# Computational Fluid Dynamics: Lecture 6
# ME EN 6720


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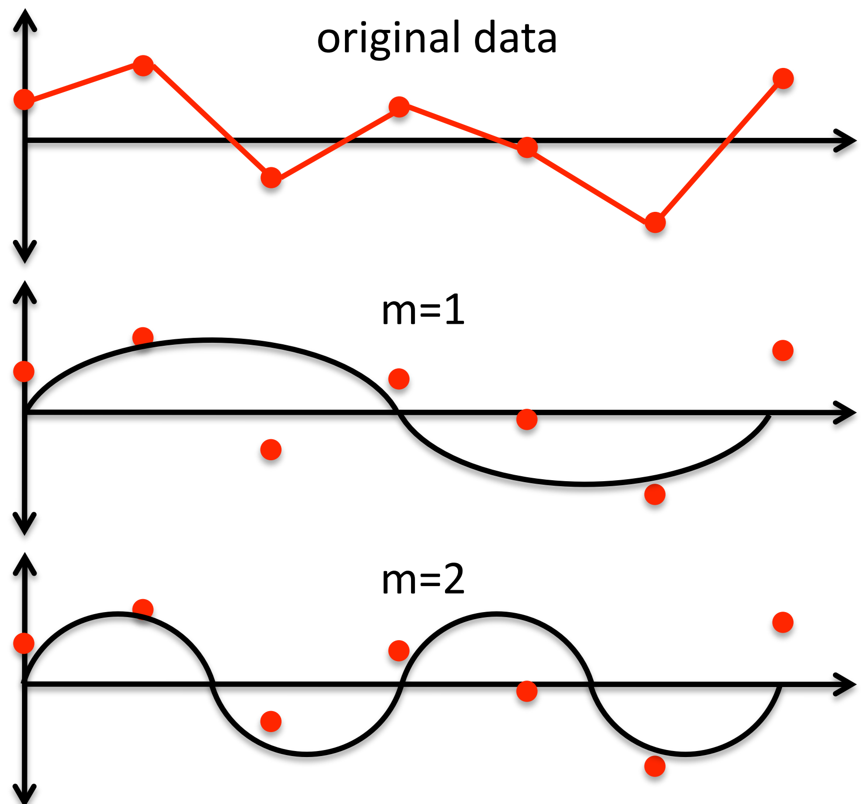
# Fourier Analysis

**A short intro to Fourier analysis:**

Fourier analysis starts with the idea that any signal can be decomposed into waves with different amplitudes and wavelengths.

• given a discrete set of data:



# Fourier Analysis

- Mathematically we can represent this process by the discrete Fourier transform pair:

Fourier transform pair

$$* \quad f(x_j) = \sum_{m=-N/2}^{N/2-1} \hat{f}(k_m) e^{ik_m x_j}$$ Backward transform

$$** \quad \hat{f}(k_m) = \frac{1}{N} \sum_{j=1}^{N} f(x_j) e^{-ik_m x_j}$$ Forward transform

Fourier coefficients (wave amplitudes)

wave number (wave period) $\rightarrow k_m = \dfrac{2\pi m}{\Delta x N}$

recall: $e^{-ik_m x_j} = \cos(k_m x_j) + i \sin(k_m x_j)$

- How is this used numerically?
  - -A Fourier series can be used to interpolate $f(x_j)$ at any point $x$ in the flow and at any time $t$
  - -If we differentiate the Fourier representation of $f(x_j)$ (equation *) with respect to $x$

$$\frac{\partial f}{\partial x} = \frac{\partial}{\partial x}\left[\sum_{m=-N/2}^{N/2-1} \hat{f}(k_m) e^{ik_m x_j}\right] = \sum_{m=-N/2}^{N/2-1} \hat{f}(k_m) \frac{\partial e^{ik_m x_j}}{\partial x} = \boxed{\sum_{m=-N/2}^{N/2-1} ik_m \hat{f}(k_m) e^{ik_m x_j}}$$

# Fourier Analysis

- if we compare this to equation ⊛ we notice that we have:

$$\frac{\partial f}{\partial x} = g = \sum_{m=-N/2}^{N/2-1} \underbrace{ik_m \hat{f}(k_m)}_{\hat{g}(k_m)} e^{ik_m x_j} = \boxed{\sum_{m=-N/2}^{N/2-1} \hat{g}(k_m) e^{ik_m x_j}}$$

- procedurally we can use this to find $\left.\frac{\partial f}{\partial x}\right|_j$ given $f(x_j)$ as follows:

  -calculate $\hat{f}(k_m)$ from by a forward transform (equation ⊛⊛)

  -multiply by $ik_m$ to get $\hat{g}(k_m)$ and then

  -perform a backward (inverse) transform using equation ⊛ to get $\left.\frac{\partial f}{\partial x}\right|_j$.

- the method easily generalizes to any order derivative

- although Fourier methods are quite attractive do to the high accuracy and near exact representation of the derivatives, they have a few important limitations.

  - $f(x_j)$ must be continuously differentiable

  - $f(x_j)$ must be periodic

  - grid spacing must be uniform

# Modified (effective) wave number analysis

- we want to use **Fourier analysis** to look at **error and stability**
- we can do this by taking finite difference schemes and comparing them to a Fourier representation

-for example: take a central difference scheme $\left.\dfrac{\partial f}{\partial x}\right|_j = \dfrac{u_{i+1} - u_{i-1}}{2\Delta x}$

-expand $u_i$ as a Fourier series using ⊛

$$\frac{\hat{f}(k_m)e^{ik_m(x_j+\Delta x)} - \hat{f}(k_m)e^{ik_m(x_j-\Delta x)}}{2\Delta x}$$

Note we can drop the summation since this must hold for all wave numbers

-we can rearrange using exponent rules and Euler's formula:

$$= \frac{\hat{f}(k_m)}{2\Delta x}\left[e^{ik_m(x_j+\Delta x)} - e^{ik_m(x_j-\Delta x)}\right]$$

$$= \frac{\hat{f}(k_m)}{2\Delta x}\left[e^{ik_m x_j}e^{ik_m\Delta x} - e^{ik_m x_j}e^{-ik_m\Delta x}\right]$$

$$= \frac{\hat{f}(k_m)}{2\Delta x}e^{ik_m x_j}\left[e^{ik_m\Delta x} - e^{-ik_m\Delta x}\right]$$

# Modified (effective) wave number analysis

$$= \frac{\hat{f}(k_m)}{2\Delta x} e^{ik_m x_j} \left[ \cos(k_m \Delta x) + i \sin(k_m \Delta x) - \cos(k_m \Delta x) + i \sin(k_m \Delta x) \right]$$

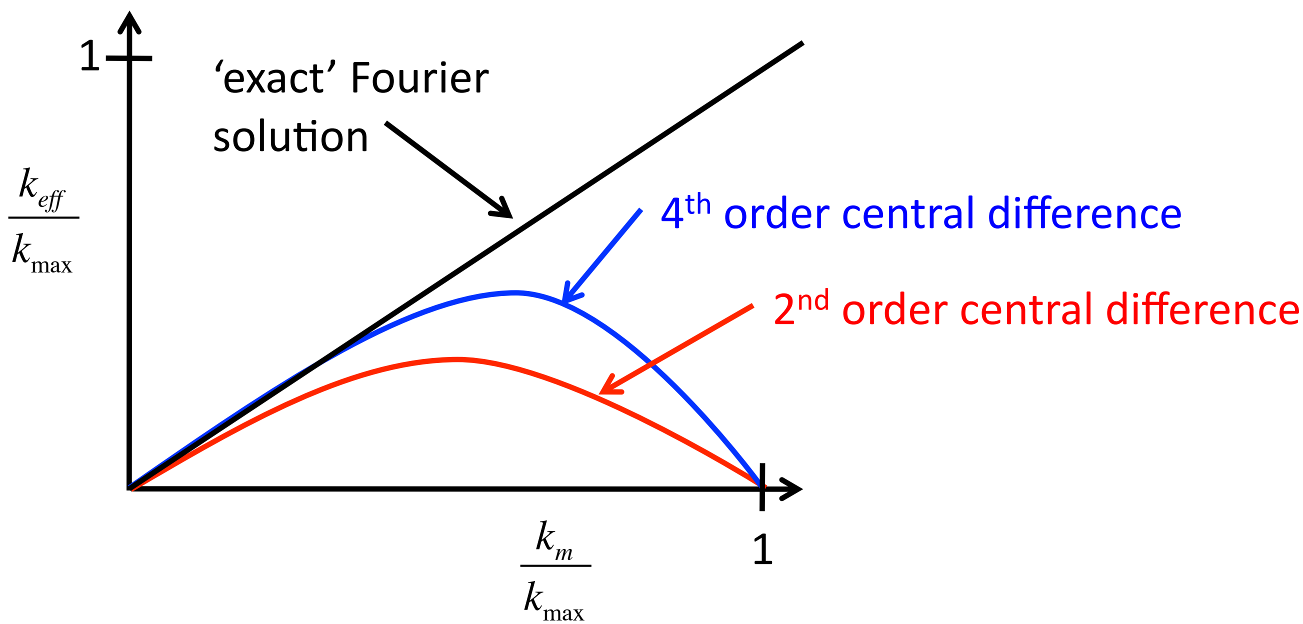
$$= \frac{\hat{f}(k_m)}{2\Delta x} e^{ik_m x_j} \left[ 2i \sin(k_m \Delta x) \right]$$

$$= \hat{f}(k_m) e^{ik_m x_j} \underbrace{\frac{\sin(k_m \Delta x)}{\Delta x}}_{\text{Modified wavenumber}} = \underbrace{i k_{eff} \hat{f}(k_m) e^{ik_m x_j}}_{\text{This is our Fourier derivative at } k_m}$$

-where $k_{eff}$ is our effective wave number for a CDS (as compared to our exact Fourier derivative)

# Modified (effective) wave number analysis

- we can look at this graphically to examine the **effect of scale on our error term**:




- for a well resolved signals (i.e. $f\left(x_j\right)$ is smooth at wave numbers resolved by the grid) central differences will do a good job (small $k_m/k_{max}$ ).

- for a $f\left(x_j\right)$ that has significant energy content at high wave numbers even a high order of accuracy won't make a solution accurate!!

# Stability analysis

- **von Neumann analysis**: example simple explicit 1-D heat equation

$$\frac{\partial u}{\partial x} = \alpha \frac{\partial^2 u}{\partial x^2} \quad \text{with} \quad \frac{u_i^{n+1} - u_i^n}{\Delta t} = \frac{\alpha}{(\Delta x)^2}\left(u_{i+1}^n - 2u_i^n + u_{i-1}^n\right)$$

- Recall that any FDE can be expressed as: $\mathrm{FDE} = D + \varepsilon$ → error
  ↳ exact
- The computed numerical solution must satisfy the FDE so plugging this into the above:

$$\frac{D_i^{n+1} + \varepsilon_i^{n+1} - D_i^n - \varepsilon_i^n}{\Delta t} = \frac{\alpha}{(\Delta x)^2}\left(D_{i+1}^n + \varepsilon_{i+1}^n - 2D_i^n - 2\varepsilon_i^n + D_{i-1}^n + \varepsilon_{i-1}^n\right)$$

- Since the solution $D$ must satisfy the FDE the error must also independently satisfy it

$$\frac{\varepsilon_i^{n+1} - \varepsilon_i^n}{\Delta t} = \frac{\alpha}{(\Delta x)^2}\left(\varepsilon_{i+1}^n - 2\varepsilon_i^n + \varepsilon_{i-1}^n\right)$$

- if we assume that we can represent the error as a Fourier series:

$$\varepsilon_m(x,t) = b_m(t)\, e^{ik_m x}$$

↳ time dependent Fourier coefficient

# Stability analysis

- Since the equation is linear, we can look at individual wavenumbers
- Here we will look for solutions of the form $z^n e^{ik_m x}$ (Fourier coefficients scale with timestep) and we will choose $z^n = e^{at}$ which is consistent for n=0, t=0 and therefore, our error should be of the form:

$$\varepsilon_m(x,t) = e^{at} e^{ik_m x}$$

- We can subsitute this into our FDE error equation

$$e^{a(t+\Delta t)} e^{ik_m x} - e^{at} e^{ik_m x} = r\left(e^{at} e^{ik_m(x+\Delta x)} - 2e^{at} e^{ik_m x} + e^{at} e^{ik_m(x-\Delta x)}\right) \text{ where } r = \frac{\alpha \Delta t}{(\Delta x)^2}$$

- If we divide the equation by $e^{at} e^{ik_m x}$

$$e^{a\Delta t} - 1 = r\left(e^{ik_m \Delta x} - 2 + e^{-ik_m \Delta x}\right)$$

- Use the trig identity: $\cos\beta = \dfrac{e^{i\beta} + e^{-i\beta}}{2}$

$$e^{a\Delta t} = 1 + 2r\left[\cos(k_m \Delta x) - 1\right]$$

- Using the trig identity: $\sin^2\dfrac{\beta}{2} = \dfrac{1-\cos(\beta)}{2}$

# Stability analysis

- Our final reduced form is:

$$\boxed{e^{a\Delta t} = 1 - 4r\sin^2\left(\frac{k_m \Delta x}{2}\right)}$$

- Note that because $\varepsilon_i^{n+1} = e^{a\Delta t}\varepsilon_i^n$ for each wavenumber in the solution, it is clear that if $\left|e^{a\Delta t}\right| \le 1$ the error will not grow in time ➔

$$\left|1 - 4r\sin^2\left(\frac{k_m \Delta x}{2}\right)\right| \le 1$$

- To evaluate this we must consider two cases:

$$1)\quad 1 - 4r\sin^2\left(\frac{k_m \Delta x}{2}\right) \ge 0 \rightarrow 4r\sin^2\left(\frac{k_m \Delta x}{2}\right) \ge 0$$

$$2)\quad 1 - 4r\sin^2\left(\frac{k_m \Delta x}{2}\right) < 0 \rightarrow 4r\sin^2\left(\frac{k_m \Delta x}{2}\right) - 1 \le 1$$

- Condition 1) is always satisfied for $r \ge 0$
- Condition 2) is only satisfied if $r \le 1/2$ this is the stability criteria for the method

# Stability analysis

- If we do the same analysis for the **simple implicit scheme**:

$$u_i^{n+1} - u_i^n = r\left(u_{i+1}^{n+1} - 2u_i^{n+1} + u_{i-1}^{n+1}\right)$$

- We can show that the amplification factor is:

$$\frac{1}{1-4r\sin^2\left(\dfrac{k_m\Delta x}{2}\right)}$$

- The stability criteria:

$$\left|\frac{1}{1-4r\sin^2\left(\dfrac{k_m\Delta x}{2}\right)}\right| \le 1$$

- Is satisfied for all $r \ge 0$ and therefore, the simple implicit scheme is unconditionally stable