Multi-Factor Commodity Price Process: Spot and Forward Price Simulation

Jake C. Fowler

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1 Introduction

This document describes a very general multi-factor price process, which is suitable use in simulating the spot and forward prices of a a commodity under the risk-neutral probability measure.

$$\frac{dF(t,T)}{F(t,T)} = \sum_{i=1}^{n} \sigma_i(T) e^{-\alpha_i (T-t)} dz_i(t) \tag{1}$$

Where $F(t,T)$ is the forward price observed at time t, for delivery over the period starting at time T.

Integrating this SDE (see Appendix I):

$$F(t_2,T) = F(t_1,t) e^{-\frac{1}{2} V(t_1,t_2,T) + I(t_1,t_2,T)} \tag{2}$$

A closed form expression exists for the integrated covariances of forward price log returns. This is derived in Appendix II. The evaluation of such covariances is essential to understand the statistical properties that the model implies for the dynamics of the forward curve. It also allows the calibration of model parameters to option market implied volatilities and historical covariances. Defining $C(t_1,t_2,T_1,T_2)$ as the integrated covariance from t_1 to t_2, between the log returns of forward contracts delivering on respective periods starting on T_1 and T_2.

$$C(t_1,t_2,T_1,T_2) = \sum_{i=1}^{n} \sigma_i(T) \sum_{j=1}^{n} \sigma_j(T) \rho_{i,j} e^{-\alpha_i T_1 - \alpha_j T_2} \frac{1}{\alpha_i + \alpha_j} (e^{t_2(\alpha_i+\alpha_j)} - e^{t_1(\alpha_i+\alpha_j)}) \tag{3}$$

2 Appendices

2.1 Appendix I - Integration of Forward Price SDE

This appendix gives the detailed step for integrated the forward price process SDE, hence getting from 1 to 2.

Using Ito's Lemma to calculate the stochastic differential of the natural logarithm of the forward price.

$$d\ln(F(t,T)) = \frac{1}{F(t,T)}dF(t,T) - \frac{1}{2}\frac{1}{F(t,T)^2}(dF(t,T))^2 \tag{4}$$

Using the properties of Brownian Motion $(dz(t))^2 = dt$ and $dtdz(t) = 0$ the forward price differential squared can be evaluated as follows.

$$(dF(t,T))^2 = F(t,T)^2 \sum_{i=1}^{n}\sum_{j=1}^{n} \sigma_i(T)\sigma_j(T)e^{-(\alpha_i+\alpha_j)(T-t)}\rho_{i,j}dt \tag{5}$$

Substituting for $(dF(t,T))^2$ and $dF(t,T)$ in 4.

$$d\ln(F(t,T)) = \sum_{i=1}^{n} \sigma_i(T)e^{-\alpha_i(T-t)}dz_i(t) - \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n} \sigma_i(T)\sigma_j(T)e^{-(\alpha_i+\alpha_j)(T-t)}\rho_{i,j}dt \tag{6}$$

The purpose of finding the stochastic differential of $ln(F(t,T))$ is to remove the $F(t,T)$ coefficient of the Brownian motions, leaving an SDE which can be integrated to a form where the Ito Integrals have non-stochastic integrands, and hence have a normal distribution with known mean and variance. Integrating 6:

$$\ln(F(t_2,T)) = \ln(F(t_1,t)) - \frac{1}{2}V(t_1,t_2,T) + I(t_1,t_2,T) \tag{7}$$

Where:

$$V(t_1,t_2,T) = \sum_{i=1}^{n} \sigma_i(T) \sum_{j=1}^{n} \sigma_j(T)\rho_{i,j} \int_{t_1}^{t_2} e^{-(\alpha_i+\alpha_j)(T-u)}du$$

$$= \sum_{i=1}^{n} \sigma_i(T) \sum_{j=1}^{n} \sigma_j(T)\rho_{i,j} \frac{1}{\alpha_i+\alpha_j}(e^{-(\alpha_i+\alpha_j)(T-t_2)} - e^{-(\alpha_i+\alpha_j)(T-t_1)})$$

And:

$$I(t_1,t_2,T) = \sum_{i=1}^{n} \sigma_i(T) \int_{t_1}^{t_2} e^{-\alpha_i(T-u)}dz_i(u) \tag{8}$$

Hence the forward price can be expressed as:

$$F(t_2, T) = F(t_1, t)e^{-\frac{1}{2}V(t_1,t_2,T)+I(t_1,t_2,T)} \tag{9}$$

2.2 Appendix II - Forward Covariances

This section derives a closed form expression for the integrated covariances of forward price log returns. Defining $C(t_1, t_2, T_1, T_2)$ as the integrated covariance from t_1 to t_2, between the log returns of forward contracts delivering on respective periods starting on T_1 and T_2.

$$C(t_1, t_2, T_1, T_2) = \mathbb{E}\left[\ln\left(\frac{F(t_2, T_1)}{F(t_1, T_1)}\right)\ln\left(\frac{F(t_2, T_2)}{F(t_1, T_2)}\right)\right] \tag{10}$$

Remove the products of stochastic and deterministic terms, as they will equal zero:

$$C(t_1, t_2, T_1, T_2) = \mathbb{E}\big[I(t_1, t_2, T_1)I(t_1, t_2, T_2)\big] \tag{11}$$

Using the known properties of Ito Integrals:

$$\begin{aligned} C(t_1, t_2, T_1, T_2) &= \sum_{i=1}^{n}\sigma_i(T)\sum_{j=1}^{n}\sigma_j(T)\rho_{i,j}e^{-\alpha_i T_1-\alpha_j T_2}\int_{t_1}^{t_2}e^{u(\alpha_i+\alpha_j)}du \\ &= \sum_{i=1}^{n}\sigma_i(T)\sum_{j=1}^{n}\sigma_j(T)\rho_{i,j}e^{-\alpha_i T_1-\alpha_j T_2}\frac{1}{\alpha_i+\alpha_j}(e^{t_2(\alpha_i+\alpha_j)} - e^{t_1(\alpha_i+\alpha_j)}) \end{aligned} \tag{12}$$