
Higher Order Categorical Semantics

Lasse Letager Hansen, 201912345

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Advisor: Bas Spitters

Abstract

in English...

Resumé

in Danish...

Acknowledgments

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*Lasse Letager Hansen,
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Chapter 1

Introduction

motivate and explain the problem to be addressed

example of a citation: [1]

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Chapter 2

M-types

2.1 Containers / Signatures

A Container (or Signature) is a pair $\mathcal{S} = (\mathcal{A}, \mathcal{B})$ of types $\vdash \mathcal{A} : \mathcal{U}$ and $a : \mathcal{A} \vdash \mathcal{B}(a) : \mathcal{U}$. From a container we can define a polynomial functor, defined for objects (types) as

$$P_{\mathcal{S}} : \mathcal{U} \rightarrow \mathcal{U}$$

$$P(\mathcal{X}) := P_{\mathcal{S}}(\mathcal{X}) = \sum_{a:\mathcal{A}} \mathcal{B}(a) \rightarrow \mathcal{X} \quad (2.1)$$

and for a function $f : \mathcal{X} \rightarrow \mathcal{Y}$ as

$$Pf : P\mathcal{X} \rightarrow P\mathcal{Y}$$

$$Pf(a, g) = (a, f \circ g) \quad (2.2)$$

As an example lets look at type for streams over the type A , defined using the container $\mathcal{S} = (\mathcal{A}, \mathbf{1})$, applying the polynomial functor we get

$$P_{\mathcal{S}}(\mathcal{X}) = \sum_{a:A} \mathbf{1} \rightarrow \mathcal{X} \quad (2.3)$$

since we are working in a Category with exponentials we get $\mathbf{1} \rightarrow \mathcal{X} \equiv \mathcal{X}^{\mathbf{1}} \equiv \mathcal{X}$, furthermore $\mathbf{1}$ and \mathcal{X} does not depend on A here, so this will be equivalent to the definition

$$P_{\mathcal{S}}(\mathcal{X}) = A \times \mathcal{X} \quad (2.4)$$

Now we define the coalgebra for this functor with type

$$\mathbf{Coalg}_{\mathcal{S}} = \sum_{C:\mathcal{U}} C \rightarrow P C \quad (2.5)$$

and morphisms

$$_ \Rightarrow _ : \mathbf{Coalg}_{\mathcal{S}} \rightarrow \mathbf{Coalg}_{\mathcal{S}}$$

$$(C, \gamma) \Rightarrow (D, \delta) = \sum_{f:C \rightarrow D} \delta \circ f = Pf \circ \gamma \quad (2.6)$$

M-types can now be defined from a container S as the type \mathbf{M} such that $(\mathbf{M}, \text{out} : \mathbf{M} \rightarrow P_S \mathbf{M})$ fulfills the property

$$\text{Final}_S := \sum_{(X, \rho) : \text{Coalg}_S(C, \gamma)} \prod_{(C, \gamma) : \text{Coalg}_S} \text{isContr}((C, \gamma) \Rightarrow (X, \rho)) \quad (2.7)$$

that is $\prod_{(C, \gamma) : \text{Coalg}_S} \text{isContr}((C, \gamma) \Rightarrow (\mathbf{M}, \text{out}))$. We denote this construction of the type \mathbf{M} , as $\mathbf{M}(A, B)$ or $\mathbf{M}S$.

If we continue our example for streams this will give us the M-type, we can see that $P_S(\mathbf{M}) = A \times \mathbf{M}$, meaning we have the following diagram, where out is an isomorphism (because of the finality of

$$\begin{array}{ccccc} A & \xleftarrow{\pi_1} & A \times \mathbf{M} & \xrightarrow{\pi_2} & \mathbf{M} \\ & \searrow \text{hd} & \uparrow \text{out} & \nearrow \text{tl} & \\ & & \mathbf{M} & & \end{array}$$

Figure 2.1: M-types of streams

the coalgebra), with inverse $\text{in} : P_S \mathbf{M} \rightarrow \mathbf{M}$. We now have a semantic for the rules we would expect for streams, if we let $\text{cons} = \text{in}$ and $\text{Stream } A = \mathbf{M}(A, \mathbf{1})$,

$$\frac{A : \mathcal{U} \quad s : \text{Stream } A}{\text{hd } s : A} E_{\text{hd}} \quad (2.8)$$

$$\frac{A : \mathcal{U} \quad s : \text{Stream } A}{\text{tl } s : \text{Stream } A} E_{\text{tl}} \quad (2.9)$$

$$\frac{A : \mathcal{U} \quad x : A \quad xs : \text{Stream } A}{\text{cons } x \ xs : \text{Stream } A} I_{\text{cons}} \quad (2.10)$$

2.2 ITrees as M-types

2.2.1 Delay Monad

We want the following rules for ITrees

$$\frac{r : R}{\text{Ret } r : \text{itree } E R} I_{\text{Ret}} \quad (2.11)$$

$$\frac{A : \mathcal{U} \quad a : E A \quad f : A \rightarrow \text{itree } E R}{\text{Vis } a \ f : \text{itree } E R} I_{\text{Vis}}. \quad (2.12)$$

Elimination rules

$$\frac{t : \text{itree } E R}{\text{Tau } t : \text{itree } E R} E_{\text{Tau}}. \quad (2.13)$$

We start by looking at **itrees** without the **Vis** constructor, this type is also know as the delay monad . We say this type is given by $S = (\mathbf{1} + R, \lambda\{\text{inl } _ \rightarrow \mathbf{1} ; \text{inr } _ \rightarrow \mathbf{0}\})$ equal to $\mathbf{M}S$, we then get the polynomial functor

$$P_S(X) = \sum_{x : \mathbf{1} + R} \lambda\{\text{inl } _ \rightarrow \mathbf{1} ; \text{inr } _ \rightarrow \mathbf{0}\} x \rightarrow X \quad (2.14)$$

check
this
state-
ment

This type is equal to the type:

$$P_S(X) = X + R \times (\mathbf{0} \rightarrow X) \quad (2.15)$$

we know that $\mathbf{0} \rightarrow X \equiv \mathbf{1}$, so we can further reduce to

$$P_S(X) = X + R \quad (2.16)$$

meaning we get the following diagram. What this diagram says is that we can define the operations

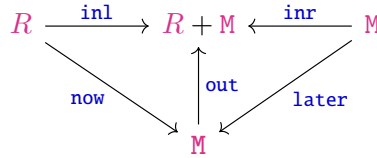


Figure 2.2: Delay monad

now and **later** using $\mathbf{in} = \mathbf{out}^{-1}$ together with the injections **inl** and **inr**.

(Later
= Tau,
Ret =
Now)

2.2.2 Tree

Now lets look at the example where we remove the **Tau** constructor. We let

$$S = \left(R + \sum_{A:\mathcal{U}} E A, \lambda\{\mathbf{inl} _ \rightarrow \mathbf{0} ; \mathbf{inr} (A, e) \rightarrow A\} \right). \quad (2.17)$$

This will give us the polynomial functor:

$$P_S(X) = \sum_{x:R + \sum_{A:\mathcal{U}} E A} \lambda\{\mathbf{inl} _ \rightarrow \mathbf{0} ; \mathbf{inr} (A, e) \rightarrow A\} x \rightarrow X \quad (2.18)$$

which simplifies to

$$P_S(X) = (R \times (\mathbf{0} \rightarrow X)) + (\sum_{A:\mathcal{U}} E A \times (A \rightarrow X)) \quad (2.19)$$

and further

$$P_S(X) = R + \sum_{A:\mathcal{U}} E A \times (A \rightarrow X) \quad (2.20)$$

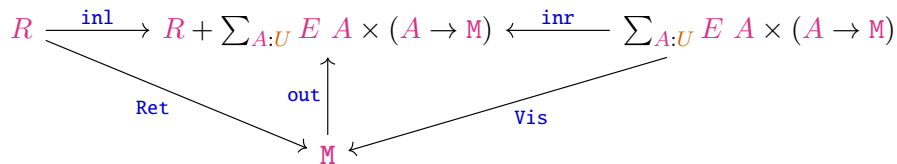


Figure 2.3: TODO: ???

Again we can define **Ret** and **Vis** using the **in** functor.

2.2.3 ITrees

Now we should have all the knowledge needed to make ITrees using **M**-types. We define ITrees by the container:

$$S = \left(\mathbf{1} + R + \sum_{A:\mathcal{U}} (E A) \ , \ \lambda \{ \text{inl} (\text{inl } _) \rightarrow \mathbf{1} ; \text{inl} (\text{inr } _) \rightarrow \mathbf{0} ; \text{inr}(A, _) \rightarrow A \} \right) \quad (2.21)$$

Then the (reduced) polynomial functor becomes

$$P_S(X) = X + R + \sum_{A:\mathcal{U}} ((E A) \times (A \rightarrow X)) \quad (2.22)$$

Giving us the diagram

$$\begin{array}{c}
 R \xrightarrow{\text{inl} \circ \text{inr}} \mathbf{M} + R + \sum_{A:\mathcal{U}} (E A \times (A \rightarrow \mathbf{M})) \xleftarrow{\text{inr}} \sum_{A:\mathcal{U}} E A \times (A \rightarrow \mathbf{M}) \\
 \searrow \text{Ret} \quad \swarrow \text{inl} \circ \text{inl} \quad \swarrow \text{out} \quad \swarrow \text{Vis} \\
 \mathbf{M} \xrightarrow{\text{Tau}} \mathbf{M}
 \end{array}$$

Figure 2.4: TODO: ???

2.3 Coinduction principle of M-types

We can now construct a bisimulation: forall coalgebras $C-\gamma : \text{Coalg}_S$, if we have a relation $\mathcal{R} : C \rightarrow C \rightarrow \mathcal{U}$, and a type $\overline{\mathcal{R}} = \sum_{a:C} \sum_{b:C} \mathcal{R} a b$, such that $\overline{\mathcal{R}}$ and $\alpha_{\overline{\mathcal{R}}} : \overline{\mathcal{R}} \rightarrow P(\overline{\mathcal{R}})$ makes a P-coalgebra $\overline{\mathcal{R}}-\alpha_{\overline{\mathcal{R}}} : \text{Coalg}_S$, such that the following diagram commutes (where \Rightarrow are P-coalgebra morphisms).

$$C-\gamma \xleftarrow{\pi_1 \overline{\mathcal{R}}} \overline{\mathcal{R}}-\alpha_{\overline{\mathcal{R}}} \xrightarrow{\pi_2 \overline{\mathcal{R}}} C-\gamma$$

Furthermore for any bisimulation over a final P-coalgebra $\mathbf{M}-\text{out} : \text{Coalg}_S$ we have the following diagram,

$$\mathbf{M}-\text{out} \xleftarrow{\pi_1 \overline{\mathcal{R}}} \overline{\mathcal{R}}-\alpha_{\overline{\mathcal{R}}} \xrightarrow{\pi_2 \overline{\mathcal{R}}} \mathbf{M}-\text{out}$$

where $\pi_1 \overline{\mathcal{R}} = ! = \pi_2 \overline{\mathcal{R}}$, which means given $r : \mathcal{R}(m, m')$ we get $m = \pi_1 \overline{\mathcal{R}}(m, m', r) = \pi_2 \overline{\mathcal{R}}(m, m', r) = m'$.

2.3.1 Bisimulation of ITrees

We define our bisimulation coalgebra from the strong bisimulation relation \mathcal{R} , defined by the following rules.

$$\frac{a, b : R \quad a \equiv_R b}{\text{Ret } a \cong \text{Ret } b} \text{EqRet} \quad (2.23)$$

$$\frac{t, u : \text{itree } E \ R \ t \cong u}{\text{Tau } t \cong \text{Tau } u} \text{EqTau} \quad (2.24)$$

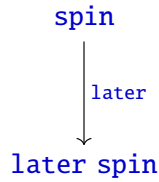
$$\frac{A : \mathcal{U} \quad e : E \ A \quad k_1, k_2 : A \rightarrow \text{itree } E \ R \ t \cong u}{\text{Vis } e \ k_1 \cong \text{Tau } e \ k_2} \text{EqVis} \quad (2.25)$$

Now we just need to define $\alpha_{\overline{R}}$. Now we have a bisimulation relation, which is equivalent to equality, using what we showed in the previous section.

define
the $\alpha_{\overline{R}}$
function

2.4 Examples of fixed points

We want to define `spin`, as being the fixed point `spin = later spin`, so that is again a final coalgebra, but of a M-type (which is a final coalgebra)



Since it is final, it also must be unique, meaning that there is just one program that spins forever, without returning a value, meaning every other program must return a value. If we just

2.5 Quotiented M-types

We can define a quotient of the polynomial functor P as a functor F , such that the following diagram commutes

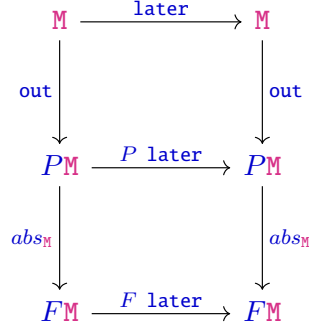
$$\begin{array}{ccc} P\alpha & \xrightarrow{Pf} & P\beta \\ \downarrow \text{abs}_\alpha & & \downarrow \text{abs}_\beta \\ F\alpha & \xrightarrow{Ff} & F\beta \end{array}$$

for all $f : \alpha \rightarrow \beta$ and α, β . This is a naturality condition for abs . As an example lets look at the quotient of the delay monad by the equality $\forall a : \mathbf{M}, \text{later } a \equiv a$. We get the diagram

Such that the (co?)limit for $F \mathbf{M}$ is $\mathbf{1} + R$, meaning

$$F \text{ later} \circ \text{abs}_M \circ \text{out} \equiv \text{abs}_M \circ \text{out} \circ \text{later} \quad (2.26)$$

$$F \text{ later} \equiv \text{later}^{-1} \quad (2.27)$$



$$\text{later} = \text{in} \circ \text{inr} \quad (2.28)$$

$$\text{out} \circ \text{later} = \text{inr} \quad (2.29)$$

$$= P \text{ later} \circ \text{out} \quad (2.30)$$

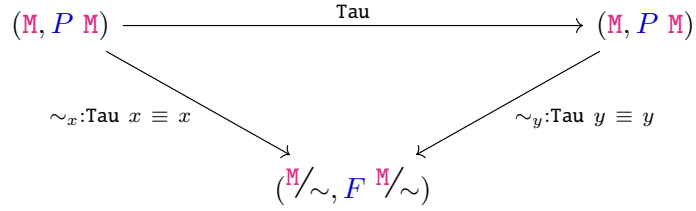
$$\text{abs}_M \circ \text{out} \circ \text{later} = \text{abs}_M \circ \text{inr} \quad (2.31)$$

$$= \text{abs}_M \circ P \text{ later} \circ \text{out} \quad (2.32)$$

$$= F \text{ later} \circ \text{abs}_M \circ \text{out} \quad (2.33)$$

$$(2.34)$$

Given the container of the final P-coalgebra for some container \mathbf{S} , we can construct the diagram:



Such that $\sim_x \equiv \sim_y \circ \text{Tau}$, where $P\mathbf{M}/\sim \equiv F(\mathbf{M}/\sim)$

Chapter 3

Conclusion

conclude on the problem statement from the introduction

Bibliography

- [1] Amin Timany and Matthieu Sozeau. Cumulative inductive types in coq. *LIPICs: Leibniz International Proceedings in Informatics*, 2018.

Appendix A

The Technical Details

