Multi-Factor Commodity Price Process: Spot and Forward Price Simulation

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Contents

1 Introduction

This document describes a very general multi-factor commodity price process model, together with some analytical results which can be used to explore the assumed price dynamics, and simulate prices computationally when performing a Monte-Carlo simulation.

1.1 Forward Price SDE

The starting point is the SDE (stochastic differential equation) for the forward price process:

$$\begin{aligned} \frac{dF(t,T)^l}{F(t,T)^l} &= \sum_{i=1}^{n^l} \sigma_i^l(T) e^{-\alpha_i^l (T-t)} dz_i^l(t) \qquad (1) \\ \alpha_i^l &\in \mathbb{R}_{\geq 0} \\ t &\in \mathbb{R}_{\geq 0} \\ T &\in \{T_0^l, T_1^l, T_2^l, \dots | T_j^l \geq t\} \\ \sigma_i^l &: \mathbb{R}_{\geq 0} \to \mathbb{R} \\ l &\in [1, m] \end{aligned}$$

Where $z_i^l(t)$ follow correlated Wiener processes with correlation $\rho_{i,j}^{x,y}$, i.e.

$$\mathbb{E}[dz_i^x(t) dz_j^y(t)] = \rho_{i,j}^{x,y} dt \qquad (2)$$

$F(t,T_j)^l$ is the forward price observed at time t, for delivery over the time interval $[T_j, T_{j+1})$ of the l^{th} of m commodity underlyings. Note for the majority of equations in the rest of this text, in order to lighten notation, the l superscript is dropped and results are given for the single-commodity case. Where relevant the l superscript will be introduced to illustrate the model behaviour in a multi-commodity context.

1.2 Comparison With Other Models

This model is almost a specific case of the general multi-factor model presented in Clewlow and Strickland (2000), which instead of a coefficients of the Wiener process being $\sigma_i(T)e^{-\alpha_i(T-t)}$ as presented here, uses the more general form $\sigma_i(t,T)$. As results presented below show, the advantage of the form used in this paper is that the future distribution of forward and spot prices can be derived analyticallly. Another difference is that the cited paper assumes the driving stochastic factors are uncorrelated Wienver processes, whereas the model presented here uses correlated Brownian motions.

The model is also similar to that presented in Warin (2012) except that the σ_i volatility in this model is a function of forward maturity date T, rather than ovservation date t in Warin (2012).

1.3 Analysis of Model

The following observations can be made about this model:

- Forward prices are lognormally distributed, which has the following practical considerations:
  - The model doesn't take into account volatility smile. This assumption is acceptable for modelling commodities which do not have a liquid market in vanilla options. It is also an acceptable to model the price dynamics as part of a pricing and/or optimisation model for a complex physical product which isn't sensitive to changes in the shape of the skew surface. For example this model probably isn't suitable for pricing Asian options on Brent crude oil as the vanilla options market is liquid enough to be able to extract a volatility surface (i.e. implied volatility by expiry and maturity) in order to calibrate a price dynamics model which take the surface into account (e.g. a stochastic volatility model) for use in pricing a relativeley simple product like Asian options. Potentially this model is suitable for the price dynamics part of a more complex model for optimising a physical oil asset such as an oil refinery, where the practical implementation of the optimisation algorithm is only possible with a relatively simple deterministic volatility model, such as the one presented here. A similar reasoning could be used to jusfify the suitability for use for pricing gas storage, swing, and power generation assets, as these require complex optimisation. But the reason would be even more justified for use with these products in in many European markets, where the vanilla options markets either does not exist, or are not liquid enough to reliably calibrate a stochastic volatility model.
  - The price of some commodities can occasionally go negative, for example power and natural gas. Whether this model is suitable depends on the frequency of such negative price occurances, and whether the use of the model is impacted by this. For example it definetely shouldn't be used to price half-hourly exercise European options on the spot power with strike price ≤ 0.
- The model is fitted to the forward curve, i.e. the expected spot price is equal to the equivalent point on the initial curve. This makes the model suitable for use under the risk-neutral probability measure, assuming deterministic interest rates.
- There are n sources of uncertainty for which the effect on $F(t, T)$ depends on the following parameters:
  - The impact of each factor's movement on forward prices decays exponentially with time to maturity (i.e. $T - t$) with the α_i parameter determining the rate of this decay. A higher α_i results in a more rapid decay, and the minimum possible value of zero for α_i means

that movements in the i^{th} stochastic factor result in non-decaying parrallel shifts in the whole forward curve, assuming $\sigma_i(T)$ returns a constant value. As the derivation of the spot price process shows below, in terms of spot price dynamics α_i translates to the mean reversion rate of the i^{th} stochastic factor driving the spot price.

– σ_i is an arbitray maturity dependent volatility function. This parameter can be used to control seasonality in the volatility and correlation of the forward prices. For example a σ_i function which only returns non-zero values for T which are in the Winter delivery periods would make the i^{th} factor only drive Winter forward prices. σ_i could also be used during parameter calibration to impose a scaling factor such that the model exactly matches at-the-money implied volatilities. A similar calibration step is described in Rebonato (2013), p. 673, although in the context of an LMM-style model.

2 Analytical Result on Distribution of Prices

2.1 Forward Price Distrubtion

Integrating this SDE (see Appendix A):

$$F(t_2, T) = F(t_1, t)e^{-\frac{1}{2}V(t_1,t_2,T)+I(t_1,t_2,T)} \tag{3}$$

Where:

$$V(t_1, t_2, T) = \sum_{i=1}^{n} \sigma_i(T) \sum_{j=1}^{n} \sigma_j(T)\rho_{i,j}\vartheta_{T-t_1,T-t_2}(\alpha_i + \alpha_j)$$

$$\vartheta_{c_1,c_2}(x) = \begin{cases} \frac{e^{-xc_2}-e^{-xc_1}}{x} & \text{for} \quad x \in \mathbb{R}_{\neq 0} \\ c_1 - c_2 & \text{for} \quad x = 0 \end{cases} \tag{4}$$

$$I(t_1, t_2, T) = \sum_{i=1}^{n} \sigma_i(T) \int_{t_1}^{t_2} e^{-\alpha_i(T-u)} dz_i(u) \tag{5}$$

2.2 Simulation of Forward Curve

A more convenient form of 3 for use in simulating the evaluation of the whole forward curve between t_1 and t_2 is:

$$F(t_2, T) = F(t_1, T)e^{-\frac{1}{2}V(t_1,t_2,T)+\sum_{i=1}^{n}\sigma_i(T)e^{-\alpha_i T}Y_i(t_1,t_2)} \tag{6}$$

Where $Y_i(t_1, t_2)$ are normally distributed, with mean zero and covariance:

$$cov(Y_i(t_1, t_2), Y_j(t_1, t_2)) = \rho_{i,j}\vartheta_{-t_1,-t_2}(\alpha_i + \alpha_j) \tag{7}$$

See Appendix B for the derivation of this, together with the equivalent result for the joint simulation of multiple commodities forward curves.

2.3 Forward Covariance Surface

A closed form expression exists for the integrated covariances of forward price log returns. This is derived in Appendix C. The evaluation of such covariances is essential to understand the statistical properties that the model implies for the dynamics of the forward curve. It also allows the calibration of model parameters to option market implied volatilities and historical covariances. Defining $C(t_1, t_2, T_1, T_2)$ as the integrated covariance from t_1 to t_2, between the log returns of forward contracts delivering on respective periods starting on T_1 and T_2.

$$C(t_1, t_2, T_1, T_2) = \sum_{i=1}^{n} \sigma_i(T) \sum_{j=1}^{n} \sigma_j(T) \rho_{i,j} e^{-\alpha_i T_1 - \alpha_j T_2} \vartheta_{-t_1, -t_2}(\alpha_i + \alpha_j) \tag{8}$$

See Appendix C for the derivation of this, together with the equivalent result for the joint simulation of multiple commodities forward curves.

2.4 Spot Price Distribution

The spot price $S(t)$ is defined as the price for delivery of the commodity in the period starting at time t, observed at time t. Appendix D derives an analytical form for the spot price conditional on the filtration $\mathcal{F}_{t_{k-1}}$:

$$S(t_k) = S(t_{k-1}) \frac{F(0, t_k)}{F(0, t_{k-1})} e^{-\frac{1}{2}(V_s(t_k) - V_s(t_{k-1})) + \sum_{i=1}^{n} \left(\sigma_i(t_k) f_i(t_k) - \sigma_i(t_{k-1}) f_i(t_{k-1}) \right)} \tag{9}$$

Where $f(t)$ is defined recursively as:

$$f_i(t_k) = e^{-\alpha_i (t_k - t_{k-1})} f_i(t_{k-1}) + Z_i(t_k) \tag{10}$$

With base case:

$$f_i(0) = 0 \tag{11}$$

$Z_i(t_k)$ is multivariate normally distributed, with mean zero and covariance:

$$cov(Z_i(t_k), Z_j(t_k)) = \rho_{i,j} \vartheta_{t_k - t_{k-1}, 0}(\alpha_i + \alpha_j) \tag{12}$$

Appendix D gives the equivalent result for the case of multiple commodities.

$f_i(t)$ can be recognised as an Ornstein–Uhlenbeck process with zero drift, mean reversion parameter α_i, and σ (coeffient of Brownian motion) of 1.

Except in the case of n being equal to one, it can be seen that the spot price is not Markovian itself, but is Markovian with respect to the n state variables f_i.

This has practical significance for two reasons. Firstly, when simulating only the spot price, it is necessary to simulate the n underlying f_i stochastic factors, before calculating the spot price from these. If simulating the price for several futures times, it will be necessary to keep track of all f_i due to their recursive nature.

Simulating just the spot price, rather than the whole forward curve, is of interest for physical energy commodity products, such as gas storage, swing contracts and power generation, as the ultimate cash flows (which are conditional upon the owners decisions) will derive from the spot price. Hence, when building certain pricing and optimisation models, the simulation of just the spot price will suffice. Of course, the whole forward curve could be simulated, with the spot price being taken as the first point on the curve. However, simulating just the spot price will be more computationally efficient, if this is all we are interested in.

Appendices

A Integration of Forward Price SDE

This appendix gives the detailed step for integrated the forward price process SDE, hence getting from 1 to 3.

Using Ito's Lemma to calculate the stochastic differential of the natural logarithm of the forward price.

$$d\ln(F(t,T)) = \frac{1}{F(t,T)}dF(t,T) - \frac{1}{2}\frac{1}{F(t,T)^2}(dF(t,T))^2 \tag{13}$$

Using the properties of Brownian Motion $(dz(t))^2 = dt$ and $dtdz(t) = 0$ the forward price differential squared can be evaluated as follows.

$$(dF(t,T))^2 = F(t,T)^2 \sum_{i=1}^{n}\sum_{j=1}^{n} \sigma_i(T)\sigma_j(T)e^{-(\alpha_i+\alpha_j)(T-t)}\rho_{i,j}dt \tag{14}$$

Substituting for $(dF(t,T))^2$ and $dF(t,T)$ in 13.

$$d\ln(F(t,T)) = \sum_{i=1}^{n}\sigma_i(T)e^{-\alpha_i(T-t)}dz_i(t) - \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n}\sigma_i(T)\sigma_j(T)e^{-(\alpha_i+\alpha_j)(T-t)}\rho_{i,j}dt \tag{15}$$

The purpose of finding the stochastic differential of $ln(F(t,T))$ is to remove the $F(t,T)$ coefficient of the Brownian motions, leaving an SDE which can be integrated to a form where the Ito Integrals have non-stochastic integrands, and hence have a normal distribution with known mean and variance. Integrating 15:

$$\ln(F(t_2,T)) = \ln(F(t_1,t)) - \frac{1}{2}V(t_1,t_2,T) + I(t_1,t_2,T) \tag{16}$$

Where:

$$V(t_1,t_2,T) = \sum_{i=1}^{n}\sigma_i(T)\sum_{j=1}^{n}\sigma_j(T)\rho_{i,j}\int_{t_1}^{t_2} e^{-(\alpha_i+\alpha_j)(T-u)}du$$

$$= \sum_{i=1}^{n}\sigma_i(T)\sum_{j=1}^{n}\sigma_j(T)\rho_{i,j}\frac{1}{\alpha_i+\alpha_j}(e^{-(\alpha_i+\alpha_j)(T-t_2)} - e^{-(\alpha_i+\alpha_j)(T-t_1)}) \tag{17}$$

And:

$$I(t_1,t_2,T) = \sum_{i=1}^{n}\sigma_i(T)\int_{t_1}^{t_2} e^{-\alpha_i(T-u)}dz_i(u) \tag{18}$$

There is a problem in that $V(t_1, t_2, T)$ is not defined for the case where $\alpha_i + \alpha_j = 0$. A continuous extension is required in order for this case to be allowed. Consider the function ψ which is only defined for the domain $\mathbb{R}_{\neq 0}$, the set real numbers, excluding zero.

$$\psi_{c_1,c_2}(x) = \frac{e^{-xc_2} - e^{-xc_1}}{x} \quad x \in \mathbb{R}_{\neq 0} \tag{19}$$

The first step to defining the continous extension is to write ψ as a Taylor Series about zero.

$$\psi_{c_1,c_2}(x) = \frac{1 - xc_2 - 1 + xc_1 + O(x^2)}{x} \tag{20}$$

Taking the limit of $\psi(x)$ towards zero, the $O(x^2)$ becomes negligible, hence:

$$\begin{aligned} \lim_{x \to 0} \psi_{c_1,c_2}(x) = \frac{1 - xc_2 - 1 + xc_1}{x} \\ = c_1 - c_2 \end{aligned} \tag{21}$$

Using this, we can define ϑ as the continous extension of ψ:

$$\vartheta_{c_1,c_2}(x) = \begin{cases} \psi_{c_1,c_2}(x) & \text{for} \quad x \in \mathbb{R}_{\neq 0} \\ c_1 - c_2 & \text{for} \quad x = 0 \end{cases} \tag{22}$$

Which in the case of equation for V is:

$$\vartheta_{T-t_1,T-t_2}(x) = \begin{cases} \psi_{T-t_1,T-t_2}(x) & \text{for} \quad x \in \mathbb{R}_{\neq 0} \\ t_2 - t_1 & \text{for} \quad x = 0 \end{cases} \tag{23}$$

Using this, we can redefine $V(t_1, t_2, T)$ to include the case of α_i and α_j both being equal to zero:

$$V(t_1, t_2, T) = \sum_{i=1}^{n} \sigma_i(T) \sum_{j=1}^{n} \sigma_j(T) \rho_{i,j} \vartheta_{T-t_1,T-t_2}(\alpha_i + \alpha_j)$$

Hence the forward price can be expressed as:

$$F(t_2, T) = F(t_1, t)e^{-\frac{1}{2}V(t_1,t_2,T)+I(t_1,t_2,T)} \tag{24}$$

B Simulation of Forward Curve

B.1 Single Commodity

To illustrate how the whole forward curve can be simulated, first we shall introduce $Y(t_1, t_2)$:

$$Y_i(t_1, t_2) = \int_{t_1}^{t_2} e^{\alpha_i u} dz_i(u) \tag{25}$$

This Ito integral can be recognised as one of the stochastic factors from the definition of $I(t_1, t_2, T)$, with the forward maturity T factored out. The foward price can then be expressed as:

$$F(t_2, T) = F(t_1, T) e^{-\frac{1}{2}V(t_1,t_2,T) + \sum_{i=1}^{n} \sigma_i(T) e^{-\alpha_i T} Y_i(t_1,t_2)} \tag{26}$$

Where $Y_i(t_1, t_2)$ are normally distributed, with mean zero and covariance:

$$cov(Y_i(t_1, t_2), Y_j(t_1, t_2)) = \rho_{i,j} \frac{1}{\alpha_i + \alpha_j} (e^{(\alpha_i + \alpha_j)t_2} - e^{(\alpha_i + \alpha_j)t_1}) \tag{27}$$

Redefining this with a continuous extension to allow $\alpha_i + \alpha_j$ to equal zero:

$$cov(Y_i(t_1, t_2), Y_j(t_1, t_2)) = \rho_{i,j} \vartheta_{-t_1,-t_2}(\alpha_i + \alpha_j) \tag{28}$$

Using the above form it can be seen that the stochastic evolution of the whole forward curve from t_1 to t_2 can be achieved as a function of n generated random numbers.

B.2 Multiple Commodities

The simulation of the whole forward curve for multiple commodities can be illustrated by a simple extension to the single commodity case:

$$F^l(t_2, T) = F^l(t_1, T) e^{-\frac{1}{2}V^l(t_1,t_2,T) + \sum_{i=1}^{n} \sigma_i^l(T) e^{-\alpha_i^l T} Y_i^l(t_1,t_2)} \tag{29}$$

Where $V^l(t_1, t_2, T)$ is defined as being identical to $V(t_1, t_2, T)$, except with n and α variables being superscripted with l, and ρ is superscripted with l, l. $Y_i^l(t_1, t_2)$ are normally distributed, with mean zero and covariance:

$$cov(Y_i^x(t_1, t_2), Y_j^y(t_1, t_2)) = \rho_{i,j}^{x,y} \vartheta_{-t_1,-t_2}(\alpha_i^x + \alpha_j^y) \tag{30}$$

C Integrated Forward Covariances

This section derives a closed form expression for the integrated covariances of forward price log returns, a quantity is referred to as "terminal covariance" in Rebonato (2013).

C.1 Single Commodity

Defining $C(t_1, t_2, T_1, T_2)$ as the integrated covariance from t_1 to t_2, between the log returns of forward contracts delivering on respective periods starting on T_1 and T_2.

$$C(t_1, t_2, T_1, T_2) = \mathbb{E}\left[\ln\left(\frac{F(t_2, T_1)}{F(t_1, T_1)}\right)\ln\left(\frac{F(t_2, T_2)}{F(t_1, T_2)}\right)\right] \tag{31}$$

Removing the products of stochastic and deterministic terms, as they will be equal to zero:

$$C(t_1, t_2, T_1, T_2) = \mathbb{E}\big[I(t_1, t_2, T_1)I(t_1, t_2, T_2)\big] \tag{32}$$

Using the known properties of Ito Integrals:

$$\begin{aligned} C(t_1, t_2, T_1, T_2) &= \sum_{i=1}^{n}\sigma_i(T)\sum_{j=1}^{n}\sigma_j(T)\rho_{i,j}e^{-\alpha_i T_1 - \alpha_j T_2}\int_{t_1}^{t_2} e^{u(\alpha_i+\alpha_j)}du \\ &= \sum_{i=1}^{n}\sigma_i(T)\sum_{j=1}^{n}\sigma_j(T)\rho_{i,j}e^{-\alpha_i T_1 - \alpha_j T_2}\frac{1}{\alpha_i+\alpha_j}(e^{t_2(\alpha_i+\alpha_j)} - e^{t_1(\alpha_i+\alpha_j)}) \end{aligned} \tag{33}$$

Allowing for the case of α_i and α_j both being equal to zero, as in the previous appendix, a continuous extension is made to redefine the function C:

$$C(t_1, t_2, T_1, T_2) = \sum_{i=1}^{n}\sigma_i(T)\sum_{j=1}^{n}\sigma_j(T)\rho_{i,j}e^{-\alpha_i T_1 - \alpha_j T_2}\vartheta_{-t_1,-t_2}(\alpha_i+\alpha_j) \tag{34}$$

C.2 Multiple Commodities

Defining $C(t_1, t_2, T_1, T_2, x, y)$ as the integrated covariance from t_1 to t_2, between the log returns of forward contracts delivering on respective periods starting on T_1 for commodity x, and T_2 for commodity y, the results in the previous subsection can be updated to show:

$$C(t_1, t_2, T_1, T_2, x, y) = \sum_{i=1}^{n^x}\sigma_i^x(T)\sum_{j=1}^{n^y}\sigma_j^y(T)\rho_{i,j}^{x,y}e^{-\alpha_i^x T_1 - \alpha_j^y T_2}\vartheta_{-t_1,-t_2}(\alpha_i^x+\alpha_j^y) \tag{35}$$

D Spot Price Process Distribution and Simulation

D.1 Single Commodity

Defining the spot price $S(t)$ as the price for delivery of the commodity in the period starting at time t, observed at time t. Using the equivalence between

$S(t)$ and $F(t,t)$ and equation 3:

$$S(t) = F(0,t)e^{-\frac{1}{2}V(0,t,t)+I(0,t,t)} \tag{36}$$

When simulating a spot price path it is necessary to know the relationship between $S(t_k)$ and $S(t_{k-1})$, where $0 \leq t_{k-1} < t_k$. Defining $V_s(t) = V(0,t,t)$ and $I_s(t) = I(0,t,t)$:

$$S(t_k) = S(t_{k-1})\frac{F(0,t_k)}{F(0,t_{k-1})}e^{-\frac{1}{2}(V_s(t_k)-V_s(t_{k-1}))+I_s(t_k)-I_s(t_{k-1})} \tag{37}$$

Focussing on the stochastic term $I_s(t_k) - I_s(t_{k-1})$:

$$I_s(t_k) - I_s(t_{k-1}) = \sum_{i=1}^{n}\Bigg(\sigma_i(t_k)\int_0^{t_k} e^{-\alpha_i(t_k-u)}dz_i(u) - \sigma_i(t_{k-1})\int_0^{t_{k-1}} e^{-\alpha_i(t_{k-1}-u)}dz_i(u)\Bigg) \tag{38}$$

Defining:

$$f_i(t) = \int_0^t e^{-\alpha_i(t-u)}dz_i(u) \tag{39}$$

Substitution this into the above equation:

$$I_s(t_k) - I_s(t_{k-1}) = \sum_{i=1}^{n}\Bigg(\sigma_i(t_k)f_i(t_k) - \sigma_i(t_{k-1})f_i(t_{k-1})\Bigg) \tag{40}$$

When simulating, this expression will be adaped to the filtration $\mathcal{F}_{t_{k-1}}$, hence the $f_i(t_{k-1})$ will have been realised. $f_i(t_k)$ will be part realised, hence it is instructive to split this into it's deterministic and stochastic parts, as of t_{k-1}. First making the substituting $t_k = t_{k-1} + t_k - t_{k-1}$:

$$\begin{aligned} f_i(t_k) &= e^{-\alpha_i(t_k-t_{k-1})}\int_0^{t_k} e^{-\alpha_i(t_{k-1}-u)}dz_i(u) \\ &= e^{-\alpha_i(t_k-t_{k-1})}\Bigg(\int_0^{t_{k-1}} e^{-\alpha_i(t_{k-1}-u)}dz_i(u) + \int_{t_{k-1}}^{t_k} e^{-\alpha_i(t_{k-1}-u)}dz_i(u)\Bigg) \end{aligned} \tag{41}$$

Noticing that the first integral is the definition of $f_i(t_{k-1})$, the function f_i can be defined recursively:

$$f_i(t_k) = e^{-\alpha_i(t_k-t_{k-1})}f_i(t_{k-1}) + \int_{t_{k-1}}^{t_k} e^{-\alpha_i(t_k-u)}dz_i(u) \tag{42}$$

For any $t_k > t_{k-1}$ and base case:

$$f_i(0) = 0 \tag{43}$$

In this form $f_i(t)$ can be recognised as an Ornstein–Uhlenbeck process with zero drift, mean reversion parameter α_i, and σ (coeffient of Brownian motion) of 1.

When simulating the spot price for an increasing sequence of n future times $t_1, t_2 \ldots t_{n-1}, t_n$ where $0 < t_{k-1} < t_k$, for each of these times the first step is to calculate $f_i(t_k)$ recursively, using the above relationship for $i = 1, \ldots, n$.

$$f_i(t_k) = e^{-\alpha_i(t_k - t_{k-1})} f_i(t_{k-1}) + Z_i(t_k) \tag{44}$$

Where, using the property of Ito Integrals $Z_i(t_k)$ is multivariate normally distributed, with mean zero and covariance:

$$cov(Z_i(t_k), Z_j(t_k)) = \frac{\rho_{i,j}}{\alpha_i + \alpha_j}(1 - e^{-(\alpha_i + \alpha_j)(t_k - t_{k-1})}) \tag{45}$$

Similar to before, we have a situation where the covariance is not defined for the case where both α_i and α_j are equal to zero. This case can be allowed, as before, by redefining $cov(Z_i(t_k), Z_j(t_k))$ to it's continuous extension:

$$cov(Z_i(t_k), Z_j(t_k)) = \rho_{i,j} \vartheta_{t_k - t_{k-1}, 0}(\alpha_i + \alpha_j) \tag{46}$$

The spot price can then be calculated recursively from the values of $f_i(t_k)$ for $k > 1$:

$$S(t_k) = S(t_{k-1}) \frac{F(0, t_k)}{F(0, t_{k-1})} e^{-\frac{1}{2}(V_s(t_k) - V_s(t_{k-1})) + \sum_{i=1}^{n} \left(\sigma_i(t_k) f_i(t_k) - \sigma_i(t_{k-1}) f_i(t_{k-1})\right)} \tag{47}$$

With the special case where $k = 1$:

$$S(t_1) = F(0, t_1) e^{-\frac{1}{2} V_s(t_1) + \sum_{i=1}^{n} \sigma_i(t_1) f_i(t_1)} \tag{48}$$

D.2 Multiple Commodities

For multiple commodities, simulation of m commodity underlying spot prices is performed for $l \in [1, m]$ using the following:

$$S^l(t_k) = S^l(t_{k-1}) \frac{F^l(0, t_k)}{F^l(0, t_{k-1})} e^{-\frac{1}{2}(V_s^l(t_k) - V_s^l(t_{k-1})) + \sum_{i=1}^{n^l} \left(\sigma_i^l(t_k) f_i^l(t_k) - \sigma_i^l(t_{k-1}) f_i^l(t_{k-1})\right)} \tag{49}$$

Where V_s^l is defined as being the same as V_s, except with n and all α and σ items being superscripted with l. f_i^l is defined recursively as:

$$f_i^l(t_k) = e^{-\alpha_i^l(t_k - t_{k-1})} f_i^l(t_{k-1}) + Z_i^l(t_k) \tag{50}$$

$Z_i^l(t_k)$ is multivariate normally distributed, with mean zero and covariance:

$$cov(Z_i^x(t_k), Z_j^y(t_k)) = \rho_{i,j}^{x,y} \vartheta_{t_k - t_{k-1}, 0}(\alpha_i^x + \alpha_j^y) \tag{51}$$

D.3 Markov Property of Spot Price

Using the definition of f_i, 47 can be rewritten as:

$$S(t_k) = S(t_{k-1}) \frac{F(0,t_k)}{F(0,t_{k-1})} e^{-\frac{1}{2}(V_s(t_k)-V_s(t_{k-1}))+\sum_{i=1}^{n}\left(\sigma_i(t_{k-1}Z_i(t_k))+f_i(t_{k-1})(\sigma_i(t_k)e^{-\alpha_i(t_k-t_{k-1})}-\sigma_i(t_{k-1}))\right)} \tag{52}$$

All terms in the right hand side of the above equation are deterministic, apart from Z_i, $S(t_{k-1})$ and $f_i(t_{k-1})$, and given knowledge of the filtration $\mathcal{F}_{t_{k-1}}$ only Z_i is random.

Defining the vectors $\mathbf{z}$ and $\mathbf{f}_{t_k}$, both containing n elements, with the i^{th} element being equal to $Z_i(t_k)$ and $f_i(t_k)$ respectively.

$$S(t_k) = \omega(\mathbf{z}, \mathbf{f}_{t_{k-1}}) \tag{53}$$

Note that $S(t_{k-1})$ has been omitted from the above, as it itself a deterministic function of the f_i terms. Given knowledge of the filtration $\mathcal{F}_{t_{k-1}}$ the expectation of an arbitray function g can be written as:

$$\mathbb{E}[g(S(t_k))|\mathcal{F}_{t_{k-1}}] = \int_{\mathbb{R}} \dots \int_{\mathbb{R}} \omega(\mathbf{u}, \mathbf{f}_{t_{k-1}}) h(\mathbf{u}) du_1 \dots du_n \tag{54}$$

Where h is the probability density function of the multivariate normal distribution with covariance given by 51.

$$\mathbb{E}[g(S(t_k))|\mathcal{F}_{t_{k-1}}] = \mathbb{E}[g(S(t_k))|\mathbf{f}_{t_{k-1}}] \tag{55}$$

The intuitive meaning of this is that, at any point in time t, given knowledge of the vector of mean reverting factors $\mathbf{f}_t$, we know the exact distribution of the spot price for any time greater than t.

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