CS6210: Homework 5

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1. An $n \times n$ linear system of equations $A\mathbf{x} = \mathbf{b}$ is modified in the following manner: For each $i = \{1, \ldots, n\}$, the value b_i on the right-hand side of the i^{th} equation is replaced by $b_i - x_i^3$. Obviously, the modified system of equations (for the unknowns x_i) is now nonlinear.

   (a) Find the corresponding Jacobian Matrix
   Solution:

   (b) Given that A is strictly diagonally dominant with positive elements on its diagonal, state whether or not it is guarenteed that the Jacobian matrix at each iterate is nonsingular.
   Solution:

   (c) Suppose that A is symmetric positive definition (not necessarily diagonally dominant) and that Newton's method is applied to solve the nonlinear system. Is it guarenteed to converge?
   Solution:

2.

   (a) Suppose Newton's method is applied to a linear system $A\mathbf{x} = \mathbf{b}$. How does the iterative formula look and how many iterations does it take to converge?
   Solution:

   (b) Suppose the Jacobian matrix is singular at the solution of a nonlinear system of equations. Speculate what can occur in terms of convergence and the rate of convergence. Specifically, is it possible to have a situation where the newton iteration converges but convergence is not quadratic?
   Solution:

3. Consider minimizing the function $\phi(\mathbf{x}) = \mathbf{c}^T\mathbf{x} + \frac{1}{2}\mathbf{x}^T H\mathbf{x}$, where $\mathbf{c} = (5.04, -59.4, 146.4, -96.6)^T$ and

$$H = \begin{pmatrix} 0.16 & -1.2 & 2.4 & -1.4 \\ -1.2 & 12.0 & -27.0 & 16.8 \\ 2.4 & -27.0 & 64.8 & -42.0 \\ -1.4 & 16.8 & -42.0 & 28.0 \end{pmatrix}$$

Try both Newton and BFGS methods, starting from $\mathbf{x}_0 = (-1, 3, 3, 0)^T$. Explain why the BFGS method requires significantly more iterations than Newton's.

Solution:

4. Wit the notation of Exercise 21, for $p = 2$ the problem can be solved as in Example 9.15 using SVD. But it can also be solved using thge techniques of constrained optimization:

   (a) Explain why it is justified to replace the objective function by $\frac{1}{2}||y||_2^2$.
   Solution:

   (b) Form the KKT conditions of the resulting constrained optimization problem, obtaining a linear system of equations.
   Solution:

   (c) Devise an example and solve it using the method developed as well as the one from Example 9.15. Compare and discuss.
   Solution:

5. Given the four data points $(-1, 1), (0, 1), (1, 2), (2, 0)$, determine the interpolating cubic polynomial

   - Using the monomial basis
   - Using the Lagrange basis
   - Using the Newton basis

   Show that the three representations give the same polynomial.

   Solution:

6. Suppose we are given 137 uniformly spaced data pairs at distinct abscissae: $(x_i, y_i), i = 0, 1, \ldots, 136$. These data are thought to represent a function which is piecewise smooth; that is, the unknown function $f(x)$ which gave rise to these data values has many bounded derivatives everywhere expect for a few points where it jups discontinuously. (Imagine drawing a curve smoothly from left to right, mixed with lifting the pen and moving it vertically a few times.) For each sub interval $[x_{i-1}, x_i]$ we wanit to pass the hopefully best cubic $p_3(x)$ for an accurate interpolation of $f(x)$ at points $x_{i-1} < x < x_i$. This involves choosing good neighbors to interpolate at.

   Propose an algorithm for this task. Justify.

   Solution:

7. A popular technique arising in methods for minimizing functions in several variables involves a *weak line search*, where an approximate minimum $\widetilde{x}$ is found for a function in one variable, $f(x)$, for whcih the values of $f(0)$, $f'(0)$, and $f(1)$ are given. The function $f(x)$ is defined for all nonnegative x, has a continuous second derivative, and satisfies $f(0) < f(1)$ and $f'(0) < 0$. We then interpolate the given values by a quadratic polynomial and set $\widetilde{x}$ as the minimum of the interpolant.

(a) Find $\widetilde{x}$ for the values of $f(0) = 1,\ f'(0) = -1,\ f(1) = 2$

Solution:

(b) Show that the quadratic interpolant has a unique minimum satisfying $0 < \widetilde{x} < 1$. Can you show the same for the function f itself?

Solution:

8. Let $f \in C^3[a, b]$ be given at equidistant points $x_i = a + ih,\ i = \{0, 1, \ldots, n\}$, where $nh = b - a$. Assume further that $f'(a)$ is given as well.

(a) Construct an algorithm for C^1 piecewise quadratic interpolation of the given values. Thus, the interpolating function is written as

$$v(x) = s_i(s) = a_i + b_i\,(x - x_i) + c_i\,(x - x_i)^2\,, \quad x_i \le x \le x_{i+1}$$

for $i = \{0, \ldots, n-1\}$, and your job is to specify an algorithm for determining the $3n$ coefficients a_i, b_i, and c_i.

Solution:

(b) How accurate do you expect this approximation to be as a function of h? Justify.

Solution:

9. Derive a B-spline basis representation for piecewise linear interplation and for piecewise Hermite cubic interpolation.

Solution:

10. Consider interpolating the data $(x_0, y_0), \ldots, (x_6, y_6)$ given by

x	0.1	0.15	0.2	0.3	0.35	0.5	0.75
y	3.0	2.0	1.2	2.1	2.0	2.5	2.5

Construct the five interpolants specified below (you may use available software for this), evaluate them at the points $\{0.05, 0.06, \ldots, 0.80\}$, plot, and comment on their respective properties:

(a) Polynomial Interpolant

Solution:

(b) Cubic Spline Interpolant

Solution:

(c) The Interpolant

$$v(x) = \sum_{j=0}^{n} c_j \phi_j(x)$$

where $n = 7$, $\phi_0(x) = 1$, and

$$\phi_j(x) = \sqrt{\left(x - x_{j-1}\right)^2 + \epsilon^2} - \epsilon$$

In addition to the n interpolation requirements, the condition $c_0 = -\sum_{j=1}^{n} c_j$ is imposed. Construct this interpolant with

i. $\epsilon = 0.1$
   Solution:
ii. $\epsilon = 0.01$
   Solution:
iii. $\epsilon = 0.001$
   Solution:

Make as many observations as you can. What will happen if we let $\epsilon \to 0$?

Solution: