Multi-Factor Commodity Price Process: Spot and Forward Price Simulation

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1 Introduction

This document describes a very general multi-factor price process, which is suitable use in simulating the spot and forward prices of a a commodity under the risk-neutral probability measure.

$$\frac{dF(t,T)}{F(t,T)} = \sum_{i=1}^{n} \sigma_i(T) e^{-\alpha_i(T-t)} dz_i(t) \tag{1}$$

Where $F(t,T)$ is the forward price observed at time t, for delivery over the period starting at time T.

Integrating this SDE (see Appendix I):

$$F(t_2,T) = F(t_1,t) e^{-\frac{1}{2}V(t_1,t_2,T)+I(t_1,t_2,T)} \tag{2}$$

A closed form expression exists for the integrated covariances of forward price log returns. This is derived in Appendix II. The evaluation of such covariances is essential to understand the statistical properties that the model implies for the dynamics of the forward curve. It also allows the calibration of model parameters to option market implied volatilities and historical covariances. Defining $C(t_1,t_2,T_1,T_2)$ as the integrated covariance from t_1 to t_2, between the log returns of forward contracts delivering on respective periods starting on T_1 and T_2.

$$C(t_1,t_2,T_1,T_2) = \sum_{i=1}^{n} \sigma_i(T) \sum_{j=1}^{n} \sigma_j(T) \rho_{i,j} e^{-\alpha_i T_1 - \alpha_j T_2} \vartheta_{-t_1,-t_2}(\alpha_i + \alpha_j) \tag{3}$$

2 Appendices

2.1 Appendix I - Integration of Forward Price SDE

This appendix gives the detailed step for integrated the forward price process SDE, hence getting from 1 to 2.

Using Ito's Lemma to calculate the stochastic differential of the natural logarithm of the forward price.

$$d\ln(F(t,T)) = \frac{1}{F(t,T)}dF(t,T) - \frac{1}{2}\frac{1}{F(t,T)^2}(dF(t,T))^2 \tag{4}$$

Using the properties of Brownian Motion $(dz(t))^2 = dt$ and $dtdz(t) = 0$ the forward price differential squared can be evaluated as follows.

$$(dF(t,T))^2 = F(t,T)^2 \sum_{i=1}^{n}\sum_{j=1}^{n} \sigma_i(T)\sigma_j(T)e^{-(\alpha_i+\alpha_j)(T-t)}\rho_{i,j}dt \tag{5}$$

Substituting for $(dF(t,T))^2$ and $dF(t,T)$ in 4.

$$d\ln(F(t,T)) = \sum_{i=1}^{n} \sigma_i(T)e^{-\alpha_i(T-t)}dz_i(t) - \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n} \sigma_i(T)\sigma_j(T)e^{-(\alpha_i+\alpha_j)(T-t)}\rho_{i,j}dt \tag{6}$$

The purpose of finding the stochastic differential of $ln(F(t,T))$ is to remove the $F(t,T)$ coefficient of the Brownian motions, leaving an SDE which can be integrated to a form where the Ito Integrals have non-stochastic integrands, and hence have a normal distribution with known mean and variance. Integrating 6:

$$\ln(F(t_2,T)) = \ln(F(t_1,t)) - \frac{1}{2}V(t_1,t_2,T) + I(t_1,t_2,T) \tag{7}$$

Where:

$$V(t_1,t_2,T) = \sum_{i=1}^{n} \sigma_i(T) \sum_{j=1}^{n} \sigma_j(T)\rho_{i,j} \int_{t_1}^{t_2} e^{-(\alpha_i+\alpha_j)(T-u)}du$$

$$= \sum_{i=1}^{n} \sigma_i(T) \sum_{j=1}^{n} \sigma_j(T)\rho_{i,j} \frac{1}{\alpha_i+\alpha_j}(e^{-(\alpha_i+\alpha_j)(T-t_2)} - e^{-(\alpha_i+\alpha_j)(T-t_1)}) \tag{8}$$

And:

$$I(t_1,t_2,T) = \sum_{i=1}^{n} \sigma_i(T) \int_{t_1}^{t_2} e^{-\alpha_i(T-u)}dz_i(u) \tag{9}$$

There is a problem in that $V(t_1, t_2, T)$ is not defined for the case where $\alpha_i + \alpha_j = 0$. A continuous extension is required in order for this case to be allowed. Consider the function ψ which is only defined for the domain $\mathbb{R}_{\neq 0}$, the set real numbers, excluding zero.

$$\psi_{c_1,c_2}(x) = \frac{e^{-xc_2} - e^{-xc_1}}{x} \quad x \in \mathbb{R}_{\neq 0} \tag{10}$$

The first step to defining the continous extension is to write ψ as a Taylor Series about zero.

$$\psi_{c_1,c_2}(x) = \frac{1 - xc_2 - 1 + xc_1 + O(x^2)}{x} \tag{11}$$

Taking the limit of $\psi(x)$ towards zero, the $O(x^2)$ becomes negligible, hence:

$$\begin{aligned}\lim_{x\to 0} \psi_{c_1,c_2}(x) = \frac{1 - xc_2 - 1 + xc_1}{x} \\ = c_1 - c_2\end{aligned} \tag{12}$$

Using this, we can define ϑ as the continous extension of ψ:

$$\vartheta_{c_1,c_2}(x) = \begin{cases} \psi_{c_1,c_2}(x) & \text{for} \quad x \in \mathbb{R}_{\neq 0} \\ c_1 - c_2 & \text{for} \quad x = 0 \end{cases} \tag{13}$$

Which in the case of equation for V is:

$$\vartheta_{T-t_1,T-t_2}(x) = \begin{cases} \psi_{T-t_1,T-t_2}(x) & \text{for} \quad x \in \mathbb{R}_{\neq 0} \\ t_2 - t_1 & \text{for} \quad x = 0 \end{cases} \tag{14}$$

Using this, we can redefine $V(t_1, t_2, T)$ to include the case of α_i and α_j both being equal to zero:

$$V(t_1, t_2, T) = \sum_{i=1}^{n} \sigma_i(T) \sum_{j=1}^{n} \sigma_j(T) \rho_{i,j} \vartheta_{T-t_1,T-t_2}(\alpha_i + \alpha_j)$$

Hence the forward price can be expressed as:

$$F(t_2, T) = F(t_1, t) e^{-\frac{1}{2}V(t_1,t_2,T) + I(t_1,t_2,T)} \tag{15}$$

2.2 Appendix II - Forward Covariances

This section derives a closed form expression for the integrated covariances of forward price log returns. Defining $C(t_1, t_2, T_1, T_2)$ as the integrated covariance from t_1 to t_2, between the log returns of forward contracts delivering on respective periods starting on T_1 and T_2.

$$C(t_1, t_2, T_1, T_2) = \mathbb{E}\left[\ln\left(\frac{F(t_2, T_1)}{F(t_1, T_1)}\right)\ln\left(\frac{F(t_2, T_2)}{F(t_1, T_2)}\right)\right] \tag{16}$$

Remove the products of stochastic and deterministic terms, as they will equal zero:

$$C(t_1, t_2, T_1, T_2) = \mathbb{E}\big[I(t_1, t_2, T_1)I(t_1, t_2, T_2)\big] \tag{17}$$

Using the known properties of Ito Integrals:

$$C(t_1, t_2, T_1, T_2) = \sum_{i=1}^{n}\sigma_i(T)\sum_{j=1}^{n}\sigma_j(T)\rho_{i,j}e^{-\alpha_i T_1-\alpha_j T_2}\int_{t_1}^{t_2} e^{u(\alpha_i+\alpha_j)}du$$

$$= \sum_{i=1}^{n}\sigma_i(T)\sum_{j=1}^{n}\sigma_j(T)\rho_{i,j}e^{-\alpha_i T_1-\alpha_j T_2}\frac{1}{\alpha_i+\alpha_j}(e^{t_2(\alpha_i+\alpha_j)} - e^{t_1(\alpha_i+\alpha_j)}) \tag{18}$$

Allowing for the case of α_i and α_j both being equal to zero, as in the previous appendix, a continuous extension is made to redefine the function C:

$$C(t_1, t_2, T_1, T_2) = \sum_{i=1}^{n}\sigma_i(T)\sum_{j=1}^{n}\sigma_j(T)\rho_{i,j}e^{-\alpha_i T_1-\alpha_j T_2}\vartheta_{-t_1,-t_2}(\alpha_i+\alpha_j) \tag{19}$$

2.3 Appendix III - Spot Price Process

Defining the spot price $S(t)$ as the price for delivery of the commodity in the period starting at time t, observed at time t. Using the equivalence between $S(t)$ and $F(t,t)$ and equation 2:

$$S(t) = F(0,t)e^{-\frac{1}{2}V(0,t,t)+I(0,t,t)} \tag{20}$$

When simulating a spot price path it is necessary to know the relationship between $S(t_k)$ and $S(t_{k-1})$, where $0 \leq t_{k-1} < t_k$. Defining $V_s(t) = V(0,t,t)$ and $I_s(t) = I(0,t,t)$:

$$S(t_k) = S(t_{k-1})\frac{F(0,t_k)}{F(0,t_{k-1})}e^{-\frac{1}{2}(V_s(t_k)-V_s(t_{k-1}))+I_s(t_k)-I_s(t_{k-1})} \tag{21}$$

Focussing on the stochastic term $I_s(t_k) - I_s(t_{k-1})$:

$$I_s(t_k) - I_s(t_{k-1}) = \sum_{i=1}^{n} \Big(\sigma_i(t_k) \int_0^{t_k} e^{-\alpha_i(t_k - u)} dz_i(u) - \sigma_i(t_{k-1}) \int_0^{t_{k-1}} e^{-\alpha_i(t_{k-1} - u)} dz_i(u) \Big) \tag{22}$$

Defining:

$$F_i(t) = \int_0^t e^{-\alpha_i(t-u)} dz_i(u) \tag{23}$$

Substitution this into the above equation:

$$I_s(t_k) - I_s(t_{k-1}) = \sum_{i=1}^{n} \Big(\sigma_i(t_k) F_i(t_k) - \sigma_i(t_{k-1}) F_i(t_{k-1}) \Big) \tag{24}$$

When simulating, this expression will be adaped to the filtration $\mathcal{F}_{t_{k-1}}$, hence the $F_i(t_{k-1})$ will have been realised. $F_i(t_k)$ will be part realised, hence it is instructive to split this into it's deterministic and stochastic parts, as of t_{k-1}. First making the substituting $t_k = t_{k-1} + t_k - t_{k-1}$:

$$F_i(t_k) = e^{-\alpha_i(t_k - t_{k-1})} \int_0^{t_k} e^{-\alpha_i(t_{k-1} - u)} dz_i(u)$$

$$= e^{-\alpha_i(t_k - t_{k-1})} \Big(\int_0^{t_{k-1}} e^{-\alpha_i(t_{k-1} - u)} dz_i(u) + \int_{t_{k-1}}^{t_k} e^{-\alpha_i(t_{k-1} - u)} dz_i(u) \Big) \tag{25}$$

Noticing that the first integral is the definition of $F_i(t_{k-1})$, the function F_i can be defined recursively:

$$F_i(t_k) = e^{-\alpha_i(t_k - t_{k-1})} F_i(t_{k-1}) + \int_{t_{k-1}}^{t_k} e^{-\alpha_i(t_k - u)} dz_i(u) \tag{26}$$

For any $t_k > t_{k-1}$ and base case:

$$F_i(0) = 0 \tag{27}$$

In this form $F_i(t)$ can be recognised as an Ornstein–Uhlenbeck process with zero drift, mean reversion parameter α_i, and σ (coeffient of Brownian motion) of 1.

When simulating the spot price for an increasing sequence of n times $t_{k-1}, t_2 \ldots t_{n-1}, t_n$ where $0 < t_{k-1} < t_k$, for each of these times the first step is to calculate $F_i(t_k)$ recursively, using the above relationship for $i = 1, \ldots, n$.

$$F_i(t_k) = e^{-\alpha_i(t_k - t_{k-1})} F_i(t_{k-1}) + Z_i(t_k) \tag{28}$$

Where, using the property of Ito Integrals $Z_i(t_k)$ is multivariate normally distributed, with mean zero and covariance:

$$cov(Z_i, Z_j) = \frac{\rho_{i,j}}{\alpha_i + \alpha_j}(1 - e^{-(\alpha_i+\alpha_j)(t_k - t_{k-1})}) \tag{29}$$

The spot price can then be calculated from the values of $F_i(t_k)$:

$$S(t_k) = S(t_{k-1}) \frac{F(0, t_k)}{F(0, t_{k-1})} e^{-\frac{1}{2}(V_s(t_k) - V_s(t_{k-1})) + \sum_{i=1}^{n} \left(\sigma_i(t_k) F_i(t_k) - \sigma_i(t_{k-1}) F_i(t_{k-1}) \right)} \tag{30}$$