CS6210: Homework 1

Christopher Mertin

September 6, 2016

1. Carry out calculations similar to those of Example 1.3 for approximating the derivative of the function $f(x) = e^{-2x}$ evaluated at $x_0 = 0.5$. Observe similarities and differences by comparing your graph against that in Figure 1.3.

2. Following Example 1.5, assess the conditioning of the problem of evaluating

$$g(x) = \tanh(cx) = \frac{e^{cx} - e^{-cx}}{e^{cx} + e^{-cx}}$$

near $x = 0$ as the positive parameter c grows.

Solution:

The condition number measures how sensitive the output of a function is to small changes in the input. It can be calculated by:

$$\text{Condition Number}(g) = x\frac{g'(x)}{g(x)}$$

which for the instance of this problem gives us

$$\begin{aligned} g(x) &= \frac{e^{cx} - e^{-cx}}{e^{cx} + e^{-cx}} = \frac{\frac{e^{cx} - e^{-cx}}{2}}{\frac{e^{cx}+e^{-cx}}{2}} \\ &= \frac{\sinh(x)}{\cosh(x)} \\ g'(x) &= c\,\text{sech}^2(cx) \\ &= \frac{c}{\cosh^2(cx)} \end{aligned}$$

with using these formulas, we can plug it into the definition of the condition number we get

$$\text{Condition Number}(g) = x\frac{g'(x)}{g(x)} = \frac{cx}{\cosh^2(x)\cdot\tanh(cx)}$$
$$= cx\frac{\cosh(cx)}{\sinh(cx)} = cx\cdot\coth(cx)$$

Where we can take the limit of the condition number as x approaches 0, giving

$$\lim_{x\to 0} cx\cdot\coth(cx) = 1$$

This shows that the funcion's sensitivity is not dependent upon c and is well conditioned.

3. The function $f_1(x_0, h) = \sin(x_0 + h) - \sin(x_0)$ can be transformed into another form, $f_2(x_0, h)$, using the trigonometric formula

$$\sin(\phi) - \sin(\psi) = 2\cos\left(\frac{\phi+\psi}{2}\right)\sin\left(\frac{\phi-\psi}{2}\right)$$

Thus f_1 and f_2 have the same values, in exact arithmetic, for any given argument values x_0 and h.

   (a) Derive $f_2(x_0, h)$

   (b) Suggest a formula that avoids cancellation errors for computing the approximation $((f(x_0+h) - f(x_0))/h)$ to the derivative of $f(x) = \sin(x)$ at $x = x_0$. Write a MATLAB program that implements your formula and computes an approximation of $f'(1.2)$ for $h = \{10^{-20}, 10^{-19}, \dots, 1\}$.

   (c) Explain the difference in accuracy between your results and the results reported in Example 1.3.

4. The function $f_1(x, \delta) = \cos(x+\delta) - \cos(x)$ can be transformed into another form, $f_2(x, \delta)$, using the trigonometric formula

$$\cos(\phi) - \cos(\psi) = -2\sin\left(\frac{\phi+\psi}{2}\right)\sin\left(\frac{\phi-\psi}{2}\right)$$

Thus, f_1 and f_2 have the same values, in exact arithmetic, for any given argument values x and δ.

   (a) Show that, analytically, $f_1(x,\delta)/\delta$ or $f_2(x,\delta)/\delta$ are effective approximations of the function $(-\sin(x))$ for δ sufficiently small.

   (b) Derive $f_2(x, \delta)$

   (c) Write a MATLAB script which will calculate $g_1(x,\delta) = f_1(x,\delta)/\delta + \sin(x)$ and $g_2(x,\delta) = f_2(x,\delta)/\delta + \sin(x)$ for $x = 3$ and $\delta = 10^{-11}$.

   (d) Explain the difference in the results of the two calculations.

5. Consider the approximation of the first derivative

$$f'(x) \approx \frac{f(x+h) - f(x)}{h}$$

The *truncation error* for this formula is $\mathcal{O}(h)$. Suppose that the absolute error in evaluating the function f is bounded by ϵ and let us ignore the errors generated in basic arithmetic operations.

   (a) Show that the total computational error (truncation error and rounding combined) is bounded by

   $$\frac{Mh}{2} + \frac{2\epsilon}{h}$$

   where M is a bound on $|f''(x)|$.

   (b) What is the value of h for which the above is minimized?

   (c) The rounding unit we employ is approximately equal to 10^{-16}. Use this to explain the behavior of the graph in Example 1.3. Make sure to explain the shape of the graph as well as the value where the apparent minimum is attained.

   (d) It is not difficult to show, using Taylor expansions, that $f'(x)$ can be approximated more accurately (in terms of truncation error) by

   $$f'(x) \approx \frac{f(x+h) - f(x-h)}{2h}$$

   For this approximation, the truncation error is $\mathcal{O}\left(h^2\right)$. Generate a graph similar to Figure 1.3 for the same function and the same value of x, namely, for $\sin(1.2)$, and compare the two graphs. Explain the meaning of your results.

6. In the statistical treatment of data one often needs to compute the quantities

$$\bar{x} = \frac{1}{n}\sum_{i=1}^{n} x_i, \quad \sigma^2 = \frac{1}{n}\sum_{i=1}^{n}(x_i - \bar{x})^2$$

where $\{x_1, x_2, \ldots, x_n\}$ are the given data. Assume that n is large, say, $n = 10000$. It is easy to see that σ^2 can also be written as

$$\sigma^2 = \frac{1}{n}\sum_{i=1}^{n} x_i^2 - \bar{x}^2$$

   (a) Which of the two methods to calculate σ^2 is cheaper in terms of overall computational cost? Assume $\bar{x}$ has already been calculated and give the operation counts for these two options.

   (b) Which of the two methods is expected to give more accurate results for σ^2 in general?

   (c) Give a small example, using a decimal system with precision $t = 2$ and numbers of your choice, validate your claims.