Computational Fluid Dynamics: Lecture 16 (ME EN 6720)

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Solutions to the N-S Equations

- Incompressible Momentum Equation:

$$\frac{\partial u_i}{\partial t} + A(u_i) = -\frac{\partial P}{\partial x_i} + \frac{1}{\mathrm{Re}}\frac{\partial^2 u_i}{\partial x_j}$$

$$\text{and} \qquad \frac{\partial u_i}{\partial x_i} = 0$$

$$\text{where } A(u_i) = \text{ nonlinear convection term} \Rightarrow$$

$$\mathrm{A}(u_i) = \begin{cases} u_j \dfrac{\partial u_i}{\partial x_j} & \text{(nonconservative)} \\ \dfrac{\partial u_i u_j}{\partial x_j} & \text{(conservative)} \end{cases}$$

Solutions to the N-S Equations

- **<u>General Solution Methods</u>:**

  - unsteady N-S
    - -Projection methods
    - -fractional step method
    - grid
      - -collocated
      - -staggered
  - steady N-S
    - -artificial compressibility
    - -etc (see Ferziger Chp 7)
    - grid
      - -collocated
      - -staggered

- here we will focus mostly on unsteady flows solutions and the projection method

- **<u>Projection Method</u>:**
  - for a simple Euler in time advancement scheme (e.g. a forward in time) we use:

$$\frac{\partial f}{\partial t} = a + b \begin{cases} \dfrac{f^{*} - f^{n}}{\Delta t} = a \\[2ex] \dfrac{f^{n+1} - f^{*}}{\Delta t} = b \end{cases}$$

(this is modified for other time advancement schemes, more later)

Solutions to the N-S Equations

- Application of Projection method to Navier-Stokes Equation

Step I)

1 $$\frac{\vec{V}^* - \vec{V}^n}{\Delta t} + A\left(\vec{V}^n\right) = \frac{1}{\mathrm{Re}} \nabla^2 \vec{V}^n$$

In the first step we ignore pressure effects

Step II)

2 $$\frac{\vec{V}^{n+1} - \vec{V}^*}{\Delta t} = -\vec{\nabla} P^{n+1}$$

3 $$\vec{\nabla} \cdot \vec{V}^{n+1} = 0$$

In this form this is difficult to solve

Recall how we linked these equations last class (Lecture 16)

-take the divergence of 2 and apply 3

$$\frac{1}{\Delta t}\left(\vec{\nabla} \cdot \vec{V}^{n+1} - \vec{\nabla} \cdot \vec{V}^*\right) = -\nabla^2 P^{n+1}$$

0 by continuity 3, this is our link between pressure and continuity

4 $$\nabla^2 P^{n+1} = \frac{1}{\Delta t} \vec{\nabla} \cdot \vec{V}^*$$

Solutions to the N-S Equations

- <u>In Summary our Steps are:</u>

    Step I) Calculate $\vec{V}^*$ from equation 1

    Step II) Calculate P^{n+1} from equation 4

    Step III) Calculate $\vec{V}^{n+1}$ from equation 2

- Issues still open:

    -Boundary Conditions for
    -grid (choice of the location of variables and linked to BCs)

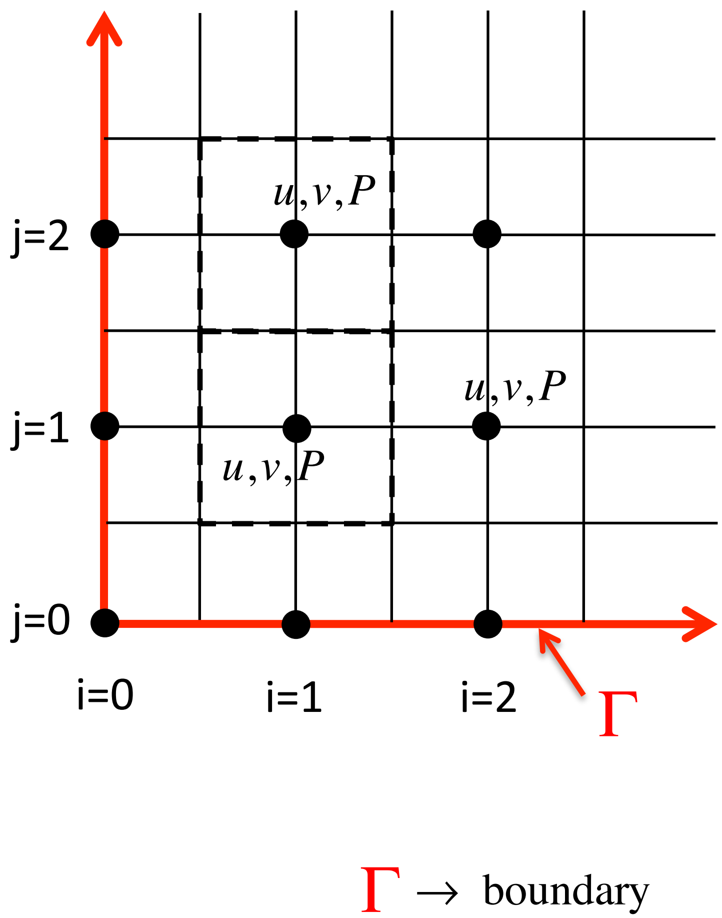
- **Choice of Variable Arrangements**

    - For basic equations (scalar equations) the grid structure wasn't overly important except to determine order of accuracy (certain choices make it harder to keep)

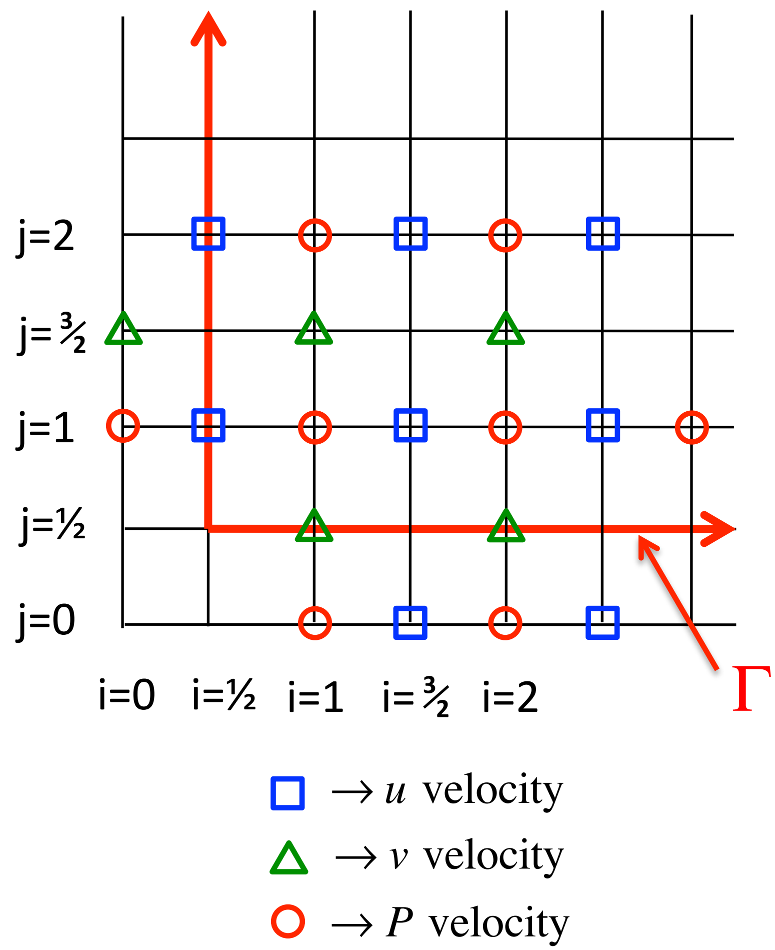
    - We have two possibilities: **Collocated** and **Staggered**

Solutions to the N-S Equations

- **Collocated**



- **Staggered (MAC grid)**



Discrete Example Solution to the N-S

• **Steps in Calculating Incompressible Momentum Equation**:

1. Calculate $\vec{V}^*$
2. Calculate P^{n+1}
3. Update $\vec{V}^{n+1}$

Application of Projection Method to a 2D problem:

- using staggered (MAC) grid
- with 2nd order finite difference approximation in space
- Euler (1st order) in time (forward in time scheme)

STEP I: solving $\boxed{\dfrac{\vec{V}^* - \vec{V}^n}{\Delta t} + A\left(\vec{V}^n\right) = \dfrac{1}{\mathrm{Re}}\nabla^2 \vec{V}^n}$

$$\text{for } u \Rightarrow \frac{1}{\Delta t}\left(u^*_{i+1/2,j} - u^n_{i+1/2,j}\right) + a^n_{i+1/2,j} = \frac{1}{\mathrm{Re}}\nabla^2 u^n_{i+1/2,j}$$

$$\text{for v} \Rightarrow \frac{1}{\Delta t}\left(v^*_{i,j+1/2} - v^n_{i,j+1/2}\right) + b^n_{i,j+1/2} = \frac{1}{\mathrm{Re}}\nabla^2 v^n_{i,j+1/2}$$

Discrete Example Solution to the N-S

- in our **step I** *u* and *v* equations, we have two different sets of nodes and our discrete equations must line up accordingly. This is especially important for the convective term

  - u-velocity nodes:

$$a^n_{i+1/2,j} = u^n_{i+1/2,j}\Delta_x u^n_{i+1/2,j} + \hat{v}^n_{i+1/2,j}\Delta_y u^n_{i+1/2,j}$$

where in general our derivative operators are:

$$\Delta_x f_{l,m} = \frac{1}{2\Delta x}\left(f_{l+1,m} - f_{l-1,m}\right) \quad \text{a CDS in the x-direction}$$

$$\Delta_y f_{l,m} = \frac{1}{2\Delta y}\left(f_{l,m+1} - f_{l,m-1}\right) \quad \text{a CDS in the y-direction}$$

  - v-velocity nodes:

$$b^n_{i,j+1/2} = \hat{u}^n_{i,j+1/2}\Delta_x v^n_{i,j+1/2} + v^n_{i,j+1/2}\Delta_y v^n_{i,j+1/2}$$

where we find $\hat{u}^n_{i,j+1/2}$ and $\hat{v}^n_{i+1/2,j}$ from their nearest neighbors:

$$\hat{u}^n_{i,j+1/2} = \frac{1}{4}\left(u^n_{i+1/2,j} + u^n_{i+1/2,j+1} + u^n_{i-1/2,j+1} + u^n_{i-1/2,j}\right)$$

$$\hat{v}^n_{i+1/2,j} = \frac{1}{4}\left(v^n_{i+1,j+1/2} + v^n_{i,j+1/2} + v^n_{i,j-1/2} + v^n_{i+1,j-1/2}\right)$$

Discrete Example Solution to the N-S

and for our Laplace operator, in general we have:

$$\nabla^n f_{l,m} = \Delta_{xx} f_{l,m} + \Delta_{yy} f_{l,m} = \frac{1}{(\Delta x)^2}\left(f_{l+1,m} - 2f_{l,m} + f_{l-1,m}\right) + \frac{1}{(\Delta y)^2}\left(f_{l,m+1} - 2f_{l,m} + f_{l,m-1}\right)$$

STEP II: solving $\boxed{\nabla^2 P^{n+1} = \frac{1}{\Delta t}\left(\vec{\nabla} \cdot \vec{V}^*\right)}$

This equation will be subject to different BCs depending on the problem (more later)

Out discrete equation is:

$$\frac{1}{(\Delta x)^2}\left(P^{n+1}_{i+1,j} - 2P^{n+1}_{i,j} + P^{n+1}_{i-1,j}\right) + \frac{1}{(\Delta y)^2}\left(P^{n+1}_{i,j+1} - 2P^{n+1}_{i,j} + P^{n+1}_{i,j-1}\right) =$$

$$\frac{1}{\Delta t}\left[\frac{1}{\Delta x}\left(u^*_{i+1/2,j} - u^*_{i-1/2,j}\right) + \frac{1}{\Delta y}\left(v^*_{i,j+1/2} - v^*_{i,j-1/2}\right)\right]$$

This equation can be solved using one of our iterative solvers (G-S, SOR with G-S, Multigrid G-S, Conjugate Gradient Methods, etc) to obtain P^{n+1}

Discrete Example Solution to the N-S

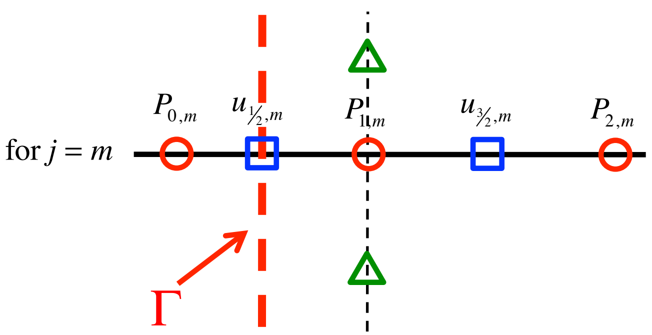
<u>What about Boundary Conditions? (</u>Example, for a wall)

If we project our original pressure equation $\dfrac{\vec{V}^{n+1}-\vec{V}^{*}}{\Delta t}=-\vec{\nabla}P^{n+1}$ perpendicularly onto the boundary:

$$\left(\frac{\partial P}{\partial n}\right)_{\Gamma}=-\frac{1}{\Delta t}\left(\vec{V}_{\Gamma}^{n+1}-\vec{V}_{\Gamma}^{*}\right)\cdot\vec{n}\qquad\begin{cases}\Gamma\Rightarrow\ \text{boundary value}\\ \vec{n}\Rightarrow\ \text{unit vector normal to surface}\end{cases}$$

we can show that the $\vec{V}_{\Gamma}^{*}$ value doesn't matter for our num. press. solution!

Why! Lets look at our grid and discrete equation at a Boundary (e.g. left vertical wall)



$$\begin{cases}\dfrac{1}{\Delta x}\left[\dfrac{P_{2,m}^{n+1}-P_{1,m}^{n+1}}{\Delta x}-\dfrac{P_{1,m}^{n+1}-P_{0,m}^{n+1}}{\Delta x}\right]+\dfrac{1}{\Delta y}\left[\dfrac{P_{1,m+1}^{n+1}-P_{1,m}^{n+1}}{\Delta y}-\dfrac{P_{1,m}^{n+1}-P_{1,m-1}^{n+1}}{\Delta y}\right]=\\ \dfrac{1}{\Delta t}\left[\dfrac{u_{3/2,m}^{*}-u_{1/2,m}^{*}}{\Delta x}+\dfrac{v_{1,m+1/2}^{*}-v_{1,m-1/2}^{*}}{\Delta y}\right]\end{cases}$$

for our Neumann BC (project equation above): $\dfrac{P_{1,m}^{n+1}-P_{0,m}^{n+1}}{\Delta x}=-\dfrac{1}{\Delta t}\left(u_{1/2,m}^{n+1}-u_{1/2,m}^{*}\right)$

u_{Γ}^{*} cancels out! ➔ it does not matter what u_{Γ}^{*} value we choose!

Typically $u_{\Gamma}^{*}=u_{\Gamma}^{n+1}$ implying that our BC is: $\boxed{\left(\dfrac{\partial P}{\partial n}\right)_{\Gamma}=0}$ (note this is <u>purely numerical</u>!)

Discrete Example Solution to the N-S

STEP III: solving $\boxed{\dfrac{\vec{V}^{n+1}-\vec{V}^{*}}{\Delta t}=\vec{\nabla}P^{n+1}}$ for $\vec{V}^{n+1}$

in discrete form using our results from **steps I and II** we have:

$$u_{i+1/2,j}^{n+1}=u_{i+1/2,j}^{*}-\frac{\Delta t}{\Delta x}\left(P_{i+1,j}^{n+1}-P_{i,j}^{n+1}\right)$$

$$v_{i,j+1/2}^{n+1}=v_{i,j+1/2}^{*}-\frac{\Delta t}{\Delta x}\left(P_{i,j+1}^{n+1}-P_{i,j}^{n+1}\right)$$

BCs for u and v?

For u and v at solid walls (e.g., the project), we can use mirror conditions in our staggered direction and a no-slip condition in the other direction (when grid nodes correspond to the boundary)