Multi-Factor Commodity Price Process: Spot and Forward Price Simulation

Jake C. Fowler

January 2020

1 Introduction

This document describes a very general multi-factor price process, which is suitable use in simulating the spot and forward prices of a a commodity under the risk-neutral probability measure.

$$\frac{dF(t,T)}{F(t,T)} = \sum_{i=1}^{n} \sigma_i(T) e^{-\alpha_i (T-t)} dz_i(t) \tag{1}$$

Where $F(t,T)$ is the forward price observed at time t, for delivery over the period starting at time T.

$$F(t,T) = \tag{2}$$

2 Appendices

2.1 Appendix I - Integration of Forward Price SDE

This appendix gives the detailed step for integrated the forward price process SDE, hence getting from 1 to 2.

Using Ito's Lemma to calculate the stochastic differential of the natural logarithm of the forward price.

$$d\ln(F(t,T)) = \frac{1}{F(t,T)} dF(t,T) - \frac{1}{2}\frac{1}{F(t,T)^2}(dF(t,T))^2 \tag{3}$$

Using the properties of Brownian Motion $(dz(t))^2 = dt$ and $dtdz(t) = 0$ the forward price differential squared can be evaluated as follows.

$$(dF(t,T))^2 = F(t,T)^2 \sum_{i=1}^{n} \sum_{j=1}^{n} \sigma_i(T)\sigma_j(T) e^{-(\alpha_i + \alpha_j)(T-t)} \rho_{i,j} dt \tag{4}$$

Substituting for $(dF(t,T))^2$ and $dF(t,T)$ in 3.

$$d\ln(F(t,T)) = \sum_{i=1}^{n} \sigma_i(T) e^{-\alpha_i(T-t)} dz_i(t) - \frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} \sigma_i(T)\sigma_j(T) e^{-(\alpha_i+\alpha_j)(T-t)} \rho_{i,j} dt \tag{5}$$

The purpose of finding the stochastic differential of $ln(F(t,T))$ is to remove the $F(t,T)$ coefficient of the Brownian motions, leaving an SDE which can be integrated to a form where the Ito Integrals have non-stochastic integrands, and hence have a normal distribution with known mean and variance. Integrating 5:

$$\ln(F(t_2,T)) = \ln(F(t_1,t)) - \frac{1}{2} V(t_1,t_2,T) + I(t_1,t_2,T) \tag{6}$$

Where:

$$V(t_1,t_2,T) = \sum_{i=1}^{n} \sigma_i(T) \sum_{j=1}^{n} \sigma_j(T) \rho_{i,j} \int_{t_1}^{t_2} e^{-(\alpha_i+\alpha_j)(T-u)} du$$

$$= \sum_{i=1}^{n} \sigma_i(T) \sum_{j=1}^{n} \sigma_j(T) \rho_{i,j} \frac{1}{\alpha_i + \alpha_j} (e^{-(\alpha_i+\alpha_j)(T-t_2)} - e^{-(\alpha_i+\alpha_j)(T-t_1)})$$

And:

$$I(t_1,t_2,T) = \sum_{i=1}^{n} \sigma_i(T) \int_{t_1}^{t_2} e^{-\alpha_i(T-u)} dz_i(u) \tag{7}$$

Hence the forward price can be expressed as:

$$F(t_2,T) = F(t_1,t) e^{-\frac{1}{2} V(t_1,t_2,T) + I(t_1,t_2,T)} \tag{8}$$