Gradient Descent

LING 282/482: Deep Learning for Computational Linguistics

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Supervised Learning Basics

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- Goal: learn the function that **best matches the dataset**

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  - Solution: learn the weights of a **parameterized function**

Parameterized Functions

Parameterized Functions

- A learning searches for a function f in a space of **possible functions**
- Parameters define a **family** of functions that share a common form
  - θ: general symbol for parameters/weights (usually represents **several**)
  - $\hat{y} = f(x; \theta)$: the function $f(x)$, **given parameters** θ
- Example: the **family of linear functions** $f(x) = mx + b$
  - $\theta = \{m, b\}$
  - This defines **all possible lines** (with different slopes and intercepts)
- Later: Neural Networks define their own family of functions

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- We always want to **minimize the loss/error**
  - This is a type of **optimization problem**, which is a huge subfield of math

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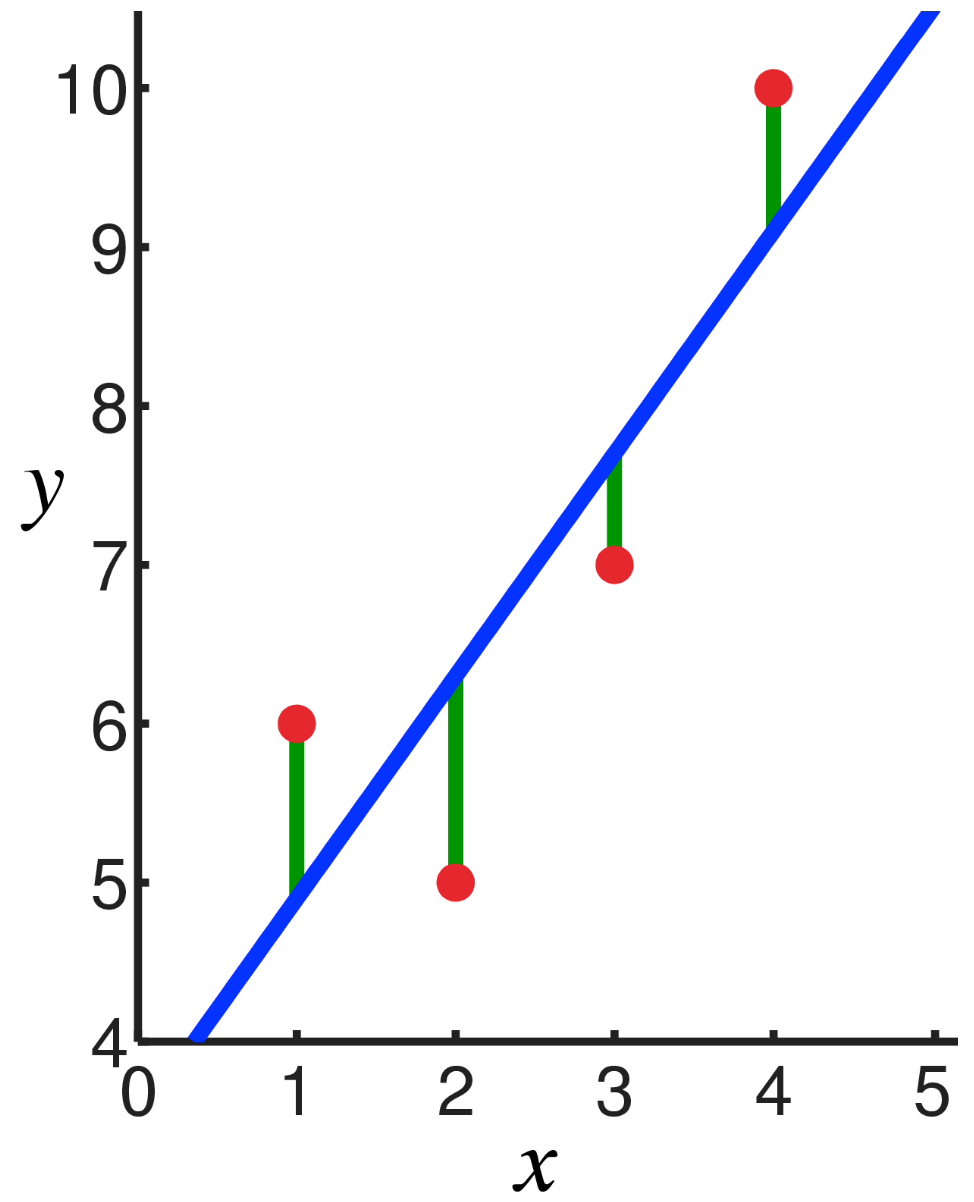
- Example: **Linear Regression** ("Least-Squares" method)

$$m^*, b^* = \arg\min_{m,b} \sum_i ((mx_i + b) - y_i)^2$$



Example: Secret Number Game

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  - $\hat{y} = f(x) = x + \theta$

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- Lastly **learn the optimal value of θ** (i.e. the value that **minimizes the loss**)

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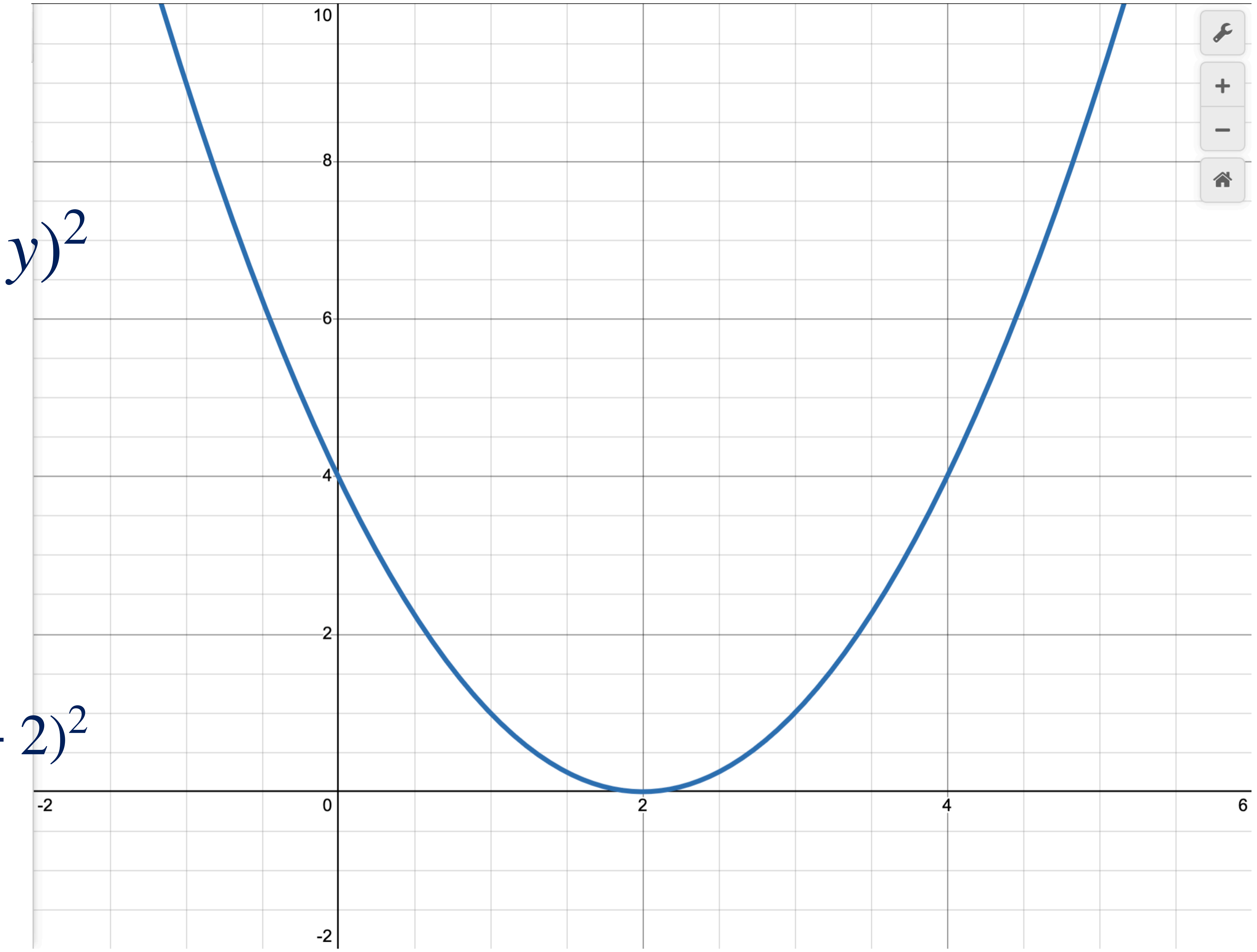

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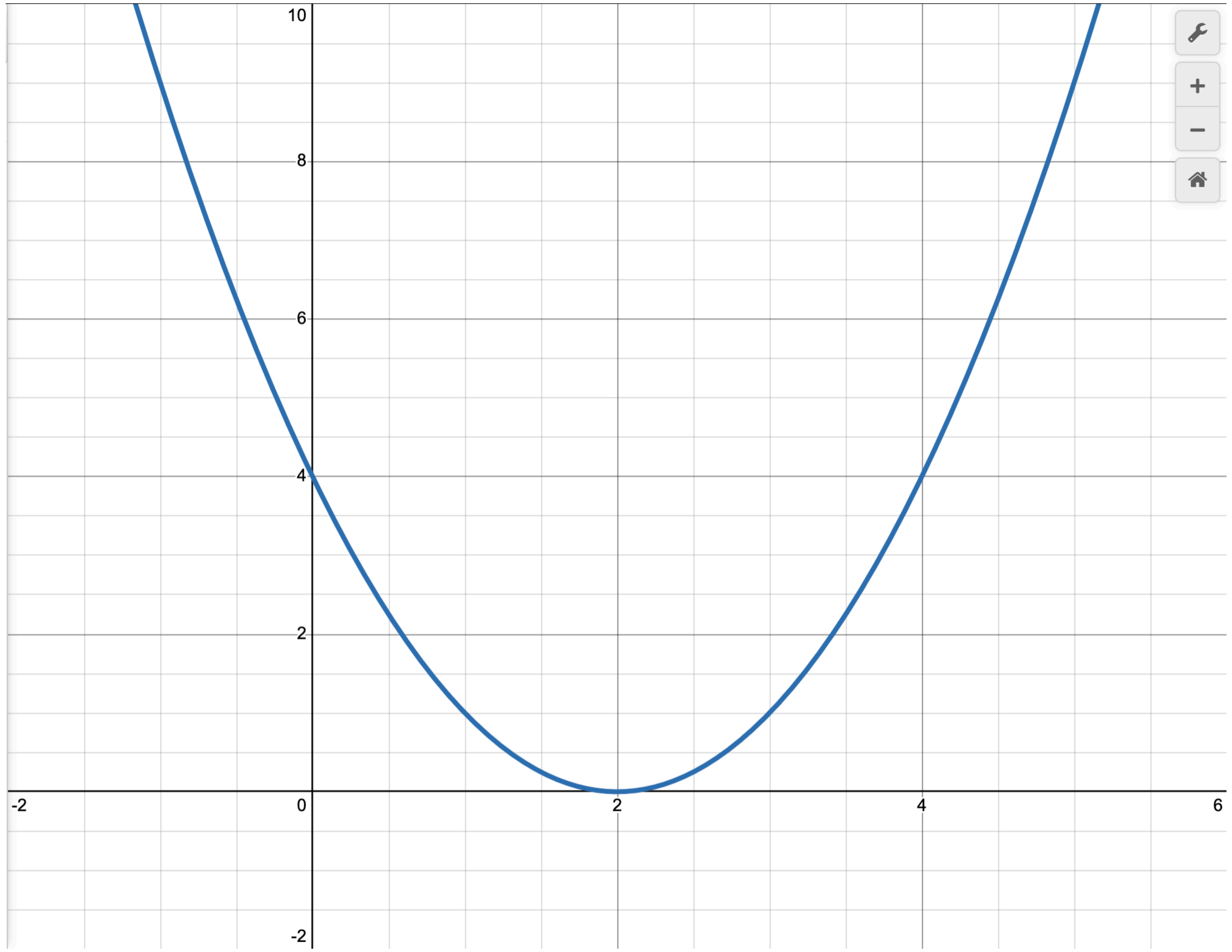

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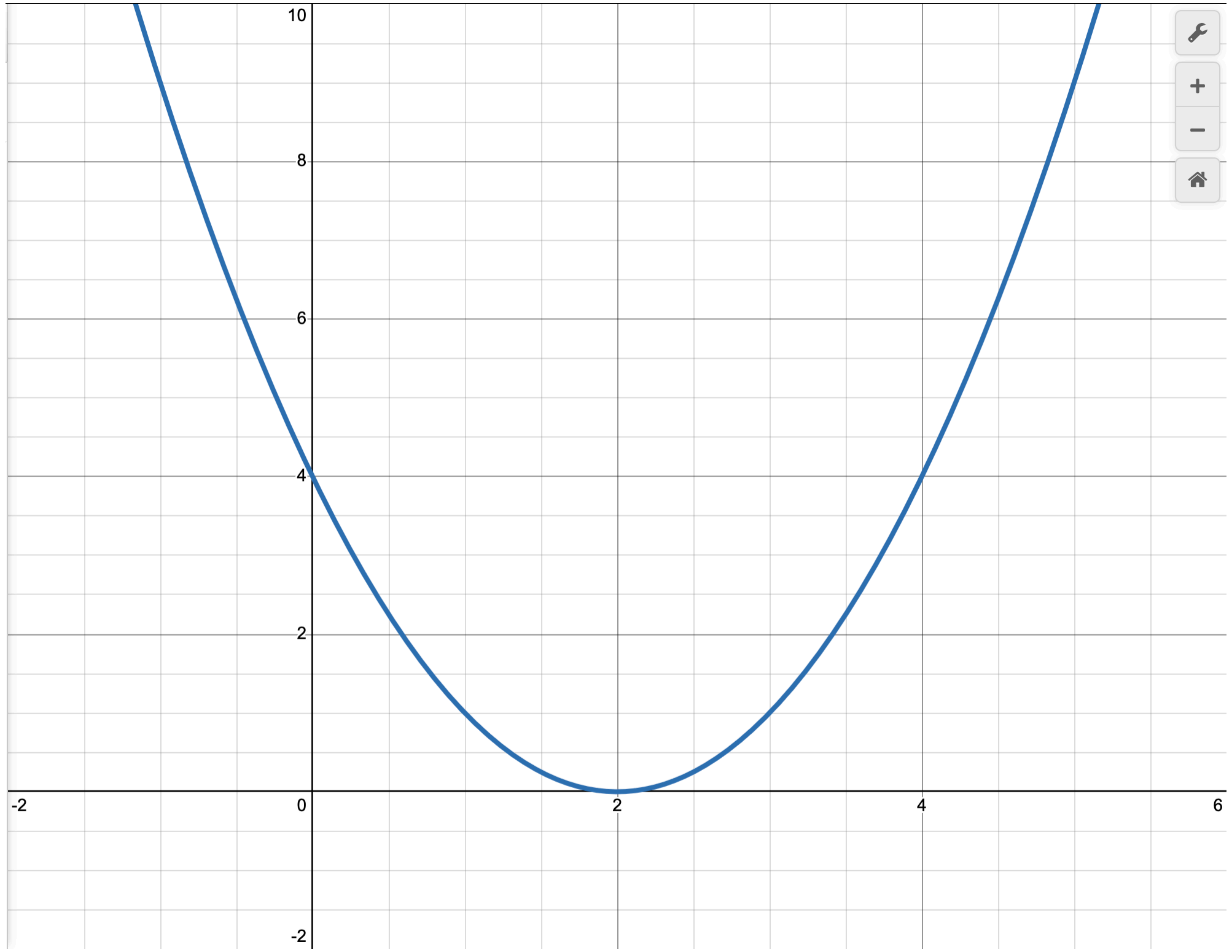

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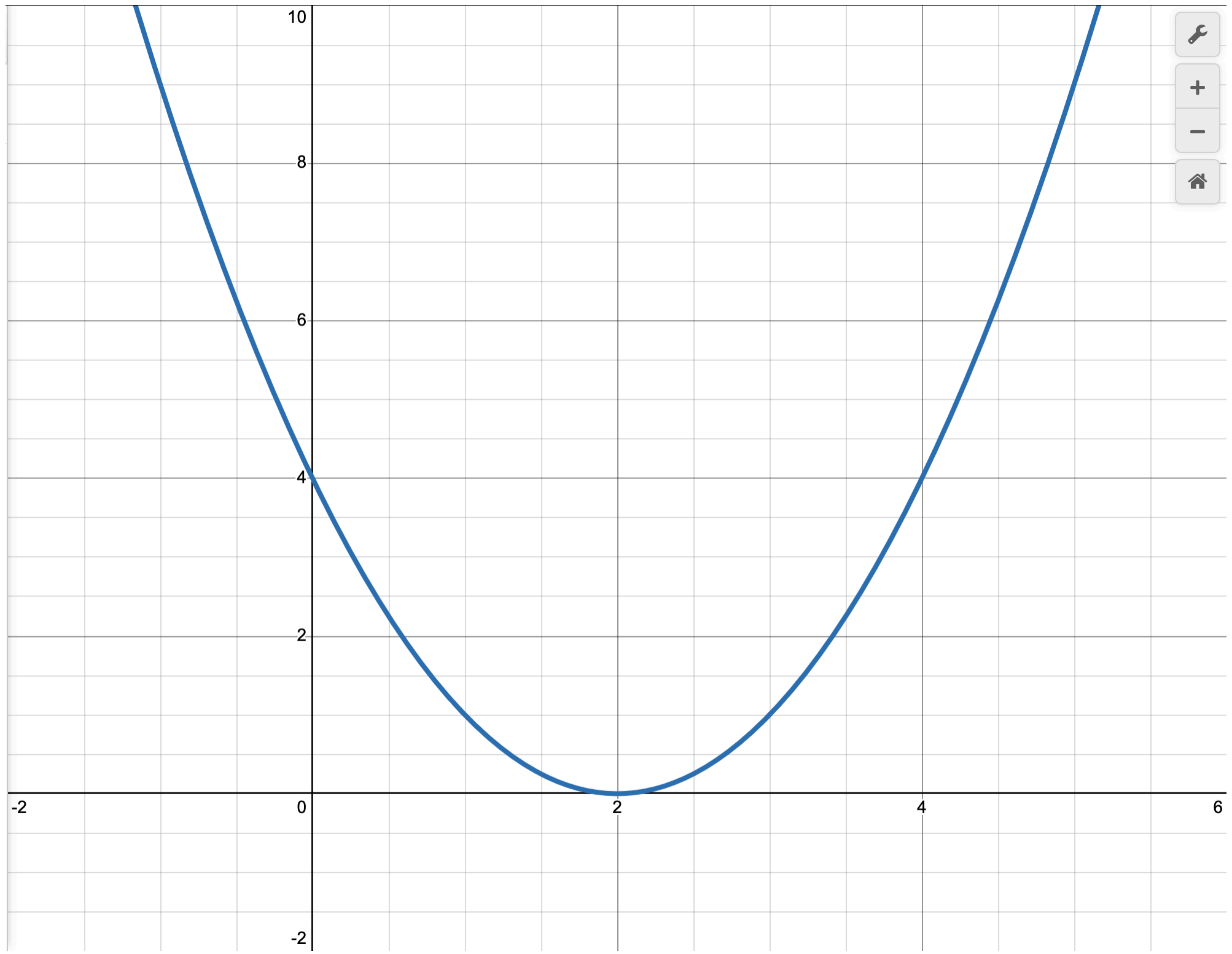

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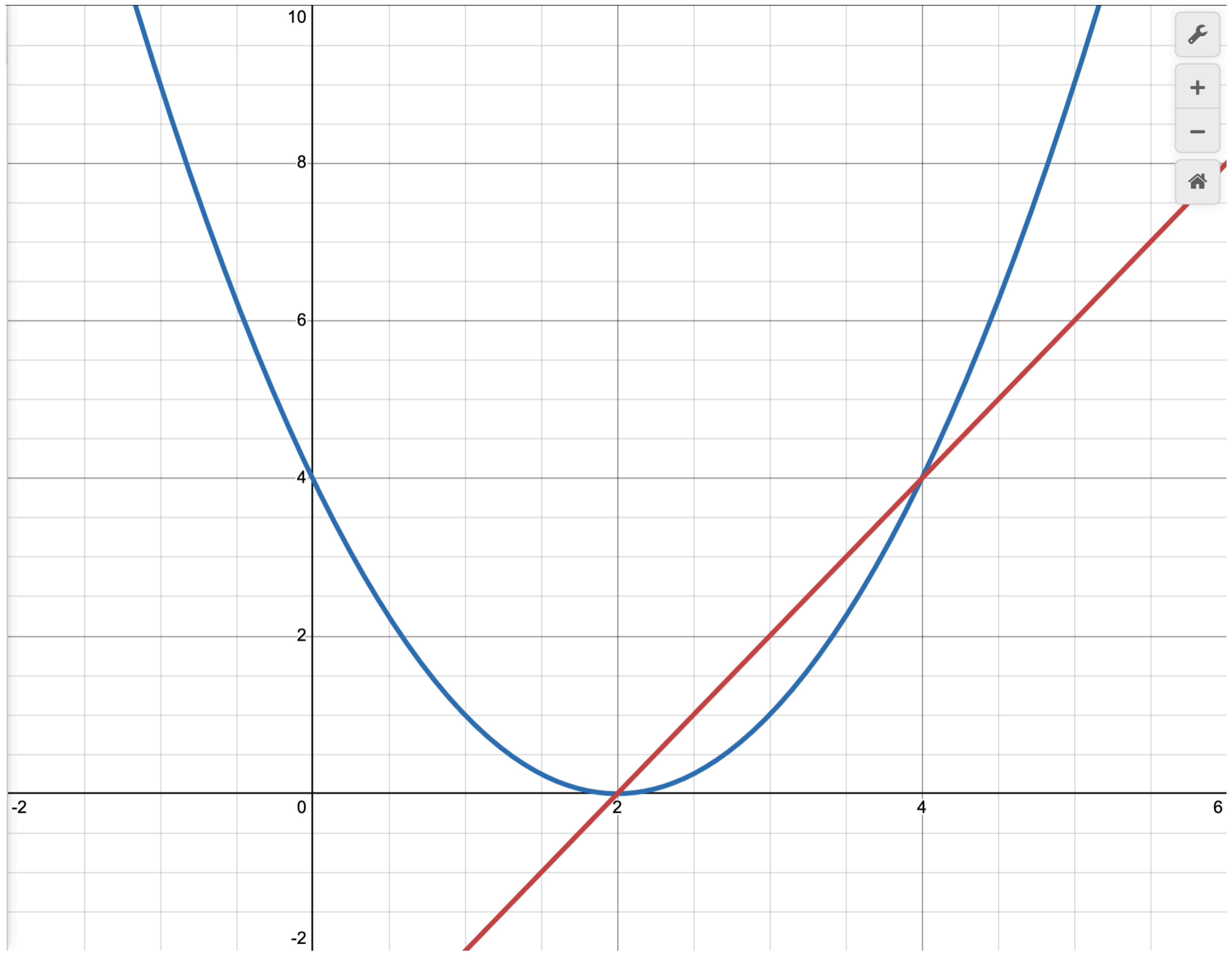

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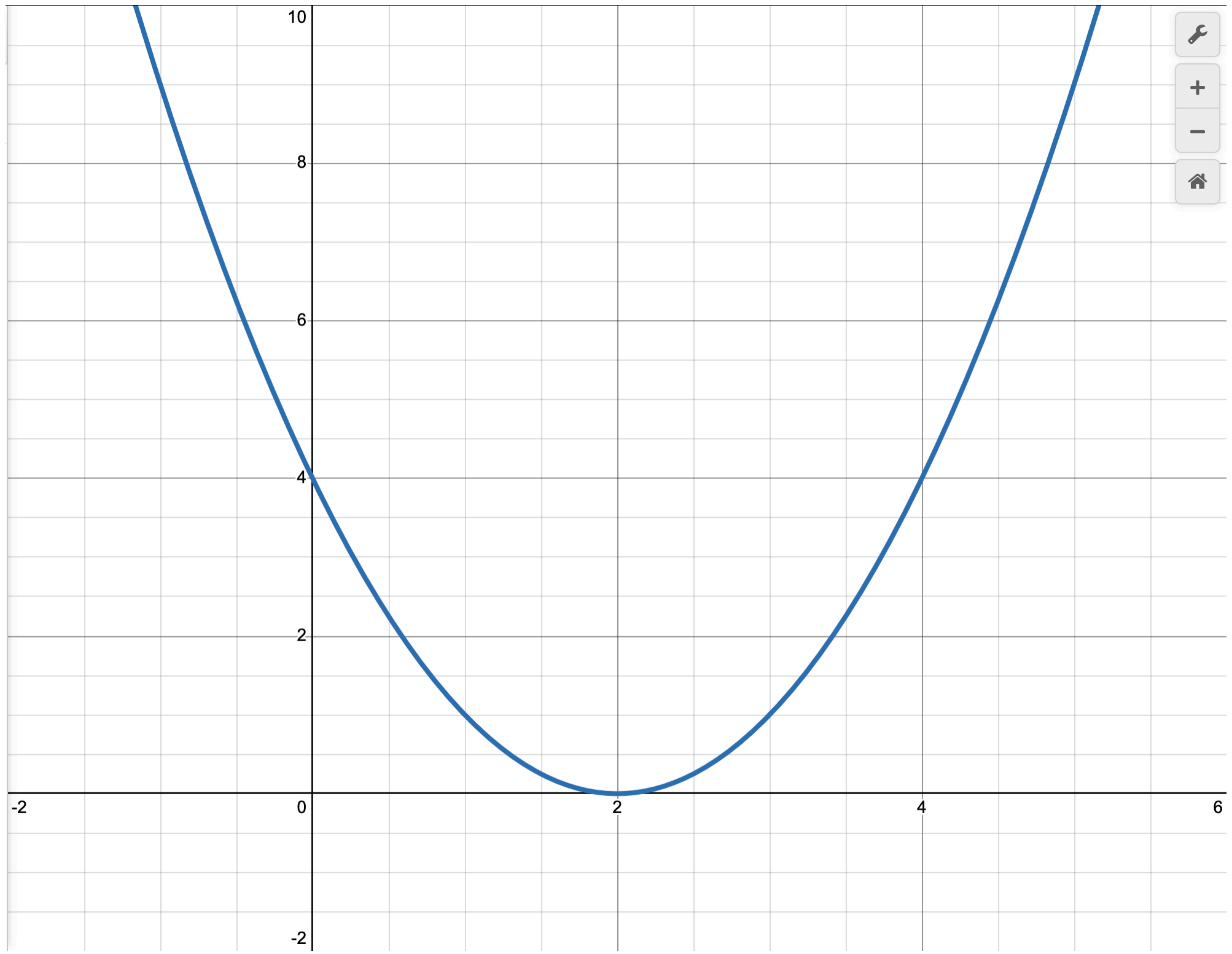

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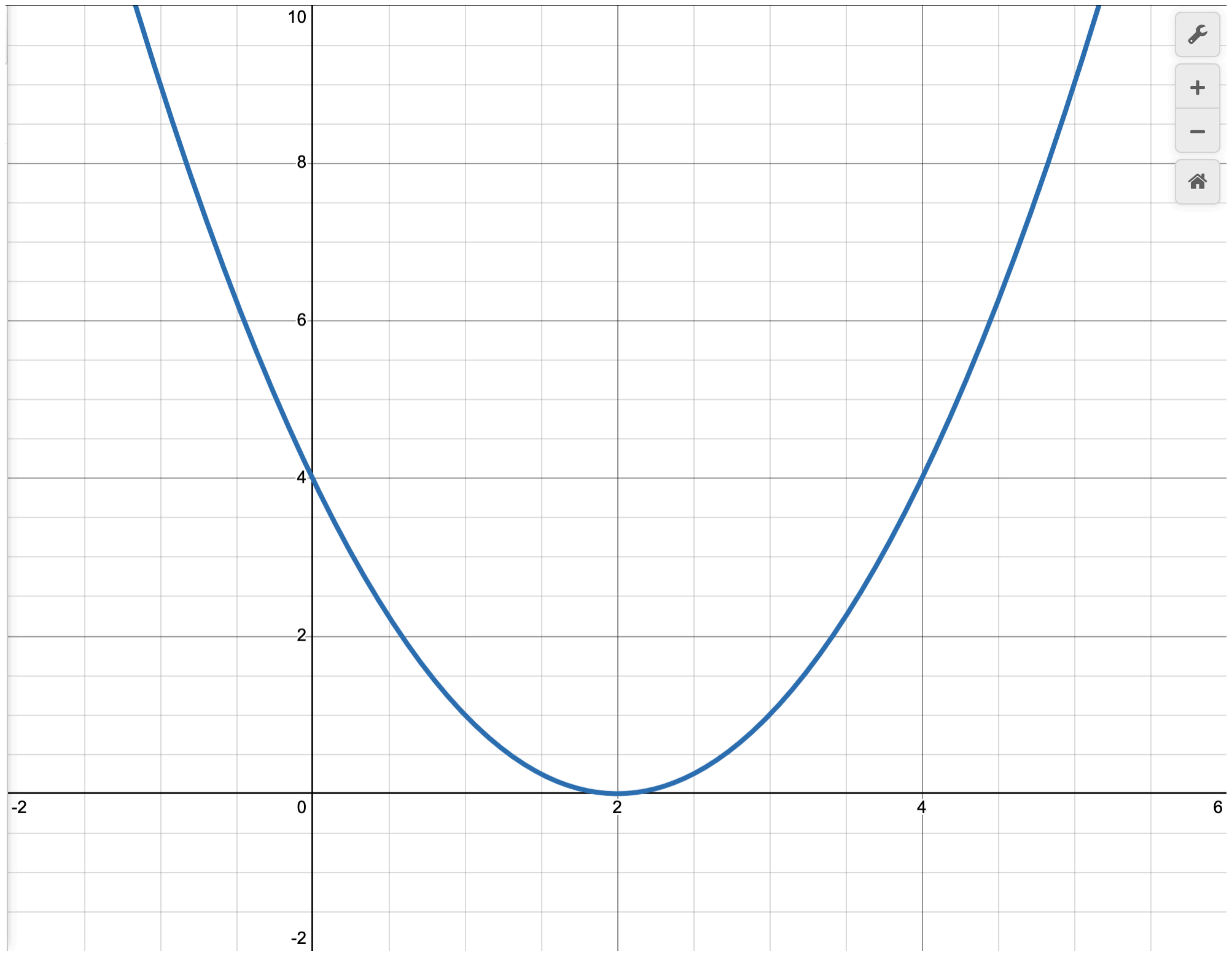

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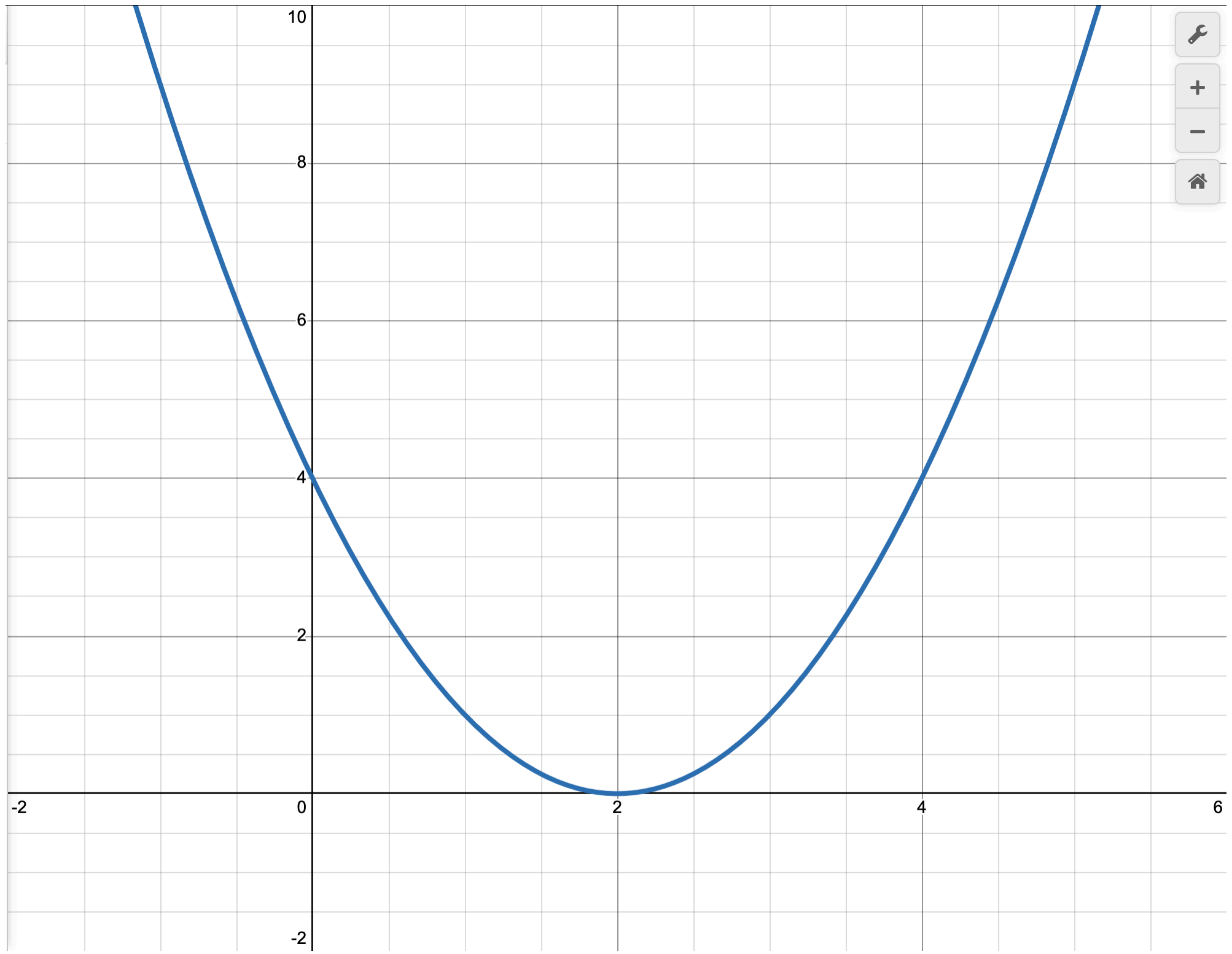

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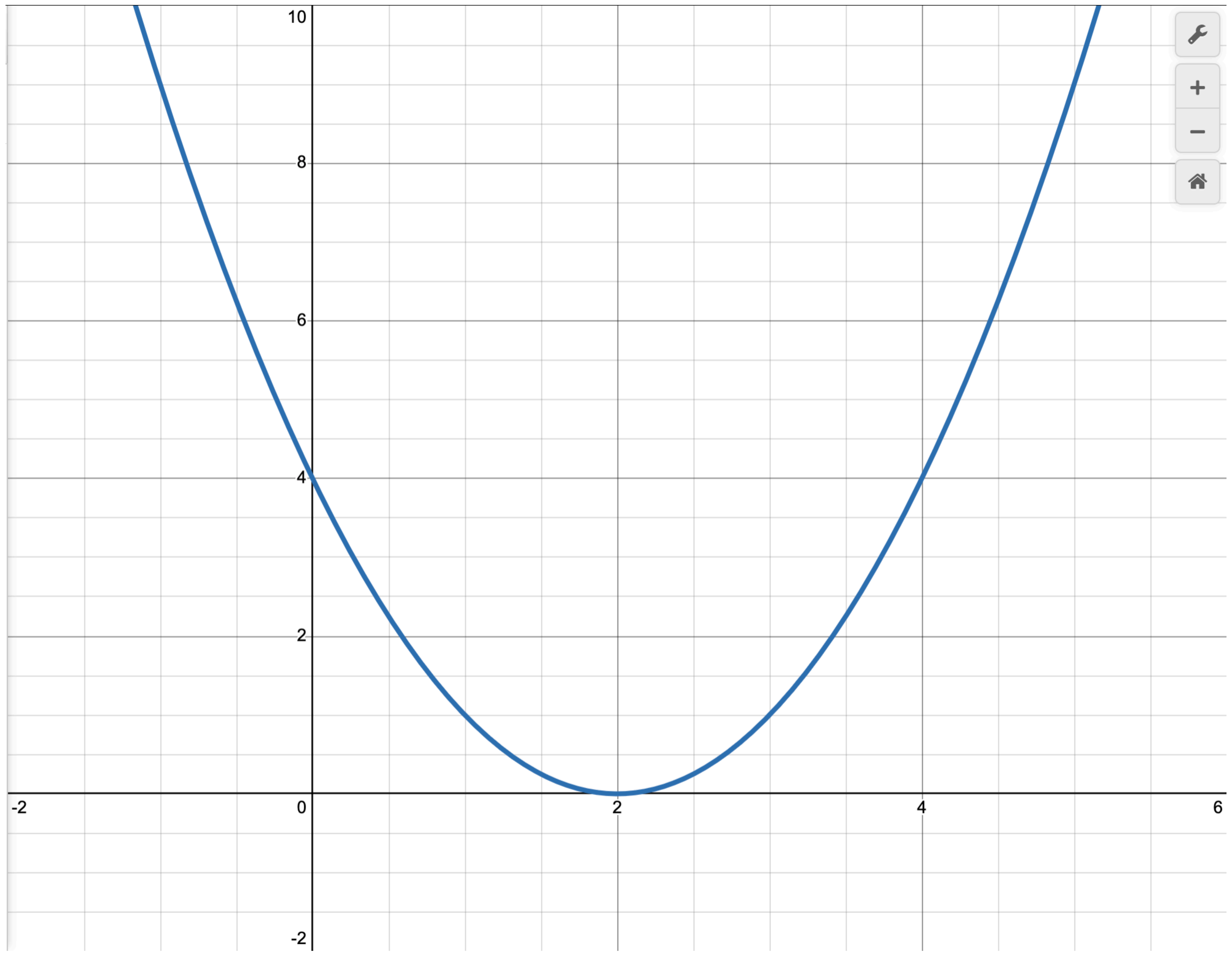

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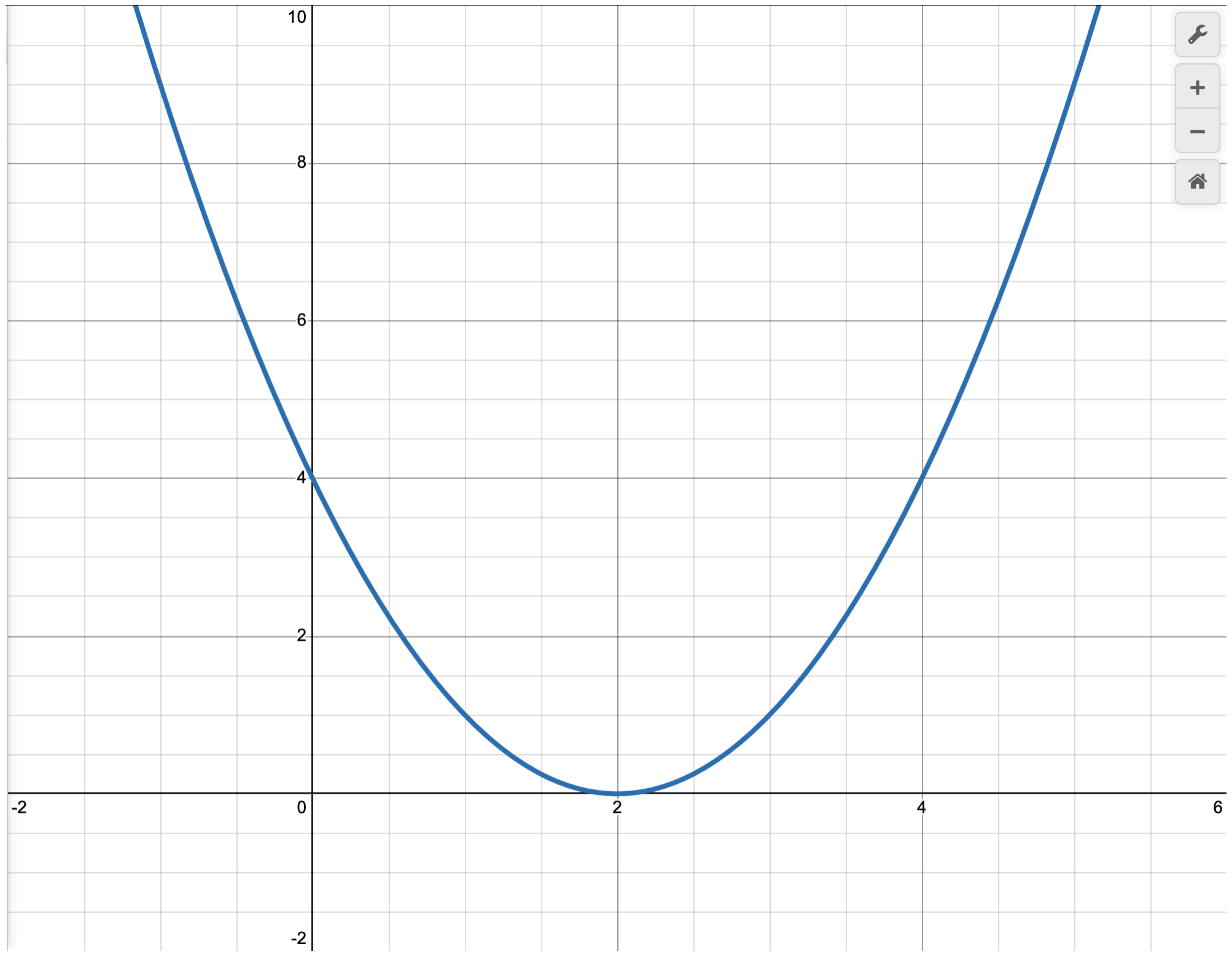

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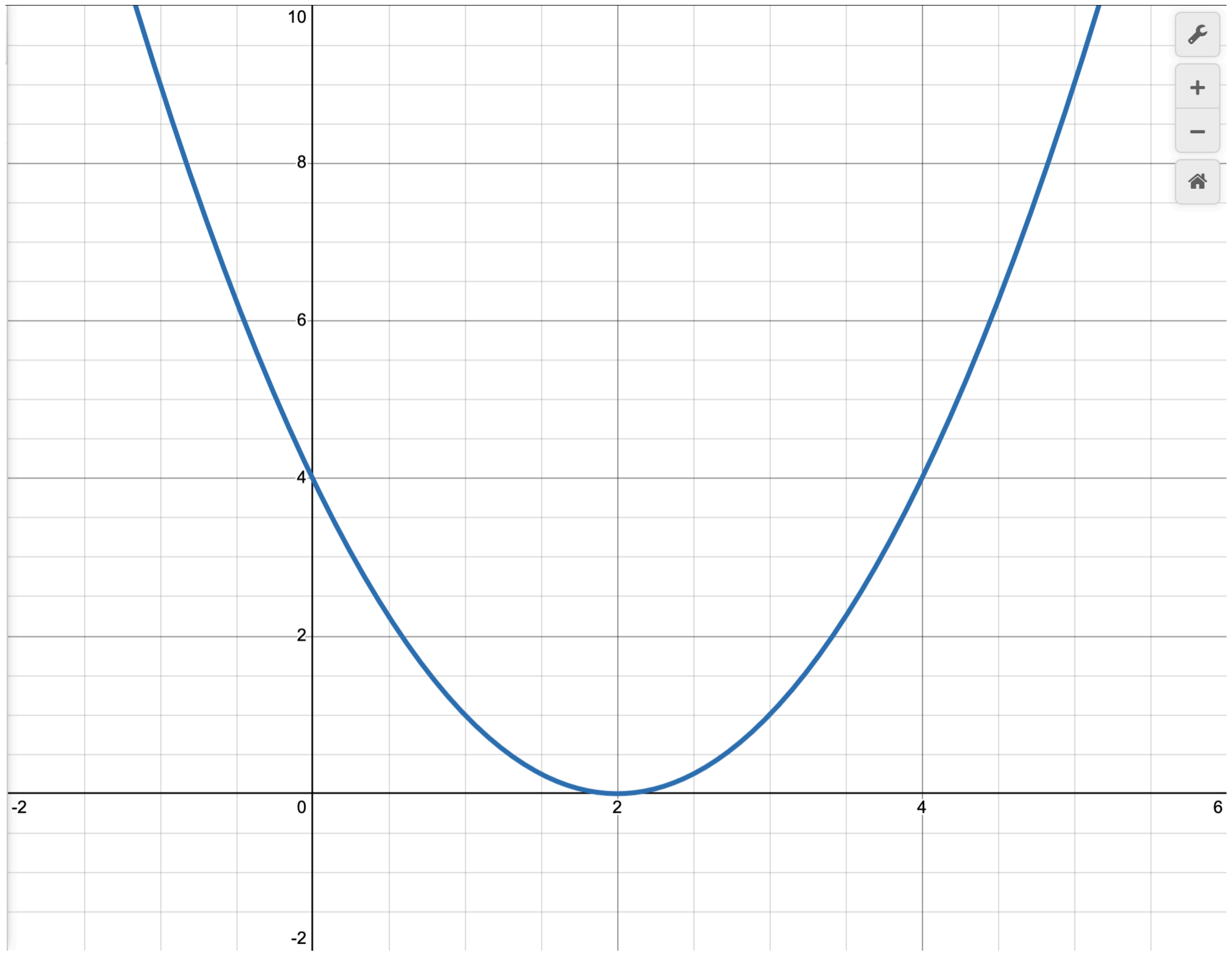

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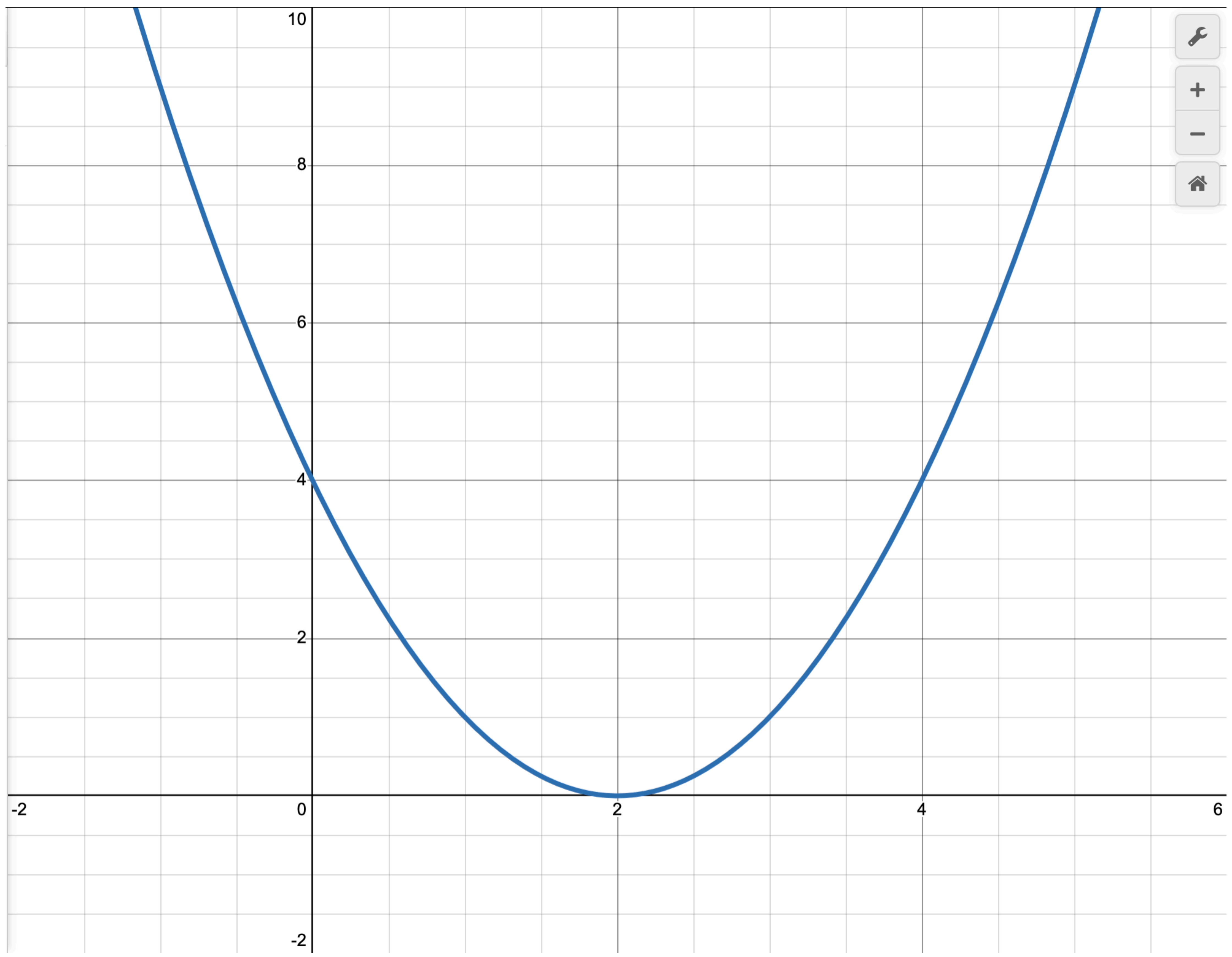

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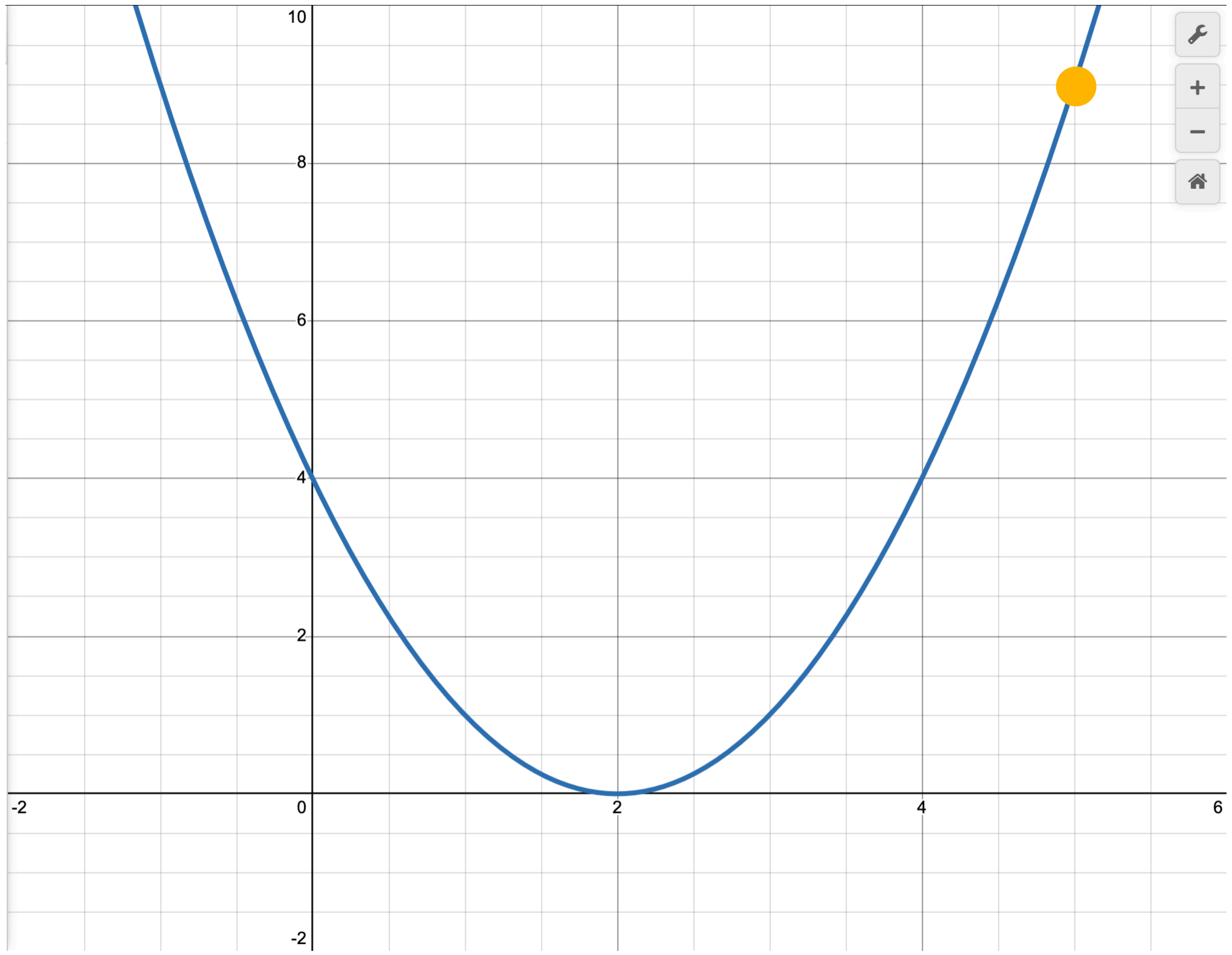

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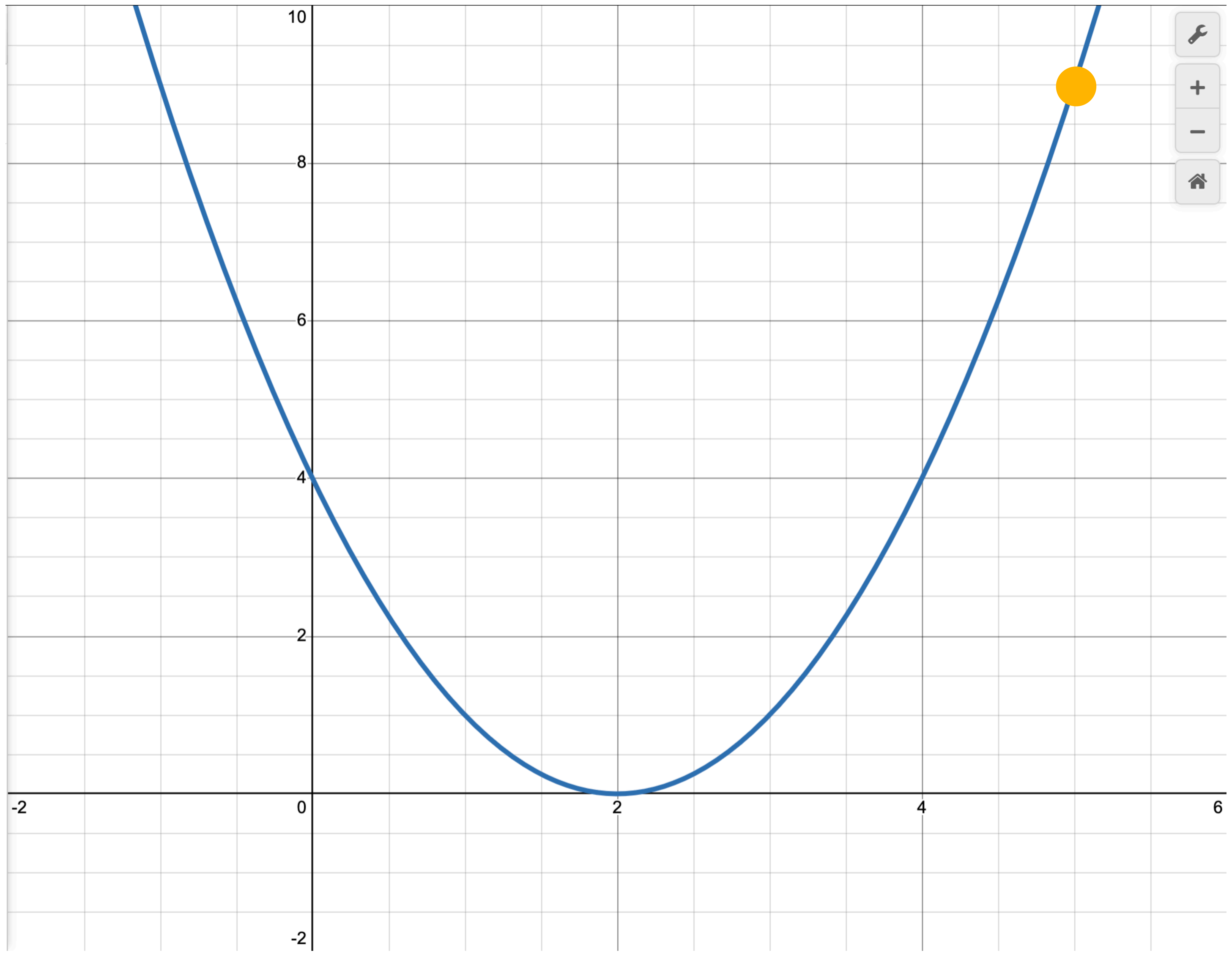

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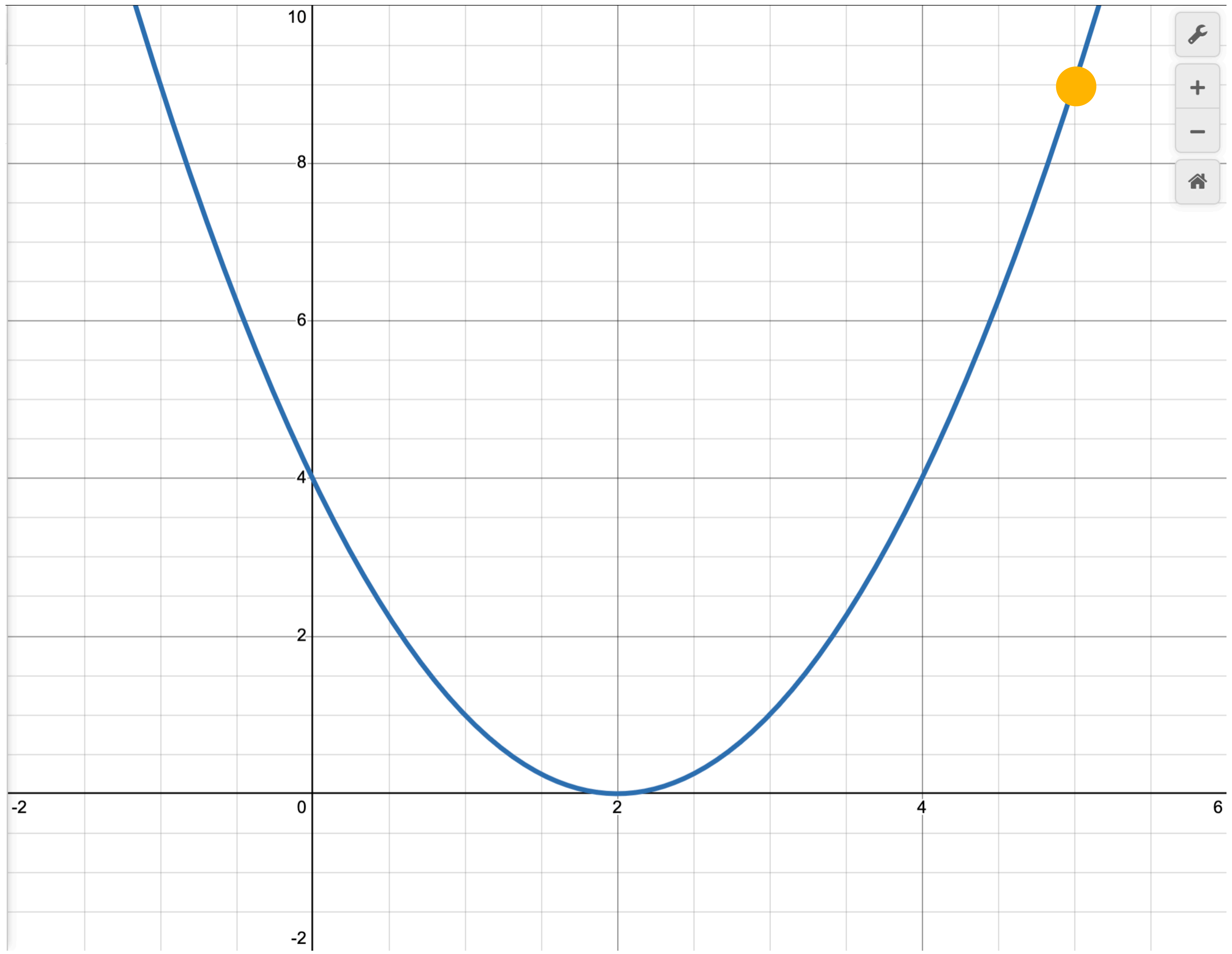

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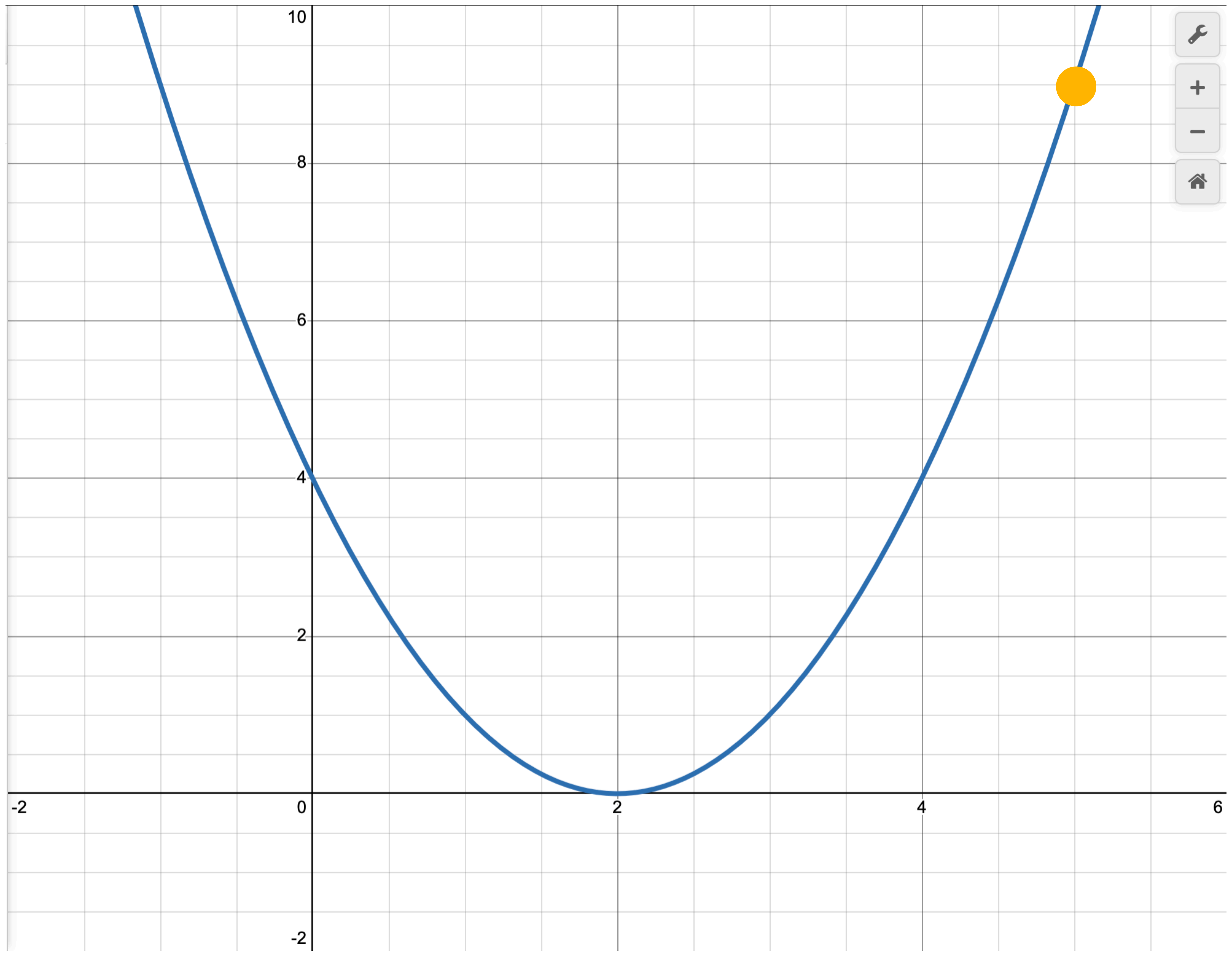

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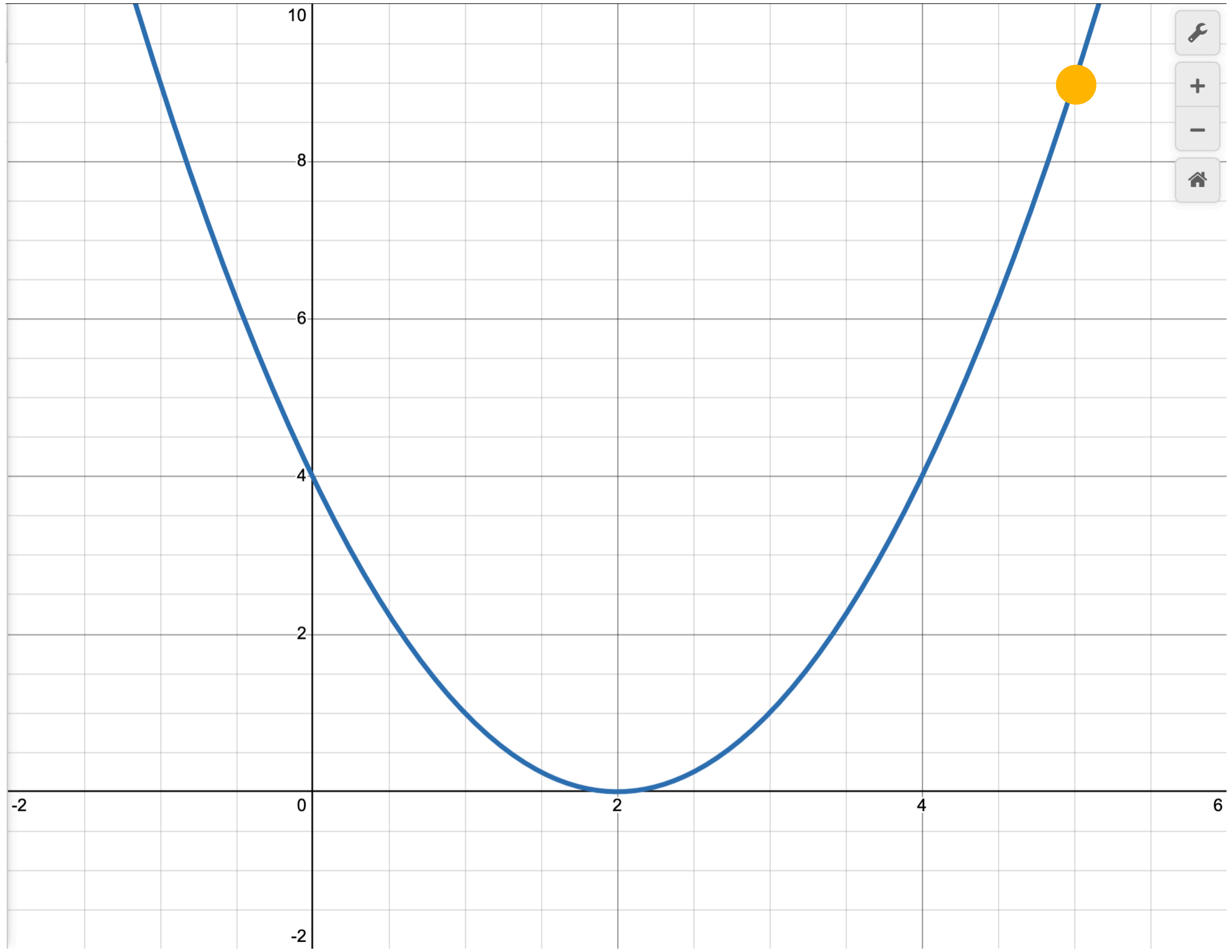

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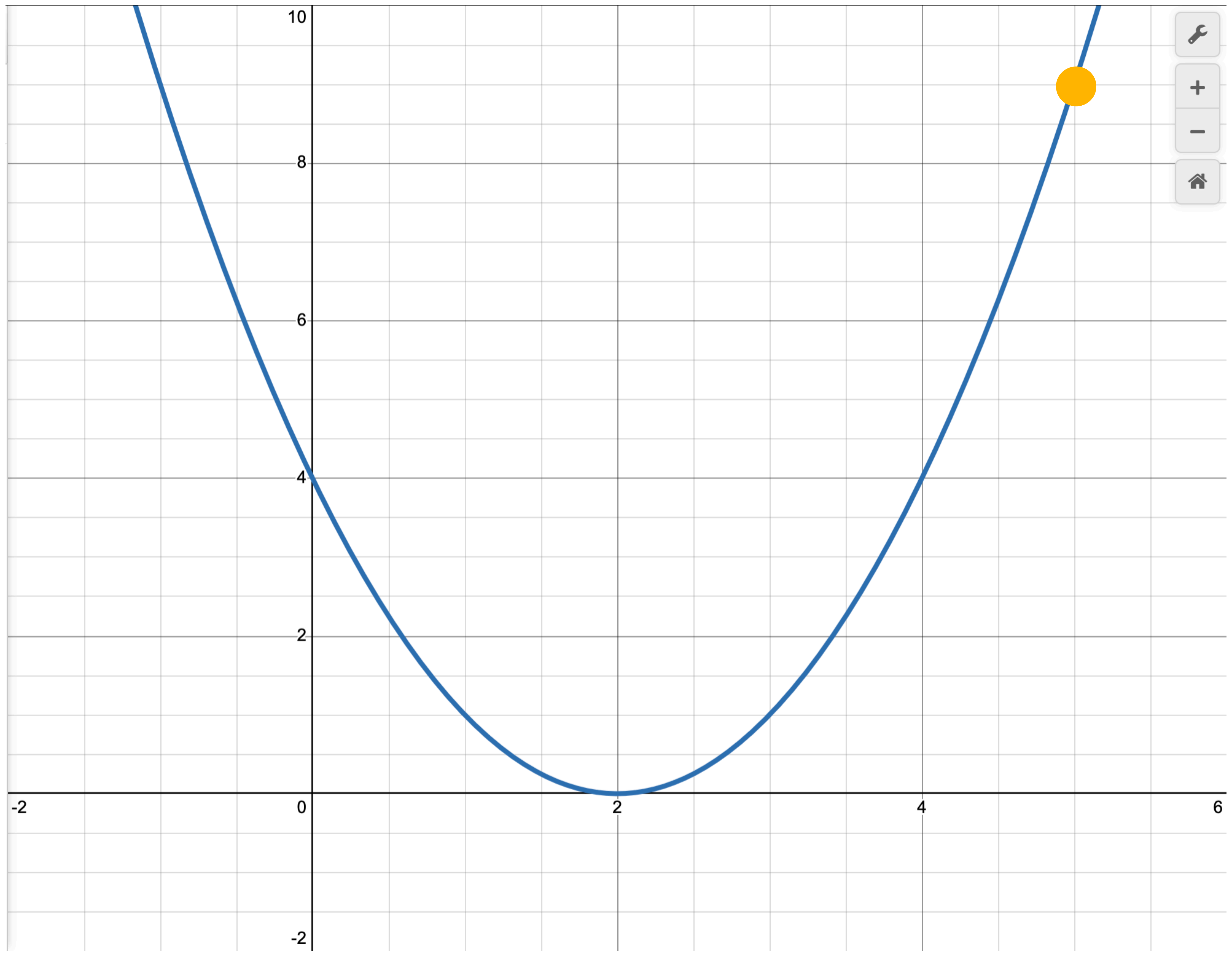

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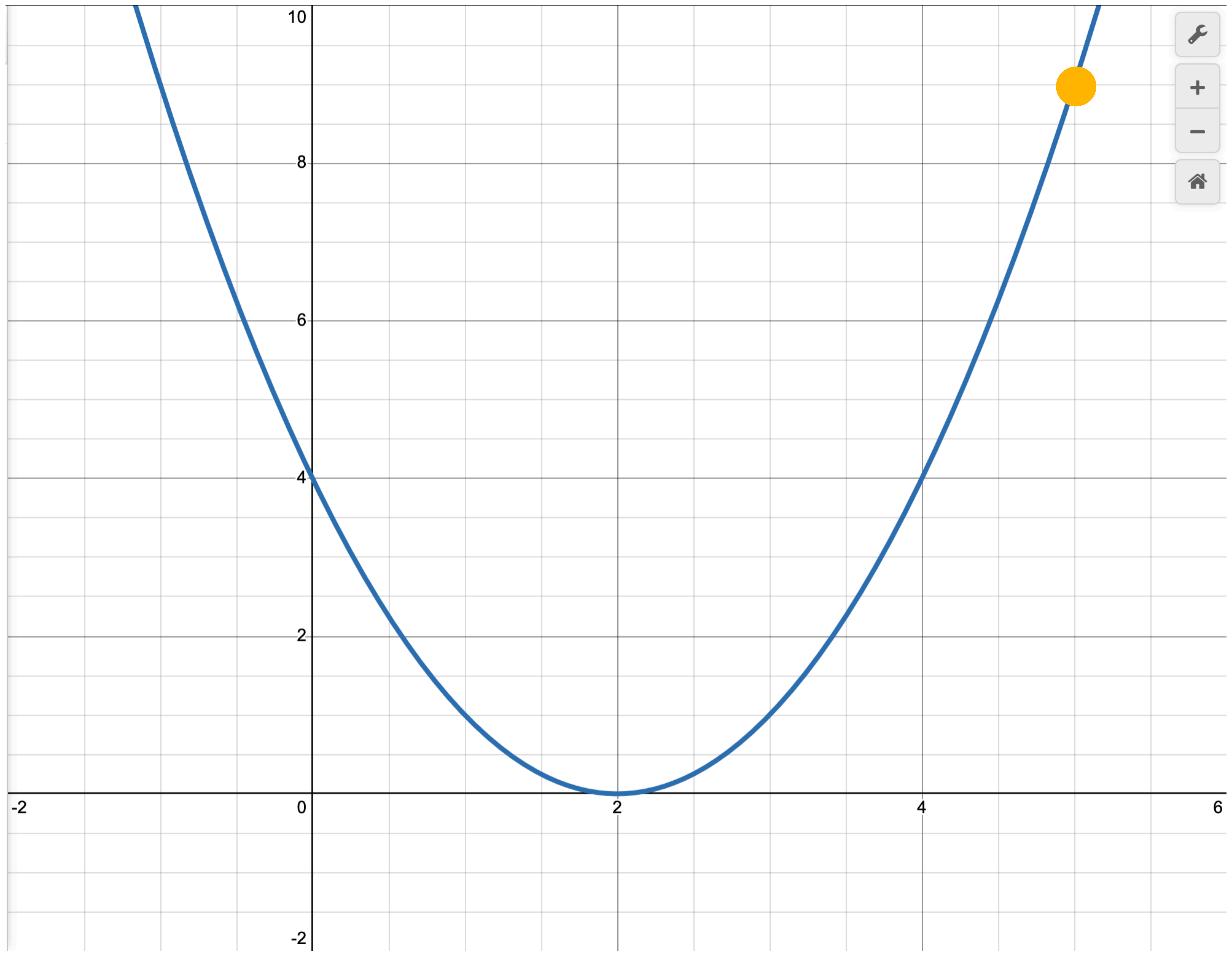

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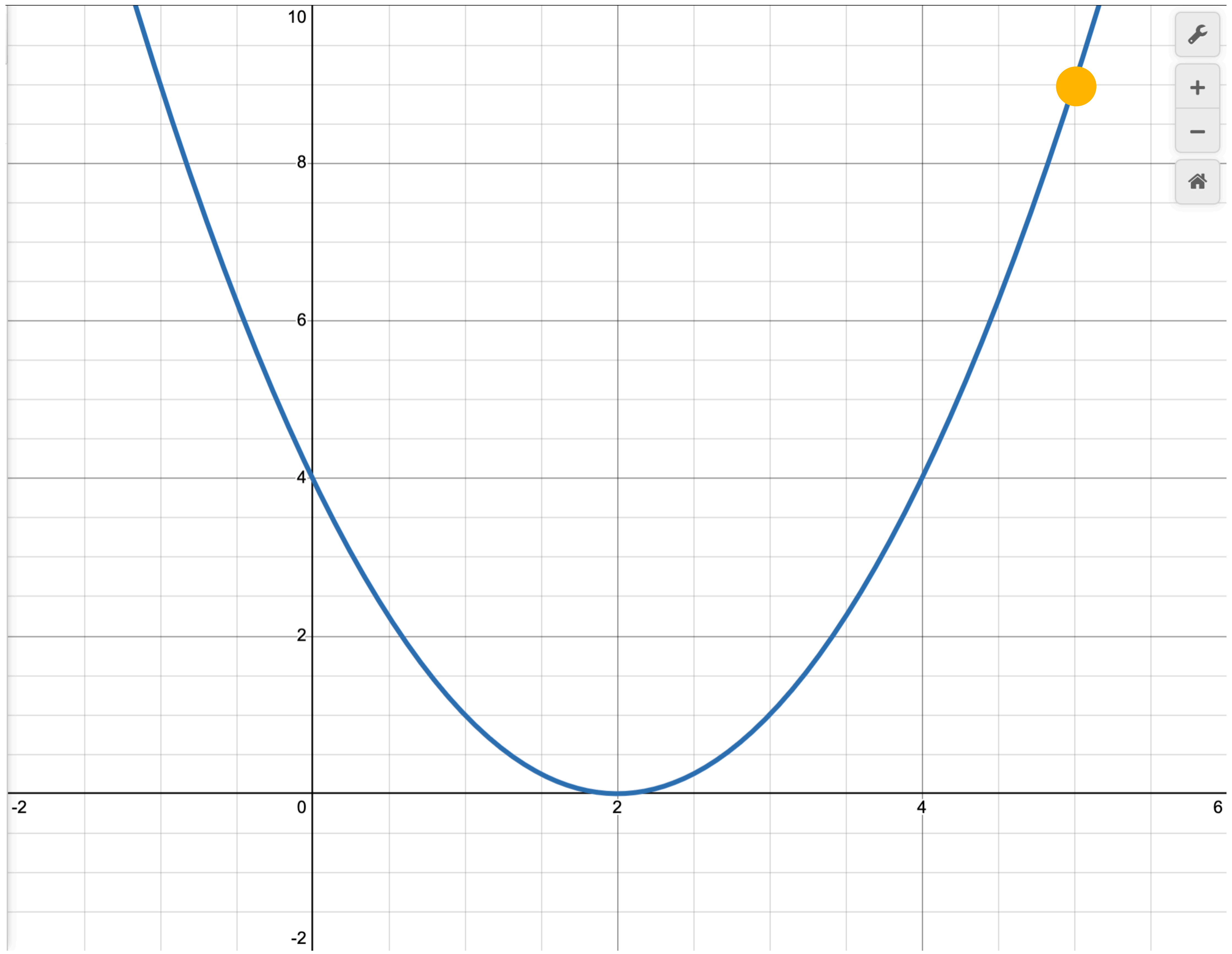

Gradient Descent (first try)

- At each step of the algorithm:
  - Calculate the **slope at the current point**
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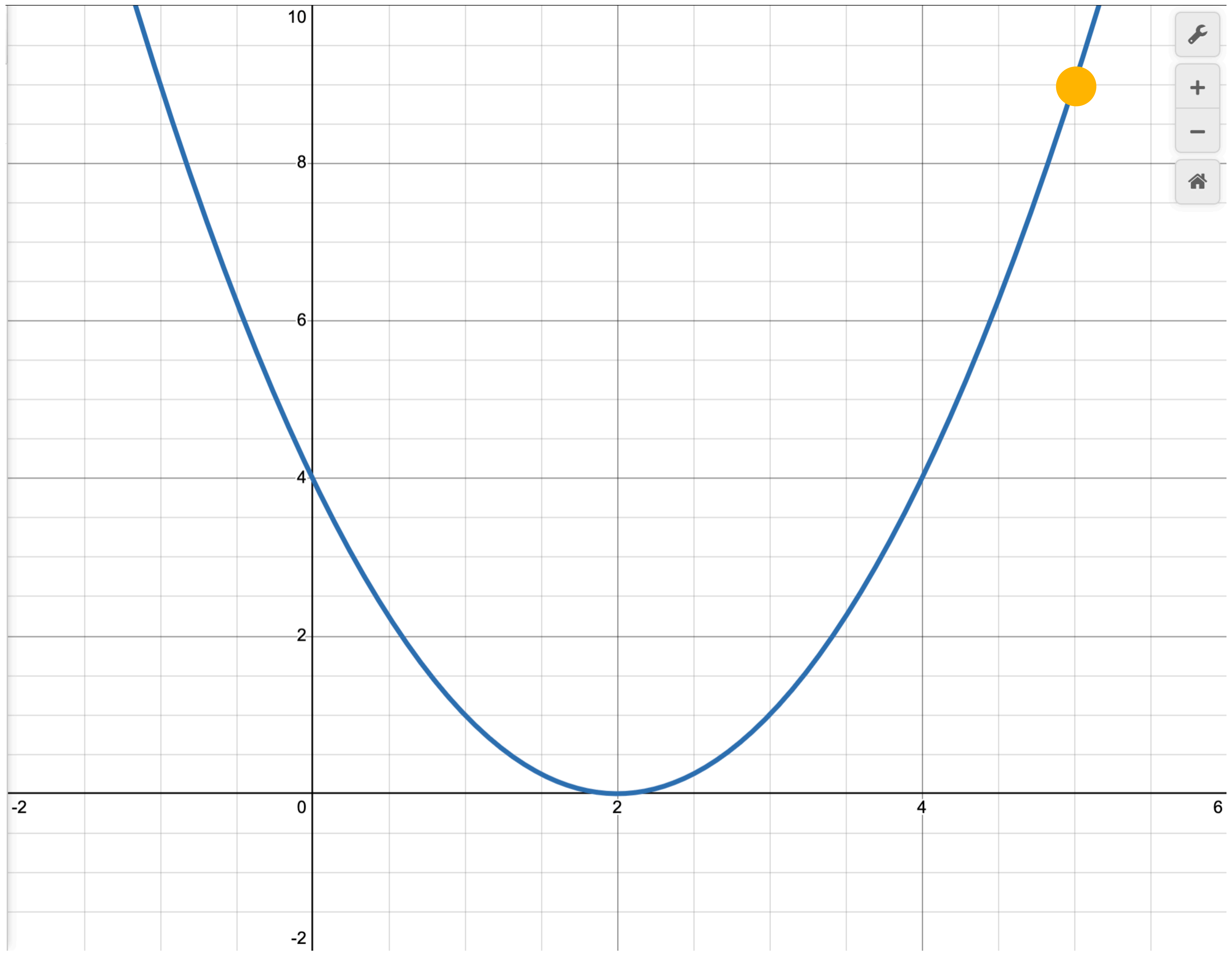

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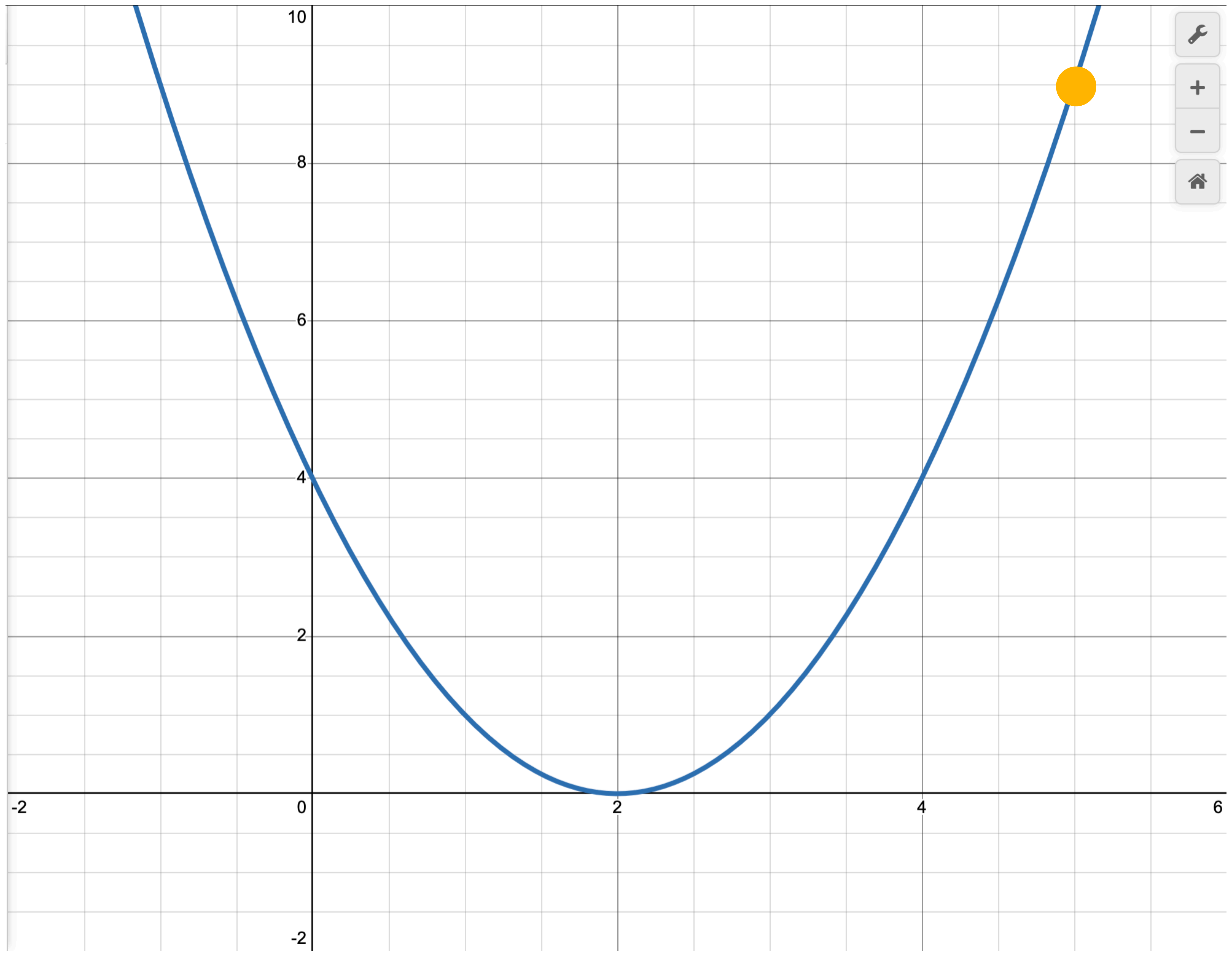

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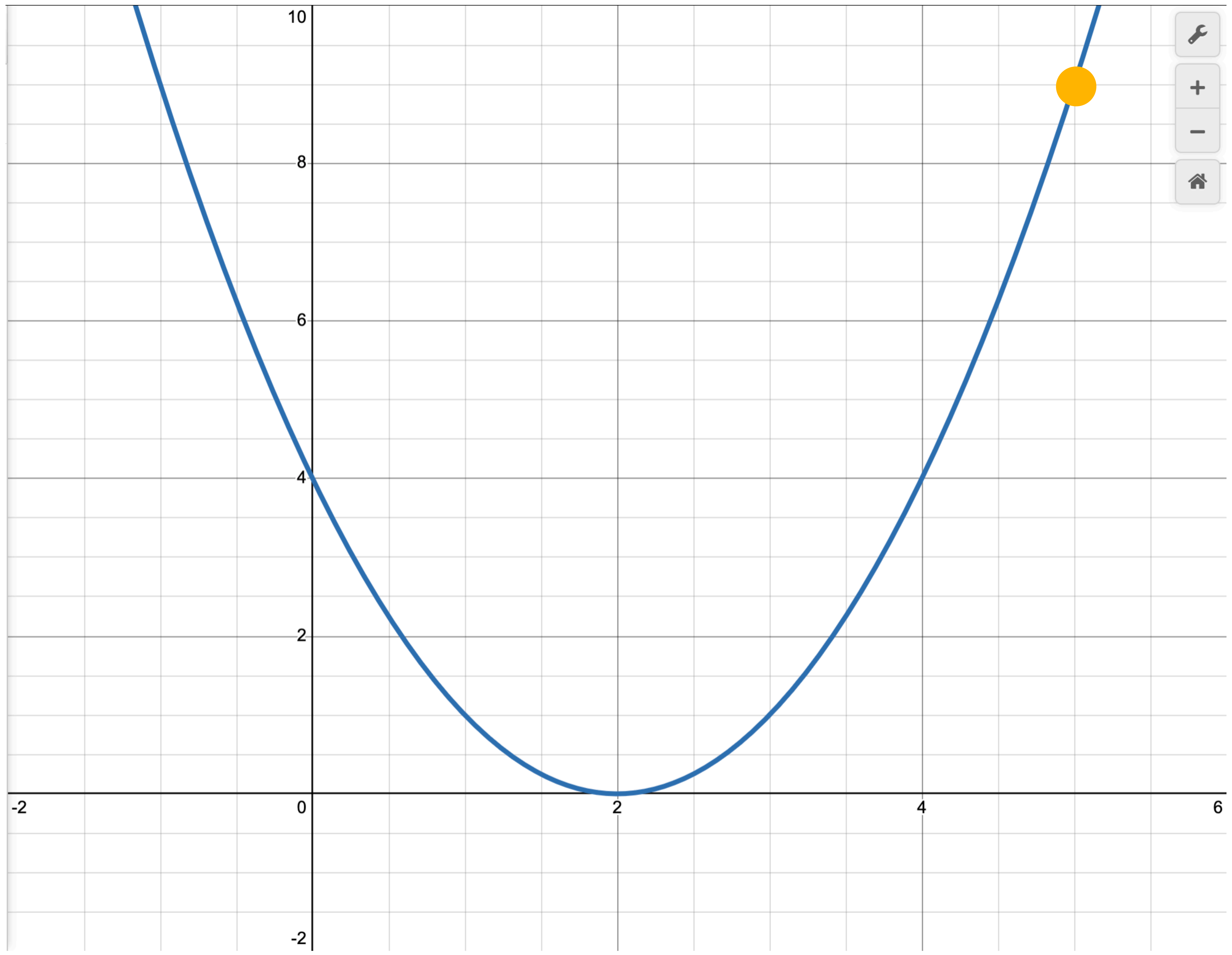

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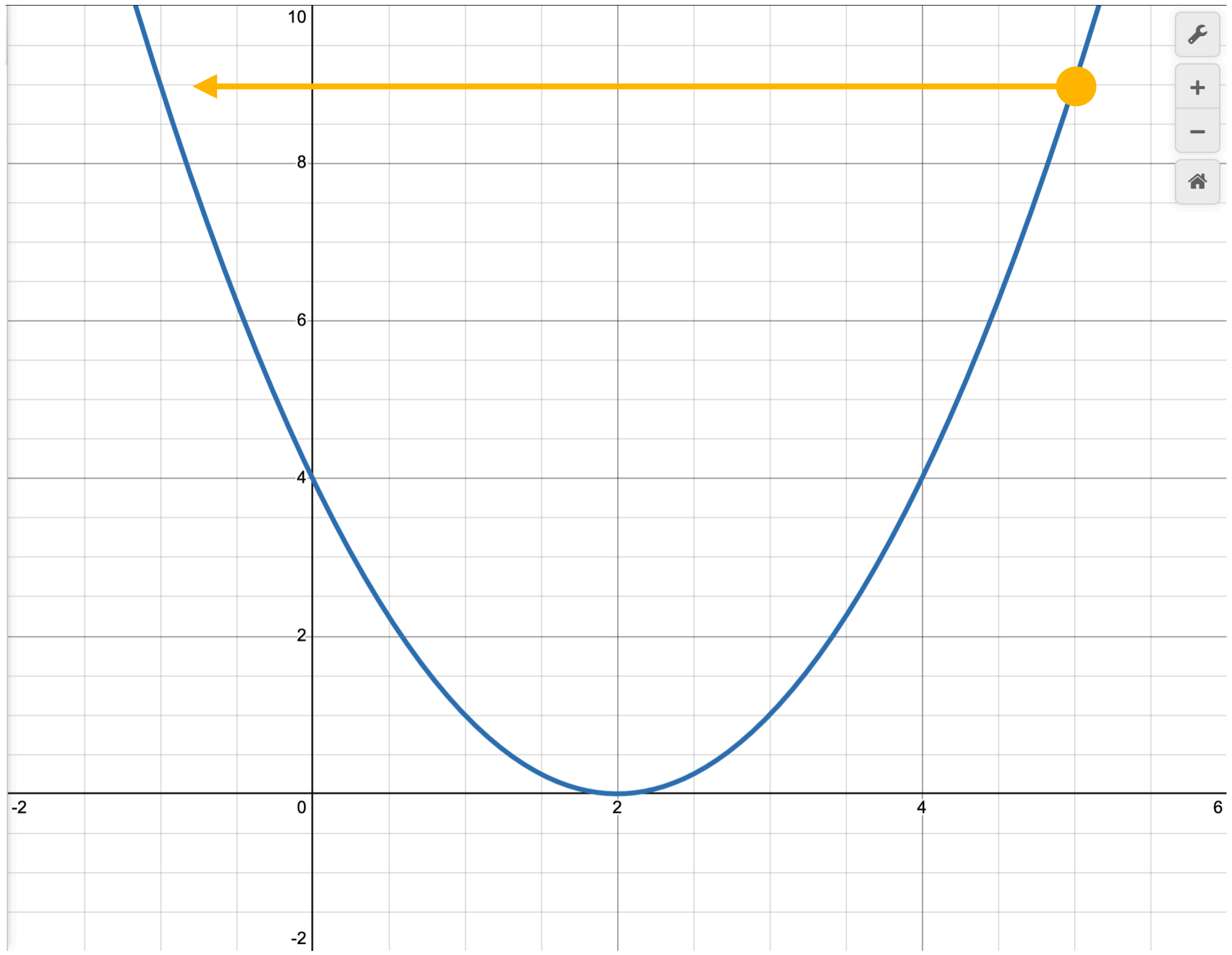

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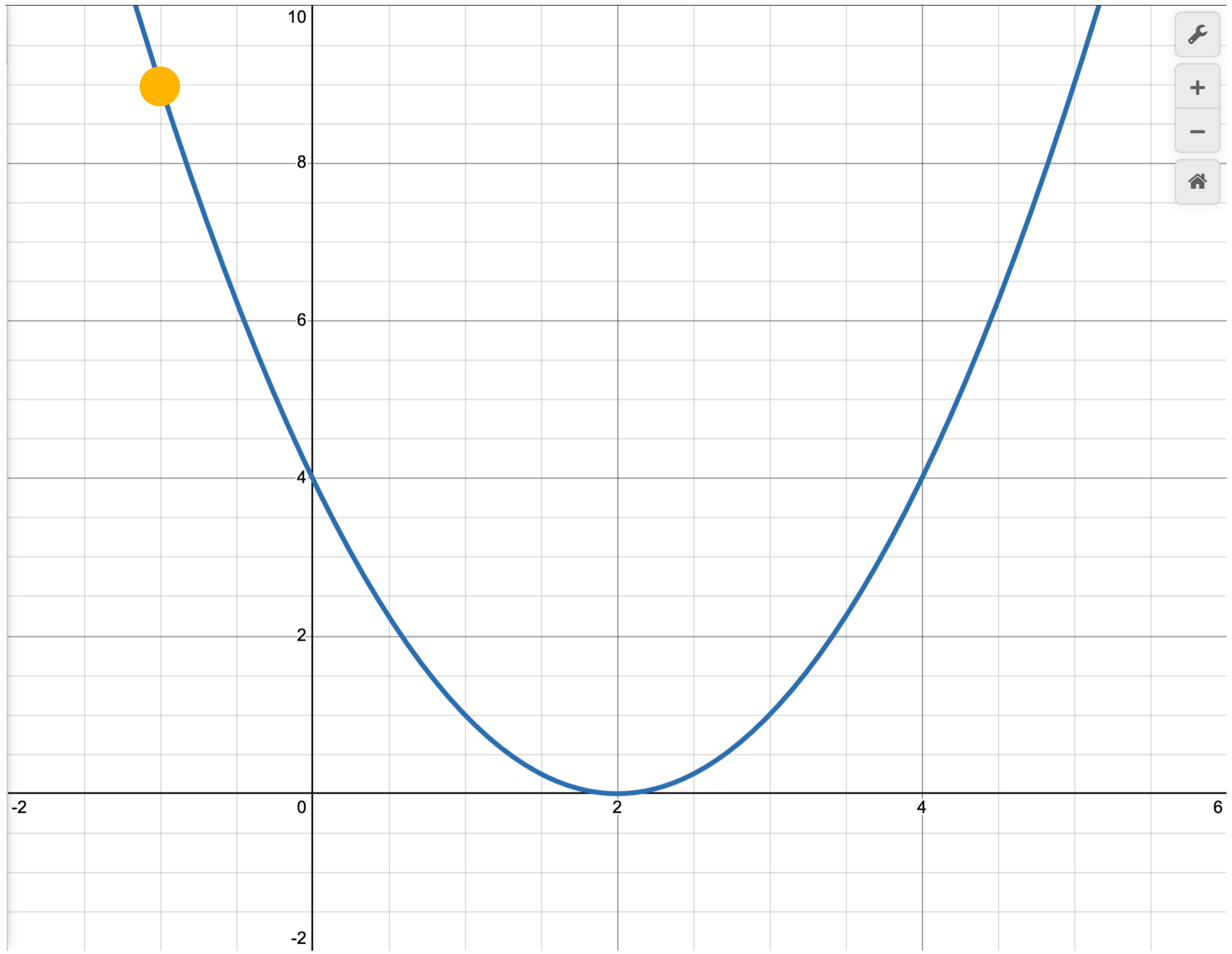

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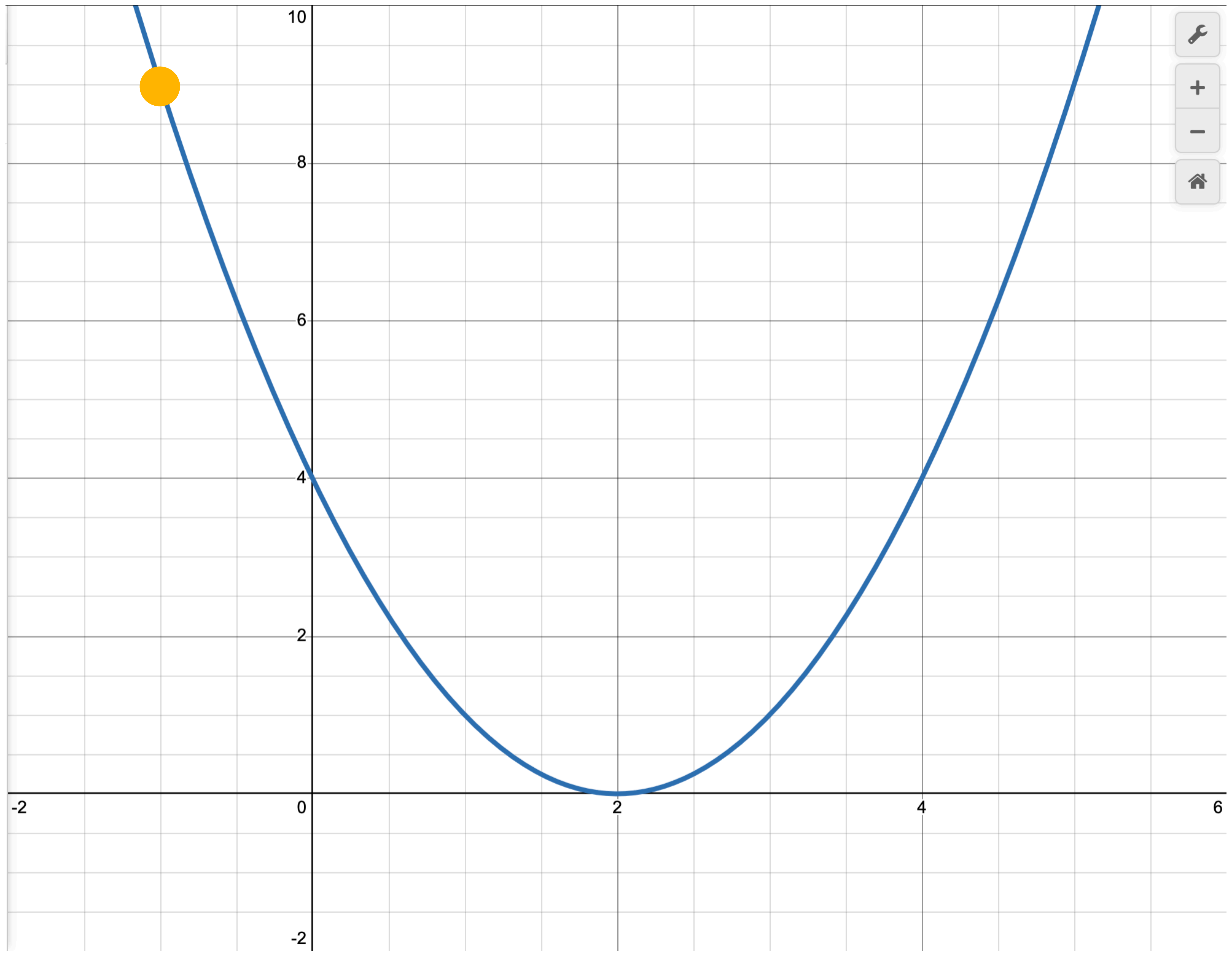

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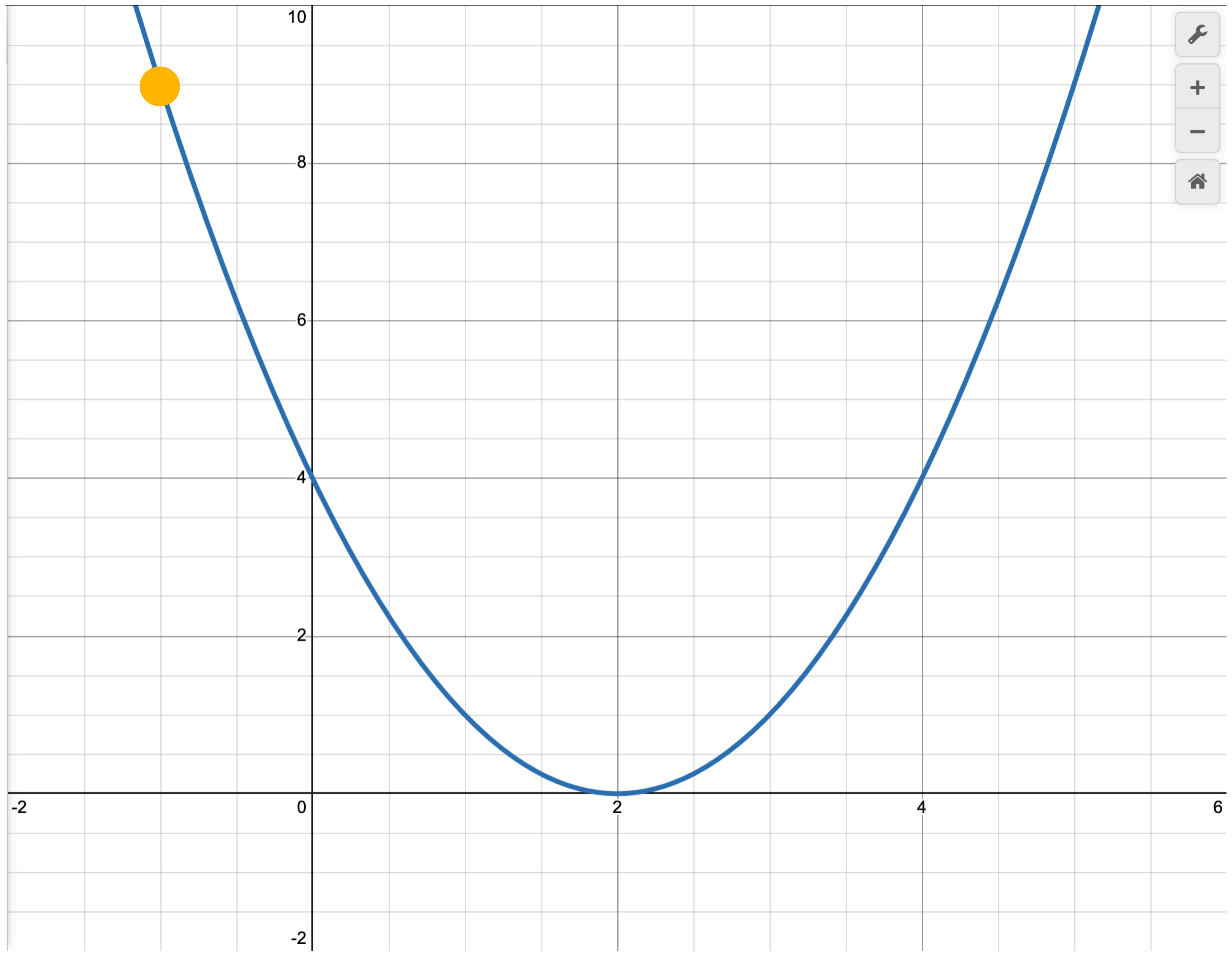

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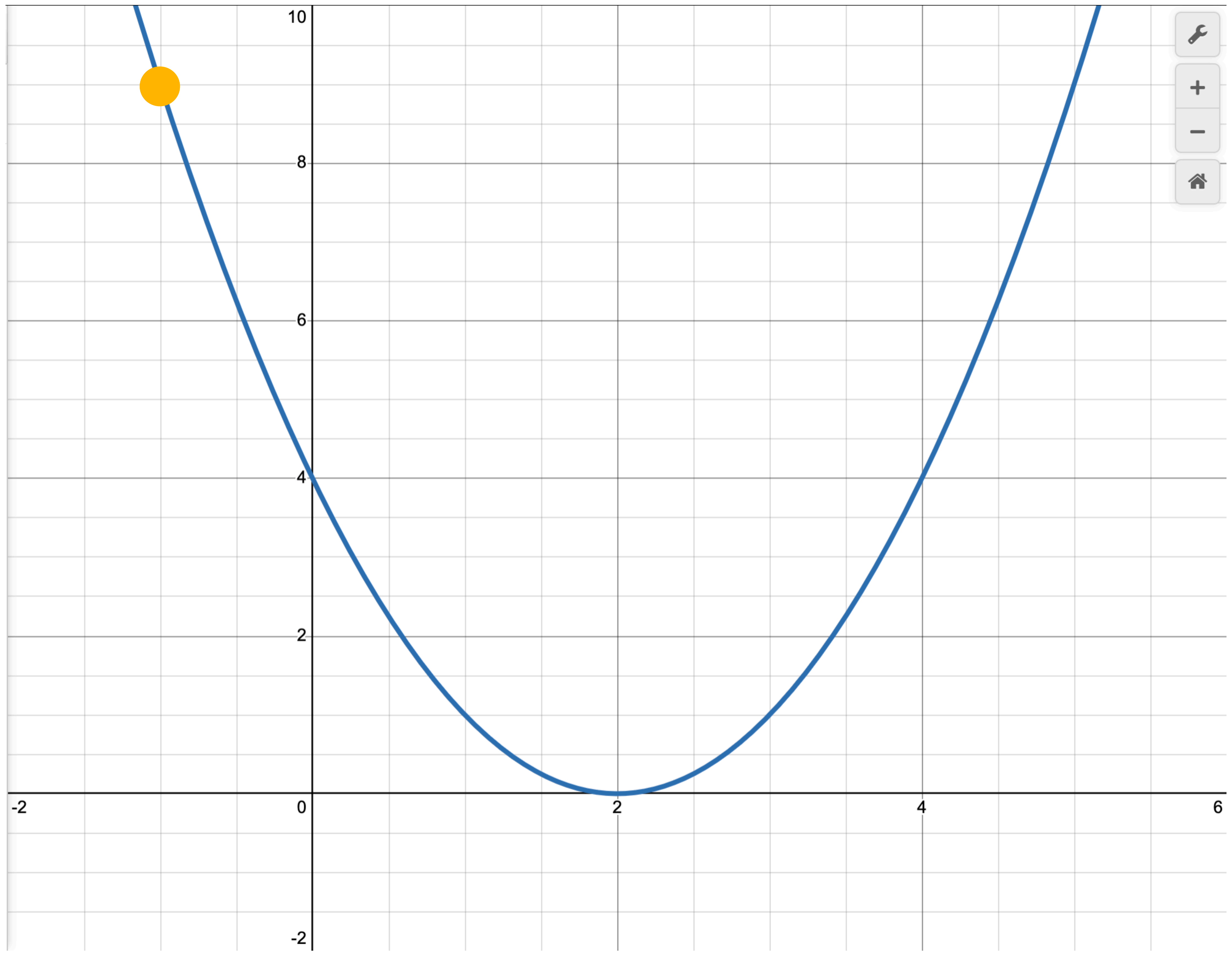

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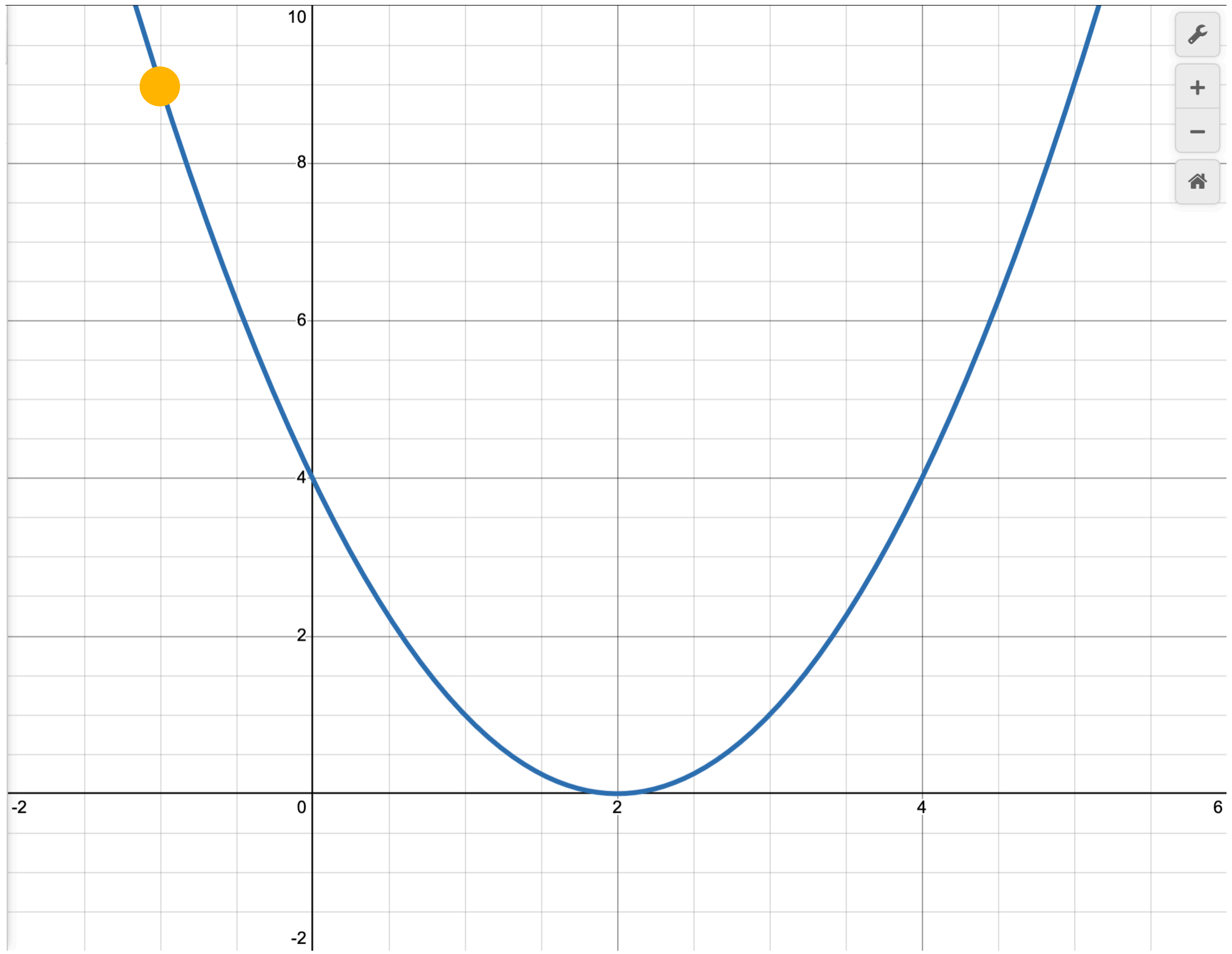

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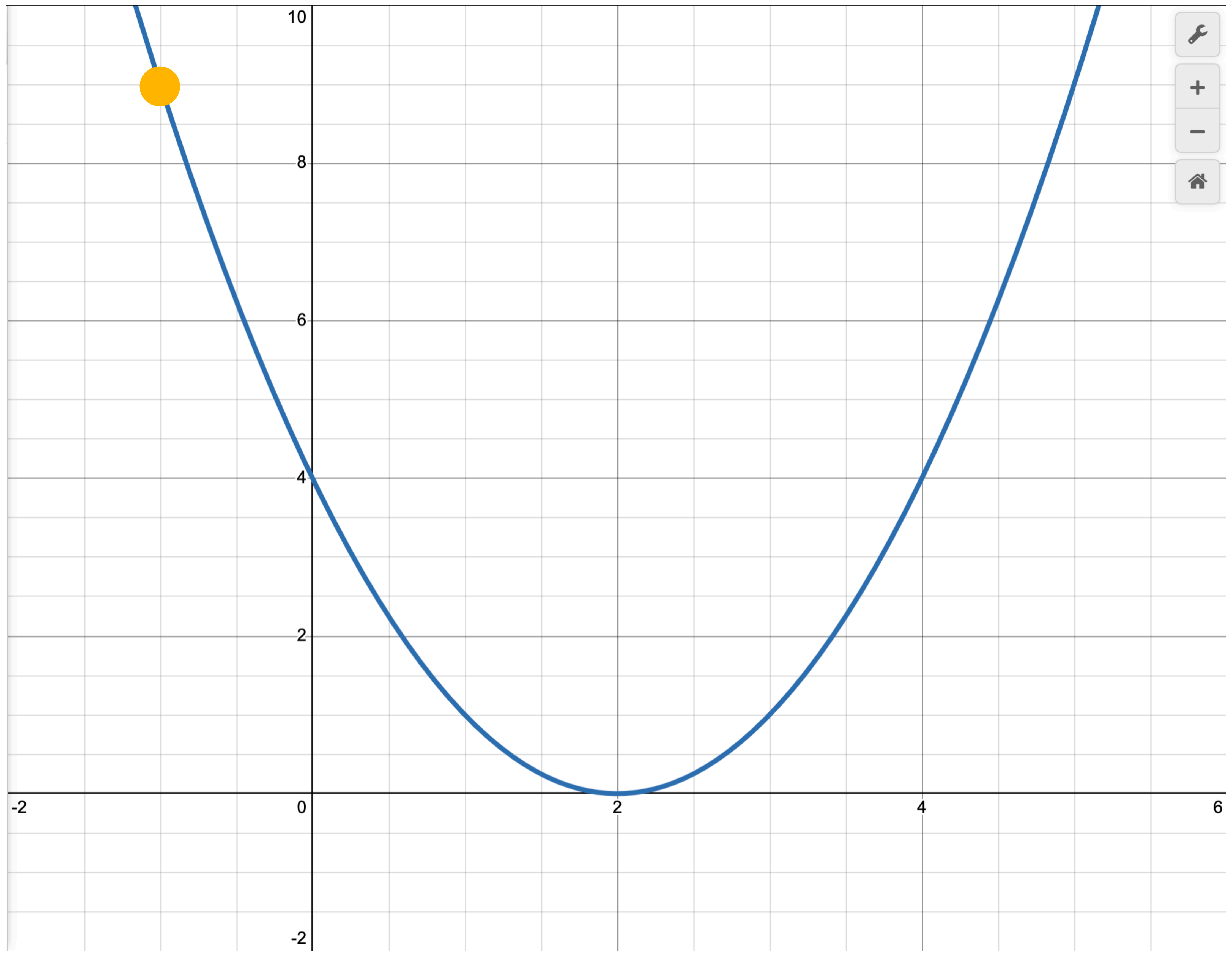

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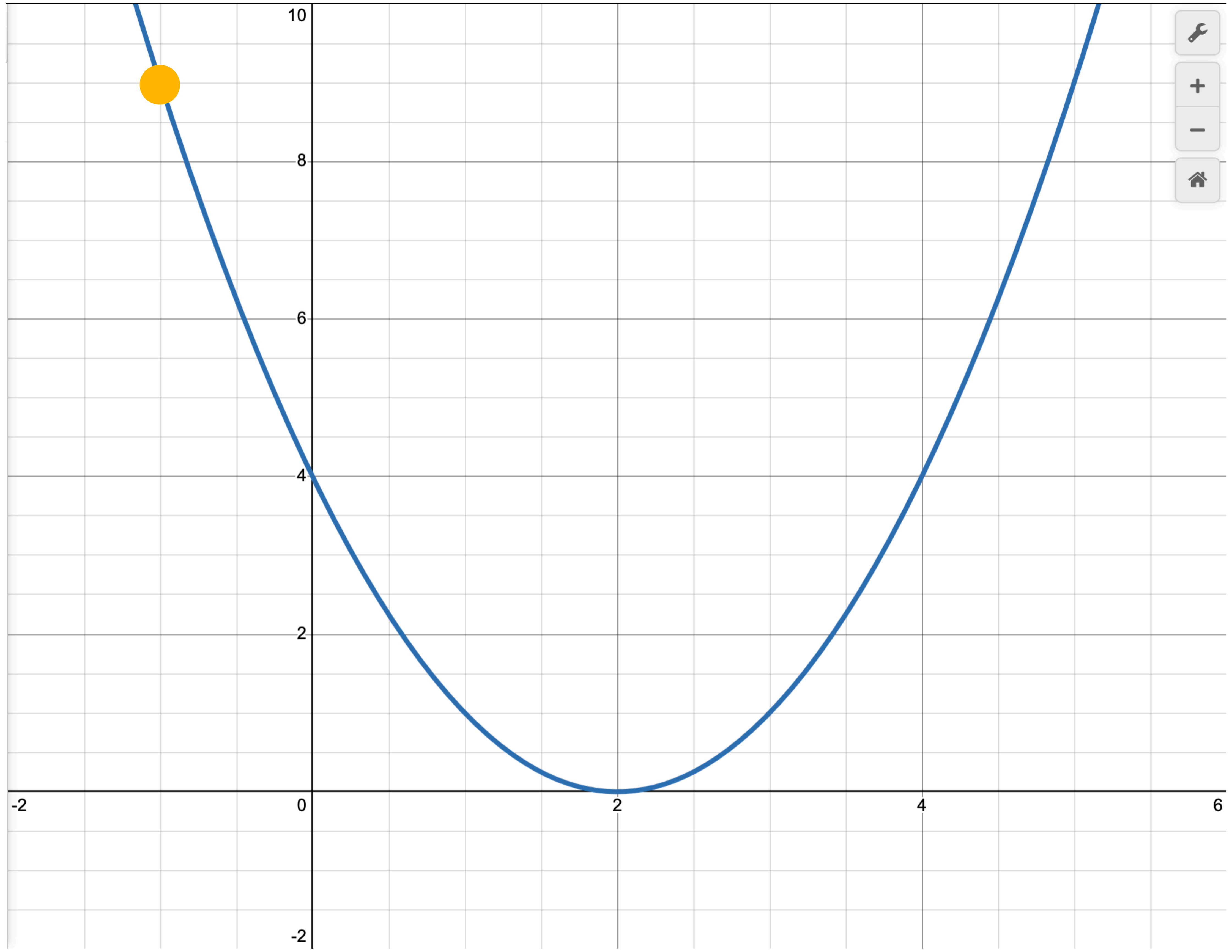

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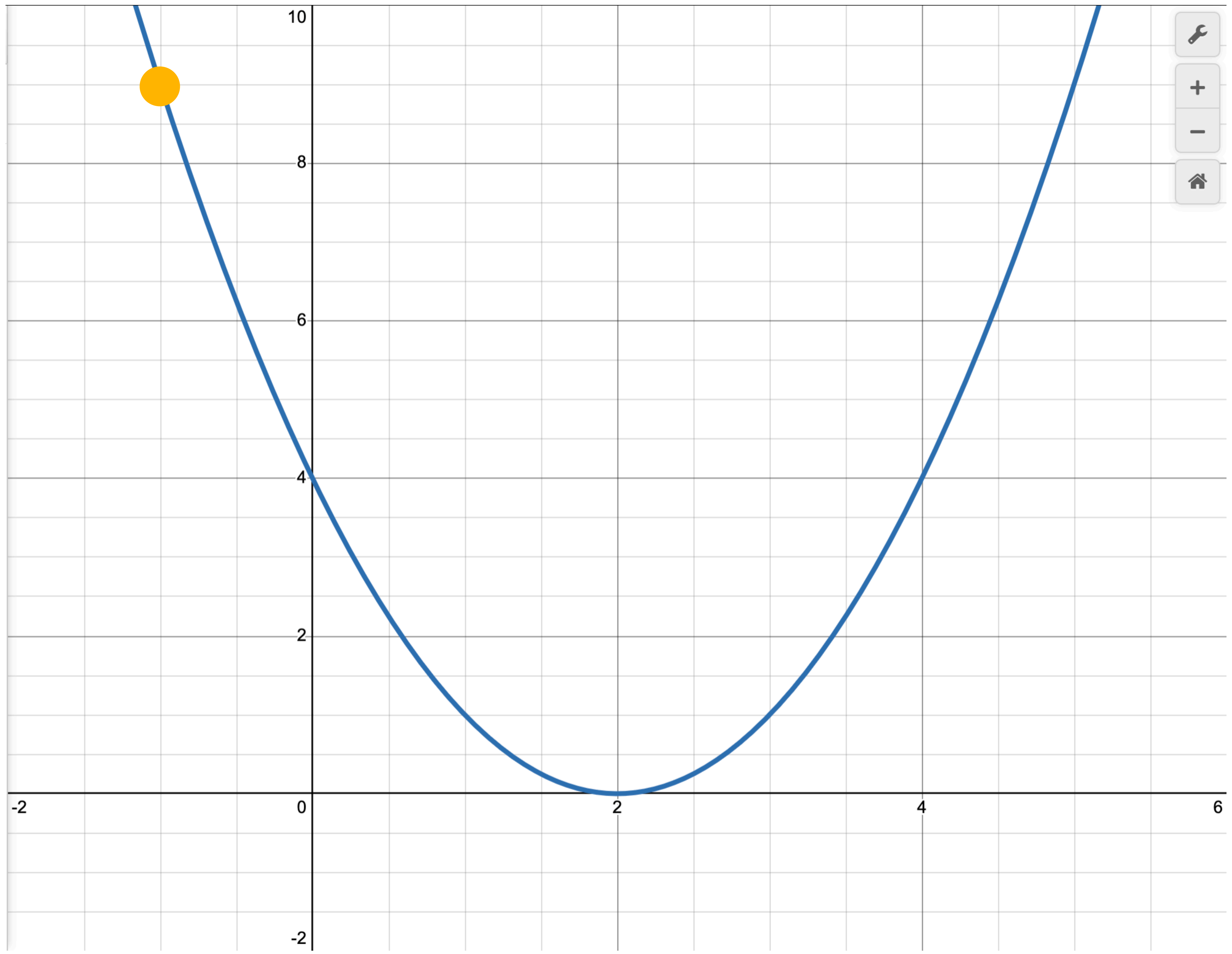

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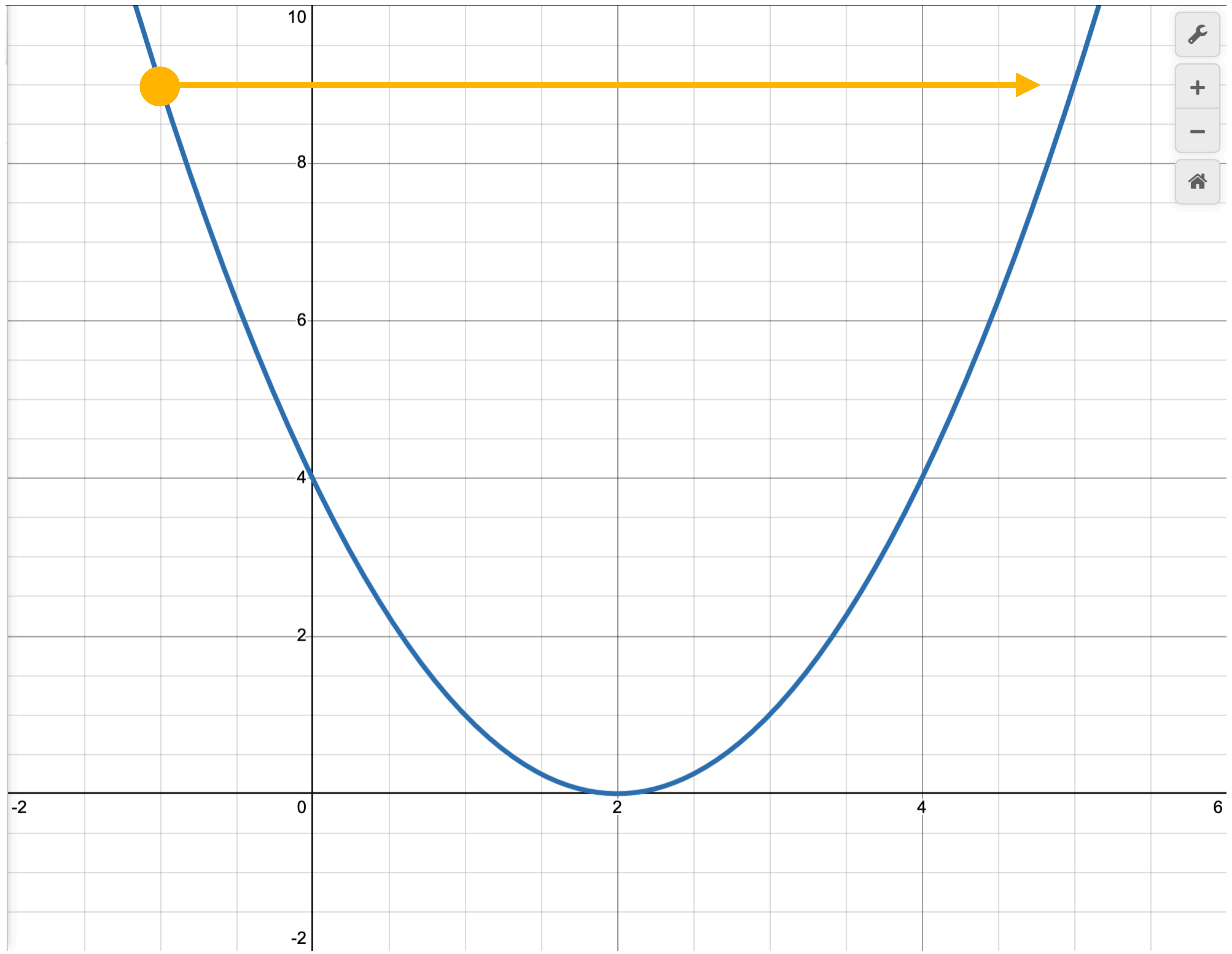

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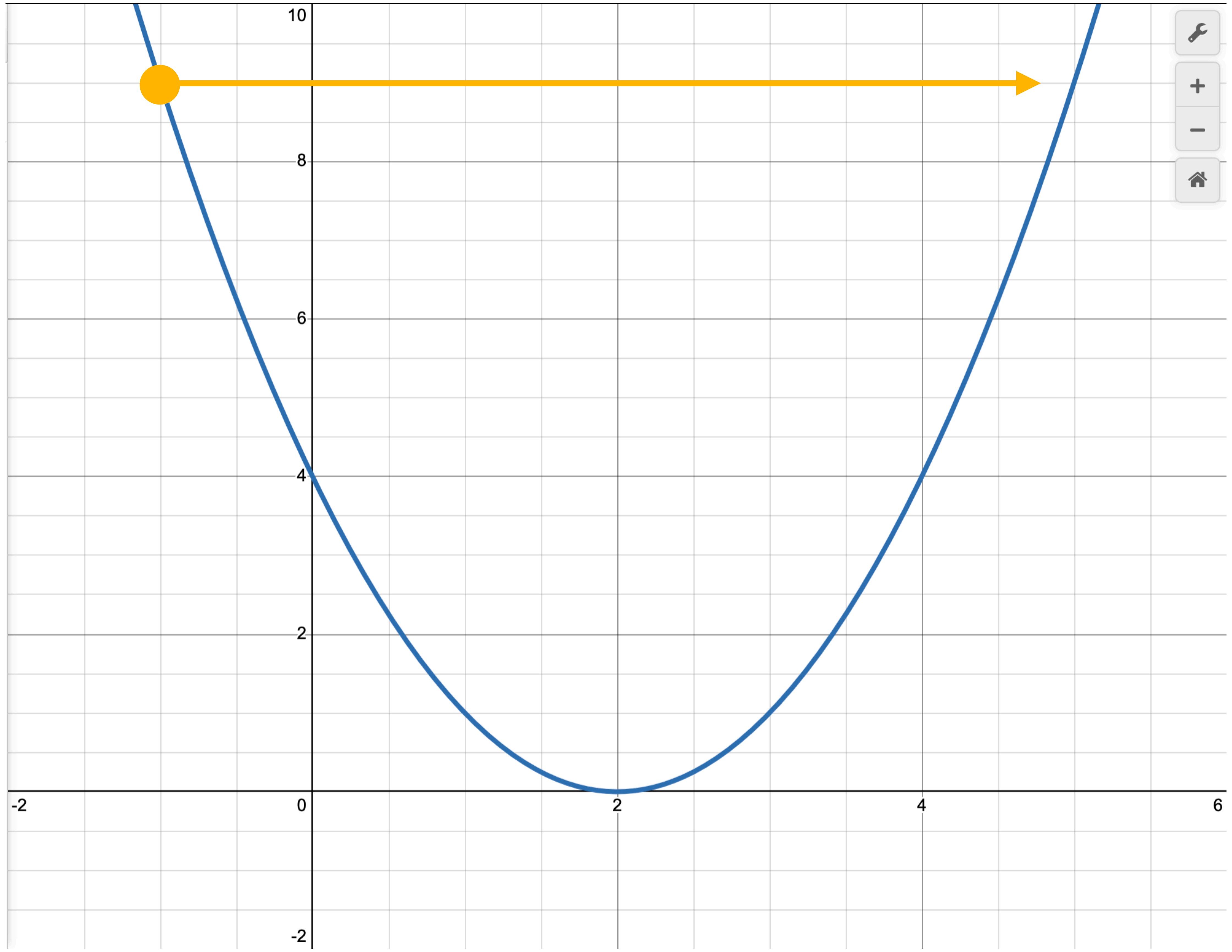

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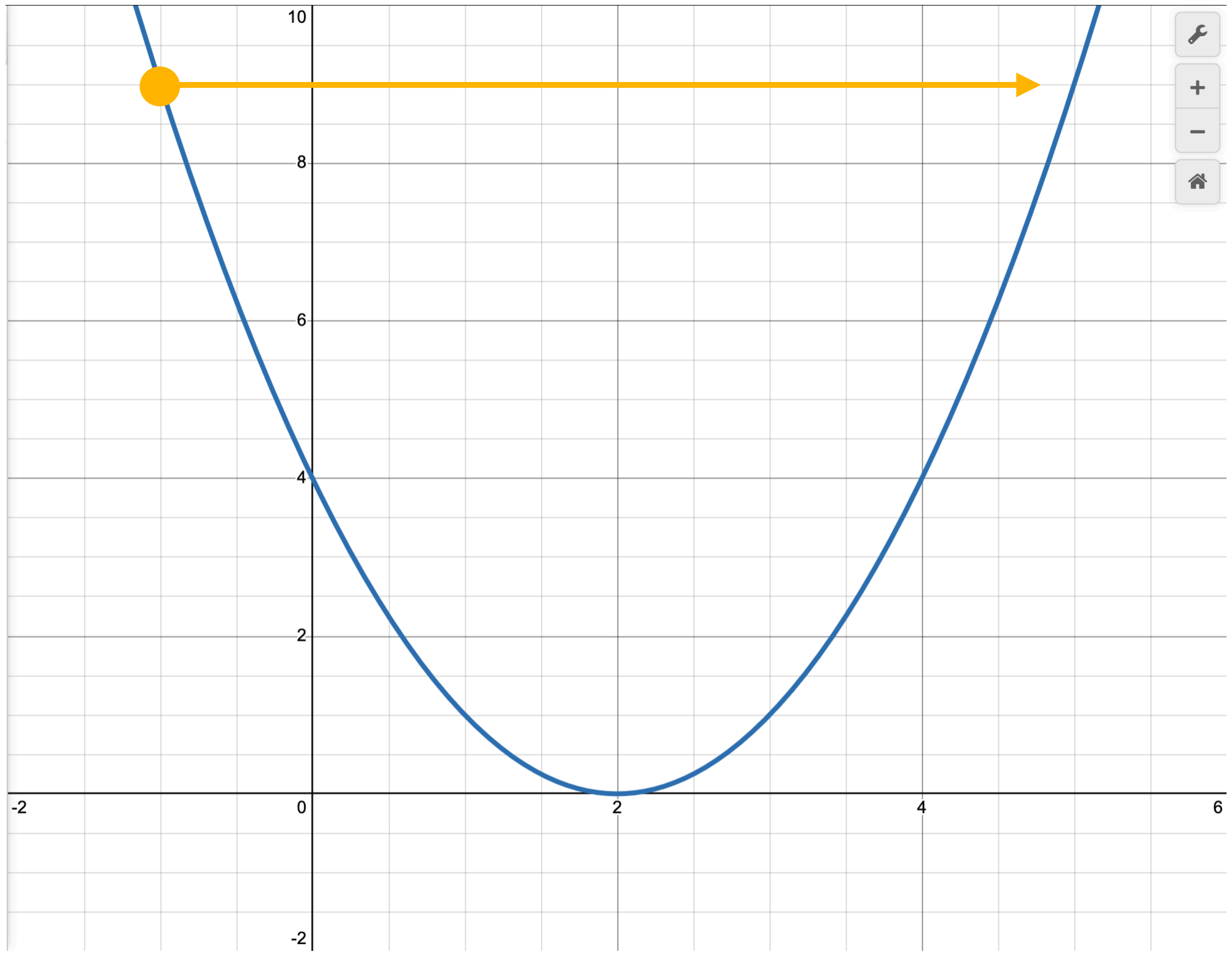
  - This process would **"bounce back and forth"** forever!



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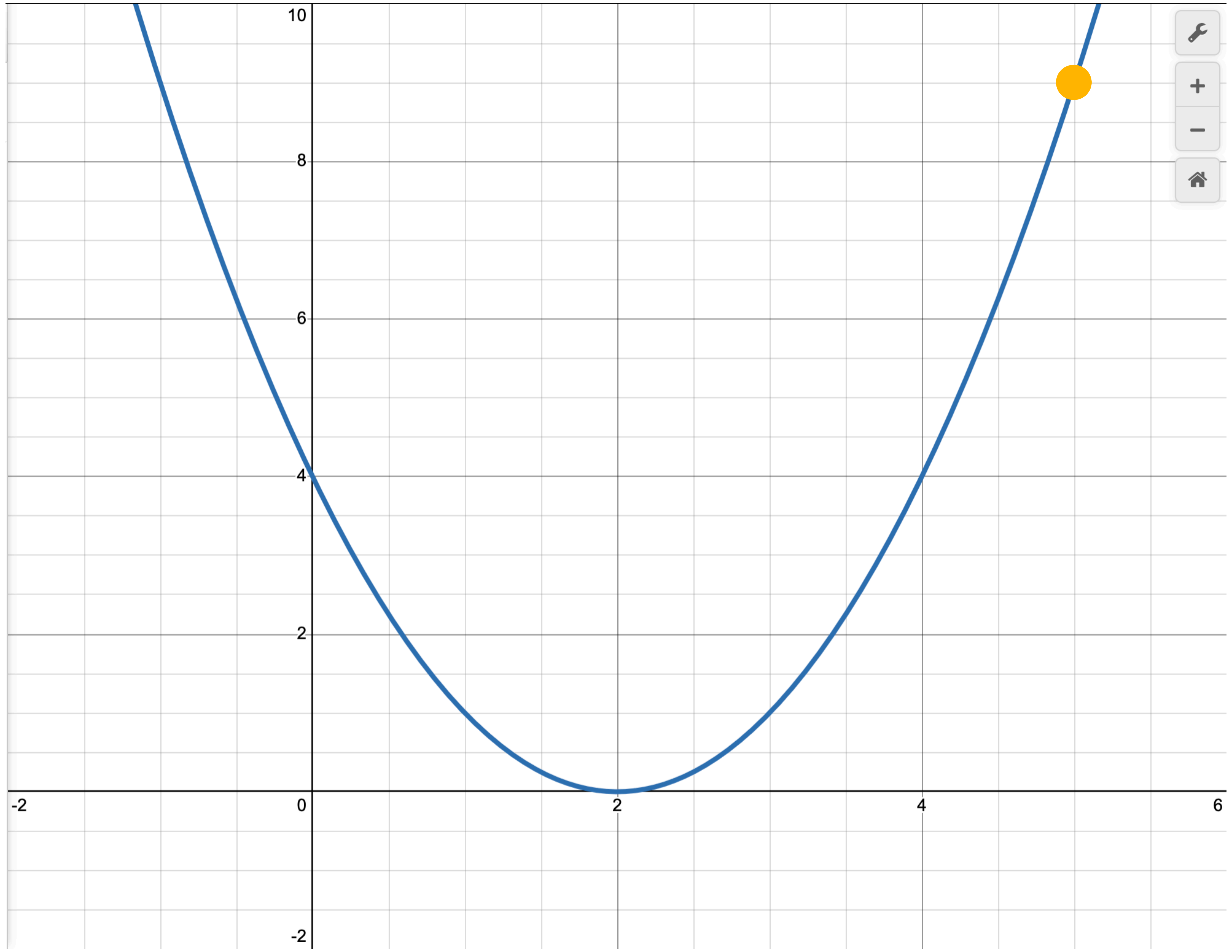

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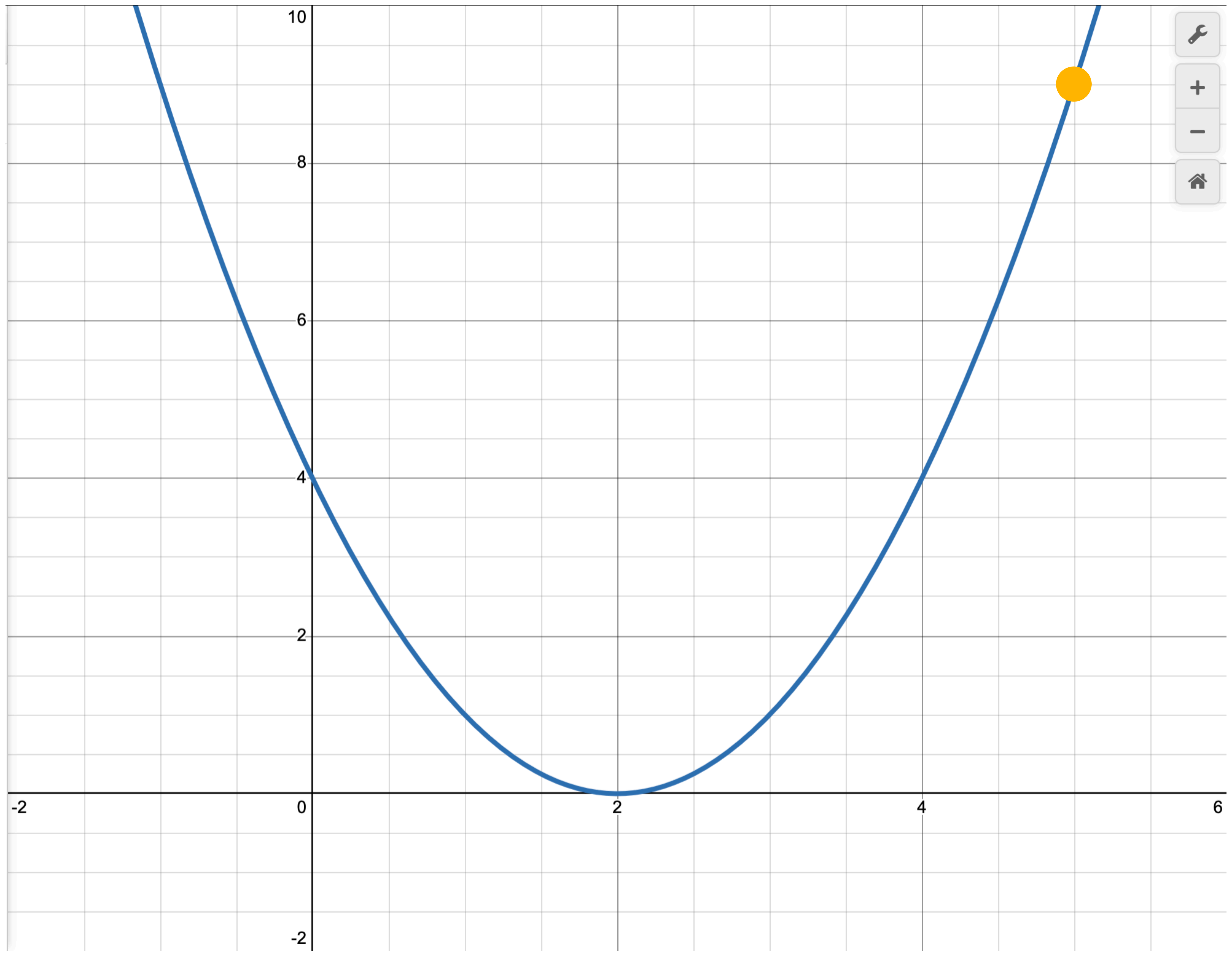

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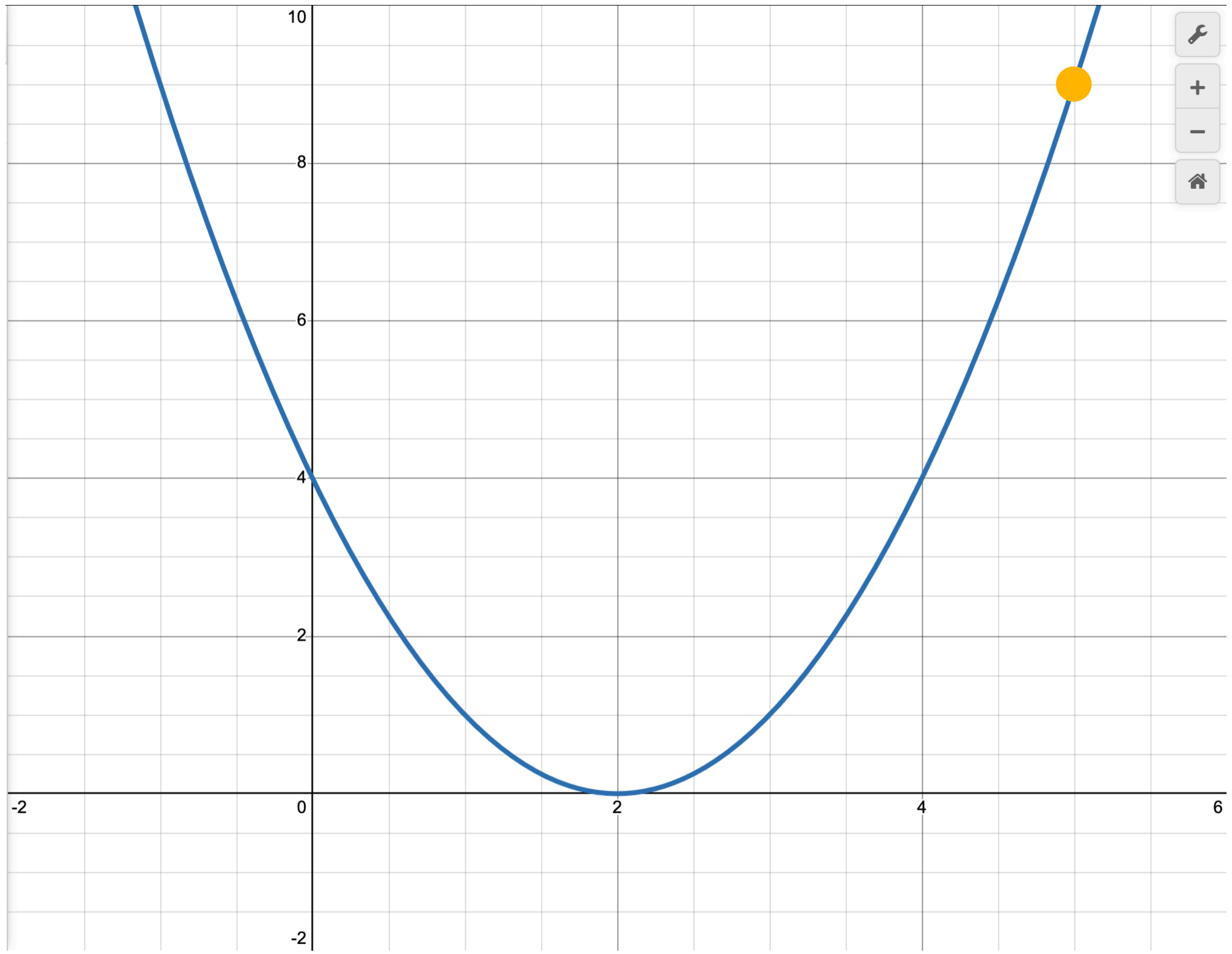

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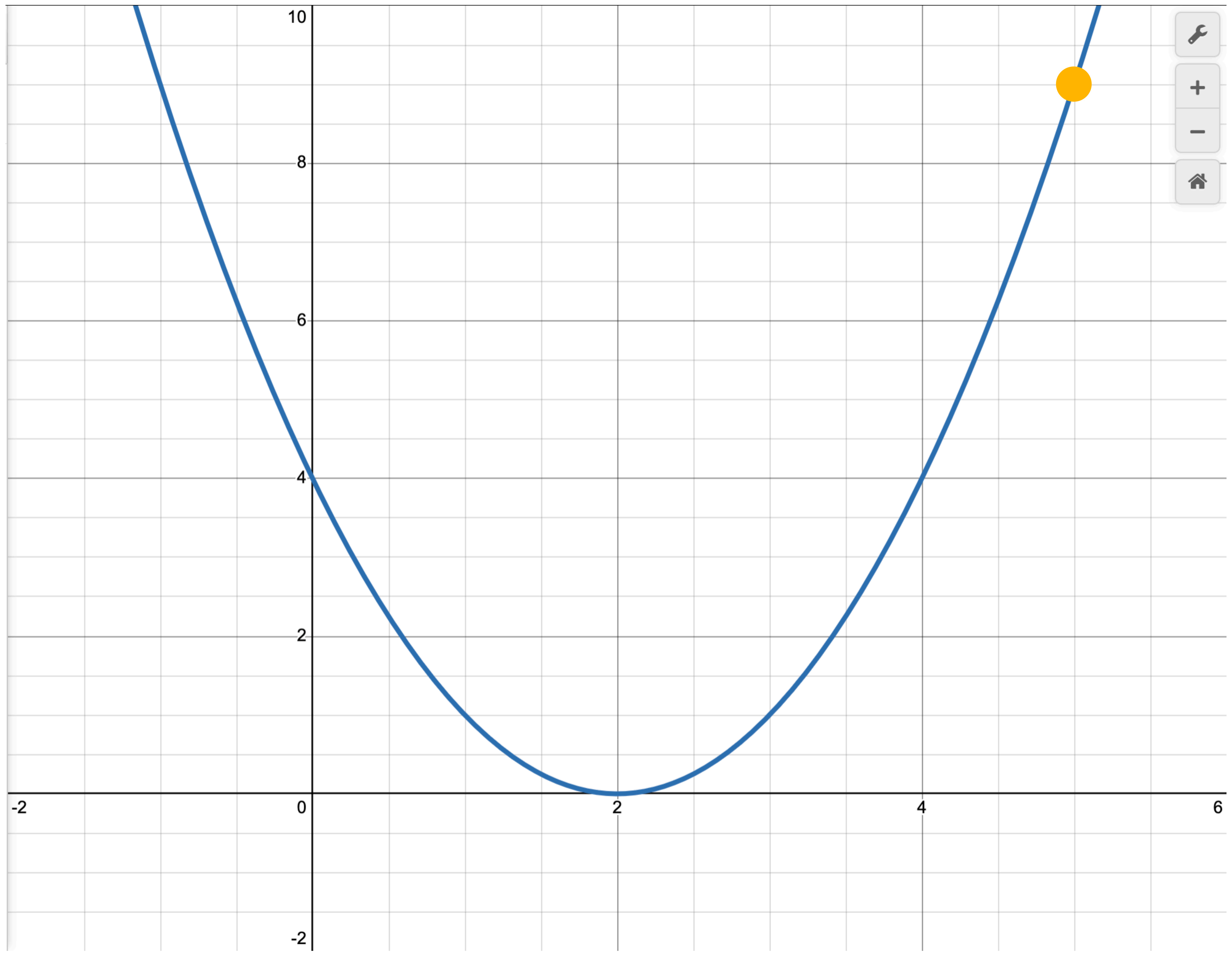

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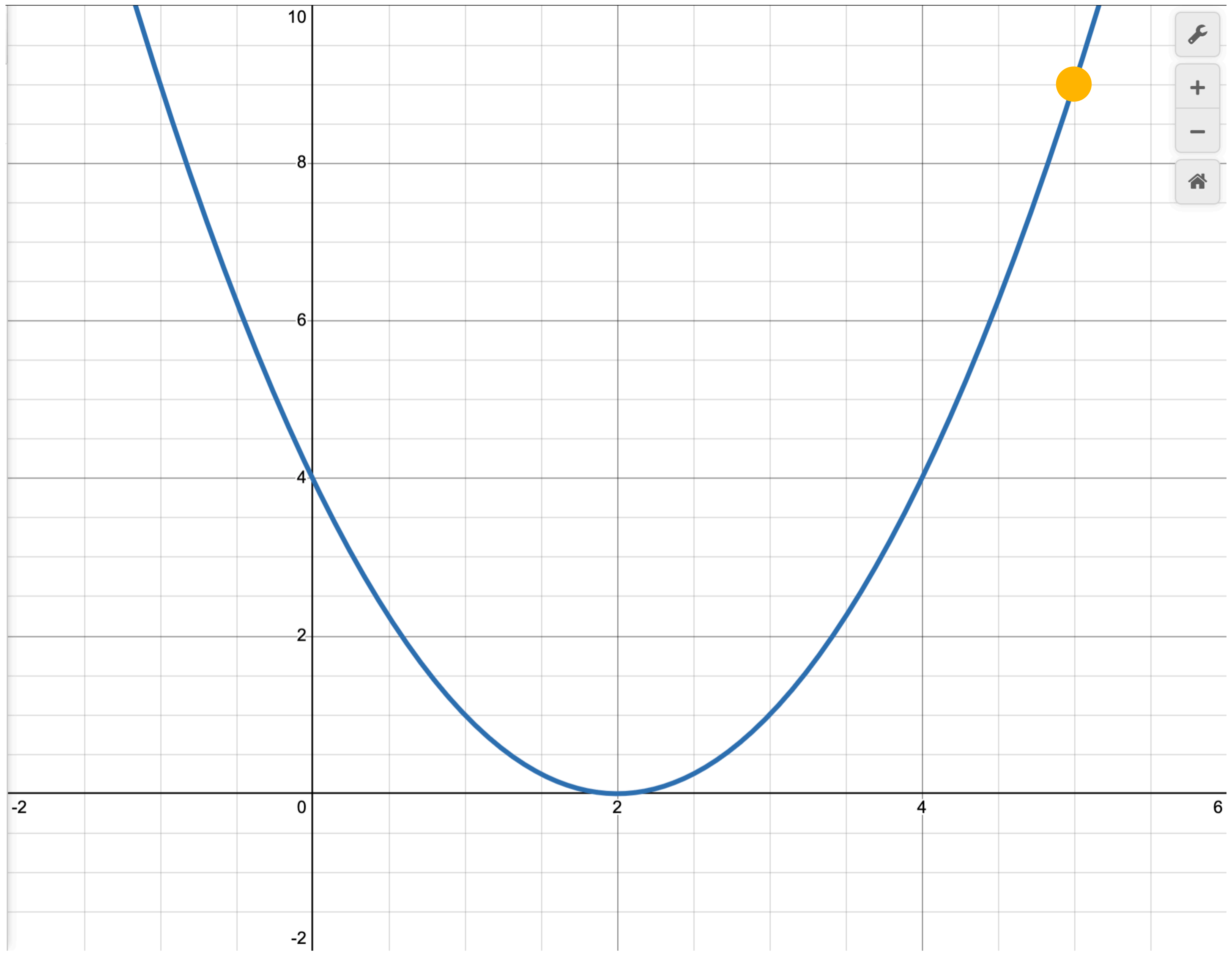

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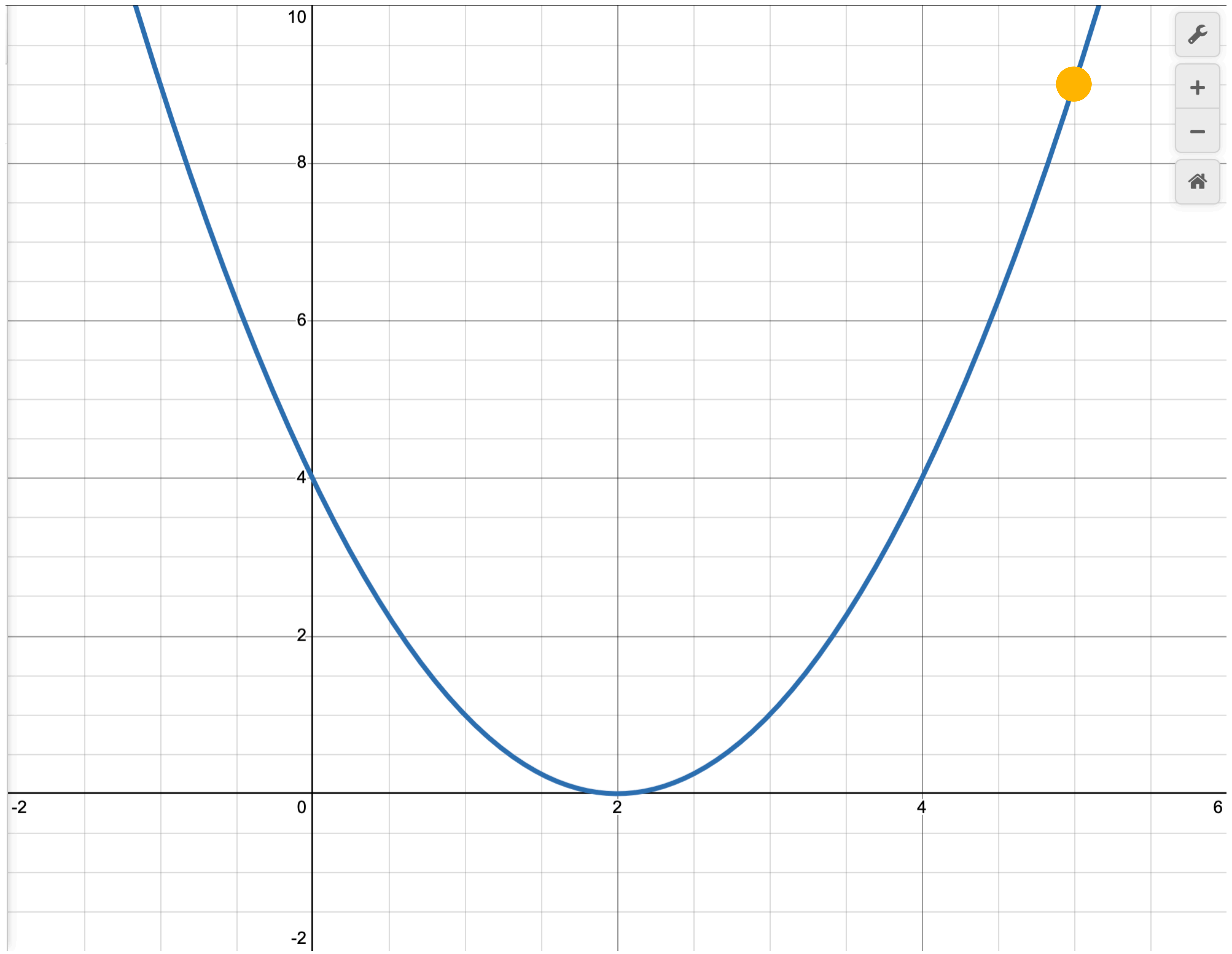

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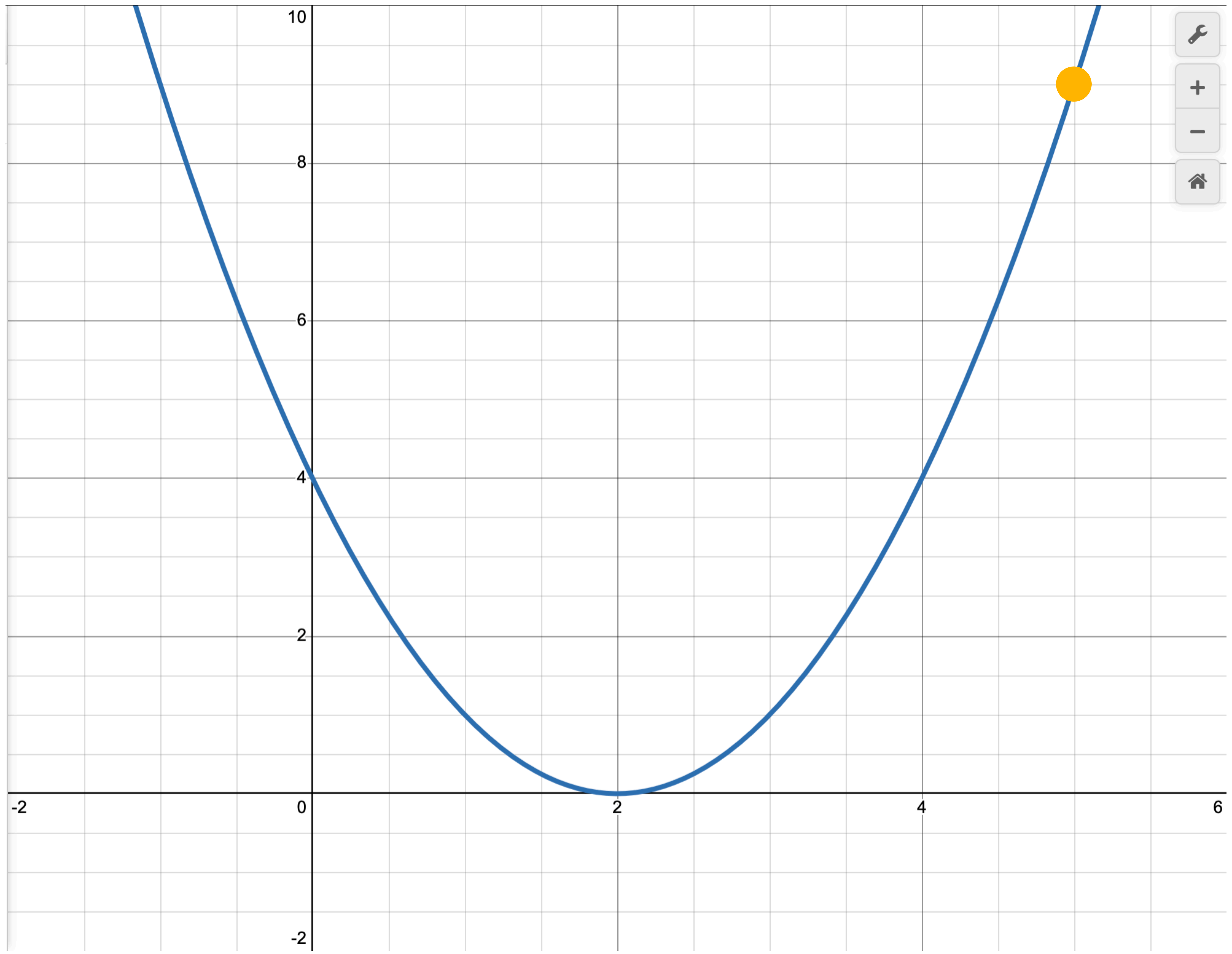

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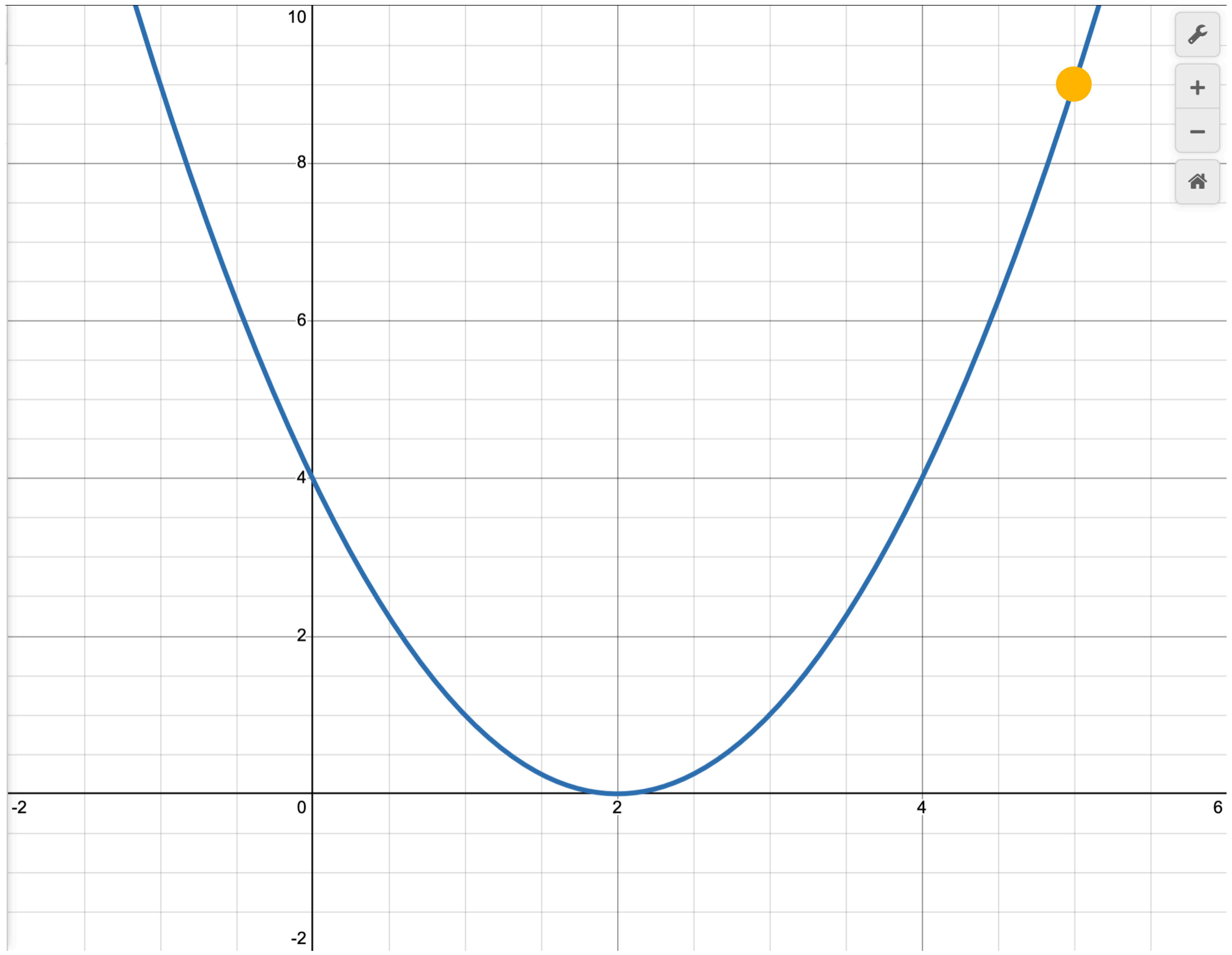

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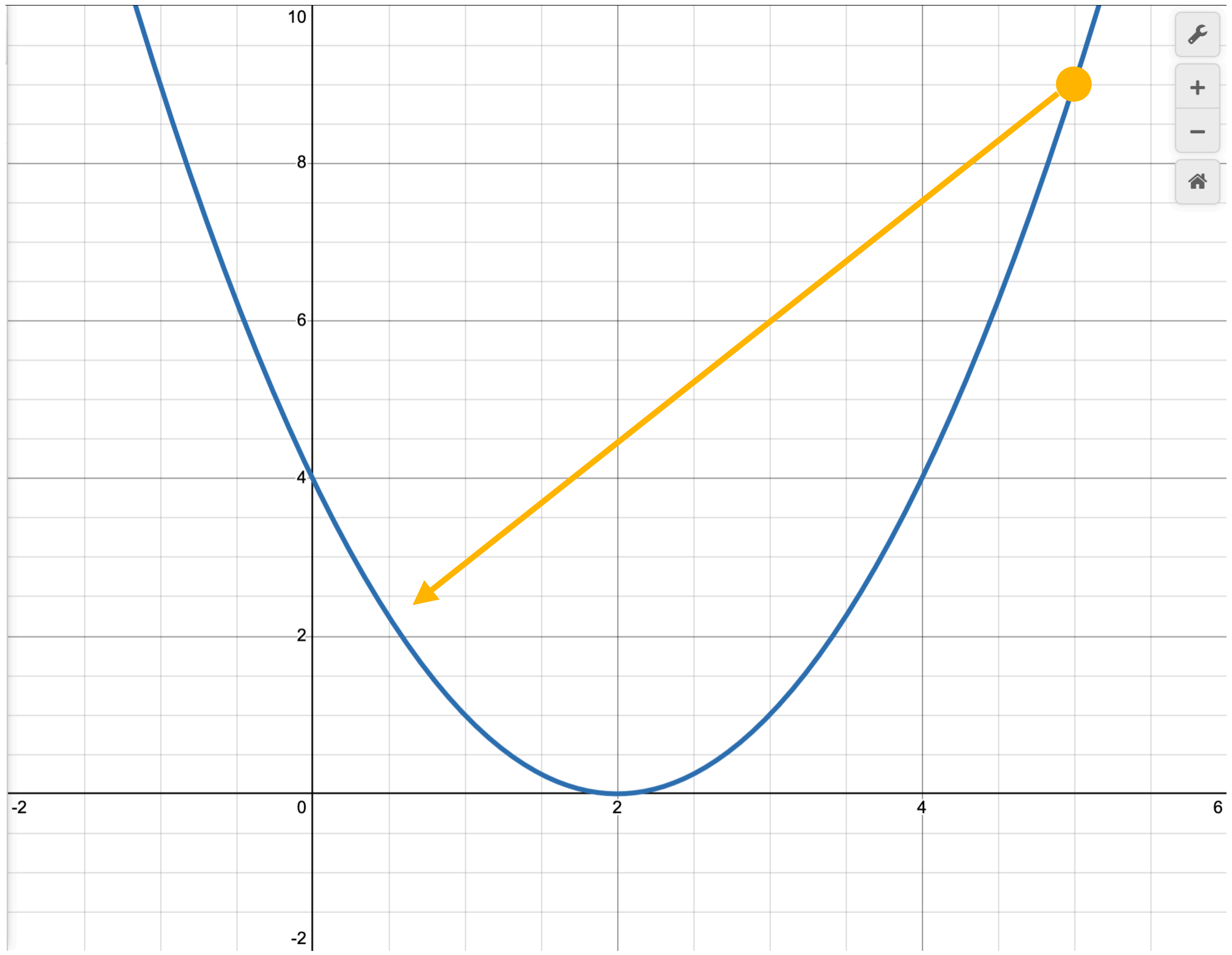

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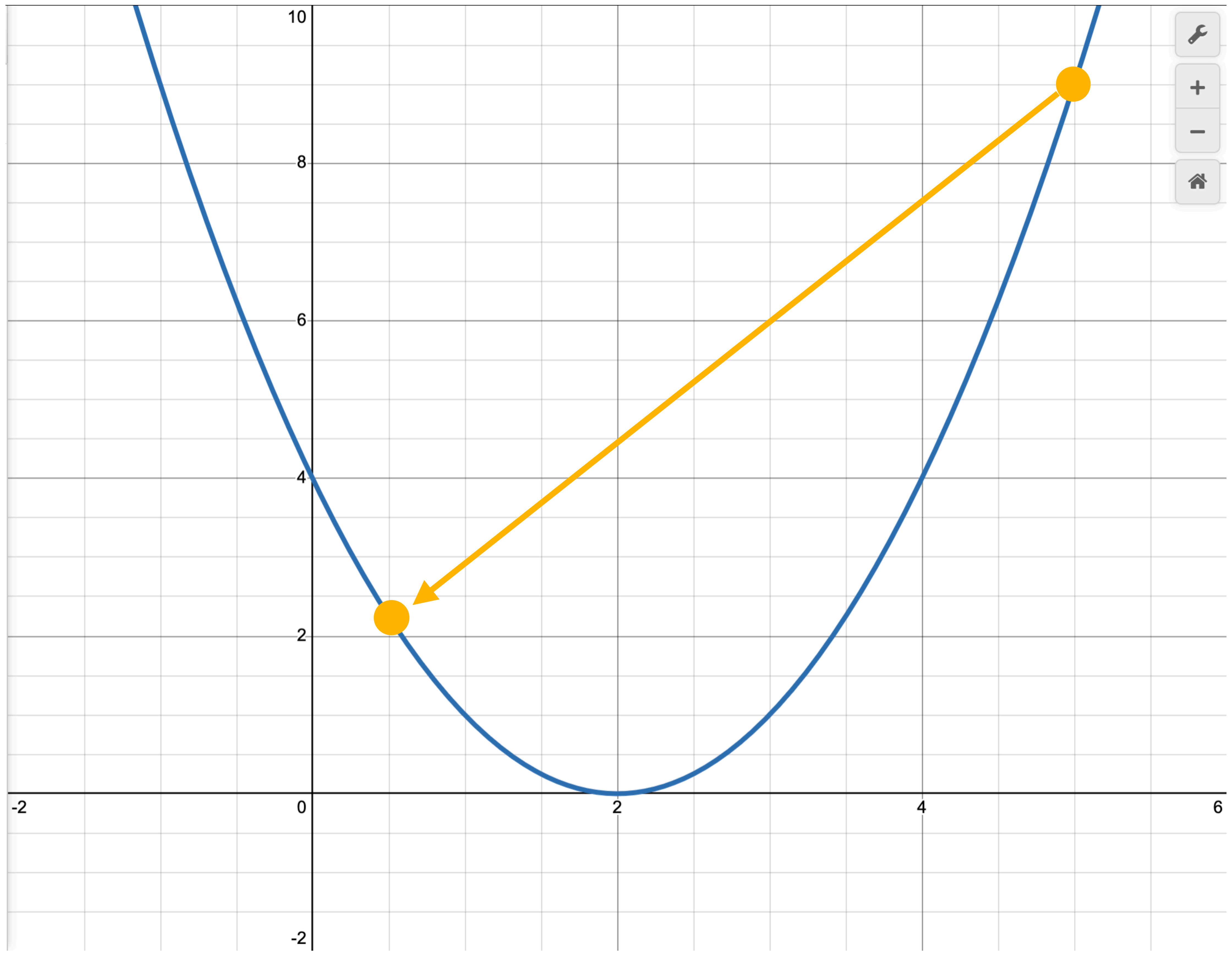

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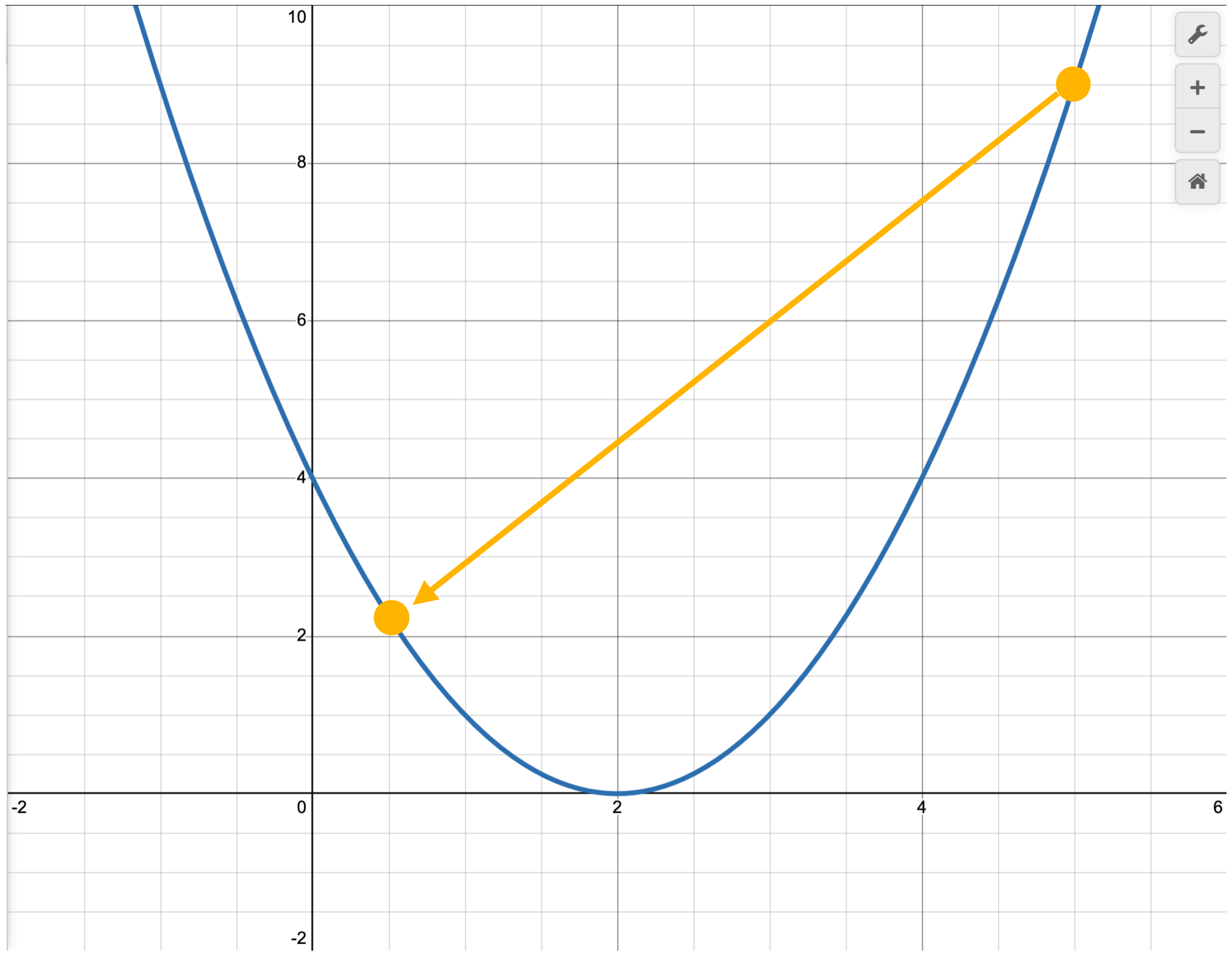

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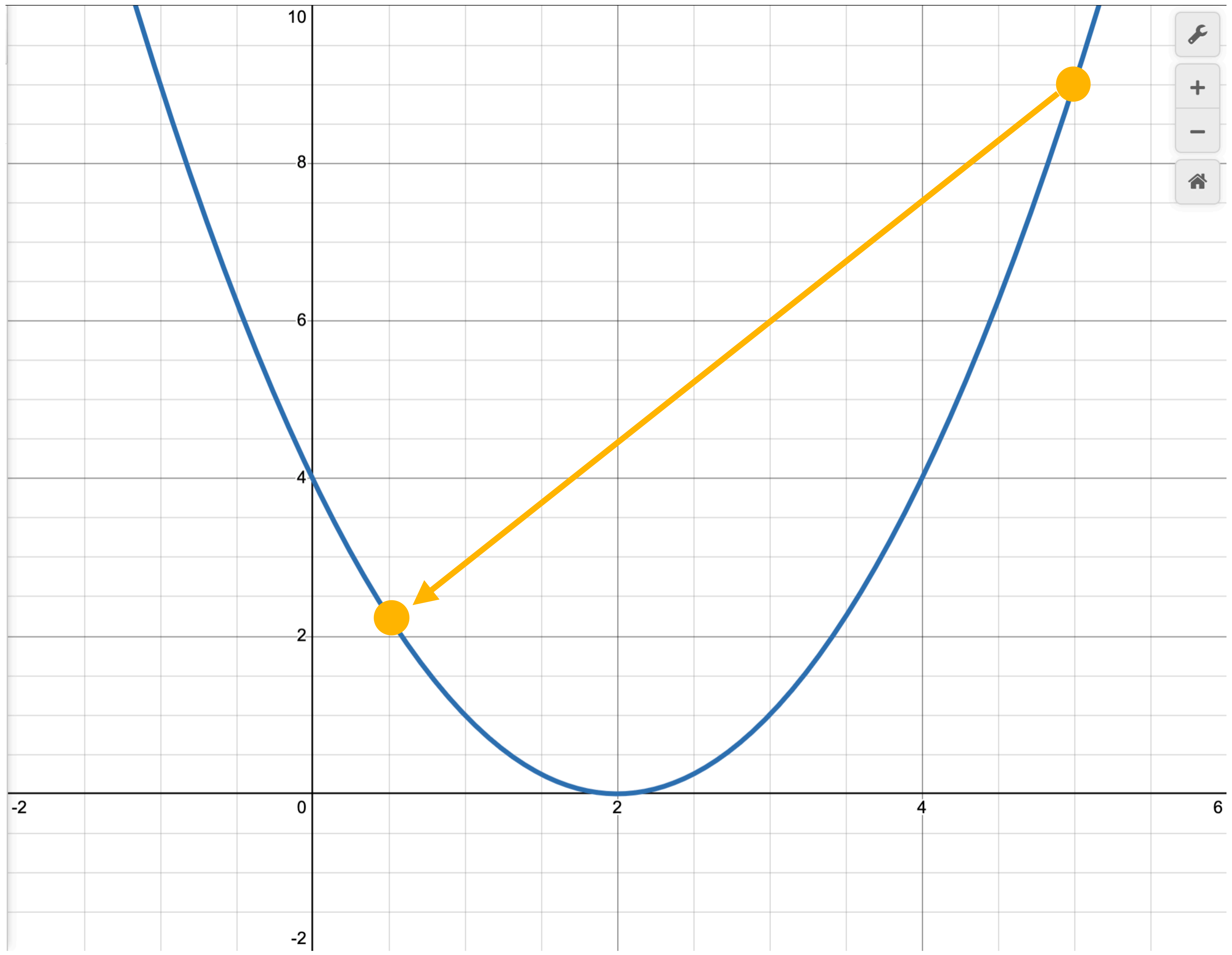

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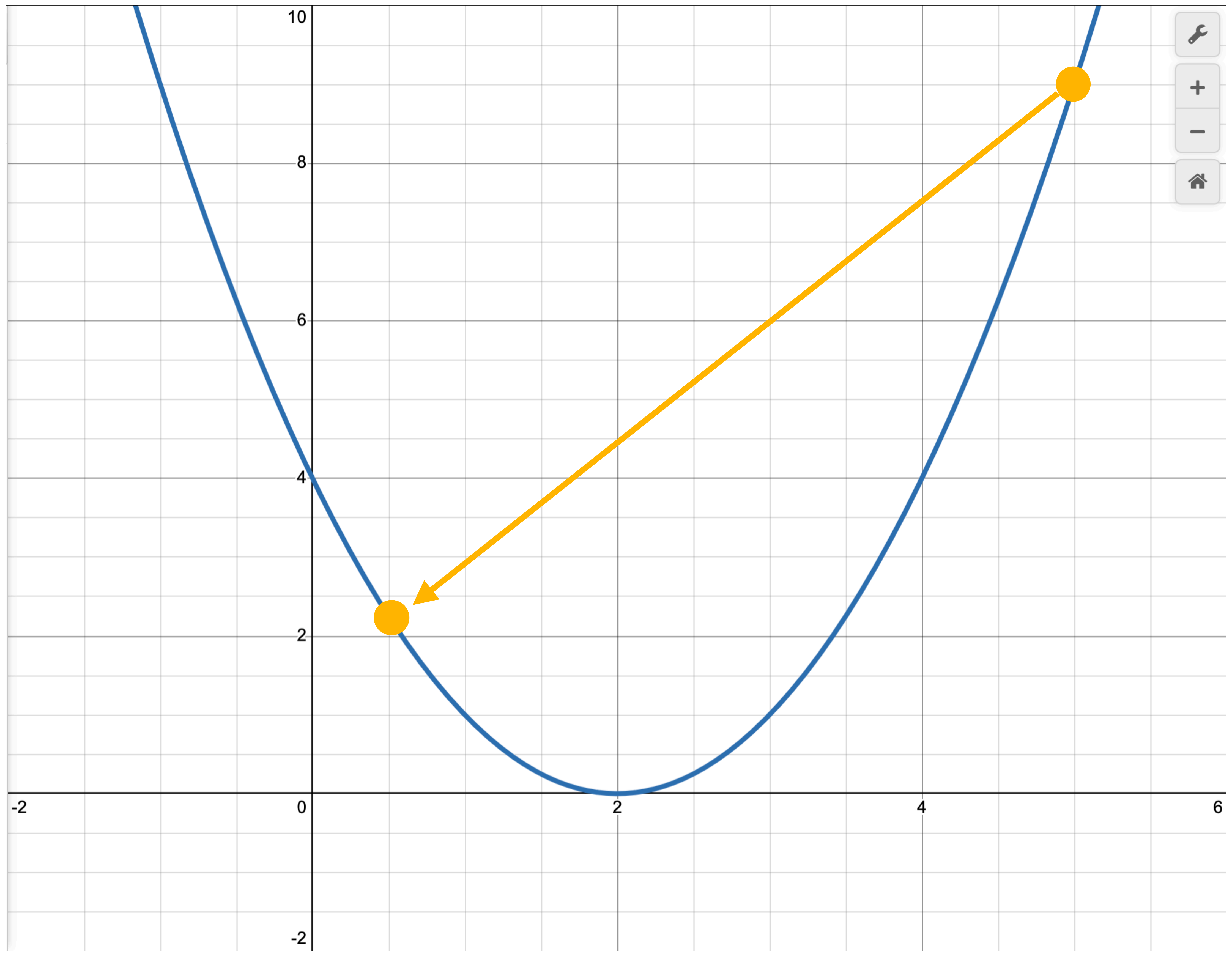

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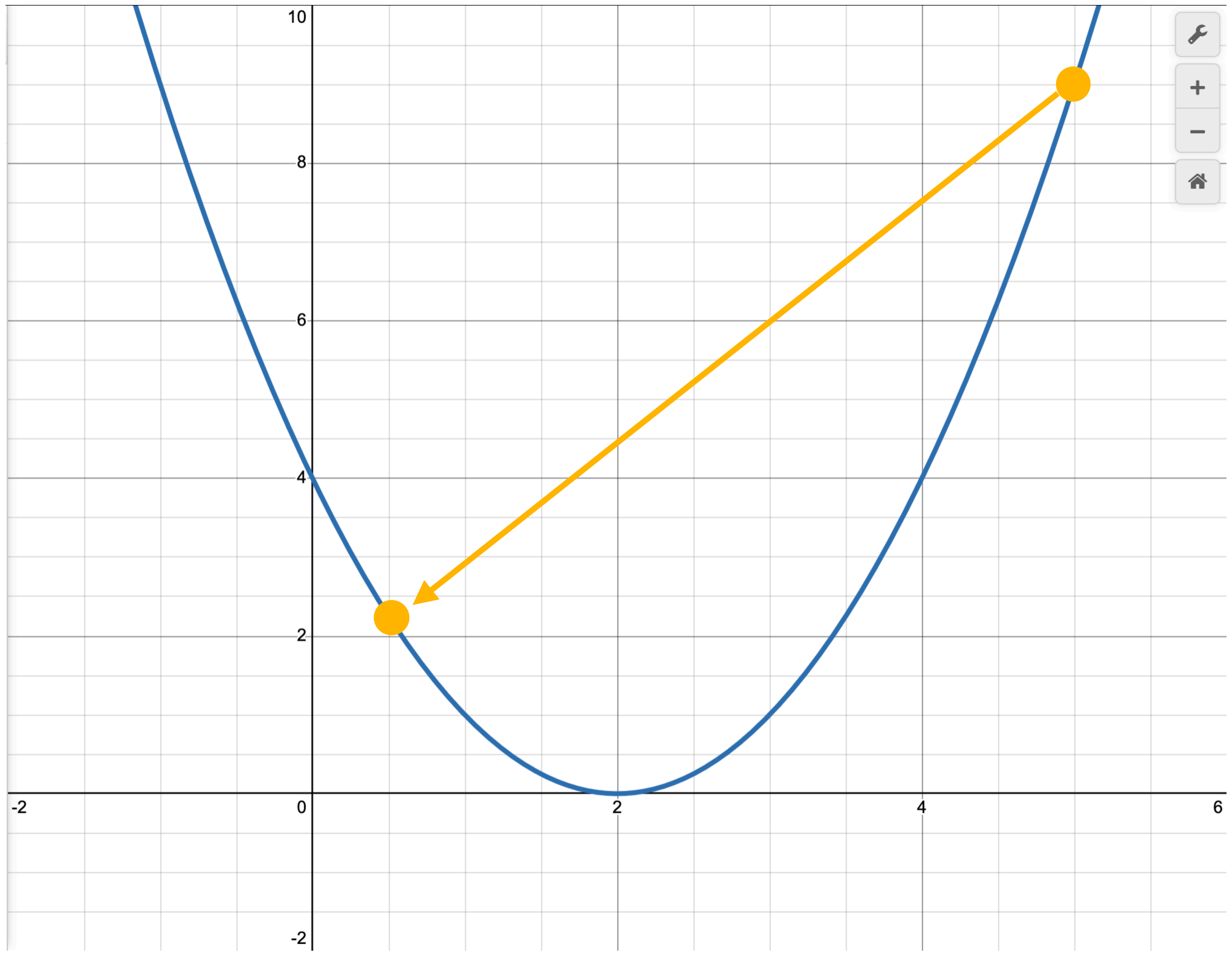

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- Second step:
  - Plug in θ_1 to derivative:
    - $2 \cdot 0.5 - 4 = -3$
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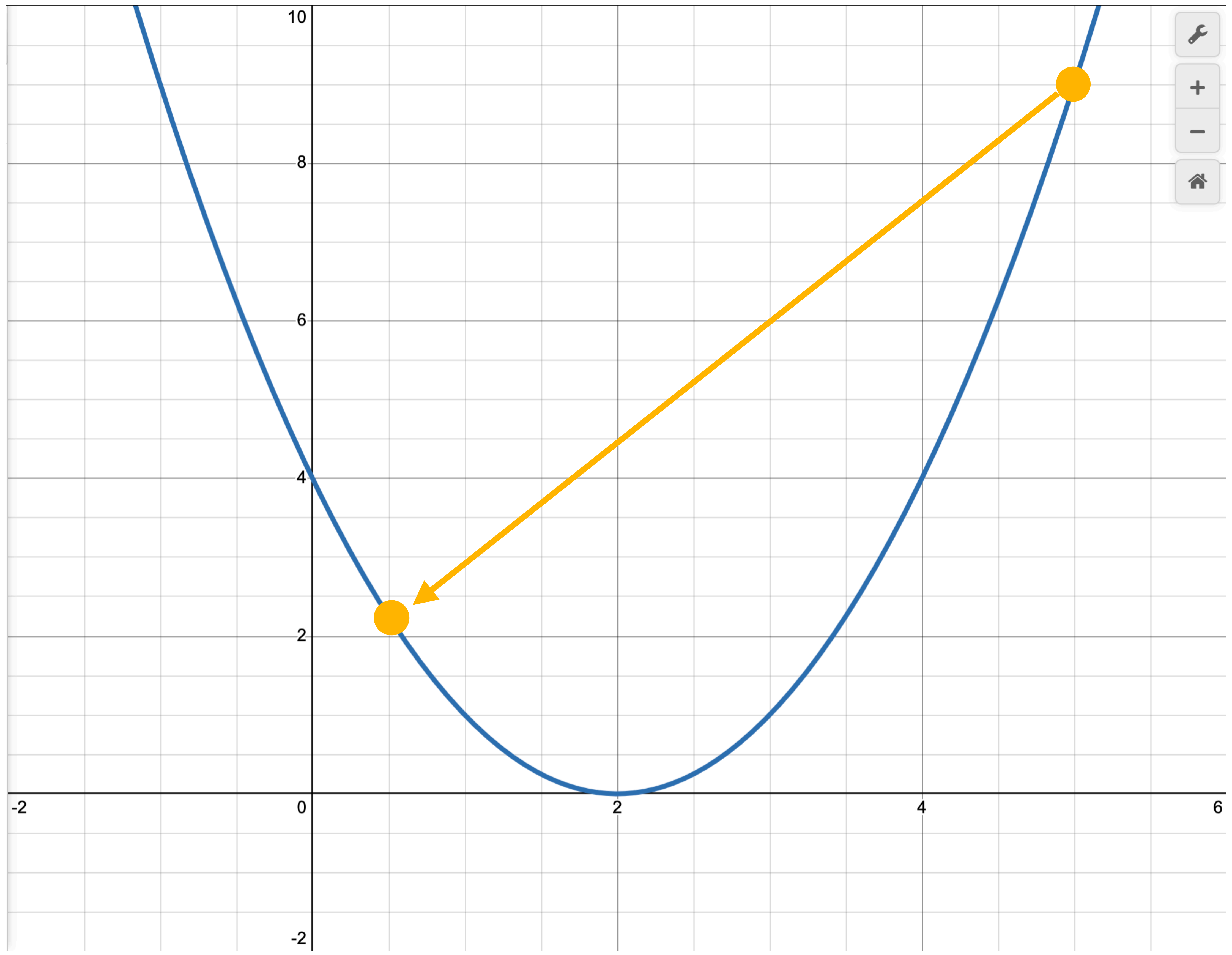

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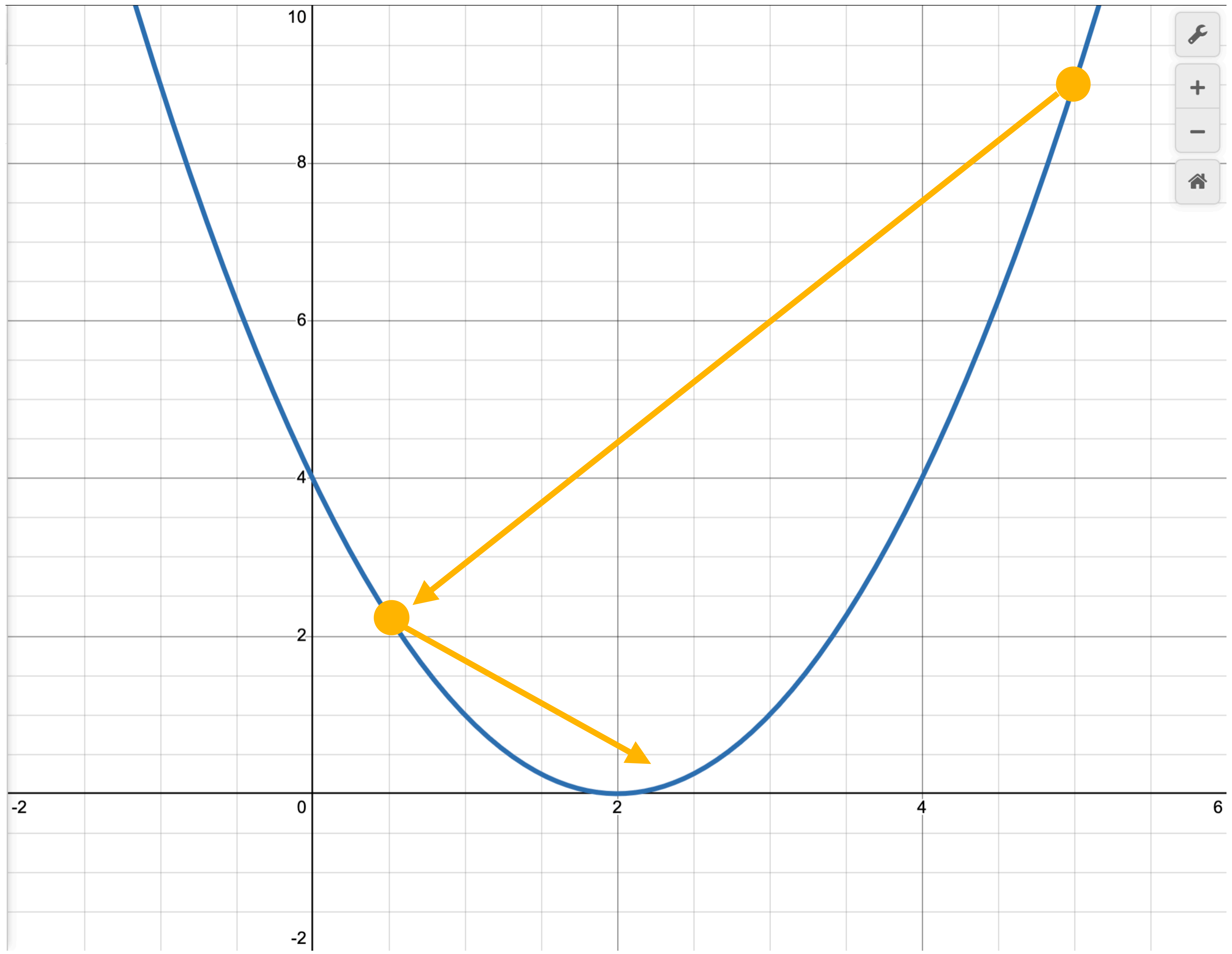

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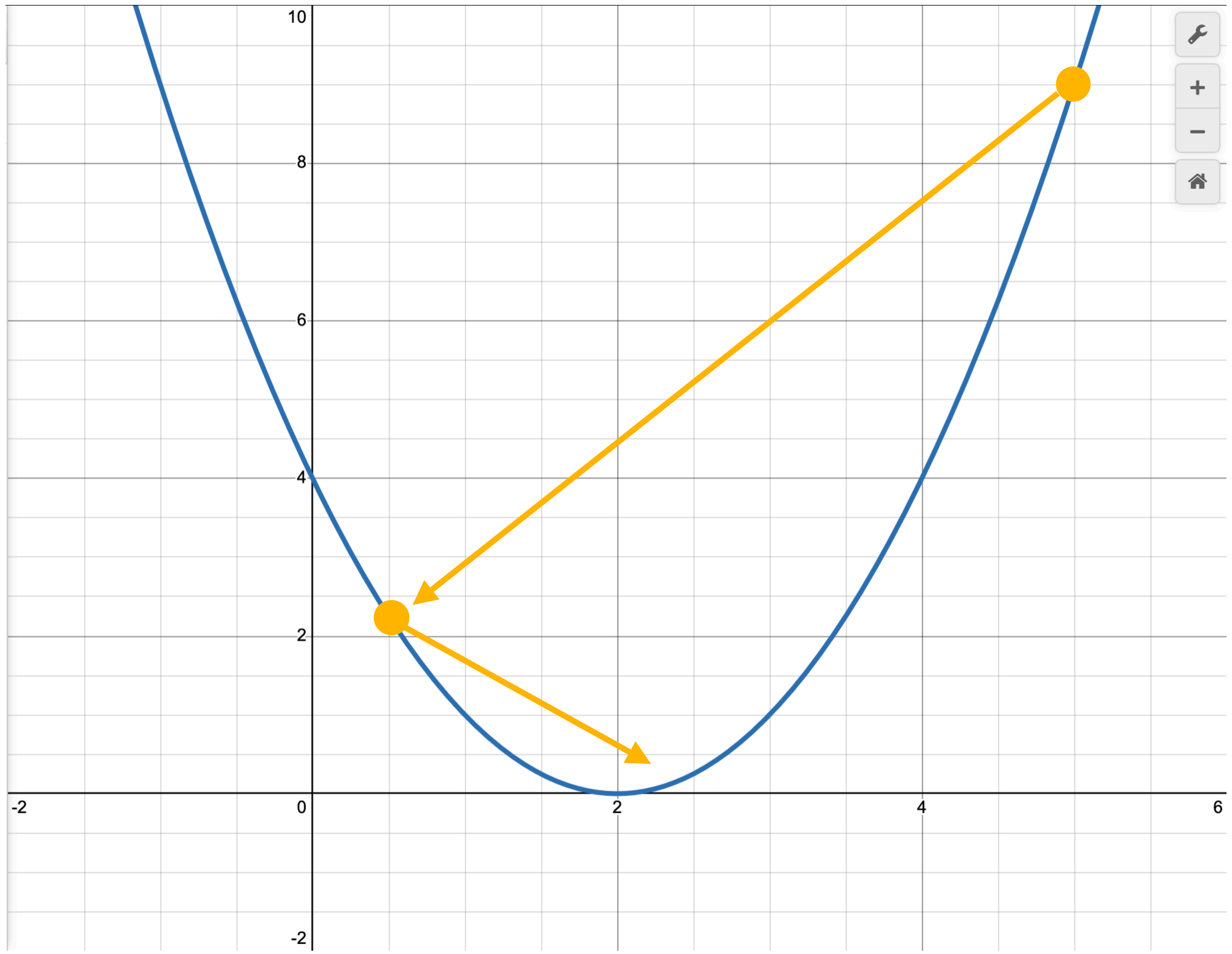
- (This is looking better)



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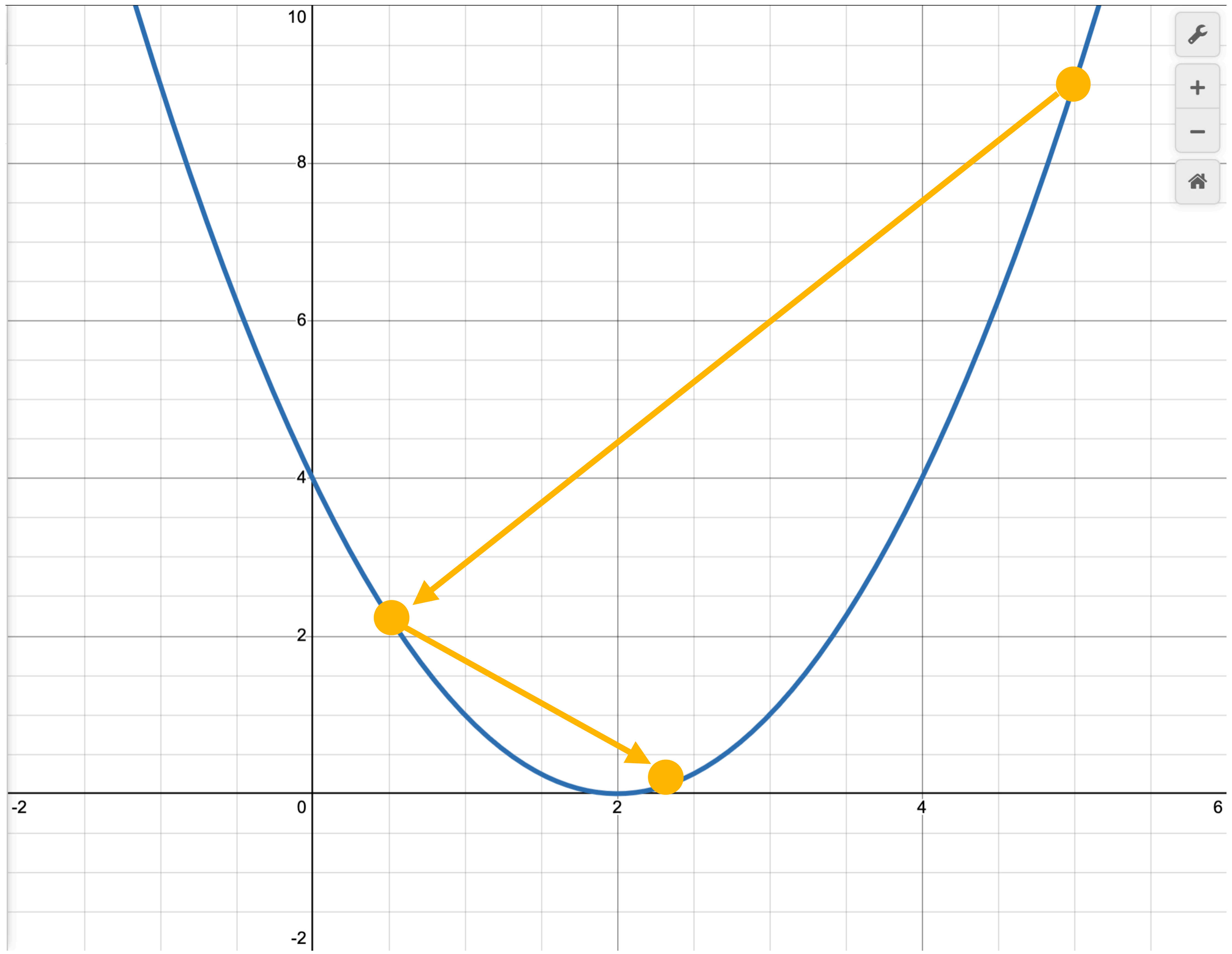

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- Third step:
  - Plug in θ_2 to derivative:
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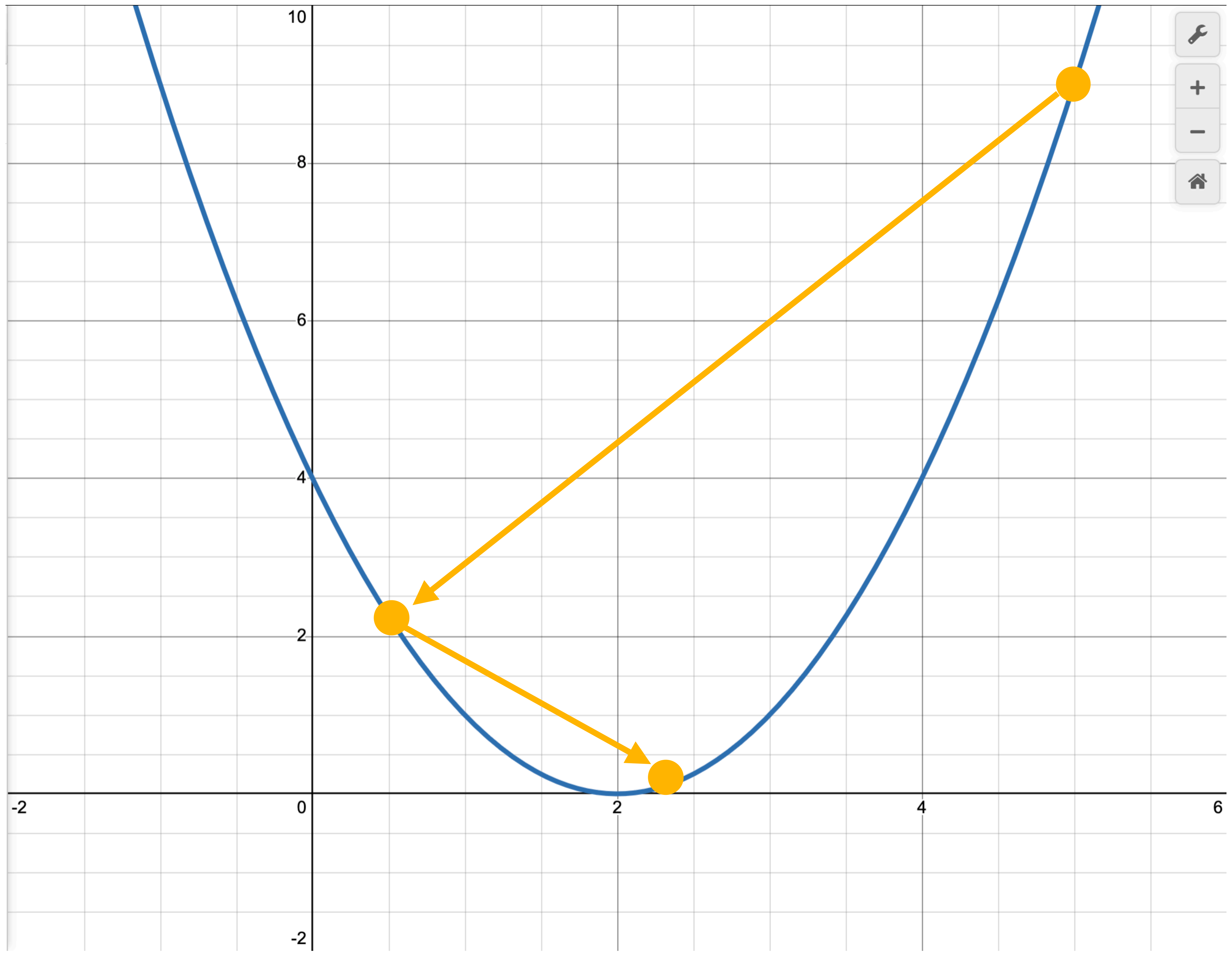

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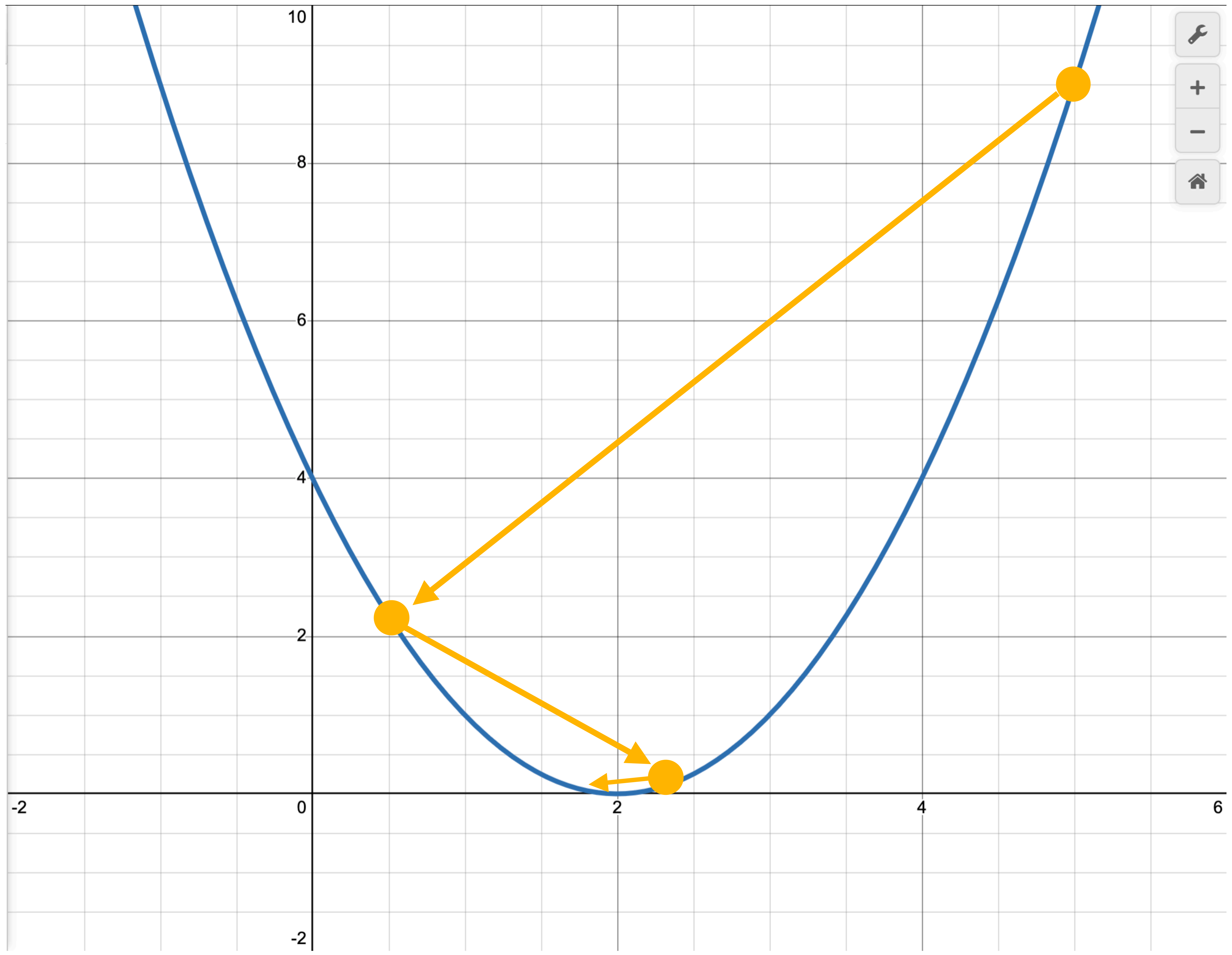

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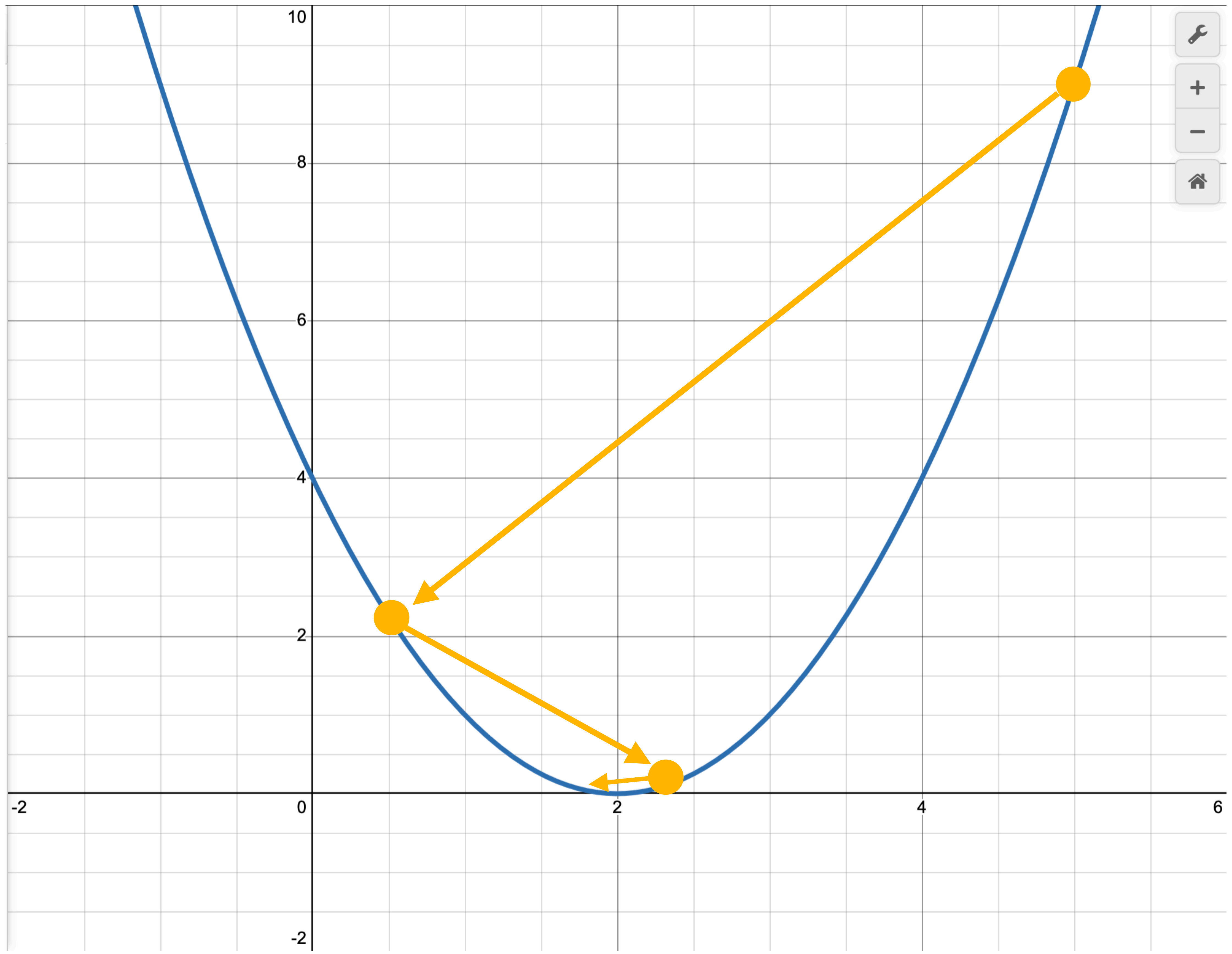

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- **Loss gets lower with every step!**
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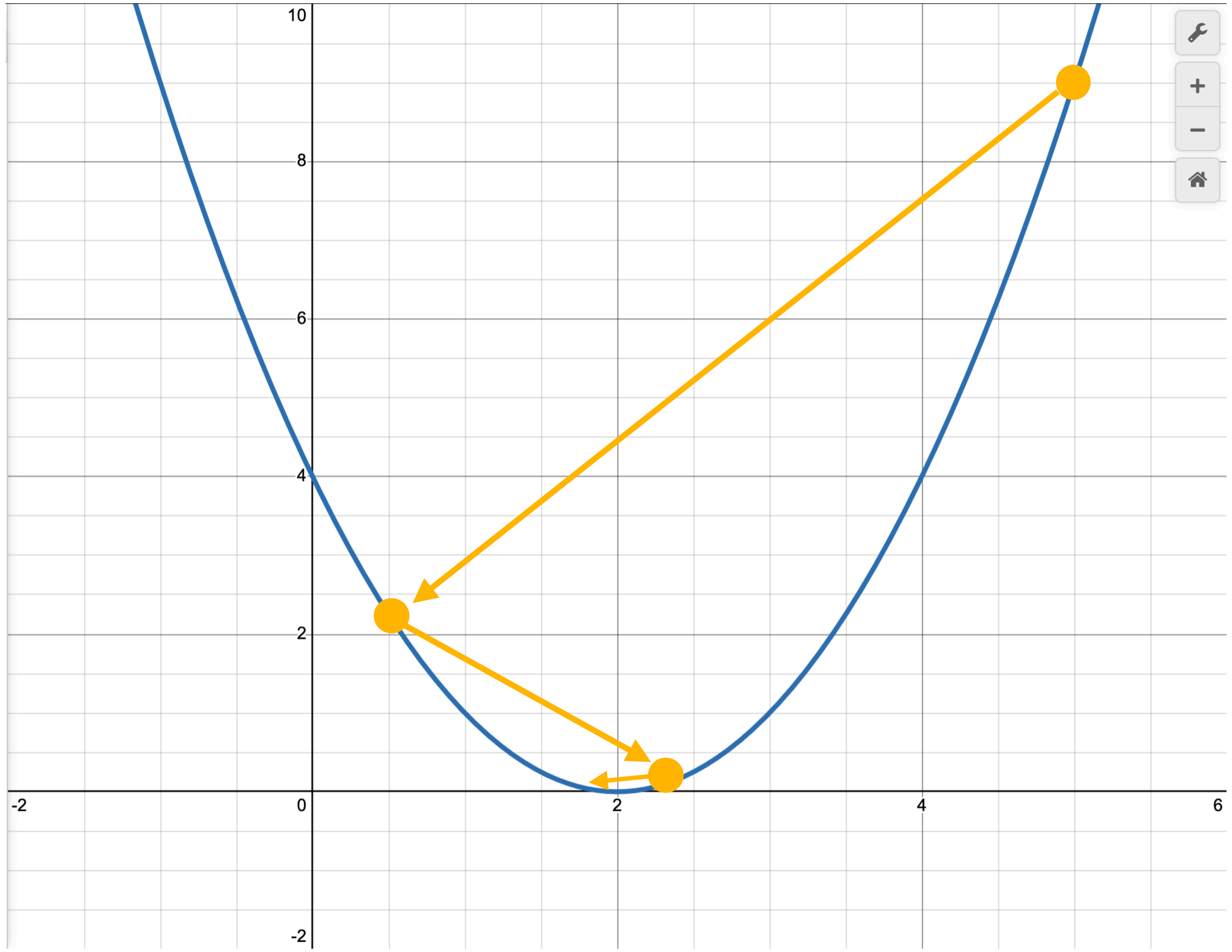

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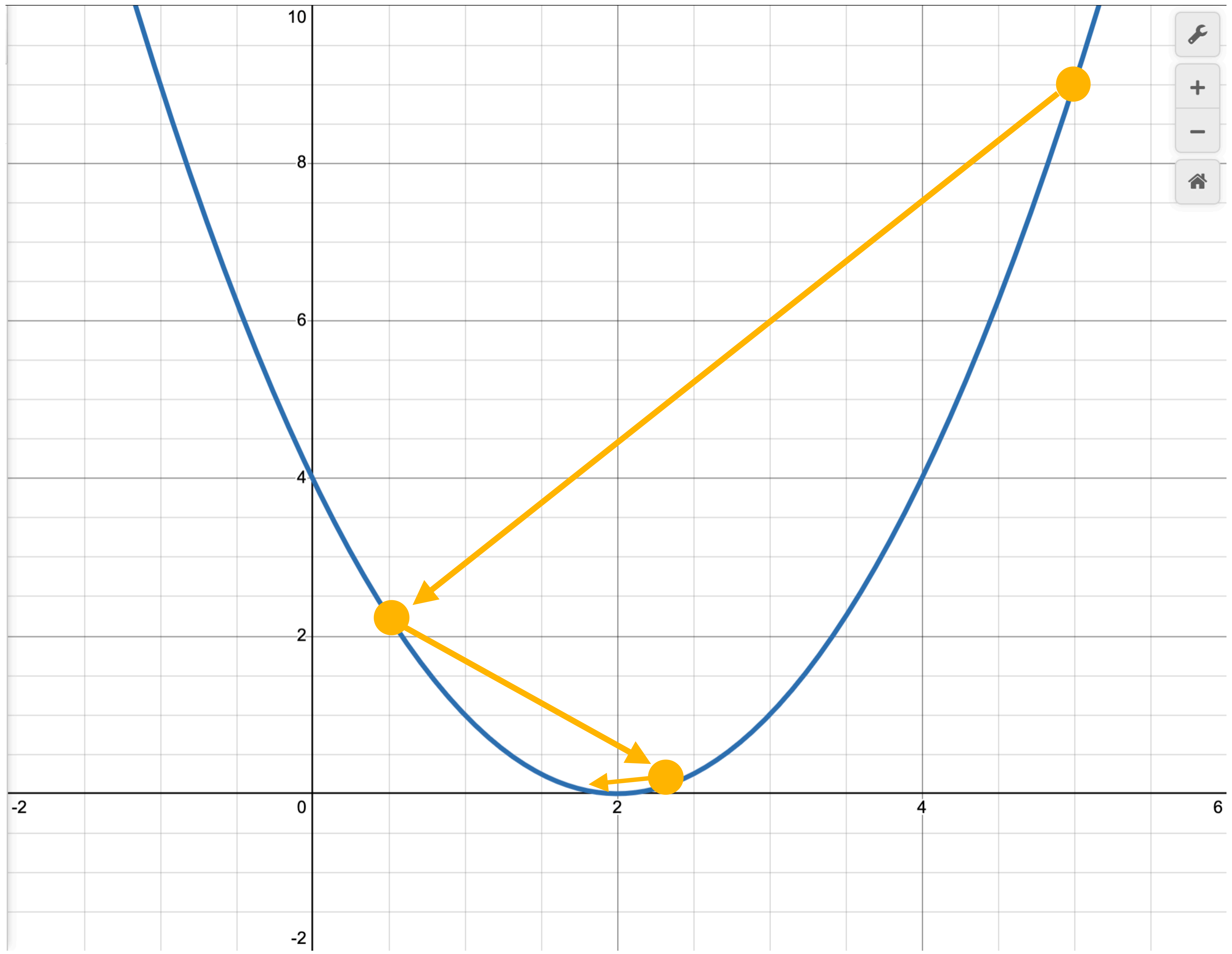

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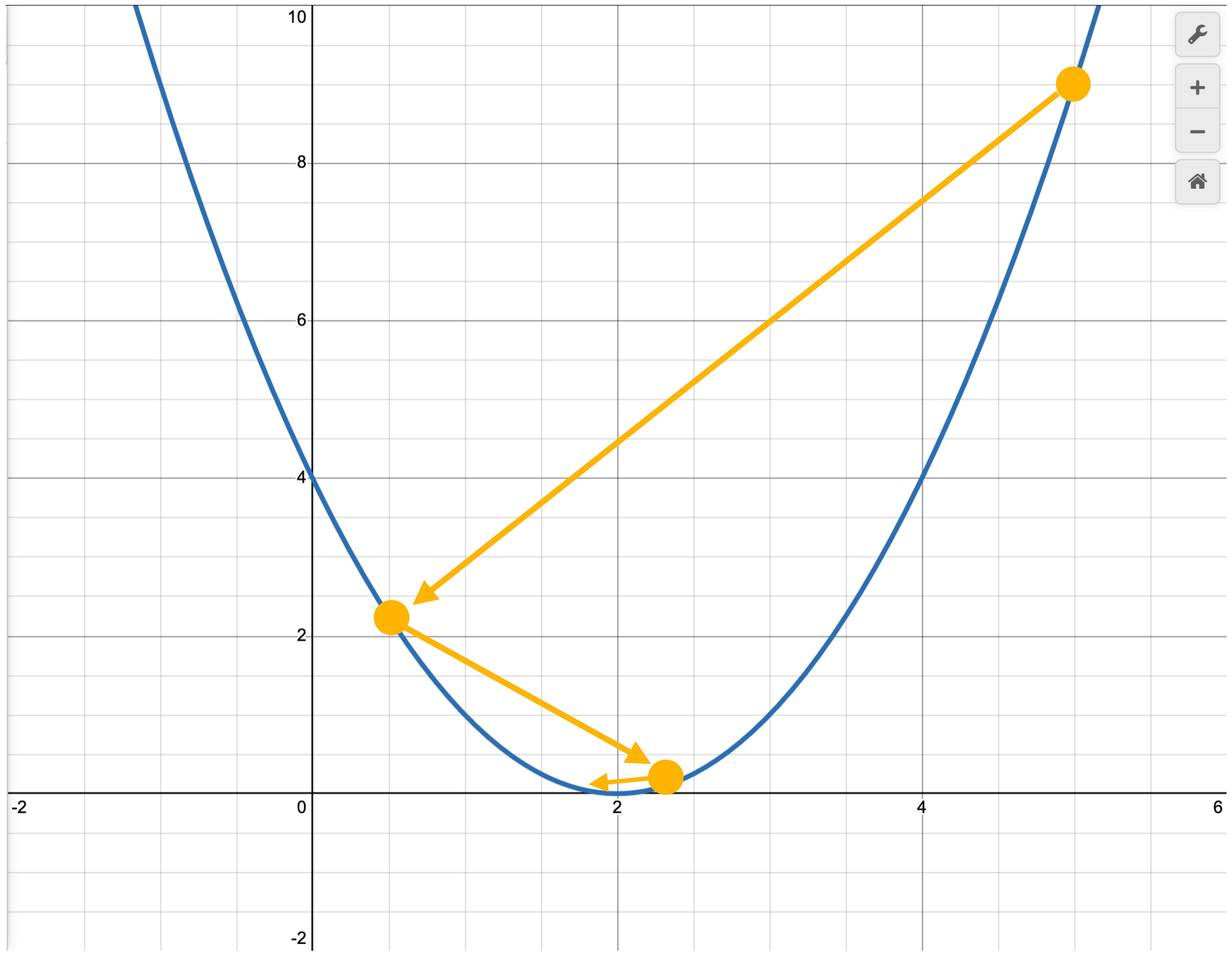

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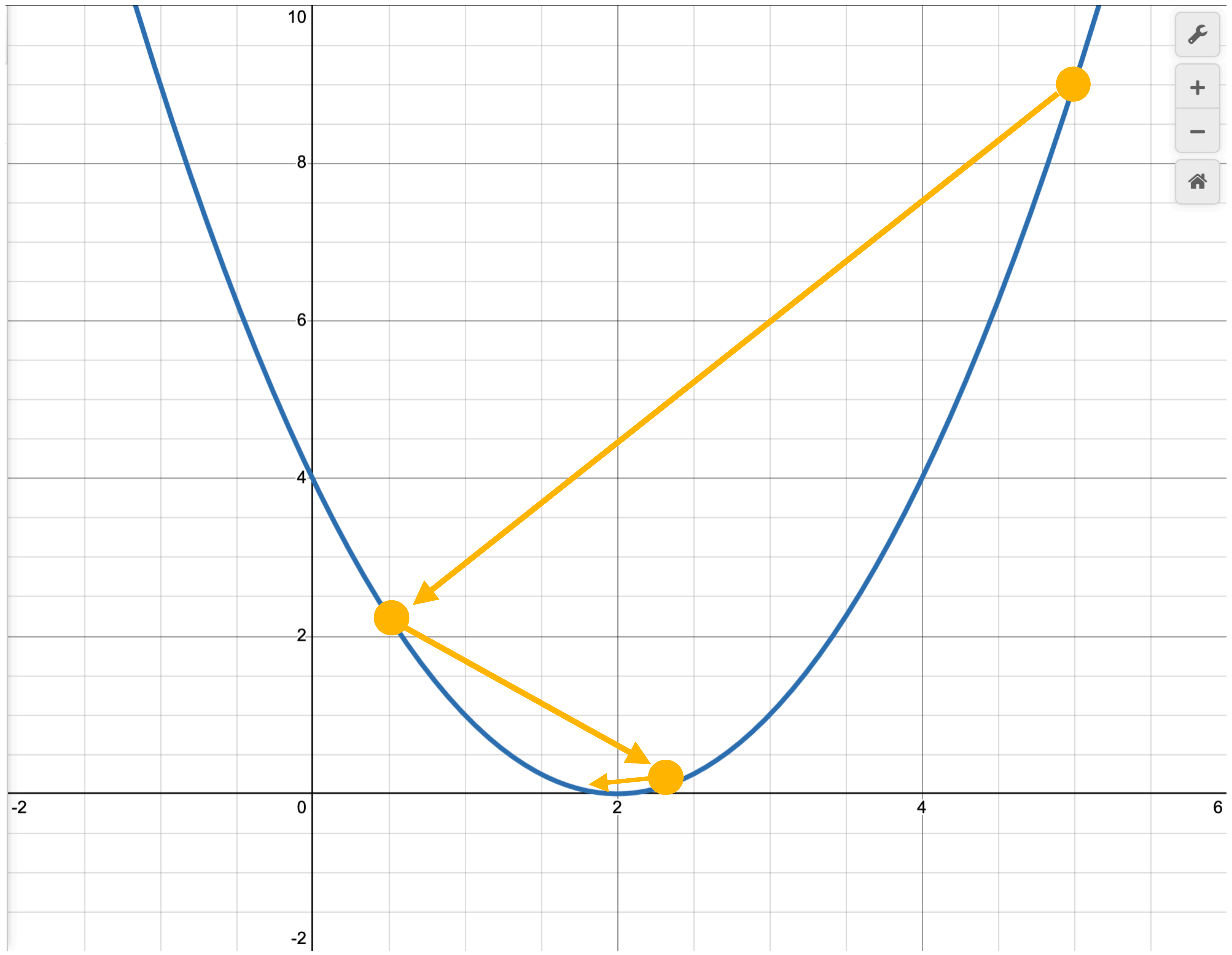

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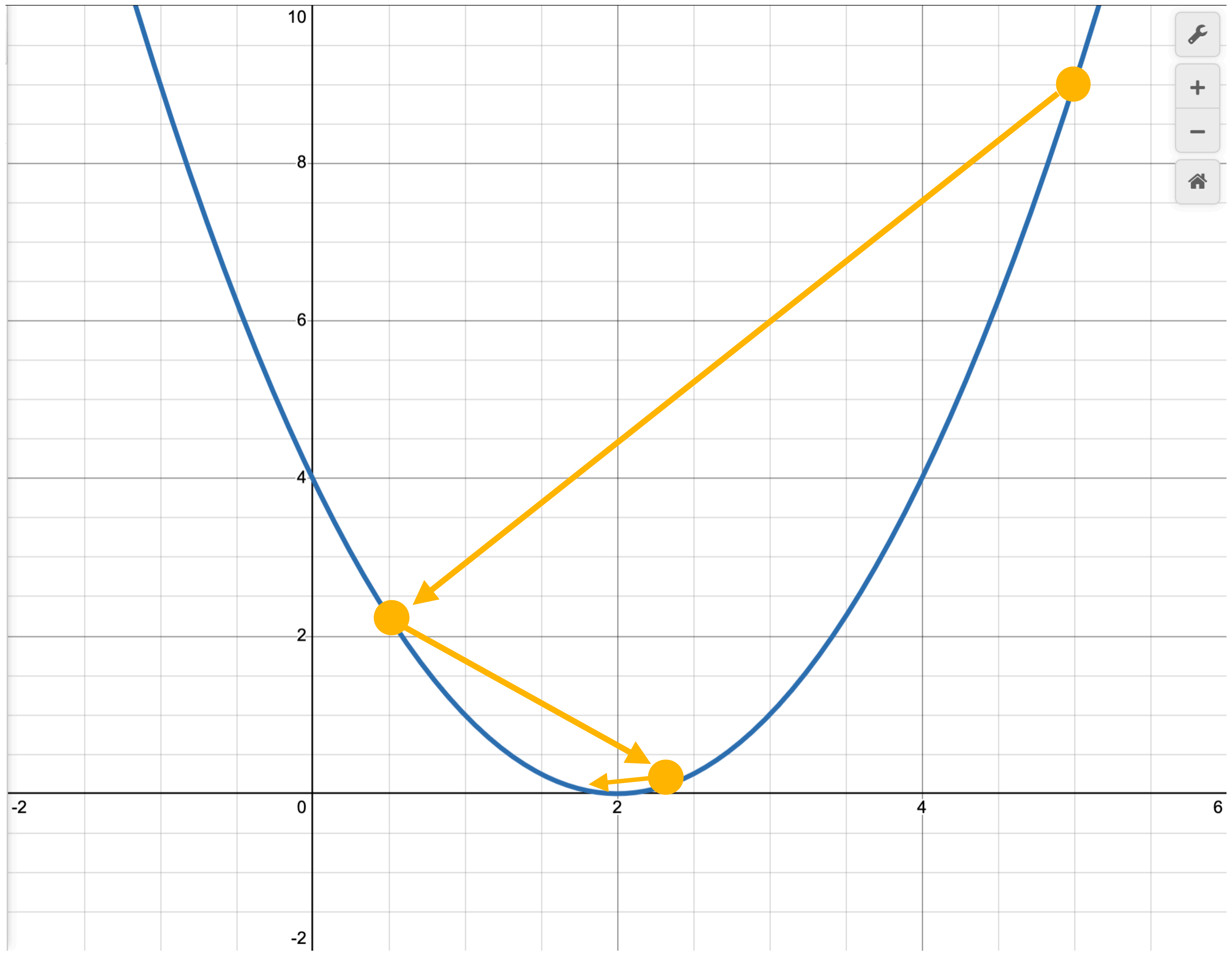
- Risks of different values
  - Too high → **"bouncing around"** and missing an optimum
  - Too low → taking **many steps** to reach the optimum



Noisy Secret Number Game

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- Gradient Descent allows us to make an **optimal estimate** given the data
- Unlike the previous example, we need to **define the loss function** over the **entire dataset**

Global loss function

$$\mathscr{L}(f(X,\theta),Y) = \frac{1}{N}\sum_{i=1}^{N} \ell(f(x_i,\theta),y_i)$$

loss over **all datapoints**

average

loss for a **single datapoint**

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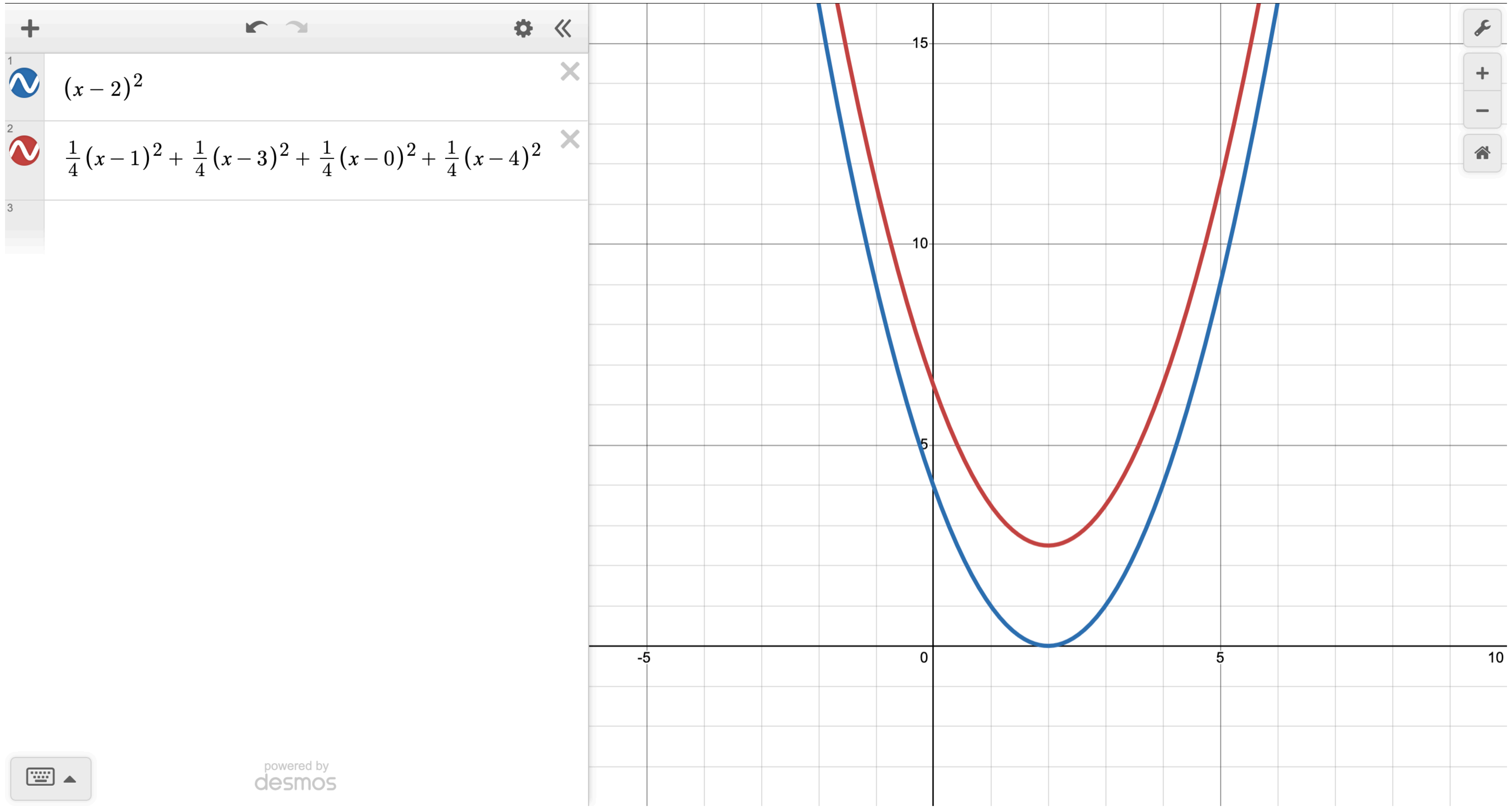
- This ensures our learned parameter(s) θ are **optimized for the dataset as a whole**, rather than for any single example
- **Any guesses** what the optimal value of θ is for our **noisy dataset?**
  - $D = \{(2, 3), (3, 6), (5, 5), (8, 12)\}$
  - Hint, look at the **difference between each pair**

Batch Gradient Descent

- The optimal θ is still 2! (The **average** input/output difference)
- Note that the **optimal loss is > 0** (i.e. there is some error left over!)



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  - This is an **iterative process** that sometimes needs a tuned **learning rate**

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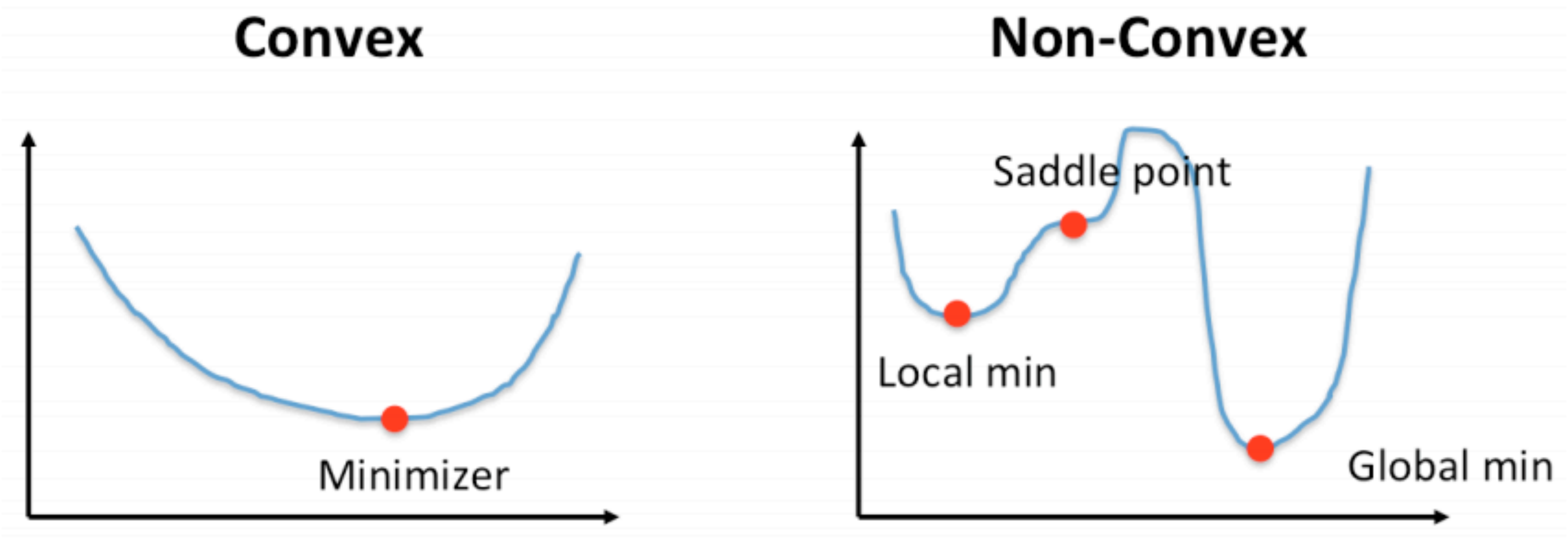
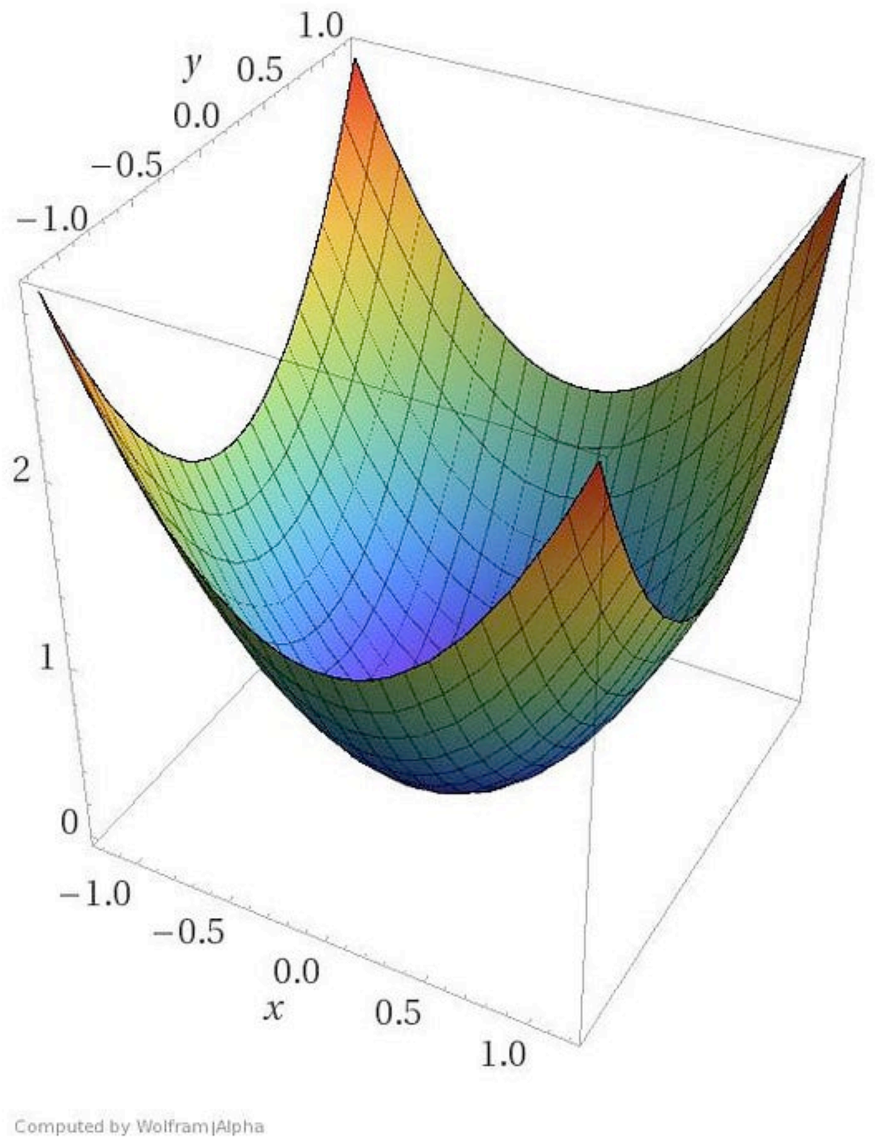
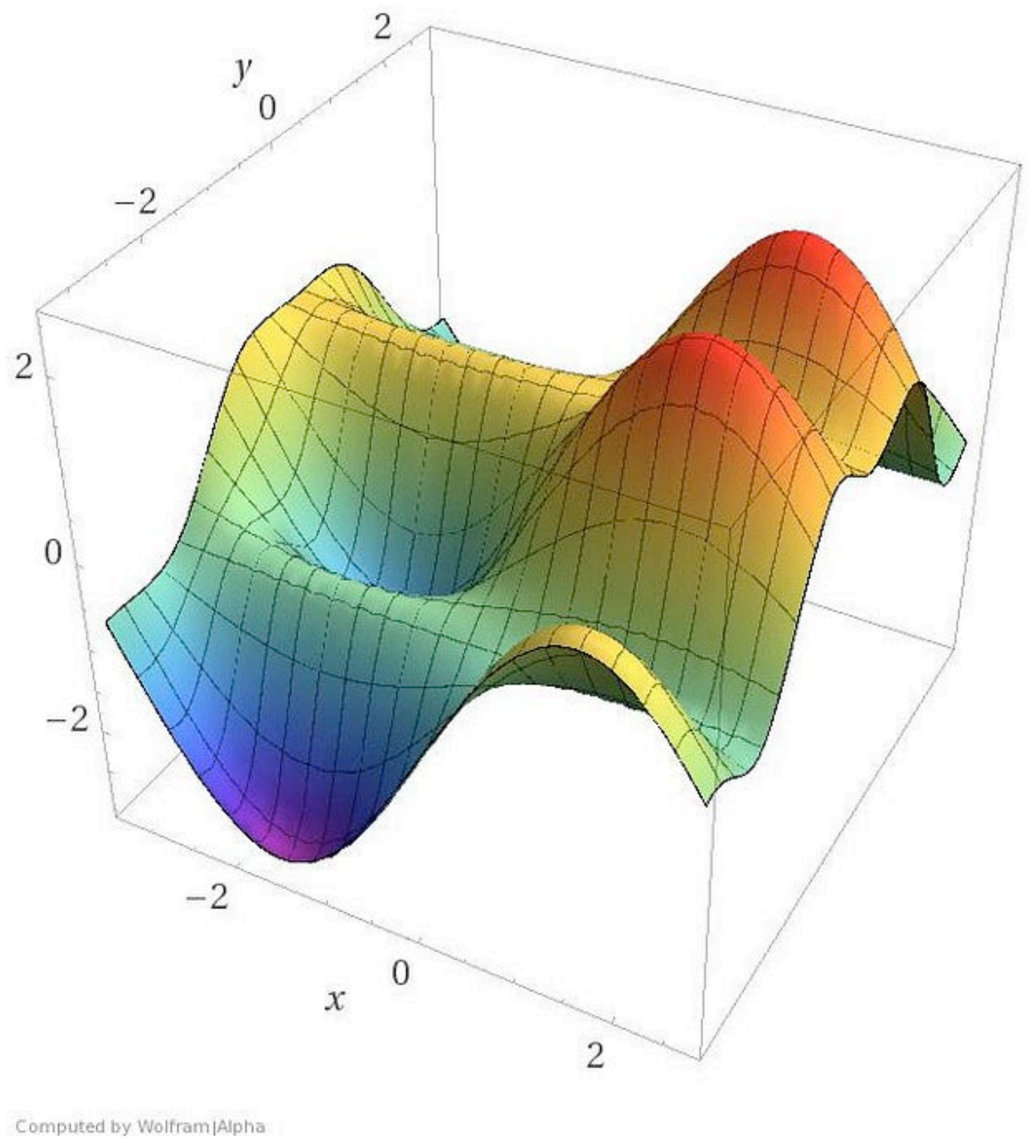
  - You can have **as many parameters as you want!**

Convex vs. Non-convex

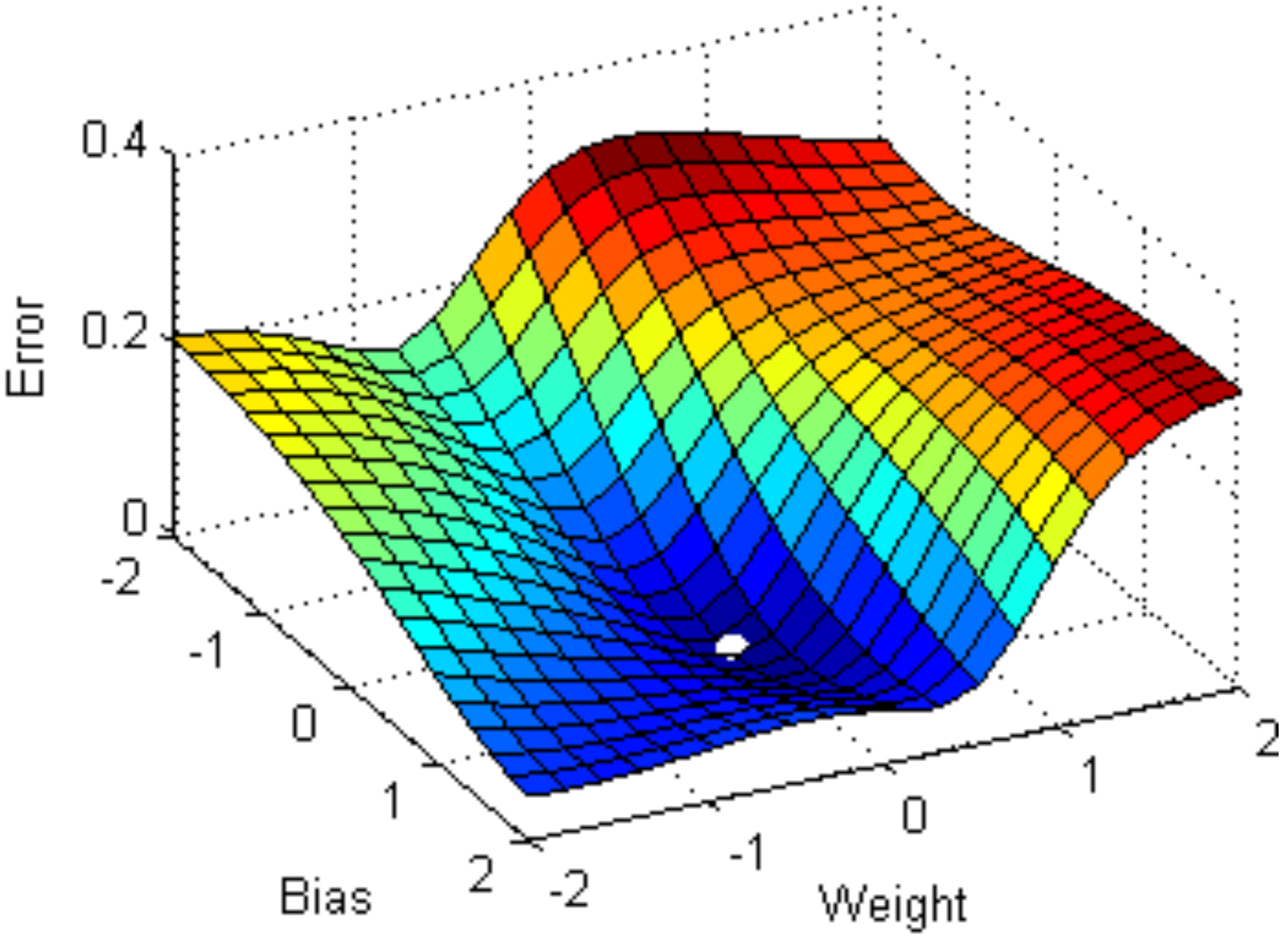






Multi-variable Functions / Gradients

Gradient Descent: Intuition



source

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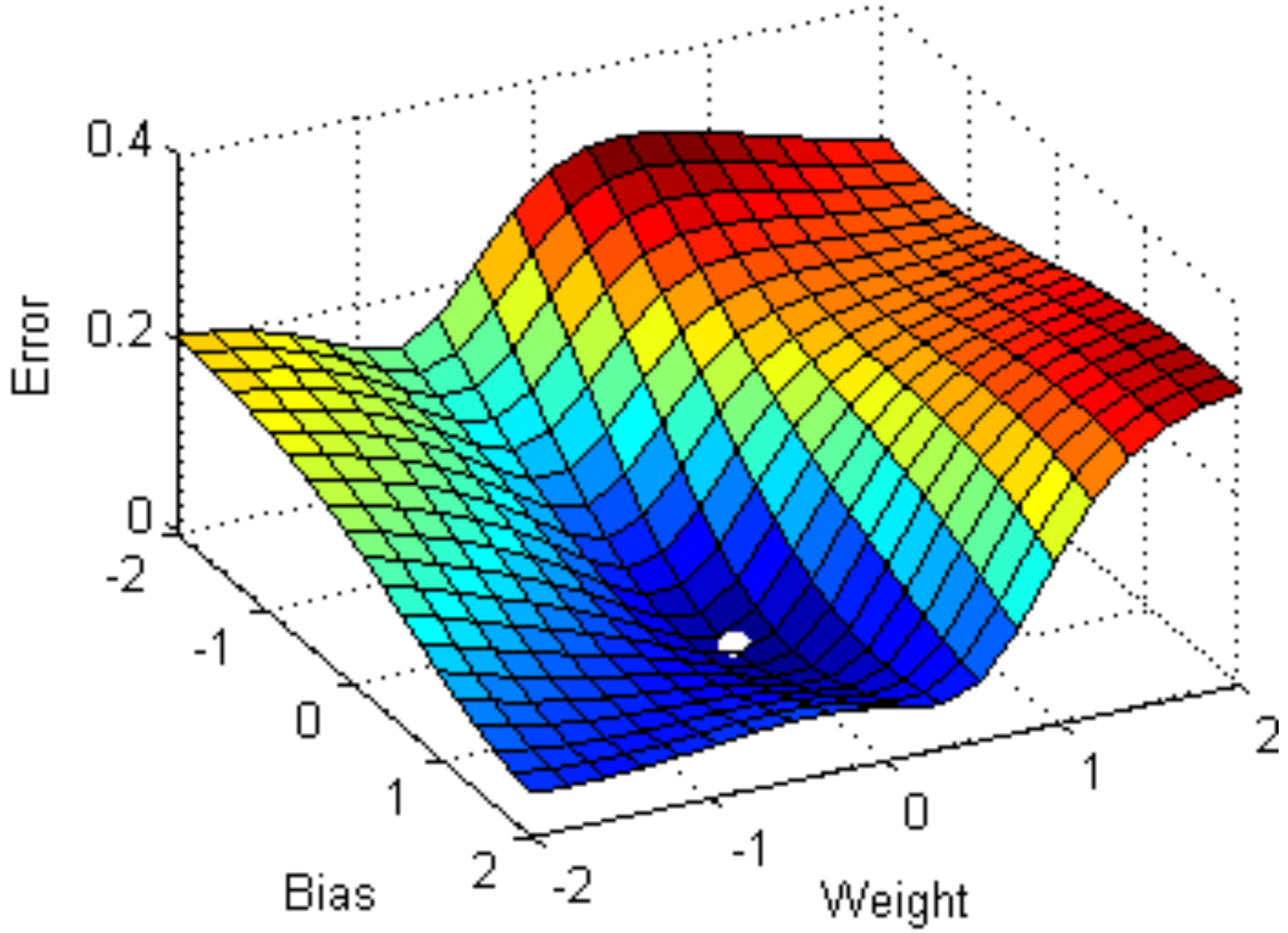

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$$\frac{\partial f}{\partial y} = 20x^3y + 15xy^2 + 1$$

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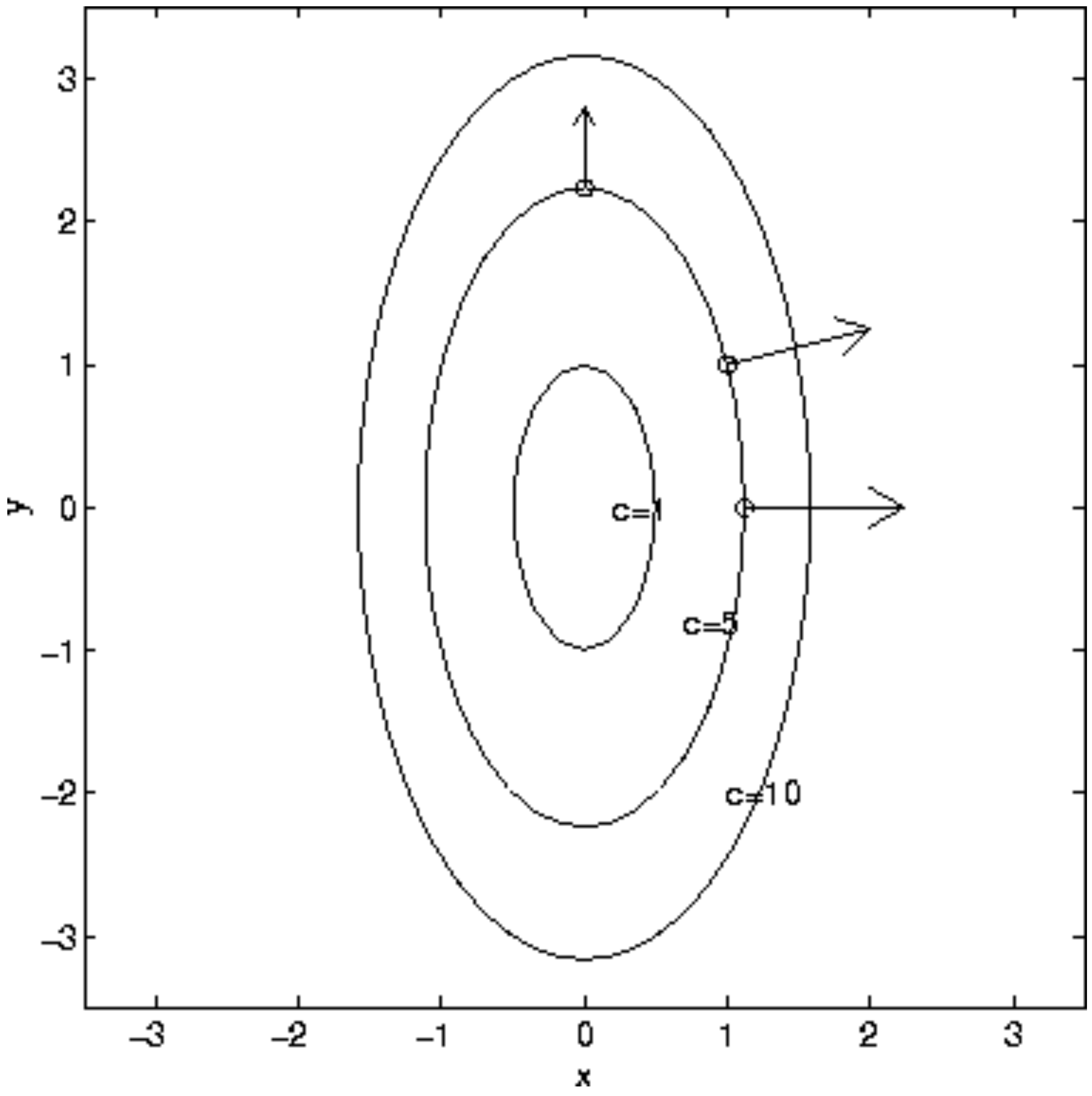
- The gradient points in the direction of **greatest increase** of f

Gradient and Level Curves



Level curves: $f(x, y) = c$

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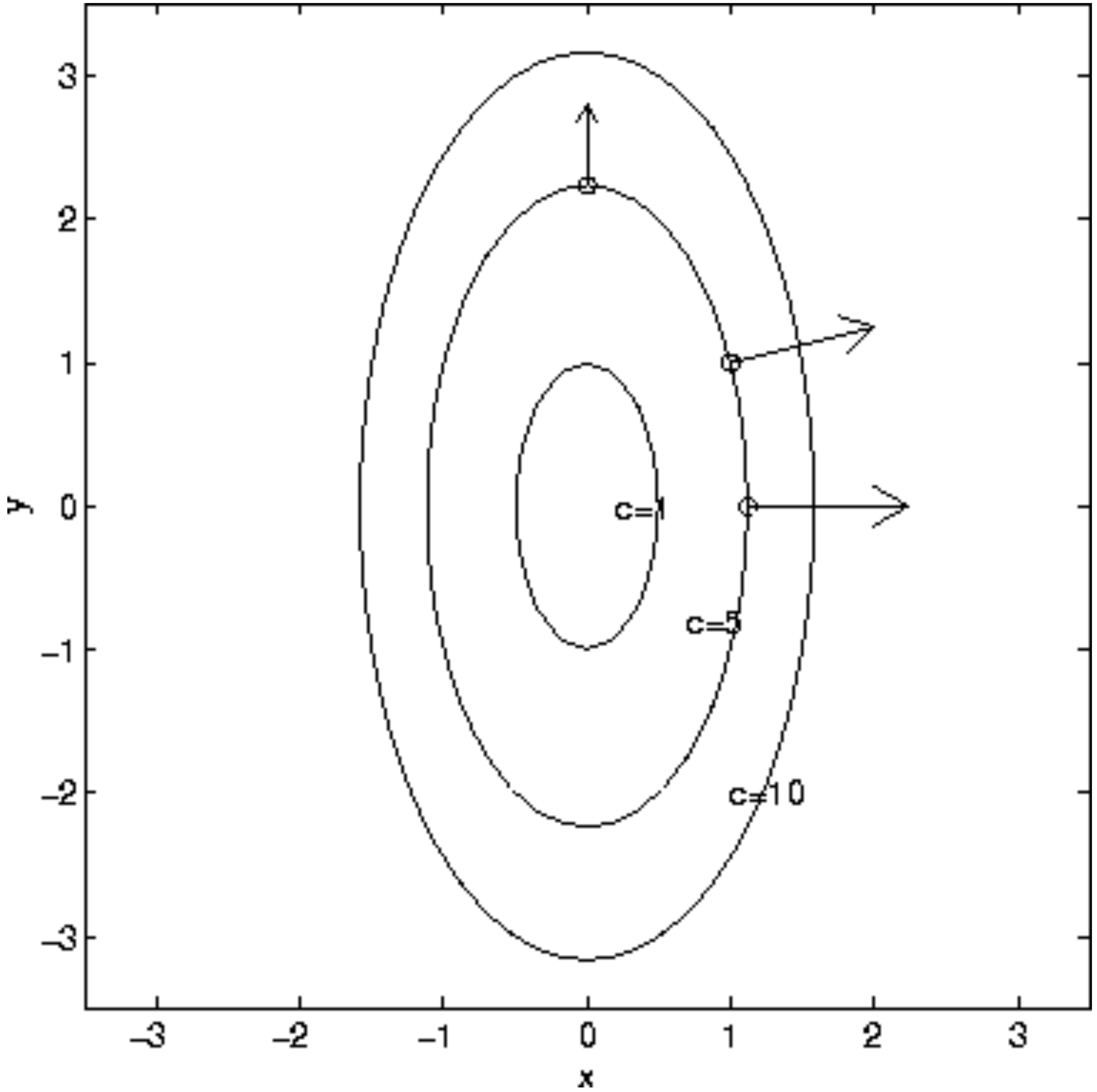

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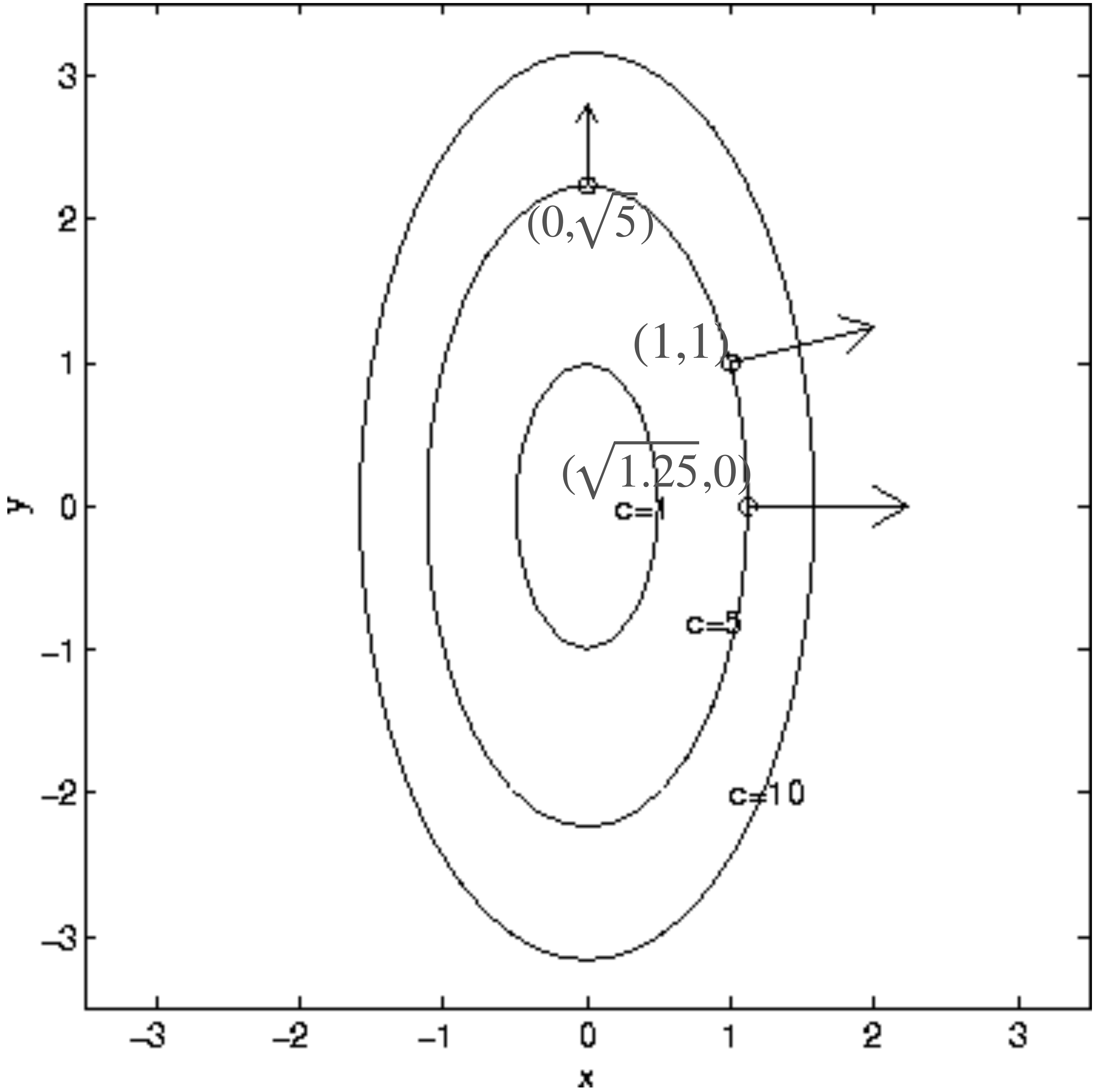

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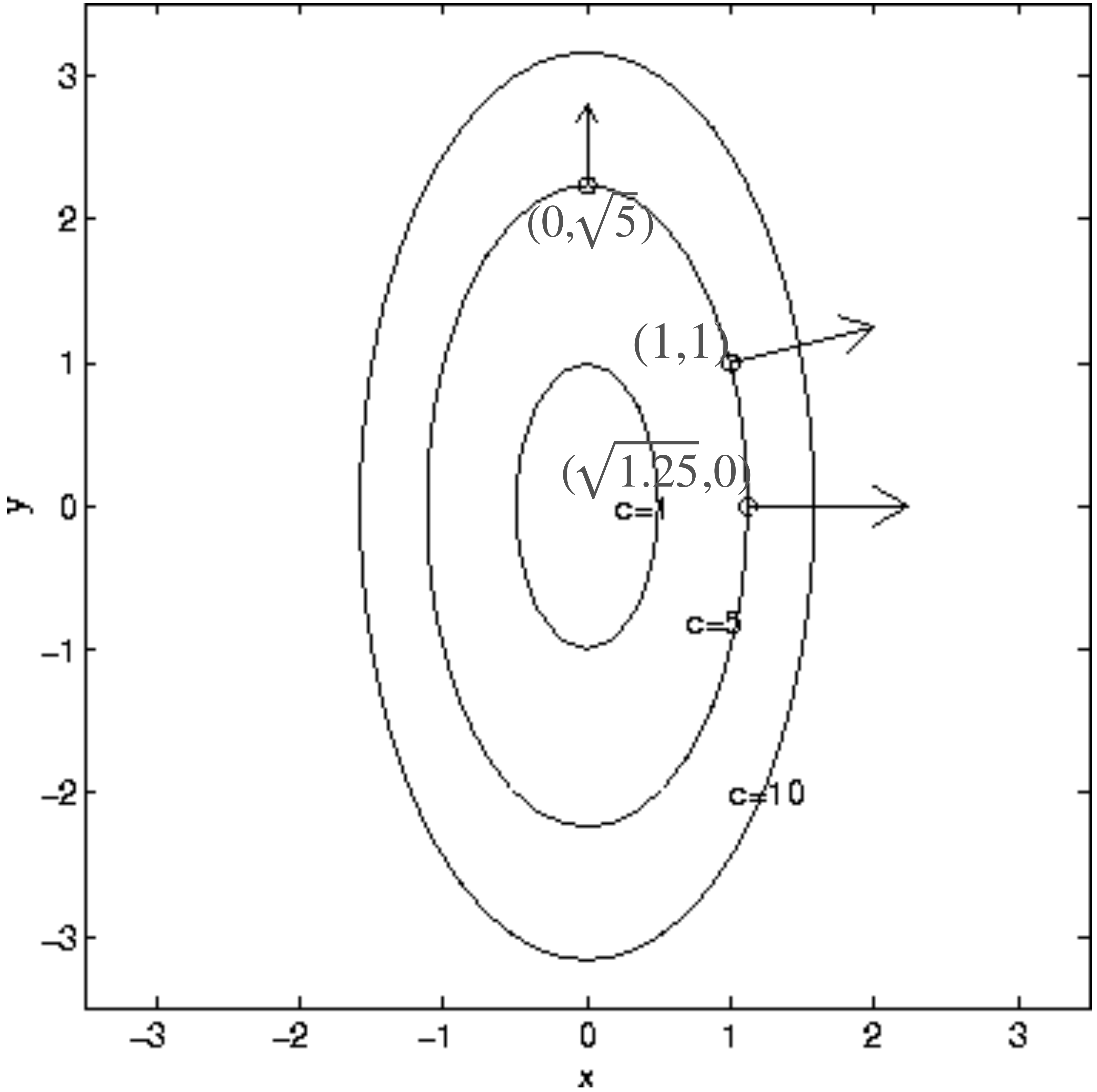

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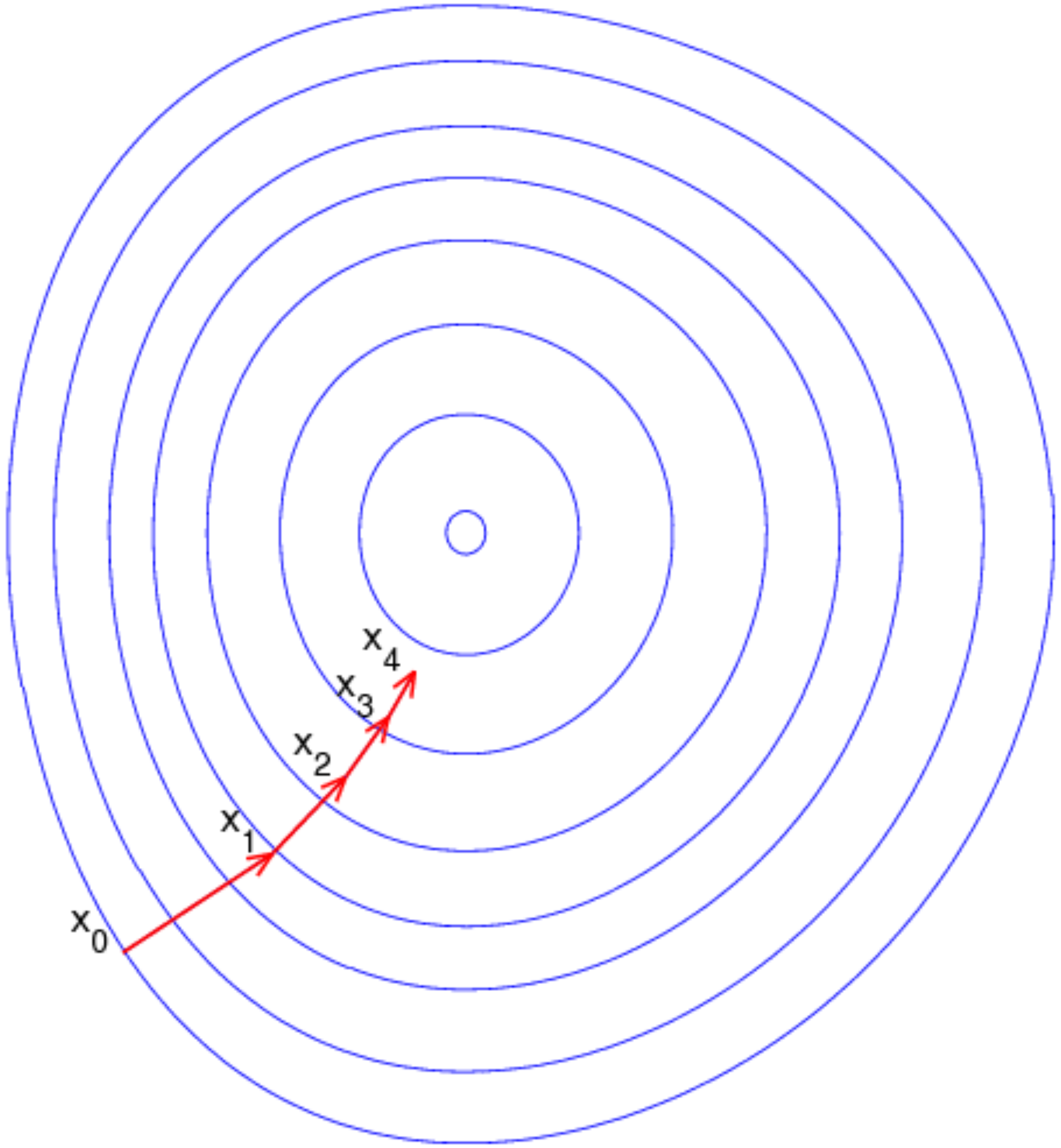
Q: what are the actual gradients at those points?

Gradient Descent and Level Curves



source

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- **Epoch**: one pass through the whole training data

Stochastic Gradient Descent

```
initialize parameters / build model

for each epoch:

 data = shuffle(data)
 batches = make_batches(data)

 for each batch in batches:

  outputs = model(batch)
  loss = loss_fn(outputs, true_outputs)
  compute gradients
  update parameters
```