
(Q)M-types and Coinduction in HoTT / CTT

Lasse Letager Hansen, 201912345

Master's Thesis, Computer Science

June 6, 2020

Advisor: Bas Spitters

Abstract

We present a construction of \mathbf{M} -types from containers in a cubical type theory (CTT). We show how the containers construct a coalgebra, for which we can define a coinduction principle, making strong bisimulation imply equality. We then show constructions of \mathbf{M} -types, and how they can be quotiented to construct what we call \mathbf{QM} -types. The problem with \mathbf{QM} -types is that in general assuming that we can lift function types of equivalence classes is equivalent to the axiom of choice [6], but this can be solved by defining the quotienting relation and the type at the same time as a quotient inductive-inductive type (QIIT), which assuming the axiom of (countable) choice, is equal to the \mathbf{QM} -type. We construct multisets and the partiality monad as some examples of quotiented \mathbf{M} -types and show the equivalence between the two constructions. We then discuss the pros and cons of the two constructions, and relate them to an alternative construction, which defines quotient polynomial functors (QPF), and define the quotiented \mathbf{M} -type as the final coalgebra for the QPF. However this construction requires the axiom of choice, so it is still not a suitable construction. We conclude with some examples of how to use \mathbf{M} -types and some properties. All work is formalized in Cubical Agda, and the work on defining \mathbf{M} -types has been accepted to the Cubical Agda github repository.

complete

Resumé

I denne afhandling vil vi give en konstruktion af \mathbf{M} -typer ved brug af "containers". Formaliseringen af dette bliver udført i en "Cubical Type Theory". Given en "container" kan vi konstruere en polynomisk funktor, og ved gentaget anvendelse af denne på enhedstypen, får vi en sekvens af typer, som vi kan tage grænsen af. Denne grænse giver os en endelig coalgebra, hvilket vi definere \mathbf{M} -type som. Vi giver nogle eksempler på \mathbf{M} -typer. Vi beskriver tre mulige definitioner for kvotient \mathbf{M} -typer, kaldet en \mathbf{QM} -type, den første er at bruge set trunkeret kvotienter givet en relation, problemet ved denne definition er at vi ikke kan løfte konstruktørerne for \mathbf{M} -typerne til \mathbf{QM} -typen. Vores anden definition er kvotient inductive-inductive typer (QIITs), hvor relationen og typen defineres samtidig. Vi giver nogle eksempler på denne konstruktion, og viser eksempler på ækvivalence mellem de konstruktioner, dog kræver disse antagelsen af udvalgsaksiomet. Den tredje konstruktion er at lave en kvotient polynomisk funktor, og tage grænsen af denne, for at få en \mathbf{QM} -type, men igen kræver det udvalgsaksiomet, siden vi skal løfte operationerne fra kvotienten.

complete

Acknowledgments

I would like to thank my supervisor Bas Spitters for some enlightening discussions and quick responses and for taking the time to do weekly meetings regarding this masters thesis. I would also like to thank my fellow students, who made this challenging but fun time less lonely despite the Corona pandemic.

*Lasse Letager Hansen,
Aarhus, June 6, 2020.*

Contents

Abstract	iii
Resumé	v
Acknowledgments	vii
1 Introduction	1
1.1 Overview	1
1.2 Expectations of reader	1
1.3 Background Theory	1
1.4 Notation	4
2 M-types	7
2.1 Containers / Signatures	7
2.1.1 Construction of M -type	9
2.2 Coinduction Principle for M -types	12
3 Examples of M-types	15
3.1 ITrees as M -types	15
3.1.1 Delay Monad	15
3.1.2 <i>R</i> -valued E-event Trees	16
3.1.3 ITrees	17
3.2 General rules for constructing M -types	18
3.3 Wacky M -type	19
4 QM-types	21
4.1 Quotienting and Constructors	21
4.2 QM-types and Quotient inductive-inductive types (QIITs)	22
4.2.1 Multiset	22
4.2.2 Partiality monad	24
4.3 How should QM-types be defined	30
4.3.1 Lifting Quotient Construction from Containers	30
4.3.2 Better behaving QM-types	32
5 Conclusion	33
5.1 Future Work	33

Bibliography	35
A The Technical Details	37

Chapter 1

Introduction

1.1 Overview

Inductive types in homotopy type theory (HoTT) / cubical type theory have been studied a lot, one body of work is W-types, which has been shown equal to inductive types. However there has not been much work done on the dual concept, namely coinductive types in HoTT defined as \mathbf{M} -types. The goal of this thesis is to get an understanding of \mathbf{M} -types and extend on the existing theory. We construct some examples of \mathbf{M} -types, as an attempt to make the theory of \mathbf{M} -types more accessible. A useful technique for defining a type with some properties, is quotienting a free objects, as such we will look at quotients of \mathbf{M} -types in the setting of cubical type theory and the problems with axiom of choice that arise from this construction.

In the rest of this chapter we will introduce some of the background theory and notation used in this thesis. In Chapter 2 we construct \mathbf{M} -types from containers, and define a coinduction principle for the \mathbf{M} -types. In Chapter 3 we give some example constructions of \mathbf{M} -types. In Chapter 4 we introduce quotiented \mathbf{M} -types (\mathbf{QM} -types), and show equalities between these and quotient inductive-inductive types (QIITs), we show the construction of multisets and the partiality monad as example, and discuss how quotients of \mathbf{M} -types should be constructed. Finally in Chapter 5 we discuss future research and conclude on the work done.

1.2 Expectations of reader

WRITE

1.3 Background Theory

We start by giving some background theory and summarize important concepts used in the rest of this thesis.

We will be using **type theory** [15] as the basis for mathematics. In type theory every term x is an element of some type A , written $x : A$. A paradigm in type theory is the idea propositions as types [14], where propositions are types, meaning proofs boils down to showing that there exists an element of some type representing a proposition. Specifically proofs of equality is a construction of

an element of the equality type. The type theory we are working in is build on **Martin L f Type Theory (MLTT)** / Intuitionistic type Theory (ITT) [13], which is designed on the principles of mathematical constructivism, where any existence proof must contain a witness. Meaning a proof of existence, can be converted into an algorithm that finds the element making the statement true. MLTT is built from the three finite types **0**, **1** and **2**, and type constructors Σ , Π and $=$. The types Σ and Π can be read as exists and for all. There is only a single way to make terms of $=$ -type, and that is **refl** : $\prod_{a:A} (a = a)$. MLTT also has universes $\mathcal{U}_0, \mathcal{U}_1, \dots$, however we will leave the index implicit and write \mathcal{U} . We will say that a type is given inductively (coinductively) from a set of constructors and/or destructors, meaning we either take the smallest (largest) type for which these rules are true. A constructor for a type A is a function, that takes some arguments, and returns an element $a : A$, dually a destructor of A will take an element $a : A$ and return something. Inductive types are defined by giving rules for some base cases and then some inductive cases, an example is the natural numbers, defined as

$$\frac{}{0 : \mathbb{N}} \quad (1.1) \qquad \frac{n : \mathbb{N}}{\text{succ } n : \mathbb{N}} \quad (1.2)$$

with base case 0, and inductive case **succ** n , which says that $n + 1$ is a natural number if n is. If we take the type defined coinductively over the same set of constructors, we get an element where **succ** is applied infinitely, namely $\infty = \text{succ } \infty$, which is not an element of \mathbb{N} . This type is also know as **coN**. A way to look at inductive definitions is that the rules describe the structure of the type, while for a coinduction definition, the rules describe the observable behavior of the type.

We define an equivalence relation $\sim_{\mathbb{N}}$ on the natural numbers inductively, since the natural numbers are defined inductively. That is equivalence on the natural numbers follows the structure defined by the constructors

$$\frac{}{0 = 0} \sim_0 \quad (1.3) \qquad \frac{n \sim_{\mathbb{N}} m}{\text{succ } n \sim_{\mathbb{N}} \text{succ } m} \sim_{\text{succ}} \quad (1.4)$$

so two natural numbers are equal, if they both are zero, or if they are equal when we subtract one from each of them. Equivalence relations defined this way for inductive types implies equality, meaning if two elements are related then they are equal. Lets try and do a similar construction for the coinductive type of streams. A stream is an infinite sequence of elements. Streams can be defined from the two destructors **hd** and **tl**, where **hd** represents the first element, and **tl** represents the rest of the sequence.

$$\frac{s : \text{stream } A}{\text{hd } s : A} \quad (1.5) \qquad \frac{s : \text{stream } A}{\text{tl } s : \text{stream } A} \quad (1.6)$$

We can again define an equivalence relation \sim_{stream} , but this time coinductively, focusing on the observable behavior instead

$$\frac{\text{hd } s = \text{hd } t \quad \text{tl } s \sim_{\text{stream}} \text{tl } t}{s \sim_{\text{stream}} t} \quad (1.7)$$

This equivalence relation does not give an equality in MLTT, we just get bisimilarity meaning elements "behave" the same, but they are not equivalent. We solve this problem by working in a type theory, where the univalence axiom holds, that is we use a **Univalent Foundations (UF)** for mathematics in this work. The **univalence axiom** says that equality is equivalent to equivalence

$$(A = B) \simeq (A \simeq B) \quad (1.8)$$

meaning if two types are equivalent, then there is an equality between them. This makes (strong) bisimilarity imply equality, solving the problem for \sim_s `tream`. The univalent foundations we will be using is **Homotopy type theory (HoTT)** [16]. HoTT is an intensional dependent type theory (built on MLTT) with the univalence axiom and higher inductive types. In HoTT the identity types form path spaces, so proofs of identity are not just `refl` as is the case in MLTT. Types are seen as "spaces", and we think of $a : A$ as a being a point in the space A , similarly functions are regarded as continuous maps from one space to another [12]. As such we can construct higher inductive types, with point constructors as well as equality constructors. One of the problems with "plain" HoTT is that the univalence axiom is just postulated, meaning it is not constructive. To remedy this we will be working in a **cubical type theory (CTT)** [8], where the univalence axiom is not an axiom, but a statement that can be proven, meaning we constructivity of univalence axiom and application thereof [11]. The reason for the name cubical type theory, is because composition is defined by square, that is given three sides of a square we get the last one, see Figure 1.1.

$$\begin{array}{ccc} A & \xrightarrow{p \cdot q \cdot r} & B \\ p^{-1} \uparrow & & \uparrow r \\ C & \xrightarrow{q} & D \end{array}$$

Figure 1.1: Composition square

algebra for functor ? W-types? should there be a decription of these?

If you are used to working in set theory, then working in HoTT will take some getting used to. Homotopy type theory is proof relevant, which means that there might be multiple proofs of one statement, and these proofs might not be interchangeable (equal). This can be explained by the use of H-level in HoTT, which describes how equality behaves. We start from H-level (-2) which are contractible types, meaning inhabited types, where all element are equal to. Then there is (-1)-types which are mere propositions or `hProp`, where all elements of the type are equal, but types might not be inhabited. If the type is inhabited, then we say the proposition is true. The 0-types are the `hSets`, where all equalities between two elements x, y are equal. For 1-types (1-groupoids) we get equalities of equalities are equal, and so on for homotopy n -types. Any n -type is also a $n+1$ -type, but with trivial equalities at the $n+1$ level. If we don't want to do proof relevant mathematics we can do propositional truncation (denoted $\|\cdot\|$), converting types to -1 -types, meaning we ignore the difference in proofs by just look at whether a type is inhabited or not. However doing this we loose some of the reasoning power of HoTT. One of the tools we get using the full power of HoTT is **Higher order inductive types (HITs)**, which defines a type by point constructors and equality constructors. An example is the propositional truncation we just described, defined as a constructor taking any element to it its truncated version, and then defining equality between all propositional truncated elements

$$\frac{x : A}{|x| : \|A\|} \quad (1.9)$$

$$\frac{x, y : \|A\|}{\text{squash } x \ y : x \equiv y} \quad (1.10)$$

Another useful example is set truncated quotients. Given some free type, we can quotient it by a relation, to get a type with equalities that respect the relation. A set truncated quotient of a type

A with a relation R is given by the HIT

$$\frac{x : A}{[x] : A/\mathcal{R}} \quad (1.11)$$

$$\frac{x, y : A/\mathcal{R} \quad r : x \mathcal{R} y}{\text{eq}/ x y r : x \equiv y} \quad (1.12)$$

$$\frac{}{\text{squash}/ : \text{isSet } (A/\mathcal{R})} \quad (1.13)$$

When working with these two HITs, you might need to use the axiom of choice (AC), which is defined as follows in HoTT

$$\prod_{(x:X)} \|Y x\| \rightarrow \left\| \prod_{(x:X)} Y x \right\| \quad (1.14)$$

where Y is a type family over X [16, Section 3.8]. Using the axiom of choice is problematic in a constructive type theory, since they do not have a constructive interpretation. To maintain the computational aspects of HoTT and CTT, we try to avoid AC [16, Introduction].

We have formalized most of Chapter 2*, Chapter 3 and some of Chapter 4 in the proof assistant / programming language Cubical Agda. A **proof assistant** helps with verifying proofs, while making the process of making proofs interactive. **Cubical Agda** [17] is an implementation of a cubical type theory (inspired by CCHM [9]) made by extending the proof assistant Agda. One of the main additions is the interval and path types. The **interval** \mathbb{I} can be thought of as elements in the interval $[0, 1]$. When working with the interval, we can only access the left and right endpoint **i0** and **i1** or some unspecified point in the middle i , keeping with the intuition of a continuous interval. Cubical agda also generalizes transporting, given a type line $A : \mathbb{I} \rightarrow \mathcal{U}$, and the endpoint **A i0** you get a line from **A i0** to **A i1**. Using these tools we can make the univalence axiom compute and it eases the formalization, since proofs become more computational.

1.4 Notation

The following is the notation / fonts used to denote specific definitions / concepts We do a lot of casing on the natural numbers, so we will also introduce some notation, to make this more readable

Definition 1.4.1. Useful notation for casing on natural numbers.

$$\Downarrow x, \text{f} \Downarrow = \lambda n, \begin{cases} x & n = 0 \\ \text{f } m & n = m + 1 \end{cases} \quad (1.15)$$

We will also define some equalities $A \equiv B$ using an isomorphism, that is we define functions **fun** : $A \rightarrow B$ and **inv** : $B \rightarrow A$, and show there are right and left identities **rinv** : **fun** \circ **inv** \equiv **id**. We will introduce further notation as we go.

*Accepted to the cubical agda github, pull request: <https://github.com/agda/cubical/pull/245>

\mathcal{U}_i or \mathcal{U}	Universe
$B : A \rightarrow \mathcal{U}$	Type
$B : A \rightarrow \mathcal{U}$	Type former
$T : \Pi_{(a:A)} B \ a$	Dependent product type
$T : \sum_{(a:A)} B \ a$	Dependent sum type
$x : A$	Term of a type
$c : A$	Constant
$f : A \rightarrow C$	Function
\mathbf{f}	Constructor
\mathbf{f}	Destructor
$p : A \equiv C$	Homogeneous path
$q : A \equiv_p C$	Heterogeneous path, denoted \equiv_* if p is clear from context
$R : A \rightarrow A \rightarrow \mathcal{U}$	Relation, elements denoted $x \ R \ y$
$\mathbf{1}$	Unit type
$\mathbf{0}$	Empty / Zero / Bottom type
\mathbf{P}	Functor
π_1, π_2	First and second projection for dependent type $\sum_{(a:A)} B \ a$

Table 1.1: Notation

Chapter 2

M-types

In this chapter we will introduce containers (aka. signatures), and use them to construct **M**-types and operations **in** and **out** on the **M**-types (Theorem 2.1.9) and showing that **M-out** is a final coalgebra (Theorem 2.1.11). We conclude the chapter by proving a coinduction principle for **M**-types (Theorem 2.2.2) [3].

2.1 Containers / Signatures

We start by introducing the notion of containers and polynomial functors for these containers.

Definition 2.1.1. A Container (or signature) is a dependent pair $S = (A, B)$ for the types $A : \mathcal{U}$ representing shape and $B : A \rightarrow \mathcal{U}$ defining branching based on the shape.

Definition 2.1.2. A polynomial functor P_S for a container $S = (A, B)$ is defined, for types as

$$\begin{aligned} P_S &: \mathcal{U} \rightarrow \mathcal{U} \\ P_S X &= \sum_{a:A} B(a) \rightarrow X \end{aligned} \tag{2.1}$$

and for a function $f : X \rightarrow Y$ as

$$\begin{aligned} P_S f &: P_S X \rightarrow P_S Y \\ P_S f (a, g) &= (a, f \circ g). \end{aligned} \tag{2.2}$$

If the container is clear from context we will just write P .

Example 1. The polynomial functor for streams over the type A is defined by the container $S = (A, \lambda _, \mathbf{1})$, we get

$$P_S X = \sum_{a:A} \mathbf{1} \rightarrow X. \tag{2.3}$$

We can simplify this expression using $\mathbf{1} \rightarrow X \equiv X^{\mathbf{1}} \equiv X$. Furthermore $\mathbf{1}$ and X does not depend on A , so (2.3) is equivalent to

$$P_S X = A \times X. \tag{2.4}$$

We define the P_S -coalgebra for a polynomial functor P_S , and the morphisms between coalgebras.

Definition 2.1.3. A P_S -coalgebra is defined as

$$\text{Coalg}_S = \sum_{C:\mathcal{U}} C \rightarrow P_S C. \quad (2.5)$$

where we denote a P_S -coalgebra given by C and γ as $C-\gamma$. Coalgebra morphisms are defined as

$$\begin{aligned} \cdot \Rightarrow \cdot : \text{Coalg}_S &\rightarrow \text{Coalg}_S \\ C-\gamma \Rightarrow D-\delta &= \sum_{f:C \rightarrow D} \delta \circ f = P f \circ \gamma \end{aligned} \quad (2.6)$$

Definition 2.1.4 (Final Coalgebra / M-type). Given a container S , we define M-types as the type, making the coalgebra given by M_S and $\text{out} : M_S \rightarrow P_S M_S$ fulfill the property

$$\text{Final}_S := \sum_{(X-\rho:\text{Coalg}_S)} \prod_{(C-\gamma:\text{Coalg}_S)} \text{isContr}(C-\gamma \Rightarrow X-\rho). \quad (2.7)$$

That is $\prod_{(C-\gamma:\text{Coalg}_S)} \text{isContr}(C-\gamma \Rightarrow M_S-\text{out})$. We denote the M-type as $M_{(A,B)}$ or M_S or just M when the container is clear from the context. When writing $\text{isContr } A$, we mean A is of H-level (-2) , that is $\sum_{x:A} \prod_{y:A} y \equiv x$ or equivalently $A \equiv \mathbf{1}$.

Continuing our example we now construct streams as an M-type.

Example 2. We define streams over the type A as the M-type over the container $(A, \lambda _, \mathbf{1})$. If we apply the polynomial functor to the M-type, then we get $P_{(A, \lambda _, \mathbf{1})} M = A \times M_{(A, \lambda _, \mathbf{1})}$, illustrated in Figure 2.1. We will show that out is an isomorphism with inverse $\text{in} : P_S M \rightarrow M$ later in this

$$\begin{array}{ccccc} A & \xleftarrow{\pi_1} & A \times M_{(A, \lambda _, \mathbf{1})} & \xrightarrow{\pi_2} & M_{(A, \lambda _, \mathbf{1})} \\ & \searrow \text{hd} & \uparrow \text{out} & \nearrow \text{tl} & \\ & & M_{(A, \lambda _, \mathbf{1})} & & \end{array}$$

Figure 2.1: M-types of streams

section. We now have a semantic for the rules, we would expect for streams, if we let $\text{cons} = \text{in}$ and $\text{stream } A = M_{(A, \lambda _, \mathbf{1})}$,

$$\frac{A:\mathcal{U} \quad s:\text{stream } A}{\text{hd } s:A} E_{\text{hd}} \quad (2.8)$$

$$\frac{A:\mathcal{U} \quad s:\text{stream } A}{\text{tl } s:\text{stream } A} E_{\text{tl}} \quad (2.9)$$

$$\frac{A:\mathcal{U} \quad x:A \quad xs:\text{stream } A}{\text{cons } x \ xs:\text{stream } A} I_{\text{cons}} \quad (2.10)$$

or more precisely $\text{hd} = \pi_1 \circ \text{out}$ and $\text{tl} = \pi_2 \circ \text{out}$.

$$X_0 \xleftarrow{\pi_{(0)}} X_1 \xleftarrow{\pi_{(1)}} \cdots \xleftarrow{\pi_{(n-1)}} X_n \xleftarrow{\pi_{(n)}} X_{n+1} \xleftarrow{\pi_{(n+1)}} \cdots$$

Figure 2.2: Chain of types / functions

2.1.1 Construction of \mathbf{M} -type

We will show that \mathbf{M} -types can be defined as the limit of a chain defined by repeatedly applying \mathbf{P} to the unit type $\mathbf{1}$.

Definition 2.1.5. We define a chain as a family of morphisms $\pi_{(n)} : X_{n+1} \rightarrow X_n$, over a family of types X_n . See Figure 2.2.

Definition 2.1.6. The limit of a chain is given as

$$\mathcal{L} = \sum_{(x:\prod_{(n:\mathbb{N})} X_n)} \prod_{(n:\mathbb{N})} (\pi_{(n)} x_{n+1} \equiv x_n) \quad (2.11)$$

Lemma 2.1.7. Given $\ell : \prod_{(n:\mathbb{N})} (X_n \rightarrow X_{n+1})$ and $y : \sum_{(x:\prod_{(n:\mathbb{N})} X_n)} x_{n+1} \equiv \ell_n x_n$ the chain collapses as the equality $\mathcal{L} \equiv X_0$.

Proof. We define this collapse by the isomorphism

$$\text{fun}_{\mathcal{L}\text{collapse}}(x, r) = x_0 \quad (2.12)$$

$$\text{inv}_{\mathcal{L}\text{collapse}} x_0 = (\lambda n, \ell^{(n)} x_0), (\lambda n, \text{refl}_{(\ell^{(n+1)} x_0)}) \quad (2.13)$$

$$\text{rinv}_{\mathcal{L}\text{collapse}} x_0 = \text{refl}_{x_0} \quad (2.14)$$

where $\ell^{(n)} = \ell_n \circ \ell_{n-1} \circ \cdots \circ \ell_1 \circ \ell_0$. To define $\text{linv}_{\mathcal{L}\text{collapse}}(x, r)$, we first define a fiber (X, z, ℓ) over \mathbb{N} given some $z : X_0$. Then any element of the type $\sum_{(x:\prod_{(n:\mathbb{N})} X_n)} x_{n+1} \equiv \ell_n x_n$ is equal to a section over the fiber we defined. This means y is equal to a section. Since the sections are defined over \mathbb{N} , which is an initial algebra for the functor $\mathbf{G}\mathbf{Y} = \mathbf{1} + \mathbf{Y}$, we get that sections are contractible, meaning $y \equiv \text{inv}_{\mathcal{L}\text{collapse}}(\text{fun}_{\mathcal{L}\text{collapse}} y)$, since both are equal to sections over \mathbb{N} . \square

Lemma 2.1.8. For all coalgebras $\mathbf{C}-\gamma$ define over the container \mathbf{S} , we get $\mathbf{C} \rightarrow \mathbf{M}_{\mathbf{S}} \equiv \text{Cone}_{\mathbf{C}-\gamma}$, where $\text{Cone} = \sum_{(\mathbf{f}:\prod_{(n:\mathbb{N})} \mathbf{C} \rightarrow X_n)} \prod_{(n:\mathbb{N})} \pi_{(n)} \circ \mathbf{f}_{n+1} \equiv \mathbf{f}_n$ illustrated in Figure 2.3.

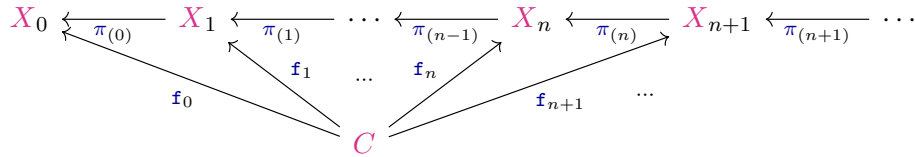


Figure 2.3: Cone

Proof. We define an isomorphism from $\mathbf{C} \rightarrow \mathbf{M}_{\mathbf{S}}$ to $\text{Cone}_{\mathbf{C}-\gamma}$

$$\text{fun}_{\text{collapse}} \mathbf{f} = (\lambda n z, \pi_1 (\mathbf{f} z) n), (\lambda n i a, \pi_2 (\mathbf{f} a) n i) \quad (2.15)$$

$$\text{inv}_{\text{collapse}} (\mathbf{u}, \mathbf{q}) z = (\lambda n, \mathbf{u} n z), (\lambda n i, \mathbf{q} n i z) \quad (2.16)$$

$$\mathbf{rinv}_{collapse}(\mathbf{u}, \mathbf{q}) = \mathbf{refl}_{(\mathbf{u}, \mathbf{q})} \quad (2.17)$$

$$\mathbf{linv}_{collapse} \mathbf{f} = \mathbf{refl}_{\mathbf{f}} \quad (2.18)$$

□

Theorem 2.1.9. *Given the container (A, B) , we define a chain as the repeated application of P to the unit element $X_n = P^n \mathbf{1}$, and $\pi_{(n)} = P^n !$ where $! : A \rightarrow \mathbf{1}$ is the unique function into the unit type. Then there is an equality*

$$shift : \mathcal{L} \equiv P\mathcal{L} \quad (2.19)$$

Proof. The proof is done using the two helper lemmas

$$\alpha : \mathcal{L}^P \equiv P\mathcal{L} \quad (2.20)$$

$$\mathcal{L}unique : \mathcal{L} \equiv \mathcal{L}^P \quad (2.21)$$

where \mathcal{L}^P is the limit of the shifted chain defined as $X'_n = X_{n+1}$ and $\pi'_{(n)} = \pi_{(n+1)}$. We can then define

$$shift = \alpha \cdot \mathcal{L}unique. \quad (2.22)$$

We start by showing $\mathcal{L}unique$, by the isomorphism

$$\mathbf{fun}_{\mathcal{L}unique}(\mathbf{a}, \mathbf{b}) = \langle \star, \mathbf{a} \rangle, \langle \mathbf{refl}_{\star}, \mathbf{b} \rangle \quad (2.23)$$

$$\mathbf{inv}_{\mathcal{L}unique}(\mathbf{a}, \mathbf{b}) = \mathbf{a} \circ \mathbf{succ}, \mathbf{b} \circ \mathbf{succ} \quad (2.24)$$

$$\mathbf{rinv}_{\mathcal{L}unique}(\mathbf{a}, \mathbf{b}) = \mathbf{refl}_{(\mathbf{a}, \mathbf{b})} \quad (2.25)$$

$$\mathbf{linv}_{\mathcal{L}unique}(\mathbf{a}, \mathbf{b}) = \mathbf{refl}_{(\mathbf{a}, \mathbf{b})} \quad (2.26)$$

By $\mathcal{L}unique$ we get that the limit of the chain is unique, so it just remains to show that, the limit of shifted chain is equal to $P\mathcal{L}$, which is α given by the equalities

$$\mathcal{L}^P \equiv \sum_{(x : \prod_{(n:\mathbb{N})} X_{n+1})} \prod_{(n:\mathbb{N})} \pi_{(n+1)} x_{n+1} \equiv x_n \quad (2.27)$$

$$\equiv \sum_{(x : \prod_{(n:\mathbb{N})} \sum_{(a:A)} B a \rightarrow X_n)} \prod_{(n:\mathbb{N})} \pi_{(n+1)} x_{n+1} \equiv x_n \quad (2.28)$$

$$\equiv \sum_{((a,p) : \sum_{(a:\prod_{(n:\mathbb{N})} A)} \prod_{(n:\mathbb{N})} a_{n+1} \equiv a_n)} \sum_{(u : \prod_{(n:\mathbb{N})} B a_n \rightarrow X_n)} \prod_{(n:\mathbb{N})} \pi_{(n)} \circ u_{n+1} \equiv_* u_n \quad (2.29)$$

$$\equiv \sum_{(a:A)} \sum_{(u : \prod_{(n:\mathbb{N})} B a \rightarrow X_n)} \prod_{(n:\mathbb{N})} \pi_{(n)} \circ u_{n+1} \equiv u_n \quad (2.30)$$

$$\equiv \sum_{(a:A)} B a \rightarrow \mathcal{L} \quad (2.31)$$

$$\equiv P\mathcal{L} \quad (2.32)$$

The equality from (2.28) to (2.29) is rearranging terms, and adding a trivial point to the start of for the shifted chain. The equality from (2.29) and (2.30) is done with Lemma 2.1.7 to collapse $\sum_{(a:\prod_{(n:\mathbb{N})} A)} \prod_{(n:\mathbb{N})} a_{n+1} \equiv a_n$ to A . For the equality between (2.30) and (2.31), we use Lemma 2.1.8. The rest of the equalities are trivial, completing the chain of equalities for the proof. □

Definition 2.1.10. For the equality *shift* we denote the functions back and forth as

$$\text{out} = \text{transport } \text{shift} \quad (2.33)$$

$$\text{in} = \text{transport } (\text{shift}^{-1}). \quad (2.34)$$

Theorem 2.1.11. The \mathbf{M} -type \mathbf{M}_S is defined as the limit for a polynomial functor \mathbf{P}_S . This definition fulfills the requirement for the \mathbf{M} -type given in Definition 2.1.4 namely $\mathbf{Final}_S \mathcal{L}$.

Proof. Unfolding the definition we need to show that $\prod_{(C-\gamma : \mathbf{Coalg}_S)} \text{isContr } (C-\gamma \Rightarrow \mathcal{L}\text{-out})$, so we assume we are given some $C-\gamma : \mathbf{Coalg}_S$ to show contractability by $(C-\gamma \Rightarrow \mathcal{L}\text{-out}) \equiv \mathbf{1}$. We get the following equalities,

$$C-\gamma \Rightarrow \mathcal{L}\text{-out} \quad (2.35)$$

$$\equiv \sum_{(f : C \rightarrow \mathcal{L})} (\text{out} \circ f \equiv \text{Pf} \circ \gamma) \quad (2.36)$$

$$\equiv \sum_{(f : C \rightarrow \mathcal{L})} (\text{in} \circ \text{out} \circ f \equiv \text{in} \circ \text{Pf} \circ \gamma) \quad (2.37)$$

$$\equiv \sum_{(f : C \rightarrow \mathcal{L})} (f \equiv \text{in} \circ \text{Pf} \circ \gamma) \quad (2.38)$$

since in and out are each others inverse, and in is part of an equality and therefore an embedding, which implies $(\text{in} \circ a \equiv \text{in} \circ b) \equiv (a \equiv b)$. We let $\psi = \text{in} \circ \text{Pf} \circ \gamma$, which simplifies the expression to $\sum_{(f : C \rightarrow \mathcal{L})} (f \equiv \psi f)$. We define e to be the function from right to left for the equality in Lemma 2.1.8, we then get the equality

$$\sum_{(f : C \rightarrow \mathcal{L})} (f \equiv \psi f) \equiv \sum_{(c : \mathbf{Cone}_{C-\gamma})} (e c \equiv \psi (e c)) \quad (2.39)$$

If we define a function $\phi : \mathbf{Cone}_{C-\gamma} \rightarrow \mathbf{Cone}_{C-\gamma}$ as $\phi(u, g) = (\phi_0 u, \phi_1 u g)$ where

$$\phi_0 u = \langle \lambda _, \star \rangle, \text{Pf} \circ \gamma \circ u \rangle \quad (2.40)$$

$$\phi_1 u g = \langle \text{funExt } \lambda _, \text{refl}_\star, \text{ap } (\text{Pf} \circ \gamma) \circ g \rangle \quad (2.41)$$

then we get the commuting square in Figure 2.4, which says $\psi(e c) = e(\phi c)$, so we can continue the simplification

$$\sum_{(c : \mathbf{Cone}_{C-\gamma})} (e c \equiv e(\phi c)) \quad (2.42)$$

$$\begin{array}{ccc} \mathbf{Cone}_{C-\gamma} & \xrightarrow{e} & (C \rightarrow \mathcal{L}) \\ \downarrow \phi & & \downarrow \psi \\ \mathbf{Cone}_{C-\gamma} & \xrightarrow{e} & (C \rightarrow \mathcal{L}) \end{array}$$

Figure 2.4: commuting square

We know that \mathbf{e} is part of an equality (namely Lemma 2.1.8), so it is an embedding, that is for all a, b the equality $\mathbf{e} \ a \equiv \mathbf{e} \ b$ implies $a \equiv b$, using this and unfolding the definition of ϕ , we get the equalities

$$\sum_{(c:\mathbf{Cone}_{C-\gamma})} (c \equiv \phi \ c) \quad (2.43)$$

$$\equiv \sum_{((u,g):\mathbf{Cone}_{C-\gamma})} ((u, g) \equiv (\phi_0 \ u, \phi_1 \ u \ g)) \quad (2.44)$$

$$\equiv \sum_{((u,g):\mathbf{Cone}_{C-\gamma})} \sum_{(p:u \equiv \phi_0 \ u)} g \equiv_* \phi_1 \ u \ g \quad (2.45)$$

We rearrange and unfold the definition of \mathbf{Cone} to get

$$\sum_{((u,p):\sum_{(u:\prod_{(n:\mathbb{N})} C \rightarrow X_n)} u \equiv \phi_0 \ u)} \sum_{(g:\prod_{(n:\mathbb{N})} \pi_{(n)} \circ u_{n+1} \equiv u_n)} g \equiv_* \phi_1 \ u \ g \quad (2.46)$$

We define the following two equalities from Lemma 2.1.7

$$\mathbf{1} \equiv \left(\sum_{(u:\prod_{(n:\mathbb{N})} C \rightarrow X_n)} u \equiv \phi_0 \ u \right) \quad (2.47)$$

$$\mathbf{1} \equiv_* \left(\sum_{(g:\prod_{(n:\mathbb{N})} \pi_{(n)} \circ u_{n+1} \equiv u_n)} g \equiv_* \phi_1 \ u \ g \right) \quad (2.48)$$

using these two equalities the proof becomes showing $\sum_{\star:1} \mathbf{1} \equiv \mathbf{1}$, which is trivial. \square

2.2 Coinduction Principle for \mathbf{M} -types

We can now construct a coinduction principle for \mathbf{M} -types given a (strong) bisimulation relation.

Definition 2.2.1. For all coalgebras $C-\gamma : \mathbf{Coalg}_S$, then a relation $\mathcal{R} : C \rightarrow C \rightarrow \mathcal{U}$ is a (strong) bisimulation relation, if the type $\overline{\mathcal{R}} = \sum_{(a:C)} \sum_{(b:C)} a \ \mathcal{R} \ b$ and the function $\alpha_{\mathcal{R}} : \overline{\mathcal{R}} \rightarrow \mathbf{P}_S \overline{\mathcal{R}}$ forms a P-coalgebra $\overline{\mathcal{R}}-\alpha_{\mathcal{R}} : \mathbf{Coalg}_S$, that makes the diagram in Figure 2.5 commute (\Rightarrow represents P-coalgebra morphisms). That is $\gamma \circ \pi_1^{\overline{\mathcal{R}}} \equiv \mathbf{P} \pi_1^{\overline{\mathcal{R}}} \circ \alpha_{\mathcal{R}}$, and similarly for $\pi_2^{\overline{\mathcal{R}}}$.

$$C-\gamma \xleftarrow{\pi_1^{\overline{\mathcal{R}}}} \overline{\mathcal{R}}-\alpha_{\mathcal{R}} \xrightarrow{\pi_2^{\overline{\mathcal{R}}}} C-\gamma$$

Figure 2.5: Bisimulation for a coalgebra

Theorem 2.2.2 (Coinduction principle). *Given a relation \mathcal{R} , that is a bisimulation for an \mathbf{M} -type, then for all x, y if $x \ \mathcal{R} \ y$ then they are equal $x \equiv y$.*

Proof. Given a relation \mathcal{R} , that is part of a bisimulation over a final P-coalgebra $\mathbf{M-out} : \mathbf{Coalg}_S$ we get the diagram in Figure 2.6. By the finality of $\mathbf{M-out}$, we get a function $!$, such that $\pi_1^{\overline{\mathcal{R}}} \equiv ! \equiv \pi_2^{\overline{\mathcal{R}}}$.

$$\mathbf{M}\text{-out} \xleftarrow{\pi_1^{\overline{\mathcal{R}}}} \overline{\mathcal{R}}\text{-}\alpha_{\mathcal{R}} \xrightarrow{\pi_2^{\overline{\mathcal{R}}}} \mathbf{M}\text{-out}$$

Figure 2.6: Bisimulation principle for final coalgebra

Now if we are given $r : x \mathcal{R} y$, we can construct the equality

$$x \equiv \pi_1^{\overline{\mathcal{R}}}(x, y, r) \equiv \pi_2^{\overline{\mathcal{R}}}(x, y, r) \equiv y. \quad (2.49)$$

Giving us the coinduction principle for \mathbf{M} -types. □

We therefore get nice equalities for \mathbf{M} -types, where we can define a bisimulation. For example the bisimilarity given for streams in the introduction (1.7), will give us equality principle for streams by this coinduction principle.

Chapter 3

Examples of **M**-types

In this section we show some examples of coinductive types that can be constructed as **M**-types, and show how to define the constructors/destructors of these types. We conclude this chapter with some general observation, and define some rules for how to construct **M**-types.

Is there anything else that is show for each **M**-type?

3.1 ITrees as **M**-types

Interaction trees (ITrees) [18] are used to model effectful behavior, where computations can interact with an external environment by events. ITrees are defined by the following constructors

$$\frac{r : R}{\text{Ret } r : \text{itree } E R} I_{\text{Ret}} \quad (3.1)$$

$$\frac{A : \mathcal{U} \quad a : E A \quad f : A \rightarrow \text{itree } E R}{\text{Vis } a f : \text{itree } E R} I_{\text{Vis}}. \quad (3.2)$$

$$\frac{t : \text{itree } E R}{\text{Tau } t : \text{itree } E R} E_{\text{Tau}}. \quad (3.3)$$

where R is the type for returned values, while E is a dependent type for events representing external interactions. We will try and give some intuition on how to construct this type, by constructing types with two out of three of these constructors.

3.1.1 Delay Monad

We start by looking at ITrees without the **Vis** constructor, this type is also know as the delay monad. It can be used to model delayed computations, either returning immediately given by the constructor **now** = **Ret**, or delayed some (possibly infinite) number of steps by the constructor **later** = **Tau**. We construct this type as an **M**-type.

Theorem 3.1.1. *The delay monad can be defined as the **M**-type.*

Proof. We want to construct a container, that will give us two constructors, one that returns immediately and one that delays the computation, that is the

$$S = \left(R + 1, \begin{cases} 0 & \text{inl } r \\ 1 & \text{inr } \star \end{cases} \right) \quad (3.4)$$

which gives us the polynomial functor

$$\mathbf{P}_S X = \sum_{(x:R+1)} \begin{cases} \mathbf{0} & x = \text{inl } r \rightarrow X, \\ \mathbf{1} & x = \text{inr } \star \end{cases} \quad (3.5)$$

since this definition is not dependent, we can simplify it to the following

$$\mathbf{P}_S X = R \times (\mathbf{0} \rightarrow X) + X. \quad (3.6)$$

from the property that $(\mathbf{0} \rightarrow X) \equiv \mathbf{1}$, we can simplify the definition further to

$$\mathbf{P}_S X = R + X \quad (3.7)$$

meaning we get diagram in Figure 3.1, by applying \mathbf{P}_S to \mathbf{M}_S and using the functions **in** and **out** defined by the equality *shift*. We can define the constructors **now** and **later** using **in** function for \mathbf{M} -

$$\begin{array}{ccccc} R & \xrightarrow{\text{inl}} & R + M & \xleftarrow{\text{inr}} & M \\ & \searrow \text{now} & \downarrow \text{in} & \swarrow \text{later} & \\ & & M & & \end{array}$$

Figure 3.1: Delay monad

types, together with the injections **inl** and **inr**, such that the diagram in Figure 3.1 commutes. \square

3.1.2 R -valued E-event Trees

Now lets look at the example, where we remove the **Tau** constructor. This gives us a type of tree, with leaves given by **Ret**, and nodes given by **Vis** branching based on some type A , for an event $a : \mathbf{E} A$.

Theorem 3.1.2. *We can define R -valued E-event trees as an \mathbf{M} -type*

Proof. We want a container, with a leaf constructor, that returning immediately with a value, and a node constructor, that branches on some event. We defined this by the container

$$S = \left(R + \sum_{(A:\mathcal{U})} \mathbf{E} A, \begin{cases} \mathbf{0} & \text{inl } r \\ A & \text{inr } (A, e) \end{cases} \right). \quad (3.8)$$

for which we get the polynomial functor

$$\mathbf{P}_S X = \sum_{(x:R+\sum_{(A:\mathcal{U})} \mathbf{E} A)} \begin{cases} \mathbf{0} & x = \text{inl } r \\ A & x = \text{inr } (A, e) \end{cases} \rightarrow X, \quad (3.9)$$

we can again split the two case into a sum

$$\mathbf{P}_S X = (R \times (\mathbf{0} \rightarrow X)) + \left(\sum_{A:\mathcal{U}} \mathbf{E} A \times (A \rightarrow X) \right), \quad (3.10)$$

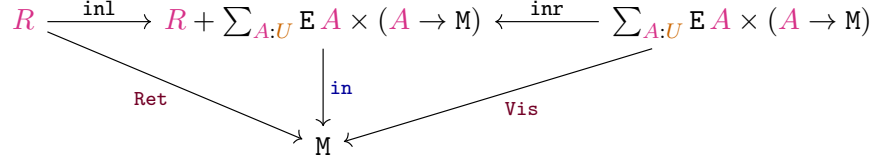


Figure 3.2: Tree Constructors

and simplify further using $(\mathbf{0} \rightarrow X) \equiv \mathbf{1}$, to get the definition

$$\mathbf{P}_S X = R + \sum_{A:\mathcal{U}} \mathbf{E} A \times (A \rightarrow X). \quad (3.11)$$

By applying this polynomial functor to the \mathbf{M} -type, we get the diagram in Figure 3.2. We can again define **Ret** and **Vis** using the **in** function together with injections. We can see that **now** and **Ret** has the same structure, both in the diagram and in how they were defined in the container. \square

3.1.3 ITrees

Get the correct equivalence for ITrees (Part of project description?)

Now we should have all the knowledge needed to make ITrees using \mathbf{M} -types.

Theorem 3.1.3. *We can define the type of ITrees as the \mathbf{M} -type*

Proof. We combine the constructions of the delay monad and R -valued \mathbf{E} -event Trees, into the container

$$S = \left(R + \mathbf{1} + \sum_{A:\mathcal{U}} (\mathbf{E} A), \begin{cases} \mathbf{0} & \text{inl } r \\ \mathbf{1} & \text{inl } (\text{inl } \star) \\ A & \text{inr } (\text{inr } (A, e)) \end{cases} \right). \quad (3.12)$$

For which the reduced polynomial functor is

$$\mathbf{P}_S X = R + X + \sum_{(A:\mathcal{U})} (\mathbf{E} A \times (A \rightarrow X)) \quad (3.13)$$

Applying this polynomial functor to the \mathbf{M} -type, for the container, we get the diagram in Figure 3.3, from which the constructors of the ITrees type can be defined using **in** and injections. \square

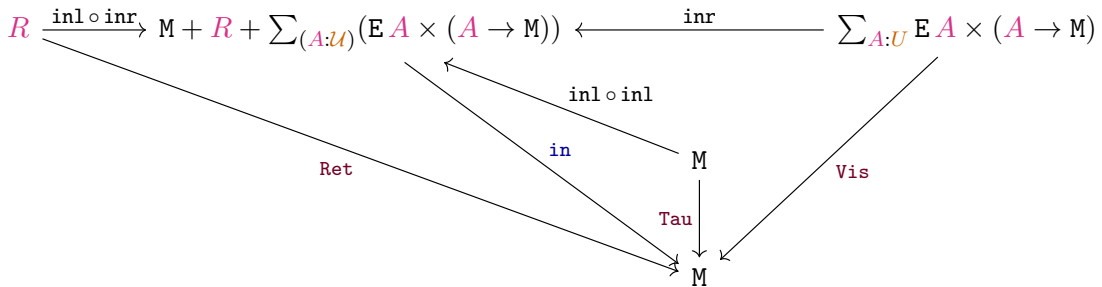


Figure 3.3: ITree constructors

3.2 General rules for constructing \mathbf{M} -types

We want to create a rule set for how to define coinductive types as \mathbf{M} -types. We would like to be able to define a type from a given set of constructor / destructors. If we for example is given the rule

$$\frac{a : A}{\text{ret } a : T} \quad (3.14)$$

we get that it corresponds to the \mathbf{M} -type for the container $(A, \lambda _, \mathbf{0})$, while if we have something that produces an element of it self as

$$\frac{a : T}{\text{tl } a : T} \quad (3.15)$$

the container is $(\mathbf{1}, \lambda _, \mathbf{1})$. If we want a type with both these rules, then we just take the disjoint union of the two containers

$$\left(A + \mathbf{1}, \begin{cases} \mathbf{0} & \text{inl } a \\ \mathbf{1} & \text{inr } \star \end{cases} \right) \quad (3.16)$$

which is the delay type. We can define some more involved constructors, that build on the type itself

$$\frac{a : A \rightarrow T}{\text{node } a : T} \quad (3.17)$$

has container $(\mathbf{1}, A)$. A type for the destructor

$$\frac{a : T}{\text{hd } a : A} \quad (3.18)$$

can be constructed by the container $(A, \lambda _, \mathbf{0})$, but this container is the same as the one for ret , so do we get both? Whether we get a constructor or destructor depends on how we combine the containers with the containers for other constructors / destructors. In general adding containers (A, B) and (C, D) for two constructors together is done by

$$\left(A + C, \begin{cases} B a & \text{inl } a \\ D c & \text{inr } c \end{cases} \right) \quad (3.19)$$

whereas adding containers for two destructors is done by

$$(A \times C, \lambda _, \lambda (a, c), B a + D c) \quad (3.20)$$

however combining destructors and constructors is not as simple. Any type T that can be defined using a (coinductive) record (except for higher inductive types), will also be definable as an \mathbf{M} -type. Given a record, which is a list of fields $f_1 : F_1, f_2 : F_2, \dots, f_n : F_n$, we can construct the \mathbf{M} -types by the container

$$(F_1 \times F_2 \times \dots \times F_n, \lambda _, \mathbf{0}) \quad (3.21)$$

where each destructor $\text{d}_n : T \rightarrow F_n$ for the field f_n will be defined as $\text{d}_n t = \pi_n (\text{out } t)$. However fields in a coinductive record may depend on previous defined fields. That is we can have the list of fields $f_1 : F_1, f_2 : F_2, \dots, f_n : F_n$, where each field depends on all the previous once. We can define this as the \mathbf{M} -type for the container

$$\left(\sum_{(f_1 : F_1)} \sum_{(f_2 : F_2)} \dots \sum_{(f_{n-1} : F_{n-1})} F_n, \lambda _, \mathbf{0} \right) \quad (3.22)$$

however, if any of the fields are non dependent, then they can be added as a product (\times) instead of a dependent sum (Σ). Fields may also coinductively construct an element of the record T , however other fields may not refer to such a field, since it will break the strictness requirements of the record / coinductive type. As an example let f_1 be a type and f_2 be the function with type $F_2 = f_1 \rightarrow (f_1 \rightarrow A) \rightarrow M$, which by currying is equal to $f_1 \times (f_1 \rightarrow A) \rightarrow M$, we can then define by the container

$$\left(\sum_{(f_1: \mathcal{U})} \left(\mathbf{1} \times \sum_{(f_3: F_3)} \cdots \sum_{(f_{n-1}: F_{n-1})} F_n \right), \lambda(f_1, \star, f_3, \dots), F_2 \right) \quad (3.23)$$

where F_2 have been moved to the last part of the container, we can even leave out the " $\mathbf{1} \times$ " from the container. Another case is that the type of a field is dependent $F_2 = (x : f_1) \rightarrow Bx \rightarrow M$, but again by currying we can get $F_2 : \sum_{(x: f_1)} Bx \rightarrow M$ which can be defined by the container

$$\left(\sum_{(f_1: \mathcal{U})} \sum_{(f_3: F_3)} \cdots \sum_{(f_{n-1}: F_{n-1})} F_n, \lambda(f_1, f_3, \dots), \sum_{x: f_1} (Bx) \right) \quad (3.24)$$

so we would also expect that a type defined as a (coinductive) record is equal to the version defined as a M -type. However this has not been shown formally.

3.3 Wacky M -type

We end this chapter by exploring a wacky M -type, that utilizes the power of containers to the fullest. We define a wacky M -type by the following container

$$\left(\mathbb{N} + \mathbb{N}, \begin{cases} \mathbb{N} & \text{inl } 0 \\ \mathbf{0} & \text{inl } x \wedge x \text{ is odd} \\ \mathbf{0} & \text{inr } x \wedge x \text{ is even} \\ \mathbf{1} & o.w. \end{cases} \right) \quad (3.25)$$

this container gives the M -type and constructors / destructors shown in Figure 3.4 The type can be

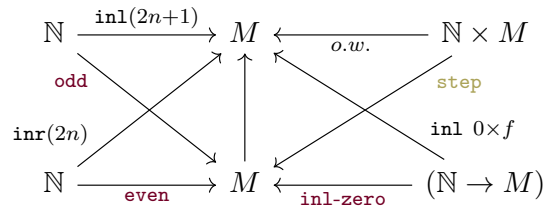


Figure 3.4: wacky M -type

interpreted as a stream of coproducts of natural numbers, that terminates whenever it is the left injection and even, or the right injection and odd, and whenever the left injection is zero, it splits in a branches indexed by the natural numbers. This show that we can define coinductive types, with rather complex structure, but a type being complex does not make it useful.

Chapter 4

QM-types

In this chapter we will introduce rules for set truncated quotients, and show some ways of how to construct quotiented **M**-types which we call **QM**-types. We show examples of how you can be constructed a quotient inductive-inductive type (QIIT) that is equal to the **QM**-type assuming axiom of choice. The examples are multisets and the partiality monad, defined as quotients with weak bisimilarity relations. We conclude the chapter by discussion, how we should define **QM**-types.

4.1 Quotienting and Constructors

We start by defining the recursor and some eliminators for the set truncated quotient, which will be used in the upcoming sections.

Definition 4.1.1 (Recursor for quotient). For all elements $x, y : A$, functions $f : A \rightarrow B$ and relations $g : x \mathrel{R} y \rightarrow f x \equiv f y$, then if B is a set, we get a function from A/R to B , defined by case as

$$\begin{aligned} \text{rec } [a] &= f a \\ \text{rec } (\text{eq}/ _ _ r i) &= g r i \\ \text{rec } (\text{squash}/ a b p q i j) &= B_{\text{set}} (\text{rec } a) (\text{rec } b) (\text{ap rec } p) (\text{ap rec } q) i j \end{aligned} \tag{4.1}$$

Definition 4.1.2 (Propositional eliminator for quotient). Given a dependent type $P : A/R \rightarrow \mathcal{U}$, that is a proposition at all points $P_{\text{prop}} : \prod_{(x:A/R)} \text{isProp } (P x)$, then given a proof for the base case $f : \prod_{(a:A)} P [a]$ we get a function from $x : A/R$ to $P x$, defined as

$$\begin{aligned} \text{elimProp } [a] &= f a \\ \text{elimProp } (\text{eq}/ a b i) &= P_{\text{prop}} (\text{elimProp } a) (\text{elimProp } b) (\text{eq}/ a b) i \\ \text{elimProp } (\text{squash}/ a b p q i j) &= \\ &\quad \text{isSet} \rightarrow \text{isSetDep } (\text{isProp} \rightarrow \text{isSet} \circ P_{\text{prop}}) \\ &\quad (\text{elimProp } a) (\text{elimProp } b) (\text{ap elimProp } p) (\text{ap elimProp } q) \\ &\quad (\text{squash}/ a b p q) i j \end{aligned} \tag{4.2}$$

where $\text{isSet} \rightarrow \text{isSetDep}$ takes a function $\prod_{a:A} \text{isSet } (B a)$ to the dependent version $\text{isSetDep } A B$. Using this eliminator we can to do propositional elimination of a quotient, by supplying the base case $P [a]$.

Definition 4.1.3 (Eliminator for quotients). Given a dependent type $P : A/R \rightarrow \mathcal{U}$ which is a set $P_{\text{set}} : \prod_{(x:A/R)} \text{isSet } (P x)$, a proof for the base case $f : \prod_{(a:A)} P [a]$ and a proof for the equality case $f_{\text{eq}} : \prod_{(a,b:A)} \prod_{(r:A/R)} f a \equiv_* f b$ for the equality constructor $\text{eq}/ a b r$, we get a function $\text{elim} : (x : A/R) \rightarrow P x$ as follows

$$\begin{aligned} \text{elim } [a] &= f a \\ \text{elim } (\text{eq}/ a b r i) &= f_{\text{eq}} a b r i \\ \text{elim } (\text{squash}/ a b p q i j) &= \\ \text{isSet} \rightarrow \text{isSetDep } P_{\text{set}} (\text{elim } a) (\text{elim } b) (\text{ap } \text{elim } p) (\text{ap } \text{elim } q) (\text{squash}/ a b p q) i j \end{aligned} \quad (4.3)$$

which is the eliminator for quotients.

We can now construct some more interesting data types, by set quotienting \mathbf{M} -types, which we call \mathbf{QM} -types, some examples follow in the following section.

preserves
the
H-level

4.2 \mathbf{QM} -types and Quotient inductive-inductive types (QIITs)

In this section we will show some examples of quotiented \mathbf{M} -type, defined using set truncated quotients. We will construct the same type using quotient inductive-inductive types instead, and show these two construction gives equal types, assuming the axiom of choice. A quotient inductive-inductive type (QIIT) is a type that is defined at the same time as a relation over that type and set truncated. We believe, but do not show, that this equality between the two ways of constructing \mathbf{QM} -types is general. However QIITs are more general than our \mathbf{QM} -types, so we cannot expect every QIIT to have a corresponding \mathbf{QM} -type.

4.2.1 Multiset

In this subsection we define infinite trees, where the order of subtrees does not matter also known as multisets [4][6][10], we construct the type of multisets as a \mathbf{QM} -type and as a QIIT and show these are equal (Theorem 4.2.8). We start by defining the tree type we want to quotient.

Definition 4.2.1. We define A -valued \mathbb{N} -branching trees $T A$ as the \mathbf{M} -type defined by the container

$$\left(A + \mathbf{1}, \begin{cases} \mathbf{0} & \text{inl } a \\ \mathbb{N} & \text{inr } \star \end{cases} \right) \quad (4.4)$$

For which we get the constructors

$$\frac{a : A}{\text{leaf } a : T A} \quad (4.5)$$

$$\frac{f : \mathbb{N} \rightarrow T A}{\text{node } f : T A} \quad (4.6)$$

Trees where the permutation of subtrees does not matter is multisets. We can define equality between permutation of subtrees by the following rule

$$\frac{f : \mathbb{N} \rightarrow T A \quad g : \mathbb{N} \rightarrow \mathbb{N} \quad \text{isIso } g}{\text{node } f \equiv \text{node } (f \circ g)} \text{ perm} \quad (4.7)$$

One way to define this type is as the \mathbf{QM} -type given by quotienting $T A$ by the relation \sim_T defined by the constructors

$$\frac{x \equiv y}{\text{leaf } x \sim_{\text{T}} \text{leaf } y} \sim_{\text{leaf}} \quad (4.8)$$

$$\frac{\prod_{(n:\mathbb{N})} \text{f } n \sim_{\text{T}} \text{g } n}{\text{node f } \sim_{\text{T}} \text{node g}} \sim_{\text{node}} \quad (4.9)$$

$$\frac{\text{f} : \mathbb{N} \rightarrow \text{T } \textcolor{violet}{A} \quad \text{g} : \mathbb{N} \rightarrow \mathbb{N} \quad \text{isIso g}}{\text{node f } \sim_{\text{T}} \text{node (f } \circ \text{ g)}} \sim_{\text{perm}} \quad (4.10)$$

We have the problem that we cannot lift the definition of **node** to the quotiented type, without the use of the axiom of choice. To solve this problem we can define multisets as a QIIT denoted **MS** *A*, with the constructors **leaf**, **node** and **perm** and set truncation constructor **MS-isSet**. We show these two ways of constructing a type for multisets are equal assuming the axiom of (countable) choice.

Definition 4.2.2. There is a function from **T** *A* to **MS** *A*

$$\begin{aligned} \text{T} \rightarrow \text{MS} (\text{leaf}_{\text{T}} x) &= \text{leaf}_{\text{MS}} x \\ \text{T} \rightarrow \text{MS} (\text{node}_{\text{T}} f) &= \text{node}_{\text{MS}} (\text{T} \rightarrow \text{MS} \circ f) \end{aligned} \quad (4.11)$$

This function takes weakly bisimilar objects to equal once.

Lemma 4.2.3. If $x, y : \text{T } \textcolor{violet}{A}$ are weakly bisimilar $p : x \sim_{\text{T}} y$ then $\text{T} \rightarrow \text{MS } x \equiv \text{T} \rightarrow \text{MS } y$.

Proof. We do the proof by casing on the weak bisimilarity.

$$\begin{aligned} \text{T} \rightarrow \text{MS} \sim \rightarrow \equiv (\sim_{\text{leaf}} p) &= \text{ap leaf}_{\text{MS}} p \\ \text{T} \rightarrow \text{MS} \sim \rightarrow \equiv (\sim_{\text{node}} k) &= \text{ap node}_{\text{MS}} (\text{funExt } (\text{T} \rightarrow \text{MS} \sim \rightarrow \equiv \circ k)) \\ \text{T} \rightarrow \text{MS} \sim \rightarrow \equiv (\sim_{\text{perm}} f g e) &= \text{perm } (\text{T} \rightarrow \text{MS} \circ f) g e \end{aligned} \quad (4.12)$$

□

With this lemma, we can lift the function **T** \rightarrow **MS** to the quotient.

Definition 4.2.4. There is a function **T**/ $\sim \rightarrow$ **MS** from **T** *A*/ \sim_{T} to **MS** *A*, defined using the recursor for quotients defined in Definition 4.1.1, with **f** = **T** \rightarrow **MS** and **g** = **T**/ $\sim \rightarrow$ **MS** $\sim \rightarrow \equiv$ and **MS** is a set by **MS-isSet**

Lemma 4.2.5. Given an equality **T** \rightarrow **MS** $x \equiv \text{T} \rightarrow \text{MS } y$ then $x \sim_{\text{T}} y$.

Proof. If x and y are leaves with values a and b then $a \equiv b$ by the injectivity of the constructor **leaf**_{MS}, making a bisimilarity using \sim_{leaf} . If x and y are nodes defined by functions f and g , then by the injectivity of the constructor **node**_{MS}, we get $\prod_{(n:\mathbb{N})} \text{f } n \equiv \text{g } n$ and by induction we get $\prod_{(n:\mathbb{N})} \text{f } n \sim_{\text{T}} \text{g } n$, making us able to use \sim_{node} to construct the bisimilarity. We do not get any other equalities, since **leaf**_{MS} and **node**_{MS} are disjoint. □

Lemma 4.2.6. The function **T**/ $\sim \rightarrow$ **MS** is injective, meaning **T**/ $\sim \rightarrow$ **MS** $x \equiv \text{T} \rightarrow \text{MS } y$ implies $x \equiv y$.

Proof. We show injectivity by doing propositional elimination defined in Definition 4.1.2, with

$$\text{P} = (\lambda x, \text{T} \rightarrow \text{MS } x \equiv \text{T} \rightarrow \text{MS } y \rightarrow x \equiv y) \quad (4.13)$$

which is a proposition for all x by $\mathbf{P}_{\text{prop}} = \lambda x, \text{isProp}\Pi (\lambda _, \text{squash}/ x y)$. We do it twice, first for x and then for y where we define \mathbf{P} and \mathbf{P}_{prop} similarly, but with $x = [a]$ since it has already been propositionally eliminated.

$$\text{elimProp} (\lambda a, \text{elimProp}(\lambda b, \text{eq}/ a b \circ \mathbf{T} \rightarrow \mathbf{MS} \equiv \rightarrow \sim a b) y) x \quad (4.14)$$

where $\mathbf{T} \rightarrow \mathbf{MS} \equiv \rightarrow \sim$ is Lemma 4.2.5. \square

Lemma 4.2.7. *The function $\mathbf{T}/\sim \rightarrow \mathbf{MS}$ is surjective $(\prod_{(b:\mathbf{MS}), \|\Sigma_{(x:\mathbf{T})} \mathbf{T}/\sim \rightarrow \mathbf{MS} x \equiv b\|})$, assuming the axiom of choice.*

Proof. We only need to look at the point constructors of \mathbf{MS} . For the leaf case, we have the simple equality $\mathbf{T}/\sim \rightarrow \mathbf{MS} [\text{leaf}_{\mathbf{T}} x] \equiv \text{leaf}_{\mathbf{MS}} x$. For the node case with $\text{node } f$ then by induction, surjection of $[\cdot]$ and the axiom of choice, we get $g : \mathbb{N} \rightarrow \mathbf{T} A$ such that $\prod_{(n:\mathbb{N})} \mathbf{T}/\sim \rightarrow \mathbf{MS} [g n] \equiv f n$, making $\mathbf{T}/\sim \rightarrow \mathbf{MS} [\text{node } g] \equiv \text{node } f$. \square

Theorem 4.2.8. *There is an equality between the types $\mathbf{T} A/\sim_{\mathbf{T}}$ and $\mathbf{MS} A$, assuming the axiom of choice.*

Proof. Since the function $\mathbf{T}/\sim \rightarrow \mathbf{MS}$ is injective and surjective, it becomes an equality. \square

4.2.2 Partiality monad

In this subsection we will define the partiality monad (see below) and show that it is equal to the delay monad quotiented by weak bisimilarity, assuming the axiom of choice. We do this proof by defining an intermediate representation of sequences, and then first showing the delay monad equal to this type (Subsection 4.2.2.1), and then showing that sequences quotiented by weak are equal to the partiality monad (Subsection 4.2.2.2) [5].

Definition 4.2.9 (Partiality Monad). A simple example of a quotient inductive-inductive type is the partiality monad $(-)_\perp$ over a type R , defined by the constructors

$$\frac{}{R_\perp : \mathcal{U}} \quad (4.15)$$

$$\frac{}{\perp : R_\perp} \quad (4.16)$$

$$\frac{a : R}{\eta a : R_\perp} \quad (4.17)$$

and a relation $(\cdot \sqsubseteq_\perp \cdot)$ indexed twice over R_\perp , with properties

$$\frac{s : \mathbb{N} \rightarrow R_\perp \quad b : \prod_{(n:\mathbb{N})} s_n \sqsubseteq_\perp s_{n+1}}{\bigsqcup (s, b) : R_\perp} \quad (4.18)$$

$$\frac{x, y : R_\perp \quad p : x \sqsubseteq_\perp y \quad q : y \sqsubseteq_\perp x}{\alpha_\perp p q : x \equiv y} \quad (4.19)$$

$$\frac{x : R_\perp}{x \sqsubseteq_\perp x} \sqsubseteq_{\text{refl}} \quad (4.20)$$

$$\frac{x \sqsubseteq_\perp y \quad y \sqsubseteq_\perp z}{x \sqsubseteq_\perp z} \sqsubseteq_{\text{trans}} \quad (4.21)$$

$$\frac{x : R_\perp}{\perp \sqsubseteq_\perp x} \sqsubseteq_{\text{never}} \quad (4.22)$$

$$\frac{s : \mathbb{N} \rightarrow R_\perp \quad b : \prod_{(n:\mathbb{N})} s_n \sqsubseteq_\perp s_{n+1}}{\prod_{(n:\mathbb{N})} s_n \sqsubseteq_\perp \bigsqcup (s, b)} \quad (4.23)$$

$$\frac{\prod_{(n:\mathbb{N})} s_n \sqsubseteq_\perp x}{\bigsqcup (s, b) \sqsubseteq_\perp x} \quad (4.24)$$

and finally set truncated

$$\frac{p, q : x \sqsubseteq_\perp y}{p \equiv q} (-)_\perp\text{-isSet} \quad (4.25)$$

4.2.2.1 Delay monad to Sequences

We define sequences, and show they are equal to the delay monad (Theorem 4.2.18). We then extend this equality to an equality between the types quotient by weak bisimilarity (Corollary 4.2.2.1). We start by defining the type of sequences.

Definition 4.2.10. We define

$$\text{Seq}_R = \sum_{(\mathbf{s}:\mathbb{N} \rightarrow R+1)} \text{isMon } \mathbf{s} \quad (4.26)$$

where

$$\text{isMon } \mathbf{s} = \prod_{(n:\mathbb{N})} (\mathbf{s}_n \equiv \mathbf{s}_{n+1}) + ((\mathbf{s}_n \equiv \text{inr } \star) \times (\mathbf{s}_{n+1} \not\equiv \text{inr } \star)) \quad (4.27)$$

meaning a sequences is $\text{inr } \star$ until it reaches a point where it switches to $\text{inl } r$ for some value r . There are also the special cases of already terminated, meaning only $\text{inl } r$ and never terminating meaning only $\text{inr } \star$.

For each index in a sequence, the element at that index \mathbf{s}_n is either not terminated $\mathbf{s}_n \equiv \text{inr } \star$, which we denote as $\mathbf{s}_n \uparrow_{R+1}$, or it is terminated $\mathbf{s}_n \equiv \text{inl } r$ with some value r , denoted by $\mathbf{s}_n \downarrow_{R+1} r$ or just $\mathbf{s}_n \downarrow_{R+1}$ to mean $\mathbf{s}_n \not\equiv \text{inr } \star$. Thus we can write isMon as

$$\text{isMon } \mathbf{s} = \prod_{(n:\mathbb{N})} (\mathbf{s}_n \equiv \mathbf{s}_{n+1}) + ((\mathbf{s}_n \uparrow_{R+1}) \times (\mathbf{s}_{n+1} \downarrow_{R+1})) \quad (4.28)$$

We also introduce notation for the two special cases of sequences given above

$$\text{now}_{\text{Seq}} r = (\lambda _, \text{inl } r), (\lambda _, \text{inl refl}) \quad (4.29)$$

$$\text{never}_{\text{Seq}} = (\lambda _, \text{inr } \star), (\lambda _, \text{inl refl}) \quad (4.30)$$

Definition 4.2.11. We can shift a sequence (\mathbf{s}, \mathbf{q}) by inserting an element (and an equality) (z_s, z_q) at $n = 0$,

$$\text{shift } (\mathbf{s}, \mathbf{q}) (z_s, z_q) = \begin{cases} z_s & n = 0 \\ \mathbf{s}_m & n = m + 1 \end{cases}, \begin{cases} z_q & n = 0 \\ \mathbf{q}_m & n = m + 1 \end{cases}, \quad (4.31)$$

Definition 4.2.12. We can unshift a sequence by removing the first element of the sequence

$$\text{unshift } (\mathbf{s}, \mathbf{q}) = \mathbf{s} \circ \text{succ}, \mathbf{q} \circ \text{succ}. \quad (4.32)$$

Lemma 4.2.13. *The function*

$$\text{shift-unshift } (\mathbf{s}, \mathbf{q}) = \text{shift } (\text{unshift } (\mathbf{s}, \mathbf{q})) (\mathbf{s}_0, \mathbf{q}_0) \quad (4.33)$$

is equal to the identity function.

Proof. Unshifting a value followed by a shift, where we reintroduce the value we just remove, gives the sequence we started with. \square

Lemma 4.2.14. *The function*

$$\text{unshift-shift } (\mathbf{s}, \mathbf{q}) = \text{unshift } (\text{shift } (\mathbf{s}, \mathbf{q}) _) \quad (4.34)$$

is equal to the identity function.

Proof. If we shift followed by an unshift, we just introduce a value to instantly remove it, meaning the value does not matter. \square

Lemma 4.2.15 ($\text{inl} \neq \text{inr}$). *For any two elements $x = \text{inl } a$ and $y = \text{inr } b$ then $x \neq y$.*

Proof. The constructors inl and inr are disjoint, so there does not exist a path between them, meaning constructing one is a contradiction. \square

We now define an equivalence between $\text{Delay } R$ and Seq_R , where later are equivalent to shifts, and $\text{now } r$ is equivalent terminated sequence with value r . We do this by defining equivalence functions, and the left and right identities.

Definition 4.2.16. We define a function from $\text{Delay } R$ to Seq_R

$$\begin{aligned} \text{Delay} \rightarrow \text{Seq} (\text{now } r) &= \text{now}_{\text{Seq}} r \\ \text{Delay} \rightarrow \text{Seq} (\text{later } x) &= \\ &\text{shift } (\text{Delay} \rightarrow \text{Seq } x) \left(\text{inr } \star, \begin{cases} \text{inr } (\text{refl}, \text{inl} \neq \text{inr}) & x = \text{now } _ \\ \text{inl } \text{refl} & x = \text{later } _ \end{cases} \right) \end{aligned} \quad (4.35)$$

Definition 4.2.17. We define function from Seq_R to $\text{Delay } R$

$$\text{Seq} \rightarrow \text{Delay} (s, q) = \begin{cases} \text{now } r & s_0 = \text{inl } r \\ \text{later } (\text{Seq} \rightarrow \text{Delay} (\text{unshift } (s, q))) & s_0 = \text{inr } \star \end{cases} \quad (4.36)$$

Theorem 4.2.18. *The type $\text{Delay } R$ is equal to Seq_R*

Proof. We define right and left identity, saying that for any sequence (s, q) , we get

$$\text{Delay} \rightarrow \text{Seq} (\text{Seq} \rightarrow \text{Delay} (s, q)) \equiv (s, q) \quad (4.37)$$

defined by cases analysis on s_0 , if $s_0 = \text{inl } r$ then we need to show

$$\text{now}_{\text{Seq}} r \equiv (s, q) \quad (4.38)$$

This is true, since (s, q) is a monotone sequence and $\text{inl } r$ is the top element of the order, then all elements of the sequence are $\text{inl } r$. If $s_0 = \text{inr } \star$ then, we need to show

$$\text{shift } (\text{Delay} \rightarrow \text{Seq} (\text{Seq} \rightarrow \text{Delay} (\text{unshift } (s, q)))) (\text{inr } \star, _) \equiv (s, q) \quad (4.39)$$

by the induction hypothesis we get

$$\text{Delay} \rightarrow \text{Seq} (\text{Seq} \rightarrow \text{Delay} (\text{unshift } (s, q))) \equiv \text{unshift } (s, q) \quad (4.40)$$

since shift and unshift are inverse, we get the needed equality. For the left identity, we need to show that for any delay monad t we get

$$\text{Seq} \rightarrow \text{Delay} (\text{Delay} \rightarrow \text{Seq } t) \equiv t \quad (4.41)$$

We do case analysis on t , if $t = \text{now } a$ then the equality is refl . If $t = \text{later } x$ then we need to show

$$\text{later } (\text{Seq} \rightarrow \text{Delay} (\text{unshift } (\text{shift } (\text{Delay} \rightarrow \text{Seq } x)))) \equiv \text{later } x \quad (4.42)$$

from unshift and shift being inverse and the induction hypothesis, we get the wanted equality. We have now defined a left and right identity function, so we get the wanted equality. \square

Corollary. The types $\text{Delay } R/\sim$ and Seq_R/\sim are equal.

Show
this

Proof. We show that if $a \sim_{\text{delay}} b$ then $\text{Delay} \rightarrow \text{Seq } a \sim_{\text{Seq}} \text{Delay} \rightarrow \text{Seq } b$,

and we show if $x \sim_{\text{Seq}} y$ then $\text{Seq} \rightarrow \text{Delay } x \sim_{\text{Seq}} \text{Seq} \rightarrow \text{Delay } y$, □

Show
this

4.2.2.2 Sequence to Partiality Monad

In this section we will show that assuming the axiom of choice, we get an equivalence between sequences quotiented by weak bisimilarity and the partiality monad (Theorem 4.2.34). We start by defining some relation on sequences.

Definition 4.2.19 (Sequence Termination). The following relations says that a sequence $(\mathbf{s}, \mathbf{q}) : \text{Seq}_R$ terminates with a given value $r : R$,

$$(\mathbf{s}, \mathbf{q}) \downarrow_{\text{Seq}} r = \sum_{(n:\mathbb{N})} \mathbf{s}_n \downarrow_{R+1} r. \quad (4.43)$$

Definition 4.2.20 (Sequence Ordering). We can define an ordering of sequences as

$$(\mathbf{s}, \mathbf{q}) \sqsubseteq_{\text{Seq}} (\mathbf{t}, \mathbf{p}) = \prod_{(a:R)} (\|\mathbf{s} \downarrow_{\text{Seq}} a\| \rightarrow \|\mathbf{t} \downarrow_{\text{Seq}} a\|) \quad (4.44)$$

where $\|\cdot\|$ is propositional truncation.

Definition 4.2.21. There is a function from $R + 1$ to the partiality monad R_{\perp}

$$\begin{aligned} \text{Maybe} \rightarrow (-)_{\perp} (\text{inl } r) &= \eta \ r \\ \text{Maybe} \rightarrow (-)_{\perp} (\text{inr } \star) &= \perp \end{aligned} \quad (4.45)$$

Definition 4.2.22 (Maybe Ordering). We can define an ordering relation on $R + 1$ as

$$x \sqsubseteq_{R+1} y = (x \equiv y) + ((x \downarrow_{R+1}) \times (y \uparrow_{R+1})) \quad (4.46)$$

This ordering definition is basically isMon at a specific index, so we can again rewrite isMon as

$$\text{isMon } \mathbf{s} = \prod_{(n:\mathbb{N})} \mathbf{s}_n \sqsubseteq_{R+1} \mathbf{s}_{n+1} \quad (4.47)$$

This rewriting confirms that if $\text{isMon } \mathbf{s}$, then \mathbf{s} is monotone, and therefore a sequence of partial values.

Lemma 4.2.23. The function $\text{Maybe} \rightarrow (-)_{\perp}$ is monotone, that is, if $x \sqsubseteq_{R+1} y$, for some x and y , then $(\text{Maybe} \rightarrow (-)_{\perp} x) \sqsubseteq_{\perp} (\text{Maybe} \rightarrow (-)_{\perp} y)$.

Proof. We do the proof by case.

$$\begin{aligned} \text{Maybe} \rightarrow (-)_{\perp} \text{-mono } (\text{inl } p) &= \\ \text{subst } (\lambda a, \text{Maybe} \rightarrow (-)_{\perp} x \sqsubseteq_{\perp} \text{Maybe} \rightarrow (-)_{\perp} a) \ p \ (\sqsubseteq_{\text{refl}} (\text{Maybe} \rightarrow (-)_{\perp} x)) \\ \text{Maybe} \rightarrow (-)_{\perp} \text{-mono } (\text{inr } (p, _)) &= \\ \text{subst } (\lambda a, \text{Maybe} \rightarrow (-)_{\perp} a \sqsubseteq_{\perp} \text{Maybe} \rightarrow (-)_{\perp} y) \ p^{-1} (\sqsubseteq_{\text{never}} (\text{Maybe} \rightarrow (-)_{\perp} y)) \end{aligned} \quad (4.48)$$

□

there exists non-monotone sequences, it just follows our definition of a sequence.

Definition 4.2.24. There is a function taking a sequence to an increasing sequence of partiality monads

$$\text{Seq} \rightarrow \text{incSeq} (s, q) = \text{Maybe} \rightarrow (-)_{\perp} \circ s, \text{Maybe} \rightarrow (-)_{\perp} \text{-mono} \circ q \quad (4.49)$$

Definition 4.2.25. There is a function taking a sequence to the partiality monad

$$\begin{aligned} \text{Seq} \rightarrow (-)_{\perp} &: \text{Seq}_R \rightarrow R_{\perp} \\ \text{Seq} \rightarrow (-)_{\perp} (g, q) &= \bigsqcup \circ \text{Seq} \rightarrow \text{incSeq} \end{aligned} \quad (4.50)$$

Lemma 4.2.26. The function $\text{Seq} \rightarrow (-)_{\perp}$ is monotone.

$$\text{Seq} \rightarrow (-)_{\perp} \text{-mono} : (x \ y : \text{Seq}_R) \rightarrow x \sqsubseteq_{\text{seq}} y \rightarrow \text{Seq} \rightarrow (-)_{\perp} x \sqsubseteq_{\perp} \text{Seq} \rightarrow (-)_{\perp} y \quad (4.51)$$

Proof. Given two sequences, where one is smaller than the another, then taking the least upper bounds of each sequence respect the ordering. \square

Lemma 4.2.27. If two sequences x, y are weakly bisimilar, then $\text{Seq} \rightarrow (-)_{\perp} x \equiv \text{Seq} \rightarrow (-)_{\perp} y$

Proof. We know that the $\text{Seq} \rightarrow (-)_{\perp}$ is monotone, so we can use this monotonicity principle on $p : x \sqsubseteq_{\text{seq}} y$ and $q : y \sqsubseteq_{\text{seq}} x$ given by the bisimilarity, to get $\text{Seq} \rightarrow (-)_{\perp} x \sqsubseteq_{\perp} \text{Seq} \rightarrow (-)_{\perp} y$ and $\text{Seq} \rightarrow (-)_{\perp} y \sqsubseteq_{\perp} \text{Seq} \rightarrow (-)_{\perp} x$ by using α_{\perp} we get the equality $\text{Seq} \rightarrow (-)_{\perp} x \equiv \text{Seq} \rightarrow (-)_{\perp} y$. We denote this construction as the function $\text{Seq} \rightarrow (-)_{\perp} \sim \rightarrow \equiv$ \square

The recursor for quotients Definition 4.1.1 allows us to lift the function $\text{Seq} \rightarrow (-)_{\perp}$ to the quotient

Definition 4.2.28. We can define a function $\text{Seq}/\sim \rightarrow (-)_{\perp}$ from Seq_R to R_{\perp}

$$\text{Seq}/\sim \rightarrow (-)_{\perp} = \text{rec Seq} \rightarrow (-)_{\perp} (\text{Seq} \rightarrow (-)_{\perp} \sim \rightarrow \equiv) (-)_{\perp} \text{-isSet} \quad (4.52)$$

Lemma 4.2.29. Given two sequences s and t , if $\text{Seq} \rightarrow (-)_{\perp} s \equiv \text{Seq} \rightarrow (-)_{\perp} t$, then $s \sim_{\text{seq}} t$.

Proof. We can reduce the burden of the proof, since

$$s \sim_{\text{seq}} t = \left(\prod_{(r:R)} \|x \downarrow_{\text{seq}} r\| \rightarrow \|y \downarrow_{\text{seq}} r\| \right) \times \left(\prod_{(r:R)} \|y \downarrow_{\text{seq}} r\| \rightarrow \|x \downarrow_{\text{seq}} r\| \right) \quad (4.53)$$

meaning we can show one part and get the other for free by symmetry. We assume $\|x \downarrow_{\text{seq}} r\|$, to show $\|y \downarrow_{\text{seq}} r\|$. By the mapping property of propositional truncation, we reduce the proof to defining a function $x \downarrow_{\text{seq}} r \rightarrow y \downarrow_{\text{seq}} r$. Since $x \downarrow_{\text{seq}} r$, then $\eta \ r \sqsubseteq_{\perp} \text{Seq} \rightarrow (-)_{\perp} x$, but we have assumed $\text{Seq} \rightarrow (-)_{\perp} x \equiv \text{Seq} \rightarrow (-)_{\perp} y$, so we get $\eta \ r \sqsubseteq_{\perp} \text{Seq} \rightarrow (-)_{\perp} y$, and thereby $y \downarrow_{\text{seq}} r$. \square

Lemma 4.2.30. The function $\text{Seq}/\sim \rightarrow (-)_{\perp}$ is injective.

Proof. We use propositional elimination of quotients (see Definition 4.1.2) to show injectivity, meaning for all $x, y : \text{Seq}_R/\sim_{\text{seq}}$ we get $\text{Seq}/\sim \rightarrow (-)_{\perp} x \equiv \text{Seq}/\sim \rightarrow (-)_{\perp} y \rightarrow x \equiv y$. We do propositional eliminating of x , followed by propositional elimination of y . We use Definition 4.1.2, with

$$P = \text{Seq}/\sim \rightarrow (-)_{\perp} x \equiv \text{Seq}/\sim \rightarrow (-)_{\perp} y \rightarrow x \equiv y \quad (4.54)$$

$$P_{\text{prop}} = \text{isProp}\Pi (\lambda _, \text{squash}/ x \ y) \quad (4.55)$$

should this be formalized entirely (There is alot of work here.. but not interesting.)

removed "isSet A", where is it used?

where we index by x for the first elimination, and by y for the second with $x = [a]$. The proof is then completed by

$$\text{elimProp } (\lambda a, \text{elimProp } (\lambda b, (\text{eq}/ a b) \circ (\text{Seq} \rightarrow (-)_\perp \dashv \dashv \dashv \sim a b)) y) x \quad (4.56)$$

where $\text{Seq} \rightarrow (-)_\perp \dashv \dashv \dashv \sim$ is (4.2.29), \square

Lemma 4.2.31. *For all constant sequences s , where all elements have the same value v , we get $\text{Seq} \rightarrow (-)_\perp s \equiv \text{Maybe} \rightarrow (-)_\perp v$.*

Proof. The left side of the equality reduces to $\text{Maybe} \rightarrow (-)_\perp$ applied on the least upper bound of the constant sequence, which is exactly the right hand side of the equality. \square

Definition 4.2.32 (Propositional eliminator for $(-)_\perp$). Given a dependent type $P : R_\perp \rightarrow \mathcal{U}$ that is a proposition at all points $P_{\text{prop}} : \prod_{(x : R_\perp)} \text{isProp } (P x)$, then given a proof for the base cases $P_\perp : P \perp$ and $P_\eta : \prod_{(a : R)} P (\eta a)$, and a proof P_\sqcup that for all $(s, q) : \sum_{(s : \mathbb{N} \rightarrow R_\perp)} \prod_{(n : \mathbb{N})} s n \sqsubseteq_\perp s (n+1)$ we get $(\prod_{(n : \mathbb{N})} P (s n)) \rightarrow P \sqcup(s, q)$. Then we get a function from any $x : R_\perp$ to $P x$

$$\begin{aligned} \text{elimProp}_\perp \perp &= P_\perp \\ \text{elimProp}_\perp (\eta a) &= P_\eta a \\ \text{elimProp}_\perp \sqcup(s, q) &= P_\sqcup(s, q) (\text{elimProp}_\perp \circ s) \\ \text{elimProp}_\perp (\alpha p q i) &= \\ &\quad \text{isProp} \rightarrow \text{isDepProp } P_{\text{prop}} (\text{elimProp}_\perp x) (\text{elimProp}_\perp y) (\alpha p q) i \\ \text{elimProp}_\perp (\perp \text{-isSet } x y p q i j) &= \\ &\quad \text{isSet} \rightarrow \text{isDepSet } (\text{isProp} \rightarrow \text{isSet} \circ P_{\text{prop}}) (\text{elimProp}_\perp x) (\text{elimProp}_\perp y) \\ &\quad (\text{ap elimProp}_\perp p) (\text{ap elimProp}_\perp q) (\perp \text{-isSet } x y p q) i j \end{aligned} \quad (4.57)$$

where x and y in the α case are the endpoints of the paths p and q .

Lemma 4.2.33. *The function $\text{Seq} \rightarrow (-)_\perp$ is surjective, assuming the axiom of choice.*

Proof. We do the proof by using the propositional eliminator for the partiality monad, with

$$P = \lambda y, \left\| \sum_{(x : \text{Seq}_R)} \text{Seq} \rightarrow \perp x \equiv y \right\| \quad (4.58)$$

and P_{prop} follows from the propositional truncation. For \perp and ηa , we use $\text{now}_{\text{seq}} a$ and $\text{never}_{\text{seq}}$ respectively, and by (4.2.31) we get $P \perp$ and $P (\eta a)$. For the least upper bound $\sqcup(s, q)$, we may assume $\prod_{(n : \mathbb{N})} P (s n)$ which gives us a pointwise equivalence

$$\prod_{(n : \mathbb{N})} \left\| \sum_{(f : \mathbb{N} \rightarrow \text{Seq}_R)} \text{Seq} \rightarrow \perp (f n) \equiv s n \right\| \quad (4.59)$$

which by the axiom of choice becomes

$$\left\| \prod_{(n : \mathbb{N})} \sum_{(f : \mathbb{N} \rightarrow \text{Seq}_R)} \text{Seq} \rightarrow \perp (f n) \equiv s n \right\| \quad (4.60)$$

Since we have a pointwise equivalence, we get an equivalence for the upper bound

$$\left\| \sum_{(x:\text{Seq}_R)} \text{Seq} \rightarrow \perp x \equiv \bigsqcup (s, q) \right\| \quad (4.61)$$

Thus we have shown all the cases needed to invoke the propositional eliminator for the partiality monad, completing the proof of surjectivity. \square

Corollary. *The function $\text{Seq}/\sim \rightarrow (-)_\perp$ is surjective assuming the axiom of choice.*

Proof. Since $\text{Seq} \rightarrow (-)_\perp$ is surjective, we can map into the quotient with $[\cdot]$. \square

Theorem 4.2.34. *Sequences quotiented by weak bisimilarity is equivalent to the partiality monad $\text{Seq}_R/\sim_{\text{seq}} \equiv R_\perp$, assuming the axiom of choice.*

Proof. The function $\text{Seq}/\sim \rightarrow (-)_\perp$ is injective (Lemma 4.2.30) and surjective (Corollary 4.2.2.2) assuming the axiom of choice, meaning we get an equivalence, since we are working in \mathbf{hSets} . \square

4.3 How should QM-types be defined

We want to define what a QM-type means in general, we draw inspiration from QW-types [10], quotient containers [2] and the quotient polynomial functors (QPF) [7]. From what we have seen as examples in the previous section, equality construction between Set Quotiented M-types and QIITs are given as

- A function $\text{QM} \rightarrow \text{QIIT}$ from QM to QIIT
- A proof $\text{QM} \rightarrow \text{QIIT} \sim \rightarrow \equiv$ that $x \sim_{\text{QM}} y$ implies $\text{QM} \rightarrow \text{QIIT } x \equiv \text{QM} \rightarrow \text{QIIT } y$
- Lifting $\text{QM} \rightarrow \text{QIIT}$ to $\text{QM}/\sim \rightarrow \text{QIIT}$ using the quotient recursor with $\text{QM} \rightarrow \text{QIIT} \sim \rightarrow \equiv$ and QIIT-isSet
- Showing injectivity by using propositional elimination of the quotient together with the inverse of $\text{QM} \rightarrow \text{QIIT} \sim \rightarrow \equiv$, namely $\text{QM} \rightarrow \text{QIIT-injective}$ saying that $\text{QM} \rightarrow \text{QIIT } x \equiv \text{QM} \rightarrow \text{QIIT } y$ implies $x \sim_{\text{QM}} y$.
- Lastly we show surjectivity by induction using the eliminator of QIIT and the axiom of choice and the surjectivity of $[\cdot]$ [16, 6.10.2].

4.3.1 Lifting Quotient Construction from Containers

An alternative to directly set quotienting the M-types we have defined, is to do the quotienting on the underlying container [2] / polynomial functor [7], and then do the fixed point construction we did for polynomial functors, but on the quotiented functor instead. We start by defining quotients on containers.

Definition 4.3.1. Given a container (A, B) , and a relation \sim we can form a quotiented container $(A, B)/R$

Which gives us the following definition of a polynomial functor.

"The category of containers lacks good coequalisers" - [1]

...

Complete definition

Should I go into more depth about point-wise equivalence?

Definition 4.3.2. Given a container (A, B) and for all X a family of relations $\sim_a : (B a \rightarrow X) \rightarrow (B a \rightarrow X) \rightarrow \mathcal{U}$ indexed by A , we can define a quotiented polynomial functor (QPF). We define it for types as

$$F X = \sum_{a:A} ((B a \rightarrow X) / \sim_a) \quad (4.62)$$

and for a function $f : X \rightarrow Y$, we use the quotient eliminator from Definition 4.1.3, with $P = \lambda _, (B a \rightarrow Y) / \sim_a$ which is a set **squash**/, since we are using set truncated quotients. The base case $f = \lambda _, [f \circ g]$ and equality case $f_{eq} = \lambda x y r, eq / (f \circ x) (f \circ y) (\sim_{ap} f r)$, where \sim_{ap} says that given $x \sim_a y$ and a function f then $f \circ x \sim_a f \circ y$. With this the definition for the quotient polynomial functor for functions is

$$F f (a, g) = (a, elim g) \quad (4.63)$$

completing the definition of a quotiented polynomial functor.

If there is a function $abs_X : P X \rightarrow F X$ that makes the diagram in Figure 4.1 commute (as in [7]), then we can construct the final F-coalgebra, to get another notion of quotiented M-type. If **abs**

$$\begin{array}{ccc} P X & \xrightarrow{P f} & P Y \\ \text{abs}_X \downarrow & & \downarrow \text{abs}_Y \\ F X & \xrightarrow{F f} & F Y \end{array}$$

Figure 4.1: Quotiented polynomial function

is surjective, then the square will commute, so we just have to define a relation that has the \sim_{ap} property, and a surjective function **abs**_X to construct a quotient M-type. The quotiented M-type has a surjective function **QM-intro** : $M_S \rightarrow M_S / \sim$, given as $F^\infty \circ \text{abs}_M$, where F^∞ is the function into the limit. As an example, lets try and construct the QM-type and the QPF for multisets, with this alternative approach

Example 3. We have the following polynomial functor for A -valued N-branching trees

$$P X = \sum_{a:A+1} \begin{cases} 0 \rightarrow X & a = \text{inl } r \\ \mathbb{N} \rightarrow X & a = \text{inr } \star \end{cases} \quad (4.64)$$

for which we define the family of relations $(\sim_{PT})_{(a:A+1)}$, if $a = \text{inl } r$ then the equality relations is just the trivial equalities, since the relation is between elements of type $0 \rightarrow X$, which is contractive. On the other hand if $a = \text{inr } \star$, then we define the relation as

$$\frac{f, h : \mathbb{N} \rightarrow X \quad g : \mathbb{N} \rightarrow \mathbb{N} \quad \text{isIso } g \quad f \circ g \equiv h}{f \sim_{PT} h} \quad (4.65)$$

we can see with this approach, we only need to define the non-trivial equalities, meaning those that are not just reflexivity. This relations fulfills the \sim_{ap} property, since if $a = \text{inl } r$ then it holds trivially, and if $a = \text{inr } \star$ then we have $f \sim_{PT} h$ and want to show $k \circ f \sim_{PT} k \circ h$ for any function k , which just boils down to showing $k \circ f \circ g \equiv k \circ h$ given $p : f \circ g \equiv h$, which is done by **ap** k p . We therefore have a QPF F . Now we want to define **abs**,

$$\begin{aligned} \text{abs } (\text{inl } r, \lambda ()) &= (\text{inl } r, [\lambda ()]) \\ \text{abs } (\text{inr } \star, f) &= (\text{inr } \star, [f]) \end{aligned} \quad (4.66)$$

and show that it is surjective, which follows directly from the surjectivity of $[\cdot]$. Taking the limit we get the commuting square in Figure 4.2.

$$\begin{array}{ccc}
 T \ A & \xrightarrow{\text{out}} & P \ (T \ A) \\
 \text{QM-intro} \downarrow & & \downarrow \text{abs}_P \ (T \ A) \\
 MS \ A & \xrightarrow{\text{out}_{QM}} & F \ (MS \ A)
 \end{array}$$

Figure 4.2: QPF and limit diagram for multiset construction

We have just postulated that the quotiented \mathbf{M} -type, is an \mathbf{F} -coalgebra, however this is not necessarily the case, since this requires us to show that given $\text{fq} : (\mathbf{B} \ a \rightarrow \mathbf{X})/\mathbf{R}$ then we can construct a function $(\mathbf{B} \ a \rightarrow \mathbf{X})$.

again we run into problems with the axiom of choice. The construction of \mathbf{QW} types got around this problem, by doing the construction as a \mathbf{QIIT} , we will try and follow the same ideas and show what that would look like for \mathbf{M} -types.

4.3.2 Better behaving \mathbf{QM} -types

If we take inspiration from the construction of \mathbf{QW} -types, and define the quotiented types as \mathbf{QIIT} s, by using a set of rules, we will get better behaving quotients, meaning we can construct the type, without assuming the axiom of choice. The rules for constructing the \mathbf{QIIT} s are

•

...

Complete construction of \mathbf{QW} inspired \mathbf{QM} -types

\mathbf{QIIT} s

Chapter 5

Conclusion

We have described the construction of \mathbf{M} -types as the limit of a sequence defined from repeated application of a polynomial functor for a container. We have showed that \mathbf{M} -types are a final coalgebra, and used this to define a coinduction principle for \mathbf{M} -types, giving us an equality relation from (strong) bisimulation. This work has been formalized in the cubical agda proof assistant, and part of the formalization has been accepted to the Cubical Agda github repository. We have used this formalization to construct a couple of examples of \mathbf{M} -types, to make the theory of \mathbf{M} -types more accessible, and given some rules for how to construct a container, that will give an \mathbf{M} -type, with the wanted set of constructors/destructors. We have also explored the concept of quotiented \mathbf{M} -types (\mathbf{QM} -types), and shown examples where we need the axiom of choice, to lift the constructor to the quotient. We have shown this problem can be solved by using quotient inductive-inductive types (QIITs), and shown that the type that arise from this construction is equal to the quotiented \mathbf{M} -type, if we assume the axiom of choice. We show an alternative construction of \mathbf{QM} -types, where we quotient the polynomial functor before taking the limit, however this construction also encounters the same problem as quotienting the \mathbf{M} -type directly, namely it needs the axiom of choice to work.

conclude on the problem statement from the introduction

5.1 Future Work

In this work we have described \mathbf{M} -types, however we have not looked at the notion of indexed \mathbf{M} -types, which are coinductive types indexed by another type. We hope that what is done in this thesis can be generalized to work for indexed \mathbf{M} -types, but formally showing this is future work. We similarly only discussed that types defined as (coinductive) records should be equal to types defined as \mathbf{M} -types, but we did not discuss how exactly to formulate this equality and proof it. One thing to do is should the construction of \mathbf{M} -types equal to a construction of \mathbf{M} -types as a record. This would give us the external interpretation of constructors in records, to be able to use more features of the termination checker and the likes in the Cubical Agda proof assistant.

cite indexed types

When looking at \mathbf{QM} -types, we did not formalize the alternative construction of quotiented \mathbf{M} -types as the limit of a quotiented polynomial functor, this should follow the same lines as the construction of \mathbf{M} -types from polynomial functors, however it is not enough, since we need the axiom of choice for this construction. Formalizing this and comparing it to the directly quotiented version

of \mathbf{QM} -types would be interesting. We would also like to get a formal description/proof of how to construct the quotient inductive-inductive types representing the quotient of a given \mathbf{M} -type over a given relation.

Remove red text

We have not proof the equality between types defined as (coinductive) records and \mathbf{M} -types.

All the work done here, should generalize to indexed \mathbf{M} -types, which would be nice to have formalized.

Building the weak bisimulation on the \mathbf{M} -type as a \mathbf{M} -type - Is this possible? Yes! Should it be included?

Define the weak bisimilarity relations as \mathbf{M} -types, since these are also coinductive types. However this seems to need something more general than "Index \mathbf{M} -types"

We have removed the section of strong bisimulations, should it be re-added. Should the section about properties of \mathbf{M} -types (stream) be re-added?

Building the Partiality Monad as a limit (Dialgebra?) - Is this possible?

Cofree Coalgebra / Dialgebra – Is this relevant?

Bibliography

- [1] Michael Gordon Abbott. *Categories of containers*. PhD thesis, University of Leicester, England, UK, 2003.
- [2] Michael Gordon Abbott, Thorsten Altenkirch, Neil Ghani, and Conor McBride. Constructing polymorphic programs with quotient types. In *Mathematics of Program Construction, 7th International Conference, MPC 2004, Stirling, Scotland, UK, July 12-14, 2004, Proceedings*, pages 2–15, 2004.
- [3] Benedikt Ahrens, Paolo Capriotti, and Régis Spadotti. Non-wellfounded trees in homotopy type theory. In *13th International Conference on Typed Lambda Calculi and Applications, TLCA 2015, July 1-3, 2015, Warsaw, Poland*, pages 17–30, 2015.
- [4] Thorsten Altenkirch, Paolo Capriotti, Gabe Dijkstra, and Fredrik Nordvall Forsberg. Quotient inductive-inductive types. *CoRR*, abs/1612.02346, 2016.
- [5] Thorsten Altenkirch, Nils Anders Danielsson, and Nicolai Kraus. Partiality, revisited. In Javier Esparza and Andrzej S. Murawski, editors, *Foundations of Software Science and Computation Structures*, pages 534–549, Berlin, Heidelberg, 2017. Springer Berlin Heidelberg.
- [6] Thorsten Altenkirch and Ambrus Kaposi. Type theory in type theory using quotient inductive types. In *Proceedings of the 43rd Annual ACM SIGPLAN-SIGACT Symposium on Principles of Programming Languages, POPL 2016, St. Petersburg, FL, USA, January 20 - 22, 2016*, pages 18–29, 2016.
- [7] Jeremy Avigad, Mario M. Carneiro, and Simon Hudon. Data types as quotients of polynomial functors. In *10th International Conference on Interactive Theorem Proving, ITP 2019, September 9-12, 2019, Portland, OR, USA*, pages 6:1–6:19, 2019.
- [8] Cyril Cohen, Thierry Coquand, Simon Huber, and Anders Mörtberg. Cubical type theory: A constructive interpretation of the univalence axiom. *FLAP*, 4(10):3127–3170, 2017.
- [9] Cyril Cohen, Thierry Coquand, Simon Huber, and Anders Mörtberg. Cubical type theory: A constructive interpretation of the univalence axiom. *FLAP*, 4(10):3127–3170, 2017.
- [10] Marcelo Fiore, Andrew M. Pitts, and S. C. Steenkamp. Constructing infinitary quotient-inductive types. *CoRR*, abs/1911.06899, 2019.
- [11] nLab authors. cubical type theory. <http://ncatlab.org/nlab/show/cubical%20type%20theory>, May 2020. Revision 15.

- [12] nLab authors. homotopy type theory. <http://ncatlab.org/nlab/show/homotopy%20type%20theory>, May 2020. Revision 111.
- [13] nLab authors. Martin-Löf dependent type theory. <http://ncatlab.org/nlab/show/Martin-L%C3%B6f%20dependent%20type%20theory>, June 2020. Revision 22.
- [14] nLab authors. propositions as types. <http://ncatlab.org/nlab/show/propositions%20as%20types>, June 2020. Revision 40.
- [15] nLab authors. type theory. <http://ncatlab.org/nlab/show/type%20theory>, June 2020. Revision 121.
- [16] The Univalent Foundations Program. *Homotopy Type Theory: Univalent Foundations of Mathematics*. <https://homotopytypetheory.org/book>, Institute for Advanced Study, 2013.
- [17] Andrea Vezzosi, Anders Mörtberg, and Andreas Abel. Cubical Agda: A Dependently Typed Programming Language with Univalence and Higher Inductive Types. Preprint available at <http://www.cs.cmu.edu/~amoertbe/papers/cubicalagda.pdf>, 2019.
- [18] Li-yao Xia, Yannick Zakowski, Paul He, Chung-Kil Hur, Gregory Malecha, Benjamin C. Pierce, and Steve Zdancewic. Interaction trees: representing recursive and impure programs in coq. *Proc. ACM Program. Lang.*, 4(POPL):51:1–51:32, 2020.

Appendix A

The Technical Details

