Computational Fluid Dynamics: Lecture 13 (ME EN 6720)

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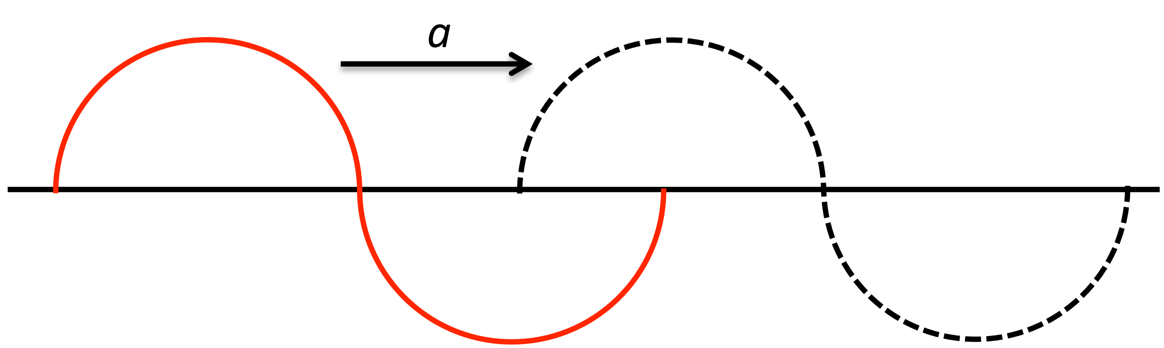
Spring 2014

Hyperbolic Equations

- So far we have looked at two difference components of the Navier-Stokes equations
  - Diffusion processes (viscosity)
  - Pressure (Poisson equations)
- We have one more important process: **advection**
  - simplified problems of this type are hyperbolic equations (strictly only for linear)
- **Linear 1st-order wave equation**:

$$\frac{\partial u}{\partial t} + a\frac{\partial u}{\partial x} = 0 \quad (\text{for } a > 0)$$

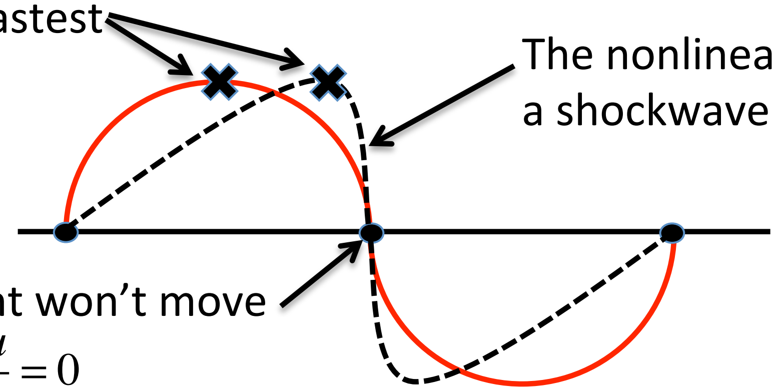


- the wave is translated downstream with no diffusion (pure advection) at the speed *a*.

Hyperbolic Equations

- **<u>Non-linear 1st-order wave equation</u>:** (inviscid Burger's equation)

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = 0$$



Discretizations for hyperbolic equations

- **Explicit Methods:**
  - Our simplest discretization is to use a forward in time, forward in space method:

$$\frac{u_i^{n+1}-u_i^n}{\Delta t}+a\frac{u_{i+1}^n-u_i^n}{\Delta x}=0$$

$TE \sim O(\Delta t, \Delta x)$; stability: using von Neumann analysis we can show that this method is **unconditionally unstable**!

  - A little better discretization might help: Forward-Time, centered space

$$\frac{u_i^{n+1}-u_i^n}{\Delta t}+a\frac{u_{i+1}^n-u_{i-1}^n}{2\Delta x}=0$$

$TE \sim O(\Delta t, \Delta x^2)$; stability is still unconditionally unstable.

What is going on here?

Discretizations for hyperbolic equations

- **Upwind differencing:**
  - Our simplest discretization is to use a forward in time, forward in space method:

$$\frac{u_i^{n+1}-u_i^n}{\Delta t}+a\frac{u_i^n-u_{i-1}^n}{\Delta x}=0 \qquad *$$

$TE \sim O(\Delta t, \Delta x)$; stability: using von Neumann analysis:

$$A(t+\Delta t)e^{ikx}-A(t)e^{ikx}+\frac{a\Delta t}{\Delta x}\left[A(t)e^{ikx}-A(t)e^{ik(x-\Delta x)}\right]=0$$

divide out $A(t)e^{ikx}$ and use our amplification factor G

$$G-1+\frac{a\Delta t}{\Delta x}\left(1-e^{-ik\Delta x}\right)=0$$

$$G=1-\frac{a\Delta t}{\Delta x}(1-\cos\beta+i\sin\beta)$$

if we look at G^2 so that $1\le\left[1-\nu(1-\cos\beta)\right]^2+\left[\nu\sin\beta\right]^2$ with $\nu=\dfrac{a\Delta t}{\Delta x}$

Our stability criteria from this is that: $C\le 1$ where the Courant number $C=\dfrac{a\Delta t}{\Delta x}$

Equivalent difference equation

- **Modified Equations (aka Equivalent difference eq):**
  - If we use a Taylor series expansion with equation * for each term $u_i^{n+1}, u_i^n, u_{i-1}^n$ we get:

$$\underbrace{\frac{\partial u_i^n}{\partial t} + a\frac{\partial u_i^n}{\partial x}}_{\text{PDE}} = \underbrace{\frac{\Delta t}{2}\frac{\partial^2 u_i^n}{\partial t^2} + \frac{a\Delta x}{2}\frac{\partial^2 u_i^n}{\partial x^2} - \frac{(\Delta t)^2}{6}\frac{\partial^3 u_i^n}{\partial t^3} - \frac{a(\Delta x)^2}{6}\frac{\partial^3 u_i^n}{\partial x^3}}_{\text{TE} \sim \mathbf{O}\left[\Delta x, \Delta t\right]} + \cdots \text{H.O.T.}$$

  - we can use our PDE to eliminate the $\partial/\partial t$ terms in the TE (recall homework #2)

$$\Rightarrow \frac{\partial}{\partial t} = -a\frac{\partial}{\partial x}; \frac{\partial^2}{\partial t^2} = a^2\frac{\partial^2}{\partial x^2} \text{ and so forth}$$

  - subbing these into our equation:

$$\frac{\partial u}{\partial t} + a\frac{\partial u}{\partial x} = \underbrace{\frac{1}{2}a\frac{\partial u^2}{\partial x^2}(\Delta x - a\Delta t)}_{\text{leading TE term}} + \mathbf{O}\left(\Delta x^2\right) + \mathbf{O}\left(\Delta t^2\right)$$

Equivalent difference equation

- or we can rewrite this as:

$$\frac{\partial u}{\partial t} + a\frac{\partial u}{\partial x} = \nu_{eff}\frac{\partial u^2}{\partial x^2} + \text{H.O.T.}$$

where ν_{eff} = effective viscosity/diffusivity due to numerics

recall for stability $C = a\dfrac{\Delta t}{\Delta x} \le 1 \Rightarrow a\Delta t \le \Delta x$

and from our equation $\nu_{eff} = \frac{1}{2} a\left(\Delta x - a\Delta t\right)$

which has two basic cases

$$\Rightarrow \begin{cases} C > 1\left(a\Delta t > \Delta x\right) \to \nu_{eff} < 0 \to \text{ effect of } \nu_{eff} \text{ is an} \\ \text{exponentially growing solution} \\ C \le 1\left(a\Delta t \le \Delta x\right) \to \nu_{eff} \ge 0 \to \text{ and the solution is smooth} \\ \text{note that for } C = 1 \to \nu_{eff} = 0 \text{ and the TE goes away!} \end{cases}$$

The Navier-Stokes Equations

- One of the stated goals for the class is to develop solutions to the full Navier-Stokes equations. This lecture focuses on that task

- **<u>Review</u>:** Navier-Stokes (N-S) Equations

  - In Integral form using mixed notation we have:

$$\frac{\partial}{\partial t}\int_{\mathrm{CV}} \rho u_i d\forall + \int_{\mathrm{CS}} \rho u_i \vec{V} \bullet d\vec{A} = \int_{\mathrm{CS}} \left(\tau_{ij} I_j\right) \bullet d\vec{A} - \int_{\mathrm{CS}} P I_i \bullet d\vec{A}$$

where for a Newtonian fluid $\tau_{ij} = 2\mu D_{ij} - \frac{2}{3}\mu\delta_{ij}\frac{\partial u_k}{\partial x_k}$ and $D_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)$

  - The index notation differential form is:

$$\frac{\partial \rho u_i}{\partial t} + \frac{\partial \rho u_i u_j}{\partial x_j} = -\frac{\partial P}{\partial x_i} + \frac{\partial \tau_{ij}}{\partial x_j}$$

  - Along with the momentum equation we must also solve the conservation of mass and depending on the problem (e.g. compressible) conservation of energy

  - Conservation of Mass reads: $\frac{\partial \rho}{\partial t} + \frac{\partial \rho u_i}{\partial x_i} = 0$

Discretization methods for the N-S

- In integral form we have:

$$\int_{CV} \rho u_i d\forall \text{ (if body forces are included they result in a similar term)}$$

  - This can be discretized using our standard methods from Lecture 8 and includes using a cell centered estimate or including more points.

- Convective Terms:

$$\frac{\partial\left(\rho u_i u_j\right)}{\partial x_j} \text{ or } \int_{CS} \rho u_i \vec{V} \bullet d\vec{A}$$

  - We treat these terms by generic methods similar to the wave equation. For integral form we treat this as a flux through the cell surfaces/faces.
  - These terms are many times treated explicitly (especially for incompressible flow)