Higher Order Categorical Semantics

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Abstract

in English…

Resumé

in Danish…

Acknowledgments

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Introduction

motivate and explain the problem to be addressed

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Chapter 2

M-types

2.1 Containers / Signatures

A Container (or Signature) is a pair $S = (A, B)$ of types $\vdash A : \mathcal{U}$ and $a : A \vdash B(a) : \mathcal{U}$. From a container we can define a polynomial functor, defined for objects (types) as

$$\begin{aligned} P_S &: \mathcal{U} \to \mathcal{U} \\ P(X) := P_S(X) &= \sum_{a:A} B(a) \to X \end{aligned} \tag{2.1}$$

and for a function $f : X \to Y$ as

$$\begin{aligned} Pf &: PX \to PY \\ Pf(a, g) &= (a, f \circ g) \end{aligned} \tag{2.2}$$

As an example lets look at type for streams over the type A, defined using the container $S = (A, \mathbf{1})$, applying the polynomial functor we get

$$P_S(X) = \sum_{a:A} \mathbf{1} \to X \tag{2.3}$$

since we are working in a Category with exponentials we get $\mathbf{1} \to X \equiv X^{\mathbf{1}} \equiv X$, furthermore $\mathbf{1}$ and X does not depend on A here, so this will be equivalent to the definition

$$P_S(X) = A \times X \tag{2.4}$$

Now we define the coalgebra for this functor with type

$$\texttt{Coalg}_S = \sum_{C:\mathcal{U}} C \to PC \tag{2.5}$$

and morphisms

$$\begin{aligned} \_ \Rightarrow \_ &: \texttt{Coalg}_S \to \texttt{Coalg}_S \\ (C, \gamma) \Rightarrow (D, \delta) &= \sum_{f:C\to D} \delta \circ f = Pf \circ \gamma \end{aligned} \tag{2.6}$$

M-types can now be defined from a container S as the type $\mathtt{M}$ such that $(\mathtt{M}, \mathtt{out} : \mathtt{M} \to P_S \mathtt{M})$ fulfills the property

$$\mathtt{Final}_S := \sum_{(X,\rho):\mathtt{Coalg}_S} \prod_{(C,\gamma):\mathtt{Coalg}_S} \mathtt{isContr}((C,\gamma) \Rightarrow (X,\rho)) \tag{2.7}$$

that is $\prod_{(C,\gamma):\mathtt{Coalg}_S} \mathtt{isContr}((C,\gamma) \Rightarrow (\mathtt{M}, \mathtt{out}))$. We denote this construction of the type $\mathtt{M}$, as $\mathtt{M}(A, B)$ or $\mathtt{M}S$.

If we continue our example for streams this will give us the $\mathtt{M}$-type, we can see that $P_S(\mathtt{M}) = A \times \mathtt{M}$, meaning we have the following diagram, where $\mathtt{out}$ is an isomorphism (because of the finality of

$$\begin{tikzcd}
A & A \times \mathtt{M} \arrow[l, "\pi_1"'] \arrow[r, "\pi_2"] & \mathtt{M} \\
& \mathtt{M} \arrow[ul, "\mathtt{hd}"] \arrow[u, "\mathtt{out}"'] \arrow[ur, "\mathtt{tl}"'] &
\end{tikzcd}$$

Figure 2.1: M-types of streams

the coalgebra), with inverse $\mathtt{in} : P_S\mathtt{M} \to \mathtt{M}$. We now have a semantic for the rules we would expect for streams, if we let $\mathtt{cons} = \mathtt{in}$ and $\mathtt{Stream}\ A = \mathtt{M}(A, \mathbf{1})$,

$$\frac{A : \mathcal{U} \quad s : \mathtt{Stream}\ A}{\mathtt{hd}\ s : A}\ \mathrm{E}_{\mathtt{hd}} \tag{2.8}$$

$$\frac{A : \mathcal{U} \quad s : \mathtt{Stream}\ A}{\mathtt{tl}\ s : \mathtt{Stream}\ A}\ \mathrm{E}_{\mathtt{tl}} \tag{2.9}$$

$$\frac{A : \mathcal{U} \quad x : A \quad xs : \mathtt{Stream}\ A}{\mathtt{cons}\ x\ xs : \mathtt{Stream}\ A}\ \mathrm{I}_{\mathtt{cons}} \tag{2.10}$$

2.2 ITrees as M-types

2.2.1 Delay Monad

We want the following rules for ITrees

$$\frac{r : R}{\mathtt{Ret}\ r : \mathtt{itree}\ E\ R}\ \mathrm{I}_{\mathtt{Ret}} \tag{2.11}$$

$$\frac{A : \mathcal{U} \quad a : E\ A \quad f : A \to \mathtt{itree}\ E\ R}{\mathtt{Vis}\ a\ f : \mathtt{itree}\ E\ R}\ \mathrm{I}_{\mathtt{Vis}}. \tag{2.12}$$

Elimination rules

$$\frac{t : \mathtt{itree}\ E\ R}{\mathtt{Tau}\ t : \mathtt{itree}\ E\ R}\ \mathrm{E}_{\mathtt{Tau}}. \tag{2.13}$$

We start by looking at $\mathtt{itree}$s without the $\mathtt{Vis}$ constructor, this type is also know as the delay monad . We say this type is given by $S = (\mathbf{1} + R, \lambda\{\mathtt{inl}\ \_ \to \mathbf{1}\ ;\ \mathtt{inr}\ \_ \to \mathbf{0}\})$ equal to $\mathtt{M}S$, we then get the polynomial functor

check this statement

$$P_S(X) = \sum_{x:\mathbf{1}+R} \lambda\{\mathtt{inl}\ \_ \to \mathbf{1}; \mathtt{inr}\ \_ \to \mathbf{0}\}\ x \to X \tag{2.14}$$

This type is equal to the type:

$$P_S(X) = X + R \times (\mathbf{0} \to X) \tag{2.15}$$

we know that $\mathbf{0} \to X \equiv \mathbf{1}$, so we can further reduce to

$$P_S(X) = X + R \tag{2.16}$$

meaning we get the following diagram. What this diagram says is that we can define the operations

$$\begin{tikzcd}
R \arrow[r, "\texttt{inl}"] \arrow[dr, "\texttt{now}"'] & R + \texttt{M} & \texttt{M} \arrow[l, "\texttt{inr}"'] \arrow[dl, "\texttt{later}"] \\
& \texttt{M} \arrow[u, "\texttt{out}"'] &
\end{tikzcd}$$

Figure 2.2: Delay monad

$\texttt{now}$ and $\texttt{later}$ using $\texttt{in} = \texttt{out}^{-1}$ together with the injections $\texttt{inl}$ and $\texttt{inr}$. (Later = Tau, Ret = Now)

2.2.2 Tree

Now lets look at the example where we remove the **Tau** constructor. We let

$$S = \left(R + \sum_{A:\mathcal{U}} E\ A\ ,\ \lambda\{\texttt{inl}\ \_ \to \mathbf{0}\ ;\ \texttt{inr}\ (A, e) \to A\} \right). \tag{2.17}$$

This will give us the polynomial functor:

$$P_S(X) = \sum_{x : R + \sum_{A:\mathcal{U}} E\ A} \lambda\{\texttt{inl}\ \_ \to \mathbf{0}\ ;\ \texttt{inr}\ (A, e) \to A\}\ x \to X \tag{2.18}$$

which simplifies to

$$P_S(X) = (R \times (\mathbf{0} \to X)) + (\sum_{A:\mathcal{U}} E\ A \times (A \to X)) \tag{2.19}$$

and further

$$P_S(X) = R + \sum_{A:\mathcal{U}} E\ A \times (A \to X) \tag{2.20}$$

$$\begin{tikzcd}
R \arrow[r, "\texttt{inl}"] \arrow[dr, "\texttt{Ret}"'] & R + \sum_{A:U} E\ A \times (A \to \texttt{M}) & \sum_{A:U} E\ A \times (A \to \texttt{M}) \arrow[l, "\texttt{inr}"'] \arrow[dl, "\texttt{Vis}"] \\
& \texttt{M} \arrow[u, "\texttt{out}"'] &
\end{tikzcd}$$

Figure 2.3: TODO: ???

Again we can define **Ret** and **Vis** using the **in** functor.

2.2.3 ITrees

Now we should have all the knowledge needed to make ITrees using $\mathtt{M}$-types. We define ITrees by the container:

$$S = \left(\mathbf{1} + R + \sum_{A:\mathcal{U}} (E\ A)\ \ ,\ \ \lambda\,\{\mathtt{inl}\ (\mathtt{inl}\ \_) \to \mathbf{1}\ ;\ \mathtt{inl}\ (\mathtt{inr}\ \_) \to \mathbf{0}\ ;\ \mathtt{inr}(A, \_) \to A\}\right) \tag{2.21}$$

Then the (reduced) polynomial functor becomes

$$P_S(X) = X + R + \sum_{A:\mathcal{U}} ((E\ A) \times (A \to X)) \tag{2.22}$$

Giving us the diagram

$$\begin{tikzcd}
R \arrow[r, "\mathtt{inl} \circ \mathtt{inr}"] \arrow[ddrr, "\mathtt{Ret}"'] & \mathtt{M} + R + \sum_{A:\mathcal{U}} (E\ A \times (A \to \mathtt{M})) & & \sum_{A:U} E\ A \times (A \to \mathtt{M}) \arrow[lll, "\mathtt{inr}"'] \arrow[ddl, "\mathtt{Vis}"] \\
 & & \mathtt{M} \arrow[ul, "\mathtt{inl} \circ \mathtt{inl}"'] \arrow[d, "\mathtt{Tau}"'] & \\
 & & \mathtt{M} \arrow[uul, "\mathtt{out}"] &
\end{tikzcd}$$

Figure 2.4: TODO: ???

2.3 Coinduction priciple of M-types

We can now construct a bisimulation: forall coalgebras $C\text{-}\gamma : \mathtt{Coalg}_S$, if we have a relation $\mathcal{R} : C \to C \to \mathcal{U}$, and a type $\overline{\mathcal{R}} = \sum_{a:C} \sum_{b:C} \mathcal{R}\ a\ b$, such that $\overline{\mathcal{R}}$ and $\alpha_{\overline{\mathcal{R}}} : \overline{\mathcal{R}} \to P(\overline{\mathcal{R}})$ makes a P-coalgebra $\overline{\mathcal{R}}\text{-}\alpha_{\overline{\mathcal{R}}} : \mathtt{Coalg}_S$, such that the following diagram commutes (where $\Rightarrow$ are P-coalgebra morphisms).

$$C\text{-}\gamma \xLeftarrow{{\pi_1}^{\overline{\mathcal{R}}}} \overline{\mathcal{R}}\text{-}\alpha_{\overline{\mathcal{R}}} \xRightarrow{{\pi_2}^{\overline{\mathcal{R}}}} C\text{-}\gamma$$

Furthermore for any bisimulation over a final P-coalgebra $\mathtt{M}\text{-}\mathtt{out} : \mathtt{Coalg}_S$ we have the following diagram,

$$\mathtt{M}\text{-}\mathtt{out} \xLeftarrow{{\pi_1}^{\overline{\mathcal{R}}}} \overline{\mathcal{R}}\text{-}\alpha_{\overline{\mathcal{R}}} \xRightarrow{{\pi_2}^{\overline{\mathcal{R}}}} \mathtt{M}\text{-}\mathtt{out}$$

where $\pi_1^{\overline{\mathcal{R}}} = ! = \pi_2^{\overline{\mathcal{R}}}$, which means given $r : \mathcal{R}(m, m')$ we get $m = \pi_1^{\overline{\mathcal{R}}}(m, m', r) = \pi_2^{\overline{\mathcal{R}}}(m, m', r) = m'$.

2.3.1 Bisimulation of ITrees

We define our bisimulation coalgebra from the strong bisimulation relation $\mathcal{R}$, defined by the following rules.

$$\frac{a, b : R \quad a \equiv_R b}{\mathtt{Ret}\ a \cong \mathtt{Ret}\ b}\ \mathtt{EqRet} \tag{2.23}$$

$$\frac{t, u : \texttt{itree}\ E\ R \quad t \cong u}{\texttt{Tau}\ t \cong \texttt{Tau}\ u}\ \texttt{EqTau} \tag{2.24}$$

$$\frac{A : \mathcal{U} \quad e : E\ A \quad k_1, k_2 : A \to \texttt{itree}\ E\ R \quad t \cong u}{\texttt{Vis}\ e\ k_1 \cong \texttt{Tau}\ e\ k_2}\ \texttt{EqVis} \tag{2.25}$$

Now we just need to define $\alpha_{\overline{\mathcal{R}}}$. Now we have a bisimulation relation, which is equivalent to equality, using what we showed in the previous section.

define the $\alpha_{\overline{R}}$ function

2.4 Examples of fixed points

We want to define spin, as being the fixed point $\texttt{spin} = \texttt{later spin}$, so that is again a final coalgebra, but of a M-type (which is a final coalgebra)

$$\begin{tikzcd}
\texttt{spin} \arrow[d, "\texttt{later}"] \\
\texttt{later spin}
\end{tikzcd}$$

Since it is final, it also must be unique, meaning that there is just one program that spins forever, without returning a value, meaning every other program must return a value. If we just

2.5 Quotiented M-types

We can define a quotient of the polynomial functor P as a functor F, such that the following diagram commutes

$$\begin{tikzcd}
P\alpha \arrow[r, "Pf"] \arrow[d, "abs_\alpha"'] & P\beta \arrow[d, "abs_\beta"] \\
F\alpha \arrow[r, "Ff"] & F\beta
\end{tikzcd}$$

for all $f : \alpha \to \beta$ and α, β. This is a naturallity condition for abs. As an example lets look at the quotient of the delay monad by the equality $\forall a : \texttt{M}, \texttt{later}\ a \equiv a$. We get the diagram
Such that the (co?)limit for $F\ \texttt{M}$ is $\mathbf{1} + R$, meaning

$$F\ later \circ abs_M \circ out \equiv abs_M \circ out \circ later \tag{2.26}$$

$$F\ later \equiv later^{-1} \tag{2.27}$$

$$\begin{tikzcd}
\mathtt{M} \arrow[r, "\mathtt{later}"] \arrow[d, "\mathtt{out}"'] & \mathtt{M} \arrow[d, "\mathtt{out}"] \\
P\mathtt{M} \arrow[r, "P\ \mathtt{later}"] \arrow[d, "abs_{\mathtt{M}}"'] & P\mathtt{M} \arrow[d, "abs_{\mathtt{M}}"] \\
F\mathtt{M} \arrow[r, "F\ \mathtt{later}"] & F\mathtt{M}
\end{tikzcd}$$

$$\begin{aligned}
later &= in \circ inr && (2.28)\\
out \circ later &= inr && (2.29)\\
&= P\ later \circ out && (2.30)\\
abs_M \circ out \circ later &= abs_M \circ inr && (2.31)\\
&= abs_M \circ P\ later \circ out && (2.32)\\
&= F\ later \circ abs_M \circ out && (2.33)\\
& && (2.34)
\end{aligned}$$

Given the container of the final P-coalgebra for some container S, we can construct the diagram:

$$\begin{tikzcd}
(\mathtt{M}, P\ \mathtt{M}) \arrow[rr, "\mathtt{Tau}"] \arrow[dr, "\sim_x : \mathtt{Tau}\ x\ \equiv\ x"'] & & (\mathtt{M}, P\ \mathtt{M}) \arrow[dl, "\sim_y : \mathtt{Tau}\ y\ \equiv\ y"] \\
& ({}^{\mathtt{M}}\!/\!\sim, F\ {}^{\mathtt{M}}\!/\!\sim) &
\end{tikzcd}$$

Such that $\sim_x \equiv \sim_y \circ \mathtt{Tau}$, where $P\ {}^{\mathtt{M}}\!/\!\sim\ \equiv F\ ({}^{\mathtt{M}}\!/\!\sim)$

Chapter 3

Conclusion

conclude on the problem statement from the introduction

Bibliography

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Appendix A

The Technical Details

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