Supplementary Materials - MultIHeaTS : an Open Source Implicit Thermal Solver for 1D Multilayered Surfaces

C. Mergny[1*] and F. Schmidt[1,2]

[1*]GEOPS, Paris-Saclay University, Orsay, France.
[2] Institut Universitaire de France, paris, France.

*Corresponding author(s). E-mail(s):
cyril.mergny@universite-paris-saclay.fr;
Contributing authors: frederic.schmidt@universite-paris-saclay.fr;

1 Dirichlet Boundary Conditions

The temperature is fixed at the boundaries which gives us at the top boundary $x = 0$:

$$\forall t, T(0,t) = T_0(t) \implies \begin{cases} b_0^i = 1 \\ c_0^i = 0 \\ s_0^i = T_0^i, \end{cases} \tag{1}$$

at at the bottom boundary $x = b$,

$$\forall t, T(b,t) = T_{nx-1}(t) \implies \begin{cases} a_{nx-1}^i = 0 \\ b_{nx-1}^i = 1 \\ s_{nx-1}^i = T_{nx-1}^i. \end{cases} \tag{2}$$

2 Analytic Solution of a Step Fonction

2.0.1 Fourier series

Any periodical function f in $\mathbb{R}$ of period P can be written as a Fourier series:

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{+\infty} a_{nx-1} \cos\left(\frac{2\pi nx}{P}\right) + b_n \sin\left(\frac{2\pi nx}{P}\right), \tag{3}$$

with the coefficients a_n and b_n given by the expressions:

$$a_n = \frac{2}{P} \int_P f(x) \cos\left(\frac{2\pi nx}{P}\right) dx, \tag{4}$$

$$b_n = \frac{2}{P} \int_P f(x) \sin\left(\frac{2\pi nx}{P}\right) dx. \tag{5}$$

The expressions are obtained by integrating equation (3) as demonstrated here:

$$\begin{aligned}
\int_P f(x) \cos\left(\frac{2\pi mx}{P}\right) dx &= \int_P \cos\left(\frac{2\pi mx}{P}\right) \left[\frac{a_0}{2} + \sum_{n=1}^{+\infty} a_n \cos\left(\frac{2\pi nx}{P}\right) + b_n \sin\left(\frac{2\pi nx}{P}\right)\right] dx \\
&= \sum_{n=1}^{+\infty} \int_P a_n \cos\left(\frac{2\pi nx}{P}\right) \cos\left(\frac{2\pi mx}{P}\right) dx \\
&= a_m \frac{P}{2}
\end{aligned}$$

2.0.2 Fourier series of a step function

Every function on a interval of length P can be extended to a periodic function in $\mathbb{R}$ of period P. First we will extend function H on the interval $[-\frac{L}{2}, \frac{3L}{2}]$ such that the derivative at $x = 0$ and $x = L$ are equal to zero (Boundary flux condition). The step function is defined on the interval $[-\frac{L}{2}, \frac{3L}{2}]$.

We use equations (4) and (5) with $P = 2L$ the interval length to find the coefficient a_n and b_n:

$$\begin{aligned}
a_n &= \frac{1}{L} \int_{-\frac{L}{2}}^{\frac{3L}{2}} H(x) \cos\left(\frac{\pi nx}{L}\right) dx \\
&= \frac{1}{L} \int_{\frac{L}{2}}^{\frac{3L}{2}} \cos\left(\frac{\pi nx}{L}\right) dx \\
&= \frac{-2}{\pi n} \sin\left(\frac{\pi n}{2}\right)
\end{aligned}$$

and a similar integration for b_n leads to $b_n = 0$. The step function can then be written as a Fourier series using equation (3):

$$H(x) = \frac{1}{2} - \sum_{n=1}^{+\infty} \frac{4}{\pi n} \sin\left(\frac{\pi n}{2}\right) \cos\left(\frac{\pi n x}{L}\right). \tag{6}$$

2.0.3 Solving the heat equation by separation of variables

The solution of the 1D heat equation with a step function initial condition, can be obtained by separation of variables. Let's assume that the solution can be written as

$$T(x,t) = v(x)w(t) \tag{7}$$

with v and w functions of x and respectively t. This expression injected int the heat equation results to:

$$v(x)\frac{\partial w(t)}{\partial t} = \alpha \frac{\partial^2 v(x)}{\partial x^2} w(t) \tag{8}$$

This first order differential equation as for w the general solution:

$$w(t) = w(0)e^{\alpha \frac{v''(x)}{v(x)} t} \tag{9}$$

where v'' is the second derivative of v with respect to x. If we take $v(x) = T_n(x,0) = f_n(x)$ the initial function decomposed in a Fourier series, then we have:

$$v''(x) = -\left(\frac{2\pi n}{P}\right)^2 f_n(x) \tag{10}$$

and

$$\begin{aligned} T(x,0) &= v(x)w(0) = f(x) \\ v(x) &= f(x) \implies w(0) = 1 \end{aligned}$$

which give us the exact express of the solution of w with:

$$w(t) = e^{-\alpha\left(\frac{2\pi n}{P}\right)^2 t} \tag{11}$$

Hence the solution is given by:

$$T(x,t) = \frac{a_0}{2} + \sum_{n=1}^{+\infty} \left(a_n \cos\left(\frac{2\pi n x}{P}\right) + b_n \sin\left(\frac{2\pi n x}{P}\right)\right) e^{-\alpha\left(\frac{2\pi n}{P}\right)^2 t} \tag{12}$$