# Computational Fluid Dynamics: Lecture 5
# ME EN 6720

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# Discretized equations

**Example of a discretized equation**

- 1-D heat equation: $$\frac{\partial u}{\partial t} = \alpha \frac{\partial^2 u}{\partial x^2}$$

- two different FDE representations of this are:

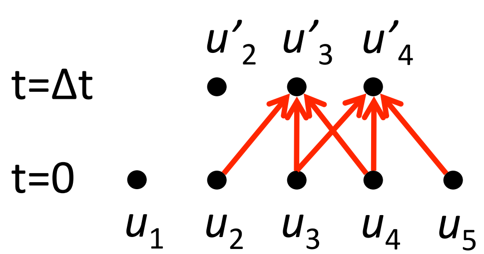
$$\frac{u_i^{n+1} - u_i^n}{\Delta t} = \alpha \frac{1}{(\Delta x)^2}\left[u_{i+1}^n - 2u_i^n + u_{i-1}^n\right] \qquad \text{(explicit)}$$

and

$$\frac{u_i^{n+1} - u_i^n}{\Delta t} = \alpha \frac{1}{(\Delta x)^2}\left[u_{i+1}^{n+1} - 2u_i^{n+1} + u_{i-1}^{n+1}\right] \qquad \text{(implicit)}$$

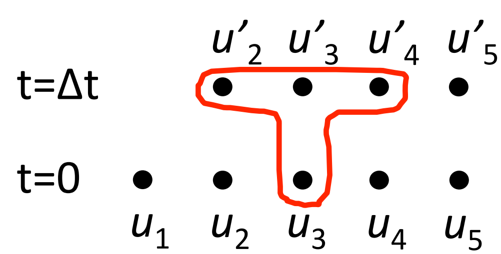
- graphically we can represent the points used by these two schemes as:

**explicit**




For example to get $u'_3$ and $u'_4$

**implicit**




For example to get $u'_3$

# Truncation error in FDEs

- We can use our FDE equation above (explicit one) to look at the total TE of our solution
  - 1st we expand each $u$ as a Taylor series about $u^n_i$:

$$u_i^{n+1} = u_i^n + \left(\frac{\partial u}{\partial t}\right)_i^n \Delta t + \left(\frac{\partial^2 u}{\partial t^2}\right)_i^n \frac{(\Delta t)^2}{2!} + \mathbf{O}\left(\Delta t^3\right)$$

$$u_{i+1}^{n} = u_i^n + \left(\frac{\partial u}{\partial x}\right)_i^n \Delta x + \left(\frac{\partial^2 u}{\partial x^2}\right)_i^n \frac{(\Delta x)^2}{2!} + \left(\frac{\partial^3 u}{\partial x^3}\right)_i^n \frac{(\Delta x)^3}{3!} + \mathbf{O}\left(\Delta x^4\right)$$

$$u_{i-1}^{n} = u_i^n - \left(\frac{\partial u}{\partial x}\right)_i^n \Delta x + \left(\frac{\partial^2 u}{\partial x^2}\right)_i^n \frac{(\Delta x)^2}{2!} - \left(\frac{\partial^3 u}{\partial x^3}\right)_i^n \frac{(\Delta x)^3}{3!} + \mathbf{O}\left(\Delta x^4\right)$$

  - substituting these into the FDE:

$$\frac{1}{\cancel{\Delta t}}\left[\cancel{u_i^n} + \left(\frac{\partial u}{\partial t}\right)_i^n \cancel{\Delta t} + \left(\frac{\partial^2 u}{\partial t^2}\right)_i^n \frac{(\Delta t)^{\cancel{2}}}{2!} + \mathbf{O}\left(\Delta t^{\overset{2}{\cancel{3}}}\right) - \cancel{u_i^n}\right] =$$

$$\frac{\alpha}{\cancel{\Delta x^2}}\left[\cancel{u_i^n} + \cancel{\left(\frac{\partial u}{\partial x}\right)_i^n \Delta x} + \left(\frac{\partial^2 u}{\partial x^2}\right)_i^n \frac{(\cancel{\Delta x})^2}{2!} + \cancel{\left(\frac{\partial^3 u}{\partial x^3}\right)_i^n \frac{(\Delta x)^3}{3!}} + \mathbf{O}\left(\Delta x^{\overset{2}{\cancel{4}}}\right) - \cancel{2u_i^n} + \cancel{u_i^n} - \cancel{\left(\frac{\partial u}{\partial x}\right)_i^n \Delta x} + \left(\frac{\partial^2 u}{\partial x^2}\right)_i^n \frac{(\cancel{\Delta x})^2}{2!} - \cancel{\left(\frac{\partial^3 u}{\partial x^3}\right)_i^n \frac{(\Delta x)^3}{3!}} + \mathbf{O}\left(\Delta x^{\overset{2}{\cancel{4}}}\right)\right]$$

  - cancelling terms ($\cancel{\ }$): $\left(\frac{\partial u}{\partial t}\right)_i^n + \left(\frac{\partial^2 u}{\partial t^2}\right)_i^n \frac{\Delta t}{2} + \mathbf{O}\left(\Delta t^2\right) = \alpha\left[\left(\frac{\partial^2 u}{\partial x^2}\right)_i^n + \mathbf{O}\left(\Delta x^2\right)\right]$
  - recovering the PDE: $\underbrace{\frac{\partial u}{\partial t} = \alpha\frac{\partial^2 u}{\partial x^2}}_{\text{PDE}} + \underbrace{\mathbf{O}\left[\Delta t, \Delta x^2\right]}_{\text{TE}}$ ➔ **PDE + TE = FDE**

- this can be done for any FDE. Mixing schemes doesn't change the TE for individual terms

# Finite difference equations

**Example of a FDE formulation:**

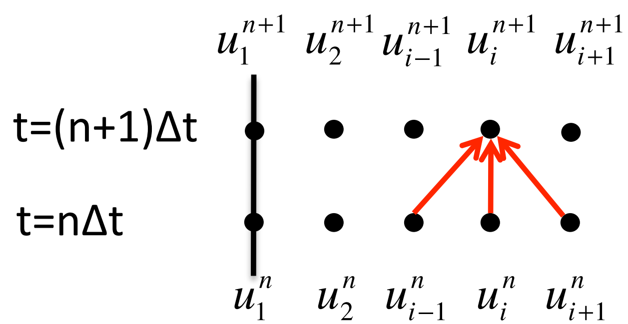
- 1-D heat equation (we have used this as an example before)

$$\frac{\partial u}{\partial t} = \alpha \frac{\partial^2 u}{\partial x^2}$$

- using a forward in time scheme ➔ $\dfrac{\partial u}{\partial t} = \dfrac{u_i^{n+1} - u_i^n}{\Delta t}$
- 2nd order central difference scheme in space ➔ $\dfrac{\partial^2 u}{\partial x^2} = \dfrac{u_{i+1}^n - 2u_i^n + u_{i-1}^n}{(\Delta x)^2}$

$$\frac{u_{i+1}^{n+1} - u_i^n}{\Delta t} = \alpha \frac{1}{(\Delta x)^2}\left[u_{i+1}^n - 2u_i^n + u_{i-1}^n\right]$$

graphically:



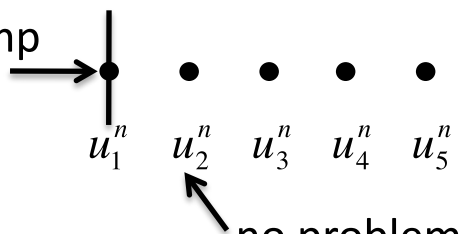

For example to get $u_i^{n+1}$

# Boundary conditions

- This problem (1-D heat equation) is the basic “heated rod” problem
- we have **<u>two basic boundary conditions</u>** for this type of problem

1) Constant temperature at the ends ($u_1$=a at x=0)

constant temp
($u_1$=a)




no problem calculating 2nd derivative at $u_2$

what if we are interested in the heat flux from the end of the rod? i.e. $\left.\frac{\partial u}{\partial x}\right|_{x=0}$

- a solution is to use a polynomial fit ➔ assume u is of the form: $u = a + bx + cx^2$

$\Rightarrow \left.\frac{\partial u}{\partial x}\right|_{x=0} = b$ and $\left.u\right|_{x=0} = a$ if we use our 1st three points:

$$\left.\begin{aligned} u_1 &= a \\ u_2 &= a + b\Delta x + c\Delta x^2 \\ u_3 &= a + 2b\Delta x + 4c\Delta x^2 \end{aligned}\right\} \quad \begin{aligned} a &= u_1 \\ b &= \frac{-3u_1 + 4u_2 - u_3}{2\Delta x} \\ c &= \frac{u_1 - 2u_2 + u_3}{2\Delta x^2} \end{aligned}$$

Using Fourier’s law the heat flux is:

$$q_x = -\alpha \left.\frac{\partial u}{\partial x}\right|_{x=0} = -\alpha b \Rightarrow q_x = \frac{\alpha}{2\Delta x}\left(3u_1 - 4u_2 + u_3\right)$$

A 2nd order accurate estimate

# Boundary conditions

2) Constant flux at the ends ($q_x$=constant at x=0). We need $u_1$ to get $\frac{\partial^2 u}{\partial x^2}$ at point 2.

- We can use the same technique as before (polynomial fit to 1st 3 points) but now *a* is an unknown and instead we know *b* ➔ $b = -q_x/\alpha$ ➔ we can solve for *a*

$$\left.\begin{aligned} u_2 &= a + b\Delta x + c\Delta x^2 \\ u_3 &= a + 2b\Delta x + 4c\Delta x^2 \end{aligned}\right\} \text{2 eqns 2 unknows} \;\Rightarrow\; \text{solve for } a$$

we can solve for a to get: $u_1 = a = \tfrac{2}{3}\frac{q_x}{\alpha}\Delta x + \tfrac{4}{3}u_2 - 3u_3$ a 2nd order accurate estimate

# Consistency and Stability

**Consistency:** a finite difference scheme is said to be consistent if as Δx (and/or Δt) ➔ 0 ➔ Truncation Error ➔ 0.

Remember we can write: P.D.E + T.E. = F.D.E
A consistent scheme will have T.E.➔0

• For our simple explicit scheme for the 1-D heat equation, we can show (using Taylor series expansion, see Lecture 4):

$$\frac{\partial u}{\partial t}+\frac{\partial^2 u}{\partial t^2}\frac{\Delta t}{2}+\mathbf{O}\left(\Delta t^2\right)=\alpha\left[\frac{\partial^2 u}{\partial x^2}+\frac{\partial^4 u}{\partial x^4}\frac{\Delta x^2}{4!}+\mathbf{O}\left(\Delta x^4\right)\right]$$

or $$\frac{\partial u}{\partial t}=\alpha\frac{\partial^2 u}{\partial x^2}+\mathbf{O}\left(\Delta t,\Delta x^2\right) \qquad \lim_{\substack{\Delta x\to 0\\ \Delta t\to 0}} \Rightarrow \text{ we recover our P.D.E.}$$

this scheme is consistent

# Consistency and Stability

- **2nd example**: Dufort-Frankel scheme for the 1-D heat equation

$$\frac{u_i^{n+1}-u_i^{n}}{2\Delta t}=\frac{\alpha}{(\Delta x)^2}\left[u_{i+1}^{n}-u_i^{n}-u_i^{n-1}+u_{i-1}^{n}\right]$$

If we expand $u_i^{n+1}, u_{i+1}^{n}, u_i^{n}, u_i^{n-1}, u_{i-1}^{n}$ with Taylor series about $u_i^n$ and then reduce and factor:

$$\frac{\partial u}{\partial t}+\alpha\frac{\partial^2 u}{\partial x^2}\left(\frac{\Delta t}{\Delta x}\right)^2=\alpha\frac{\partial^2 u}{\partial x^2}+\mathbf{O}\left(\Delta t^2,\Delta x^2\right)$$

Writing as P.D.E. + T.E. = F.D.E.

$$\frac{\partial u}{\partial t}=\alpha\frac{\partial^2 u}{\partial x^2}+\mathbf{O}\left[\Delta t^2,\Delta x^2,\left(\frac{\Delta t}{\Delta x}\right)^2\right]$$

This method is only consistent if:

$$\left.\begin{array}{l}\Delta x\to 0\\ \Delta t\to 0\\ \dfrac{\Delta t}{\Delta x}\to 0\end{array}\right\}\ \Delta x \text{ and } \Delta t \text{ must } \to 0 \text{ at the same rate}$$

# Consistency and Stability

- **Consistency** is only a necessary not a sufficient condition for a numerical scheme
- besides having a consistent scheme we also want our numerical scheme to be **stable**

  - **Stability**: (marching problems)

  -For a stable scheme, any error introduced in the F.D.E. (if stable) does not grow with the solution of the F.D.E.

  - **Convergence**: Solution of the F.D.E. approaches that of the P.D.E. as the size of the grid approaches zero

    - consistency
    - stability

    } Convergence (a consistent stable scheme will converge)