AATM 529 – Air Sea Interactions

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Homework 2

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Problem 1

Given the following instantaneous measurements of potential temperature (θ) and vertical velocity (w) in this table, fill in all the remaining blanks in the table. Also, verify with the answers from above that $\overline{w\theta} = \overline{w}\overline{\theta} + \overline{w'\theta'}$.

Solution. The table below summarizes the calculations given the measurements.

	w	θ	w'	θ'	$(w')^2$	$(\theta')^2$	$w\theta$	$w'\theta'$
	0.5	295	0.4	1	0.16	1	147.5	0.4
	-0.5	293	-0.6	-1	0.36	1	-146.5	0.6
	1	295	0.9	1	0.81	1	295	0.9
	0.8	298	0.7	4	0.49	16	238.4	2.8
	0.9	292	0.8	-2	0.64	4	262.8	-1.6
	-0.2	294	-0.3	0	0.09	0	-58.8	0
	-0.5	292	-0.6	-2	0.36	4	-146	1.2
	0	289	-0.1	-5	0.01	25	0	0.5
	-0.9	293	-1	-1	1	1	-263.7	1
	-0.1	299	-0.2	5	0.04	25	-29.9	-1
Average	0.1	294	0	0	0.396	7.8	29.88	0.48

To verify, $\overline{w\theta} = 29.88$ and $\overline{w}\overline{\theta} + \overline{w'\theta'} = 29.4 + 0.48$. Thus, $\overline{w\theta} = \overline{w}\overline{\theta} + \overline{w'\theta'}$.

□

Problem 2

Given the data in problem (2), find the biased standard deviation for w and θ, and find the linear correlation coefficient between w and θ.

Solution. The biased standard deviation σ is given by

$$\sigma_A = \left(\overline{a'^2}\right)^{1/2} \tag{2.1}$$

For w, the biased standard deviation is $\sigma_w = \left(\overline{w'^2}\right)^{1/2} = \sqrt{0.396} = 0.629$. For θ, the biased standard devation is $\sigma_\theta = \left(\overline{\theta'^2}\right)^{1/2} = \sqrt{7.8} = 2.79$. The linear correlation coefficient between two variables is given by

$$r_{AB} = \frac{\overline{a'b'}}{\sigma_a \sigma_b} \tag{2.2}$$

For w and θ,

$$r_{w\theta} = \frac{\overline{w'\theta'}}{\sigma_w \sigma_\theta} = \frac{0.48}{(0.629)(2.79)} = 0.27 \tag{2.3}$$

□

Problem 3

Using your results from problems (2) and (3), is the data characteristic of a stable, neutral, or unstable boundary layer?

Solution. To check for stability we look at $\overline{w'\theta'}$. If it's positive, we have a positive heat flux, so heat travels upward, which tends to make the lapse rate more adiabatic. If it's negative, we have a negative heat flux, so heat travels downward, which tends to make the lapse rate less adiabatic. Since $\overline{w'\theta'} = 0.48$ is positive, then we have upward heat flux, which is characteristic of an **unstable boundary layer**.

□

Problem 4

The following terms are given in summation notation. Expand them (that is, write out each term of the indicated sums).

(a)
$$\frac{\partial(\overline{u_i' u_j'})}{\partial x_j} \tag{4.1}$$

(b)
$$u_i' \frac{\partial \theta'}{\partial x_i} \tag{4.2}$$

(c)
$$\overline{U_j} \frac{\partial(\overline{u_i' u_k'})}{\partial x_j} \tag{4.3}$$

(d)
$$\overline{u_i' u_j'} \frac{\partial \overline{U_k}}{\partial x_j} \tag{4.4}$$

(e)
$$\frac{\partial(\overline{u_i' u_j' u_k'})}{\partial x_j} \tag{4.5}$$

(f)
$$\left(\frac{\partial u_i}{\partial x_j}\right)\left(\frac{\partial u_k}{\partial x_j}\right) \tag{4.6}$$

Solution.

(a) The index j repeats so we sum across $j = 1, 2, 3$.

$$\frac{\partial(\overline{u_i' u_1'})}{\partial x_1} + \frac{\partial(\overline{u_i' u_2'})}{\partial x_2} + \frac{\partial(\overline{u_i' u_3'})}{\partial x_3} \tag{4.7}$$

We then get three sets of terms for $i = 1, 2, 3$. In terms of the components of u and x, we have

$$\begin{aligned} &\frac{\partial(\overline{u'^2})}{\partial x} + \frac{\partial(\overline{u'v'})}{\partial y} + \frac{\partial(\overline{u'w'})}{\partial z} \\ &\frac{\partial(\overline{v'u'})}{\partial x} + \frac{\partial(\overline{v'^2})}{\partial y} + \frac{\partial(\overline{v'w'})}{\partial z} \\ &\frac{\partial(\overline{w'u'})}{\partial x} + \frac{\partial(\overline{w'v'})}{\partial y} + \frac{\partial(\overline{w'^2})}{\partial z} \end{aligned} \tag{4.8}$$

(b) The index i is repeating so we sum over $i = 1, 2, 3$. In terms of the components of u and x, we have

$$u'\frac{\partial\theta'}{\partial x} + v'\frac{\partial\theta'}{\partial y} + w'\frac{\partial\theta'}{\partial z} \tag{4.9}$$

(c) The index j is repeating so we sum over $j = 1, 2, 3$ to get

$$\overline{U_1}\frac{\partial(\overline{u_i'u_k'})}{\partial x_1} + \overline{U_2}\frac{\partial(\overline{u_i'u_k'})}{\partial x_2} + \overline{U_3}\frac{\partial(\overline{u_i'u_k'})}{\partial x_3} \tag{4.10}$$

In total, we should get nine sets of terms, one for each pair of i and k. In terms of the components of u and x, we get

$$\begin{aligned} &\overline{U}\frac{\partial(\overline{u'^2})}{\partial x} + \overline{V}\frac{\partial(\overline{u'^2})}{\partial y} + \overline{W}\frac{\partial(\overline{u'^2})}{\partial z} \\ &\overline{U}\frac{\partial(\overline{u'v'})}{\partial x} + \overline{V}\frac{\partial(\overline{u'v'})}{\partial y} + \overline{W}\frac{\partial(\overline{u'v'})}{\partial z} \\ &\overline{U}\frac{\partial(\overline{u'w'})}{\partial x} + \overline{V}\frac{\partial(\overline{u'w'})}{\partial y} + \overline{W}\frac{\partial(\overline{u'w'})}{\partial z} \\ &\overline{U}\frac{\partial(\overline{v'u'})}{\partial x} + \overline{V}\frac{\partial(\overline{v'u'})}{\partial y} + \overline{W}\frac{\partial(\overline{v'u'})}{\partial z} \\ &\overline{U}\frac{\partial(\overline{v'^2})}{\partial x} + \overline{V}\frac{\partial(\overline{v'^2})}{\partial y} + \overline{W}\frac{\partial(\overline{v'^2})}{\partial z} \\ &\overline{U}\frac{\partial(\overline{v'w'})}{\partial x} + \overline{V}\frac{\partial(\overline{v'w'})}{\partial y} + \overline{W}\frac{\partial(\overline{v'w'})}{\partial z} \\ &\overline{U}\frac{\partial(\overline{w'u'})}{\partial x} + \overline{V}\frac{\partial(\overline{w'u'})}{\partial y} + \overline{W}\frac{\partial(\overline{w'u'})}{\partial z} \\ &\overline{U}\frac{\partial(\overline{w'v'})}{\partial x} + \overline{V}\frac{\partial(\overline{w'v'})}{\partial y} + \overline{W}\frac{\partial(\overline{w'v'})}{\partial z} \\ &\overline{U}\frac{\partial(\overline{w'^2})}{\partial x} + \overline{V}\frac{\partial(\overline{w'^2})}{\partial y} + \overline{W}\frac{\partial(\overline{w'^2})}{\partial z} \end{aligned} \tag{4.11}$$

After removing repeating sets of terms, we are left with

$$\begin{aligned}
&\overline{U}\frac{\partial(\overline{u'^2})}{\partial x}+\overline{V}\frac{\partial(\overline{u'^2})}{\partial y}+\overline{W}\frac{\partial(\overline{u'^2})}{\partial z}\\
&\overline{U}\frac{\partial(\overline{u'v'})}{\partial x}+\overline{V}\frac{\partial(\overline{u'v'})}{\partial y}+\overline{W}\frac{\partial(\overline{u'v'})}{\partial z}\\
&\overline{U}\frac{\partial(\overline{u'w'})}{\partial x}+\overline{V}\frac{\partial(\overline{u'w'})}{\partial y}+\overline{W}\frac{\partial(\overline{u'w'})}{\partial z}\\
&\overline{U}\frac{\partial(\overline{v'^2})}{\partial x}+\overline{V}\frac{\partial(\overline{v'^2})}{\partial y}+\overline{W}\frac{\partial(\overline{v'^2})}{\partial z}\\
&\overline{U}\frac{\partial(\overline{v'w'})}{\partial x}+\overline{V}\frac{\partial(\overline{v'w'})}{\partial y}+\overline{W}\frac{\partial(\overline{v'w'})}{\partial z}\\
&\overline{U}\frac{\partial(\overline{w'^2})}{\partial x}+\overline{V}\frac{\partial(\overline{w'^2})}{\partial y}+\overline{W}\frac{\partial(\overline{w'^2})}{\partial z}
\end{aligned} \tag{4.12}$$

(d) The index j is repeating so we sum over $j = 1, 2, 3$ to get

$$\overline{u_i'u_1'}\frac{\partial\overline{U_k}}{\partial x_1}+\overline{u_i'u_2'}\frac{\partial\overline{U_k}}{\partial x_2}+\overline{u_i'u_3'}\frac{\partial\overline{U_k}}{\partial x_3} \tag{4.13}$$

In total, we should get nine sets of terms, one for each pair of i and k. In terms of the components of u and x, we get

$$\begin{aligned}
&\overline{u'^2}\frac{\partial\overline{U}}{\partial x}+\overline{u'v'}\frac{\partial\overline{U}}{\partial y}+\overline{u'w'}\frac{\partial\overline{U}}{\partial z}\\
&\overline{u'^2}\frac{\partial\overline{V}}{\partial x}+\overline{u'v'}\frac{\partial\overline{V}}{\partial y}+\overline{u'w'}\frac{\partial\overline{V}}{\partial z}\\
&\overline{u'^2}\frac{\partial\overline{W}}{\partial x}+\overline{u'v'}\frac{\partial\overline{W}}{\partial y}+\overline{u'w'}\frac{\partial\overline{W}}{\partial z}\\
&\overline{v'u'}\frac{\partial\overline{U}}{\partial x}+\overline{v'^2}\frac{\partial\overline{U}}{\partial y}+\overline{v'w'}\frac{\partial\overline{U}}{\partial z}\\
&\overline{v'u'}\frac{\partial\overline{V}}{\partial x}+\overline{v'^2}\frac{\partial\overline{V}}{\partial y}+\overline{v'w'}\frac{\partial\overline{V}}{\partial z}\\
&\overline{v'u'}\frac{\partial\overline{W}}{\partial x}+\overline{v'^2}\frac{\partial\overline{W}}{\partial y}+\overline{v'w'}\frac{\partial\overline{W}}{\partial z}\\
&\overline{w'u'}\frac{\partial\overline{U}}{\partial x}+\overline{w'v'}\frac{\partial\overline{U}}{\partial y}+\overline{w'^2}\frac{\partial\overline{U}}{\partial z}\\
&\overline{w'u'}\frac{\partial\overline{V}}{\partial x}+\overline{w'v'}\frac{\partial\overline{V}}{\partial y}+\overline{w'^2}\frac{\partial\overline{V}}{\partial z}\\
&\overline{w'u'}\frac{\partial\overline{W}}{\partial x}+\overline{w'v'}\frac{\partial\overline{W}}{\partial y}+\overline{w'^2}\frac{\partial\overline{W}}{\partial z}
\end{aligned} \tag{4.14}$$

(e) The index j is repeating so we sum over $j = 1, 2, 3$ to get

$$\frac{\partial(\overline{u_i'u_1'u_k'})}{\partial x_1}+\frac{\partial(\overline{u_i'u_2'u_k'})}{\partial x_2}+\frac{\partial(\overline{u_i'u_3'u_k'})}{\partial x_3} \tag{4.15}$$

In total, we should get nine sets of terms, one for each pair of i and k. In terms of the components of u and x, we get

$$
\begin{aligned}
&\frac{\partial(\overline{u'^3})}{\partial x}+\frac{\partial(\overline{u'^2v'})}{\partial y}+\frac{\partial(\overline{u'^2w'})}{\partial z}\\
&\frac{\partial(\overline{u'^2v'})}{\partial x}+\frac{\partial(\overline{u'v'^2})}{\partial y}+\frac{\partial(\overline{u'w'v'})}{\partial z}\\
&\frac{\partial(\overline{u'^2w'})}{\partial x}+\frac{\partial(\overline{u'v'w'})}{\partial y}+\frac{\partial(\overline{u'w'^2})}{\partial z}\\
&\frac{\partial(\overline{v'^2u'})}{\partial x}+\frac{\partial(\overline{v'^3})}{\partial y}+\frac{\partial(\overline{v'^2w'})}{\partial z}\\
&\frac{\partial(\overline{v'u'w'})}{\partial x}+\frac{\partial(\overline{v'^2w'})}{\partial y}+\frac{\partial(\overline{v'w'^2})}{\partial z}\\
&\frac{\partial(\overline{w'u'^2})}{\partial x}+\frac{\partial(\overline{w'v'u'})}{\partial y}+\frac{\partial(\overline{w'^2u'})}{\partial z}\\
&\frac{\partial(\overline{w'u'v'})}{\partial x}+\frac{\partial(\overline{w'v'^2})}{\partial y}+\frac{\partial(\overline{w'^2v'})}{\partial z}\\
&\frac{\partial(\overline{w'^2u'})}{\partial x}+\frac{\partial(\overline{w'^2v'})}{\partial y}+\frac{\partial(\overline{w'^3})}{\partial z}
\end{aligned}
\tag{4.16}
$$

After removing repeating sets of terms, we are left with

$$
\begin{aligned}
&\frac{\partial(\overline{u'^3})}{\partial x}+\frac{\partial(\overline{u'^2v'})}{\partial y}+\frac{\partial(\overline{u'^2w'})}{\partial z}\\
&\frac{\partial(\overline{u'^2v'})}{\partial x}+\frac{\partial(\overline{u'v'^2})}{\partial y}+\frac{\partial(\overline{u'w'v'})}{\partial z}\\
&\frac{\partial(\overline{u'^2w'})}{\partial x}+\frac{\partial(\overline{u'v'w'})}{\partial y}+\frac{\partial(\overline{u'w'^2})}{\partial z}\\
&\frac{\partial(\overline{v'u'^2})}{\partial x}+\frac{\partial(\overline{v'^2u'})}{\partial y}+\frac{\partial(\overline{v'w'u'})}{\partial z}\\
&\frac{\partial(\overline{v'^2u'})}{\partial x}+\frac{\partial(\overline{v'^3})}{\partial y}+\frac{\partial(\overline{v'^2w'})}{\partial z}\\
&\frac{\partial(\overline{v'u'w'})}{\partial x}+\frac{\partial(\overline{v'^2w'})}{\partial y}+\frac{\partial(\overline{v'w'^2})}{\partial z}\\
&\frac{\partial(\overline{w'^2u'})}{\partial x}+\frac{\partial(\overline{w'^2v'})}{\partial y}+\frac{\partial(\overline{w'^3})}{\partial z}
\end{aligned}
\tag{4.17}
$$

(f) The index j is repeating so we sum over $j = 1, 2, 3$ to get

$$
\left(\frac{\partial u_i}{\partial x_1}\right)\left(\frac{\partial u_k}{\partial x_1}\right)+\left(\frac{\partial u_i}{\partial x_2}\right)\left(\frac{\partial u_k}{\partial x_2}\right)+\left(\frac{\partial u_i}{\partial x_3}\right)\left(\frac{\partial u_k}{\partial x_3}\right)
\tag{4.18}
$$

In total, we should get nine sets of terms, one for each pair of i and k. In terms of the components of u and x, we get

$$
\begin{gathered}
\left(\frac{\partial x}{\partial x}\right)\left(\frac{\partial x}{\partial x}\right)+\left(\frac{\partial x}{\partial y}\right)\left(\frac{\partial x}{\partial y}\right)+\left(\frac{\partial x}{\partial z}\right)\left(\frac{\partial x}{\partial z}\right)\\
\left(\frac{\partial x}{\partial x}\right)\left(\frac{\partial y}{\partial x}\right)+\left(\frac{\partial x}{\partial y}\right)\left(\frac{\partial y}{\partial y}\right)+\left(\frac{\partial x}{\partial z}\right)\left(\frac{\partial y}{\partial z}\right)\\
\left(\frac{\partial x}{\partial x}\right)\left(\frac{\partial z}{\partial x}\right)+\left(\frac{\partial x}{\partial y}\right)\left(\frac{\partial z}{\partial y}\right)+\left(\frac{\partial x}{\partial z}\right)\left(\frac{\partial z}{\partial z}\right)\\
\left(\frac{\partial y}{\partial x}\right)\left(\frac{\partial x}{\partial x}\right)+\left(\frac{\partial y}{\partial y}\right)\left(\frac{\partial x}{\partial y}\right)+\left(\frac{\partial y}{\partial z}\right)\left(\frac{\partial x}{\partial z}\right)\\
\left(\frac{\partial y}{\partial x}\right)\left(\frac{\partial y}{\partial x}\right)+\left(\frac{\partial y}{\partial y}\right)\left(\frac{\partial y}{\partial y}\right)+\left(\frac{\partial y}{\partial z}\right)\left(\frac{\partial y}{\partial z}\right)\\
\left(\frac{\partial y}{\partial x}\right)\left(\frac{\partial z}{\partial x}\right)+\left(\frac{\partial y}{\partial y}\right)\left(\frac{\partial z}{\partial y}\right)+\left(\frac{\partial y}{\partial z}\right)\left(\frac{\partial z}{\partial z}\right)\\
\left(\frac{\partial z}{\partial x}\right)\left(\frac{\partial x}{\partial x}\right)+\left(\frac{\partial z}{\partial y}\right)\left(\frac{\partial x}{\partial y}\right)+\left(\frac{\partial z}{\partial z}\right)\left(\frac{\partial x}{\partial z}\right)\\
\left(\frac{\partial z}{\partial x}\right)\left(\frac{\partial y}{\partial x}\right)+\left(\frac{\partial z}{\partial y}\right)\left(\frac{\partial y}{\partial y}\right)+\left(\frac{\partial z}{\partial z}\right)\left(\frac{\partial y}{\partial z}\right)\\
\left(\frac{\partial z}{\partial x}\right)\left(\frac{\partial z}{\partial x}\right)+\left(\frac{\partial z}{\partial y}\right)\left(\frac{\partial z}{\partial y}\right)+\left(\frac{\partial z}{\partial z}\right)\left(\frac{\partial z}{\partial z}\right)
\end{gathered}
\tag{4.19}
$$

Simplifying this and removing repeating sets of terms, we get

$$
\begin{gathered}
1+\left(\frac{\partial x}{\partial y}\right)^2+\left(\frac{\partial x}{\partial z}\right)^2\\
\left(\frac{\partial y}{\partial x}\right)+\left(\frac{\partial x}{\partial y}\right)+\left(\frac{\partial x}{\partial z}\right)\left(\frac{\partial y}{\partial z}\right)\\
\left(\frac{\partial z}{\partial x}\right)+\left(\frac{\partial x}{\partial y}\right)\left(\frac{\partial z}{\partial y}\right)+\left(\frac{\partial x}{\partial z}\right)\\
\left(\frac{\partial y}{\partial x}\right)^2+1+\left(\frac{\partial y}{\partial z}\right)^2\\
\left(\frac{\partial y}{\partial x}\right)\left(\frac{\partial z}{\partial x}\right)+\left(\frac{\partial z}{\partial y}\right)+\left(\frac{\partial y}{\partial z}\right)\\
\left(\frac{\partial z}{\partial x}\right)+\left(\frac{\partial z}{\partial y}\right)\left(\frac{\partial x}{\partial y}\right)+\left(\frac{\partial x}{\partial z}\right)\\
\left(\frac{\partial z}{\partial x}\right)^2+\left(\frac{\partial z}{\partial y}\right)^2+1
\end{gathered}
\tag{4.20}
$$

□

Problem 5

Consider a 100 m thick layer of air at sea level, with an initial potential temperature of 290 K. If the kinematic heat flux into the bottom of this layer is 0.2 $K\ m/s$ and the flux out of the top is 0.1 $K\ m/s$, then what is the potential temperature of that layer 2 hours later? Assume that the potential temperature is constant with height in the layer.

Solution. In this problem, energy coming in from the bottom of the layer is 0.2 $K\ m/s$, and coming out is 0.1 $K\ m/s$. We can then assume that the difference of 0.1 $K\ m/s$ between the two gets absorbed by the layer. Two hours is 7200 s and the layer spans 100 m. By cancelling out the units, we get the change in heat in our layer.

$$\frac{0.1\ K\ m/s}{100\ m} \cdot 7200\ s = 7.2\ K \tag{5.1}$$

Adding this to the initial potential temperature, we get the final potential temperature to be 297.2 K.

□

Problem 6

Given the typical variation of wind speed with height within the surface layer (see Chapt 1), and using a development similar to that in section 2.7:

(a) Determine whether the net kinematic momentum flux, $\overline{u'w'}$ is positive, negative, or zero within the surface layer.

(b) Does you answer mean the momentum is being transported up or down, on the average?

(c) This momentum that is transported up or down, where does it go or where does it come from, and how would that alter the mean state of the atmosphere?

Solution.

(a) The zonal wind profile u in the surface layer is nearly logarithmic with height. The profile of u in the surface would look similar to Fig. 2.12b. Thus, for an air parcel moving up, $w' > 0$ and $u' < 0$, and for an air parcel moving down, $w' < 0$ and $u' > 0$. The average kinetic momentum flux is then $\overline{u'w'} < 0$ within the surface layer.

(b) Because $\overline{u'w'}$ is negative, kinetic momentum is transported down on the average.

(c) This downward transported momentum reaches the surface, so the rough surface acts as a momentum sink, similar to the grooves along the edges of the pipe in the *Turbulence* video by R.W. Stewart. Since momentum is on average lost to the surface, this gives the characteristic logarithmic profile of winds in the boundary layer.

□

Problem 7

Expand the Coriolis term $[-2\varepsilon_{ijk}\Omega_j U_k]$ using summation notation for the case $i = 1$. Assume that $\Omega_j = (0, \omega\cos\lambda, \omega\sin\lambda)$ are the three components of the angular velocity vector, where λ is latitude and ω is the speed of rotation of the earth $\omega = 360$ degrees in 24 hours). Assume that $W = 0$ for simplicity.

Solution. We're just looking at $i = 1$, so we use $-2\varepsilon_{1jk}\Omega_j U_k$. Here, the index j is repeated, so we sum them over $j = 1, 2, 3$ such that

$$-2\varepsilon_{1jk}\Omega_j U_k = -2\left(\varepsilon_{11k}\Omega_1 U_k + \varepsilon_{12k}\Omega_2 U_k + \varepsilon_{13k}\Omega_3 U_k\right) \tag{7.1}$$

Here, $\varepsilon_{11k} = 0$ because of repeating indices, so the first term disappears such that

$$-2\varepsilon_{1jk}\Omega_j U_k = -2\left(\varepsilon_{12k}\Omega_2 U_k + \varepsilon_{13k}\Omega_3 U_k\right) \tag{7.2}$$

The index k repeats as well, so we sum them again over $k = 1, 2, 3$. For the first term in the right hand side, we get

$$\varepsilon_{12k}\Omega_2 U_k = \varepsilon_{121}\Omega_2 U_1 + \varepsilon_{122}\Omega_2 U_2 + \varepsilon_{123}\Omega_2 U_3 = \Omega_2 U_3 \tag{7.3}$$

The first two terms vanish again because of repeating indices, and $\varepsilon_{123} = 1$ because of increasing indices. For the second term, we get

$$\varepsilon_{13k}\Omega_3 U_k = \varepsilon_{131}\Omega_3 U_1 + \varepsilon_{132}\Omega_3 U_2 + \varepsilon_{133}\Omega_3 U_3 = -\Omega_3 U_2 \tag{7.4}$$

The first and last terms vanish again because of repeating indices, and $\varepsilon_{132} = -1$ because of decreasing indices. Thus, the Coriolis term expands into

$$-2\varepsilon_{1jk}\Omega_j U_k = 2(\Omega_3 U_2 - \Omega_2 U_3) \tag{7.5}$$

Further expanding this based on the components of Ω and U, we get

$$-2\varepsilon_{1jk}\Omega_j U_k = 2(V\omega \sin\lambda - W\omega\cos\lambda) \tag{7.6}$$

Since $W = 0$ for simplicity, then $-2\varepsilon_{1jk}\Omega_j U_k = 2V\omega\sin\lambda$

□

Problem 8

Given the following variances in m^2/s^2. Where, when and for which variables is the turbulence:

(a) Stationary?

(b) Homogeneous?

(c) Isotropic?

Solution.

(a) At stationary turbulence, the variable is not statistically changing from one averaging interval to the next. Thus, the measured variance at 1000 UTC and 1100 UTC should be the same. From the given, $\overline{u'^2}$ at location A and $\overline{v'^2}$ at location B are both stationary.

(b) At homogeneous turbulence, the variable is not statistically changing from one location to the next. Thus, the measured variance at location A and location B should be the same. From the given, $\overline{u'^2}$ at 1100 UTC, $\overline{v'^2}$ at 1000 UTC, and $\overline{w'^2}$ at 1000 UTC are homogeneous.

(c) At isotropic turbulence, the variable should be directionally invariant. Thus, the measured variance at a specific time or location must be same in all directions (i.e. equal along u, v, and w). From the given, location A at 1100 UTC is isotropic, since the wind variances are equal in all directions.

□

Problem 9

Simplify the following term (assume horizontal homogeneity).

$$\delta_{k1}\varepsilon_{ijk}\frac{\partial\overline{u_i'\theta'}}{\partial x_j} \tag{9.1}$$

Solution. Because of the δ_{k1} in front, only terms with $k = 1$ are non-zero, so we'll only be looking at

$$\varepsilon_{ij1}\frac{\partial\overline{u_i'\theta'}}{\partial x_j} \tag{9.2}$$

For $i = 1$ and $j = 1$, $\varepsilon_{ij1} = 0$, so we'll only be looking at terms with $i = 2, 3$ and $j = 2, 3$. The indices i and j can't be equal as well, so the only terms that don't become zero are $i, j = 2, 3$ and $i, j = 3, 2$. For $i, j = 2, 3$, we get the term

$$\varepsilon_{231}\frac{\partial\overline{u_2'\theta'}}{\partial x_3} = \frac{\partial\overline{u_2'\theta'}}{\partial x_3} \tag{9.3}$$

For $i, j = 3, 2$, we get the term

$$\varepsilon_{321}\frac{\partial\overline{u_3'\theta'}}{\partial x_2} = -\frac{\partial\overline{u_3'\theta'}}{\partial x_2} \tag{9.4}$$

Summing these two terms we get the simplified form of the original equation as

$$\delta_{k1}\varepsilon_{ijk}\frac{\partial\overline{u_i'\theta'}}{\partial x_j} = \frac{\partial\overline{u_2'\theta'}}{\partial x_3} - \frac{\partial\overline{u_3'\theta'}}{\partial x_2} \tag{9.5}$$

Rewriting this in terms of the components of u' and x, we get

$$\delta_{k1}\varepsilon_{ijk}\frac{\partial\overline{u_i'\theta'}}{\partial x_j} = \frac{\partial\overline{v'\theta'}}{\partial z} - \frac{\partial\overline{w'\theta'}}{\partial y} \tag{9.6}$$

Assuming horizontal homogeneity, the variances should not change along x or y, so $\frac{\partial\overline{w'\theta'}}{\partial y} = 0$, simplifying this into

$$\delta_{k1}\varepsilon_{ijk}\frac{\partial\overline{u_i'\theta'}}{\partial x_j} = \frac{\partial\overline{v'\theta'}}{\partial z} \tag{9.7}$$

□