CS6210: Homework 2

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September 15, 2016

1. Consider the fixed point iteration $x_{k+1} = g\left(x_k\right),\ k = \{0, 1, \ldots\}$ and let all the assumptions of the Fixed Point Theorem hold. Use a Taylor's series expansion to show that the order of convergence depends on how many of the derivatives of g vanish at $x = \widetilde{x}$. Use your result to state how fast (at least) a fixed point iteration is expected to converge if $g' = \cdots = g^{(r)}\left(\widetilde{x}\right) = 0$, where the integer $r \geq 1$ is given.

   Solution:

   The definition of the Taylor Series Expansion for a function $g(x)$ around $x = \widetilde{x}$ is defined as:

$$
\begin{aligned}
g(x) &= \sum_{n=0}^{\infty} \frac{g^{(n)}\left(\widetilde{x}\right)}{n!}\left(x - \widetilde{x}\right)^n \\
g\left(x_k\right) &= \sum_{n=0}^{\infty} \frac{g^{(n)}\left(\widetilde{x}\right)}{n!}\left(x_k - \widetilde{x}\right)^n \\
x_{k+1} - \widetilde{x} = \xi_{k+1} &= \sum_{n=0}^{\infty} \frac{g^{(n)}\left(\widetilde{x}\right)}{n!}\xi_k^n
\end{aligned}
$$

   If $g'\left(\widetilde{x}\right) = 0$, then

$$
\xi_{k+1} = \sum_{n=2}^{\infty} \frac{g^{(n)}\left(\widetilde{x}\right)}{n!}\xi_k^n = \xi_k^2 \sum_{n=2}^{\infty} \frac{g^{(n)}\left(\widetilde{x}\right)}{n!}\xi_k^{n-2}
$$

   Which shows that $g(x)$ is *minimally* quadratically convergent. Consequently, if $g^{(i)}\left(\widetilde{x}\right) = 0$ for $i \in \{1, 2, \ldots, r\}$, then the function has a divergence of *at least* $\mathcal{O}\left(r\right)$.

2. Consider the function $g(x) = x^2 + \frac{3}{16}$

   (a) This function has two fixed points. What are they?

   Solution:

   In order to find the *fixed points* of the above function, we need to find the zeros

$$x^2 + \frac{3}{16} = x$$
$$x^2 - x + \frac{3}{16} = 0$$
$$\Rightarrow x_1 = \frac{1}{4}$$
$$\Rightarrow x_2 = \frac{3}{4}$$

(b) Consider the fixed point iteration $x_{k+1} = g(x_k)$ for this g. For which of the points you have found in (a) can you be sure that the iterations will converge to that fixed point? Briefly justify your answer. You may assume that the initial guess is sufficiently close to the fixed point.

Solution:

Since the inital guess is said to be *sufficiently close to the fixed point*, we can assume that the given range is the following.

$$\frac{1}{4} \leq x \leq \frac{3}{4}$$

We need to check to make sure the function $f(x)$ is constrained in this domain, such that it won't go "over" or "under" when getting the next x_{k+1} point. To do so

$$\frac{1}{16} \leq x^2 \leq \frac{9}{16}$$
$$\frac{1}{16} + \frac{3}{16} \leq x^2 + \frac{3}{16} \leq \frac{9}{16} + \frac{3}{16}$$
$$\frac{1}{4} \leq f(x) \leq \frac{3}{4}$$

So $f(x)$ is contained in the interval $\frac{1}{4} \leq f(x) \leq \frac{3}{4}$ for choosing an initial x in the same interval. Therefore, we can now check to see if it will converge to each of the fixed points by checking the *slope* of the function

$$g'(x) = 2x$$
$$g'\left(\frac{1}{4}\right) = \frac{1}{2}$$
$$g'\left(\frac{3}{4}\right) = \frac{3}{2}$$

Since $g'\left(\frac{1}{4}\right) < 1$, it is a decreasing slope and the function will converge for $x_1 = \frac{1}{4}$. Like wise, $g'\left(\frac{3}{4}\right) > 1$ means that it will converge to $x_2 = \frac{3}{4}$, assuming that the initial value is *sufficiently close.*

(c) For the point or points you found in (b), roughly how many iterations will be required to reduce the convergence error by a factor of 10?

Solution:

Page 49 of the text gives the equation to reduce the convergence error by a factor of 10. The number of iterations is defined as $k = \lceil 1/rate \rceil$, for which $rate = -\log_{10} |g'(\widetilde{x})|$. Using this equation, we can get the convergence rate to be

$$\begin{aligned} g'\left(\frac{1}{4}\right) &= \frac{1}{2} \\ \Rightarrow rate &= -\log_{10}\left|2^{-1}\right| = \frac{3}{10} \end{aligned}$$

So it takes $k = \lceil 1/rate \rceil$ iterations to reduce the error by more than an order of magnitude, resulting in

$$k = \left\lceil \left(\frac{3}{10}\right)^{-1} \right\rceil = 4$$

3. It is known that the order of convergence of the secant method is $p = \frac{1+\sqrt{5}}{2} \approx 1.618\ldots$ and that of Newton's method is $p = 2$. Suppose that evaluating f' costs approximately α times the cost of approximating f. Determine approximately for what values of α Newton's method is more efficient (in terms of number of function evaluations) than the secant method. You may neglect the asymptotic error constants in your calculations. Assume that both methods are starting with initial guesses of a similar quality.

Solution:

The Secant Method only requires one function evaluation, which will be denoted by ς, as $f(x_{n-1})$ can be stored from the previous computation. On the other hand, Newton's Method requires one function evaluation, and one derivative evaluation, having a total of $\varsigma(\alpha+1)$ as the total cost.

Therefore, we can determine the minimum values of α for which the computations would be better to use the Secant Method over the Newton's Method.

$$\begin{aligned} \varsigma &\leq \varsigma(\alpha+1) \\ \varsigma - \varsigma(\alpha+1) &\leq 0 \\ \varsigma(1-1-\alpha) &\leq 0 \\ \alpha &\leq 0 \end{aligned}$$

Therefore, since you cannot have *negative computational cost*, the Secant Method will always be faster than Newton's Method in the case of linear convergence, but Newton's Method would win in the case of Quadratic Convergence.

4. The function

$$f(x) = (x-1)^2 e^x$$

has a double root at $x = 1$.

(a) Derive Newton's iteration for this function. Show that the iteration is well-defined so long as $x_k \neq -1$ and that the convergence rate is expected to be similar to that of the bisection method (and certainly not quadratic).

Solution:

In order to build the methods, we need the following

$$\begin{aligned} f(x) &= (x-1)^2 e^x \\ f'(x) &= (x-1)^2 e^x + 2(x-1)e^x \\ f''(x) &= (x-1)^2 e^x + 2(x-1)e^x + 2(x-1)^2 e^x + 2e^x \end{aligned}$$

with Newton's method being defined as:

$$\begin{aligned} x_{k+1} &= x_k - \frac{f(x_k)}{f'(x_k)} \\ &= x_k - \frac{(x_k-1)^2 e^{x_k}}{2(x_k-1)e^{x_k} + (x_k-1)^2 e^{x_k}} \\ &= x_k - \frac{x_k-1}{x_k+1} \end{aligned}$$

Which shows that it is well defined *except* for $x_k = -1$. We can subtract the approximation of the root for $x = 1$ of $f(x)$ from both sides, which gives

$$\begin{aligned} x_{k+1} - \widetilde{x} &= x_k - \frac{x_k-1}{x_k+1} - \widetilde{x} \\ \xi_{k+1} &= \xi_k - \frac{x_k-1}{x_k+1} \approx \xi_k - \frac{x_k-\widetilde{x}}{x_k+1} \\ &= \xi_k - \frac{\xi_k}{x_k+1} = \xi_k\left(1 - \frac{1}{x_k+1}\right) \\ &= \xi_k\left(\frac{x_k}{x_k+1}\right) \end{aligned}$$

With $\frac{x_k}{x_k+1} < 1$, it is not quadratically convergent.

(b) Implement Newton's method and observe its performance starting from $x_0 = 2$.

Solution:

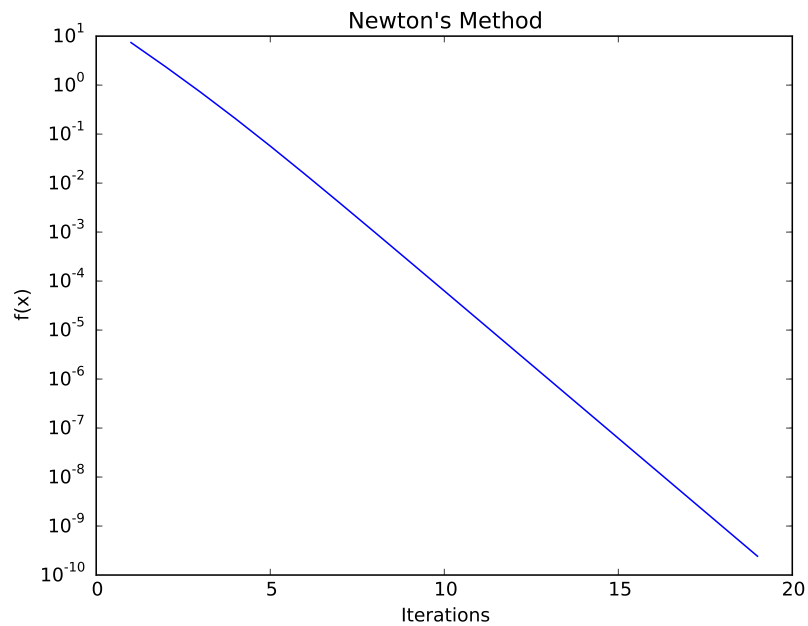


Figure 1: `prob4.py`

(c) How easy would it be to apply the bisection method? Explain.

Solution:

To use the Bisection Method, we need two points x_0 and x_1 in the domain. It is also required that $f(x_0)$ has a different sign than $f(x_1)$, which since $f(x)$ is positive over the entire domain, cannot hold true. Therefore, the Bisection Method cannot be used.

5. Given $a > 0$, we wish to compute $x = \ln(a)$ using addition, subtraction, multiplication, division, and the exponential function e^x.

(a) Suggest an iterative formula based on Newton's method, and write it in a way suitable for numerical computation.

Solution:

In order to find an iterative formula, we need an $f(x)$ such that $f(x) = 0$ at some point, and has a derivative in the domain of interest. We can do that by taking the above equation and solving for 0, thus giving the $f(x)$

$$\begin{aligned} x &= \ln(a) \\ e^x &= e^{\ln(a)} \\ \therefore f(x) &= e^x - a \\ f'(x) &= e^x \end{aligned}$$

As we have a derivative, we can define the Newton's Method for this equation as being

$$
\begin{aligned}
x_{k+1} &= x_k - \frac{f(x_k)}{f'(x_k)} \\
&= x_k - \frac{e^{x_k} - a}{e^{x_k}} \\
&= x_k - \frac{a}{e^{x_k}} - 1
\end{aligned}
$$

(b) Show that your formula converges quadratically.

Solution:

We can define $g(x) = \frac{a}{e^x} - 1$, and then take the Taylor Series of it, giving

$$
g(x) = \sum_{n=0}^{\infty} \frac{g^{(n)}\left(\widetilde{x}\right)}{n!}\left(x - \widetilde{x}\right)^n
$$

For $n = 0$ we get $g\left(\widetilde{x}\right) = 0$ and for $n = 1$ we get $g'\left(\widetilde{x}\right) = -1$, this changes our series to

$$
g\left(x_k\right) = 0 - \xi_k + \sum_{n=2}^{\infty} \frac{g^{(n)}\left(\widetilde{x}\right)}{n!}\xi_k^n
$$

where we can replace this into the iterative equation from above, giving

$$
\begin{aligned}
x_{k+1} &= x_k - \xi_k + \sum_{n=2}^{\infty} \frac{g^{(n)}\left(\widetilde{x}\right)}{n!}\xi_k^n \\
&= x_k - \xi_k + \xi_k^2 \sum_{n=2}^{\infty} \frac{g^{(n)}\left(\widetilde{x}\right)}{n!}\xi_k^{n-2}
\end{aligned}
$$

Finally, after subtracting $\widetilde{x}$ from both sides, gives

$$
\begin{aligned}
\xi_{k+1} = x_{k+1} - \widetilde{x} &= \xi_k - \xi_k + \xi_k^2 \sum_{n=2}^{\infty} \frac{g^{(n)}\left(\widetilde{x}\right)}{n!}\xi_k^{n-2} \\
&= \xi_k^2 \sum_{n=2}^{\infty} \frac{g^{(n)}\left(\widetilde{x}\right)}{n!}\xi_k^{n-2}
\end{aligned}
$$

So it is quadratically convergent.

(c) Write down an iterative formula based on the secant method.

Solution:

$$
\begin{aligned}
x_{k+1} &= x_k - \frac{f\left(x_k\right)\left(x_k - x_{k-1}\right)}{f\left(x_k\right) - f\left(x_{k-1}\right)} \\
&= x_k - \frac{\left(e^{x_k} - a\right)\left(x_k - x_{k-1}\right)}{e^{x_k} - a - \left(e^{x_{k-1}} - a\right)} \\
&= x_k - \frac{\left(e^{x_k} - a\right)\left(x_k - x_{k-1}\right)}{e^{x_k} - e^{x_{k-1}}}
\end{aligned}
$$

(d) State which of the Secant and Newton's Methods is expected to perform better in this case in terms of overall number of exponential function evaluations. Assume a fair comparison, *i.e.* same floating point system, "same quality" initial guesses, and idential convergence criterion.

Solution:

Newton's method only has two evaluations of the exponential function ($f(x)$ and $\frac{\mathrm{d}}{\mathrm{d}x} f(x)$) while the Secant Method has three. Thus, Newton's Method should perform better.

This can be reduced down to a single exponential evaluation for the Secant Method by, instead of taking the naïve approach you store the last value and only calculate e^{x_k} once instead. In that instance, the Secant Method would win out as it only has to do one function evaluation, but Newton's Method is stuck at 2.

However, in overall time, Newton's Method will most likely win out since it is quadratically convergent in this case.

6. For $x > 0$ consider the equation

$$x + \ln(x) = 0$$

It is a reformulation of the equation of Example 3.4

(a) Show analytically that there is exactly one root, $0 < \widetilde{x} < \infty$

Solution:

$\ln(x)$ is a continuous function for $x > 0$, and we can choose two points such that $x_0 = \frac{1}{2}$ and $x_1 = 1$.

$$f(x_0) \approx \frac{1}{2} - 0.69 < 0$$
$$f(x_1) = 1 + 0 > 0$$

By the *Intermediate Value Theorem*, there must exist some $f(c)$ such that $f(c) = 0$ due to the sign change of the function.

(b) Plot a graph of the function on the interval $[0.1, 1]$

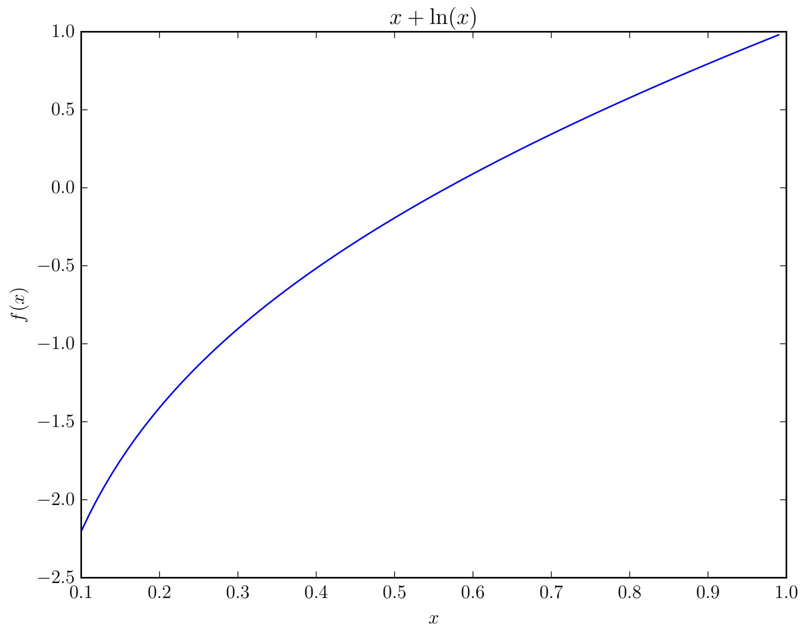
Solution:

(c) As you can see from the graph, the root is between 0.5 and 0.6. Write MATLAB routines for finding the root, using the following:

i. The bisection method, with the initial interval $[0.5, 0.6]$. Explain why this choice of the intial interval is valid.

Solution:

This interval is valid because the sign changes in the domain $[0.5, 0.6]$ and the function is everywhere continuous, so the Bisection method can be used. It was able to converge in 30 iterations, which is within the theoretical limits. For the plot, please see the last question.



ii. A linearly convergent fixed point interation, with $x_0 = 0.5$. Show that the conditions of the Fixed Point Theorem (for the function g you have selected) are satisfied.
Solution:
In determining $g(x)$, we wanted a function such that $g(\widetilde{x}) = \widetilde{x}$, which was determined in the following way

$$
\begin{aligned}
x + \ln(x) &= 0 \\
-x &= \ln(x) \\
x &= e^{-x}
\end{aligned}
$$

So $g(\widetilde{x}) = e^{\widetilde{x}}$ since $e^{\widetilde{x}} = \widetilde{x}$ for the given equation. This satisfies the conditions of the Fixed Point Method, and can be used to converge to the root. In the implementation, it took 38 iterations to converge to the root.
For the plot, please see the last question.

iii. Newton's method, with $x_0 = 0.5$.
Solution:
Newton's Method was the fastest implementation, as it converged in only 3 iterations.
For the plot, please see the last question.

iv. The Secant Method, with $x_0 = 0.5$ and $x_1 = 0.6$.
Solution:
The Secant Method was the second fastest of the 4 methods, and converged in 4 iterateions. However, since the number of computations is less, it most likely was faster than Newton's Method for overall time.

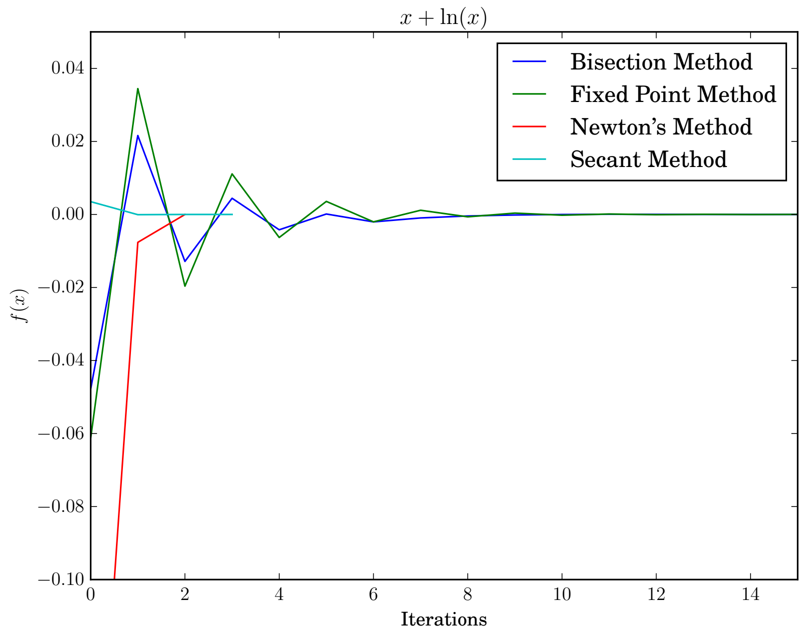


Figure 2: `prob6.py`

The Secant Method approached the root from the positive side of the function, which is what is to be expected from the definition of the secant line and how it is used in determining the root. Newton's Method on the other hand approached it from the opposite side of the root, and converged from there. On the other hand, both the Fixed Point and Bisection methods oscillated back and forth on both sides of the root, which is usually what happens when implementing either of these two methods, so it was to be expected.

For each of the methods:

- Use $|x_k - x_{k-1}| < 10^{-10}$ as convergence criterion
- Print out eh iterates and show the progress in the number of correct decimal digits throughout the iteration.
- Explain the convergence behavior and how it matches theoretical expectations