CS 6150: HW4

Due Date:

This assignment has 5 questions, for a total of 100 points. Unless otherwise specified, complete and reasoned arguments will be expected for all answers.

Question	Points	Score
Integer programs	20	
LP duals	20	
Max cardinality matching	20	
Generalized Duals	20	
Best fit line	20	
Total:	100	

Question 1: Integer programs .... **[20]**

Write down integer programs for the following problems.

(a) [**10**] Let U be a set, and let $\mathcal{C} = \{S_1, \ldots, S_n\}$ be a collection of subsets of U. Each set S has a weight w_S. Find a subcollection $\mathcal{C}' \subset \mathcal{C}$ of minimum total weight ($w(\mathcal{C}') = \sum_{S\in\mathcal{C}'} w_S$) such that the sets in $\mathcal{C}'$ *cover* U: i.e

$$\cup_{S\in\mathcal{C}'} S = U$$

Solution:

Let x_i be the variable corresponding to each S_i and ϕ_{ik} be the variable corresponding to the element $k \in U$, but it is also in S_i set. Let $W = [w_1, w_2, ..., w_n]^T$ where w_i corresponds to weight of the element S_i. Then we can write down a linear program for the given problem as follows.

$$x_i = \begin{cases} 1 & \text{if } S_i \text{ is picked} \\ 0 & \text{otherwise} \end{cases}$$

Optimization Function F(x), can be written as,

$$F(x) = \min W^T X = \sum_{i=1}^{n} w_i x_i \tag{1}$$

Constraints,

$$\phi_{ik} = x_i \tag{2}$$

$$\forall k \ \sum_{i=1}^{n} \phi_{ik} > 0 \tag{3}$$

$$x_i \in \{0, 1\} \tag{4}$$

(b) [**10**] Let U be a universe, and let $\mathcal{C} = \{S_1, \ldots, S_n\}$ be a collection of subsets of U. Each element $u \in U$ has a weight w_u. Find a subset $H \subset U$ of minimum total weight ($w(H) = \sum_{u\in H} w_u$) such that each set in $\mathcal{C}$ is *hit* by H: i.e

$$\forall S \in \mathcal{C}, H \cap S \neq \emptyset$$

Solution: Let define x_i variable for each element in the set U.We can define x_i as follows.

$$x_i = \begin{cases} 1 & \text{if } i^{th} \text{ element in U is picked} \\ 0 & \text{otherwise} \end{cases}$$

We can define ϕ_{ik} corresponding to element k in the S_i set. Using the above mentioned notation we can write a linear program as follows.

Optimization Function F(x), can be written as,

$$F(x) = \min |X|_1 = \sum_{k=1}^{|U|} x_k \tag{5}$$

We can write the constraints as follows,

$$\phi_{ik} = x_k \tag{6}$$
$$\forall i \ , \sum_{k=1}^{|S_i|} \phi_{ik} \leq 1 \tag{7}$$
$$\forall k \ , x_k > 0 \tag{8}$$

Question 2: LP duals .... **[20]**

(a) **[10]** Consider the linear program

$$\begin{aligned} \max \quad & 5x + 3y - 2z \\ \text{such that} \quad & 3x - 2y \leq 6 \\ & 4y + 2z \leq 7 \\ & -3x + 2z \leq 3 \end{aligned}$$

Write down the dual of this LP.

Solution: We can convert the above linear system to canonical form so we can write the dual LP for the above problem easily.
Here we have 3 unbounded variables x, y, z. We can introduce 6 new variables $x_1, x_2, y_1, y_2, z_1, z_2$ such that $x_1, x_2, y_1, y_2, z_1, z_2 \geq 0$ and $x = x1 - x_2, y = y_1 - y_2, z = z_1 - z_2$ and replace x, y, z using the newly introduced variables. Hence out optimization function becomes,

$$\max 5x + 3y - 2z \tag{9}$$
$$\max 5(x_1 - x_2) + 3(y_1 - y_2) - 2(z_1 - z_2) \tag{10}$$
$$\max 5x_1 - 5x_2 + 3y_1 - 3y_2 - 2z_1 + 2z_2 \tag{11}$$
$$\tag{12}$$

The given constraints can be rewrite as,

$$3x_1 - 3x_2 - 2y_1 + 2y_2 \leq 6 \tag{13}$$
$$4y_1 - 4y_2 + 2z_1 - 2z_2 \leq 7 \tag{14}$$
$$-3x_1 + 3x_2 + 2z_1 - 2z_2 \leq 3 \tag{15}$$
$$\tag{16}$$

Let $X = [x_1, x_2, y_1, y_2, z_1, z_2]^T$ $C = [5, -5, 3, -3, -2, 2]^T$,

$A = \begin{bmatrix} 3 & -3 & -2 & 2 & 0 & 0 \\ 0 & 0 & 4 & -4 & 2 & -2 \\ -3 & 3 & 0 & 0 & 2 & -2 \end{bmatrix}$ and
$b = [6, 7, 3]^T$ Then we can write the given LP as follows,

$$\max C^T X \text{ such that,} \tag{17}$$
$$AX \leq b \tag{18}$$
$$X \geq 0 \tag{19}$$

By using the definintion of the Dual LP, we can write the dual LP as follows.

Let $Y = [y_1, y_2, y_3]^T$ be the variables in the dual LP.

$$\min Y^T b \text{ such that,} \tag{20}$$
$$Y^T A \geq C^T \tag{21}$$
$$Y \geq 0 \tag{22}$$

We can simplify the above dual LP to get the following equivalent LP,

$$\min 6y_1 + 7y_2 + 3y_3 \tag{23}$$

such that,

$$3y_1 - 3y_3 \geq 5 \tag{24}$$
$$-3y_1 + 3y_3 \geq -5 \tag{25}$$
$$-2y_1 + 4y_2 + 2y_3 \geq 3 \tag{26}$$
$$2y_1 - 4y_2 - 2y_3 \geq -3 \tag{27}$$
$$2y_2 + 2_y 3 \geq -2 \tag{28}$$
$$-2y_2 - 2y_3 \geq 2 \tag{29}$$

we can simplyfy the above conditions much further,

$$3y_1 - 3y_3 = 5 \tag{30}$$
$$-2y_1 + 4y_2 + 2y_3 = 3 \tag{31}$$
$$y_2 + y_3 = -1 \tag{32}$$

The final dual LP for the above given primal LP is,

$$\min 6y_1 + 7y_2 + 3y_3 \text{ such that,} \tag{33}$$
$$3y_1 - 3y_3 = 5 \tag{34}$$
$$-2y_1 + 4y_2 + 2y_3 = 3 \tag{35}$$
$$y_2 + y_3 = -1 \tag{36}$$
$$y_1, y_2, y_3 \geq 0 \tag{37}$$

(b) **[10]** Write down the dual of the linear program obtained by relaxing the integer program from Question 1(b) above.

Solution: Let's convert the Question 1 (b) LP to canonical form, then we can use that to write the dual form of the primal LP. Let X be the variable vector, Let,

$$X = [x_1, ...x_{|U|},\ \underbrace{\phi_{11}, \phi_{12}, .., \phi_{1|S_1|}}_{\text{Variables related to } S_1}, \cdots,\ \underbrace{\phi_{n1}, \phi_{n2}, .., \phi_{n|S_n|}}_{\text{Variables related to } S_n}]^T \tag{38}$$

$$C = [-w_1, -w_2, ... - w_{|U|},\ \underbrace{0, 0...0}_{S_1 + S_2 + .. + S_n \text{ zeros}}\ ] \tag{39}$$

$$b = [\underbrace{-1, -1, ... - 1}_{\text{n times}}, \cdots,\ \underbrace{0, 0, 0, ...0}_{\forall k \in U \times \text{ number of sets contains k times}}\ ]^T \tag{40}$$

According to the constraints that we have we can write the matrix A as,

$$\tag{41}$$

$$A = \begin{bmatrix} 0 & 0 & 0\cdots & 0 & 1 & 1 & 1\cdots & 1 & 0 & 0\cdots 0 \\ \vdots \\ 0 & 0 & 0\cdots & 0 & 0 & 0 & 0\cdots & 0 & 1 & 1\cdots 1 \\ 1 & 0 & 0\cdots & & \text{some cols get -1 entries depending } x_1 \in S_1 \\ -1 & 0 & 0\cdots & & \text{some cols get 1 entries depending } x_1 \in S_1 \\ \vdots \\ 0 & 0 & 0\cdots & 1 & \text{some cols get -1 entries depending } x_{|U|} \in S_n \\ 0 & 0 & 0\cdots & -1 & \text{some cols get 1 entries depending } x_{|U|} \in S_n \end{bmatrix}$$

Hence we can write the primal LP problem as,

$$\max C^T X \tag{42}$$
$$AX \leq b \tag{43}$$
$$X \geq 0 \tag{44}$$

Hence we can write down the Dual LP for the above primal as,

$$Y = [y_1, y_2, ...y_W] \text{ where, } W = \text{number of elements in } b \tag{45}$$
$$\min Y^T b \tag{46}$$
$$Y^T A \geq C^T \tag{47}$$
$$Y \geq 0 \tag{48}$$

Question 3: Max cardinality matching .... **[20]**

(a) [**10**] Write down a linear program for computing a maximum cardinality matching in a bipartite graph. Your linear program will have one variable for each edge.

Solution: Let X_{uv} be the variable corresponding to edge u, v, where $u, v\ \in V$ and $X_{uv} = 1$ if we have picked that edge as a matching edge. Then we can write the LP for the given problem as follows.

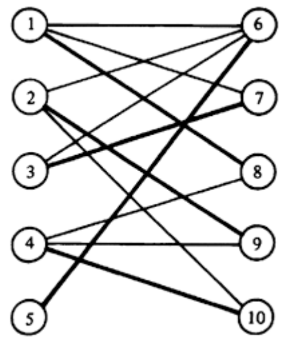
$$\max \sum_{\forall u,v} X_{uv} \tag{49}$$

$$\forall\ u, v, k \text{ such that}, u \neq v \neq k,\ X_{uv} + X_{uk} \leq 1 \tag{50}$$

$$\forall\ u, v\ X_{uv} \geq 0 \tag{51}$$

(b) [**10**] Write down the dual of this LP. What well known problem does it capture ?

Solution: In order to write the dual LP of the above primal LP let us convert it to the canonical form first. Let's consider the following bipartite graph.



Assume that we have written the primal LP for $G = (V, E)$ bipartite graph. Hence in the primal LP we have $|E|$ number of variables. How many how many constraints that we need for for a given bipartite graph. Assume that, according the the primal LP notation all the $u \in S_1$ and $v \in S_2$ where S_1, S_2 are the basic node sets. Let u_i denotes the i^{th} node of the S_1 and v_i denotes the i^{th} node of S_2.

Number of constraint in the primal LP can be written as $W = \sum_{i=1}^{|S_1|} \binom{deg(u_i)}{2}$ which is the number of combinations that we can choose edges which have the common vertex u_i. Hence for dual LP we have W number of variables. Let $Y = [y_1, y_2, ...y_W]$ denotes the variable vector of dual LP. For primal LP $C = [1, 1, 1, ...1]^T$ which is a $|E| \times 1$ vector. In the primal LP $b = [1, 1, ..., 1]^T$ which is $|W| \times 1$ vector.
Let,

$$A = \begin{bmatrix} 1 & 1 & 0 & 0\cdots & 0 & 0 & 0 \\ 1 & 0 & 1 & 0\cdots & 0 & 0 & 0 \\ 1 & 0 & 0 & 1\cdots & 0 & 0 & 0 \\ \vdots & & & & & & \\ 0 & 1 & 1 & 0\cdots & 0 & 0 & 0 \\ 0 & 1 & 0 & 1\cdots & 0 & 0 & 0 \\ \vdots & & & & & & \\ 0 & 0 & 0 & 0\cdots & 1 & 1 & 0 \\ 0 & 0 & 0 & 0\cdots & 0 & 1 & 1 \end{bmatrix} \tag{52}$$

Hence we can write the primal LP in the canonical form as follows.

$$\max C^T X \tag{53}$$
$$AX \leq b \tag{54}$$
$$X \geq 0 \tag{55}$$

Using the definition of the dual LP we can write the dual LP for the above primal as follows.

$$\min Y^T b \tag{56}$$
$$Y^T A \geq C^T \tag{57}$$
$$Y \geq 0 \tag{58}$$

The dual problem written above will try to find the min length path (visiting each vertex once,) which include all the vertices which has included in the maximum perfect matching. Hence the dual LP represent the instance of Travelling salesman problem with all the edges having a weight of 1.

Question 4: Generalized Duals .... **[20]**

We've seen that any linear program can be written in the canonical form

$$\begin{array}{rl} \max & c^\top x \\ \text{such that} & Ax \leq b \\ & x \geq 0 \end{array}$$

which gives rise to the corresponding dual

$$\begin{array}{rl} \min & y^\top b \\ \text{such that} & y^\top A \geq c \\ & y \geq 0 \end{array}$$

It turns out that first transforming a general linear program with equality and $\geq$ constraints into canonical form, and then writing the dual, can be a little inconvenient, and that it's easier to write the dual directly.

But what would this dual look like ? Let's take a general linear program that looks like this:

$$\begin{array}{rl} \max & ax + by + cz \\ \text{such that} & Ax + By + Cz \leq d \\ & Dx + Ey + Fz = e \\ & Gx + Hy + Iz \geq f \\ & x \geq 0, z \leq 0 \end{array}$$

Note that x, y, z are *vectors* and y is unconstrained (i.e the coordinates of y could be more or less than zero).

Write down the dual of this linear program. You will do this by first transforming this into the canonical setting, writing the canonical dual, and then rewriting the dual in simplified form. It will help to remember that if a and b are two variables that are both greater than zero, then $a - b$ represents a variable that could be either more or less than zero.

Solution: We need to define some new variables y_1, y_2, w such that, $y_1, y_2, w \geq 0$ to ensure write the given LP in canonical form.

$$\text{Let, } y = y_1 - y_2 \text{ and } w = -z \tag{59}$$

we can rewrite the given optimization function as follows,

$$\begin{align} \tag{60} \\ \max\ ax + by + cz \tag{61} \\ \max\ ax + b(y_1 - y_2) + c(-w) \tag{62} \\ \max\ ax + by_1 - by_2) - cw \tag{63} \end{align}$$

We can rewite the given constriants as follows.

$$\begin{align} Ax + By_1 - By_2 - Cw \leq d \tag{64} \\ Dx + Ey_1 - Ey_2 - Fw \leq e \tag{65} \\ -Dx - Ey_1 + Ey_2 + Fw \leq -e \tag{66} \\ -Gx - Hy_1 + Hy_2 + Iw \leq -f \tag{67} \end{align}$$

Then we can write $X = [x, y_1, y_2, w]^T$ Let $P = \begin{bmatrix} A & B & -B & -C \\ D & E & -E & -F \\ -D & -E & E & F \\ -G & -H & H & I \end{bmatrix}$, $b = [d, e, -e, -f]^T$ and $C = [a, b, -b, -c]^T$then we can write the given LP as follows.

$$\begin{align} \max\ C^T X \tag{68} \\ PX \leq b \tag{69} \\ X \geq 0 \tag{70} \end{align}$$

We can write down the dual LP for the above system as follows,

$$\begin{align} \text{Let, } Y = [y_1, y_2, y_3, y_4]^T \tag{71} \\ \min\ Y^T b \tag{72} \\ Y^T A \geq C^T \tag{73} \\ Y \geq 0 \tag{74} \end{align}$$

We can simplify the above LP as follows.

$$\begin{align} &\min\ y_1 d + y_2 e - y_3 e - y_4 f \text{ such that ,} \tag{75} \\ &y_1 A + y_2 D - y_3 D - y_4 G \geq a \tag{76} \\ &y_1 B + y_2 E - y_3 E - y_4 H \geq b \tag{77} \\ &-y_1 B - y_2 E + y_3 E + y_4 H \geq -b \tag{78} \\ &-y_1 C - y_2 F + y_3 F + y_4 I \geq -c \tag{79} \\ &Y \geq 0 \tag{80} \end{align}$$

Do you notice any pattern in the relation between primal constraint and dual variables (and vice versa) ?

Solution: Patterns that we can observe between the primal and dual problem are,

- Dual LP is always the opposite optimization (if primal is maximization then the dual is minimization, vice versa,) but the optimization problem of both primal and dual are different.
- Dual LP contains the number of variables which is equal to the number of constraints in the primal problems.
- Number of constraints in the dual problem is equal to the number of variables in the primal problem.
- When we introduce the variables y_1, y_2 to convert the primal LP to canonical form, in the Dual we get a equality constraint $(y_1B + y_2E - y_3E - y_4H = b)$ using the which is equal to b which is the factor associated with variable y in the original LP.

Question 5: Best fit line .... **[20]**

this is question 7(b) from the Erickson notes on LPs

You are given n points (x_i, y_i) in the plane, and you wish to find a line of *best fit*. But instead of the standard squared error norm, you will be using the ℓ_1 error: namely, for any given line $y = ax + b$, the error is given by

$$\epsilon_1(a, b) = \sum_{i=1}^{n} |y_i - (ax_i + b)|.$$

Write down a linear program to find a line that minimizes ϵ_1.

Solution:

Let $X = \begin{bmatrix} a \\ b \end{bmatrix}$ In order to write down the LP we need to remove the absolute function from the expression, for that we can you some additional variables and conditions. Let λ_i be variables corresponding to (x_i, y_i) point.

$$\lambda_i = |y_i - (ax_i + b)| \tag{81}$$

Hence, We can write down the optimization function as a linear function as follows,

$$\min \sum_{i=1}^{n} \lambda_i \tag{82}$$

$$\tag{83}$$

such that,

$$\forall i \ \lambda_i = |y_i - (ax_i + b)| \text{ Which is not a linear constraint.} \tag{84}$$

$$\tag{85}$$

we can convert that to a linear constraint as follows.

$$\forall i \ y_i - (ax_i + b) \leq \lambda_i \tag{86}$$

$$\forall i \ -y_i + (ax_i + b) \leq \lambda_i \tag{87}$$

$$\tag{88}$$

Hence we can write the LP for the given problem as follows.

$$\min \sum_{i=1}^{n} \lambda_i \text{ such that,} \tag{89}$$

$$\forall i \; y_i - (ax_i + b) \leq \lambda_i \tag{90}$$

$$\forall i \; -y_i + (ax_i + b) \leq \lambda_i \tag{91}$$

$$\tag{92}$$