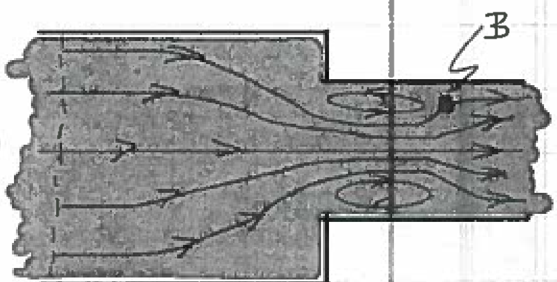
Chapter 4: Integral Analysis → (Fluids Moving)

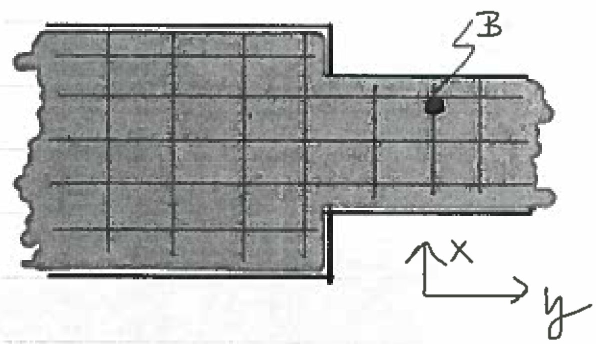
As engineers you can study the flow properties (pressure, velocity) in a flow field in two different ways:

1) Lagrangian approach: Track the trajectory of a fluid particle (not molecule) and study how this one evolves in time. ⇒ (like a pathline).

2) Eulerian approach: Solve a set of equations for the flow properties at any point in the flow field.

▷ Using a pictorial representation: let's consider a sudden contraction:





(Notes)

• which one is which?

1) Lagrangian approach 2) Eulerian approach.

if someone wanted to determine, let's say, the pressure at point B, knowing the velocity field and pressure at the inlet, it could be done in 2 different ways:

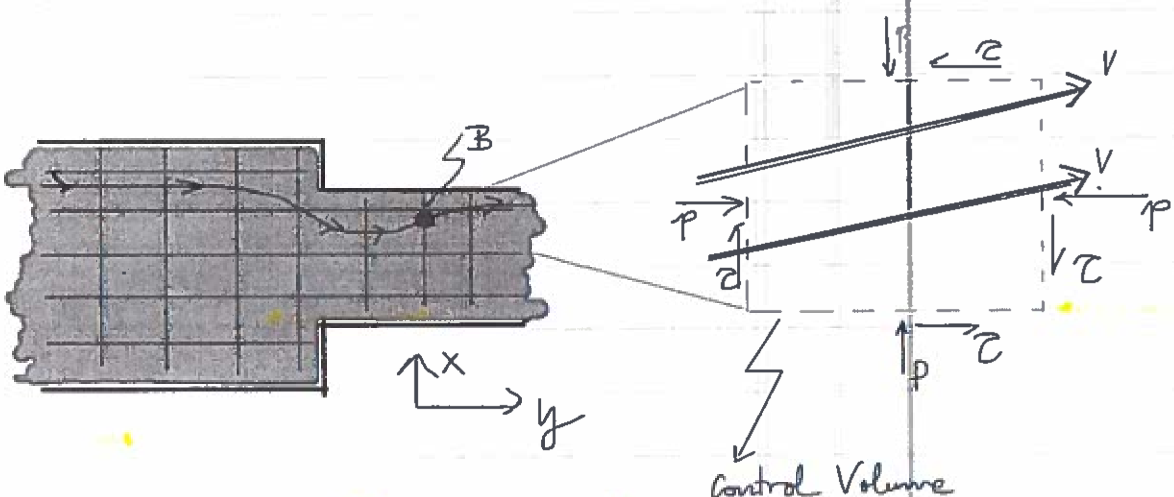
1) Find the pathline that passes by point B. As the particle moves along the pathline, the pressure changes can be tracked using Euler's equation (will see it later on). Or if the flow is steady then Bernoulli's equation can be used. ⇒ LAGRANGIAN APPROACH.

Problem: Impossible to keep track of all possible pathlines, to evaluate the fluid properties at each point. ⇒ For unsteady flows, even worse, since different pathlines will pass through the same point.

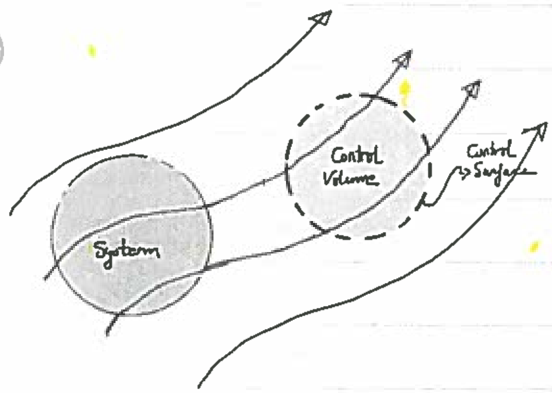
2) Develop a solution to the flow field that provides the flow properties at any point. If for example, the pressure was available as a function of location $p(x,y)$, then one could get p_B just by replacing (x_B, y_B) coordinates into the pressure solution. EULERIAN APPROACH

▷ In order to tackle the problem following an Eulerian approach, the equations of movement must be re-written accordingly. (Similar to solid mechanics, the fundamental equations are developed using the "free body" concept.)

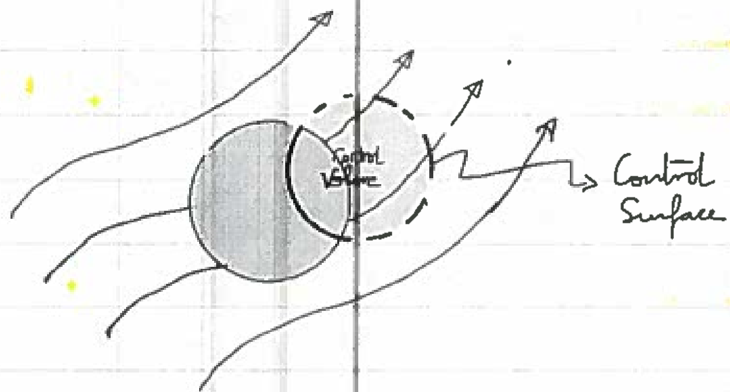
▷ Meaning that: imagine we could isolate a portion of the flow field, and the effects of the surroundings where introduced by forces acting on the limiting surfaces enclosing our portion of flow field.



The volume enclosing a point is called "control volume". The effects of the surroundings are replaced by pressure forces and shear stress acting on the surface of the control volume. Also, it must be considered that there is a flow through the control volume (CV). ⇒ Sometimes the Eulerian approach is also called as "control volume" approach.



$t_1 = t_0$



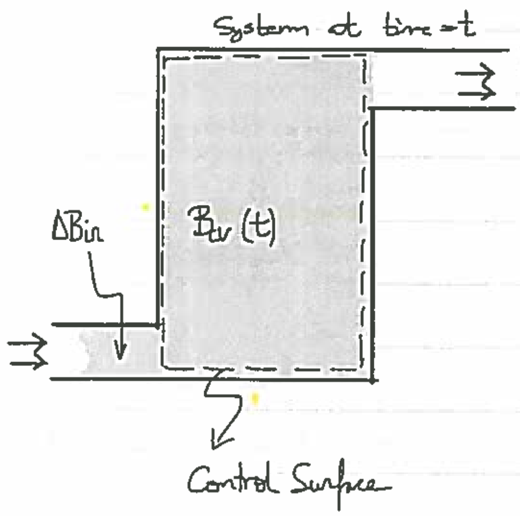
$t_2 = t_0 + \Delta t$

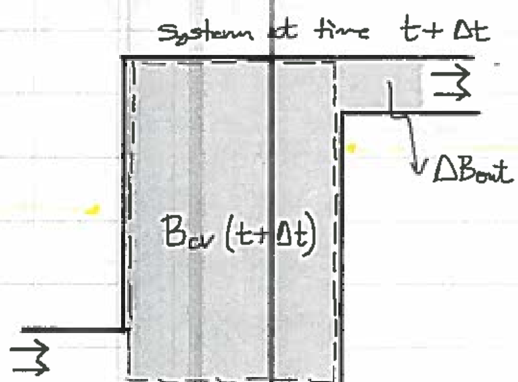
END.

○ Reynolds transport theorem

↳ it relates the Eulerian and Lagrangian approaches, from a mathematical standpoint. The Reynolds transport theorem is derived by considering the rate of change of a property (B) of a system as it passes through the control volume.

Let's consider the following progression of a system as it flows through a control volume:





The property "B" of the system can be any of the relevant variables: Mass, linear momentum, energy, angular momentum, ...

Unfortunately the basic equations (Newton's 2nd law) are applicable for a body of given mass, or a fluid particle, moving in the field ⟹ Lagrangian approach.

↓

Next, a procedure to transform the governing laws from a Lagrangian perspective to an Eulerian perspective, will be introduced. ⟹ Therefore, equations can be applied in a control volume. Later, by considering the volume enclosed in the CV tend to zero, the Eulerian forms of the differential equations will be derived.

* The Eulerian form of the equations is very important because provides the basic equations for numerically solving the flow field ⟹ CFD (computational Fluid Dynamics).

o System and Control Volume (Notes)

System ≡ is a continuous mass of fluid that always contains the same matter. The shape of the system may change with time, but the mass is constant since it always consists of the same matter.

Control Volume ≡ is a volume located in space and through which matter can pass. The system can pass through the CV. The shape and position of the CV is problem dependent (There is no unique CV).

The CV is enclosed by the control surface, through which mass enters and leaves the control volume. The CV can deform with time, move, rotate, and the mass in the CV can change.

In the previous drawing the control volume is enclosed by the control surface (dashed line). The system is identified with the blue-colored regions.

at time = t : the system consists of the material inside the CV and the material going in. Therefore the property (B) of the system can be written as:

$$B_{sys}(t) = B_{cv}(t) + \Delta B_{in}$$

at time = t+Δt, the system has moved and now consists of the material in the control volume and the material passing out, so

(Notes 2)

$$B_{sys}(t+\Delta t) = B_{cv}(t+\Delta t) + \Delta B_{out}$$

therefore, one can write the rate of change of the property B as:

$$\frac{dB_{sys}}{dt} = \lim_{\Delta t \to 0}\left[\frac{B_{sys}(t+\Delta t) - B_{sys}(t)}{\Delta t}\right]$$

which by replacing the previous terms: (Notes 2)

$$\frac{dB_{sys}}{dt} = \lim_{\Delta t \to 0}\left[\frac{B_{cv}(t+\Delta t) + \Delta B_{out} - B_{cv}(t) - \Delta B_{in}}{\Delta t}\right] =$$

upon rearrangement:

$$\frac{dB_{sys}}{dt} = \underbrace{\lim_{\Delta t \to 0}\left[\frac{B_{cv}(t+\Delta t) - B_{cv}(t)}{\Delta t}\right]}_{1} + \underbrace{\lim_{\Delta t \to 0}\frac{\Delta B_{out}}{\Delta t}}_{2} - \underbrace{\lim_{\Delta t \to 0}\frac{\Delta B_{in}}{\Delta t}}_{3}$$

(Notes)

term ① ≡ Rate of change of the property 'B' inside the control volume

$$\lim_{\Delta t \to 0} \left[\frac{B_{cv}(t+\Delta t) - B_{cv}(t)}{\Delta t} \right] = \frac{\partial B_{cv}}{\partial t}$$

term ② ≡ outflow of the property 'B' through the control surface.

(Notes)

$$\lim_{\Delta t \to 0} \frac{\Delta B_{out}}{\Delta t} = \dot{B}_{out} \quad (\equiv \text{flux of the property 'B' through the CS}).$$

term ③ ≡ inflow of the property 'B' through control surface.

(Notes)

$$\lim_{\Delta t \to 0} \frac{\Delta B_{in}}{\Delta t} = \dot{B}_{in} \quad (\equiv \text{influx of property 'B' through CS}).$$

therefore, the rate of change of property 'B' can be rewritten as follows:

$$\frac{dB_{sys}}{dt} = \frac{\partial B_{cv}}{\partial t} + \dot{B}_{out} - \dot{B}_{in} = \frac{\partial B_{cv}}{\partial t} + \dot{B}_{net}$$

At this point we just need a specific expression for 'B_{cv}' and '$\dot{B}_{net}$'.
The amount of property B contained in a control volume at a given instant is

$$B_{cv} = \int_{cv} b \, dm = \int_{cv} b \rho \, d\forall$$

where $d\forall$ is the CV differential. One could imagine "dm" as the differential of "material" or "fluid" contained inside the CV. Therefor "b" represents the amount of property "B" per unit mass (material or fluid contained in the CV).

(Note)

if B represents mass : $B = M$; $\longrightarrow b = 1$
B represents linear momentum: $B = \vec{P}$; $\longrightarrow b = \vec{V}$
B represents angular momentum: $B = \vec{H}$; $\longrightarrow b = \vec{r} \times \vec{V}$
B represents energy : $B = E$; $\longrightarrow b = e$
B represents entropy: $B = S$; $\longrightarrow b = s$

therefore one can rewrite that: $\frac{\partial B_{cv}}{\partial t} = \frac{\partial}{\partial t} \int_{cv} b \rho \, d\forall$

On the other hand the net flow rate of the property B through the control volume is given by:

$$\dot{B}_{net} = \int_{CS} b \rho \vec{V} \cdot d\vec{S}$$

Notice that here the integral is not a volume of integral of the control volume, but a surface integral of the overall control surface.

Further, here $\vec{V}$ represents the velocity vector of the fluid crossing the control surface. Thus the velocity is with respect to the velocity of the control surface. Also, $d\vec{S}$ represents the differential vector of the CS. Recall that this one points outwards of the surface

Finally the rate of change of property 'B' can be written as:

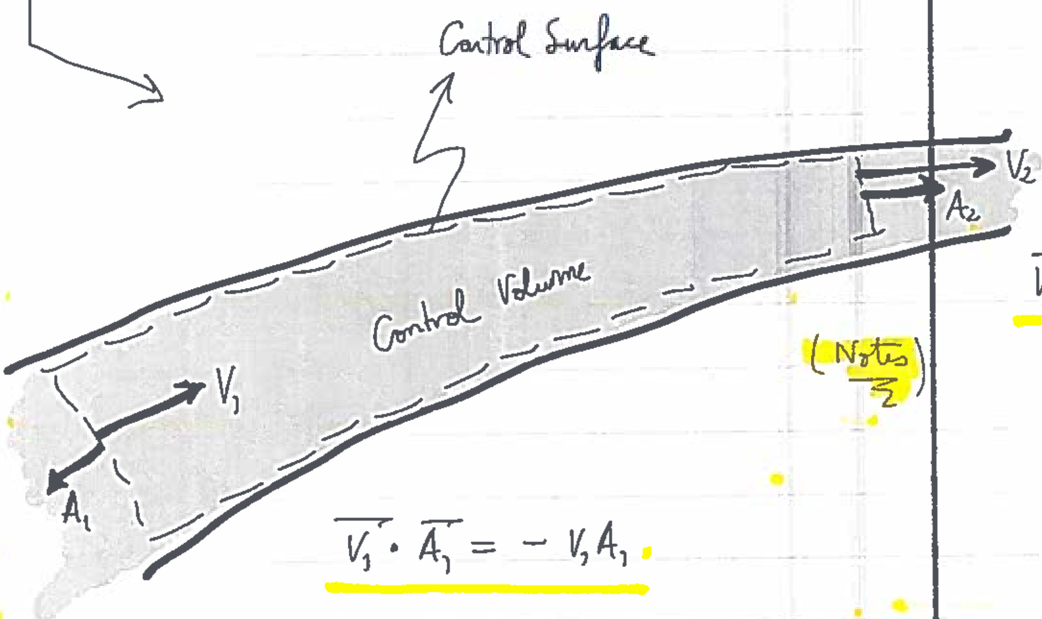
$$\boxed{\underbrace{\frac{dB_{sys}}{dt}}_{\text{Lagrangian}} = \underbrace{\frac{\partial}{\partial t}\int_{cv} b \rho \, d\forall + \int_{cs} b \rho \vec{V} \cdot d\vec{A}}_{\text{Eulerian.}}}$$

Reynolds Transport theorem

expressed in a different way: (Notes)

$$\left\{\begin{matrix}\text{Rate of change}\\ \text{of property B}\\ \text{of system}\end{matrix}\right\} = \left\{\begin{matrix}\text{Rate of change}\\ \text{of property B}\\ \text{in control volume}\end{matrix}\right\} + \left\{\begin{matrix}\text{Net outflow}\\ \text{of property B}\\ \text{through control}\\ \text{surface}\end{matrix}\right\}$$

Important: Special care should be taken when evaluating the dot product in the Reynolds transport theorem. $\vec{A}$ is always directed outwards, the dot product will be positive when $\vec{v}$ is outwards, and negative when $\vec{v}$ is inwards.



$$\overline{V_2} \cdot \overline{A_2} = V_2 A_2$$

(Notes)

$$\overline{V_1} \cdot \overline{A_1} = - V_1 A_1$$

Particular case: Conservation of Mass & Continuity equation

The continuity equation derives from the conservation of mass, which in Lagrangian form, simply states that the mass of the system is constant

$$M_{sys} = cte.$$ (Here the property $B = M$, thus $b = 1$).

The Eulerian form is derived applying the Reynolds transport theorem.

$$\left.\frac{dM}{dt}\right)_{system} = \underbrace{\frac{\partial}{\partial t}\int_{CV} \rho \, d\forall}_{①} + \underbrace{\int_{CS} \rho \vec{V} \cdot d\vec{A}}_{②} = 0$$

This is the mass conservation equation, which states that the total mass of the system doesn't change with time.

Term ① indicates the rate of change of mass within the CV.

Term ② represents the net rate of mass flux out through the control Surface.

This equation is also called <u>the continuity equation</u>, and can be rewritten as:

$$\frac{\partial}{\partial t}\int_{CV} \rho \, d\forall = -\int_{CS} \rho \vec{V} \cdot d\vec{A}$$

* In the special case where the flow is incompressible, $\rho = cte$, and it can be pulled out from the integrals, and thus simplified. Therefore:

$$\frac{\partial \forall}{\partial t} + \int_{CS} \vec{V} \cdot d\vec{A} = 0$$

* For a non-deformable control volume of fixed size and shape, $\forall = cte$. Therefore, conservation of mass becomes:

$$\int_{CS} \vec{V} \cdot d\vec{A} = 0$$

* if the velocity field can be approximated as <u>uniform</u> at the inlet and outlet of the CV, the previous equation can be further simplified to:

$$\boxed{\sum_{CS} \vec{V} \cdot \vec{A} = 0}$$

END.

Note: So far we haven't imposed the condition of the flow to be steady, we have only applied the restriction of incompressibility. Therefore:

$$\sum_{cs} \vec{v} \cdot \vec{A} = 0 \qquad \text{and}$$

$$\int_{cs} \vec{v} \cdot d\vec{A} = 0$$

state conservation of mass for an incompressible flow that can be steady or unsteady.

* The integral term: $\int_{cs} \vec{v} \cdot d\vec{A}$ is commonly called the "volume flow rate" or "volume rate of flow". This is directly related to the dimensions of the integrand: $[L^3/t]$

Thus one can define the volume flow rate 'Q', through a section of the control volume as:

$$Q = \int_{cs} \vec{v} \cdot d\vec{A}$$; see that 'Q' is a scalar quantity.

* for a steady case, the mass flow rate into a control volume must be equal to the mass flow rate out of the control volume.