# (Q)$\mathtt{M}$-types and Coinduction in HoTT / CTT

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# Abstract

We present a construction of `M`-types from containers in a cubical type theory (CTT), We show how the containers construct a coalgebra, for which we can define a coinduction principle, making strong bisimulation imply equality. We then show constructions of `M`-types, and how they can be quotiented to construct what we call `QM`-types. The problem with `QM`-types is that in general assuming that we can lift function types of equivalence classes is equivalent to the axiom of choice [3], but this can be solved by defining the quotienting relation and the type at the same time as a quotient inductive-inductive type (QIIT), which assuming the axiom of (countable) choice, is equal to the `QM`-type. We conclude with some examples of how to use `M`-types and some properties. All work is formalized in Cubical Agda, and the work on defining `M`-types has been accepted to the Cubical Agda github repository.

# Resumé

in Danish...

# Acknowledgments

I would like to thank my supervisor Bas Spitters for some enlightening discussions and quick responses and for taking the time to do weekly meetings regarding this masters thesis. I would also like to thank my fellow students, who made this challenging but fun time less lonely even in the time of the Corona pandemic.

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*Aarhus, May 29, 2020.*

# Contents

# Chapter 1

# Introduction

motivate and explain the problem to be addressed

get your bibtex entries from `https://dblp.org/`

when ... a constructor for a type $A$ is a function, that takes some arguments, and returns an element $a : A$, dually a destructor of $A$ will return something given an element $a : A$. ... we can define a type inductively from a set of constructors, for example the natural numbers $\mathbb{N}$, which are defined by the two constructors

$$\frac{}{0 : \mathbb{N}} \quad (1.1) \qquad\qquad \frac{n : \mathbb{N}}{\texttt{succ}\ n : \mathbb{N}} \quad (1.2)$$

with an equivalence relation $\sim_{\mathbb{N}}$ defined inductively, meaning we follow the structure of defined by the constructors

$$\frac{}{0 = 0}\ \sim_0 \quad (1.3) \qquad\qquad \frac{n \sim_{\mathbb{N}} m}{\texttt{succ}\ n \sim_{\mathbb{N}} \texttt{succ}\ m}\ \sim_{\texttt{succ}} \quad (1.4)$$

This relation gives equality, so if $a \sim_{\mathbb{N}} b$ then $a \equiv b$. Likewise there is coinductive construction, where the focus is on the destructors instead of the constructors, an example is streams, representing an infinite sequence of elements, that can be defined from the two destructors head ($\texttt{hd}$) and tail ($\texttt{tl}$), where head represents the first element, and tail represents the rest of the sequence. The inference rules are given as

$$\frac{s : \texttt{stream}\ A}{\texttt{hd}\ s : A} \quad (1.5) \qquad\qquad \frac{s : \texttt{stream}\ A}{\texttt{tl}\ s : \texttt{stream}\ A} \quad (1.6)$$

we can again define an equivalence relation $\sim_{\texttt{stream}}$, but this time coinductively, focusing on the structure of the destructors instead

$$\frac{\texttt{hd}\ s = \texttt{hd}\ t \quad \texttt{tl}\ s \sim_{\texttt{stream}} \texttt{tl}\ t}{s \sim_{\texttt{stream}} t} \tag{1.7}$$

This relation does not give an equality in MLTT, we just get bisimilarity meaning elements "behave" the same, but they are not equivalent. However one of the advantages of using homotopy type theory is that bisimilarity becomes equality. We will use this advantage to get the "correct" for of equality for some example of coinductive types. ... An algebra is an operator `F` with some closure relation . ... inductive types are given as the initial algebra for some functor, this can be formalized as `W`-types, dually the coinductive types that we are interested in, can be formalized as the final coalgebra for some functor. We will be looking at how to define some coinductive types, as `M`-types, and define some bisimilarity relations for these types, showing we get equality when using homotopy type theory. We will then introduce weaker notion of bisimilarity, that does not yield equality, but can be used to construct a new type, by quotienting with the relation, giving us a type where the relation gives equality. ... When working with quotiented coinductive types. ... We define the quotiented delay monad $\texttt{Delay}/\sim_{\texttt{weak}}$, and want to show that we can construct a partiality monad from this construction. A problem with the partiality operation $D(-)/\sim_D$ is that countable choice is needed to show that it is a monad, however using QIIT types we can get around this problem, furthermore we can show that assuming countable choice, these two constructions are equal.

...

cite something

...

final coalgebra is scary words, find something more "noob" friendly

### 1.0.1 A Brief Overview of the Thesis

The rest of this chapter will introduce the some background theory and notation used in the rest of the thesis. In the second chapter we construct `M`-types from containers, and define a coinduction principle for the M-types. In the third chapter we give some example constructions of `M`-types. In the fourth chapter we introduce quotiented `M`-types (QM-types), and show equalities between these and quotient inductive-inductive types (QIITs), we show the construction of the partiality monad as an example. In the fifth and sixth chapter we go through various applications of the theory developed in the first couple chapters. Finally we conclude by discussion future research and improvements.

describe quotienting and its usefullness by a couple examples

...

## 1.1 Background Theory

We start by giving some background theory / history on cubical type theory and summarize important concepts used in the rest of this thesis.

What is a monad?

...

Should a description of Set Theory be included

We will be using **type theory** as the basis for mathematics. In type theory every term $x$ is an element of some type $A$, written $x : A$. The idea in type theory is that propositions are types, so proofs boils down to showing that there exists an element of some type representing a proposition. Specifically proofs of equality becomes construction of an element of an equality type. The type theory we are working in is inspired by **Martin Löf Type Theory (MLTT)** / Intuitionistic type Theory (ITT), which is designed on the principles of mathematical constructivism, where any existence proof must contain a witness. Meaning a proof of existence, can be converted into an algorithm that finds the element making the statement true. MLTT is built from the three finite types $\mathbf{0}$, $\mathbf{1}$ and $\mathbf{2}$, and type constructors $\Sigma$, $\pi$ and $=$. There is only a single way to make terms of $=$-type, and that is $\texttt{refl} : \prod_{a:A}(a = a)$. . We will be working in a cubical type theory, where the univalence axiom holds. The **univalence axiom** says that equality is equivalent to equivalence

inductive types, universes and judgements

$$(A = B) \simeq (A \simeq B) \tag{1.8}$$

meaning if two objects are equivalent, then there is an equality between them, such that we can replace one by the other.

extensional vs intensional https://ncatlab.org/nlab/

The type theory we will be working in is called **homotopy type theory (HoTT)** [7], which is an intensional dependent type theory (built on MLTT) with the univalence axiom and higher inductive types. In HoTT the idenitity types form path spaces, so proofs of identity are not just `refl`, as for MLTT. Types are seen as "spaces", and we think of $a : A$ as $a$ being a point in the space $A$, similarly functions are regarded as continuous maps from one space to another [6]. The use of a type theory with the univalence axiom as a foundations for mathematics, is called **Univalent Foundations (UF)**, which HoTT is. We will be working in a **cubical type theory (CTT)** [4] where the univalence axiom, is not just an axiom, but a statement that can actually be proven, meaning we can reduce the use of the univalence axiom, making it easier to do proofs involving the univalence axiom [5]. The reason for the name cubical type theory, is because composition is defined by square, that is given three sides of a square we get the last one, see Figure 1.1. Most of

should this be left out?

example of what computational "axioms" mean

$$\begin{tikzcd} A \arrow[r, dashed, "p \cdot q \cdot r"] & B \\ C \arrow[u, "p^{-1}"] \arrow[r, "q"'] & D \arrow[u, "r"'] \end{tikzcd}$$

Figure 1.1: Composition square

the work in this thesis has been formalized in the proof assistant / programming languge Cubical Agda. A proof assistant helps with verifying proofs, while making the process of making proofs interactive. **Cubical Agda** is an implementation of a cubical type theory done by extending the proof assistant Agda. One of the main additions is the interval and path types. The interval can be thought of as elements in $[0, 1]$. When working with the interval, we can only access the left and right endpoint `i0` and `i1` or some unspecified point in the middle $i$, modeling the intuition of a continuous interval. Cubical agda also generalizes transporting, given a type line $\texttt{A} : \mathbb{I} \to \mathcal{U}$, and the endpoint `A i0` you get a line from `A i0` to `A i1` [8].

Axioms of cubical Agda

Homotopy type theory is proof relevant, which means that a there might be multiple proofs of one statement, and these proofs might not be interchangeable (equal). The reason is that types in HoTT have a H-level, describing how equality behaves. We start from (-2) with contractiable types, meaning there is an element which all other element are equal to. Then there is (-1)-types which are mere propositions or hProp, where all elements of the type are equal, but there might not be any. If the type is inhabited, then we say the proposition is true. The 0-types are the hSets, where all equalities between two elements $x, y$ are equal. For 1-types (1-groupoids) we get equalities of equalities are equal, and then so on for homotopy $n$-types. Any $n$-type is also a $n+1$-type, but with trivial equalities at the $n+1$ level. If we don't want to do proof relevant mathematics we can do propositional truncation, converting types to $-1$-type, meaning we ignore the difference in proofs by just look at whether a type is inhabited or not. However doing this we loose some of the reasoning power of HoTT. One of the tools we get using the full power of HoTT is **Higher order inductive types (HITs)**, where we define types with point constructors and equality constructors, an example is the propositional truncation we just described, another useful example is set truncated quotients.

Some better (intuitive) description of the interval type!!

write up constructors for propositional truncation

write up constructors for

A problem that arise when using a constructive type theory is the axiom of choice and the law of excluded middle, which does not have a computational interpretation, so to maintain the computational aspects of HoTT and CTT, we try to not use these axioms [7, Introduction].

W-types, Induction , M-types, Coinduction..., see introduction?

Coinduction is the dual concept (in a categorical manner) of induction. The induction principle is an equivalence principle for congruent elements in an initial algebra.

Cubical Agda has a hierarchy of universes

Universes

#### 1.1.0.1 UIP

**Path type** We add a type $\mathbb{I}$, which is defined to be a free de Morgan algebra on a discrete infinite set of names . The elements of $\mathbb{I}$ can be described by the grammar

cite: https://arxiv.org/pdf/1611.02108.pdf

$$r, s ::= 0 \mid 1 \mid i \mid 1 - r \mid r \wedge s \mid r \vee s \tag{1.9}$$

The set $\mathbb{I}$ has decidable equality. The elements in $\mathbb{I}$ can be thought as formal representations of elements in the unit interval $[0, 1]$. There is a special substitution with `i0` and `i1` being the endpoints of $[0, 1]$.

### 1.1.1 Problems with using AC and LEM

These axioms does not have a computational interpretation, so to maintain the computational aspects of HoTT and CTT, we try to not use these axioms. [7, Introduction]

### 1.1.2 Universes ??

## 1.2 Notation

Most of the work is formalized in the Cubical Agda proof assistant. The following is the notation / fonts used to denote specific definitions / concepts

- Universe $\mathcal{U}_i$ or $\mathcal{U}$
- Type $A : \mathcal{U}$
- A type former or dependent type $\mathtt{B} : A \to \mathcal{U}$
- A term $x : A$ or for constants $\mathtt{c} : A$
- A function $\mathtt{f} : A \to C$
- A constructor $\mathtt{f} : A \to C$
- A destructor $\mathtt{f} : A \to C$
- A path $p : A \equiv C$, heterogeneous paths are denotes $\equiv_p$ or if the path is clear from context $\equiv_*$.
- A relation $\mathtt{R} : A \to A \to \mathcal{U}$ with notation $x \; \mathtt{R} \; y$.

better description, not always a function

better description, not always a function

- The unit type is $\mathbf{1}$ while the empty type is $\mathbf{0}$.
- A functor $\mathtt{P}$
- A container is denoted as $S$ or $(A \mathbin{,} \mathtt{B})$
- A coalgebra $C$-$\gamma$
- We denote the function giving the first and second projection of a dependent pair by $\pi_1$ and $\pi_2$.

Furthermore we define some useful notation for casing on natural numbers.

**Definition 1.2.1.**

$$\rangle\!\rangle \ x \ , \ \mathtt{f} \ \langle\!\langle = \lambda\, n, \begin{cases} x & n = 0 \\ \mathtt{f} \ m & n = m + 1 \end{cases} \tag{1.10}$$

# Chapter 2

# M-types

In this chapter we will introduce containers (aka. signatures), and use them to construct M-types and operations `in` and `out` on the M-types (Theorem 2.1.8). We conclude the chapter by defining a coinduction principle for M-types [2].

Some formalization of coalgebras is missing?

## 2.1 Containers / Signatures

**Definition 2.1.1.** A Container (or signature) is a dependent pair $S = (A , \mathtt{B})$ for the types $A : \mathcal{U}$ and $\mathtt{B} : A \to \mathcal{U}$.

**Definition 2.1.2.** A polynomial functor over the container $S = (A , \mathtt{B})$ is defined for types as

$$\begin{aligned} &\mathtt{P}_S : \mathcal{U} \to \mathcal{U} \\ &\mathtt{P}_S(X) = \sum_{a:A} \mathtt{B}(a) \to X \end{aligned} \tag{2.1}$$

and for a function $\mathtt{f} : X \to Y$ as

$$\begin{aligned} &\mathtt{P}_S\mathtt{f} : \mathtt{P}_S X \to \mathtt{P}_S Y \\ &\mathtt{P}_S\mathtt{f}(a, \mathtt{g}) = (a, \mathtt{f} \circ \mathtt{g}). \end{aligned} \tag{2.2}$$

**Example 1.** The polynomial functor for streams over the type $A$ is defined by the container $S = (A , \lambda\_, \mathbf{1})$, we get

$$\mathtt{P}_S(X) = \sum_{a:A} \mathbf{1} \to X. \tag{2.3}$$

Since we are working in a logic with exponentials, we get $\mathbf{1} \to X \equiv X^{\mathbf{1}} \equiv X$. Furthermore $\mathbf{1}$ and $X$ does not depend on $A$, so (2.3) is equivalent to

$$\mathtt{P}_S(X) = A \times X. \tag{2.4}$$

We now construct the $\mathtt{P}_S$-coalgebra for a polynomial functor $\mathtt{P}_S$.

**Definition 2.1.3.** A $\mathtt{P}_S$-coalgebra is defined as

$$\mathtt{Coalg}_S = \sum_{C:\mathcal{U}} C \to \mathtt{P}_S C. \tag{2.5}$$

We denote a $\mathtt{P}_S$-coalgebra given by $C$ and $\gamma$ as $C$-$\gamma$. Coalgebra morphisms are defined as

$$\begin{aligned} \cdot \Rightarrow \cdot : \mathtt{Coalg}_S \to \mathtt{Coalg}_S \\ C\text{-}\gamma \Rightarrow D\text{-}\delta = \sum_{\mathtt{f}:C \to D} \delta \circ \mathtt{f} = \mathtt{Pf} \circ \gamma \end{aligned} \tag{2.6}$$

We can now define $\mathtt{M}$-types.

**Definition 2.1.4.** Given a container $S$, we define $\mathtt{M}$-types as the type, making the coalgebra given by $\mathtt{M}_S$ and $\mathtt{out} : \mathtt{M}_S \to \mathtt{P}_S(\mathtt{M}_S)$ fulfill the property

$$\mathtt{Final}_S := \sum_{(X\text{-}\rho:\mathtt{Coalg}_S)} \prod_{(C\text{-}\gamma:\mathtt{Coalg}_S)} \mathtt{isContr}\ (C\text{-}\gamma \Rightarrow X\text{-}\rho). \tag{2.7}$$

That is $\prod_{(C\text{-}\gamma:\mathtt{Coalg}_S)} \mathtt{isContr}(C\text{-}\gamma \Rightarrow \mathtt{M}_S\text{-}\mathtt{out})$. We denote the $\mathtt{M}$-type as $\mathtt{M}_{(A\,,\,\mathtt{B})}$ or $\mathtt{M}_S$ or just $\mathtt{M}$ when the Container is clear from the context.

Continuing our example we now construct streams as an $\mathtt{M}$-type.

**Example 2.** We define streams over the type $A$ as the $\mathtt{M}$-type over the container $(A\,,\lambda\_,\,\mathbf{1})$. If we apply the polynomial functor to the $\mathtt{M}$-type, then we get $\mathtt{P}_{(A\,,\,\lambda\_,\,\mathbf{1})}\mathtt{M} = A \times \mathtt{M}_{(A\,,\,\lambda\_,\mathbf{1})}$, illustrated in Figure 2.1. We will that $\mathtt{out}$ is an isomorphism with inverse $\mathtt{in} : \mathtt{P}_S(\mathtt{M}) \to \mathtt{M}$ later in this section.

$$\begin{tikzcd} A & A \times \mathtt{M}_{(A\,,\,\lambda\_,\mathbf{1})} \arrow[l, "\pi_1"'] \arrow[r, "\pi_2"] & \mathtt{M}_{(A\,,\,\lambda\_,\mathbf{1})} \\ & \mathtt{M}_{(A\,,\,\lambda\_,\mathbf{1})} \arrow[ul, "\mathtt{hd}"] \arrow[u, "\mathtt{out}"'] \arrow[ur, "\mathtt{tl}"'] & \end{tikzcd}$$

Figure 2.1: $\mathtt{M}$-types of streams

We now have a semantic for the rules, we would expect for streams, if we let $\mathtt{cons} = \mathtt{in}$ and $\mathtt{stream}\ A = \mathtt{M}_{(A\,,\,\lambda\_,\mathbf{1})}$,

$$\frac{A : \mathcal{U} \quad s : \mathtt{stream}\ A}{\mathtt{hd}\ s : A}\ \mathtt{E}_{\mathtt{hd}} \tag{2.8}$$

$$\frac{A : \mathcal{U} \quad s : \mathtt{stream}\ A}{\mathtt{tl}\ s : \mathtt{stream}\ A}\ \mathtt{E}_{\mathtt{tl}} \tag{2.9}$$

$$\frac{A : \mathcal{U} \quad x : A \quad xs : \mathtt{stream}\ A}{\mathtt{cons}\ x\ xs : \mathtt{stream}\ A}\ \mathtt{I}_{\mathtt{cons}} \tag{2.10}$$

or more precisely $\mathtt{hd} = \pi_1 \circ \mathtt{out}$ and $\mathtt{tl} = \pi_2 \circ \mathtt{out}$.

**Definition 2.1.5.** We define a chain as a family of morphisms $\pi_{(n)} : X_{n+1} \to X_n$, over a family of types $X_n$. See Figure 2.2.

$$X_0 \xleftarrow[\pi_{(0)}]{} X_1 \xleftarrow[\pi_{(1)}]{} \cdots \xleftarrow[\pi_{(n-1)}]{} X_n \xleftarrow[\pi_{(n)}]{} X_{n+1} \xleftarrow[\pi_{(n+1)}]{} \cdots$$

Figure 2.2: Chain of types / functions

$$
\begin{tikzcd}
X_0 & X_1 \arrow[l, "\pi_{(0)}"'] & \cdots \arrow[l, "\pi_{(1)}"'] & X_n \arrow[l, "\pi_{(n-1)}"'] & X_{n+1} \arrow[l, "\pi_{(n)}"'] & \cdots \arrow[l, "\pi_{(n+1)}"'] \\
& & C \arrow[ull, "\mathtt{f}_0"] \arrow[ul, "\mathtt{f}_1"'] \arrow[u, phantom, "\ldots"] \arrow[ur, "\mathtt{f}_n"] \arrow[urr, "\mathtt{f}_{n+1}"'] & & \ldots &
\end{tikzcd}
$$

Figure 2.3: Cone

**Lemma 2.1.6.** *For all coalgebras $C$-$\gamma$ for the container $S$, we get $C \to \mathtt{M}_S \equiv \mathtt{Cone}_{C\text{-}\gamma}$, where $\mathtt{Cone} = \sum_{(\mathtt{f}:\prod_{(n:\mathbb{N})} C \to X_n)} \prod_{(n:\mathbb{N})} \pi_{(n)} \circ (\mathtt{f}_{(n+1)}) \equiv f_n$ illustrated in Figure 2.3.*

*Proof.* We define an isomorphism from $C \to \mathtt{M}_S$ to $\mathtt{Cone}_{C\text{-}\gamma}$

$$\mathtt{fun}_{collapse}\ \mathtt{f} = (\lambda\, n\, z, \pi_1\ (\mathtt{f}\ z)\ n), (\lambda\, n\ i\ a,\ \pi_2\ (\mathtt{f}\ a)\ n\ i) \tag{2.11}$$

$$\mathtt{inv}_{collapse}\ (\mathtt{u}, \mathtt{q})\ z = (\lambda\, n,\ \mathtt{u}\ n\ z), (\lambda\, n\, i,\ \mathtt{q}\ n\ i\ z) \tag{2.12}$$

$$\mathtt{rinv}_{collapse}\ (\mathtt{u}, \mathtt{q}) = \mathtt{refl}_{(\mathtt{u},\mathtt{q})} \tag{2.13}$$

$$\mathtt{linv}_{collapse}\ \mathtt{f} = \mathtt{refl}_{\mathtt{f}} \tag{2.14}$$

□

**Lemma 2.1.7.** *Given $\ell : \prod_{(n:\mathbb{N})}(X_n \to X_{n+1})$ and $y : \sum_{(x:\prod_{(n:\mathbb{N})} X_n)} x_{n+1} \equiv \ell_n\ x_n$ the chain collapses as the equality $\mathcal{L} \equiv X_0$.*

*Proof.* We define this collapse by the isomorphism

$$\mathtt{fun}_{\mathcal{L}collapse}\ (x, r) = x_0 \tag{2.15}$$

$$\mathtt{inv}_{\mathcal{L}collapse}\ x_0 = (\lambda\, n,\ \ell^{(n)}\ x_0)\ ,\ (\lambda\, n,\ \mathtt{refl}_{(\ell^{(n+1)}\ x_0)}) \tag{2.16}$$

$$\mathtt{rinv}_{\mathcal{L}collapse}\ x_0 = \mathtt{refl}_{x_0} \tag{2.17}$$

where $\ell^{(n)} = \ell_n \circ \ell_{n-1} \circ \cdots \circ \ell_1 \circ \ell_0$. To define $\mathtt{linv}_{\mathcal{L}collapse}\ (x, r)$, we first define a fiber $(X, z, \ell)$ over $\mathbb{N}$ given some $z : X_0$. Then any element of the type $\sum_{(x:\prod_{(n:\mathbb{N})} X_n)} x_{n+1} \equiv \ell_n\ x_n$ is equal to a section over the fiber we defined. This means $y$ is equal to a section. Since the sections are defined over $\mathbb{N}$, which is an initial algebra for the functor $\mathtt{G}Y = \mathbf{1} + Y$, we get that sections are contractible, meaning $y \equiv \mathtt{inv}_{\mathcal{L}collapse}(\mathtt{fun}_{\mathcal{L}collapse}\ y)$, since both are equal to sections over $\mathbb{N}$. □

We can now define the construction of $\mathtt{in}$ and $\mathtt{out}$.

**Theorem 2.1.8.** *Given the container $(A, \mathtt{B})$ we define the equality*

$$shift : \mathcal{L} \equiv \mathtt{P}\mathcal{L} \tag{2.18}$$

*where $\mathtt{P}\mathcal{L}$ is the limit of a shifted sequence. Then*

$$\mathtt{in} = \mathtt{transport}\ shift \tag{2.19}$$

$$\mathtt{out} = \mathtt{transport}\ (shift^{-1}). \tag{2.20}$$

*Proof.* The proof is done using the two helper lemmas

$$\alpha : \mathcal{L}^{\mathtt{P}} \equiv \mathtt{P}\mathcal{L} \tag{2.21}$$

$$\mathcal{L}unique : \mathcal{L} \equiv \mathcal{L}^{\mathtt{P}} \tag{2.22}$$

We define $\mathcal{L}unique$ by the ismorphism

$$\mathtt{fun}_{\mathcal{L}unique}\ (\mathtt{a}, \mathtt{b}) = \rrparenthesis\ \star\ ,\ \mathtt{a}\ \llparenthesis , \rrparenthesis\ \mathtt{refl}_{\star}\ ,\ \mathtt{b}\ \llparenthesis \tag{2.23}$$

$$\mathtt{inv}_{\mathcal{L}unique}\ (\mathtt{a}, \mathtt{b}) = \mathtt{a} \circ \mathtt{succ}\ ,\ \mathtt{b} \circ \mathtt{succ} \tag{2.24}$$

$$\mathtt{rinv}_{\mathcal{L}unique}\ (\mathtt{a}, \mathtt{b}) = \mathtt{refl}_{(\mathtt{a},\mathtt{b})} \tag{2.25}$$

$$\mathtt{linv}_{\mathcal{L}unique}\ (\mathtt{a}, \mathtt{b}) = \mathtt{refl}_{(\mathtt{a},\mathtt{b})} \tag{2.26}$$

The definition of $\alpha$ is then,

$$\begin{aligned}
\mathcal{L}^{\mathtt{P}} &\equiv \sum_{(x : \prod_{(n:\mathbb{N})} \sum_{(a:A)} \mathtt{B}\, a \to X_n)} \prod_{(n:\mathbb{N})} \pi_{(n+1)}\ x_{n+1} \equiv x_n && (2.27) \\
&\equiv \sum_{(x : \sum_{(a : \prod_{(n:\mathbb{N})} A)} \prod_{(n:\mathbb{N})} a_{n+1} \equiv a_n)} \sum_{(\mathtt{u} : \prod_{(n:\mathbb{N})} \mathtt{B}\, (\pi_1\, x)_n \to X_n)} \prod_{(n:\mathbb{N})} \pi_{(n)} \circ \mathtt{u}_{n+1} \equiv_{*} \mathtt{u}_n && (2.28) \\
&\equiv \sum_{(a:A)} \sum_{(\mathtt{u} : \prod_{(n:\mathbb{N})} \mathtt{B}\, a \to X_n)} \prod_{(n:\mathbb{N})} \pi_{(n)} \circ \mathtt{u}_{n+1} \equiv \mathtt{u}_n && (2.29) \\
&\equiv \sum_{(a:A)} \mathtt{B}\ a \to \mathcal{L} && (2.30) \\
&\equiv \mathtt{P}\mathcal{L} && (2.31)
\end{aligned}$$

To collapse $\sum_{(a:\prod_{(n:\mathbb{N})} A)} \prod_{(n:\mathbb{N})} a_{n+1} \equiv a_n$ to $A$ between (2.28) and (2.29) we use Lemma 2.1.7 . We use Lemma 2.1.6 for the equality between (2.29) and (2.30). The rest of the equalities are trivial. The definition of $shift$ is

$$shift = \alpha^{-1} \cdot \mathcal{L}unique. \tag{2.32}$$

We furthermore get the definitions $\mathtt{in} = \mathtt{transport}\ shift$ and $\mathtt{out} = \mathtt{transport}\ (shift^{-1})$, since $\mathtt{in}$ and $\mathtt{out}$ are part of an equality relation $shift$, they are both surjective and embeddings. $\square$

is surjectivity and embedding important here? Describe this where relevant instead!

## 2.2 Coinduction Principle for M-types

We can now construct a coinduction principle given a bisimulation relation.

**Definition 2.2.1.** For all coalgebras $C\text{-}\gamma : \mathtt{Coalg}_S$, given a relation $\mathcal{R} : C \to C \to \mathcal{U}$ and a type $\overline{\mathcal{R}} = \sum_{(a:C)} \sum_{(b:C)} a\ \mathcal{R}\ b$, such that $\overline{\mathcal{R}}$ and $\alpha_{\mathcal{R}} : \overline{\mathcal{R}} \to \mathtt{P}_S(\overline{\mathcal{R}})$ forms a P-coalgebra $\overline{\mathcal{R}}\text{-}\alpha_{\mathcal{R}} : \mathtt{Coalg}_S$, making the diagram in Figure 2.4 commute ($\Longrightarrow$ represents P-coalgebra morphisms).

What does commute mean here?

$$C\text{-}\gamma \xLeftarrow{\ \pi_1{}^{\overline{\mathcal{R}}}\ } \overline{\mathcal{R}}\text{-}\alpha_{\mathcal{R}} \xRightarrow{\ \pi_2{}^{\overline{\mathcal{R}}}\ } C\text{-}\gamma$$

Figure 2.4: Bisimulation for a coalgebra

$$\begin{tikzcd} \texttt{M-out} & \overline{\mathcal{R}}\text{-}\alpha_{\mathcal{R}} \arrow[l, Rightarrow, "\pi_1{}^{\overline{\mathcal{R}}}"'] \arrow[r, Rightarrow, "\pi_2{}^{\overline{\mathcal{R}}}"] & \texttt{M-out} \end{tikzcd}$$

Figure 2.5: Bisimulation principle for final coalgebra

**Definition 2.2.2** (Coinduction principle)**.** Given a relation $\mathcal{R}$, that is part of a bisimulation over a final P-coalgebra $\texttt{M-out} : \texttt{Coalg}_S$ we get the diagram in Figure 2.5, where $\pi_1^{\overline{\mathcal{R}}} = ! = \pi_2^{\overline{\mathcal{R}}}$ where ! is the unique mapping property (UMP) out of the final coalgebra. Given $r : m\ \mathcal{R}\ m'$ we get the equation

what is !

$$m = \pi_1^{\overline{\mathcal{R}}}(m, m', r) = \pi_2^{\overline{\mathcal{R}}}(m, m', r) = m'. \tag{2.33}$$

What is the consequence of this?

# Chapter 3

# Instantiation of M-types

In this section we show some examples of types that can be constructed as $M$-types, and show how their constructors can be defined. We then conclude the chapter with some general observation, and define some rules for how to construct `M`-types.

Is there anything else that is show for each M-type?

## 3.1 Stream Formalization using `M`-types

As described earlier, given a type $A$ we define the stream of that type as

$$\texttt{stream}\ A := \texttt{M}_{(A\,,\,\lambda\_,\mathbf{1})} \tag{3.1}$$

this is equal to an alternative definition of streams

complete this section

...

## 3.2 ITrees as `M`-types

Interaction trees (ITrees) [9] are used to model effectful behavior, where computations can interact with an external environment by events. ITrees are defined by the following constructors

$$\frac{r : R}{\texttt{Ret}\ r : \texttt{itree E}\ R}\ \texttt{I}_{\texttt{Ret}} \tag{3.2}$$

$$\frac{A : \mathcal{U} \quad a : \texttt{E}\ A \quad f : A \to \texttt{itree E}\ R}{\texttt{Vis}\ a\ f : \texttt{itree E}\ R}\ \texttt{I}_{\texttt{Vis}}. \tag{3.3}$$

$$\frac{t : \texttt{itree E}\ R}{\texttt{Tau}\ t : \texttt{itree E}\ R}\ \texttt{E}_{\texttt{Tau}}. \tag{3.4}$$

where $R$ is the type for returned values, while `E` is a dependent type for events representing external interactions.

### 3.2.1 Delay Monad

We start by looking at ITrees without the `Vis` constructor, this type is also know as the delay monad. It can be used to model delayed computations, either returning immediately given by the constructor `now` = `Ret`, or delayed some (possibly infinite) number of steps by the constructor `later` = `Tau`. We construct this type as an `M`-type.

**Definition 3.2.1.** The delay monad can be defined as the $\mathtt{M}$-type for the container

$$S = \left( R + \mathbf{1} \, , \begin{cases} \mathbf{0} & \mathtt{inl}\ r \\ \mathbf{1} & \mathtt{inr}\ \star \end{cases} \right) \tag{3.5}$$

The polynomial functor for this container is

$$\mathtt{P}_S(X) = \sum_{(x:R+\mathbf{1})} \begin{cases} \mathbf{0} & x = \mathtt{inl}\ r \\ \mathbf{1} & x = \mathtt{inr}\ \star \end{cases} \to X, \tag{3.6}$$

which we can simplify

$$\mathtt{P}_S(X) = R \times (\mathbf{0} \to X) + X. \tag{3.7}$$

We know that $(\mathbf{0} \to X) \equiv \mathbf{1}$, so we can simplify further to

$$\mathtt{P}_S(X) = X + R \tag{3.8}$$

meaning we get diagram in Figure 3.1. We can define the constructors `now` and `later` using `in`

$$\begin{tikzcd} R \arrow[r, "\mathtt{inl}"] \arrow[dr, "\mathtt{now}"'] & R + \mathtt{M} \arrow[d, "\mathtt{in}"] & \mathtt{M} \arrow[l, "\mathtt{inr}"'] \arrow[dl, "\mathtt{later}"] \\ & \mathtt{M} & \end{tikzcd}$$

Figure 3.1: Delay monad

function for $\mathtt{M}$-types, together with the injections `inl` and `inr`.

### 3.2.2 Tree

Now lets look at the example, where we remove the `Tau` constructor. This gives us a type of tree, with leaves given by `Ret`, and nodes given by `Vis` branching based on some type $A$, for an event $a : \mathtt{E}\ A$.

**Definition 3.2.2.** We can define $R$-valued $\mathtt{E}$-event trees as the $\mathtt{M}$-type defined by the container

$$S = \left( R + \sum_{(A:\mathcal{U})} \mathtt{E}\ A \, , \begin{cases} \mathbf{0} & \mathtt{inl}\ r \\ A & \mathtt{inr}\ (A, e) \end{cases} \right). \tag{3.9}$$

The polynomial functor for this container is

$$\mathtt{P}_S(X) = \sum_{(x:R+\sum_{(A:\mathcal{U})} \mathtt{E}\ A)} \begin{cases} \mathbf{0} & x = \mathtt{inl}\ r \\ A & x = \mathtt{inr}\ (A, e) \end{cases} \to X, \tag{3.10}$$

which simplifies to

$$\mathtt{P}_S(X) = (R \times (\mathbf{0} \to X)) + (\sum_{A:\mathcal{U}} \mathtt{E}\ A \times (A \to X)), \tag{3.11}$$

and further

$$\mathtt{P}_S(X) = R + \sum_{A:\mathcal{U}} \mathtt{E}\ A \times (A \to X). \tag{3.12}$$

We get the diagram in Figure 3.2 for the $\mathtt{P}$-coalgebra. Again we can define `Ret` and `Vis` using the

$$\begin{tikzcd}
R \arrow[r, "\texttt{inl}"] \arrow[dr, "\texttt{Ret}"'] & R + \sum_{A:U} \mathtt{E}\ A \times (A \to \mathtt{M}) \arrow[d, "\texttt{in}"] & \sum_{A:U} \mathtt{E}\ A \times (A \to \mathtt{M}) \arrow[l, "\texttt{inr}"'] \arrow[dl, "\texttt{Vis}"] \\
& \mathtt{M} &
\end{tikzcd}$$

Figure 3.2: Tree Constructors

`in` function.

### 3.2.3 ITrees

Get the correct equivalence for ITrees (Part of project description?)

Now we should have all the knowledge needed to make ITrees using $\mathtt{M}$-types.

**Definition 3.2.3.** We define the type of ITrees as the $\mathtt{M}$-type given by the container

$$S = \left( R + \mathbf{1} + \sum_{A:\mathcal{U}} (\mathtt{E}\ A)\, ,\ \begin{cases} \mathbf{0} & \mathtt{inl}\ r \\ \mathbf{1} & \mathtt{inl}\ (\mathtt{inl}\ \star) \\ A & \mathtt{inr}\ (\mathtt{inr}\ (A, e)) \end{cases} \right). \tag{3.13}$$

The (reduced) polynomial functor for this container is

$$\mathtt{P}_S(X) = R + X + \sum_{(A:\mathcal{U})} (\mathtt{E}\ A \times (A \to X)) \tag{3.14}$$

Giving us the diagram in Figure 3.3, from which the constructors of the type can be defined using `in`.

$$\begin{tikzcd}
R \arrow[r, "\texttt{inl} \circ \texttt{inr}"] \arrow[ddrr, "\texttt{Ret}"'] & \mathtt{M} + R + \sum_{(A:\mathcal{U})} (\mathtt{E}\ A \times (A \to \mathtt{M})) \arrow[ddr, "\texttt{in}"] & & \sum_{A:U} \mathtt{E}\ A \times (A \to \mathtt{M}) \arrow[ll, "\texttt{inr}"'] \arrow[ddl, "\texttt{Vis}"] \\
& & \mathtt{M} \arrow[ul, "\texttt{inl} \circ \texttt{inl}"'] \arrow[d, "\texttt{Tau}"'] & \\
& & \mathtt{M} &
\end{tikzcd}$$

Figure 3.3: ITree constructors

## 3.3 Automaton

An automaton is defined as a set of state $V$ and an alphabet $\alpha$ and a transition function $\delta : V \to \alpha \to V$. This gives us the diagram in Figure 3.4

$$\begin{tikzcd}
V \times (V \to \alpha \to V) \arrow[r] \arrow[dr, "build"'] & M & M \arrow[l] \arrow[dl, "step"] \\
& M \arrow[u] &
\end{tikzcd}$$

Figure 3.4: automaton

## 3.4 General rules for constructing $\texttt{M}$-types

We want to create a calculus for defining coinductive types as $\texttt{M}$-types. We would like to be able to define a type that has a given set of constructor / destructors rules. If we for example is given the rule

$$\frac{a : A}{\texttt{ret}\ a : T} \tag{3.15}$$

we get that it corresponds to the M-type for the container $(A, \lambda\_, \mathbf{0})$, while if we have something that produces an element of it self as

$$\frac{a : T}{\texttt{tl}\ a : T} \tag{3.16}$$

why give some arguments or illustation showing these reasonings

the container is $(\mathbf{1}, \lambda\_, \mathbf{1})$. If we want a type with both these rules, then we just take the disjoint union of the two containers

$$\left(A + \mathbf{1}, \begin{cases} \mathbf{0} & \texttt{inl}\ a \\ \mathbf{1} & \texttt{inr}\ \star \end{cases}\right) \tag{3.17}$$

which is the delay type. We can also define some more involved constructors, that build on the type itself

$$\frac{a : A \to T}{\texttt{node}\ a : T} \tag{3.18}$$

has container $(\mathbf{1}, A)$. We can types with a given destructor

$$\frac{a : T}{\texttt{hd}\ a : A} \tag{3.19}$$

has container $(A, \lambda\_, \mathbf{0})$, but this is the same as for $\texttt{ret}$, and we do not always want both. The difference is how they are added to other constructors / destructors. Destructors are easily added together, take for example $\texttt{hd}$ and $\texttt{tl}$, $(A, \lambda\_, \mathbf{1})$. In general adding containers $(A, \texttt{B})$ and $(C, \texttt{D})$ for two constructors together is done by

$$\left(A + C, \begin{cases} \texttt{B}\ a & \texttt{inl}\ a \\ \texttt{D}\ c & \texttt{inr}\ c \end{cases}\right) \tag{3.20}$$

Is this correct? Seems correct..

whereas adding containers for two destructors is done by

$$(A \times C\,, \lambda\,\_,\ \lambda\,(a, c),\ \mathtt{B}\ a + \mathtt{D}\ c) \tag{3.21}$$

however combining destructors and constructors is not as simple. Anything type $T$ that is defined using a record (except higher inductive types), will also be definable as an M-type. Given a record, which is a list of fields $f_1 : F_1, f_2 : F_2, \ldots, f_n : F_n$, we can construct the M-types by the container

$$(F_1 \times F_2 \times \cdots \times F_n\,, \lambda\,\_,\ \mathbf{0}) \tag{3.22}$$

where each destructor $\mathrm{d_n} : T \to F_n$ for the field $f_n$ will be defined as $\mathrm{d_n}\ t = \pi_n\ (\mathtt{out}\ t)$. However fields in a coinductive container may depend on previous defined fields, as given by the general list of fields $f_1 : F_1, f_2 : \mathtt{F}_2, \cdots, f_n : \mathtt{F}_n$, where each field depends on all the previous once, this can be defined by the container

$$\left(\sum_{(f_1 : F_1)} \sum_{(f_2 : \mathtt{F}_2)} \cdots \sum_{(f_{n-1} : \mathtt{F}_{n-1})} \mathtt{F}_n\,, \lambda\,\_,\ \mathbf{0}\right) \tag{3.23}$$

however, if any of the destructors/fields are non dependent, then the can be added as a product $(\times)$ instead of a dependent product $(\Sigma)$. Furthermore the fields may construct an element of the type of the record $T$, however anything after that field cannot on it, since it will break the strictness requirements of the record / coinductive type. As an example let $f_1$ be a type and $\mathtt{f}_2$ be the function with type $\mathtt{F}_2 = f_1 \to (f_1 \to A) \to \mathtt{M}$, which by currying is equal to $f_1 \times (f_1 \to A) \to \mathtt{M}$, we can then define by the container

$$\left(\sum_{(f_1 : \mathcal{U})} \left(\mathbf{1} \times \sum_{(f_3 : \mathtt{F}_3)} \cdots \sum_{(f_{n-1} : \mathtt{F}_{n-1})} \mathtt{F}_n\right)\,, \lambda\,(f_1, \star, f_3, \ldots), \mathtt{F}_2\right) \tag{3.24}$$

where $\mathtt{F}_2$ have been moved to the last part of the container, we can even leave out the "$\mathbf{1}\times$" from the container. The types of the field can also be dependent $\mathtt{F}_2 = (x : f_1) \to \mathtt{B}\ x \to \mathtt{M}$, but again by currying we can get $\mathtt{F}_2 : \sum_{(x:f_1)} \mathtt{B}\ x \to \mathtt{M}$ which is defined by the container

$$\left(\sum_{(f_1 : \mathcal{U})} \sum_{(f_3 : \mathtt{F}_3)} \cdots \sum_{(f_{n-1} : \mathtt{F}_{\mathrm{n}-1})} \mathtt{F}_{\mathrm{n}}\,, \lambda\,(f_1, f_3, \ldots), \sum_{x : f_1} (\mathtt{B}\ x)\right) \tag{3.25}$$

so we would also expect that a type defined as a (coinductive) record is equal to the version defined as a M-type.

which is the case, proof needed!

But we run into problems if ...

Problem cases

## 3.5 Wacky M-type

What differes from W and M types for closure of constructors / destructors?

We end this chapter by showing of some wacky M-type, that utilizes the definition of the M-type to the fullest.

**Definition 3.5.1.** We define a wacky M-type by the following container

$$\left(\mathbb{N} + \mathbb{N}\,, \begin{cases} \mathbb{N} & \mathtt{inl}\ 0 \\ \mathbf{0} & \mathtt{inl}\ x \wedge x \text{ is odd} \\ \mathbf{0} & \mathtt{inr}\ x \wedge x \text{ is even} \\ \mathbf{1} & o.w. \end{cases}\right) \tag{3.26}$$

this container gives the $\mathtt{M}$-type and constructors / destructors shown in Figure 3.5 The type can

$$\begin{tikzcd}
\mathbb{N} \arrow[r, "\mathtt{inl}(2n{+}1)"] \arrow[dr, "\mathtt{odd}"'] & M & \mathbb{N} \times M \arrow[l, "o.w."] \arrow[dl, "\mathtt{step}"] \\
\mathbb{N} \arrow[r, "\mathtt{even}"'] \arrow[ur, "\mathtt{inr}(2n)"] & M \arrow[u] & (\mathbb{N} \to M) \arrow[l, "\mathtt{inl\text{-}zero}"] \arrow[ul, "\mathtt{inl}\ 0 \times f"']
\end{tikzcd}$$

Figure 3.5: wacky $\mathtt{M}$-type

be interpreted as a stream of coproducts of natural numbers, that terminates whenever it is the left injection and even, or the right injection and odd, and whenever the left injection is zero, it splits in a branches indexed by the natural numbers.

This show that we can define coinductive types, with rather complex structure.

define a wacky M-type showing some of the use of M-types / complexity

# Chapter 4

# QM-types

In this chapter we will introduce quotients, and show how to construct quotiented `M`-types which we call Q`M`-types. We show you can be constructed a QIIT that is equal to the Q`M`-type assuming axiom of choice.

better introduction to chapter, and reference to main points

## 4.1 Quotienting and Constructors

Describe set truncated quotients and their construction / elimination principles, and how it relates to quotienting M-types

## 4.2 Quotient `M`-type

Since we know that `M`-types preserves the H-level, we can use set-truncated quotients, to define quotient `M`-types, though we run into the problem of , as a solution we use QIIT to define the relation and type at the same type.

problem of direct quotients

## 4.3 Quotient inductive-inductive types (QIITs)

A quotient inductive-inductive type (QIIT) is a type together with a relation defined on that type. QIITs are HIITs that are set truncated.

## 4.4 Partiality monad

In this section we will define the partiality monad (see below) and show that (assuming the axiom of countable choice) the delay monad quotiented by weak bisimilarity.

**Definition 4.4.1** (Partiality Monad)**.** A simple example of a quotient inductive-inductive type is the partiality monad $(-)_\perp$ over a type $R$, defined by the constructors

$$\frac{}{R_\perp : \mathcal{U}} \quad (4.1) \qquad \frac{}{\perp : R_\perp} \quad (4.2) \qquad \frac{a : R}{\eta\ a : R_\perp} \quad (4.3)$$

Should I define what it means to be an ordering relation sepa-

and a relation $(\cdot \sqsubseteq_\perp \cdot)$ indexed twice over $R_\perp$, with properties

$$\frac{\mathtt{s} : \mathbb{N} \to R_\perp \quad \mathtt{b} : \prod_{(n:\mathbb{N})} \mathtt{s}_n \sqsubseteq_\perp \mathtt{s}_{n+1}}{\bigsqcup (\mathtt{s}, \mathtt{b}) : R_\perp} \tag{4.4}$$

$$\frac{x, y : R_\perp \quad p : x \sqsubseteq_\perp y \quad q : y \sqsubseteq_\perp x}{\alpha_\perp\ p\ q : x \equiv y} \tag{4.5}$$

$$\frac{x : R_\perp}{x \sqsubseteq_\perp x} \sqsubseteq_{\mathtt{refl}} \tag{4.6}$$

$$\frac{x \sqsubseteq_\perp y \quad y \sqsubseteq_\perp z}{x \sqsubseteq_\perp z} \sqsubseteq_{\mathtt{trans}} \tag{4.7}$$

$$\frac{x : R_\perp}{\perp \sqsubseteq_\perp x} \sqsubseteq_{\mathtt{never}} \tag{4.8}$$

$$\frac{\mathtt{s} : \mathbb{N} \to R_\perp \quad \mathtt{b} : \prod_{(n:\mathbb{N})} \mathtt{s}_n \sqsubseteq_\perp \mathtt{s}_{n+1}}{\prod_{(n:\mathbb{N})} \mathtt{s}_n \sqsubseteq_\perp \bigsqcup(\mathtt{s}, \mathtt{b})} \tag{4.9}$$

$$\frac{\prod_{(n:\mathbb{N})} \mathtt{s}_n \sqsubseteq_\perp x}{\bigsqcup(\mathtt{s}, \mathtt{b}) \sqsubseteq_\perp x} \tag{4.10}$$

and finally set truncated

$$\frac{p, q : x \sqsubseteq_\perp y}{p \equiv q} (-)_\perp\text{-}\mathtt{isSet} \tag{4.11}$$

### 4.4.1 Delay monad to Sequences

introduction to sub-section

**Definition 4.4.2.** We define

$$\mathtt{Seq}_R = \sum_{(\mathtt{s}:\mathbb{N} \to R+\mathbf{1})} \mathtt{isMon}\ \mathtt{s} \tag{4.12}$$

where

$$\mathtt{isMon}\ \mathtt{s} = \prod_{(n:\mathbb{N})} (\mathtt{s}_n \equiv \mathtt{s}_{n+1}) + ((\mathtt{s}_n \equiv \mathtt{inr}\ \star) \times (\mathtt{s}_{n+1} \not\equiv \mathtt{inr}\ \star)) \tag{4.13}$$

meaning a sequences is $\mathtt{inr}\ \star$ until it reaches a point where it switches to $\mathtt{inl}\ r$ for some value $r$. There are also the special cases of already terminated, meaning only $\mathtt{inl}\ r$ and never teminating meaning only $\mathtt{inr}\ \star$.

For each index in a sequence, the element at that index $\mathtt{s}_n$ is either not terminated $\mathtt{s}_n \equiv \mathtt{inr}\ \star$, which we denote as $\mathtt{s}_n \uparrow_{R+\mathbf{1}}$, or it is terminated $\mathtt{s}_n \equiv \mathtt{inl}\ r$ with some value $r$, denoted by $\mathtt{s}_n \downarrow_{R+\mathbf{1}} r$ or just $\mathtt{s}_n \downarrow_{R+\mathbf{1}}$ to mean $\mathtt{s}_n \not\equiv \mathtt{inr}\ \star$. Thus we can write $\mathtt{isMon}$ as

$$\mathtt{isMon}\ \mathtt{s} = \prod_{(n:\mathbb{N})} (\mathtt{s}_n \equiv \mathtt{s}_{n+1}) + ((\mathtt{s}_n \uparrow_{R+\mathbf{1}}) \times (\mathtt{s}_{n+1} \downarrow_{R+\mathbf{1}})) \tag{4.14}$$

We also introduce notation for the two special cases of sequences given above

$$\mathtt{now}_{Seq}\ r = (\lambda\_, \mathtt{inl}\ r), (\lambda\_, \mathtt{inl}\ \mathtt{refl}) \tag{4.15}$$

$$\mathtt{never}_{Seq} = (\lambda\_, \mathtt{inr}\ \star), (\lambda\_, \mathtt{inl}\ \mathtt{refl}) \tag{4.16}$$

Some comment about decidable equivalence needed to show that $\mathtt{s}_{n+1} \not\equiv \mathtt{inr}\ \star$

**Definition 4.4.3.** We can shift a sequence $(\mathtt{s}, \mathtt{q})$ by inserting an element (and an equality) $(z_s, z_q)$ at $n = 0$,

$$\mathtt{shift}\ (\mathtt{s}, \mathtt{q})\ (z_s, z_q) = \begin{cases} z_s & n = 0 \\ \mathtt{s}_m & n = m+1 \end{cases}, \begin{cases} z_q & n = 0 \\ \mathtt{q}_m & n = m+1 \end{cases}, \tag{4.17}$$

**Definition 4.4.4.** We can unshift a sequence by removing the first element of the sequence

$$\texttt{unshift}\ (\mathtt{s}, \mathtt{q}) = \mathtt{s} \circ \texttt{succ}, \mathtt{q} \circ \texttt{succ}. \tag{4.18}$$

**Lemma 4.4.5.** *The function*

$$\texttt{shift-unshift}\ (\mathtt{s}, \mathtt{q}) = \texttt{shift}\ (\texttt{unshift}\ (\mathtt{s}, \mathtt{q}))\ (\mathtt{s}_0, \mathtt{q}_0) \tag{4.19}$$

*is equal to the identity function.*

*Proof.* Unshifting a value followed by a shift, where we reintroduce the value we just remove, gives the sequence we started with. □

**Lemma 4.4.6.** *The function*

$$\texttt{unshift-shift}\ (\mathtt{s}, \mathtt{q}) = \texttt{unshift}\ (\texttt{shift}\ (\mathtt{s}, \mathtt{q})\ \_) \tag{4.20}$$

*is equal to the identity function.*

*Proof.* If we shift followed by an unshift, we just introduce a value to instantly remove it, meaning the value does not matter. □

We now define an equivalence between `delay` $R$ and $\texttt{Seq}_R$, where `later` are equivalent to shifts, and `now` $r$ is equivalent terminated sequence with value $r$. We do this by defining equivalence functions, and the left and right identities.

**Lemma 4.4.7** ($\texttt{inl}{\not\equiv}\texttt{inr}$)**.** *For any two elements $x = \texttt{inl}\ a$ and $y = \texttt{inr}\ b$ then $x \not\equiv y$.*

*Proof.* The constructors are disjoint, so there is not a path between them . □

better formulated proof

*f*

**Definition 4.4.8.** We define a function from `Delay` $R$ to $\texttt{Seq}_R$

$$\begin{aligned}
&\texttt{Delay}{\to}\texttt{Seq}\ (\texttt{now}\ r) = \texttt{now}_{Seq}\ r \\
&\texttt{Delay}{\to}\texttt{Seq}\ (\texttt{later}\ x) = \\
&\qquad \texttt{shift}\ (\texttt{Delay}{\to}\texttt{Seq}\ x) \left(\texttt{inr}\ \star, \begin{cases} \texttt{inr}\ (\texttt{refl}, \texttt{inl}{\not\equiv}\texttt{inr}) & x = \texttt{now}\ _ \\ \texttt{inl}\ \texttt{refl} & x = \texttt{later}\ _ \end{cases}\right)
\end{aligned} \tag{4.21}$$

**Definition 4.4.9.** We define function from $\texttt{Seq}_R$ to `Delay` $R$

$$\texttt{Seq}{\to}\texttt{Delay}\ (\mathtt{s}, \mathtt{q}) = \begin{cases} \texttt{now}\ r & \mathtt{s}_0 = \texttt{inl}\ r \\ \texttt{later}\ (\texttt{Seq}{\to}\texttt{Delay}\ (\texttt{unshift}\ (\mathtt{s}, \mathtt{q}))) & \mathtt{s}_0 = \texttt{inr}\ \star \end{cases} \tag{4.22}$$

**Theorem 4.4.10.** *The type* $\texttt{Seq}_R$ *is equal to* `Delay` $R$

*Proof.* We define right and left identity, saying that for any sequence $(\mathtt{s}, \mathtt{q})$, we get

$$\mathtt{Delay{\to}Seq}\ (\mathtt{Seq{\to}Delay}\ (\mathtt{s}, \mathtt{q})) \equiv (\mathtt{s}, \mathtt{q}) \tag{4.23}$$

defined by cases analysis on $\mathtt{s}_0$, if $\mathtt{s}_0 = \mathtt{inl}\ r$ then we need to show

$$\mathtt{now}_{Seq}\ r \equiv (\mathtt{s}, \mathtt{q}) \tag{4.24}$$

This is true, since $(\mathtt{s}, \mathtt{q})$ is a monotone sequence and $\mathtt{inl}\ r$ is the top element of the order, then all elements of the sequence are $\mathtt{inl}\ r$. If $\mathtt{s}_0 = \mathtt{inr}\ \star$ then, we need to show

$$\mathtt{shift}\ (\mathtt{Delay{\to}Seq}\ (\mathtt{Seq{\to}Delay}\ (\mathtt{unshift}\ (\mathtt{s}, \mathtt{q})))) \ (\mathtt{inr}\ \star, \_) \equiv (\mathtt{s}, \mathtt{q}) \tag{4.25}$$

by the induction hypothesis we get

$$\mathtt{Delay{\to}Seq}\ (\mathtt{Seq{\to}Delay}\ (\mathtt{unshift}\ (\mathtt{s}, \mathtt{q}))) \equiv \mathtt{unshift}\ (\mathtt{s}, \mathtt{q}) \tag{4.26}$$

since shift and unshift are inverse, we get the needed equality.

Shift takes two arguemnts, either clarify that its shift' that inserts inr tt or ...

For the left identity, we need to show that for any delay monad $t$ we get

$$\mathtt{Seq{\to}Delay}\ (\mathtt{Delay{\to}Seq}\ t) \equiv t \tag{4.27}$$

defined by case analysis on $t$, if $t = \mathtt{now}\ a$ then the equality is $\mathtt{refl}$. If $t = \mathtt{later}\ x$ then we need to show

$$\mathtt{later}\ (\mathtt{Seq{\to}Delay}\ (\mathtt{unshift}\ (\mathtt{shift}\ (\mathtt{Delay{\to}Seq}\ x))) \equiv \mathtt{later}\ x \tag{4.28}$$

By unshift and shift being inverse, and the induction hypothesis we get the wanted equality. Since we are able to define a left and right identity function, we get the wanted equality. □

**Corollary.** *The types* $\mathtt{Delay}/\sim$ *and* $\mathtt{Seq}/\sim$ *are equal.*

Show this

*Proof.* We show that if $a \sim_{\mathtt{delay}} b$ then $\mathtt{Delay{\to}Seq}\ a \sim_{\mathtt{Seq}} \mathtt{Delay{\to}Seq}\ b$,

Show this

and we show if $x \sim_{\mathtt{Seq}} y$ then $\mathtt{Seq{\to}Delay}\ x \sim_{\mathtt{Seq}} \mathtt{Seq{\to}Delay}\ y$, □

### 4.4.2 Sequence to Partiality Monad

In this section we will show that assuming the "Axiom of Countable Choice", we get an equivalence between sequences and the partiality monad.

**Definition 4.4.11** (Sequence Termination)**.** The following relations says that a sequence $(\mathtt{s}, \mathtt{q}) : \mathtt{Seq}_R$ terminates with a given value $r : R$,

$$(\mathtt{s}, \mathtt{q}) \downarrow_{\mathtt{Seq}} r = \sum_{(n:\mathbb{N})} \mathtt{s}_n \downarrow_{R+\mathbf{1}} r. \tag{4.29}$$

**Definition 4.4.12** (Sequence Ordering)**.**

$$(\mathtt{s}, \mathtt{q}) \sqsubseteq_{\mathtt{Seq}} (\mathtt{t}, \mathtt{p}) = \prod_{(a:R)} (\|\mathtt{s} \downarrow_{\mathtt{Seq}} a\| \to \|\mathtt{t} \downarrow_{\mathtt{Seq}} a\|) \tag{4.30}$$

where $\| \cdot \|$ is propositional truncation.

**Definition 4.4.13.** There is a conversion from $R + \mathbf{1}$ to the partiality monad $R_\perp$

$$\begin{aligned} \texttt{Maybe}{\to}(-)_\perp\ (\texttt{inl}\ r) &= \eta\ r \\ \texttt{Maybe}{\to}(-)_\perp\ (\texttt{inr}\ \star) &= \perp \end{aligned} \tag{4.31}$$

**Definition 4.4.14** (Maybe Ordering)**.** Given some $x, y : R + \mathbf{1}$, the ordering relation is defined as

$$x \sqsubseteq_{\mathtt{R+1}} y = (x \equiv y) + ((x \downarrow_{R+\mathbf{1}}) \times (y \uparrow_{R+\mathbf{1}})) \tag{4.32}$$

This ordering definition is basically `isMon` at a specific index, so we can again rewrite `isMon` as

$$\texttt{isMon s} = \prod_{(n:\mathbb{N})} \mathtt{s}_n \sqsubseteq_{\mathtt{R+1}} \mathtt{s}_{n+1} \tag{4.33}$$

This rewriting confirms that if `isMon s`, then `s` is monotone, and therefore a sequence of partial values.

there exists non-monotone sequences, it just follows our definition of a sequence.

**Lemma 4.4.15.** *The function* $\texttt{Maybe}{\to}(-)_\perp$ *is monotone, that is, if* $x \sqsubseteq_{\mathtt{A}+1} y$*, for some* $x$ *and* $y$*, then* $(\texttt{Maybe}{\to}(-)_\perp\ x) \sqsubseteq_\perp (\texttt{Maybe}{\to}(-)_\perp\ y)$*.*

*Proof.* We do the proof by case.

$$\begin{aligned} &\texttt{Maybe}{\to}(-)_\perp\texttt{-mono}\ (\texttt{inl}\ p) = \\ &\qquad \texttt{subst}\ (\lambda\, a,\ \texttt{Maybe}{\to}(-)_\perp\ x \sqsubseteq_\perp \texttt{Maybe}{\to}(-)_\perp\ a)\ p\ (\sqsubseteq_{\texttt{refl}}\ (\texttt{Maybe}{\to}(-)_\perp\ x)) \\ &\texttt{Maybe}{\to}(-)_\perp\texttt{-mono}\ (\texttt{inr}\ (p, \_)) = \\ &\qquad \texttt{subst}\ (\lambda\, a,\ \texttt{Maybe}{\to}(-)_\perp\ a \sqsubseteq_\perp \texttt{Maybe}{\to}(-)_\perp\ y)\ p^{-1}\ (\sqsubseteq_{\texttt{never}}\ (\texttt{Maybe}{\to}(-)_\perp\ y)) \end{aligned} \tag{4.34}$$

□

**Definition 4.4.16.** There is a function taking a sequence to an increasing sequence

What is an increasing sequence ??, this is not defined anywhere!!

$$\begin{aligned} &\texttt{Seq}{\to}\texttt{incSeq} \\ &\texttt{Seq}{\to}\texttt{incSeq}\ (\texttt{g}, \texttt{q}) = \texttt{Maybe}{\to}(-)_\perp \circ \texttt{g}, \texttt{Maybe}{\to}(-)_\perp\texttt{-mono} \circ \texttt{q} \end{aligned} \tag{4.35}$$

**Definition 4.4.17.** There is a function taking a sequence to the partiality monad

$$\begin{aligned} \texttt{Seq}{\to}(-)_\perp &: \texttt{Seq}_A \to A_\perp \\ \texttt{Seq}{\to}(-)_\perp\ (\texttt{g}, \texttt{q}) &= \bigsqcup \circ \texttt{Seq}{\to}\texttt{incSeq} \end{aligned} \tag{4.36}$$

**Lemma 4.4.18.** *The function* $\texttt{Seq}{\to}(-)_\perp$ *is monotone.*

$$\texttt{Seq}{\to}(-)_\perp\texttt{-mono} : \texttt{isSet}A \to (x\ y : \texttt{Seq}_A) \to x \sqsubseteq_{\texttt{seq}} y \to \texttt{Seq}{\to}(-)_\perp\ x \sqsubseteq_\perp \texttt{Seq}{\to}(-)_\perp\ y \tag{4.37}$$

*Proof.* Given two sequences, if one is smaller than the another, then the least upper bounds of each sequence respect the ordering. □

should this be formalized entirely, or should there just be a comment about monotonicity?

**Definition 4.4.19.** If two sequences $x, y$ are weakly bisimular, then $\texttt{Seq}{\to}(-)_\perp\ x \equiv \texttt{Seq}{\to}(-)_\perp\ y$

$$\texttt{Seq}{\to}(-)_\perp\texttt{-}{\approx}{\to}{\equiv}\ A_{set}\ x\ y\ (p, q) = \alpha_\perp\ (\texttt{Seq}{\to}(-)_\perp\texttt{-mono}\ A_{set}\ x\ y\ p)\ (\texttt{Seq}{\to}(-)_\perp\texttt{-mono}\ A_{set}\ y\ x\ q) \tag{4.38}$$

**Definition 4.4.20** (Recursor for Quotient). For all sequences $x, y : \mathtt{Seq}_A$, functions $\mathtt{f} : A \to B$ and relations $\mathtt{g} : x \mathrel{\mathtt{R}} y \to \mathtt{f}\ x \equiv \mathtt{f}\ y$, then if $B$ is a set $B_{set} : \mathtt{isSet}\ B$, we get a function $\mathtt{rec} : A/\mathtt{R} \to B$, defined by case as

$$\begin{aligned} &\mathtt{rec}\ [\, z \,] = \mathtt{f}\ z \\ &\mathtt{rec}\ (\mathtt{eq/}\ \_\ \_\ r\ i) = \mathtt{g}\ r\ i \\ &\mathtt{rec}\ (\mathtt{squash/}\ a\ b\ p\ q\ i\ j) = B_{set}\ (\mathtt{rec}\ a)\ (\mathtt{rec}\ b)\ (\mathtt{ap}\ \mathtt{rec}\ p)\ (\mathtt{ap}\ \mathtt{rec}\ q)\ i\ j \end{aligned} \tag{4.39}$$

is this a recursor, and for what? The quotient?

This recursor allows us to lift the function $\mathtt{Seq}{\to}(-)_\perp$ to the quotient

**Definition 4.4.21.** We can define a function $\mathtt{Seq/}{\sim}{\to}(-)_\perp$ from $\mathtt{Seq}_A$ to $A_\perp$, where $A_{set} : \mathtt{isSet}\ A$ as

$$\mathtt{Seq/}{\sim}{\to}(-)_\perp = \mathtt{rec}\ \mathtt{Seq}{\to}(-)_\perp\ (\mathtt{Seq}{\to}(-)_\perp\text{-}{\approx}{\to}{\equiv}\ A_{set})\ (-)_\perp\text{-}\mathtt{isSet} \tag{4.40}$$

**Lemma 4.4.22.** *Given two sequences $s$ and $t$, if $\mathtt{Seq}{\to}(-)_\perp\ s \equiv \mathtt{Seq}{\to}(-)_\perp\ t$, then $s \sim_{\mathtt{seq}} t$.*

*Proof.* We can reduce the burden of the proof, since

$$s \sim_{\mathtt{seq}} t = \left(\prod_{(r:R)} \|x \downarrow_{\mathtt{seq}} r\| \to \|y \downarrow_{\mathtt{seq}} r\|\right) \times \left(\prod_{(r:R)} \|y \downarrow_{\mathtt{seq}} r\| \to \|x \downarrow_{\mathtt{seq}} r\|\right) \tag{4.41}$$

so we can just show one part and get the other by symmetry. We assume $\|x \downarrow_{\mathtt{seq}} r\|$, to show $\|y \downarrow_{\mathtt{seq}} r\|$. By the mapping property of propositional truncation, we reduce the proof to defining a function $x \downarrow_{\mathtt{seq}} r \to y \downarrow_{\mathtt{seq}} r$. Since $x \downarrow_{\mathtt{seq}} r$, then $\eta\ r \sqsubseteq_\perp \mathtt{Seq}{\to}(-)_\perp\ x$, but we have assumed $\mathtt{Seq}{\to}(-)_\perp\ x \equiv \mathtt{Seq}{\to}(-)_\perp\ y$, so we get $\eta\ r \sqsubseteq_\perp \mathtt{Seq}{\to}(-)_\perp\ y$, and thereby $y \downarrow_{\mathtt{seq}} r$. ☐

**Lemma 4.4.23.** *The function $\mathtt{Seq/}{\sim}{\to}(-)_\perp$ is injective.*

Should this be formalized?

*Proof.* We use propositional elimination of quotients

$$\begin{aligned} \mathtt{elimProp} : (\mathtt{B} : \mathtt{Seq}_R/{\sim_{\mathtt{seq}}} \to \mathcal{U}) \to ((x : \mathtt{Seq}_R/{\sim_{\mathtt{seq}}}) \to \mathtt{isProp}\ (\mathtt{B}\ x)) \\ \to (\mathtt{f} : (a : \mathtt{Seq}_R) \to \mathtt{B}\ [\ a\ ]) \to (x : \mathtt{Seq}_R/{\sim_{\mathtt{seq}}}) \to \mathtt{B}\ x \end{aligned} \tag{4.42}$$

Convert to text, instead of a proof term!?

to show the injectivity, meaning for all $x\ y : \mathtt{Seq}_R/{\sim_{\mathtt{seq}}}$ we get $\mathtt{Seq/}{\sim}{\to}(-)_\perp\ x \equiv \mathtt{Seq/}{\sim}{\to}(-)_\perp\ y \to x \equiv y$. We start by eliminating $x$, followed by elimination of $y$, this gives us the proof term

$$\begin{aligned} &\mathtt{elimProp} \\ &\quad (\lambda\, a,\ \mathtt{Seq/}{\sim}{\to}(-)_\perp\ a \equiv \mathtt{Seq/}{\sim}{\to}(-)_\perp\ y \to a \equiv y) \\ &\quad (\lambda\, a,\ \mathtt{isProp\Pi}\ (\lambda\, \_, \mathtt{squash/}\ a\ y)) \\ &\quad (\lambda\, a,\ \mathtt{elimProp} \\ &\qquad\qquad (\lambda\, b,\ \mathtt{Seq}{\to}(-)_\perp\ a \equiv \mathtt{Seq/}{\sim}{\to}(-)_\perp\ b \to [\ a\ ] \equiv b) \\ &\qquad\qquad (\lambda\, b,\ \mathtt{isProp\Pi}\ (\lambda\, \_, \mathtt{squash/}\ [\ a\ ]\ b)) \\ &\qquad\qquad (\lambda\, b,\ (\mathtt{eq/}\ a\ b) \circ (\mathtt{Seq}{\to}(-)_\perp\text{-}\mathtt{isInjective}\ a\ b))) \end{aligned} \tag{4.43}$$

where $\mathtt{Seq}{\to}(-)_\perp\text{-}\mathtt{isInjective}$ is (4.4.22), ☐

**Lemma 4.4.24.** *For all constant sequences $s$, where all elements have the same value $v$, we get $\mathtt{Seq}{\to}(-)_\perp\ s \equiv \mathtt{Maybe}{\to}(-)_\perp\ v$.*

*Proof.* The left side of the equality reduces to $\texttt{Maybe}{\to}(-)_\perp$ applied on the least upper bound of the constant sequence, which is exactly the right hand side of the equality. □

**Lemma 4.4.25.** *Assuming countable choice, the function* $\texttt{Seq}{\to}(-)_\perp$ *is surjective*

*describe countable choice (and why it is needed!)*

*Proof.* We do the proof by case on $R_\perp$, if it is $\eta\ r$ or $\texttt{never}$, we convert them to the sequences $\texttt{now}_{seq}\ r$ and $\texttt{never}_{seq}$ respectively, then we are done by (4.4.24). For the least upper bound $\bigsqcup(\texttt{s},\texttt{b})$, we translate to the (increasing) sequence, defined by $(\texttt{s},\texttt{b})$ . □

describe what it means to do the surjective proof by case!

**Lemma 4.4.26.** *Assuming countable choice, the function* $\texttt{Seq}/{\sim}{\to}(-)_\perp$ *is surjective*

*Proof.* □

more precise description!

**Theorem 4.4.27.** *Assuming countable choice, we get an equivalence between sequences and the partiality monad.*

*Proof.* The function $\texttt{Seq}/{\sim}{\to}(-)_\perp$ is injective and surjective assuming countable choice, meaning we get an equivalence, since we are working in hSets. □

Complete the rest of the proof!

Complete proof

#### 4.4.2.1 Building the weak bisimulation on the M-type as a M-type

Is this possible? Yes! Should it be included?

#### 4.4.2.2 Building the Partiality Monad as a limit (Dialgebra?)

Is this possible?

## 4.5 Permutation Trees

introduction to section

### 4.5.1 Binary Trees

introduction to section

**Definition 4.5.1.** Binary trees $bT$ are given as the M-type defined by the container

$$\left(\mathbf{1}+\mathbf{1}, \begin{cases}\mathbf{0} & \texttt{inl}\ \star \\ \mathbf{1}+\mathbf{1} & \texttt{inr}\ \star\end{cases}\right) \tag{4.44}$$

For which we get the constructors

$$\frac{}{\texttt{leaf} : bT} \quad (4.45) \qquad\qquad \frac{\texttt{f} : \mathbf{1}+\mathbf{1} \to bT}{\texttt{node}\ \texttt{f} : bT} \quad (4.46)$$

We want to define trees where the permutation does not matter, that is the following rule is true

$$\frac{\mathtt{f} : \mathbf{1}+\mathbf{1} \to bT \quad \mathtt{g} : \mathbf{1}+\mathbf{1} \to \mathbf{1}+\mathbf{1} \quad \mathtt{isIso}\ \mathtt{g}}{\mathtt{node}\ \mathtt{f} \equiv \mathtt{node}\ (\mathtt{f} \circ \mathtt{g})}\ \mathtt{mix} \tag{4.47}$$

this type of tree we call $bTp$. We can either define this as a quotient $bT/\sim$, where

$$\frac{}{\mathtt{leaf} \sim \mathtt{leaf}}\ \sim_{\mathtt{leaf}} \tag{4.48}$$

$$\frac{\prod_{(x:\mathbf{1}+\mathbf{1})} \mathtt{f}\ x \sim \mathtt{g}\ x}{\mathtt{node}\ \mathtt{f} \sim \mathtt{node}\ \mathtt{g}}\ \sim_{\mathtt{node}} \tag{4.49}$$

and $\sim_{\mathtt{m}}$ $\mathtt{ix}$ is $\mathtt{mix}$ with $\sim$ replacing the equality. Another way to define the type is as a QIIT type where $\mathtt{mix}$ and $\mathtt{bTp\text{-}isSet}$ are additional constructors. We will show these two definitions are equal

**Theorem 4.5.2.** *There is an equality* $bT/\sim \equiv bTp$

*Proof.*

$$\begin{aligned} &\mathtt{bT{\to}bTp}\ \mathtt{leaf}_{bT} = \mathtt{leaf}_{bTp} \\ &\mathtt{bT{\to}bTp}\ (\mathtt{node}_{bT}\ \mathtt{f}) = \mathtt{node}_{bTp}\ (\mathtt{bT{\to}bTp} \circ \mathtt{f}) \end{aligned} \tag{4.50}$$

This definition takes weakly bisimilar elements to equivalent elements

$$\begin{aligned} &\mathtt{bT{\to}bTp\text{-}{\approx}{\to}{\equiv}}\ (\sim_{\mathtt{leaf}}) = \mathtt{refl} \\ &\mathtt{bT{\to}bTp\text{-}{\approx}{\to}{\equiv}}\ (\sim_{\mathtt{node}}\ \mathtt{p}) = \mathtt{ap}\ \mathtt{node}_{bTp}\ (\mathtt{bT{\to}bTp\text{-}{\approx}{\to}{\equiv}} \circ \mathtt{p}) \end{aligned} \tag{4.51}$$

Then we lift the definition to the quotient

$$\mathtt{bT/{\sim}{\to}bTp} = \mathtt{rec}\ \mathtt{bT{\to}bTp}\ (\mathtt{bT{\to}bTp\text{-}{\approx}{\to}{\equiv}}\ A_{set})\ \mathtt{bTp\text{-}isSet} \tag{4.52}$$

□

### 4.5.2 Silhouette Trees

We start by defining an $R$ valued $E$ branching tree, as the $\mathtt{M}$-type given by the following container

$$\left(R+\mathbf{1}, \begin{cases} \mathbf{0} & \mathtt{inl}\ a \\ E & \mathtt{inr}\ \star \end{cases}\right) \tag{4.53}$$

We get the constructors

$$\frac{a : R}{\mathtt{leaf}\ a : \mathtt{tree}\ R\ E} \tag{4.54}$$

$$\frac{\mathtt{k} : E \to \mathtt{tree}\ R\ E}{\mathtt{node}\ \mathtt{k} : \mathtt{tree}\ R\ E} \tag{4.55}$$

Then we define the weak bisimilarity relation $\sim_{\mathtt{tree}}$

$$\frac{}{\mathtt{leaf}\ x \sim_{\mathtt{tree}} \mathtt{leaf}\ y}\ \sim_{\mathtt{leaf}} \tag{4.56}$$

$$\frac{\prod_{(v:E)} \mathtt{k}_1\ v \sim_{\mathtt{tree}} \mathtt{k}_2\ v}{\mathtt{node}\ \mathtt{k}_1 \sim_{\mathtt{tree}} \mathtt{node}\ \mathtt{k}_2}\ \sim_{\mathtt{node}} \tag{4.57}$$

This is enough to define, what we call, silhouette trees, which are trees quotiented by this notion of weak bisimilarity, namely $\texttt{tree}/\!\sim_{\texttt{tree}}$. We can also construct this type directly as a QIIT, with type constructors

add all needed constructors

$$\frac{}{\texttt{leaf}_{\texttt{sTree}} : \texttt{sTree}\ E} \tag{4.58}$$

$$\frac{\texttt{k} : E \to \texttt{sTree}\ E}{\texttt{node}_{\texttt{sTree}}\ \texttt{k} : \texttt{sTree}\ E} \tag{4.59}$$

And the ordering relation $(\cdot \sqsubseteq_{\texttt{sTree}} \cdot)$ of how "defined" the trees are by the constructors

add all needed constructors

$$\frac{x \sqsubseteq_{\texttt{sTree}} y \quad y \sqsubseteq_{\texttt{sTree}} x}{\alpha_{\texttt{sTree}}\ x\ y : \texttt{sTree}\ E} \tag{4.60}$$

$$\frac{\texttt{s} : (\mathbb{N} \to E) \to \texttt{sTree}\ E}{\bigsqcup_{(\texttt{e}:\mathbb{N}\to E)} (\texttt{s}\ \texttt{e})} \tag{4.61}$$

#### 4.5.2.1 From tree to $\texttt{Seq}_{tree}$

We now want to show the equivalence between these two constructions, to do this we define an intermediate construction $\texttt{Seq}_{tree}$, where we get an ordering on the "definedness" of trees.

**Definition 4.5.3.** We define monotone increasing sequences of trees as , all breanches are monotone increasing .

ordering is container ordering not maybe?

$$\texttt{Seq}_{tree} = \prod_{(\texttt{e}:\mathbb{N}\to E)} \sum_{(\texttt{s}:\mathbb{N}\to R+1)} \prod_{(n:\mathbb{N})} \texttt{s}_n \sqsubseteq_{\texttt{R+1}} \texttt{s}_{n+1} \tag{4.62}$$

where $\sqsubseteq_{\texttt{R+1}}$ is similar to the relation defined at (4.4.14) .

**Definition 4.5.4.** We define a function to shift a $\texttt{Seq}_{tree}$, it takes $\texttt{f} : E \to \texttt{Seq}_{tree}$ as an argument. We let $\texttt{s}' = \texttt{f}\ \texttt{e}_0\ (\texttt{e} \circ \texttt{succ})$, then the definition is given as

specify branches increasing?

$$\begin{aligned} &\texttt{shift-seq}\ \texttt{f} = \\ &\quad \lambda\,\texttt{e}, \{\!|\ \texttt{inr}\ \star\ ,\ \pi_1\ \texttt{s}'\ |\!\}, \left(\lambda\, n, \begin{cases} \texttt{inr}\ (\texttt{refl}, \texttt{inl}\not\equiv\texttt{inr}) & n = 0 \wedge \pi_1\ \texttt{s}'\ 0 = \texttt{inl}\ r \\ \texttt{inl}\ \texttt{refl} & n = 0 \wedge \pi_1\ \texttt{s}'\ 0 = \texttt{inr}\ \star \\ \pi_2\ \texttt{s}'\ m & n = m + 1 \end{cases}\right) \end{aligned} \tag{4.63}$$

how does it differ? Constructors are equal type is different (trees instead of delay)

**Definition 4.5.5.** We define a function to unshift a $Seq_{tree}$

$$\texttt{unshift-seq}\ \texttt{s}\ v = \lambda\,\texttt{e},\ (\pi_1\ (\texttt{s}\ (\{\!|\ v\ ,\ \texttt{e}\ |\!\})) \circ \texttt{succ}), (\pi_2\ (\texttt{s}\ (\{\!|\ v\ ,\ \texttt{e}\ |\!\})) \circ \texttt{succ}) \tag{4.64}$$

**Lemma 4.5.6.** *Shift and unshift er inverse to each other*

*Proof.* The same reasoning as for $\square$

shift - unshift

**Definition 4.5.7.** We get a function from trees to monotone sequences

$$\begin{aligned} \texttt{tree}\!\to\!\texttt{Seq}\ (\texttt{leaf}\ r) &= \lambda\,\_, (\lambda\,\_, \texttt{inl}\ r), (\lambda\,\_, \texttt{inl}\ \texttt{refl}) \\ \texttt{tree}\!\to\!\texttt{Seq}\ (\texttt{node}\ \texttt{k}) &= \texttt{shift}\ (\texttt{tree}\!\to\!\texttt{Seq} \circ \texttt{k}) \end{aligned} \tag{4.65}$$

**Definition 4.5.8.** We get a function from monotone sequences to trees

$$\mathtt{Seq{\to}tree}\ \mathtt{s} = \begin{cases} \mathtt{leaf}\ r & \prod_{(\mathtt{e}:\mathbb{N}\to E)} \pi_1\ (\mathtt{s}\ \mathtt{e})\ 0 = \mathtt{inl}\ r \\ \mathtt{node}\ (\mathtt{Seq{\to}tree} \circ \mathtt{unshift}\ \mathtt{s}) & o.w. \end{cases} \tag{4.66}$$

**Lemma 4.5.9.** *If the first element in the sequence is terminated / a leaf, then the rest of the elements will also be terminated.*

$$\left( \prod_{e:\mathbb{N}\to E} \pi_1\ (\mathtt{s}\ \mathtt{e})\ 0 = \mathtt{inl}\ r \right) \Leftrightarrow (\mathtt{s} \equiv \lambda\_, (\lambda\_, \mathtt{inl}\ r), (\lambda\_, \mathtt{inl}\ \mathtt{refl})) \tag{4.67}$$

*Proof.* Since the sequence is monotone, and $\mathtt{inl}\ r$ is the top element of the order, if the first element is $\mathtt{inl}\ r$, then the sequence must be $\lambda\_, (\lambda\_, \mathtt{inl}\ r), (\lambda\_, \mathtt{inl}\ \mathtt{refl})$. The other direction is trivial. □

**Theorem 4.5.10.** *The types* `tree` *and* $\mathtt{Seq}_{tree}$ *are equal*

*Proof.* We construct an isomorphism by the functions $\mathtt{tree{\to}Seq}$ and $\mathtt{Seq{\to}tree}$, with right inverse given by two cases, one where the first element in the sequence is $\mathtt{inl}\ r$, meaning representing a leaf with value $r$, then we need to show that $\mathtt{s} \equiv \lambda\_, (\lambda\_, \mathtt{inl}\ r), (\lambda\_, \mathtt{inl}\ \mathtt{refl})$ which follows from Lemma 4.5.9. Otherwise we need to show that

$$\mathtt{shift}\ (\mathtt{tree{\to}Seq} \circ \mathtt{Seq{\to}tree} \circ \mathtt{unshift}\ \mathtt{s}) \equiv \mathtt{s} \tag{4.68}$$

By induction we get

$$\mathtt{tree{\to}Seq} \circ \mathtt{Seq{\to}tree} \circ \mathtt{unshift}\ \mathtt{s} \equiv \mathtt{unshift}\ \mathtt{s} \tag{4.69}$$

then by the right inverse of the equality between shift and unshift, we are done. For the left inverse we do case analysis, using induction and the left inverse of the equality between shift and unshift

$$\begin{aligned} &\mathtt{tree\text{-}Seq}\ (\mathtt{leaf}\ r) = \mathtt{refl} \\ &\mathtt{tree\text{-}Seq}\ (\mathtt{node}\ k) = \mathtt{unshift\text{-}shift}\ (\mathtt{tree{\to}Seq} \circ \mathtt{k}) \cdot \mathtt{tree\text{-}Seq}\ k \end{aligned} \tag{4.70}$$

□

We start by defining some ordering relation on $\mathtt{Seq}_{tree}$

**Definition 4.5.11** (Sequence Termination)**.** The following relations says that a branche $\mathtt{e} : \mathbb{N} \to E$ of a sequence $\mathtt{s} : \mathtt{Seq}_{tree}$ terminates at depth $n$,

$$(\mathtt{s}\ \mathtt{e}) \downarrow_{\mathtt{Seq_{tree}}} n = (\mathtt{s}\ \mathtt{e}\ n) \downarrow_{R+1} . \tag{4.71}$$

Is this defined for partiality monad?

We define weak bisimilarity relation for sequences

**Definition 4.5.12.**

$$\mathtt{s} \sim_{\mathtt{Seq_{tree}}} \mathtt{t} = \prod_{(e:E)} \prod_{(n:\mathbb{N})} (\|(\mathtt{s}\ \mathtt{e}) \downarrow_{\mathtt{Seq}} n\| \longleftrightarrow \|(\mathtt{t}\ \mathtt{e}) \downarrow_{\mathtt{Seq}} n\|) \tag{4.72}$$

where $\|\cdot\|$ is propositional truncation.

**Corollary.** *The types* $\mathtt{tree}/\!\sim_{\mathtt{tree}}$ *and* $\mathtt{Seq}_{tree}/\!\sim_{\mathtt{Seq_{tree}}}$ *are equal.*

*Proof.* We follow the same strategy as for $\mathtt{Delay}/\!\sim$ and $\mathtt{Seq}/\!\sim$ □

Introduction to Seq to tree

#### 4.5.2.2 $\texttt{Seq}_{tree}$ to $\texttt{sTree}$

**Definition 4.5.13.** We define a function converting a sequence on trees to a monotone sequence on $\texttt{sTree}$'s

$$\texttt{Seq}{\to}\texttt{incSeq} \tag{4.73}$$

**Definition 4.5.14.** There is a function from $\texttt{Seq}_{tree}$ to $\texttt{sTree}$

$$\texttt{Seq}{\to}\texttt{sTree s} = \bigsqcup_{(\texttt{e}:\mathbb{N}\to E)} (\texttt{Seq}{\to}\texttt{incSeq s e}) \tag{4.74}$$

**Lemma 4.5.15.** *Given* $a \sqsubseteq_{\texttt{Seq}_{\texttt{tree}}} b$*, then* $\texttt{Seq}{\to}\texttt{sTree}\ a \sqsubseteq_{\texttt{sTree}} \texttt{Seq}{\to}\texttt{sTree}\ b$*.*

*Proof.* . □

Complete proof

Since this definition is monotone, it can be lifted to the quotiented sequences.

**Lemma 4.5.16.** *If two elements are weakly bisimilar, then they are equal as* $\texttt{sTrees}$

describe better

$$\begin{aligned}&\texttt{Seq}{\to}\texttt{sTree-}{\approx}{\to}{\equiv}\ x\ y\ (\texttt{p},\texttt{q}) = \\ &\quad \alpha_{\texttt{sTree}}\ (\texttt{Seq}{\to}\texttt{sTree-mono}\ x\ y\ \texttt{p})\ (\texttt{Seq}{\to}\texttt{sTree-mono}\ y\ x\ \texttt{q})\end{aligned} \tag{4.75}$$

**Definition 4.5.17.** Function from $\texttt{Seq}/{\sim}$ to $\texttt{sTree}$

$$\texttt{Seq}/{\sim}{\to}\texttt{sTree} = \texttt{rec Seq}{\to}\texttt{sTree Seq}{\to}\texttt{sTree-}{\approx}{\to}{\equiv}\ \texttt{sTree-isSet} \tag{4.76}$$

**Lemma 4.5.18.** $\texttt{Seq}/{\sim}{\to}\texttt{sTree}$ *is injective*

TODO

**Lemma 4.5.19.** $\texttt{Seq}/{\sim}{\to}\texttt{sTree}$ *is surjective*

TODO

**Theorem 4.5.20.** $\texttt{Seq}/{\sim}{\to}\texttt{sTree}$ *is an equivalence.*

TODO

### 4.5.3 QM-types

We want to define sequences based on $\texttt{M}$-types, here are some examples

$$\texttt{Seq}_{\texttt{M}_{(A\,,\,\mathbf{0})}} = A \tag{4.77}$$

$$\texttt{Seq}_{\texttt{M}_{(A+\mathbf{1}\,,\,[\mathbf{0},\mathbf{1}])}} = \sum_{(\texttt{s}:\mathbb{N}\to A+\mathbf{1})} \prod_{(n:\mathbb{N})} \texttt{s}_n \sqsubseteq \texttt{s}_{n+1} \tag{4.78}$$

$$\texttt{Seq}_{\texttt{M}_{(\mathbf{1}+\mathbf{1}\,,\,[\mathbf{0},E])}} = \prod_{(\texttt{e}:\mathbb{N}\to E)} \sum_{(\texttt{s}:\mathbb{N}\to\mathbf{1}+\mathbf{1})} \prod_{(n:\mathbb{N})} \texttt{s}_n \sqsubseteq \texttt{s}_{n+1} \tag{4.79}$$

$$\texttt{Seq}_{\texttt{M}_{(A+\mathbf{1}\,,\,[\mathbf{0},E])}} = \prod_{(\texttt{e}:\mathbb{N}\to E)} \sum_{(\texttt{s}:\mathbb{N}\to A+\mathbf{1})} \prod_{(n:\mathbb{N})} \texttt{s}_n \sqsubseteq \texttt{s}_{n+1} \tag{4.80}$$

$$\texttt{Seq}_{\texttt{M}_{(A+B\,,\,[\mathbf{0},E])}} = \prod_{(\texttt{e}:\mathbb{N}\to E)} \sum_{(\texttt{s}:\mathbb{N}\to A+B)} \prod_{(n:\mathbb{N})} \texttt{s}_n \sqsubseteq \texttt{s}_{n+1} \tag{4.81}$$

$$\texttt{Seq}_{\texttt{M}_{(\mathbf{1}\,,\,\mathbf{1}+\mathbf{1})}} = \prod_{(\texttt{e}:\mathbb{N}\to\mathbf{1}+\mathbf{1})} \sum_{(\texttt{s}:\mathbb{N}\to\mathbf{1})} \prod_{(n:\mathbb{N})} \texttt{s}_n \sqsubseteq \texttt{s}_{n+1} \tag{4.82}$$

$$\mathtt{Seq}_{\mathtt{M}_{(A\,,\,\mathbf{1}+\mathbf{1})}} = \prod_{(\mathtt{e}:\mathbb{N}\to\mathbf{1}+\mathbf{1})} \sum_{(\mathtt{s}:\mathbb{N}\to A)} \prod_{(n:\mathbb{N})} \mathtt{s}_n \sqsubseteq \mathtt{s}_{n+1} \tag{4.83}$$

$$\mathtt{Seq}_{\mathtt{M}_{(A\,,\,\mathtt{B})}} = \prod_{(\mathtt{e}:(n:\mathbb{N})\to\mathtt{B}\,\mathtt{s}_{n-1})} \sum_{(\mathtt{s}:\mathbb{N}\to A)} \prod_{(n:\mathbb{N})} \mathtt{s}_n \sqsubseteq \mathtt{s}_{n+1} \tag{4.84}$$

$$\mathtt{Seq}^{(n)}_{\mathtt{M}_{(A\,,\,\mathtt{B})}}\ (\mathtt{s}_{n-1} : A) = \sum_{(\mathtt{s}_n:A)} \prod_{(n:\mathbb{N})} \prod_{(\mathtt{e}:(n:\mathbb{N})\to\mathtt{B}\,\mathtt{s}_{n-1})} \mathtt{s}_n \sqsubseteq \mathtt{s}_{n+1} \tag{4.85}$$

$$\mathtt{Seq}_{\mathtt{M}_{(A+B\,,\,[X,Y])}} = \prod_{\left(\mathtt{e}:(n:\mathbb{N})\to\begin{cases} X\ a & \mathtt{s}_{n-1}=\mathtt{inl}\ a \\ Y\ b & \mathtt{s}_{n-1}=\mathtt{inr}\ b\end{cases}\right)} \sum_{(\mathtt{s}:\mathbb{N}\to A)} \prod_{(n:\mathbb{N})} \mathtt{s}_n \sqsubseteq \mathtt{s}_{n+1} \tag{4.86}$$

$$\mathtt{Seq}_{\mathtt{M}_{(A\,,\,B)}} = \sum_{(s':\mathtt{M}_{(A\,,\,\mathtt{B})})} \sum_{(\mathtt{s}:\mathbb{N}\to\sum_{(a:A)}\mathtt{B}\,a\to\mathtt{M}_{(A\,,\,\mathtt{B})})} \left(\prod_{p:\sum_{a:A}\mathtt{B}\,a} s' \sqsubseteq (\mathtt{s}_0\ p)\right) \times \left(\prod_{(n:\mathbb{N})} \prod_{(p:\sum_{(a:A)} B\,a)} (s_n\ p) \sqsubseteq (s_{n+1}\ p)\right) \tag{4.87}$$

which other QM types can be expressed as QIITs

A QM-type is a quotiented M-type, we try to define this as a quotient on containers. We define container quotients as

## 4.6 Cofree Coalgebra / Dialgebra

Is this relevant?

# Chapter 5

# Properties of M-types?

## 5.1 Closure properties of $\mathtt{M}$-types

We want to show that $\mathtt{M}$-types are closed under simple operations, we start by looking at the product.

### 5.1.1 Product of $\mathtt{M}$-types

We start with containers and work up to $\mathtt{M}$-types.

**Definition 5.1.1.** The product of two containers is defined as [1]

$$(A, \mathtt{B}) \times (C, \mathtt{D}) \equiv (A \times C, \lambda\,(a, c), \mathtt{B}\ a \times \mathtt{D}\ c)\,. \tag{5.1}$$

We can lift this rule, through the diagram in Figure 5.1, used to define $\mathtt{M}$-types.

**Theorem 5.1.2.** *For any $n : \mathbb{N}$ the following is true*

$$\mathtt{P}_{(A,\mathtt{B})}{}^{n}\ \mathit{1} \times \mathtt{P}_{(C,\mathtt{D})}{}^{n}\ \mathit{1} \equiv \mathtt{P}_{(A,\mathtt{B})\times(C,\mathtt{D})}{}^{n}\ \mathit{1}. \tag{5.2}$$

*Proof.* We do induction on $n$, for $n = 0$, we have $\mathbf{1} \times \mathbf{1} \equiv \mathbf{1}$. For $n = m + 1$, we may assume

$$\mathtt{P}_{(A,\mathtt{B})}{}^{m}\ \mathbf{1} \times \mathtt{P}_{(C,\mathtt{D})}{}^{m}\ \mathbf{1} \equiv \mathtt{P}_{(A,\mathtt{B})\times(C,\mathtt{D})}{}^{m}\ \mathbf{1}, \tag{5.3}$$

in the following

$$\begin{aligned}
&\mathtt{P}_{(A,\mathtt{B})}{}^{m+1}\ \mathbf{1} \times \mathtt{P}_{(C,\mathtt{D})}{}^{m+1}\ \mathbf{1} && (5.4)\\
\equiv\ &\mathtt{P}_{(A,\mathtt{B})}(\mathtt{P}_{(A,\mathtt{B})}{}^{m}\ \mathbf{1}) \times \mathtt{P}_{(C,\mathtt{D})}(\mathtt{P}_{(C,\mathtt{D})}{}^{m}\ \mathbf{1}) && (5.5)\\
\equiv\ &\sum_{a:A} \mathtt{B}\ a \to \mathtt{P}_{(A,\mathtt{B})}{}^{m}\ \mathbf{1} \times \sum_{c:C} \mathtt{D}\ c \to \mathtt{P}_{(C,\mathtt{D})}{}^{m}\ \mathbf{1} && (5.6)\\
\equiv\ &\sum_{a,c:A\times C} (\mathtt{B}\ a \to \mathtt{P}_{(A,\mathtt{B})}{}^{m}\ \mathbf{1}) \times (\mathtt{D}\ c \to \mathtt{P}_{(C,\mathtt{D})}{}^{m}\ \mathbf{1}) && (5.7)\\
\equiv\ &\sum_{a,c:A\times C} \mathtt{B}\ a \times \mathtt{D}\ c \to \mathtt{P}_{(A,\mathtt{B})}{}^{m}\ \mathbf{1} \times \mathtt{P}_{(C,\mathtt{D})}{}^{m}\ \mathbf{1} && (5.8)
\end{aligned}$$

$$
\begin{tikzcd}
A & A \times C \arrow[l, "\pi_1"'] \arrow[r, "\pi_2"] & C \\
\mathtt{P}(\mathtt{M}_{(A\,,\,\mathtt{B})}) \arrow[u, "\pi_1"'] & \mathtt{P}(\mathtt{M}_{(A\,,\,\mathtt{B})\times(C\,,\,\mathtt{D})}) \arrow[l] \arrow[r] \arrow[u, "\pi_1"'] & \mathtt{P}(\mathtt{M}_{(C\,,\,\mathtt{D})}) \arrow[u, "\pi_1"'] \\
\mathtt{M}_{(A\,,\,\mathtt{B})} \arrow[u, "\mathtt{out}"'] \arrow[d, "\pi_{(!)}"'] & \mathtt{M}_{(A\,,\,\mathtt{B})\times(C\,,\,\mathtt{D})} \arrow[l] \arrow[r] \arrow[u, "\mathtt{out}"'] \arrow[d, "\pi_{(!)}"'] & \mathtt{M}_{(C\,,\,\mathtt{D})} \arrow[u, "\mathtt{out}"'] \arrow[d, "\pi_{(!)}"'] \\
\mathtt{P}_{(A\,,\,\mathtt{B})}{}^{n}\ \mathbf{1} \arrow[d, "\pi_{(n)}"'] & \mathtt{P}_{(A\,,\,\mathtt{B})\times(C\,,\,\mathtt{D})}{}^{n}\ \mathbf{1} \arrow[l] \arrow[r] \arrow[d, "\pi_{(n)}"'] & \mathtt{P}_{(C\,,\,\mathtt{D})}{}^{n}\ \mathbf{1} \arrow[d, "\pi_{(n)}"'] \\
\vdots \arrow[d, "\pi_{(1)}"'] & \vdots \arrow[d, "\pi_{(1)}"'] & \vdots \arrow[d, "\pi_{(1)}"'] \\
\mathtt{P}_{(A\,,\,\mathtt{B})}\ \mathbf{1} \arrow[d, "\pi_{(0)}"'] & \mathtt{P}_{(A\,,\,\mathtt{B})\times(C\,,\,\mathtt{D})}\ \mathbf{1} \arrow[l] \arrow[r] \arrow[d, "\pi_{(0)}"'] & \mathtt{P}_{(C\,,\,\mathtt{D})}\ \mathbf{1} \arrow[d, "\pi_{(0)}"'] \\
\mathbf{1} & \mathbf{1} \arrow[l, "\mathtt{refl}"] \arrow[r, "\mathtt{refl}"'] & \mathbf{1}
\end{tikzcd}
$$

Figure 5.1: Diagram for products of chains

$$\equiv \sum_{a,c:A\times C} \mathtt{B}\ a \times \mathtt{D}\ c \to \mathtt{P}_{(A\,,\,\mathtt{B})\times(C\,,\,\mathtt{D})}{}^{m}\ \mathbf{1} \tag{5.9}$$

$$\equiv \mathtt{P}_{(A\,,\,\mathtt{B})\times(C\,,\,\mathtt{D})}(\mathtt{P}_{(A\,,\,\mathtt{B})\times(C\,,\,\mathtt{D})}{}^{m}\ \mathbf{1}) \tag{5.10}$$

$$\equiv \mathtt{P}_{(A\,,\,\mathtt{B})\times(C\,,\,\mathtt{D})}{}^{m+1}\ \mathbf{1} \tag{5.11}$$

taking the limit of (5.2) we get

$$\mathtt{M}_{(A\,,\,\mathtt{B})} \times \mathtt{M}_{(C\,,\,\mathtt{D})} \equiv \mathtt{M}_{(A\,,\,\mathtt{B})\times(C\,,\,\mathtt{D})}. \tag{5.12}$$

□

**Example 3.** For streams we get

$$\mathtt{stream}\ A \times \mathtt{stream}\ B \equiv \mathtt{M}_{(A\,,\,\lambda\ \_,\mathbf{1})} \times \mathtt{M}_{(B\,,\,\lambda\ \_,\mathbf{1})} \equiv \mathtt{M}_{(A\,,\,\lambda\ \_,\mathbf{1})\times(B\,,\,\lambda\ \_,\mathbf{1})} \equiv \mathtt{stream}\ (A \times B) \tag{5.13}$$

as expected. Transporting along (5.13) gives us a definition for `zip`.

### 5.1.2 Co-product

Coproducts?

### 5.1.3 ...

The rest of the closures defined in "Categories of Containers" [1]

# Chapter 6

# Examples of M-types

## 6.1 TODO: Place these subsections

What makes a relation a bisimulation? Is bisim and equality equal.

### 6.1.1 Identity Bisimulation

Lets start with a simple example of a bisimulation namely the one given by the identity relation for any M-type.

**Lemma 6.1.1.** *The identity relation $(\cdot \equiv \cdot)$ is a bisimulation for any final coalgebra* $\mathtt{M}_S$-$\mathtt{out}$ *defined over an* M*-type.*

*Proof.* We first define the function

$$\begin{aligned} \alpha_{\equiv} &: \equiv \to \mathtt{P}(\equiv) \\ \alpha_{\equiv}(x, y) &:= \pi_1\ (\mathtt{out}\ x)\ ,\ (\lambda\, b, (\pi_2\ (\mathtt{out}\ x)\ b\, , \mathtt{refl}_{(\pi_2\, (\mathtt{out}\, x)\, b)})) \end{aligned} \tag{6.1}$$

and the two projections

$$\pi_1^{\equiv} = (\pi_1, \mathtt{funExt}\ \lambda\, (a, b, r), \mathtt{refl}_{\mathtt{out}\, a}) \tag{6.2}$$

$$\pi_2^{\equiv} = (\pi_2, \mathtt{funExt}\ \lambda\, (a, b, r), \mathtt{cong}_{\mathtt{out}}\ (r^{-1})). \tag{6.3}$$

This defines the bisimulation, given by the diagram in Figure 6.1. $\square$

$$\mathtt{M}\text{-}\mathtt{out} \xLeftarrow{\pi_1^{\equiv}} \equiv\text{-}\alpha_{\equiv} \xRightarrow{\pi_2^{\equiv}} \mathtt{M}\text{-}\mathtt{out}$$

Figure 6.1: Identity bisimulation

### 6.1.2 Bisimulation of Streams

TODO

### 6.1.3 Bisimulation of Delay Monad

We want to define a strong bisimulation relation $\sim_{\texttt{delay}}$ for the delay monad,

**Definition 6.1.2.** The relation $\sim_{\texttt{delay}}$ is defined by the following rules

$$\frac{R : U \quad r : R}{\texttt{now}\ r \sim_{\texttt{delay}} \texttt{now}\ r : \mathcal{U}}\ \texttt{now}\sim \tag{6.4}$$

$$\frac{R : \mathcal{U} \quad t : \texttt{delay}\ R \quad u : \texttt{delay}\ R \quad t \sim_{\texttt{delay}} u : \mathcal{U}}{\texttt{later}\ t \sim_{\texttt{delay}} \texttt{later}\ u : \mathcal{U}}\ \texttt{later}\sim \tag{6.5}$$

**Theorem 6.1.3.** *The relation* $\sim_{\texttt{delay}}$ *is a bisimulation for* $\texttt{delay}\ R$*.*

*Proof.* First we define the function

$$\begin{aligned}
\alpha_{\sim_{\texttt{delay}}} &: \overline{\sim_{delay}} \to \texttt{P}(\overline{\sim_{delay}}) \\
\alpha_{\sim_{\texttt{delay}}}\ (a, b, \texttt{now}\sim\ r) &:= (\texttt{inr}\ r, \lambda\,(\,)) \\
\alpha_{\sim_{\texttt{delay}}}\ (a, b, \texttt{later}\sim\ x\ y\ q) &:= (\texttt{inl}\ \star, \lambda\,_, (x, y, q))
\end{aligned} \tag{6.6}$$

then we define the projections

$$\pi_1^{\overline{\sim_{delay}}} = \left(\pi_1 \quad , \quad \texttt{funExt}\ \lambda\,(a, b, p), \begin{cases} (\texttt{inr}\ r, \lambda\,(\,)) & p = \texttt{now}\sim\ r \\ (\texttt{inl}\ \star, \lambda\,\_, x) & p = \texttt{later}\sim\ x\ y\ q \end{cases}\right) \tag{6.7}$$

$$\pi_2^{\overline{\sim_{delay}}} = \left(\pi_2 \quad , \quad \texttt{funExt}\ \lambda\,(a, b, p), \begin{cases} (\texttt{inr}\ r, \lambda\,(\,)) & p = \texttt{now}\sim\ r \\ (\texttt{inl}\ \star, \lambda\,\_, y) & p = \texttt{later}\sim\ x\ y\ q \end{cases}\right) \tag{6.8}$$

$$\tag{6.9}$$

This defines the bisimulation, given by the diagram in Figure 6.2. ☐

$$\texttt{delay}\ R\texttt{-out} \xLeftarrow{\pi_1^{\overline{\sim_{delay}}}} \overline{\sim_{\texttt{delay}}}\text{-}\alpha_{\sim_{\texttt{delay}}} \xRightarrow{\pi_2^{\overline{\sim_{delay}}}} \texttt{delay}\ R\texttt{-out}$$

Figure 6.2: Strong bisimulation for delay monad

### 6.1.4 Bisimulation of ITrees

We define our bisimulation coalgebra from the strong bisimulation relation $\mathcal{R}$, defined by the following rules.

$$\frac{a, b : R \quad a \equiv_R b}{\texttt{Ret}\ a \cong \texttt{Ret}\ b}\ \texttt{EqRet} \tag{6.10}$$

$$\frac{t, u : \texttt{itree E}\ R \quad t \cong u}{\texttt{Tau}\ t \cong \texttt{Tau}\ u}\ \texttt{EqTau} \tag{6.11}$$

$$\frac{A : \mathcal{U} \quad e : \texttt{E}\ A \quad k_1, k_2 : A \to \texttt{itree E}\ R \quad t \cong u}{\texttt{Vis}\ e\ k_1 \cong \texttt{Tau}\ e\ k_2}\ \texttt{EqVis} \tag{6.12}$$

Now we just need to define $\alpha_{\mathcal{R}}$

define the $\alpha_{\mathrm{R}}$ function

. Now we have a bisimulation relation, which is equivalent to equality, using what we showed in the previous section.

### 6.1.5 Zip Function

We want the diagram in Figure 6.3 to commute, meaning we get the computation rules

$$
\begin{tikzcd}
 & A \times B & \\
\mathtt{stream}\ A \times \mathtt{stream}\ B \arrow[ur, "\mathtt{hd}\times\mathtt{hd}"] \arrow[rr, "\mathtt{zip}"] \arrow[d, "\mathtt{tl}\times\mathtt{tl}"] & & \mathtt{stream}\ (A \times B) \arrow[ul, "\mathtt{hd}"'] \arrow[d, "\mathtt{tl}"] \\
\mathtt{stream}\ A \times \mathtt{stream}\ B \arrow[rr, "\mathtt{zip}"] & & \mathtt{stream}\ (A \times B)
\end{tikzcd}
$$

Figure 6.3: TODO

$$(\mathtt{hd} \times \mathtt{hd}) \equiv \mathtt{hd} \circ \mathtt{zip} \tag{6.13}$$

$$\mathtt{zip} \circ (\mathtt{tl} \times \mathtt{tl}) \equiv \mathtt{tl} \circ \mathtt{zip} \tag{6.14}$$

we can define the zip function as we did in the end of the last section. Another way to define the zip function is more directly, using the following lifting property of $\mathtt{M}$-types

$$\begin{aligned} &\mathtt{lift}_{\mathtt{M}}\ \left(x : \prod_{n:\mathbb{N}}(A \to \mathtt{P}_{S}{}^{n}\ \mathbf{1})\right)\ \left(u : \prod_{n:\mathbb{N}}(A \to \pi_n(x_{n+1}a) \equiv x_n a)\right) (a : A) : \mathtt{M}\ S := \\ &\quad (\lambda\, n, x\ n\ a), (\lambda\, n\ i, p\ n\ a\ i). \end{aligned} \tag{6.15}$$

To use this definition, we first define some helper functions

$$\mathtt{zip}_X\ n\ (x, y) = \begin{cases} \mathbf{1} & \mathtt{if}\ n = 0 \\ (\mathtt{hd}\ x, \mathtt{hd}\ y), (\lambda\, \_, \mathtt{zip}_X\ m\ (\mathtt{tl}\ x, \mathtt{tl}\ y)), & \mathtt{if}\ n = m + 1 \end{cases} \tag{6.16}$$

$$\mathtt{zip}_\pi\ n\ (x, y) = \begin{cases} \mathtt{refl} & \mathtt{if}\ n = 0 \\ \lambda\, i, (\mathtt{hd}\ x, \mathtt{hd}\ y), (\lambda\, \_, \mathtt{zip}_\pi\ m\ (\mathtt{tl}\ x, \mathtt{tl}\ y)\ i), & \mathtt{if}\ n = m + 1 \end{cases}, \tag{6.17}$$

we can then define

$$\mathtt{zip}_{lift}\ (x, y) := \mathtt{lift}_{\mathtt{M}}\ \mathtt{zip}_X\ \mathtt{zip}\ (x, y). \tag{6.18}$$

#### 6.1.5.1 Equality of Zip Definitions

We would expect that the two definitions for zip are equal

$$\begin{aligned} \mathtt{transport}_?\ a &\equiv \mathtt{zip}_{lift}\ a && (6.19) \\ &\equiv \mathtt{lift}_{\mathtt{M}}\ \mathtt{zip}_X\ \mathtt{zip}_\pi\ (x, y) && (6.20) \\ &\equiv (\lambda\, n, \mathtt{zip}_X\ n\ (x, y)), (\lambda\, n\ i, \mathtt{zip}_\pi\ n\ (x, y)\ i) && (6.21) \end{aligned}$$

zero case $X$

$$\mathtt{zip}_X\ 0\ (x,y) \equiv \mathbf{1} \tag{6.22}$$

Successor case $X$

$$\begin{aligned}
\mathtt{zip}_X\ (m+1)\ (x,y) &\equiv (\mathtt{hd}\ x, \mathtt{hd}\ y), (\lambda_, \mathtt{zip}_X\ m\ (\mathtt{tl}\ x, \mathtt{tl}\ y)) && (6.23)\\
&\equiv (\mathtt{hd}\ x, \mathtt{hd}\ y), (\lambda_, ?\ (\mathtt{tl}\ a)) && (6.24)\\
&\equiv (\mathtt{hd}\ (\mathtt{transport}_? a)), (\lambda_, \mathtt{transport}_? (\mathtt{tl}\ a)) && (6.25)\\
&\equiv \mathtt{transport}_?\ a && (6.26)\\
& && (6.27)
\end{aligned}$$

Zero case $\pi$: $(\lambda\, i, \mathtt{zip}_\pi\ 0\ (x,y)\ i \equiv refl)$.

$$\begin{aligned}
&\equiv (), (\lambda\, i, \mathtt{zip}_\pi\ 0\ (x,y)\ i) && (6.28)\\
&\equiv \mathbf{1}, \mathtt{refl} && (6.29)\\
& && (6.30)
\end{aligned}$$

successor case

$$\begin{aligned}
&\equiv (\mathtt{zip}_X\ (m+1)\ (x,y)), (\lambda\, i, \mathtt{zip}_\pi\ (m+1)\ (x,y)\ i) && (6.31)\\
&\equiv ((\mathtt{hd}\ x, \mathtt{hd}\ y), (\lambda_, \mathtt{zip}_X\ m\ (\mathtt{tl}\ x, \mathtt{tl}\ y))), (\lambda\, i, (\mathtt{hd}\ x, \mathtt{hd}\ y), (\lambda_, \mathtt{zip}_\pi\ m\ (\mathtt{tl}\ x, \mathtt{tl}\ y)\ i)) && (6.32)
\end{aligned}$$

Complete this proof

### 6.1.6 Examples of Fixed Points

#### 6.1.6.1 Zeros

Let us try to define the zero stream, we do this by lifting the functions

$$\mathtt{const_X}\ (n:\mathbb{N})\ (c:\mathbb{N}) := \begin{cases} \mathbf{1} & n = 0 \\ (c, \lambda\_, \mathtt{const_X}\ m\ c) & n = m+1 \end{cases} \tag{6.33}$$

$$\mathtt{const}_\pi\ (n:\mathbb{N})\ (c:\mathbb{N}) := \begin{cases} \mathtt{refl} & n = 0 \\ \lambda\, i, (c, \lambda\_, \mathtt{const}_\pi\ m\ c\ i) & n = m+1 \end{cases} \tag{6.34}$$

to get the definition of zero stream

$$\mathtt{zeros} := \mathtt{lift_M}\ \mathtt{const_X}\ \mathtt{const}_\pi\ 0. \tag{6.35}$$

We want to show that we get the expected properties, such as

$$\mathtt{hd}\ \mathtt{zeros} \equiv 0 \tag{6.36}$$

$$\mathtt{tl}\ \mathtt{zeros} \equiv \mathtt{zeros} \tag{6.37}$$

#### 6.1.6.2 Spin

We want to define spin, as being the fixed point `spin` = `later spin`, so that is again a final coalgebra, but of a `M`-type (which is a final coalgebra)
Since it is final, it also must be unique, meaning that there is just one program that spins forever, without returning a value, meaning every other program must return a value. If we just

$$\begin{tikzcd} \texttt{spin} \arrow[d, "\texttt{later}"] \\ \texttt{later spin} \end{tikzcd}$$

Figure 6.4: TODO

# Chapter 7

# Conclusion

conclude on the problem statement from the introduction

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## Appendix A

# Additions to the Cubical Agda Library

# Appendix B

# The Technical Details

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