# Computational Fluid Dynamics: Lectures 8 (ME EN 6720)

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# Finite Volumes

**Finite Volume Approach:** (Ferziger Ch. 4)

• finite difference equations are based on the differential form of the conservation laws:

$$\frac{\partial(\rho\phi)}{\partial t}+\frac{\partial\left(\rho u_j\phi\right)}{\partial x_j}=\frac{\partial}{\partial x_j}\left(\alpha\frac{\partial\phi}{\partial x_j}\right)+q_\phi$$

• finite volume approaches instead start from the integral or **control volume (CV)** formulation of the conservation laws:

$$\frac{\partial}{\partial t}\int_{CV}\rho\phi d\forall+\int_{CS}\rho\phi\vec{V}\bullet d\vec{A}=\int_{CS}\alpha\vec{\nabla}V\bullet d\vec{A}+\int_{CV}q_\phi d\forall$$

• instead of point differences (FDE) with a grid composed of nodes at which we estimate gradients we break our domain up into a series of finite size control volumes

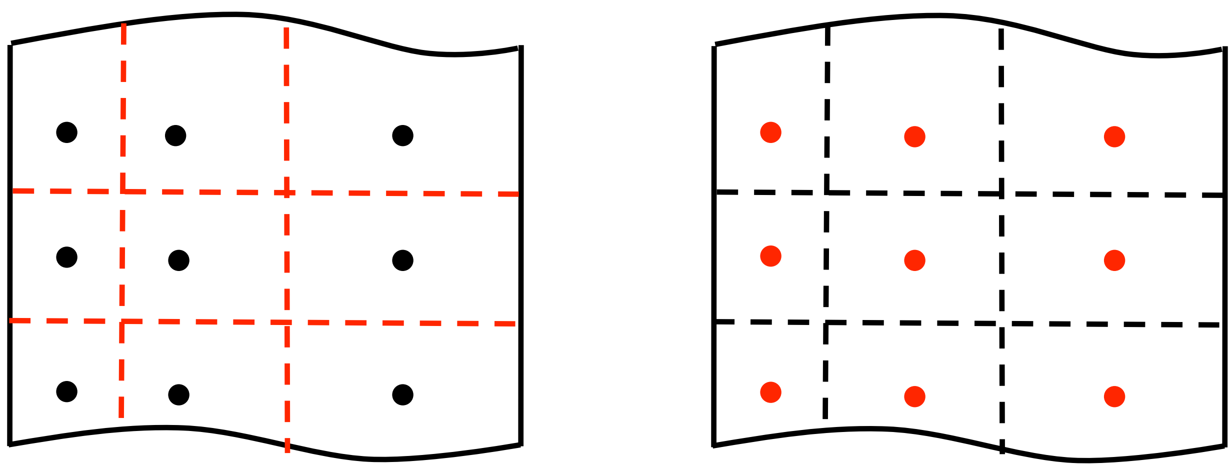
• The integral (CV) equations are evaluated over each CV

-this explicitly acknowledges the discrete character of our problem (in contrast to FDEs where we assumed “smooth” functions between grid points)

-The equation can be applied to any region (single volume or entire domain) ➔ automatically control volume formulations satisfy global conservation

# Finite Volumes grids

- we build up our finite volume grid in one of two ways:
    1. Define node points throughout the domain and construct CV faces at the midpoint between nodes.
    2. Define control volume grid and place the nodes at the center of each volume



- for uniform grids the two approaches are equivalent
- for stretched grids 1) offers easier approximations for gradients at faces while 2) has the advantage that it represents the volume average more accurately (since the nodes are at CV centroids)

# Finite Volume Surface Integrals

- we have two types of terms to evaluate:
  - surface integrals
  - volume integrals

- **Surface Integrals**

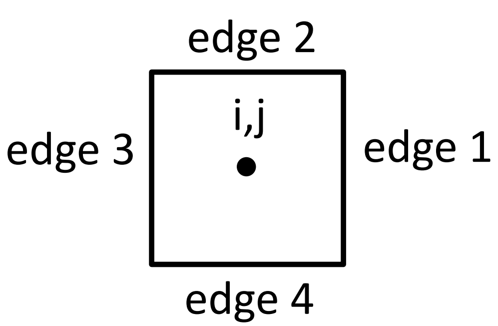
  - we need to represent $\int_{CS} f\, d\vec{A}$ or $\sum_k \int_{CS_k} f_k\, dA_k$ (where we have k faces or edges)

  - we need the net flux through the CV boundaries

  - for 2D we have 4 "faces" (6 for 3D)

  - *f* is the function that must be integrated ( $\rho\phi\vec{V}$ or $\alpha\vec{\nabla}\phi$ ) normal to the face

  - exact calculation would require knowledge of *f* everywhere on each face




○ instead of exact calculations we approximate the integrals using one (or more) values on the volume faces (or edges for 2D)

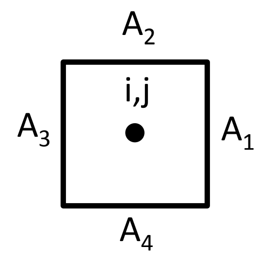
○ we get the face values in terms of the node values (through interpolation)

# Finite Volume Surface Integrals

- **<u>Approximations for Surface Integrals</u>**:

-**midpoint rule**: integrand is the product of the value at the volume-face center and the volume-face area:

$$\int_{CS_k} f_k \, dA_k = \overline{f}_k A_k \approx f_k A_k \text{ where } A_k \text{ is the face (or edge) area}$$




-**trapezoid rule**:

$$\int_{CS_1} f_1 \, dA_1 \approx \frac{A_1}{2}\left(f_{i+\frac{1}{2},j-\frac{1}{2}} + f_{i+\frac{1}{2},j+\frac{1}{2}}\right) \text{ for face (or edge) } A_1$$

-**Simpson's rule**:

$$\int_{CS_1} f_1 \, dA_1 \approx \frac{A_1}{6}\left(f_{i+\frac{1}{2},j-\frac{1}{2}} + 4 f_{i+\frac{1}{2},j} + f_{i+\frac{1}{2},j+\frac{1}{2}}\right) \text{ for face (or edge) } A_1$$

-All of these approximations require *f* on the volume faces

-In general we don't have these values!

# Finite Volume Volume Integrals

- **Approximation of Volume Integrals**

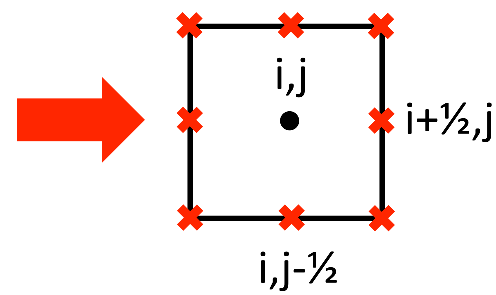
-we also need to approximate $\int_{CV} q d\forall$

- simplest approximation for this is to use the value of *q* at the volumes center:

$$\int_{CV} q d\forall = \bar{q}\Delta\forall \approx q_{i,j}\Delta x \Delta y$$

- for $q_{i,j}$ centered in the volume this is a 2nd order accurate estimate

- higher order estimates can be obtained by including more points (in general, this needed for unstructured meshes)




-Example of higher-order scheme (Ferziger page 75):

-2D bi-quadratic interpolation (for Cartesian grids)

$$\int_{CV} q d\forall \approx \frac{\Delta x \Delta y}{36}\left(16q_{i,j} + 4q_{i,j-\frac{1}{2}} + 4q_{i,j+\frac{1}{2}} + 4q_{i-\frac{1}{2},j} + 4q_{i+\frac{1}{2},j} + q_{i+\frac{1}{2},j+\frac{1}{2}} + q_{i-\frac{1}{2},j+\frac{1}{2}} + q_{i+\frac{1}{2},j-\frac{1}{2}} + q_{i-\frac{1}{2},j-\frac{1}{2}}\right)$$

(we have a weighted average of the points in the figure)

# Interpolation for Finite Volume Integrals

- **Interpolation Methods**:

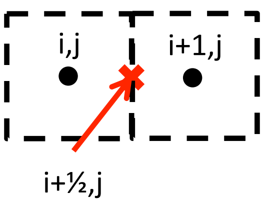
Estimating volume and surface integrals requires values of variables at volume faces

-Upwind Interpolation: approximate $\phi_{i+1/2,j}$ by its upwind value

$$\phi_{i+1/2,j} = \begin{cases} \phi_i \text{ if } u > 0 \\ \phi_{i+1} \text{ if } u < 0 \end{cases}$$

This is a 1st order scheme (use Taylor series expansion of $\phi_{i+1/2,j}$ about $\phi_{i,j}$) and is equivalent to our forward (or backwards) difference schemes.

-Linear Interpolation: interpolate values of $\phi_{i+1/2,j}$ using the adjacent node points




$$\phi_{i+1/2,j} = \phi_{i,j}\lambda + \phi_{i+1,j}(1-\lambda) \text{ where } \lambda = \frac{x_{i+1/2,j} - x_{i,j}}{x_{i+1,j} - x_{i,j}}$$

-Linear interpolation also gives a nice way to estimate derivatives at i+½:

$$\left.\frac{\partial \phi}{\partial x}\right|_{i+1/2,j} = \frac{\phi_{i+1,j} - \phi_{i,j}}{x_{i+1,j} - x_{i,j}}$$

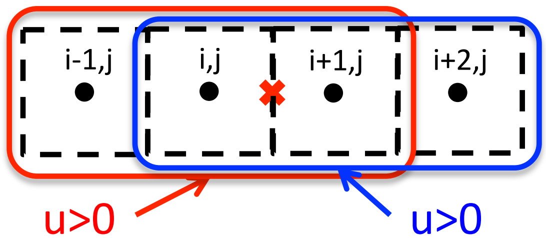
for a uniform grid this is $\Delta x$ and equivalent to our finite difference CDS

# Interpolation for Finite Volume Integrals

-Quadratic Upwind Interpolation: (QUICK)

- a better estimate for interpolating can be developed by adding more points
- in the QUICK method we add a third point
- one problem is which point to add (i+½ won't be centered with 3 points)?

-we can add a point depending on the flow direction:




-this ends up being a 3rd order 1-sided estimate that changes points based on flow direction and is given by:

$$\phi_{i+\frac{1}{2},j} = \phi_{i,j} + g_1\left(\phi_{i+1,j} - \phi_{i,j}\right) + g_2\left(\phi_{i,j} - \phi_{i-1,j}\right)$$

$$\text{where } g_1 = \frac{\left(x_{i+\frac{1}{2},j} - x_{i,j}\right)\left(x_{i+\frac{1}{2},j} - x_{i-1,j}\right)}{\left(x_{i+1,j} - x_{i,j}\right)\left(x_{i+1,j} - x_{i-1,j}\right)} \text{ and } g_2 = \frac{\left(x_{i+\frac{1}{2},j} - x_{i,j}\right)\left(x_{i+1,j} - x_{i+\frac{1}{2},j}\right)}{\left(x_{i,j} - x_{i-1,j}\right)\left(x_{i+1,j} - x_{i-1,j}\right)}$$

-we can find a 1st (or second) derivative to go with this interpolation scheme by differentiating the interpolating function

# Finite Volume Example

- **2D Heat Conduction in a Solid by Finite Volumes**:

- governed by the conservation equation:

$$\frac{\partial}{\partial t}\int_{CV} T d\forall + \int_{CS} \vec{q} \bullet d\vec{A} = 0$$

→ Fourier's law: $\vec{q} = -\alpha \vec{\nabla} T$

- we need to estimate two terms: a **volume integral** and a **surface integral**

-Volume integral:

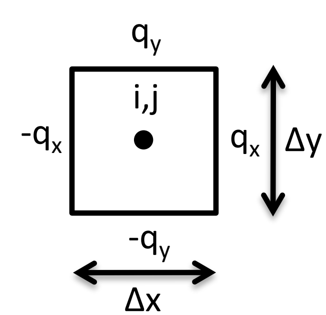
•Use the simple center value approximation ➔ $\int_{CV} T d\forall = T_{i,j}\Delta x \Delta y$

•We can use a forward in time for $\frac{\partial}{\partial t}$ ➔ $\frac{\partial}{\partial t}\int_{CV} T d\forall \approx \boxed{\frac{T_{i,j}^{n+1} - T_{i,j}^{n}}{\Delta t}\Delta x \Delta y}$

-Surface Integral:

•We need to approximate $\int_{CS} \vec{q} \bullet d\vec{A}$ by calculating the total flux through each cell edge.

•Remember that: $q_x = -\alpha \frac{\partial T}{\partial x}$ and $q_y = -\alpha \frac{\partial T}{\partial y}$



# Finite Volume Example

•Using the **midpoint estimate** and summing up all of our contributions:

$$\int_{CS} \vec{q} \bullet d\vec{A} \approx -\alpha \Delta y \left.\frac{\partial T}{\partial x}\right|_{i+½,j} - \alpha \Delta x \left.\frac{\partial T}{\partial y}\right|_{i,j+½} + \alpha \Delta y \left.\frac{\partial T}{\partial x}\right|_{i-½,j} + \Delta x \alpha \left.\frac{\partial T}{\partial y}\right|_{i,j-½}$$

•For the derivatives at the ½ points we can use a **centered difference scheme** (derivative of linear interpolation) at time level *n* e.g.

$$-\alpha \, \Delta y \left.\frac{\partial T}{\partial x}\right|_{i+½,j} \approx \boxed{-\alpha \Delta y \frac{T_{i+1,j}^{n} - T_{i,j}^{n}}{\Delta x}}$$

•Putting this together with our volume integral term estimate we get:

$$\frac{T_{i,j}^{n+1} - T_{i,j}^{n}}{\Delta t} \Delta x \Delta y + \alpha \Delta y \frac{T_{i,j}^{n} - T_{i+1,j}^{n}}{\Delta x} + \alpha \Delta x \frac{T_{i,j}^{n} - T_{i,j+1}^{n}}{\Delta y} + \alpha \Delta y \frac{T_{i,j}^{n} - T_{i-1,j}^{n}}{\Delta x} + \alpha \Delta x \frac{T_{i,j}^{n} - T_{i,j-1}^{n}}{\Delta y} = 0$$

•Dividing by ΔxΔy and rearranging:

$$\boxed{\frac{T_{i,j}^{n+1} - T_{i,j}^{n}}{\Delta t} = \alpha \left[ \frac{T_{i+1,j}^{n} - 2T_{i,j}^{n} + T_{i-1,j}^{n}}{(\Delta x)^2} + \frac{T_{i,j+1}^{n} - 2T_{i,j}^{n} + T_{i,j-1}^{n}}{(\Delta y)^2} \right]}$$

•This is the same formulation we would end up with if we used finite differences with a forward  in time a centered in space scheme