Computational Fluid Dynamics: Lectures 12 (ME EN 6720)

Prof. Rob Stoll
Department of Mechanical Engineering
University of Utah

Spring 2014

Vorticity and the Streamfunction

- Vorticity: $\vec{\omega} \equiv$ twice the angular velocity

$$\vec{\omega} = \vec{\nabla} \times \vec{V} \text{ where } \vec{\nabla} = \frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial x}\hat{k}$$

- In a 2D flow

$$\vec{\omega} = \omega_z = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}$$

- We also can define a **streamfunction** ψ which is mathematically (and physically) linked to the **streamlines** which are lines of constant ψ (you can think of ψ as the potential of the velocity field)

- **Streamlines** are lines that are everywhere tangent to the velocity field and in **2D**:

$$u = \frac{\partial \psi}{\partial y} \text{ and v=-} \frac{\partial \psi}{\partial x}$$

Vorticity transport equation (2D)

- General strategy: differentiate the *x*-dir and *y*-dir Navier-Stokes equations with respect to *y* and *x*, respectively.
- recall N-S in 2D:

$$\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}=-\frac{1}{\rho}\frac{\partial P}{\partial x}+\nu\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}\right)$$

$$\frac{\partial v}{\partial t}+u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}=-\frac{1}{\rho}\frac{\partial P}{\partial y}+\nu\left(\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}\right)$$

- Now if we differentiate:

1 $$\frac{\partial u}{\partial y\partial t}+\frac{\partial u}{\partial y}\frac{\partial u}{\partial x}+u\frac{\partial u}{\partial x\partial y}+\frac{\partial v}{\partial y}\frac{\partial u}{\partial y}+v\frac{\partial^2 u}{\partial y^2}=-\frac{1}{\rho}\frac{\partial P}{\partial y\partial x}+\nu\left(\frac{\partial^3 u}{\partial y\partial x^2}+\frac{\partial^3 u}{\partial y^3}\right)$$

2 $$\frac{\partial v}{\partial x\partial t}+\frac{\partial u}{\partial x}\frac{\partial v}{\partial x}+u\frac{\partial^2 u}{\partial x^2}+\frac{\partial v}{\partial x}\frac{\partial v}{\partial y}+v\frac{\partial^2 v}{\partial x\partial y}=-\frac{1}{\rho}\frac{\partial P}{\partial y\partial x}+\nu\left(\frac{\partial^3 v}{\partial x\partial y^2}+\frac{\partial^3 v}{\partial x^3}\right)$$

Vorticity transport equation (2D)

- if we subtract 1 from 2

$$\frac{\partial}{\partial t}\left(\frac{\partial v}{\partial x}-\frac{\partial u}{\partial y}\right)+u\frac{\partial}{\partial x}\left(\frac{\partial v}{\partial x}-\frac{\partial u}{\partial y}\right)+v\frac{\partial}{\partial y}\left(\frac{\partial v}{\partial x}-\frac{\partial u}{\partial y}\right)+\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\right)\left(\frac{\partial v}{\partial x}-\frac{\partial u}{\partial y}\right)=$$

$$\nu\left[\frac{\partial^2}{\partial x^2}\left(\frac{\partial v}{\partial x}-\frac{\partial u}{\partial y}\right)+\frac{\partial^2}{\partial y^2}\left(\frac{\partial v}{\partial x}-\frac{\partial u}{\partial y}\right)\right]$$

- by continuity $\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\right)=0$
- and $\left(\frac{\partial v}{\partial x}-\frac{\partial u}{\partial y}\right)=\omega_z$ (or just ω for short)
- the pressure term is eliminated by the subtraction!

$$\rightarrow \quad \boxed{\frac{\partial \omega}{\partial t}+u\frac{\partial \omega}{\partial x}+v\frac{\partial \omega}{\partial y}=\nu\left[\frac{\partial^2 \omega}{\partial x^2}+\frac{\partial^2 \omega}{\partial y^2}\right]}$$

Which is the **vorticity transport equation** in 2D

Vorticity transport equation (2D)

- At steady state this reduces to:

$$u\frac{\partial\omega}{\partial x}+v\frac{\partial\omega}{\partial y}=\nu\left[\frac{\partial^2\omega}{\partial x^2}+\frac{\partial^2\omega}{\partial y^2}\right]$$

- and for both stead and unsteady out velocity can be found from:

$$u=\frac{\partial\psi}{\partial y};\ \mathrm{v}=-\frac{\partial\psi}{\partial x}\ \text{and}\ \omega=\left(\frac{\partial v}{\partial x}-\frac{\partial u}{\partial y}\right)$$

- which combined is:

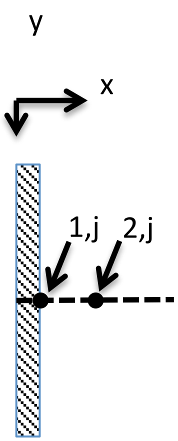
$$\boxed{\frac{\partial^2\psi}{\partial x^2}+\frac{\partial^2\psi}{\partial y^2}=-\omega}$$

the streamfunction equation

- Unknowns: ω and ψ with two governing equations:
  - Vorticity transport (parobilic or elliptical)
  - Streamfunction (elliptical)
- **No pressure term** (to get *u, v, P* we must compute using *u, v = f(ψ)* and a Poisson equation for the pressure)
-
- One problem with this is that the **extension to 3D isn't very useful** and or straightforward (so its appropriate for 2D problems only)

Vorticity boundary conditions

a) **Wall Boundaries**:



- At any boundary we have: $\left(\frac{\partial^2\psi}{\partial x^2}+\frac{\partial^2\psi}{\partial y^2}\right)_{i,j} = -\omega_{i,j}$
- Along any surface we can always say that **a solid surface is 'a streamline'** and the streamfunction is therefore constant (same streamline)
- The value for this streamfunction is arbitrary ➔

$$\psi_{\text{wall}} = \psi_1$$

- From this we can deduce that (for the vertical wall example):

$$\psi_{1,j} = \psi_1 \Rightarrow \left.\frac{\partial^2\psi}{\partial y^2}\right|_{1,j} = 0$$

- Using a Taylor series expansion: (point 2 from point 1)

$$\psi_{2,j} = \psi_{1,j} + \left.\frac{\partial\psi}{\partial x}\right|_{1,j}\Delta x + \left.\frac{\partial^2\psi}{\partial x^2}\right|_{1,j}\frac{\Delta x^2}{2} + \mathbf{O}\left(\Delta x^3\right)$$

Vorticity boundary conditions

a) **Wall Boundaries**:

- Also on this wall we know by definition that:

$$v_{1,j} = -\left.\frac{\partial \psi}{\partial x}\right|_{1,j} = 0$$

- And therefore:

$$\psi_{2,j} = \psi_{1,j} + \left.\frac{\partial^2 \psi}{\partial x^2}\right|_{1,j} \frac{\Delta x^2}{2} + \mathbf{O}\left(\Delta x^3\right) \text{ or rearranging}$$

$$\left.\frac{\partial^2 \psi}{\partial x^2}\right|_{1,j} = \frac{2\left(\psi_{2,j} - \psi_{1,j}\right)}{\Delta x^2} + \mathbf{O}\left(\Delta x^3\right)$$

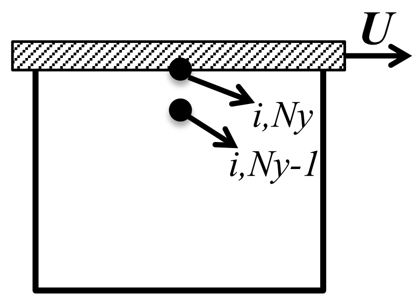
- Going back to our streamfunction equation:

$$\text{Our vorticity B.C is} \Rightarrow -\left.\frac{\partial^2 \psi}{\partial x^2}\right|_{1,j} - \left.\frac{\partial^2 \psi}{\partial y^2}\right|_{1,j} = \boxed{\frac{2\left(\psi_{1,j} - \psi_{2,j}\right)}{\Delta x^2}}$$

- Other walls are specified in the same way

Vorticity boundary conditions

b) <u>A Moving Boundary</u>: e.g. lid driven cavity ➔



- Taylor series for the example:

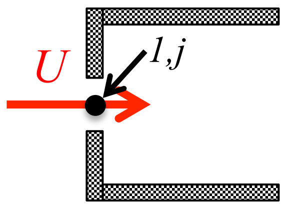
$$\psi_{i,Ny-1} = \psi_{i,Ny} - \underbrace{\left.\frac{\partial \psi}{\partial y}\right|_{i,Ny}}_{u=\frac{\partial \psi}{\partial y}=U} \Delta y + \left.\frac{\partial^2 \psi}{\partial x^2}\right|_{i,Ny} \frac{\Delta y^2}{2} + \mathbf{O}\left(\Delta y^3\right) \text{ or rearranging}$$

$$\psi_{i,Ny-1} = \psi_{i,Ny} - U\Delta y + \left.\frac{\partial^2 \psi}{\partial x^2}\right|_{i,Ny} \frac{\Delta y^2}{2} + \mathbf{O}\left(\Delta y^3\right) \Rightarrow$$

$$\boxed{\omega_{i,Ny} = \frac{2\left(\psi_{i,Ny} - \psi_{i,Ny-1}\right)}{\Delta y^2} - \frac{2U}{\Delta y}} + \mathbf{O}(\Delta y)$$

Vorticity boundary conditions

c) **Inflow conditions**: e.g. Jet into a cavity ➔



- At the inflow $\frac{\partial \psi}{\partial x} = 0$ (or an outflow)
- We can use this with a forward difference to get an inflow condition
- Recall in general $\frac{\partial f}{\partial x} = \frac{1}{2\Delta x}\left(-f_{i+2} + 4f_{i+1} - 3f_i\right)$ a one-sided approx that is $\sim\Delta x^2$
- Using this:

$$\frac{\partial \psi}{\partial x} = 0 \Rightarrow \psi_{1,j} = \frac{1}{3}\left(4\psi_{2,j} - \psi_{3,j}\right)$$

and if we combine with the streamfunction equation

$$\boxed{\omega_{1,j} = \frac{2\left(\psi_{1,j} - 2\psi_{2,j}\right)}{\Delta x^2} - \frac{\psi_{1,j+1} - 2\psi_{1,j} + \psi_{1,j-1}}{\Delta y^2}}$$

- Outflow BCs are basically formulated in the same manner