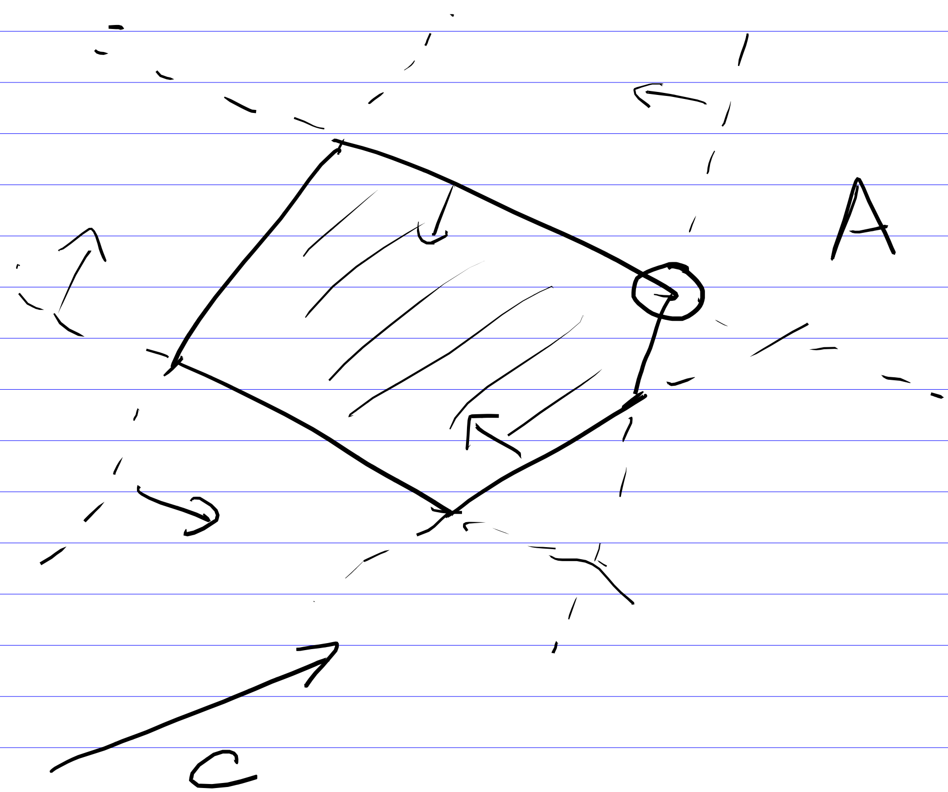
LPs. continued

$$\left.\begin{aligned} \max \quad & \sum c_i x_i = c \cdot x \\ & Ax \leq b \\ & x \geq 0 \end{aligned}\right\} \text{ Linear program}$$



Primal LP	Dual LP
$\max \; c \cdot x$	$\min \; y \cdot b$
$A \in \mathbb{R}^{m \times n}$ $\quad Ax \leq b$	$y^T A \geq c^T$
$x \geq 0$	$y \geq 0$

(1) $\forall$ feasible x, y (weak duality)

$$c \cdot x \leq y \cdot b$$

$$y^T A \geq c^T$$

$$y^T A x \geq c^T x = c \cdot x$$

$$y^T A x \leq \underbrace{y^T b}_{y \cdot b} \quad \therefore \quad c \cdot x \leq y \cdot b$$

(2) $c \cdot x^* = y^* \cdot b$ (strong duality)

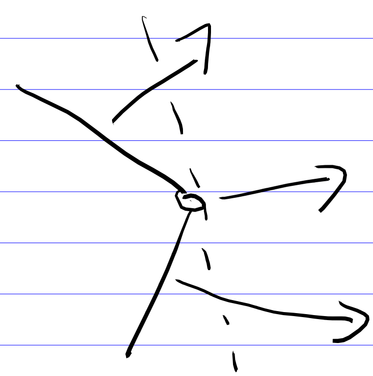
Show that given x is optimal

(1) Check $Ax \leq b$

(2) Ask for feasible y for dual LP

(3) Check $y^T A \geq c$

(4) $c \cdot x = y \cdot b$?



$\max \ c^T x$	$\min \ y^T b$
$Ax \leq b$	$y^T A \geq c^T$
$x \geq 0$	$y \geq 0$
m constraints	n constraints
n variables	m variables

primal var $\longrightarrow$ dual constraint

primal constraint $\longrightarrow$ dual variable

$$\max \quad x_1 + x_2$$

s.t.

$$x_1 + 2x_2 \leq 1$$

$$-x_1 + x_2 \leq 0$$

$$x_1 + 0.x_2 \leq 3$$

$c = (1, 1)$

$b = (1, 0, 3)$

$$A = \begin{pmatrix} 1 & 2 \\ -1 & 1 \\ 1 & 0 \end{pmatrix}$$

$$\min \quad y_1 + 3y_3$$

$$(y_1 \; y_2 \; y_3) \begin{pmatrix} & & \\ & A & \\ & & \end{pmatrix} \geq \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$y_1 - y_2 + y_3 \geq 1$$

$$2y_1 + y_2 \geq 1$$

Feasible soln: $(0, 0)$

Cost $c.x = 0$

Feasible soln $(1, 0)$

Cost $= 1$

$$\max \quad x_1 + x_2$$

s.t.

$$x_1 + 2x_2 \le 1 \quad (1)$$

$$-x_1 + x_2 \le 0 \quad (2)$$

$$x_1 + 0.x_2 \le 3 \quad (3)$$

$$x \geq 0$$

Add inequalities

$$x_1 + 3x_2 \le 4$$

$$\max \quad x_1 + x_2$$

$$\Downarrow$$

$$x_1 + x_2 \le 4$$

$$\frac{1}{3}(1) + \frac{1}{3}(2) + (3)$$

$$x_1 + x_2 \le 3\frac{1}{3}$$

$$y_1(1) + y_2(2) + y_3(3)$$

$$y^T A \geq c$$

$$\min \quad y^T b$$

$$y^T A \geq c$$

$$y \geq 0$$

$$\max \sum_{e=(s,v)} x_e - \sum_{e=(u,s)} x_e$$

$$= c \cdot x \qquad c_e = \begin{cases} 1 & e=(s,v) \\ -1 & e=(u,s) \\ 0 & \text{otherwise} \end{cases}$$

$$\left.\begin{array}{ll}
(1) & \forall e \quad f_e \le c_e \\[2ex]
(2) & \forall v \ne s,t \quad \displaystyle\sum_{e=(u,v)} f_e - \sum_{e=(v,u)} f_e \le 0 \\[2ex]
(2') & \forall v \ne s,t \quad \displaystyle\sum_{e=(v,u)} f_e - \sum_{e=(u,v)} f_e \le 0
\end{array}\right\}
\begin{array}{ll}
A_1 & b_1 \\[2ex]
A_2 & b_2 \\[2ex]
A_3 & b_3
\end{array}$$

$$f_e \ge 0$$

$f_e \le c_e$

$$A_1 \begin{bmatrix} 1 & & & & 0 \\ & 1 & & & \\ & & 1 & & \\ & & & 1 & \\ 0 & & & & 1 \end{bmatrix} \qquad b_1 \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_e \end{bmatrix} \Big\} y_i$$

in edge

$$A_2 \quad v \begin{bmatrix} 1 & -1 & 1 & 0 & 0 \end{bmatrix} \qquad b_2 = \begin{bmatrix} 0 \end{bmatrix} \Big\} z_j$$

out edge

$$A_3 = -A_2 \qquad b_3 = \begin{bmatrix} 0 \end{bmatrix} \Big\} u_j$$

$$\min \sum_e c_e y_e \quad \leftarrow \text{dual objective}$$

$$(\,-\,y\,-\;\;-\,z\,-\;\;-\,u\,-\,)\begin{array}{cc} & \begin{array}{cc} \downarrow s,v & \downarrow (u,v) \end{array} \\ \begin{array}{c} \\ \\ u \rightarrow \\ v \rightarrow \\ \\ \end{array} & \left[\begin{array}{cc} A_1 & \cdot \\ \hline & +1 \\ A_2 & -1 \\ \hline A_3 & \end{array}\right] \end{array} \qquad \begin{array}{cc} \left[\begin{array}{c} +1 \\ c \\ 0 \\ -1 \end{array}\right] & \begin{array}{l} e = (s,v) \\ \\ \leftarrow e = (u,v) \\ e = (u,s) \end{array} \end{array}$$

$$y_{uv} + z_u - z_v - u_u + u_v \geq 0$$

$$y_{sv} + (-z_v) + (u_v) \geq 1$$

$$y_{us} + z_u - u_u \geq -1$$