CS6210: Homework 1

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1. Carry out calculations similar to those of Example 1.3 for approximating the derivative of the function $f(x) = e^{-2x}$ evaluated at $x_0 = 0.5$. Observe similarities and differences by comparing your graph against that in Figure 1.3.

2. Following Example 1.5, assess the conditioning of the problem of evaluating

$$g(x) = \tanh(cx) = \frac{e^{cx} - e^{-cx}}{e^{cx} + e^{-cx}}$$

near $x = 0$ as the positive parameter c grows.

Solution:

The condition number measures how sensitive the output of a function is to small changes in the input. It can be calculated by:

$$\text{Condition Number}(g) = x\frac{g'(x)}{g(x)}$$

which for the instance of this problem gives us

$$\begin{aligned}
g(x) &= \frac{e^{cx} - e^{-cx}}{e^{cx} + e^{-cx}} = \frac{\frac{e^{cx} - e^{-cx}}{2}}{\frac{e^{cx} + e^{-cx}}{2}} \\
&= \frac{\sinh(x)}{\cosh(x)} \\
g'(x) &= c\,\text{sech}^2(cx) \\
&= \frac{c}{\cosh^2(cx)}
\end{aligned}$$

with using these formulas, we can plug it into the definition of the condition number we get

$$\begin{aligned}\text{Condition Number}(g) &= x\frac{g'(x)}{g(x)} = \frac{cx}{\cosh^2(x)\cdot\tanh(cx)}\\ &= cx\frac{\cosh(cx)}{\sinh(cx)} = cx\cdot\coth(cx)\end{aligned}$$

Where we can take the limit of the condition number as x approaches 0, giving

$$\lim_{x\to 0} cx\cdot\coth(cx) = 1$$

This shows that the funcion's sensitivity is not dependent upon c and is well conditioned.

3. The function $f_1(x_0,h) = \sin(x_0+h) - \sin(x_0)$ can be transformed into another form, $f_2(x_0,h)$, using the trigonometric formula

$$\sin(\phi) - \sin(\psi) = 2\cos\left(\frac{\phi+\psi}{2}\right)\sin\left(\frac{\phi-\psi}{2}\right)$$

Thus f_1 and f_2 have the same values, in exact arithmetic, for any given argument values x_0 and h.

(a) Derive $f_2(x_0,h)$

Solution:

$$f_2(x_0,h) = \sin(x_0+h) - \sin(x_0) = 2\cos\left(x_0+\frac{h}{2}\right)\sin\left(\frac{h}{2}\right)$$

(b) Suggest a formula that avoids cancellation errors for computing the approximation $((f(x_0+h)-f(x_0))/h)$ to the derivative of $f(x)=\sin(x)$ at $x=x_0$. Write a MATLAB program that implements your formula and computes an approximation of $f'(1.2)$ for $h=\{10^{-20},10^{-19},\ldots,1\}$.

Solution:

For h significantly small, the values of $\sin(x_0+h)$ and $\sin(x_0)$ are close as $\sin(x)$ is a smooth trigonometric function. Therefore, there is a possibiltiy of cancellation occurring for small values of h. This can be avoided by omitting the subtraction and using the equivalent equation stated above. Therefore, we have

$$\frac{\sin(x_0+h)-\sin(x_0)}{h} = \frac{2\cos\left(x_0+\frac{h}{2}\right)\sin\left(\frac{h}{2}\right)}{h}$$

(c) Explain the difference in accuracy between your results and the results reported in Example 1.3.

Solution:

In comparing the two figures, it shows that using the above equation to avoid the cancellation for small values of h produced correct results as they both agree at every point. In Figure 1.3, the two curves diverge.

4. The function $f_1(x,\delta) = \cos(x+\delta) - \cos(x)$ can be transformed into another form, $f_2(x,\delta)$, using the trigonometric formula

$$\cos(\phi) - \cos(\psi) = -2\sin\left(\frac{\phi+\psi}{2}\right)\sin\left(\frac{\phi-\psi}{2}\right)$$

Thus, f_1 and f_2 have the same values, in exact arithmetic, for any given argument values x and δ.

(a) Show that, analytically, $f_1(x,\delta)/\delta$ or $f_2(x,\delta)/\delta$ are effective approximations of the function "$-\sin(x)$" for δ sufficiently small.

Solution:

This can be done by taking the limit of both functions as x approaches 0, which gives

$$\begin{aligned}
\lim_{x\to 0}\frac{\cos(x+\delta)-\cos(x)}{\delta} &= \lim_{x\to 0}\frac{\cos(x)\cos(\delta)-\sin(x)\sin(\delta)-\cos(x)}{\delta} \\
&= \lim_{x\to 0}\frac{\cos(x)\left(\cos(\delta)-1\right)-\sin(x)\sin(\delta)}{\delta} \\
&= \lim_{x\to 0}\frac{\cos(x)\left(\cos(\delta)-1\right)}{\delta} - \lim_{x\to 0}\frac{\sin(x)\sin(\delta)}{\delta} \\
&= \cos(x)\lim_{x\to 0}\frac{\cos(\delta)-1}{\delta} - \sin(x)\lim_{x\to 0}\frac{\sin(\delta)}{\delta} \\
&= 0\cdot\cos(x) - 1\cdot\sin(x) = -\sin(x)
\end{aligned}$$

(b) Derive $f_2(x,\delta)$

Solution:

$$f_2(x,\delta) = \cos(x+\delta) - \cos(x) = -2\sin\left(\frac{2x+\delta}{2}\right)\sin\left(\frac{\delta}{2}\right)$$

(c) Write a MATLAB script which will calculate $g_1(x,\delta) = f_1(x,\delta)/\delta + \sin(x)$ and $g_2(x,\delta) = f_2(x,\delta)/\delta + \sin(x)$ for $x = 3$ and $\delta = 10^{-11}$.

(d) Explain the difference in the results of the two calculations.

Solution:

The values of g_1 and g_2 should converge to zero as δ goes to zero. As can be seen in the results, g_2 is closer to zero so it therefore provides a better approximation of the derivative of $\cos(x)$.

5. Consider the approximation of the first derivative

$$f'(x) \approx \frac{f(x+h)-f(x)}{h}$$

The *truncation error* for this formula is $\mathcal{O}(h)$. Suppose that the absolute error in evaluating the function f is bounded by ϵ and let us ignore the errors generated in basic arithmetic operations.

(a) Show that the total computational error (truncation error and rounding combined) is bounded by

$$\frac{Mh}{2} + \frac{2\epsilon}{h}$$

where M is a bound on $|f''(x)|$.

Solution:

In order to solve this, the following terms need to be defined:

- f': The *exact* derivative of f
- $\widetilde{f'}$: The value of f' with the discretization error
- $\widetilde{f'_\star}$: The value of f' with discretization *and* rounding error
- $\widetilde{f_\star}$: The value of f with discretization *and* rounding error

Now that the notation is defined. We can approximate the first derivative of a function with a Taylor Series expansion about 0. In doing so, we get

$$\begin{aligned} f\left(x_0+h\right) &= f\left(x_0\right) + hf'\left(x_0\right) + \frac{h^2}{2} f''\left(x_0\right) + \ldots \\ \left| f'\left(x_0\right) - \frac{f\left(x_0+h\right) - f\left(x_0\right)}{h} \right| &\leq h\frac{f''\left(x_0\right)}{2} \leq h\frac{M}{2} \end{aligned}$$

which can be rewritten using the new notation defined above as

$$\left| f' - \widetilde{f'} \right| \leq h\frac{M}{2}$$

From the definition of the *abolsute error* (ξ), we get the absolute error on f being bounded by ϵ

$$\xi = \left| f - \widetilde{f_\star} \right| < \epsilon$$

Following this, we have

$$\begin{aligned} \left| f' - \widetilde{f'_\star} \right| &= \left| f' - \widetilde{f'} + \widetilde{f'} - \widetilde{f'_{star}} \right| \\ &< \left| f' - \widetilde{f'} \right| + \left| \widetilde{f'} - \widetilde{f'_\star} \right| \\ &< h\frac{M}{2} + \left| \widetilde{f'} - \widetilde{f'^\star} \right| \end{aligned}$$

If we can say $\left| \widetilde{f'} - \widetilde{f'^\star} \right| < \frac{2\epsilon}{h}$ holds true, then the proof is complete.

(b) What is the value of h for which the above is minimized?

Solution:

In finding the extremas, we can take the derivative of the funciton, set it equal to zero, and then solve the system

$$\frac{d}{dh}\left(h\frac{M}{2} + \frac{2\epsilon}{h}\right) = \frac{M}{2} - \frac{2\epsilon}{h^2}$$
$$\frac{M}{2} - \frac{2\epsilon}{h^2} = 0$$
$$h = 2\sqrt{\frac{\epsilon}{M}}$$

(c) The rounding unit we employ is approximately equal to 10^{-16}. Use this to explain the behavior of the graph in Example 1.3. Make sure to explain the shape of the graph as well as the value where the apparent minimum is attained.

Solution:

Example 1.3 gives $f(x) = \sin(x)$, so $f''(x) = -\sin(x)$. Since M is a bound for $|-\sin(x)|$, it is 1. Using the given tolerance value of ϵ, we can solve for h, giving

$$\epsilon = 10^{-16}$$
$$h = 2\sqrt{\frac{\epsilon}{M}} = 2 \cdot 10^{-8}$$

(d) It is not difficult to show, using Taylor expansions, that $f'(x)$ can be approximated more accurately (in terms of truncation error) by

$$f'(x) \approx \frac{f(x+h) - f(x-h)}{2h}$$

For this approximation, the truncation error is $\mathcal{O}(h^2)$. Generate a graph similar to Figure 1.3 for the same function and the same value of x, namely, for $\sin(1.2)$, and compare the two graphs. Explain the meaning of your results.

Solution:

In comparing Figure 1.3 and this one below

6. In the statistical treatment of data one often needs to compute the quantities

$$\bar{x} = \frac{1}{n}\sum_{i=1}^{n} x_i, \quad \sigma^2 = \frac{1}{n}\sum_{i=1}^{n}(x_i - \bar{x})^2$$

where $\{x_1, x_2, \ldots, x_n\}$ are the given data. Assume that n is large, say, $n = 10\,000$. It is easy to see that σ^2 can also be written as

$$\sigma^2 = \frac{1}{n}\sum_{i=1}^{n} x_i^2 - \bar{x}^2$$

(a) Which of the two methods to calculate σ^2 is cheaper in terms of overall computational cost? Assume $\bar{x}$ has already been calculated and give the operation counts for these two options.

Solution:

The first equation of σ^2 requires n squares, n subtractions, n additions, and 1 division. This results in an overall computational cost of $\mathcal{O}(3n+1)$, while the second equation, assuming $\bar{x}^2$ is *not* precomputed, of σ^2 requires $n+1$ squares, 1 subtraction, n additions, and 1 division, resulting in an overall complexity of $\mathcal{O}(2n+2)$.

(b) Which of the two methods is expected to give more accurate results for σ^2 in general?

Solution:

The first equation has n subtractions while there is only a single subtraction in the second equation. Therefore, there will be less cancellation error in the second equation, so it should produce a more accurate answer.

(c) Give a small example, using a decimal system with precision $t = 2$ and numbers of your choice, validate your claims.

Solution:

Work goes here