Multi-Factor Commodity Price Process: Spot and Forward Price Simulation

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1 Introduction

This document describes a very general multi-factor price process, which is suitable use in simulating the spot and forward prices of a a commodity under the risk-neutral probability measure.

$$\frac{dF(t,T)}{F(t,T)} = \sum_{i=1}^{n} \sigma_i(T) e^{-\alpha_i(T-t)} dz_i(t) \tag{1}$$

$$\alpha_i \in \mathbb{R}_{\geq 0}$$
$$t \in \mathbb{R}_{\geq 0}$$
$$T \in \{T_0, T_1, T_2, \dots | T_j \geq t\}$$
$$\sigma_i : \mathbb{R}_{\geq 0} \to \mathbb{R}$$

Where $z_i(t)$ follow correlated Wiener processes with correlation $\rho_{i,j}$, i.e.

$$\mathbb{E}[dz_i(t)dz_j(t)] = \rho_{i,j}dt \tag{2}$$

$F(t,T)$ is the forward price observed at time t, for delivery over the period starting at time T. The following observations can be made about this model.

- Forward prices are lognormally distributed, which has the following practical considerations:
  - The model doesn't take into account volatility smile. This assumption is fine for the situation of there not being a liquid market in vanilla options and/or the model is being used for a complex product which isn't sensitive to changes in the shape of the skew surface. For example this model probably isn't suitable for pricing Asian options on Brent crude oil as the vanilla options market is liquid enough to be able to extract a volatility surface (i.e. implied volatility by expiry and maturity) in order to calibrate a model to take the surface into

account (e.g. a stochastic volatility model) and price a relativeley simple product like an Asian option. Maybe the model is suitable as part of a more complex model for optimising a physical oil asset such as an oil refinery, where the practical implementation of the optimisation algorithm is only possible with a relatively simple deterministic volatilities model such as this one. A similar reasoning could be used to jusfify the suitability for use for pricing gas storage, swing, and power generation assets, as these require complex optimisation. But the reason would be even more compelling for such assets as, in many Europe markets, vanilla options markets either do not exist, or are not liquid enough to reliably calibrate a stochastic volatility model.

  – The price of some commodities can occasionally go negative, for example power and natural gas. Whether this model is suitable depends on the frequency of such negative price occurances, and whether the use of the model is impacted by this.

- The model is fitted to the forward curve in that the expected spot price is equal to the equivalent point on the initial curve. This makes the model suitable for use assuming the risk-neutral probability measure with deterministic interest rates.

- There are n sources of uncertainty for which the effect on $F(t, T)$ depends on the following parameters:

  – The impact of each factor's movement on forward prices decays exponentially with time to maturity (i.e. $T - t$) with the α_i parameter determining the rate of this decay. A higher α_i results in a more rapid decay, and the minimum possible value of zero for α_i means that the i^{th} factor result in non-decaying parrallel shifts in the whole forward curve, before the $\sigma_i(T)$ parameter is taken into account. As the derivation of the spot price process shows below, in terms of spot price dynamics α_i translates to the mean reversion rate of the i^{th} stochastic factor driving the spot price.

  – σ_i is an arbitray maturity dependent volatility function. This parameter can be used to control seasonality in the volatility and correlation of the forward prices. For example a σ_i function which only returns non-zero values for T which are in the Winter delivery periods would make the i^{th} factor only drive Winter forward prices. σ_i could also be used during parameter calibration to impose a scaling factor such that the model exactly matches at-the-money implied volatilities.

Integrating this SDE (see Appendix I):

$$F(t_2, T) = F(t_1, t) e^{-\frac{1}{2} V(t_1, t_2, T) + I(t_1, t_2, T)} \tag{3}$$

A closed form expression exists for the integrated covariances of forward price log returns. This is derived in Appendix II. The evaluation of such covariances is essential to understand the statistical properties that the model implies for the dynamics of the forward curve. It also allows the calibration of model parameters to option market implied volatilities and historical covariances. Defining $C(t_1, t_2, T_1, T_2)$ as the integrated covariance from t_1 to t_2, between the log returns of forward contracts delivering on respective periods starting on T_1 and T_2.

$$C(t_1, t_2, T_1, T_2) = \sum_{i=1}^{n} \sigma_i(T) \sum_{j=1}^{n} \sigma_j(T) \rho_{i,j} e^{-\alpha_i T_1 - \alpha_j T_2} \vartheta_{-t_1, -t_2}(\alpha_i + \alpha_j) \tag{4}$$

2 Appendices

2.1 Appendix I - Integration of Forward Price SDE

This appendix gives the detailed step for integrated the forward price process SDE, hence getting from 1 to 3.

Using Ito's Lemma to calculate the stochastic differential of the natural logarithm of the forward price.

$$d\ln(F(t,T)) = \frac{1}{F(t,T)} dF(t,T) - \frac{1}{2}\frac{1}{F(t,T)^2}(dF(t,T))^2 \tag{5}$$

Using the properties of Brownian Motion $(dz(t))^2 = dt$ and $dtdz(t) = 0$ the forward price differential squared can be evaluated as follows.

$$(dF(t,T))^2 = F(t,T)^2 \sum_{i=1}^{n}\sum_{j=1}^{n} \sigma_i(T)\sigma_j(T) e^{-(\alpha_i+\alpha_j)(T-t)} \rho_{i,j} dt \tag{6}$$

Substituting for $(dF(t,T))^2$ and $dF(t,T)$ in 5.

$$d\ln(F(t,T)) = \sum_{i=1}^{n} \sigma_i(T) e^{-\alpha_i(T-t)} dz_i(t) - \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n} \sigma_i(T)\sigma_j(T) e^{-(\alpha_i+\alpha_j)(T-t)} \rho_{i,j} dt \tag{7}$$

The purpose of finding the stochastic differential of $ln(F(t,T))$ is to remove the $F(t,T)$ coefficient of the Brownian motions, leaving an SDE which can be integrated to a form where the Ito Integrals have non-stochastic integrands, and hence have a normal distribution with known mean and variance. Integrating 7:

$$\ln(F(t_2,T)) = \ln(F(t_1,t)) - \frac{1}{2}V(t_1,t_2,T) + I(t_1,t_2,T) \tag{8}$$

Where:

$$V(t_1,t_2,T) = \sum_{i=1}^{n}\sigma_i(T)\sum_{j=1}^{n}\sigma_j(T)\rho_{i,j}\int_{t_1}^{t_2} e^{-(\alpha_i+\alpha_j)(T-u)}du$$

$$= \sum_{i=1}^{n}\sigma_i(T)\sum_{j=1}^{n}\sigma_j(T)\rho_{i,j}\frac{1}{\alpha_i+\alpha_j}(e^{-(\alpha_i+\alpha_j)(T-t_2)} - e^{-(\alpha_i+\alpha_j)(T-t_1)}) \tag{9}$$

And:

$$I(t_1,t_2,T) = \sum_{i=1}^{n}\sigma_i(T)\int_{t_1}^{t_2} e^{-\alpha_i(T-u)}dz_i(u) \tag{10}$$

There is a problem in that $V(t_1,t_2,T)$ is not defined for the case where $\alpha_i + \alpha_j = 0$. A continuous extension is required in order for this case to be allowed. Consider the function ψ which is only defined for the domain $\mathbb{R}_{\neq 0}$, the set real numbers, excluding zero.

$$\psi_{c_1,c_2}(x) = \frac{e^{-xc_2} - e^{-xc_1}}{x} \quad x \in \mathbb{R}_{\neq 0} \tag{11}$$

The first step to defining the continous extension is to write ψ as a Taylor Series about zero.

$$\psi_{c_1,c_2}(x) = \frac{1 - xc_2 - 1 + xc_1 + O(x^2)}{x} \tag{12}$$

Taking the limit of $\psi(x)$ towards zero, the $O(x^2)$ becomes negligible, hence:

$$\begin{aligned}\lim_{x\to 0}\psi_{c_1,c_2}(x) = \frac{1 - xc_2 - 1 + xc_1}{x} \\ = c_1 - c_2\end{aligned} \tag{13}$$

Using this, we can define ϑ as the continous extension of ψ:

$$\vartheta_{c_1,c_2}(x) = \begin{cases}\psi_{c_1,c_2}(x) & \text{for} \quad x \in \mathbb{R}_{\neq 0} \\ c_1 - c_2 & \text{for} \quad x = 0\end{cases} \tag{14}$$

Which in the case of equation for V is:

$$\vartheta_{T-t_1,T-t_2}(x) = \begin{cases}\psi_{T-t_1,T-t_2}(x) & \text{for} \quad x \in \mathbb{R}_{\neq 0} \\ t_2 - t_1 & \text{for} \quad x = 0\end{cases} \tag{15}$$

Using this, we can redefine $V(t_1,t_2,T)$ to include the case of α_i and α_j both being equal to zero:

$$V(t_1,t_2,T)=\sum_{i=1}^{n}\sigma_i(T)\sum_{j=1}^{n}\sigma_j(T)\rho_{i,j}\vartheta_{T-t_1,T-t_2}(\alpha_i+\alpha_j)$$

Hence the forward price can be expressed as:

$$F(t_2,T)=F(t_1,t)e^{-\frac{1}{2}V(t_1,t_2,T)+I(t_1,t_2,T)} \tag{16}$$

2.2 Appendix II - Forward Covariances

This section derives a closed form expression for the integrated covariances of forward price log returns. Defining $C(t_1,t_2,T_1,T_2)$ as the integrated covariance from t_1 to t_2, between the log returns of forward contracts delivering on respective periods starting on T_1 and T_2.

$$C(t_1,t_2,T_1,T_2)=\mathbb{E}\Big[\ln\Big(\frac{F(t_2,T_1)}{F(t_1,T_1)}\Big)\ln\Big(\frac{F(t_2,T_2)}{F(t_1,T_2)}\Big)\Big] \tag{17}$$

Remove the products of stochastic and deterministic terms, as they will equal zero:

$$C(t_1,t_2,T_1,T_2)=\mathbb{E}\big[I(t_1,t_2,T_1)I(t_1,t_2,T_2)\big] \tag{18}$$

Using the known properties of Ito Integrals:

$$\begin{aligned} C(t_1,t_2,T_1,T_2)&=\sum_{i=1}^{n}\sigma_i(T)\sum_{j=1}^{n}\sigma_j(T)\rho_{i,j}e^{-\alpha_iT_1-\alpha_jT_2}\int_{t_1}^{t_2}e^{u(\alpha_i+\alpha_j)}du\\ &=\sum_{i=1}^{n}\sigma_i(T)\sum_{j=1}^{n}\sigma_j(T)\rho_{i,j}e^{-\alpha_iT_1-\alpha_jT_2}\frac{1}{\alpha_i+\alpha_j}(e^{t_2(\alpha_i+\alpha_j)}-e^{t_1(\alpha_i+\alpha_j)}) \end{aligned} \tag{19}$$

Allowing for the case of α_i and α_j both being equal to zero, as in the previous appendix, a continuous extension is made to redefine the function C:

$$C(t_1,t_2,T_1,T_2)=\sum_{i=1}^{n}\sigma_i(T)\sum_{j=1}^{n}\sigma_j(T)\rho_{i,j}e^{-\alpha_iT_1-\alpha_jT_2}\vartheta_{-t_1,-t_2}(\alpha_i+\alpha_j) \tag{20}$$

2.3 Appendix III - Spot Price Process

Defining the spot price $S(t)$ as the price for delivery of the commodity in the period starting at time t, observed at time t. Using the equivalence between $S(t)$ and $F(t,t)$ and equation 3:

$$S(t)=F(0,t)e^{-\frac{1}{2}V(0,t,t)+I(0,t,t)} \tag{21}$$

When simulating a spot price path it is necessary to know the relationship between $S(t_k)$ and $S(t_{k-1})$, where $0 \leq t_{k-1} < t_k$. Defining $V_s(t) = V(0,t,t)$ and $I_s(t) = I(0,t,t)$:

$$S(t_k) = S(t_{k-1})\frac{F(0,t_k)}{F(0,t_{k-1})}e^{-\frac{1}{2}(V_s(t_k)-V_s(t_{k-1}))+I_s(t_k)-I_s(t_{k-1})} \tag{22}$$

Focussing on the stochastic term $I_s(t_k) - I_s(t_{k-1})$:

$$I_s(t_k)-I_s(t_{k-1}) = \sum_{i=1}^{n}\left(\sigma_i(t_k)\int_0^{t_k} e^{-\alpha_i(t_k-u)}dz_i(u)-\sigma_i(t_{k-1})\int_0^{t_{k-1}} e^{-\alpha_i(t_{k-1}-u)}dz_i(u)\right) \tag{23}$$

Defining:

$$F_i(t) = \int_0^t e^{-\alpha_i(t-u)}dz_i(u) \tag{24}$$

Substitution this into the above equation:

$$I_s(t_k) - I_s(t_{k-1}) = \sum_{i=1}^{n}\left(\sigma_i(t_k)F_i(t_k) - \sigma_i(t_{k-1})F_i(t_{k-1})\right) \tag{25}$$

When simulating, this expression will be adaped to the filtration $\mathcal{F}_{t_{k-1}}$, hence the $F_i(t_{k-1})$ will have been realised. $F_i(t_k)$ will be part realised, hence it is instructive to split this into it's deterministic and stochastic parts, as of t_{k-1}. First making the substituting $t_k = t_{k-1} + t_k - t_{k-1}$:

$$F_i(t_k) = e^{-\alpha_i(t_k-t_{k-1})}\int_0^{t_k} e^{-\alpha_i(t_{k-1}-u)}dz_i(u)$$
$$= e^{-\alpha_i(t_k-t_{k-1})}\left(\int_0^{t_{k-1}} e^{-\alpha_i(t_{k-1}-u)}dz_i(u) + \int_{t_{k-1}}^{t_k} e^{-\alpha_i(t_{k-1}-u)}dz_i(u)\right) \tag{26}$$

Noticing that the first integral is the definition of $F_i(t_{k-1})$, the function F_i can be defined recursively:

$$F_i(t_k) = e^{-\alpha_i(t_k-t_{k-1})}F_i(t_{k-1}) + \int_{t_{k-1}}^{t_k} e^{-\alpha_i(t_k-u)}dz_i(u) \tag{27}$$

For any $t_k > t_{k-1}$ and base case:

$$F_i(0) = 0 \tag{28}$$

In this form $F_i(t)$ can be recognised as an Ornstein–Uhlenbeck process with zero drift, mean reversion parameter α_i, and σ (coeffient of Brownian motion) of 1.

When simulating the spot price for an increasing sequence of n times $t_{k-1}, t_2 \dots t_{n-1}, t_n$ where $0 < t_{k-1} < t_k$, for each of these times the first step is to calculate $F_i(t_k)$ recursively, using the above relationship for $i = 1, \dots, n$.

$$F_i(t_k) = e^{-\alpha_i(t_k - t_{k-1})} F_i(t_{k-1}) + Z_i(t_k) \tag{29}$$

Where, using the property of Ito Integrals $Z_i(t_k)$ is multivariate normally distributed, with mean zero and covariance:

$$cov(Z_i, Z_j) = \frac{\rho_{i,j}}{\alpha_i + \alpha_j}(1 - e^{-(\alpha_i + \alpha_j)(t_k - t_{k-1})}) \tag{30}$$

The spot price can then be calculated from the values of $F_i(t_k)$:

$$S(t_k) = S(t_{k-1}) \frac{F(0, t_k)}{F(0, t_{k-1})} e^{-\frac{1}{2}(V_s(t_k) - V_s(t_{k-1})) + \sum_{i=1}^{n} \left(\sigma_i(t_k) F_i(t_k) - \sigma_i(t_{k-1}) F_i(t_{k-1})\right)} \tag{31}$$