Long-Horizon Returns

Eugene F. Fama
Booth School of Business, University of Chicago

Kenneth R. French
Amos Tuck School of Business, Dartmouth College

We use bootstrap simulations to examine the properties of long-horizon U.S. stock market returns. We document the rate at which continuously compounded market returns converge toward normal distributions as we extend the horizon from 1 month to 30 years, and the rate at which dollar payoffs converge toward lognormal. We also verify that, though largely irrelevant at short horizons, uncertainty about the expected market return has a substantial impact on uncertainty about long-horizon payoffs. (*JEL* G11, G12, G17, C15, C58)

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There is much research on the predictability of long-horizon returns and optimal portfolio allocations for long-horizon investors (e.g., Merton 1969; Samuelson 1969; Fama and French 1988, 1989; Brennan, Schwartz, and Lagnado 1997; Barberis 2000; Viceira 2001; Campbell and Viceira 2002; Campbell, Chan, and Viceira 2003; Boudoukh, Richardson, and Whitelaw 2008.)

We also know a lot about the characteristics of short-horizon stock returns. Distributions of daily and monthly returns, for example, are leptokurtic relative to the normal distribution (Fama 1965). Because we have fewer observations, we know less about distributions of long-horizon returns. It seems likely, however, that the characteristics of long-horizon returns are of central interest to investors saving for distant payoffs. If so, it is likely that the characteristics of long-horizon returns affect asset pricing in ways missed by our models. Evidence on distributions of long-horizon returns is a logical first step in remedying this deficiency.

We use bootstrap simulations to study distributions of U.S. stock returns for horizons up to 30 years. We focus on the value-weight market portfolio of

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U.S. stocks (henceforth Market) and ask two questions. First, how does the distribution of investment payoffs change as we extend the horizon? Second, how does uncertainty about the expected return affect the distribution of long-horizon payoffs?

Investment payoffs for longer horizons are products of payoffs for shorter periods, which are months in our tests:

$$1 + R_T = \Pi_{t=1}^{T} (1 + r_t). \tag{1}$$

In this equation, r_t is the simple return for month t, R_T is the simple return for T months, and $1 + R_T$ is the payoff. Equivalently, continuously compounded (CC) returns for longer horizons, C_T, are sums of monthly CC returns, c_t:

$$C_T \equiv \ln(1 + R_T) = \sum_{t=1}^{T} \ln(1 + r_t) = \sum_{t=1}^{T} c_t. \tag{2}$$

Our bootstrap simulations construct CC cumulative returns, C_T, by summing T continuously compounded monthly returns from observed Market returns. We randomly select monthly returns with replacement, so they are independent and identically distributed (IID). They also have finite variance. Thus, the central limit theorem says bootstrapped CC cumulative returns converge toward normal distributions as the investment horizon increases. If CC returns converge toward normal, payoffs converge toward lognormal. We use the bootstrap simulations to examine the rate at which distributions of CC returns and payoffs approach normal and lognormal distributions.

Uncertainty about the expected monthly return can have a large effect on distributions of long-horizon payoffs. To examine this issue, we add uncertainty about expected monthly returns—estimated by the standard error of the average monthly Market return for 1926–2016 or 1963–2016—to the bootstrap simulations. The results reinforce the central message of the more structured analysis in Pastor and Stambaugh (2012). The error in the estimate of the expected return for next month or next year is tiny relative to the standard deviation of the unexpected return. Because the error persists over the full holding period, however, imprecision in the estimate of the expected monthly return has a larger effect on the possible dispersion of long horizon payoffs. Simulations for 1963–2016 say that incorporating this imprecision increases uncertainty by about one-third for 20-year payoffs and two-thirds for 30-year payoffs.

The bootstrap simulations treat our samples of monthly returns for 1963–2016 or 1926–2016 as populations of monthly returns. As the raw material for constructing distributions of 20- or 30-year returns, samples of 642 (1963–2016) or 1086 (1926–2016) monthly returns are small. We worry that our inferences about distributions of longer horizon returns are affected by differences between the distribution of monthly returns we observe and the underlying population.

We use a novel bootstrap of bootstraps (BB) approach to address this issue. In our simulations that use the actual returns of 1963–2016, for example, we sample the 642 monthly returns with replacement to produce 100,000 returns for each longer return horizon. To judge how random differences between the base sample of actual monthly returns and the underlying population of monthly returns might affect the results, we draw 1,000 base samples of 642 monthly returns from a normal distribution with the same mean and standard deviation as actual 1963–2016 monthly returns. Each base sample is used to generate 100,000 returns for each of the longer return horizons we examine. In this way, we produce 1,000 replications of the bootstrap simulations that allow us to judge how random features of the observed monthly returns can affect distributions of simulated returns for longer horizons.

Like the bootstrap simulations that use actual returns, each of the 1,000 replications of the bootstrap of bootstraps uses a fixed sample of 642 or 1,086 monthly returns to generate 100,000 returns for each return horizon. Much of the variation in the results across the replications is caused by differences in the fixed samples (FS) of monthly returns. For perspective, we show results for 1,000 replications of a simplified bootstrap of bootstraps in which there are no base samples and every monthly return is a new drawing from a normal distribution with the same mean and standard deviation as actual monthly returns. We label these NED (new each draw) simulations to distinguish them from the fixed sample (FS) bootstraps of bootstraps that more closely replicate the bootstrap procedure for actual returns. The perspective provided by the FS and NED simulations about the simulations that use actual returns is among the more interesting contributions of the paper.

1. The Bootstrap Procedure

We examine distributions of monthly, annual, and 3-, 5-, 10-, 20-, and 30-year returns on the CRSP Market portfolio of U.S. stocks for July 1926 to December 2016 (henceforth 1926–2016) and July 1963 to December 2016 (henceforth 1963–2016). Market is a value-weight portfolio of NYSE, AMEX (after 1962), and NASDAQ (after 1972) stocks. July 1963 is the start date in many of our previous papers. Return volatility is high during the 1930s, and one can argue that this period is a different regime, unlikely to repeat. But the salient characteristics of return distributions are similar for 1963–2016 and 1926–2016.

Sample sizes are large for monthly returns, but shrink quickly as the return horizon increases. We use bootstrap simulations to fill in details of long-horizon return distributions. The simulations treat the 642 returns of 1963–2016 or the 1,086 monthly returns of 1926–2016 as the population. To simulate the distribution of annual returns, for example, we draw 100,000 samples of 12 monthly returns with replacement from the 1963–2016 or 1926–2016 population and compound the returns in each sample.

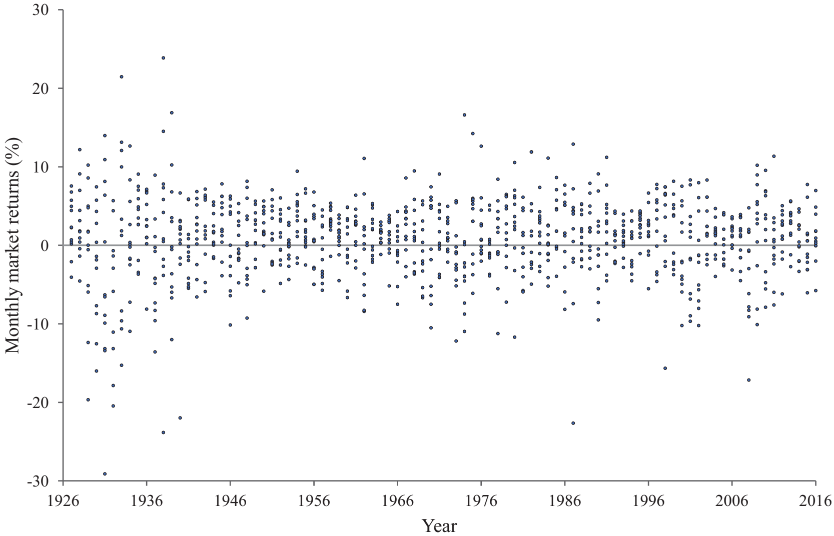

Figure 1
Monthly market returns plotted by year, 1927–2016

Because the simulations randomly mix the monthly returns of 1963–2016 or 1926–2016, our inferences about long-horizon returns assume monthly returns are independent and identically distributed. Independence is a good approximation. Autocorrelations of monthly returns are close to zero. Identically distributed is more tenuous, at least for the full sample period. Figure 1, which plots the monthly returns of 1927–2016 by year, shows that Market returns are heteroskedastic. The 1930s are a period of high return volatility. For example, during 1930–1940 there are four monthly returns below -20% and five above 20%. Thereafter, there is one monthly return below -20% (October 1987) and none above 20%. In contrast, 1941–1962 is a period of relatively low return volatility, with one monthly return below -10% (barely) and one above 10% (again barely). Thereafter, monthly returns above 10% and below -10% are common.

As a rough assessment of time-varying return volatility, we estimate autocorrelations of squared monthly returns and absolute monthly returns. Skipping the details, the autocorrelations for 1926–2016 are around 0.20 and show little tendency to decay for longer lags. This is consistent with the long-term shifts in volatility suggested by Figure 1. For 1963–2016, the autocorrelations are close to zero, which suggests that, for our purposes, constant volatility is a reasonable assumption for this period. At a minimum, the 1963–2016 results suggest that during this period variation in expected volatility is small relative to the total volatility of monthly returns. The simulations based on 1963–2016 monthly returns are also arguably more relevant for return prospects going forward. The text focuses on 1963–2016 simulation results. The appendix provides results for 1926–2016.

2. Simulating Bootstrap Simulations

Our conclusions about how fast CC returns and payoffs converge toward normal and lognormal distributions as the horizon increases are from bootstrap simulations that repeatedly sample from a set of realized monthly returns. The simulation results are affected by the way the distribution of observed monthly returns differs (albeit randomly) from the underlying population distribution and by the fact that the same sample of monthly returns is the raw material for all simulated returns of all horizons. For perspective we run two experiments. Both replicate our bootstrap procedure 1,000 times, but they replace realized monthly CC returns with draws from a normal distribution. The mean and standard deviation of the normal distribution match those of the 642 realized CC returns of 1963–2016.

The fixed sample bootstrap of bootstraps mimics the simulations that use actual 1963–2016 returns. We start each of 1,000 FS replications by drawing 642 independent normal random returns. We sample these monthly returns with replacement to generate each horizon's 100,000 cumulative CC returns. The variation in results across 1,000 FS replications is our evidence on sampling variation due to random specifics of the base samples of 642 monthly returns.

The second, new each draw, bootstrap of bootstraps experiment draws a new IID random variable for every monthly CC return in the 100,000 cumulative CC returns for each horizon. Comparing FS and NED BB simulations provides evidence of how sampling with replacement from a fixed sample of monthly returns (the standard bootstrap procedure) differs from an ideal scenario in which every monthly return used is a new draw.

The monthly CC returns in our FS and NED BB simulations are from a normal distribution with the same mean and standard deviation as the actual monthly Market returns of 1963–2016. Actual returns are almost surely nonnormal. One goal of the BB simulations, however, is general perspective on inference issues when a fixed sample of monthly returns is used to simulate distributions of returns for longer horizons. Our view is that the features of BB results that are our focus on this issue are robust to the deviations of actual monthly returns from normal. More important, the variation across 1,000 FS BB replications in parameter estimates for T-period CC return distributions is evidence on sampling variation when return distributions are normal. We use this sampling variation to judge convergence to normal of distributions of T-month returns from the bootstrap simulations that use actual monthly returns.

Table 1 summarizes the FS and NED simulations of CC returns. For each return horizon, panel A shows average values across 1,000 replications of the averages and standard deviations of the returns in a replication. These

Table 1
Simulations of bootstrap simulations, 1963–2016

A. Average across 1,000 replications

	Moments of CC returns				$S_{Ti} = (C_{Ti} - Mean_T) / SD_T$								
	Mean	*SD*	*Skew*	*Kurt*	1%	5%	10%	25%	50%	75%	90%	95%	99%
Unit normal					−2.326	−1.645	−1.282	−0.674	0.000	0.674	1.282	1.645	2.326
NED: New normal monthly return for each draw													
1 month	0.0080	0.0444	−0.000	3.000	−2.326	−1.645	−1.282	−0.674	0.000	0.675	1.281	1.645	2.326
1 year	0.0962	0.1537	−0.000	3.000	−2.327	−1.645	−1.282	−0.675	0.000	0.675	1.282	1.645	2.326
3 years	0.2885	0.2664	−0.000	3.000	−2.326	−1.645	−1.281	−0.675	−0.000	0.674	1.282	1.645	2.326
5 years	0.4808	0.3438	0.000	2.999	−2.326	−1.645	−1.282	−0.675	−0.000	0.674	1.282	1.645	2.326
10 years	0.9616	0.4862	0.000	3.000	−2.326	−1.645	−1.282	−0.674	0.000	0.674	1.281	1.645	2.326
20 years	1.9231	0.6877	0.000	3.000	−2.326	−1.645	−1.282	−0.674	−0.000	0.675	1.281	1.645	2.327
30 years	2.8847	0.8421	−0.000	3.000	−2.326	−1.645	−1.281	−0.674	0.000	0.675	1.281	1.645	2.327
FS: Sampling from a fixed sample of 642 normal CC returns for all trials and horizons in a replication													
1 month	0.0080	0.0443	0.001	2.986	−2.319	−1.645	−1.282	−0.677	0.000	0.676	1.283	1.647	2.319
1 year	0.0955	0.1536	0.000	3.000	−2.326	−1.645	−1.282	−0.675	−0.000	0.675	1.282	1.645	2.326
3 years	0.2866	0.2660	0.001	3.000	−2.326	−1.644	−1.281	−0.675	−0.000	0.675	1.282	1.645	2.327
5 years	0.4776	0.3434	0.000	3.000	−2.326	−1.645	−1.282	−0.674	−0.000	0.675	1.282	1.645	2.327
10 years	0.9553	0.4855	−0.000	2.999	−2.326	−1.645	−1.282	−0.675	0.000	0.675	1.281	1.645	2.326
20 years	1.9107	0.6867	−0.000	3.000	−2.327	−1.645	−1.282	−0.674	0.000	0.675	1.282	1.645	2.326
30 years	2.8660	0.8410	−0.000	3.000	−2.327	−1.645	−1.282	−0.674	0.000	0.675	1.282	1.645	2.326

(continued)

B. Standard deviations across 1,000 replications

	Moments of CC returns				$S_{Ti} = (C_{Ti} - Mean_T) \ / \ SD_T$								
	Mean	*SD*	*Skew*	*Kurt*	1%	5%	10%	25%	50%	75%	90%	95%	99%
NED: New normal monthly return for each draw													
1 month	0.0001	0.0001	0.008	0.015	0.010	0.005	0.003	0.002	0.002	0.002	0.003	0.005	0.010
1 year	0.0005	0.0003	0.007	0.016	0.010	0.004	0.003	0.003	0.002	0.002	0.003	0.005	0.010
3 years	0.0008	0.0006	0.008	0.015	0.010	0.005	0.003	0.002	0.002	0.003	0.003	0.004	0.010
5 years	0.0011	0.0008	0.008	0.016	0.010	0.005	0.003	0.003	0.002	0.003	0.003	0.004	0.010
10 years	0.0016	0.0011	0.008	0.016	0.010	0.004	0.003	0.002	0.002	0.002	0.003	0.005	0.010
20 years	0.0022	0.0015	0.007	0.016	0.010	0.005	0.003	0.003	0.002	0.003	0.003	0.005	0.010
30 years	0.0027	0.0018	0.008	0.015	0.010	0.005	0.003	0.002	0.002	0.002	0.003	0.005	0.010
FS: Sampling from a fixed sample of 642 normal CC returns for all trials and horizons in a replication													
1 month	0.0017	0.0012	0.098	0.198	0.123	0.058	0.043	0.031	0.029	0.032	0.042	0.056	0.126
1 year	0.0207	0.0043	0.029	0.023	0.023	0.009	0.005	0.004	0.005	0.004	0.005	0.009	0.023
3 years	0.0622	0.0075	0.018	0.017	0.016	0.007	0.004	0.003	0.004	0.003	0.004	0.007	0.016
5 years	0.1036	0.0096	0.015	0.016	0.014	0.006	0.004	0.003	0.003	0.003	0.004	0.006	0.014
10 years	0.2072	0.0136	0.012	0.016	0.012	0.005	0.003	0.003	0.003	0.003	0.003	0.005	0.012
20 years	0.4145	0.0192	0.010	0.015	0.011	0.005	0.003	0.003	0.003	0.003	0.003	0.005	0.011
30 years	0.6217	0.0236	0.009	0.015	0.011	0.005	0.003	0.003	0.003	0.003	0.003	0.005	0.010

The new each draw (NED) and fixed sample (FS) results summarize 1000 replications of 100,000 simulated CC returns for each horizon. In the NED simulations, every monthly return used is a new drawing from a normal distribution with the same mean and standard deviation as actual monthly returns of July 1963 to December 2016 (1963–2016). Each FS replication constructs 100,000 CC returns for each horizon by drawing monthly returns with replacement from a fixed sample of 642 normal random variables with the same mean and standard deviation as the monthly returns of 1963–2016. Panel A summarizes averages, across the 1,000 replications, of parameters of the distibutions of the 100,000 returns for each horizon. The parameters are the *Mean*, standard deviation (*SD*), kurtosis (*Kurt*), and skewness (*Skew*) of the 100,000 simulated returns for a horizon, and percentiles of the 100,000 standardized returns. For comparison, unit normal percentiles are in the first line of each block of panel A. *Skew* is the third moment about the *Mean* divided by SD^3. *Kurt* is the fourth moment about the *Mean* divided by SD^4. Panel B shows standard deviations of the 1,000 replication values of the parameters.

average values from the NED and FS simulations are close to the average returns and standard deviations from the bootstrap simulations that use actual returns, presented later. Panel A of Table 1 also shows averages across 1,000 replications of the skewness (*Skew*) and kurtosis (*Kurt*) of each replication's CC returns. The average values of *Skew* and *Kurt* are almost all 0.000 and 3.000, their values when return distributions are normal. Finally, panel A of Table 1 shows average values across replications of the standardized percentiles of a replication. Define $Mean_T$, SD_T, and C_{Ti} as the sample average, standard deviation, and ith percentile of the 100,000 T-month CC returns of a replication. The standardized value of C_{Ti} is

$$S_{Ti} = (C_{Ti} – Mean_T)\ /SD_T. \tag{3}$$

The average values of the standardized percentiles from the simulations are never more than 0.001 from the corresponding percentiles of the unit normal distribution. In short, panel A of Table 1 just confirms that T-month returns in the FS and NED simulations are built from normally distributed monthly returns with the same means and standard deviations as the actual monthly returns.

A major purpose of the FS and NED simulations, however, is to gain perspective on the differences between repeated sampling from a fixed sample of 642 monthly returns (the FS simulations) and IID sampling from an unlimited population of monthly returns (the NED simulations). Relevant results are in panel B of Table 1, which shows standard deviations across 1,000 replications of the parameters of FS and NED bootstrap distributions of T-month CC returns.

The general result in panel B of Table 1 is that parameter estimates vary more from replication to replication in FS simulations. This is especially true for means of T-month returns. The standard deviation of the average 1-month returns from the 1,000 FS simulations, 0.0017, is 17 times the standard deviation of the average 1-month returns in the NED simulations, 0.0001. The standard deviation of the 30-year average returns, 0.6217, is 230 times the standard deviation from the NED simulations, 0.0027.

Standard deviations of average CC returns are much larger and grow much faster as the return horizon increases in the FS simulations than in the NED simulations. In the NED simulations, every T-month CC return is the sum of T IID normal monthly CC returns and the average T-month CC return for a replication is T times the average of 100,000*T IID normal monthly CC returns. If σ_1 is the standard deviation of monthly returns, the standard deviation of the average of the 100,000 independent T-month returns in an NED replication is $\sigma_1\sqrt{T/100,000}$.

In contrast, when we sample with replacement from the same 642 normal monthly returns to produce 100,000 returns for each horizon in an FS replication, the average of the 100,000 T-month returns must be almost exactly T times the average of the 642 monthly returns. But the average of each replication's 642 monthly returns differs randomly from the expected value of the monthly draws, the average value of the actual monthly *CC* Market returns. The standard deviation of this random variation is $\sigma_1/\sqrt{642}$, and the standard deviation of an FS replication's average T-month CC return is $\sigma_1 T/\sqrt{642}$. Since the NED T-month standard deviation is $\sigma_1\sqrt{T/100,000}$, the FS value is bigger at 1 month and grows more quickly as T increases.

The differences between the FS and NED standard deviations arise because effective sample sizes are bigger in the NED simulations. Consider 1-month returns. The average monthly return in each NED replication summarizes 100,000 distinct monthly draws, so the cross-replication standard deviation of the NED averages is $\sigma_1/\sqrt{100,000}$. The average in an FS replication also summarizes 100,000 monthly returns, but these are all drawn with replacement from a relatively small set of 642 independent monthly returns. Thus, the effective sample size is at best 642 distinct returns, and the cross-replication standard deviation of the FS averages must be at least $\sigma_1/\sqrt{642}$. (This is a lower bound for the standard deviation because an equal weight average of the 642 months is the most efficient estimate of the true mean, but some months will appear more frequently than others in each replication's sample of 100,000.) For longer return horizons, the difference between the FS and NED standard deviations of average 1-month returns is magnified by the fact that the FS standard deviation grows like T and the NED standard deviation grows like $\sqrt{T}$.

Our primary concern is whether the higher sampling variation of parameters from FS simulations clouds inferences about convergence to normal of distributions of actual CC returns for longer horizons. The FS simulations suggest that this is not the case.

Inferences about convergence to normal center on *Skew*, *Kurt*, and the percentiles of standardized T-month CC returns. In the NED simulations, all CC returns are sums of independent draws of normal monthly returns, so each replication's 100,000 CC returns for a horizon are also IID normal variables. As a result, the standard error of each replication's estimated *Skew* is $\sqrt{6/100,000} = 0.0077$, and the standard deviation of *Kurt* is $\sqrt{24/100,000} = 0.0155$ (Kendall and Stuart 1977, p. 258). The tiny cross-replication standard deviations of *Skew*, 0.007 and 0.008, and *Kurt*, 0.015 and 0.016, for the NED simulations (panel B of Table 1) bracket these standard errors. Most standard deviations of S_{Ti} in panel B are even smaller, especially in the middle of the distribution, and like *Skew* and *Kurt*, they vary little across horizons.

In the FS simulations, the same 642 monthly returns are used to construct the 100,000 returns of every horizon of a replication. As a result, cross-replication variation in the parameters is bigger in the FS simulations (panel B of Table 1). The standard deviations of *Skew* and *Kurt* for the fixed samples of 642 monthly returns of the 1,000 FS replications are 0.098 and 0.198, respectively, versus 0.008 and 0.015 in the NED simulations. On a purely random basis, the fixed samples of monthly returns in the FS simulations are skewed left or right, and they are more or less kurtotic than the normal distribution from which they are drawn. Similarly, standard deviations across 1,000 replications of standardized percentiles of monthly returns are much larger in the FS simulations, for example, 0.126 for the 99th percentile in the FS simulations versus 0.010 in the NED simulations.

The good news is that for longer return horizons, the standard deviations of *Skew*, *Kurt*, and the standardized percentiles in the FS simulations shrink toward the tiny standard deviations of the NED simulations. This is the central limit theorem at work. The random deviations from normal in the fixed samples of 642 monthly returns become less important as we combine random draws into longer horizon returns. As a result, return distributions approach normal for longer horizons. Panel B of Table 1 shows that convergence toward normal of each FS replication's longer horizon returns results in cross-replication standard deviations of *Skew*, *Kurt*, and standardized percentiles that approach the tiny standard deviations of the NED simulations.

The simulations that use actual monthly returns, examined next, are like one replication of the 1,000 FS BB simulations. We lean on the cross-section of FS BB replications for inferences about convergence to normal of distributions of T-month CC returns. Specifically, we use the standard deviations of *Skew*, *Kurt*, and standardized percentiles in panel B of Table 1 to judge when the estimates from the simulations that use actual returns signal statistically reliable deviations from the values predicted when return distributions are normal.

3. Bootstrap Simulations of Long-Horizon CC Returns

Table 2 summarizes the distribution of actual monthly CC Market returns for 1963–2016 and distributions of bootstrapped CC Market returns for longer horizons. These simulations are like one replication of the FS BB simulations. Each of the 100,000 bootstrapped CC returns for horizon T, C_T, is the sum of T independent and identically distributed draws (with replacement) from the 642 monthly CC returns of 1963–2016. The central limit theorem says bootstrapped C_T converge toward normal distributions as T increases. Convergence is apparent in the *Skew* and *Kurt* estimates in panel A of Table 2. Monthly CC Market returns for 1963–2016 are skewed left (*Skew* $= -0.768$) and leptokurtic (*Kurt* $= 5.814$ vs. 3.0 for the normal distribution). *Skew* shrinks to -0.222 at 1 year, -0.057 at 10, and -0.041 at 30. Similarly, *Kurt* shrinks to 3.270 at 1 year, 3.022 at 10, and 2.999 at 30.

Table 2

Actual and bootstrapped continuously compounded returns, 1963–2016

A. Summary statistics for 100,000 bootstrap trials using actual returns

	Moments of CC returns				$S_{Ti} = (C_{Ti} - Mean_T) / SD_T$								
	Mean	*SD*	*Skew*	*Kurt*	1%	5%	10%	25%	50%	75%	90%	95%	99%
Unit normal					−2.326	−1.645	−1.282	−0.674	0.000	0.674	1.282	1.645	2.326
1 month	0.0080	0.0444	−0.768	5.814	−2.801	−1.707	−1.217	−0.571	0.093	0.671	1.112	1.438	2.250
1 year	0.0951	0.1544	−0.222	3.270	−2.525	−1.703	−1.285	−0.638	0.032	0.678	1.246	1.574	2.223
3 years	0.2888	0.2667	−0.138	3.093	−2.456	−1.683	−1.292	−0.657	0.023	0.679	1.261	1.615	2.234
5 years	0.4812	0.3441	−0.107	3.056	−2.418	−1.673	−1.288	−0.660	0.021	0.680	1.268	1.613	2.262
10 years	0.9628	0.4865	−0.057	3.022	−2.388	−1.658	−1.287	−0.668	0.011	0.678	1.273	1.622	2.300
20 years	1.9284	0.6885	−0.041	3.026	−2.367	−1.659	−1.282	−0.669	0.008	0.676	1.274	1.628	2.295
30 years	2.8873	0.8434	−0.041	2.999	−2.353	−1.654	−1.289	−0.670	0.006	0.677	1.277	1.631	2.296

B. Normal cumulative distribution function for values of S_{Ti}

	CDF(S_{Ti})								
	1%	5%	10%	25%	50%	75%	90%	95%	99%
1 month	0.25	4.39	11.18	28.39	53.69	74.88	86.70	92.48	98.78
1 year	0.58	4.43	9.94	26.18	51.28	75.10	89.36	94.23	98.69
3 years	0.70	4.62	9.82	25.57	50.92	75.14	89.64	94.68	98.73
5 years	0.78	4.71	9.89	25.46	50.84	75.17	89.76	94.66	98.82
10 years	0.85	4.86	9.90	25.20	50.46	75.12	89.85	94.76	98.93
20 years	0.90	4.85	9.99	25.18	50.34	75.03	89.86	94.82	98.91
30 years	0.93	4.91	9.86	25.15	50.26	75.09	89.92	94.86	98.92

The 1-month results summarize monthly continuously compounded (CC) returns for 1963–2016. The results for 1 to 30 years summarize 100,000 bootstrapped CC returns, obtained by sampling with replacement from the 642 monthly returns of 1963–2016. In panel A, *Mean* is the average; *SD* is the standard deviation; *Skew* is the skew; *Kurt* is the kurtosis of the actual or bootstrapped returns for each horizon; and S_{Ti} are the standardized percentiles of the CC returns for each horizon, along with (in the first line of the panel) the corresponding unit normal percentiles. Panel B maps the standardized percentiles in panel A to their location in the normal cumulative distribution function, CDF(S_{Ti}).

Skipping the details, since each long-horizon return is the sum of T IID draws from the population of 642 monthly CC returns, the expected value of *Skew* shrinks toward 0.0 at the rate $1/\sqrt{T}$ as the horizon T increases:

$$\mathrm{E}(Skew_T) = Skew_1/\sqrt{T}\ . \tag{4}$$

The expected value of *Kurt* converges to 3.0 more quickly, at the rate $1/T$:

$$\mathrm{E}(Kurt_T - \ 3.0) = (Kurt_1 - \ 3.0)\ /T. \tag{5}$$

The rates of decay for $Skew_T$ and $Kurt_T$ in Table 2 deviate slightly from these expectations because of sampling error.

The cross-replication standard deviations of $Skew_T$ and $Kurt_T$ from the FS BB simulations in panel B of Table 1 also decay at roughly the expected rates for $Skew_T$ and $Kurt_T$ in Equations (4) and (5). This happens because a replication's $Skew_T$ at longer horizons is $Skew_1/\sqrt{T}$ plus sampling error and its excess kurtosis is $(Kurt_1 - 3)/T$ plus sampling error. This means the dispersion of $Skew_T$ across FS replications shrinks toward the dispersion of *Skew* in the NED BB simulations at roughly the rate $1/\sqrt{T}$ as T increases, and the dispersion of $Kurt_T$ shrinks toward the dispersion of *Kurt* in the NED BB simulations at roughly the rate 1/T. The bottom line is that the standard deviations of *Skew* and *Kurt* in the FS BB simulations (panel B of Table 1) are tiny for longer horizons, approaching those of the NED BB simulations.

Using the standard deviations from the FS simulations in panel B of Table 1, all estimates of *Skew* in the simulations that use actual returns (panel A of Table 2) are more than four standard deviations from 0.0, even for 20- and 30-year returns, where *Skew* (-0.041) seems trivial. For *Kurt*, convergence to 3.0 is more complete, and the 10-, 20-, and 30-year estimates, 3.022, 3.026, and 2.999, are less than two FS BB standard deviations from 3.0. *Kurt* for the 100,000 five-year bootstrapped CC returns, 3.056, is close to 3.0, but its tiny standard deviation, 0.016, leaves it 3.5 standard deviations from 3.0. In short, *Skew* and *Kurt* suggest that Market returns for horizons of 10 years or more are close to normal with a bit of residual left skew.

The percentiles of standardized CC Market returns in panel A of Table 2 provide additional perspective on this inference. The left skew of actual monthly returns is apparent. The 1st, 5th, 75th, 90th, 95th, and 99th percentiles of standardized monthly returns are less than their unit normal counterparts (shown in the first line of panel A of Table 2), and the 10th, 25th, and 50th percentiles are above the corresponding unit normal percentiles. As the return horizon increases, the percentiles of simulated returns built from actual returns move toward their unit normal counterparts (panel A of Table 2), but again the standard deviations of the percentiles shrink to tiny values (panel B of Table 1). As a result, there is a small but statistically reliable residual left skew even in bootstrapped 30-year returns.

Table 3
Actual and bootstrapped payoffs, 1963–2016

						Percentiles								
	Mean	*SD*	*Skew*	*Kurt*	*Neg*	1%	5%	10%	25%	50%	75%	90%	95%	99%
1 month	1.01	0.04	−0.50	4.95	38.5	0.89	0.93	0.96	0.98	1.01	1.04	1.06	1.08	1.11
1 year	1.11	0.17	0.28	3.28	25.7	0.75	0.85	0.90	1.00	1.11	1.22	1.33	1.40	1.55
3 years	1.38	0.37	0.69	3.82	13.9	0.69	0.85	0.95	1.12	1.34	1.60	1.87	2.05	2.42
5 years	1.72	0.60	0.98	4.79	8.3	0.70	0.91	1.04	1.29	1.63	2.04	2.50	2.82	3.52
10 years	2.95	1.50	1.61	8.08	2.6	0.82	1.17	1.40	1.89	2.63	3.64	4.87	5.77	8.02
20 years	8.70	6.69	2.76	19.56	0.3	1.35	2.20	2.85	4.34	6.92	10.95	16.53	21.10	33.39
30 years	25.51	25.40	3.88	37.59	0.1	2.47	4.45	6.05	10.20	18.04	31.77	52.67	71.03	124.42

The 1-month results describe monthly payoffs (1 + simple monthly returns) for 1963–2016. The results for 1 to 30 years summarize 100,000 bootstrapped payoffs, obtained by sampling with replacement from the 642 monthly returns of 1963–2016. *Mean* is the average; *SD* is the standard deviation; *Skew* is the skew; and *Kurt* is the kurtosis of the actual or bootstrapped payoffs for each horizon. *Neg* is the percentage of cumulative returns that are negative. The table also shows payoff percentiles for each horizon.

A final perspective on convergence to normal of bootstrap CC Market returns is in panel B of Table 2. It shows where the percentiles of standardized returns in panel A sit in the unit normal distribution. For example, the standardized fifth percentile of actual monthly returns, −1.707, is the 4.39 percentile of the unit normal, and so to the left of the fifth unit normal percentile, −1.645. As the return horizon increases, the lines in panel B move toward the unit normal percentiles in the panel's header, but though close at longer horizons, they never quite get there. For example, the first percentile of standardized actual monthly returns is the 0.25 percentile of the unit normal, rising to the 0.93 percentile of the unit normal for 30-year returns, but never quite reaching the first percentile. The median standardized monthly return is the 53.69 percentile of the unit normal. The median for standardized 30-year bootstrap returns is the 50.26 unit normal percentile, which is close to but still above the 50th unit normal percentile.

4. Bootstrap Simulations of Long-Horizon Payoffs

Investors consume dollar payoffs, not CC returns. Table 3 summarizes actual and bootstrap distributions of compounded Market returns for 1963–2016. For each return horizon, the table shows the average, standard deviation, *Skew*, and *Kurt* of the payoffs; the percentage of negative returns (*Neg*); and percentiles of the payoffs. The monthly row summarizes the distribution of actual monthly payoffs. Each remaining row describes the distribution of 100,000 simulated payoffs for a longer horizon.

Table 3 offers few surprises. The average payoff and uncertainty about the payoff grow as the horizon increases. *Skew* also increases, from -0.50 for 1963–2016 monthly payoffs and 0.28 for simulated annual payoffs to 3.88 for 30-year payoffs. Despite the greater dispersion in payoffs for longer horizons, the increases in the average payoff and *Skew* combine to pull the left tail rightward as the horizon increases. The frequency of negative compounded returns, for example, declines from 38.5% for monthly payoffs of 1963–2016 to 0.1% for bootstrapped 30-year payoffs. *Kurt* is above 3.0 for all horizons; it declines from 1 month to 1 year and then increases for longer return horizons. *Kurt* is 4.95 for monthly payoffs of 1963–2016, 3.28 for simulated annual payoffs, and 37.59 for 30-year payoffs. *Skew* and *Kurt* are even more extreme for long-horizon payoffs of 1926–2016 (Appendix Table A3). In short, distributions of payoffs for longer horizons are increasingly leptokurtic relative to normal distributions, but increasing right skew means outliers are primarily in the right tail.

Does the distribution of Market payoffs approach lognormal as the horizon increases? Yes. If C_T approaches normal as T increases, $1 + R_T$ must approach lognormal. The relation linking the payoff and the CC return for horizon T is monotonic, $1 + R_T = \exp(C_T)$. As a result, the ith percentile of a horizon's CC return, C_{Ti}, must be the ith percentile of the horizon's payoffs, R_{Ti}. If C_{Ti} is the jth percentile of the normal distribution with the same mean and standard deviation as C_T, then R_{Ti} is the jth percentile of the corresponding lognormal distribution. Thus, panel B of Table 2 can be interpreted as mapping the percentiles of continuously compounded returns to normal percentiles or as mapping the percentiles of payoffs to lognormal percentiles. This is the sense in which convergence or lack of convergence of C_T to normal is the same as convergence or lack of convergence of R_T to lognormal.

5. Uncertainty about the Expected Return

Pastor and Stambaugh (2012) emphasize that the return distribution for a long-horizon investment has two sources of dispersion: the unexpected return, which creates the dispersion in the bootstrap distributions above, and imprecision in the estimate of the expected return. Dispersion caused by the unexpected return is roughly proportional to the square root of the investment horizon. In contrast, if the true expected monthly return and the estimate of the expected monthly return are constant over the next T months, the error in the forecast of the cumulative return R_T due to uncertainty about the expected return is roughly proportional to T. (The stated relations between horizon and dispersion are not exact because a long-horizon return is a nonlinear function of monthly returns.) Uncertainty about the expected return has relatively little impact on the payoff from a 1-month investment in the Market portfolio; the error in any reasonable estimate of the expected monthly return is tiny relative to the standard deviation of the unexpected return. We modify the bootstrap simulations to explore how uncertainty about the expected return affects assessments of payoff distributions for longer horizons. Our results reinforce the conclusions of Pastor and Stambaugh (2012) from their more analytical approach.

We take the perspective of an investor forecasting long-horizon payoffs at the end of 2016 who assumes past and future monthly returns are generated by the same process. Thus, the standard error of the average monthly Market

Table 4
Bootstrapped payoffs, incorporating uncertainty about the expected return, 1963–2016

						Percentiles								
	Mean	*SD*	*Skew*	*Kurt*	*Neg*	1%	5%	10%	25%	50%	75%	90%	95%	99%
1 month	1.01	0.04	−0.54	5.04	38.6	0.89	0.93	0.96	0.98	1.01	1.04	1.06	1.07	1.11
1 year	1.11	0.17	0.29	3.28	25.9	0.74	0.84	0.90	1.00	1.11	1.22	1.34	1.41	1.56
3 years	1.39	0.38	0.72	3.89	14.6	0.69	0.84	0.94	1.12	1.34	1.61	1.89	2.08	2.47
5 years	1.72	0.63	1.04	4.99	9.4	0.68	0.89	1.01	1.28	1.63	2.07	2.56	2.89	3.65
10 years	3.01	1.69	1.76	8.53	3.7	0.74	1.09	1.32	1.84	2.63	3.75	5.15	6.21	8.87
20 years	9.50	8.98	3.55	30.42	0.9	1.03	1.80	2.43	3.99	6.89	11.86	19.33	25.75	44.21
30 years	31.14	43.80	6.54	104.39	0.3	1.51	3.12	4.63	8.81	18.01	36.59	68.65	101.29	207.78

For each of a horizon's 100,000 simulation runs, we draw a normally distributed random error with mean zero and standard deviation equal to the standard error of the average monthly Market return for 1963–2016, 0.17%. We then add the random error to each of the simulation run's monthly returns and compute the payoff by compounding the adjusted monthly returns. *Mean* is the average; *SD* is the standard deviation; *Skew* is the skewness; and *Kurt* is the kurtosis of the bootstrapped payoffs for each horizon. *Neg* is the percentage of cumulative returns that are negative. The table also shows percentiles of the payoffs (1 + simple cumulative returns) for each horizon.

return for 1963–2016, 0.17%, measures uncertainty about expected monthly returns. For each of a horizon's 100,000 simulation runs, we draw a normally distributed random error with mean zero and standard deviation equal to 0.17%. We add the random error to each of the simulation run's monthly returns and compound the adjusted monthly returns. Table 4 provides the resulting bootstrap distributions.

Uncertainty about the expected Market return has little effect on distributions of short-horizon payoffs. The random error increases the standard deviation of bootstrapped three-year returns for 1963–2016 by only about 3%, from 0.37 (Table 3) to 0.38 (Table 4). Uncertainty about the mean increases the standard deviation by 13% for 10-year returns, 34% for 20-year returns, and 72% for 30-year returns. Skew and kurtosis also increase at longer horizons. At 20 years, for example, *Skew* is 2.76 and *Kurt* is 19.56 if we ignore imprecision in the expected return estimate (Table 3) versus 3.55 and 30.42 if we include the imprecision (Table 4). The importance of this source of return uncertainty for long-horizon investors is also apparent in the percentiles of return distributions. Uncertainty about the mean lowers the 10th percentile of 20-year payoffs from 285% (Table 3) to 243% (Table 4) and raises the 90th percentile from 1,653% to 1,933%. The results (in the appendix) are similar if we use the monthly returns for 1926–2016 and the standard error of the 1926–2016 average return (0.16%) in the simulations. In short, imprecision in the estimated expected return is a minor source of uncertainty for short-horizon payoffs, but it is progressively more important in long-horizon payoffs.

6. Conclusions

We use bootstrap simulations to explore the properties of long-horizon Market returns. The central limit theorem suggests that sums of continuously compounded monthly returns converge to normal distributions as we extend the return horizon. This implies that distributions of investment payoffs converge to lognormal. By construction, the monthly CC returns in the bootstrap samples are independent, they are identically distributed, and they have finite variance, so bootstrapped CC long-horizon returns must converge toward normal distributions. Deviations from normal are large for short horizons, but they quickly shrink as the return horizon increases. For 10-, 20-, and 30-year horizons, kurtosis (*Kurt*) is statistically indistinguishable from 3.0, its value for normal distributions. The left skew of monthly CC returns ($Skew = -0.768$) also tends to disappear for longer return horizons. But like all the parameters of distributions of standardized CC returns, the precision of estimates of *Skew* increases for longer returns horizons, and the tiny remaining left skew in 20- and 30-year CC returns ($Skew = -0.041$) is more than four standard deviations from zero. In short, distributions of CC returns are close to, but not quite normal for return horizons of 10 years or more.

Investment payoffs are monotonic transformations of CC returns ($1+ R_T = exp(C_T)$), and percentiles of C_T correspond to the same percentiles of R_T; the observation at the fifth percentile of the C_T distribution is also at the fifth percentile of the $1 + R_T$ distribution. And if the standardized value of that observation's CC return is the fourth percentile of the unit normal distribution, its standardized payoff is the fourth percentile of the lognormal distribution produced by a unit normal. In this sense, convergence of C_T toward normal is equivalent to convergence of R_T toward lognormal. To the extent that convergence of C_T is incomplete even for horizons out to 30 years, convergence of R_T is also incomplete.

Deviations of long-horizon returns from lognormal can have important implications for investors. Investment professionals often use simulations to estimate the distributions of their clients' distant payoffs. One common procedure simulates long-horizon returns by drawing random monthly or annual CC returns from a normal distribution with constant mean and variance. Our evidence that distributions of even 30-year payoffs do not completely converge to lognormal suggests that bootstrap simulations from actual monthly returns provide better descriptions of potential investment outcomes, especially in the extreme tails. Of course, the importance of this issue can only be decided in the context of how deviations from lognormal affect expected utility.

Like Pastor and Stambaugh (2012), we find that uncertainty about the expected monthly return can have a large impact on uncertainty about long-horizon payoffs. Noise in the estimate of a monthly or annual expected return is dwarfed by uncertainty about the unexpected return, but imprecision in the estimate of the expected return has a big impact on the dispersion of possible payoffs from a 20-year investment. Investment implications are obvious: investors can improve assessments of distributions of distant payoffs by including uncertainty about expected returns in simulations.

Appendix

Table A1
Simulations of bootstrap simulations, 1926–2016

A. Average across 1,000 replications

	Moments of CC returns				$S_{Ti} = (C_{Ti} - Mean_T) \mid SD_T$								
	Mean	*SD*	*Skew*	*Kurt*	1%	5%	10%	25%	50%	75%	90%	95%	99%
Unit normal					−2.326	−1.645	−1.282	−0.674	0.000	0.674	1.282	1.645	2.326
NED: New normal monthly return for each draw													
1 month	0.0078	0.0535	−0.000	3.000	−2.326	−1.645	−1.282	−0.674	0.000	0.675	1.281	1.645	2.326
1 year	0.0941	0.1855	−0.000	3.000	−2.327	−1.645	−1.282	−0.675	0.000	0.675	1.282	1.645	2.326
3 years	0.2822	0.3213	−0.000	3.000	−2.326	−1.645	−1.281	−0.675	−0.000	0.674	1.282	1.645	2.326
5 years	0.4704	0.4148	0.000	2.999	−2.326	−1.645	−1.282	−0.675	−0.000	0.674	1.282	1.645	2.326
10 years	0.9407	0.5866	0.000	3.000	−2.326	−1.645	−1.282	−0.674	0.000	0.674	1.281	1.645	2.326
20 years	1.8813	0.8295	0.000	3.000	−2.326	−1.645	−1.282	−0.674	−0.000	0.675	1.281	1.645	2.327
30 years	2.8221	1.0158	−0.000	3.000	−2.326	−1.645	−1.281	−0.674	0.000	0.675	1.281	1.645	2.327
FS: Sampling from a fixed sample of 642 normal CC returns for all trials and horizons in a replication													
1 month	0.0078	0.0536	−0.005	2.993	−2.328	−1.649	−1.284	−0.674	0.001	0.675	1.282	1.647	2.321
1 year	0.0931	0.1857	−0.002	3.000	−2.327	−1.645	−1.282	−0.674	0.000	0.675	1.282	1.644	2.325
3 years	0.2793	0.3216	−0.001	3.000	−2.327	−1.645	−1.282	−0.674	0.000	0.675	1.281	1.644	2.326
5 years	0.4655	0.4151	−0.000	3.002	−2.327	−1.645	−1.282	−0.674	0.000	0.674	1.281	1.645	2.326
10 years	0.9312	0.5871	−0.000	3.000	−2.327	−1.645	−1.282	−0.674	0.000	0.674	1.281	1.645	2.326
20 years	1.8622	0.8303	−0.000	3.000	−2.327	−1.645	−1.282	−0.675	0.000	0.674	1.282	1.645	2.326
30 years	2.7933	1.0169	0.000	2.999	−2.327	−1.645	−1.281	−0.675	−0.000	0.675	1.282	1.645	2.326

B. Standard deviations across 1,000 replications

	Moments of CC returns				$S_{Ti} = (C_{Ti} - Mean_T) \ / \ SD_T$								
	Mean	*SD*	*Skew*	*Kurt*	1%	5%	10%	25%	50%	75%	90%	95%	99%
NED: New normal monthly return for each draw													
1 month	0.0002	0.0001	0.008	0.015	0.010	0.005	0.003	0.002	0.002	0.002	0.003	0.005	0.01
1 year	0.0006	0.0004	0.007	0.016	0.010	0.004	0.003	0.003	0.002	0.002	0.003	0.005	0.01
3 years	0.0010	0.0007	0.008	0.015	0.010	0.005	0.003	0.002	0.002	0.003	0.003	0.004	0.01
5 years	0.0013	0.0009	0.008	0.016	0.010	0.005	0.003	0.003	0.002	0.003	0.003	0.004	0.01
10 years	0.0019	0.0013	0.008	0.016	0.010	0.004	0.003	0.002	0.002	0.002	0.003	0.005	0.01
20 years	0.0026	0.0018	0.007	0.016	0.010	0.005	0.003	0.003	0.002	0.003	0.003	0.005	0.01
30 years	0.0033	0.0022	0.008	0.015	0.010	0.005	0.003	0.002	0.002	0.002	0.003	0.005	0.01
FS: Sampling from a fixed sample of 1,086 normal CC returns for all trials and horizons in a replication													
1 month	0.0016	0.0011	0.076	0.152	0.098	0.045	0.032	0.025	0.023	0.025	0.032	0.043	0.098
1 year	0.0193	0.0039	0.023	0.020	0.019	0.008	0.004	0.003	0.004	0.003	0.004	0.008	0.019
3 years	0.0580	0.0069	0.015	0.017	0.013	0.006	0.004	0.003	0.003	0.003	0.004	0.006	0.014
5 years	0.0967	0.0088	0.012	0.016	0.012	0.005	0.003	0.003	0.003	0.003	0.003	0.005	0.013
10 years	0.1935	0.0124	0.010	0.016	0.012	0.005	0.003	0.002	0.003	0.003	0.003	0.005	0.011
20 years	0.3868	0.0176	0.009	0.015	0.011	0.005	0.003	0.003	0.003	0.003	0.003	0.005	0.011
30 years	0.5802	0.0215	0.009	0.015	0.011	0.005	0.003	0.003	0.002	0.003	0.003	0.005	0.011

The new each draw (NED) and fixed sample (FS) results summarize 1,000 replications of 100,000 simulated CC returns for each horizon. In the NED simulations, every monthly return used is a new drawing from a normal distribution with the same mean and standard deviation as actual monthly returns of July 1926 to December 2016 (1926–2016). Each FS replication constructs 100,000 CC returns for each horizon by drawing monthly returns with replacement from a fixed sample of 1,086 normal random variables with the same mean and standard deviation as the monthly returns of 1926–2016. Panel A summarizes averages, across the 1,000 replications, of parameters of the distibutions of the 100,000 returns for each horizon. The parameters are the *Mean* and standard deviation (*SD*) of the 100,000 simulated returns for a horizon, skewness and kurtosis parameters (*Skew* and *Kurt*) for the 100,000 returns, and percentiles of the 100,000 standardized returns for comparison with the unit normal percentiles in the first line of each block of panel A. *Skew* is the third moment about the *Mean* divided by SD^3. *Kurt* is the fourth moment about the *Mean* divided by SD^4. Panel B shows standard deviations of the 1,000 replication values of the parameters.

Table A2

Actual and bootstrapped continuously compounded returns, 1926–2016

A. Summary statistics for 100,000 bootstrap trials using actual returns

	Moments of CC returns				$S_{Ti} = (C_{Ti} - Mean_T) / SD_T$								
	Mean	*SD*	*Skew*	*Kurt*	1%	5%	10%	25%	50%	75%	90%	95%	99%
Unit normal					−2.326	−1.645	−1.282	−0.674	0.000	0.674	1.282	1.645	2.326
1 month	0.0078	0.0535	−0.549	9.844	−3.108	−1.606	−1.113	−0.482	0.088	0.581	0.977	1.241	2.143
1 year	0.0945	0.1848	−0.173	3.587	−2.551	−1.687	−1.268	−0.625	0.033	0.656	1.212	1.568	2.343
3 years	0.2823	0.3219	−0.089	3.205	−2.437	−1.669	−1.275	−0.655	0.015	0.671	1.251	1.604	2.323
5 years	0.4705	0.4138	−0.064	3.102	−2.398	−1.662	−1.281	−0.665	0.010	0.676	1.267	1.622	2.301
10 years	0.9424	0.5842	−0.034	3.052	−2.362	−1.655	−1.287	−0.664	0.002	0.673	1.276	1.633	2.310
20 years	1.8807	0.8253	−0.026	3.021	−2.341	−1.653	−1.281	−0.675	0.005	0.674	1.275	1.632	2.312
30 years	2.8209	1.0146	−0.044	3.003	−2.369	−1.654	−1.288	−0.671	0.008	0.679	1.274	1.633	2.305

B. Normal cumulative distribution function for values of S_{Ti}

		CDF(S_{Ti})							
	1%	5%	10%	25%	50%	75%	90%	95%	99%
1 month	0.09	5.41	13.28	31.50	53.52	71.93	83.56	89.28	98.40
1 year	0.54	4.58	10.23	26.58	51.31	74.39	88.73	94.16	99.04
3 years	0.74	4.76	10.12	25.61	50.62	74.89	89.46	94.56	98.99
5 years	0.83	4.83	10.02	25.32	50.38	75.04	89.74	94.76	98.93
10 years	0.91	4.90	9.90	25.34	50.08	74.94	89.91	94.88	98.96
20 years	0.96	4.92	10.01	24.98	50.21	74.99	89.89	94.87	98.96
30 years	0.89	4.90	9.89	25.11	50.34	75.15	89.87	94.88	98.94

The 1-month results summarize monthly continuously compounded (CC) returns for 1926–2016. The results for 1 to 30 years summarize 100,000 bootstrapped CC returns, obtained by sampling with replacement from the 1,086 monthly returns of 1926–2016. In panel A *Mean* is the average; *SD* is the standard deviation; *Skew* is the skew; *Kurt* is the kurtosis of the actual or bootstrapped returns for each horizon; and S_{Ti} are the standardized percentiles of the CC returns for each horizon, along with (in the first line of the panel) the corresponding unit normal percentiles. Panel B maps the standardized percentiles in panel A to their location in the normal cumulative distribution function, CDF(S_{Ti}).

Table A3
Actual and bootstrapped payoffs, 1926–2016

						Percentiles								
	Mean	*SD*	*Skew*	*Kurt*	*Neg*	1%	5%	10%	25%	50%	75%	90%	95%	99%
1 month	1.01	0.05	0.16	10.76	37.7	0.85	0.92	0.95	0.98	1.01	1.04	1.06	1.08	1.13
1 year	1.12	0.21	0.55	4.29	28.7	0.69	0.81	0.87	0.98	1.11	1.24	1.38	1.47	1.70
3 years	1.40	0.46	1.03	5.28	18.6	0.61	0.78	0.88	1.07	1.33	1.65	1.98	2.22	2.80
5 years	1.74	0.75	1.36	6.80	12.7	0.59	0.81	0.94	1.22	1.61	2.12	2.70	3.13	4.15
10 years	3.04	1.93	2.12	11.46	5.5	0.65	0.98	1.21	1.74	2.57	3.80	5.41	6.66	9.89
20 years	9.20	8.97	3.63	28.91	1.2	0.95	1.68	2.28	3.76	6.59	11.44	18.78	25.22	44.20
30 years	27.88	35.89	5.10	56.10	0.3	1.52	3.13	4.55	8.50	16.94	33.45	61.19	88.08	174.11

The 1-month results describe monthly payoffs (1 + simple monthly returns) for 1926–2016. The results for 1 to 30 years summarize 100,000 bootstrapped payoffs, obtained by sampling with replacement from the 1,086 monthly returns of 1926–2016. *Mean* is the average; *SD* is the standard deviation; *Skew* is the skew; and *Kurt* is the kurtosis of the actual or bootstrapped payoffs for each horizon. *Neg* is the percentage of cumulative returns that are negative. The table also shows payoff percentiles for each horizon.

Table A4
Bootstrapped payoffs, incorporating uncertainty about the expected return, 1926–2016

						Percentiles								
	Mean	*SD*	*Skew*	*Kurt*	*Neg*	1%	5%	10%	25%	50%	75%	90%	95%	99%
1 month	1.01	0.05	0.20	10.62	37.9	0.86	0.92	0.95	0.98	1.01	1.04	1.06	1.08	1.13
1 year	1.12	0.21	0.55	4.29	28.9	0.69	0.80	0.87	0.98	1.11	1.24	1.38	1.47	1.70
3 years	1.40	0.47	1.06	5.40	19.1	0.60	0.77	0.87	1.07	1.33	1.65	2.00	2.24	2.84
5 years	1.75	0.77	1.42	7.11	13.3	0.58	0.79	0.93	1.21	1.61	2.13	2.74	3.20	4.25
10 years	3.10	2.08	2.21	11.87	6.4	0.60	0.93	1.16	1.70	2.57	3.89	5.65	7.07	10.68
20 years	9.90	11.02	4.14	35.46	2.0	0.78	1.45	2.03	3.55	6.58	12.15	21.13	29.18	53.99
30 years	32.78	52.45	7.00	114.15	0.9	1.07	2.42	3.69	7.65	16.94	37.00	74.76	112.84	246.17

For each of a horizon's 100,000 simulation runs, we draw a normally distributed random error with mean zero and standard deviation equal to the standard error of the average monthly Market return for 1926–2016, 0.16%. We then add the random error to each of the simulation run's monthly returns and compute the payoff by compounding the adjusted monthly returns. *Mean* is the average; *SD* is the standard deviation; *Skew* is the skewness; and *Kurt* is the kurtosis of the bootstrapped payoffs for each horizon. *Neg* is the percentage of cumulative returns that are negative. The table also shows percentiles of the payoffs (1 + simple cumulative returns) for each horizon.

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