Homework 1

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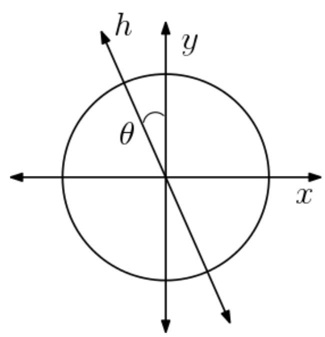
CS6966: Theory of Machine Learning

February 13, 2017

1. Consider the problem of classifying points in the two-dimensional plane, *i.e.*, $\chi = \mathbb{R}^2$. Suppose that the (unknown) true label of a point (x, y) is given by $\text{sign}(x)$ (we define $\text{sign}(0) = \pm 1$ for convenience). Suppose the input distribution $\mathcal{D}$ is the uniform distribution over the unit circle centered at the origin.

   (a) Consider the hypothesis h as shown in the figure below (h classifies all the points on the right of the line as $+1$ and all the points to the left as -1). Compyte the risk $L_{\mathcal{D}}(h)$, as a function of θ (which is, as is standard, given in radians).



Solution:

$$Area_{circle} = \pi r^2$$
$$Area_{slice} = \cancel{\pi} r^2 \frac{\theta}{2\cancel{\pi}} = \frac{\theta}{2} r^2$$

For a unit circle, the failure percentage for something *completey* random on the unit circle ($r = 1$) would be

$$\mathcal{L}_{\mathcal{D}}(h) = \frac{\theta r^2}{\pi r^2} = \frac{\theta}{\pi}$$

   (b) Suppose we obtain $1/\theta$ (whcih is given to be an integer ≥ 2) training samples (*i.e.*, samples from $\mathcal{D}$, along with their true labels). What is the probability that we find a point whose label is "inconsistent" with h? Can you bound this probability by a constant independent of θ?

Solution:

(c) Give an example of a distribution $\mathcal{D}$ under which h has risk zero.

Solution:

$$\text{Distribution} = \begin{cases} y = 1 & x \geq 0 \\ y = -1 & x < 0 \end{cases}$$

2. Suppose $A_1, A_2, \ldots, A_n$ are events in a probability space.

(a) Suppose $\Pr[A_i] = \frac{1}{2n}$ for all i. Then, show that the probability that none of the A_i's occur is at least 1/2.

Solution:

$$P(A_i) = \frac{1}{2n}$$

If independent, the probability of choosing all of them is

$$P(A_{all}) = \left(\frac{1}{2n}\right)\left(\frac{1}{2n}\right)\cdots\left(\frac{1}{2n}\right)$$
$$P(A_{all}) = \prod_{i=1}^{n}\left(\frac{1}{2n}\right) = \left(\frac{1}{2n}\right)^n$$

Therefore, the probability of not choosing any

$$P(A_{none}) = 1 - \left(\frac{1}{2n}\right)^n$$

We can place a lower bound for $n = 1$

$$P(A_1) = 1 - \frac{1}{2} = \frac{1}{2}$$

Therefore, the probability of not being chosen for a given n

$$P(A_i) = 1 - \left(\frac{1}{2n}\right)^n \geq \frac{1}{2}$$

(b) Give a concrete example of events A_i for which $\Pr[A_i] = \frac{1}{n-1}$ for all i, and the probability that none of them occur is zero.

Solution:

Probability of not choosing an integer at random on the infinite domain.

(c) Suppose $n \geq 3$, and $\Pr[A_i] = \frac{1}{n-1}$, but the events are all *independent*. Show that the probability that none of them occur is $\geq 1/8$.

Solution:

Probability of not choosing one

$$P(A_i) = \left(1 - \frac{1}{n-1}\right)$$

probability of not choosing n independent

$$P(A_1, A_2, \ldots, A_n) = \left(1 - \frac{1}{n-1}\right)^n = \left(-\frac{2-n}{n-1}\right)^n$$

We can bound it by using $n = 3$, which gives

$$P(A_1, A_2, A_3) = \frac{1}{8}$$

3. In our proof of the no-free lunch theorem, we assumed the algorithm A to be deterministic. Let us now see how to allow randomized algorithms. Let A be a randomized map from set X to set Y. Formally, this means that for every $x \in X$, $A(x)$ is a random variable, that takes values in Y. Suppose $|X| < c\,|Y|$, for some constant $c < 1$.

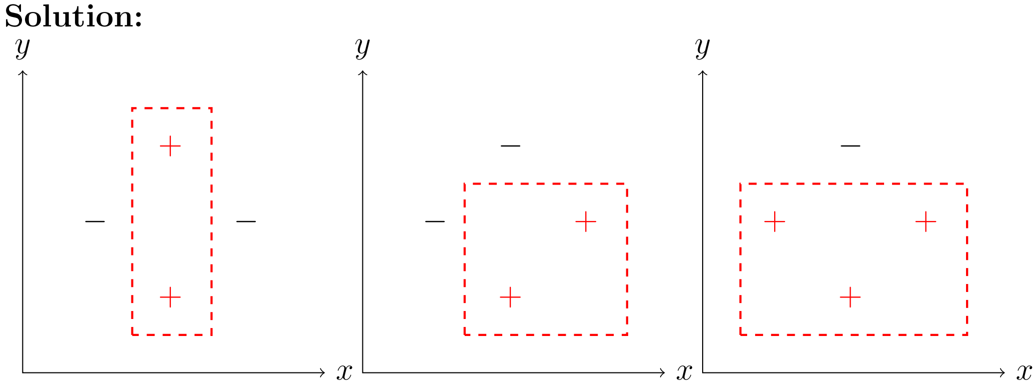
   (a) Show that there exists $y \in Y$ such that $\max_{x \in X} \Pr[A(x) = y] \leq c$.
   Solution:

   (b) Show that this implies that for any distribution $\mathcal{D}$ over X, $\Pr_{x \sim \mathcal{D}}(A(x) = y) \leq c$.
   Solution:

4. Recall that the VC dimension of a hypothesis class $\mathcal{H}$ is the size fo the largest set that it can "shatter."

   (a) Consider the task of classifying points on a 2D plane, and let $\mathcal{H}$ be the class of axis parallel rectangles (points inside the rectangle are "+" and the poitns outside are "−"). Prove that the VC dimension of $\mathcal{H}$ is 4.
   In order to prove VC dimension, we need to prove two things

      i. There exists ($d = 4$) points which can be shattered
      Solution:

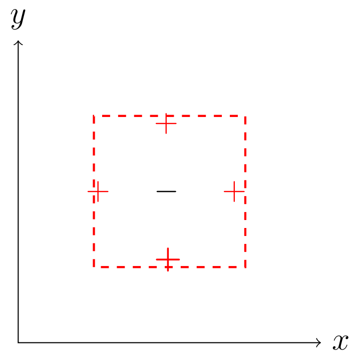


With the last case of all 4 being postive being trivial. These are also trivially applicable when flipping between $+$'s and $-$'s.

ii. No set of 5 points can be shattered

Solution:

The minimum enclosing rectangle is defined where 1 point is each edge



Therefore, the 5^{th} point must lie on the edge or inside of the rectangle.

(b) This time, let $\chi = \mathbb{R}^d \backslash \{0\}$ (origin excluded), and let $\mathcal{H}$ be the set of all hyperplanes through the origin (points on one side are "+" and the other side are "−"). Prove that the VC dimension of $\mathcal{H}$ is $\leq d$.

Hint: Consider *any* set of $d+1$ points. They need to be linearly dependent. Now, could it happen that u, v are "+", but $\alpha u + \beta v$ is "−" for α, $\beta \geq 0$? Can you generalize this?

Solution:

Consider d unit base vectors in $\mathbb{R}^d$

$$(1, 0, \ldots, 0)(0, 1, 0, \ldots, 0), \ldots, (0, \ldots, 0, 1)$$

It is trivially seen that this set can be defined by d hyper planes through the origin. We can now show that there are no $d+1$ vectors in $\mathbb{R}^d$ that can be shattered by hyperplanes through the origin. We can do this by *proof by contradiction.*

Suppose that $u_1, \ldots, u_{d+1}$ can be shattered. This implies that there exists 2^{d+1} vectors $a_i \in \mathbb{R}^d, i = \{1, \ldots, 2^{d+1}\}$ such that the matrix of inner products, denoted by $z_{i,j} = u_i^T a_j$ has columns with all possible combination of signs. Therefore, we have the matrix inner products of

$$A = \begin{pmatrix} z_{1,1} & \cdots & z_{1,2^{d+1}} \\ \vdots & \ddots & \vdots \\ z_{(d+1),1} & \cdots & z_{(d+1),2^{d+1}} \end{pmatrix}$$

which has all 2^{d+1} possible combinations of signs.

$$\text{sign}(A) = \begin{pmatrix} - & - & \cdots & - & + \\ - & \cdot & \cdots & \cdot & + \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ - & + & \cdots & \cdot & + \end{pmatrix}$$

Then, the rows of A are linearly independent as there are no constants c such that $\sum_{i=1}^{d+1} c_i z_{i,\forall} = 0$ as for any value of c_i there is a column with the same sign, which makes it always non-zero. This implies that $d+1$ vectors in $\mathbb{R}^d$ are linearly independent but it is a false statement. This contradiction proves there are no $d+1$ vectors in $\mathbb{R}^d$ that can be shattered by hyperplaces through the origin. Thus, the VC dimension is d

(c) **(BONUS)** Let χ be the points on the real line, and let $\mathcal{H}$ be the class of hypotheses of the form $\text{sign}(p(x))$, where $p(x)$ is a polynomial of degree at most d (for convenience, define $\text{sign}(0) = +1$). Prove that the VC dimension of this class is $d + 1$.

Hint: The tricky part is the uppoer bound. Here, suppose $d = 2$, and suppose we consider any four points $x_1 < x_2 < x_3 < x_4$. Can the sign pattern $+$, $-$, $+$, $-$ arise from a degree 2 polynomial?

5. In the examples above (and in general), a good rule of thumb for VC dimension of a function class is the *number of parameters* involved in defining a function in that class. However, this is not universally true, as illustrated in this problem: Let χ be the points on the real line, and define $\mathcal{H}$ to be the class of functions of the form $h_\theta \coloneqq \text{sign}(\sin(\theta x))$, for $\theta \in \mathbb{R}$. Note that each hypothesis is defined by the single parameter θ.

   Prove that the VC dimension of $\mathcal{H}$ is infinity.

   Solution:

   So where does the "complexity" of the function class come from? **(BONUS)** Prove that if we restrict θ to be a rational number whose numerator and denominator have at most n bits, then the VC dimension is $\mathcal{O}(n)$.