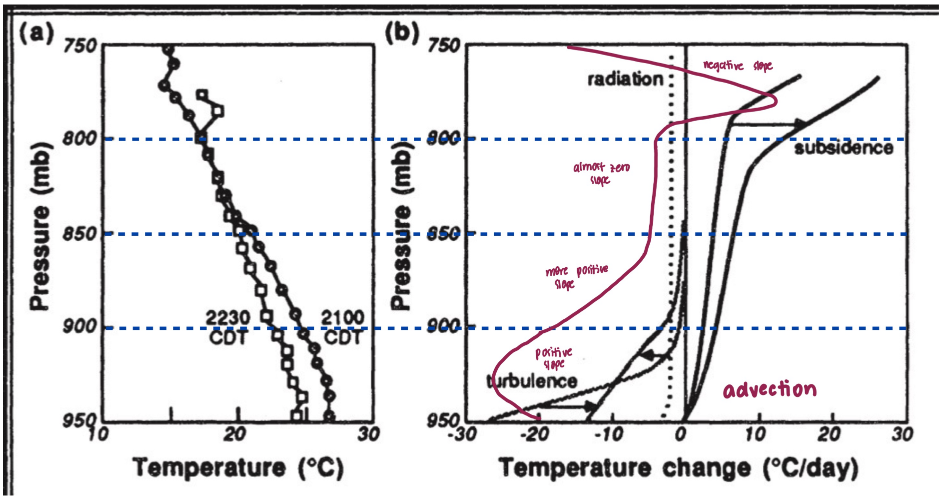
Solution. Here, we use the equation for the conservation of heat in turbulent motion (Eqn. 3.5.3f in Stull). There is no latent heat exchange here, so we can ignore the third term to get

$$\frac{\partial \overline{\theta}}{\partial t} = -\frac{1}{\overline{\rho} C_p}\left(\frac{\partial \overline{Q}_j^*}{\partial x_j}\right) - \frac{\partial\left(\overline{u_j'\theta'}\right)}{\partial x_j} - \overline{U}_j \frac{\partial \overline{\theta}}{\partial x_j} \tag{6.1}$$

In the right hand side, the first term is radiation, the second is the turbulent heat flux divergence, and the third is the advection. The radiation has an almost vertical profile (slope is almost zero), so we can just focus on the contributions of turbulence and advection to the cooling temperatures with height.

$$\frac{\partial \overline{\theta}}{\partial t} \approx -\frac{\partial\left(\overline{u_j'\theta'}\right)}{\partial x_j} - \overline{U}_j \frac{\partial \overline{\theta}}{\partial x_j} \tag{6.2}$$

Near the surface, the turbulence has a high positive slope, while subsidence is also almost zero, so turbulence dominates advection, causing the cooling temperatures. At around 900 hPa, the turbulence starts to have a nearly vertical profile, so the contributions of the advection become more dominant. From our equation, the slope of advection must be positive for temperatures to cool, and vice versa. Thus, from 900 hPa to 850 hPa, advection has a slight positive slope. From 850 hPa to 800 hPa advection has an almost zero slope, since temperatures don't change much between 2100 and 2230 CDT. Finally, from 800 hPa onward, advection has a negative slope, since temperatures are warming.



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