CS 5350/6350: Machine Learining Fall 2015

Homework 2
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Handed out: Sep 17, 2015
Due date: Oct 1, 2015

General Instructions

- You are welcome to talk to other members of the class about the homework. I am more concerned that you understand the underlying concepts. However, you should write down your own solution. Please keep the class collaboration policy in mind.
- Feel free ask questions about the homework with the instructor or the TAs.
- Your written solutions should be brief and clear. You need to show your work, not just the final answer, but you do *not* need to write it in gory detail. Your assignment should be **no more than 10 pages**. Every extra page will cost a point.
- Handwritten solutions will not be accepted.
- The homework is due by midnight of the due date. Please submit the homework on Canvas.
- Some questions are marked **For 6350 students**. Students who are registered for CS 6350 should do these questions. Of course, if you are registered for CS 5350, you are welcome to do the question too, but you will not get any credit for it.

1 Warmup: Boolean Functions

1. [3 points] Table 1 shows several data points (the x's) along with corresponding labels (y). (That is, each row is an example with a label.) Write down three different Boolean functions, all of which can produce the label y when given the inputs x.

y	$x1$	$x2$	$x3$	$x4$
0	0	1	1	0
0	1	1	1	0
1	0	1	1	1

Table 1: Initial data set

(a) $f_a(x) = x_2 \wedge x_3 \wedge x_4$

(b) $f_b(x) = \tilde{x_1} \wedge x_4$

(c) $f_c(x) = x_1 \wedge x_2 \wedge x_3 \wedge \tilde{x_4}$

2. [5 points] Now the Table 1 is expanded to Table 2 by adding more data points. How many errors will each of your functions from the previous questions make on the expanded data set.

y	$x1$	$x2$	$x3$	$x4$	$f_a(x)$	$f_b(x)$	$f_c(x)$
0	0	1	1	0	1	1	0
0	1	1	1	0	1	1	0
1	0	1	1	1	1	1	0
1	1	0	1	1	0	0	0
0	0	1	1	0	1	1	1
1	1	1	0	1	0	0	0

Table 2: Expanded data set

3. [7 points] Is the function in Table 2 linearly separable? If so, write down a linear threshold function that classifies the data. If not, prove that there is no linear threshold function that can classify the data.

- $f(x) = x_4$

2 Mistake-bound learning

1. Let us define a concept class C of Boolean functions from $\{0,1\}^n$ to $\{0,1\}$. Every function in this class is an indicator function for a particular element in the input space. That is, for every $\mathbf{z} \in \{0,1\}^n$, there is a function $f_\mathbf{z}$ in C defined as follows:

$$f_\mathbf{z}(\mathbf{x}) = \begin{cases} 1 & \text{if } \mathbf{z} = \mathbf{x} \\ 0 & \text{otherwise} \end{cases} \tag{1}$$

(a) [2 points] How many functions does C contain?

- 2^n

(b) [10 points] Suppose we use the Halving algorithm to learn this concept class. How many mistakes will the algorithm make? Write a short proof for your answer. (Hint: The bound we proved in class doesn't directly apply.)

2^n possible combinations

$$
\begin{aligned}
1z &\leq \frac{1}{2}|C_{n-1}| && (2)\\
&\leq \frac{1}{2}\cdot\frac{1}{2}|C_{n-2}| && (3)\\
&\vdots && (4)\\
&\leq \frac{1}{2^n}\cdot\frac{1}{2^n} = \frac{1}{2^n}^{\,n} \geq 1 && (5)\\
&\Rightarrow \mathcal{O}(n\log(n)) && (6)
\end{aligned}
$$

(c) [1 point] Is Halving a mistake-bound algorithm for this concept class?

- No

2. [12 points] Recall from class that the Halving algorithm assumes that there is the true (hidden) function is in the concept class C with N elements and tries to find it. In this setting, we know the number of mistakes made by the algorithm is $O(\log N)$.

   Another way to think about this setting is that we are trying to predict with expert advice. That is, we have a pool of N so called experts, only one of whom is perfect. As the halving algorithm proceeds, it cuts down this pool by at least half each time a mistake is made.

   Suppose, instead of one perfect expert, we have M perfect experts in our pool. Show that the mistake bound of the same Halving algorithm in this case is $O(\log \frac{N}{M})$.

   (*Hint*: To show this, consider the stopping condition of the algorithm. At what point, will the algorithm stop making mistakes?)

   Would only make $\frac{N}{M}$ mistakes

$$
\begin{aligned}
1 = |C_n| &< \frac{1}{2}|C_{n-1}| && (7)\\
&< \frac{1}{2}\cdot\frac{1}{2}|C_{n-2}| && (8)\\
&< \vdots && (9)\\
&< \frac{1}{2^n}|C_n| = \frac{1}{2^n}\frac{N}{M} && (10)
\end{aligned}
$$

   Which comes about because the base case C_0 would be reacing the expert in $\frac{N}{M}$ tests

$$
\begin{aligned}
2^n &\leq \frac{N}{M} && (11)\\
&\log\left(\frac{N}{M}\right) && (12)
\end{aligned}
$$

3 The Perceptron Algorithm and its Variants

For this question, you will experiment with the Perceptron algorithm and some variants on a large collection of data sets.

Data

The training and test data sets are available on the assignments page of the class website. There are two collections of data, named `data0` and `data1`. Both have files with the following naming convention: `trainI.K` and `testI.K`, where K represents the number of dimensions/features in each data point. So for example `train0.10` is a training set in `data0` with 11 features (ten features plus a constant bias feature) and the corresponding test data set is `test0.10`.

One popular format for representing labeled data is the `libSVM` format (described below). All the data for this assignment is presented in this format. Each row in the file is a single example. The format of the each row in the data is

`<label> <index1>:<value1> <index2>:<value2> ···`

Here `<label>` denotes the label for that example. The rest of the elements of the row is a sparse vector denoting the feature vector. For example, if the original feature vector is $[0, 0, 1, 2, 0, 3]$, this would be represented as `3:1 4:2 6:3`. That is, only the non-zero entries of the feature vector are stored.

Algorithms

You will implement two variants of the Perceptron algorithm. Note that each variant has different hyper-parameters, as described below.

- **Perceptron**: This is the simple version of Perceptron as described in the class. An update will be performed on an example $(\mathbf{x}, y)$ if $y(\mathbf{w}^T\mathbf{x}) \leq 0$.

  Hyper-parameters: The learning rate r.

  Two things bear additional explanation. First, note that in the formulation above, the bias term b is *not* explicitly mentioned. This is because the features in the data folder include a bias feature. (See the class lectures for more information.) Second, if all elements of $\mathbf{w}$ and the bias term are initialized with zero, then the learning rate will have no effect. To see this, recall the Perceptron update:

  $$\mathbf{w}_{new} \leftarrow \mathbf{w}_{old} + ry\mathbf{x}$$

  Now, if $\mathbf{w}$ are initialized with zeroes and a learning rate r is used, then we can show that the final parameters will be equivalent to having a learning rate 1. The final weight vector and the bias term will be scaled by r compared to the unit learning rate case.

  For this assignment, you should initialize the weight vector and the bias randomly and tune the learning rate parameter. We recommend trying small values less than one. (eg. 1, 0.1, 0.01, etc.)

- **Margin Perceptron:** This variant of Perceptron will perform an update on an example$(\mathbf{x}, y)$ if $y(\mathbf{w}^T\mathbf{x}) \leq \mu$, where μ is an additional positive hyper-parameter, specified by the user. Note that because μ is positive, this algorithm can update the weight vector even when the current weight vector does not make a mistake on the current example.

  Hyper-parameters: Learning rate r and the margin μ. We recommend setting the value of μ between 0 and 5.0.

Experiments

For all your experiments, you may choose whatever hyper-parameters you like, but we suggest that you informally experiment with them before submitting the results. (You can use cross-validation to find the hyper-parameters as you did in the previous homework. Note that we did not partition the data into parts, so you should do that if you want to find hyper-parameters using cross-validation.)

1. [Sanity check, 5 points] Run the simple Perceptron algorithm on the data in Table 2 (one pass only, without shuffling) and report the weight vector that the algorithm returns. How many mistakes does it make?

   ```
   Sanity Check. Running Perceptron Algorithm on Table 2

   Number of Updates: 4

   Weight Vector: [ 0.   0.1  0.1  0.1]
   ```

   The given weight vector accuracy classifies the data. It does require $x_2,\ x_3,\ x_4$ to be greater than 0 to be classified when the actual function only requires x_4 to be positive. Thus it included extra "features" to the data which don't really effect the final result since the classification is only dependent upon the value of x_4.

2. [Online Perceptron, 15 points] Choose the 10 dimensional training set (`train0.10`) from the `data0` folder and its corresponding test dataset. Run both the Perceptron algorithm and the margin Perceptron on this dataset for one pass. Report the number of updates (or equivalently mistakes) made by each algorithm and the accuracy of the final weight vector on both the training and the test set. Once again, you will require some playing with the algorithm hyper-parameters. You will see that the hyper-parameters will make a difference and so try out different values.

   ```
   Running over the 10-Dimensional data

   Normal Perceptron

   Updates: 35 Accuracy: 1.0

   Margin Perceptron

   Updates: 33 Accuracy: 1.0
   ```

   For this instance, the Perceptron and the Margin Perceptron algorithm were used on the 10 dimensional data set in the `d0` file. For each of these functions, there was an

initial weight vector w that was randomly generated to use as the initial starting point of the weight vector for each algorithm.

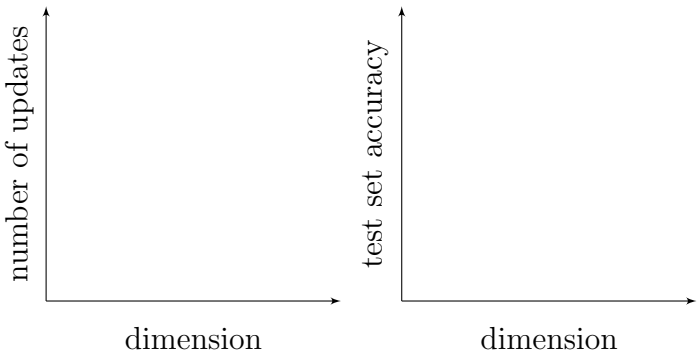
For the Normal Perceptron Algorithm, what was done was a list of the learning rate r was given in the range from $[0.1, 1]$. Each value was tested with the Perceptron Algorithm, and the value for r that gave the least number of updates was returned. The least number of updats was choosen because this implies that the weight vector that was created would also not need as many updates when classifying future data (test data) if it's relatively the same. As can be seen in the above output, the Normal Perceptron Algorithm needed `35` updates and was able to correctly classify all of the data in the test file.

When doing the Margin Perceptron Algorithm, a list of learning rates for r and μ was choosen, where r was in the range $[0.1, 1]$ and μ was constrained to $[0, 0.1]$. The ranges for both of these were determind by multiple trials and seeing which one classified better. The same technique was used for choosing the hyper-parameters, except in this instance each combination of r and μ were looked at. After determining the hyper-parameters, the Margin Perceptron algorithm was run with them and the weight vector to classify the data was returned. As can be seen above, the Margin Perceptron Algorithm did it in slightly less number of updates, `33` when compared to `35`, and also successfully classified all of the test cases.

3. [Batch Perceptron on all datasets, 40 points] Now, on each train/test dataset in the `data0` and `data1` folders, run the Perceptron and margin Perceptron algorithms for ten epochs. *Do not forget to randomly shuffle the data at the start of each epoch.*

   Record the the number of updates on the train datasets and the accuracy on the test data sets.

   So, for example, use the `data0/train0.10` file to train your Perceptron and test it on `data0/test0.10` and record the number of updates/mistakes and accuracy. Then, repeat this for all other datasets. This constitutes one experiment. Once you have the above data, plot two sets of graphs for each experiment (`data0`/`data1` + Perceptron/margin Perceptron):



   For the results of this part, please look at question 4 as I decided to include the Agressive Margin Perceptron in the results as well.

4. **(For 6350 Students)** [Aggressive Perceptron with Margin, 15 points] This algorithm is an extension of the margin Perceptron which performs an aggressive update as follows:

   If $y(\mathbf{w}^T\mathbf{x}) \leq \mu$, then update
   (a) $w_{new} \leftarrow w_{old} + \eta yx$

   Unlike the standard Perceptron algorithm, here the learning rate η is given by

$$\eta = \frac{\mu - y(\mathbf{w}^T\mathbf{x})}{\mathbf{x}^T\mathbf{x} + 1}$$

   As with the margin perceptron, there is an additional positive parameter μ.

   Repeat the experiment 3 with the aggressive update. Note that You should report two sets of results (one for `data0` and one for `data1`).

```
Running Perceptron over the d0 data
Dimension: 10
Dimension: 20
Dimension: 30
Dimension: 40
Dimension: 50
Dimension: 60
Dimension: 70
Dimension: 80
Dimension: 90
Dimension: 100
Plots d0_updates.pdf and d0_accuracy.pdf saved

Running Perceptron over the d1 data
Dimension: 10
Dimension: 20
Dimension: 30
Dimension: 40
Dimension: 50
Dimension: 60
Dimension: 70
Dimension: 80
```

```
Dimension: 90

Dimension: 100

Plots d1_updates.pdf and d1_accuracy.pdf saved
```

For each of the Normal Perceptron, Margin Perceptron, and Agressive Margin Perceptron algorithms, each one was ran 10 times to average out the randomly generated error that came about from using a random initial weight vector. For each of the 10 iterations, the accuracy and the number of updates are summed and then after the 10 iterations they are averaged.

For the Normal Perceptron algorithm, for each of the dimensions and epochs, the best value for the hyper-paramter r was chosen to be the lowest number of updates for the given weight vector. This weight vector was returned and then this was used to test the number of correct classifications on the test data set.

For the Margin Perceptron Algorithm, the hyper-parameters r and μ were chosen by the same technique as was used for the Normal Perceptron algorithm, where the values that were chosen were to minimize the number of updates. The average accuracy and average number of updates for the 10 epochs per dimension were recorded and used in the plots below.

Finally, for the Agressive Perceptron Algorithm, only two hyper-parameters (r, μ) just as with the Margin Perceptron which were optimized, but there was a third hyper-parameter η which was done too. However, as η is used in calculating how to change the new weight vector such as $w_{i+1} = w_i + \eta * y * x$, but this isn't done until after choosing a value for μ. Therefore, μ is a function of η, so choosing μ was explicit and η was implicit. So two list of values were fed in for μ and r and this was used to optimize the data to the least number of updates, just as the other two algorithms. For each of the 10 epochs per dimensionality, the average number of updates and the average accuracy was recorded for comparison with the other two algorithms.

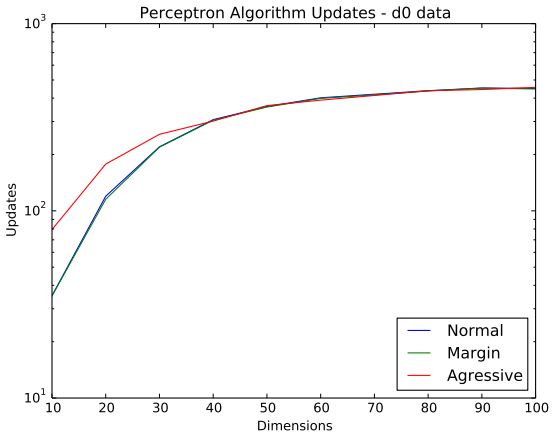


Figure 1: Updates for `data0`

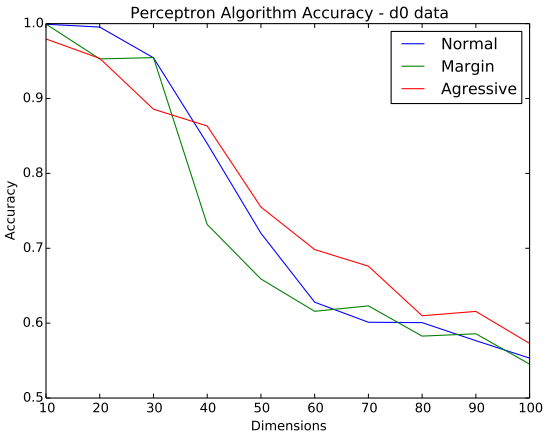


Figure 2: Accuracy for `data0`

Figure 1 shows the number of updates per algorithm for each of the different Perceptron

algorithms vs the dimensionality. This is only for the dataset in `data0`. As can be seen in the figures, the Margin Perceptron and Normal Perceptron performed roughly the same, while the Agressive Perceptron required more updates than either of the others, after which it performed roughly the same.

Seeing how this effects the accuracy in Figure 2, up through 40 dimensions the Agressive Perceptron performs worse than the first two algorithms. However, even though this is the case, the overall accuracy of the Agressive algorithm drops off roughly linearly with the number of dimensions, while the other algorithms drop off much quicker in accuracy for the dimensionality greater than 40.

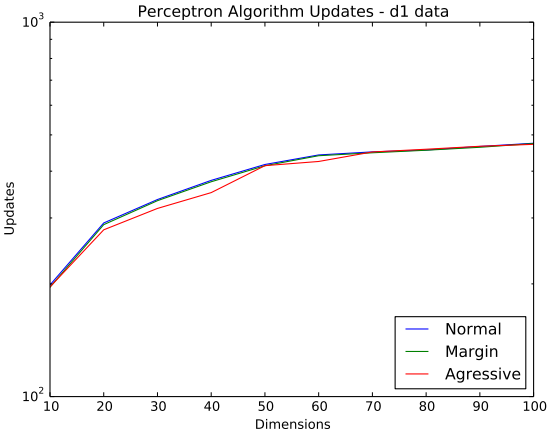


Figure 3: Updates for `data1`

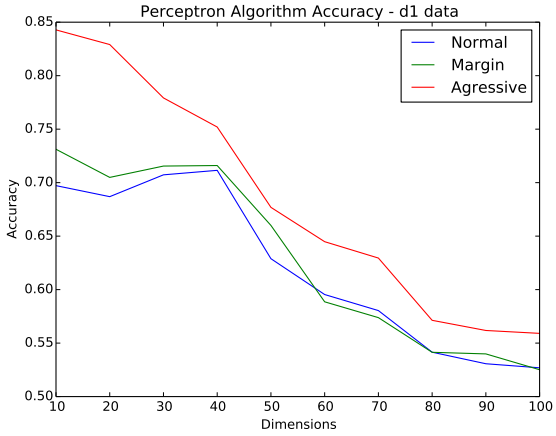


Figure 4: Accuracy for `data1`

Figure 3 shows the number of updates for each of the different Perceptron algorithms vs dimensionality for `data1`. The difference between this set of data and the first set is that for the number of updates the Agressive Perceptron algorithm seems to do better when looking at the number of updates. The Margin Perceptron and Normal Perceptron perform relatively the same.

The results of this shows up in 4. When compared to the accuracy that is in Figure 2 from `data0`, all of the algorithms perform much less accurate than their counterparts in Figure 2. However, the drop in the amount of accuracy is much more for both the Margin Perceptron and the Normal Perceptron Algorithms. In other words, the Agressive Perceptron algorithm performed much better than the other two in this data set, which was not the case for `data0`.

Overall, out of the 3 different algorithms, the Agressive Perceptron algorithm performs better for higher dimensional data when compared to either of the other two. This is most likely because when updating the weight vector with η, you're dividing by the norm of x which allows for it to be less susceptible to noise. On top of this, the Agressive Perceptron also did better than both in `data1` than `data0` and did not loose as much in accuracy. Again, this is most likely because of there being more noise in the second set of data and the agressive persceptron is less sensitive to this. The Agressive

Perceptron Algorithm seems to drop of linearly with respect to the dimensions while the other two are more sensitive to the number of dimensions.

Explanation of the update We call this the aggressive update because the learning rate is derived from the following optimization problem:

When we see that $y(\mathbf{w}^T\mathbf{x}) \leq \mu$, we try to find new values of $\mathbf{w}$ such that $y(\mathbf{w}^T\mathbf{x}) = \mu$ using

$$\begin{aligned} \min_{\mathbf{w}_{new}} \quad & \tfrac{1}{2}||\mathbf{w}_{new} - \mathbf{w}_{old}||^2 \\ \text{such that} \quad & y(\mathbf{w}^T x) = \mu. \end{aligned}$$

That is, the goal is to find the smallest change in the weights so that the current example is on the right side of the weight vector.

By substituting (a) from above into this optimization problem, we will get a single variable optimization problem whose solution gives us the η defined above. You can think of this algorithm as trying to tune the weight vector so that the current example is correctly classified right after the update.

Submission Guidelines

1. The report should detail your experiments. For each step, explain in no more than a paragraph or so how your implementation works.

2. *Your code should run on the CADE machines.* You should include a shell script, `run.sh`, that will execute your code in the CADE environment. Your code should produce similar output to what you include in your report.

   You are responsible for ensuring that the grader can execute the code using only the included script. If you are using an esoteric programming language, you should make sure that its runtime is available on CADE.

3. Please do not hand in binary files! We will *not* grade binary submissions.