(Q)M-types and Coinduction in HoTT / CTT

Master's Thesis, Computer Science

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Overview

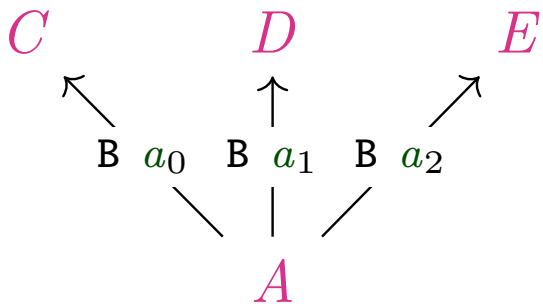
Goals

- Formalize coinductive types as M-types
- Define equality for coinductive types
- Explore ways to define quotiented M-types

Containers and Polynomial functors

Definition

A Container (or signature) is a dependent pair $S = (A\,, \mathtt{B})$ for the types $A : \mathcal{U}$ and $\mathtt{B} : A \to \mathcal{U}$.



Containers and Polynomial functors

Definition

A polynomial functor $\mathtt{P}_S$ (or extension) for a container $S = (A \,, \mathtt{B})$ is defined, for types as

$$\mathtt{P}_S\, X = \sum_{(a:A)} \mathtt{B}a \to X \tag{1}$$

and for a function $\mathtt{f} : X \to Y$ as

$$\mathtt{P}_S\, \mathtt{f}\ (a, \mathtt{g}) = (a, \mathtt{f} \circ \mathtt{g}). \tag{2}$$

Chain

Definition (Chain)

We define a chain as a family of morphisms $\pi_{(n)} : X_{n+1} \to X_n$, over a family of types X_n. See figure.

$$X_0 \xleftarrow[\pi_{(0)}]{} X_1 \xleftarrow[\pi_{(1)}]{} \cdots \xleftarrow[\pi_{(n-1)}]{} X_n \xleftarrow[\pi_{(n)}]{} X_{n+1} \xleftarrow[\pi_{(n+1)}]{} \cdots$$

Definition

The limit of a chain is given as

$$\mathcal{L} = \sum_{(x:\prod_{(n:\mathbb{N})} X_n)} \prod_{(n:\mathbb{N})} (\pi_{(n)}\ x_{n+1} \equiv x_n) \tag{3}$$

We let $\mathtt{M}_S$ be the limit, for a chain defined by $X_n = \mathrm{P}^n\,\mathbf{1}$, and $\pi_{(n)} = \mathrm{P}^n\,!$

Equality between $\mathcal{L}$ and $\mathtt{P}\,\mathcal{L}$

Theorem

There is an equality

$$shift : \mathtt{M} \equiv \mathtt{P}\,\mathtt{M} \tag{4}$$

from which we can define helper functions

$$\mathtt{in} : \mathtt{P}\,\mathtt{M} \to \mathtt{M} \qquad \mathtt{out} : \mathtt{M} \to \mathtt{P}\,\mathtt{M} \tag{5}$$

Proof structure

The proof is done using the two helper lemmas

$$\alpha : \mathcal{L}^{\mathtt{P}} \equiv \mathtt{P}\,\mathcal{L} \tag{6}$$

$$\mathcal{L}unique : \mathcal{L} \equiv \mathcal{L}^{\mathtt{P}} \tag{7}$$

where $\mathcal{L}^{\mathtt{P}}$ is the limit of the shifted chain defined as ${X'}_n = X_{n+1}$ and ${\pi'}_{(n)} = \pi_{(n+1)}$. With these two lemmas we get $shift = \alpha \cdot \mathcal{L}unique$.

Coalgebra

M-types are final coalgebras

Definition

We define a P-coalgebra as

$$\sum_{(C:\mathcal{U})} C \to \mathtt{P}\, C \tag{8}$$

which we denote $C\text{-}\gamma$. We define P-coalgebra morphisms as

$$C\text{-}\gamma \Rightarrow D\text{-}\delta = \sum_{(\mathtt{f}:C\to D)} \delta \circ \mathtt{f} \equiv \mathtt{P}\mathtt{f} \circ \gamma \tag{9}$$

A coalgebra is final, if the following is true

$$\sum_{(D\text{-}\rho)} \prod_{(C\text{-}\gamma)} \mathtt{isContr}\ (C\text{-}\gamma \Rightarrow D\text{-}\rho) \tag{10}$$

M-types are final coalgebras

Theorem

The M-type $\mathtt{M}_S$ *is defined as the limit for a polynomial functor* $\mathtt{P}_S$*. This definition fulfills the requirement that* $\mathtt{Final}_S\ \mathcal{L}$*.*

Proof structure

The definition of finality is

$$\prod_{(C\text{-}\gamma:\mathtt{Coalg}_S)} \mathtt{isContr}\ (C\text{-}\gamma \Rightarrow \mathcal{L}\text{-}\mathtt{out}) \tag{11}$$

which we show by $(C\text{-}\gamma \Rightarrow \mathcal{L}\text{-}\mathtt{out}) \equiv \mathbf{1}$.

Overview

Example: Delay Monad

A delay monad is defined by the two constructors

$$\frac{r : R}{\texttt{now}\ r : \texttt{Delay}\ R} \quad (12) \qquad \frac{t : \texttt{Delay}\ R}{\texttt{later}\ t : \texttt{Delay}\ R} \quad (13)$$

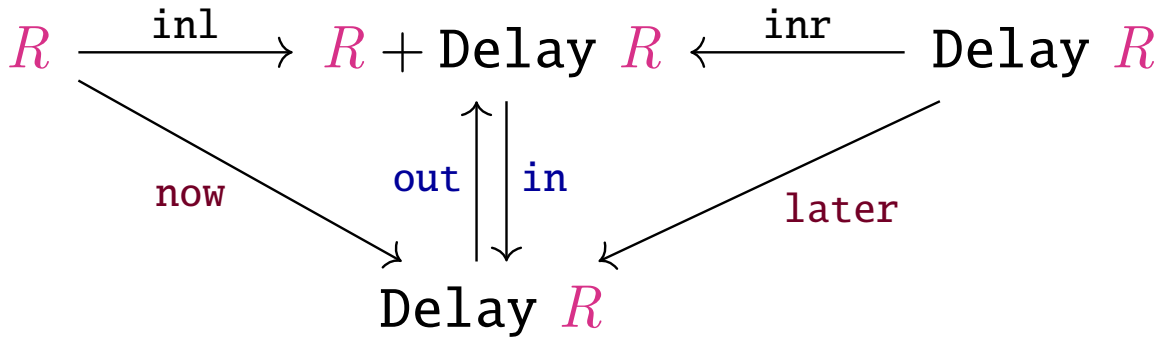
We define a container

$$(R + \mathbf{1}\,, [\mathbf{0}, \mathbf{1}]) \quad (14)$$

and a polynomial functor

$$\mathtt{P}\, X = \sum_{(x : R + \mathbf{1})} \begin{cases} \mathbf{0} & x = \texttt{inl}\ r \\ \mathbf{1} & x = \texttt{inr}\ \star \end{cases} \to X \quad = R + X, \quad (15)$$

such that we get the diagram



Rules for Constructing M-types

Adding containers $(A \,, \mathtt{B})$ and $(C \,, \mathtt{D})$ for two constructors together is done by

$$\left(A + C \,, \begin{cases} \mathtt{B}\, a & \mathtt{inl}\ a \\ \mathtt{D}\, c & \mathtt{inr}\ c \end{cases} \right) \tag{16}$$

whereas adding containers for two destructors is done by

$$(A \times C \,, \lambda\, (a, c),\ \mathtt{B}\, a + \mathtt{D}\, c) \tag{17}$$

However combining both destructors and constructors is not as simple. Which is similar to rules for coinductive records.

M-types and Records

We can take a coinductive record and transform it to an M-type. The types of fields in a coinductive records are

- non-dependent fields
- dependent fields
- recursive fields
- dependent and recursive fields

We will give some examples of these.

M-types and Records

Example: Record to M-type

Lets try and convert the record

$$\begin{aligned} &\texttt{record}\ \mathit{tree} : \mathcal{U}\ \texttt{where} \\ &\quad \mathit{value} : \mathbb{N} \\ &\quad \mathit{left\text{-}child} : \mathit{tree} \\ &\quad \mathit{right\text{-}child} : \mathit{tree} \end{aligned} \tag{18}$$

To an M-type defined by the container

$$(\mathbb{N}\,,\, \mathit{tree} \times \mathit{tree}) \tag{19}$$

M-types and Records

Example: Record to M-type

Lets try and convert the record

$$\begin{aligned} &\texttt{record}\ \mathit{bet} : \mathcal{U}\ \texttt{where} \\ &\quad \mathit{value}_a : \mathbb{N} \\ &\quad \mathit{value}_a : \mathbb{N} \\ &\quad \mathit{total} : \mathit{value}_a \leq \mathit{value}_b \to \mathit{bool} \end{aligned} \tag{20}$$

To an M-type defined by the container

$$\left(\sum_{(\mathit{value}_a : \mathbb{N})} \sum_{(\mathit{value}_b : \mathbb{N})} \mathit{value}_a \leq \mathit{value}_b \to \mathit{bool}\,, \mathbf{0} \right) \tag{21}$$

M-types and Records

Example: Record to M-type

Lets try and convert the record

$$\begin{aligned}&\texttt{record example } A : \mathcal{U} \texttt{ where}\\ &\quad \mathit{value} : A\\ &\quad \mathit{index\text{-}type} : \mathcal{U}\\ &\quad \mathit{continue} : \mathit{index\text{-}type} \to \texttt{example } A\end{aligned} \tag{22}$$

To an M-type defined by the container

$$(A \times \mathcal{U} \, , \lambda \, (\_, \mathit{index\text{-}type}), \mathit{index\text{-}type}) \tag{23}$$

Overview

Example: Streams

We can now define streams for a given type A, as a records

$$\begin{aligned} &\texttt{record Stream } A : \mathcal{U} \texttt{ where} \\ &\quad hd : A \\ &\quad tl : \texttt{Stream } A \end{aligned} \tag{24}$$

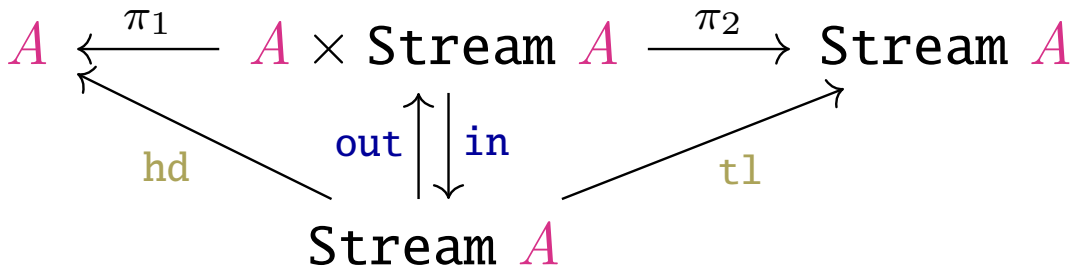
corresponding to the container

$$(A \,, \mathbf{1}) \tag{25}$$

for which we get the polynomial functor

$$\mathtt{P}\, X = A \times X \tag{26}$$

For which the we get the M-type for stream



Bisimulation for Streams

We can define an equivalence relation

$$\frac{\mathtt{hd}\ s \equiv \mathtt{hd}\ t \quad \mathtt{tl}\ s \sim_{\mathtt{stream}} \mathtt{tl}\ t}{s \sim_{\mathtt{stream}} t} \tag{27}$$

We can formalize this as a bisimulation. A (strong) bisimulation for a P-coalgebra C-γ is given by

- a relation $\mathcal{R} : C \to C \to \mathcal{U}$
- a type $\overline{\mathcal{R}} = \sum_{(a:C)} \sum_{(b:C)} a\ \mathcal{R}\ b$
- and a function $\alpha_{\mathcal{R}} : \overline{\mathcal{R}} \to \mathtt{P}_S\, \overline{\mathcal{R}}$

Such that $\overline{\mathcal{R}}$-$\alpha_{\mathcal{R}}$ is a P-coalg, making the following diagram commute.

$$C\text{-}\gamma \xleftarrow{\pi_1^{\overline{\mathcal{R}}}} \overline{\mathcal{R}}\text{-}\alpha_{\mathcal{R}} \xrightarrow{\pi_2^{\overline{\mathcal{R}}}} C\text{-}\gamma$$

In MLTT this is not enough to define an equality, however in HoTT and CTT it is.

Coinduction Principle

Theorem (Coinduction principle)

Given a relation $\mathcal{R}$, that is a bisimulation for a `M`*-type, then (strongly) bisimilar elements $x\ \mathcal{R}\ y$ are equal $x \equiv y$.*

Proof.

We get the diagram

$$\texttt{M-out} \xleftarrow{\pi_1^{\overline{\mathcal{R}}}} \overline{\mathcal{R}}\text{-}\alpha_{\mathcal{R}} \xrightarrow{\pi_2^{\overline{\mathcal{R}}}} \texttt{M-out}$$

Since `M-out` is a final coalgebra, functions into it are unique, meaning

$$\pi_1^{\overline{\mathcal{R}}} \equiv \pi_2^{\overline{\mathcal{R}}} \tag{28}$$

therefore given $r : x\ \mathcal{R}\ y$, we can construct the equality

$$x \equiv \pi_1^{\overline{\mathcal{R}}}(x, y, r) \equiv \pi_2^{\overline{\mathcal{R}}}(x, y, r) \equiv y. \tag{29}$$

□

Overview

Propositional Truncation and Set Truncated Quotient

A Higher Inductive Type (HIT) is a type defined by point constructors as well as equality constructors. We define propositional truncation by

Definition (Propositional Truncation)

$$\frac{x : A}{|\,x\,| : \|A\|} \quad (30) \qquad \frac{x, y : \|A\|}{\mathtt{squash}\ x\ y : x \equiv y} \quad (31)$$

We can define set truncated quotients as the following HIT.

Definition (Set Truncated Quotient)

$$\frac{x : A}{[\,x\,] : A/\mathcal{R}} \quad (32) \qquad \frac{x, y : A/\mathcal{R} \quad r : x\ \mathcal{R}\ y}{\mathtt{eq}/\ x\ y\ r : x \equiv y} \quad (33)$$

$$\frac{}{\mathtt{squash}/ : \mathtt{isSet}\ (A/\mathrm{R})} \quad (34)$$

Partiality monad

QM-type

We would like to model equality of programs as two programs being equal if one terminates with a value then the other also terminates with the same value a finite number of steps later. This is model by quotienting the delay monad by a relation defined by the constructors

$$\frac{x \sim y}{\texttt{later}\ x \sim y}\ \sim_{\texttt{later}_\texttt{l}} \quad (35) \qquad \frac{x \sim y}{x \sim \texttt{later}\ y}\ \sim_{\texttt{later}_\texttt{r}} \quad (36)$$

$$\frac{a \equiv b}{\texttt{now}\ a \sim \texttt{now}\ b}\ \sim_{\texttt{now}} \quad (37) \qquad \frac{x \sim y}{\texttt{later}\ x \sim \texttt{later}\ y}\ \sim_{\texttt{later}} \quad (38)$$

This gives us a construction for a QM-types.

$$\texttt{Delay}\ R/\sim \quad (39)$$

Quotient Inductive-Inductive Type (QIIT)

A QIIT is a type that is defined at the same time as a relation. That is

- Constructors may refer to previously defined constructors (recursive).
- Constructors can be a point constructor or equality constructor.
- Constructors may refer to constructors of the relation.
- The type is set truncated.

And similarly for the constructors of the relation.

Partiality Monad

QIIT

We can define it by type constructors

$$\frac{}{R_\perp : \mathcal{U}} \quad (40) \qquad \frac{}{\perp : R_\perp} \quad (41) \qquad \frac{a : R}{\eta\ a : R_\perp} \quad (42)$$

and an ordering relation $(\cdot \sqsubseteq_\perp \cdot)$ indexed twice over $R_\perp$

$$\frac{x : R_\perp}{x \sqsubseteq_\perp x}\ \sqsubseteq\texttt{refl} \quad (43) \qquad \frac{x \sqsubseteq_\perp y \quad y \sqsubseteq_\perp z}{x \sqsubseteq_\perp z}\ \sqsubseteq\texttt{trans} \quad (44)$$

$$\frac{x, y : R_\perp \quad p : x \sqsubseteq_\perp y \quad q : y \sqsubseteq_\perp x}{\alpha_\perp\ p\ q : x \equiv y} \quad (45)$$

where $\perp$ is lowest in the order

$$\frac{x : R_\perp}{\perp \sqsubseteq_\perp x}\ \sqsubseteq\texttt{never} \quad (46)$$

with an upper bound

$$\frac{\mathtt{s} : \mathbb{N} \to R_\perp \quad \mathtt{b} : \prod_{(n:\mathbb{N})} \mathtt{s}_n \sqsubseteq_\perp \mathtt{s}_{n+1}}{\bigsqcup (\mathtt{s}, \mathtt{b}) : R_\perp} \tag{47}$$

which gives a bound for a sequence

$$\frac{\mathtt{s} : \mathbb{N} \to R_\perp \quad \mathtt{b} : \prod_{(n:\mathbb{N})} \mathtt{s}_n \sqsubseteq_\perp \mathtt{s}_{n+1}}{\prod_{(n:\mathbb{N})} \mathtt{s}_n \sqsubseteq_\perp \bigsqcup(\mathtt{s}, \mathtt{b})} \tag{48}$$

and which is a least upper bound

$$\frac{\prod_{(n:\mathbb{N})} \mathtt{s}_n \sqsubseteq_\perp x}{\bigsqcup(\mathtt{s}, \mathtt{b}) \sqsubseteq_\perp x} \tag{49}$$

and finally set truncated by the constructor $(-)_\perp$-isSet

Partiality Monad

Equality

To show these two definitions are equal we

- Define an intermediate type of sequences

$$\mathsf{Seq}\ A = \sum_{(\mathsf{s}:\mathbb{N}\to A+\mathbf{1})} \prod_{(n:\mathbb{N})} \mathsf{s}_n \sqsubseteq_{\mathsf{R}+\mathbf{1}} \mathsf{s}_{n+1} \tag{50}$$

- Show that the Delay monad is equal to this intermediate type
- Define weak bisimilarty for sequences, and show it respects the equality to the Delay monad
- Show that $\mathsf{Seq}_R/\sim$ is equal to the partiality monad, by
  - Defining a function from $\mathsf{Seq}_R/\sim$ to $R_\perp$
  - Show this function is injective and surjective

However the proof for surjectivity requires the axiom of countable choice!

Simple Partialty Monad QIIT

We can see that the QIIT for the partiality monad is not trivial, however we can define quotients as QIITs rather trivially, however this would *only* give us the constructors

$$\frac{}{\texttt{now}\ a} \tag{51}$$

$$\frac{x}{\texttt{later}\ x} \tag{52}$$

$$\frac{x \equiv y}{\texttt{later}\ x \equiv y}\ \texttt{later}_{\equiv} \tag{53}$$

What do we want from a quotient M-type?

- We would like to be able to construct a quotient from an M-type and a relation.
- We should be able to take an element to its equality class without the axiom of choice
- It should be equal to the set truncated quotient, if we assume the axiom of choice?

Alternative: Quotient Polynomial Functor (QPF)

We can define quotiented $\mathtt{M}$-types from a quotient polynomial functor.

Definition (Quotient Polynomial Functor)

We define a quotient polynomial functor (QPF), for types as

$$\mathtt{F}\, X = \sum_{(a:A)} ((\mathtt{B}\, a \to X)/\sim_a) \tag{54}$$

and for a function $\mathtt{f} : X \to Y$, we use the quotient eliminator with

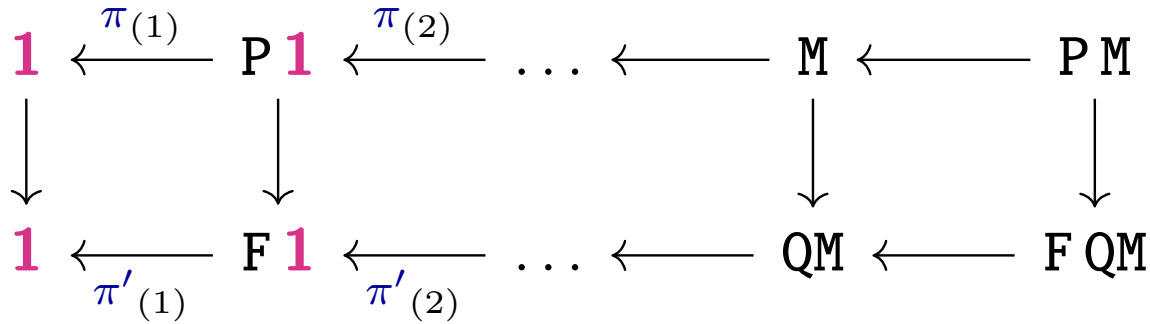
$$\mathtt{P} = \lambda \_, (\mathtt{B}\, a \to Y)/\sim_a \tag{55}$$

for which we need $\sim_{\mathtt{ap}}$, which says that given a function $\mathtt{f}$ and $\mathtt{x} \sim_a \mathtt{y}$ then $\mathtt{f} \circ \mathtt{x} \sim_a \mathtt{f} \circ \mathtt{y}$.

$$\mathtt{F}\, \mathtt{f}\ (a, \mathtt{g}) = (a, \mathtt{elim}\ \mathtt{g}) \tag{56}$$

Alternative: Quotient Polynomial Functor (QPF)

We get the diagram



For which $\mathtt{M} \equiv \mathtt{P}\,\mathtt{M}$ and we would hope $\mathtt{QM} \equiv \mathtt{F}\,\mathtt{QM}$, however this requires the axiom of choice.

Conclusion

We have

- given a formalization/semantic of M-types
- shown examples of and rules for how to construct M-types
- given a coinduction principle for M-types
- described the construction of the partiality monad as a QIIT
- discussed ways of constructing quotient M-types

Contribution

- Formalization in Cubical Agda
- Introducing QM-types

Future Work

- Indexed M-types
- Showing finality of QM-types and fully formalizing the constructions
- Equality between Coinductive records and M-types
- Explore Guarded Cubical Type Theory