CS6210: Homework 2

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1. Consider the fixed point iteration $x_{k+1} = g\left(x_k\right),\ k = \{0, 1, \ldots\}$ and let all the assumptions of the Fixed Point Theorem hold. Use a Taylor's series expansion to show that the order of convergence depends on how many of the derivatives of g vanish at $x = x^*$. Use your result to state how fast (at least) a fixed point iteration is expected to converge if $g' = \cdots = g^{(r)}\left(x^*\right) = 0$, where the integer $r \geq 1$ is given.

   Solution:

   The definition of the Taylor Series Expansion for a function $g(x)$ around $x = x^*$ is defined as:

$$
\begin{aligned}
g(x) &= \sum_{n=0}^{\infty} \frac{g^{(n)}\left(x^*\right)}{n!}\left(x - x^*\right)^n \\
g\left(x_k\right) &= \sum_{n=0}^{\infty} \frac{g^{(n)}\left(x^*\right)}{n!}\left(x_k - x^*\right)^n \\
x_{k+1} - x^* = \xi_{k+1} &= \sum_{n=1}^{\infty} \frac{g^{(n)}\left(x^*\right)}{n!}\xi_k^n
\end{aligned}
$$

2. Consider the function $g(x) = x^2 + \frac{3}{16}$
   (a) This function has two fixed points. What are they?
   (b) Consider the fixed point iteration $x_{k+1} = g\left(x_k\right)$ for this g. For which of the points you have found in (a) can you be sure that the iterations will converge to that fixed point? Briefly justify your answer. You may assume that the initial guess is sufficiently close to the fixed point.
   (c) For the point or points you found in (b), roughly how many iterations will be required to reduce the convergence error by a factor of 10?

3. It is known that the order of convergence of the secant method is $p = \frac{1+\sqrt{5}}{2} \approx 1.618\ldots$ and that of Newton's method is $p = 2$. Suppose that evaluating f' costs approximately α times the cost of approximating f. Determine approximately for what values of α Newton's method is more efficient (in terms of number of function evaluations) than the secant method. You may neglect the asymptotic error constants in your calculations. Assume that both methods are starting with initial guesses of a similar quality.

4. The function

$$f(x) = (x-1)^2 e^x$$

has a double root at $x = 1$.

   (a) Derive Newton's iteration for this function. Show that the iteration is well-defined so long as $x_k \neq -1$ and that the convergence rate is expected to be similar to that of the bisection method (and certainly not quadratic).

   (b) Implement Newton's method and observe its performance starting from $x_0 = 2$.

   (c) How easy would it be to apply the bisection method? Explain.

5. Given $a > 0$, we wish to compute $x = \ln(a)$ using addition, subtraction, multiplication, division, and the exponential function e^x.

   (a) Suggest an iterative formula based on Newton's method, and write it in a way suitable for numerical computation

   (b) Show that your formula converges quadratically

   (c) Write down an iterative formula based on the secant method

   (d) State which of the secant and Newton's methods is expected to perform better in this case in terms of overall number of exponential function evaluations. Assume a fair comparison, *i.e.* same floating point system, "same quality" initial guesses, and idential convergence criterion.

6. For $x > 0$ consider the equation

$$x + \ln(x) = 0$$

It is a reformulation of the equation of Example 3.4

   (a) Show analytically that there is exactly one root, $0 < x^* < \infty$

   (b) Plot a graph of the function on the interval $[0.1, 1]$

   (c) As you can see from the graph, the root is between 0.5 and 0.6. Write MATLAB routines for finding the root, using the following:

      i. The bisection method, with the initial interval $[0.5, 0.6]$. Explain why this choice of the intial interval is valid.

      ii. A linearly convergent fixed point interation, with $x_0 = 0.5$. Show that the conditions of the Fixed Point Theorem (for the function g you have selected) are satisfied.

      iii. Newton's method, with $x_0 = 0.5$.

      iv. The secant method, with $x_0 = 0.5$ and $x_1 = 0.6$.

   For each of the methods:

- Use $|x_k - x_{k-1}| < 10^{-10}$ as convergence criterion
- Print out eh iterates and show the progress in the number of correct decimal digits throughout the iteration.
- Explain the convergence behavior and how it matches theoretical expectations