

Exploring interpretations
of
probabilistic algorithms
using measure theory
in Coq

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1 Introduction (Finished 12.06.)

Probabilistic algorithms are widely used, but far less widely proved correct. We explore some probabilistic languages and how to embed them in the widely known and trusted proof assistant Coq. Having an interpretation of such a language in Coq makes it possible to prove facts about the algorithms in Coq's proof logic.

We explore both a functional design, as presented in [1], and an imperative one, as in [2].

Requirements to the reader:

- A rudimentary understanding of probability theory, but no knowledge of measure theory
- Section 2.2 requires an understanding of monads and how they are used in functional programming
- Section 4 requires a basic understanding of Coq

2 Theory and existing frameworks (DRAFT READY, minor additions needed; should be done by 03.06.)

As mentioned before we focused our work on developing a framework for proofs of randomised algorithms in the proof assistant Coq. Coq is known to be a reliable tool and proofs formalised with it are widely trusted. Coq is, however, also known to be notoriously difficult to code things in due to a strict type system that requires determinism and certain termination of all programs written in it. Obviously these two are not the optimal conditions for the encoding of randomised algorithms that may not terminate.

This means that we need a way to encode an interpretation general recursion (or iteration) as well as randomness in such a way that we can still reason about our programs in the proof system of Coq without having to run them.

We will in this section present a monadic interpretation into probability distribution over the outcome of a randomised program as well as a way of interpreting general recursion.

In this section we will have a look at how to use complete partial orders to interpret general recursion (Section 2.1), and how we can represent the result of a probabilistic computation using a monadic interpretation of probability measures (Section 2.2). Afterwards we will move on to presenting two different developments that have worked with probabilistic languages in Coq: the functional `Rml` (Section 2.3) and the imperative `pwhile` (Section 2.4).

2.1 Complete partial orders

A partially ordered set (poset) is a set with an associated binary ordering relation \leq which is both reflexive and transitive. The order is partial when the ordering relation is not defined on every pair of elements in the set.

There exist a number of different completeness properties that a poset can have. We will here have a look at ω -complete partial orders, which we will use in order to interpret general recursion and probabilistic programs.

Definition 1. *ω -complete partial order (ω -cpo)*

An ω -cpo is a partially ordered set that, additionally, has a distinct least element and where there exist least upper bounds on all monotonic sequences.

2.1.1 Recursive definitions as fixed point iterations

Before using ω -cpo to interpret recursion, let us first have a look at some interesting things that our definition entails.

A monotonic sequence on an ω -cpo X can be viewed as a monotonic function $f : \mathbb{N} \xrightarrow{m} X$ where $f(n)$ is the n th element of the sequence (or the least upper bound of the sequence, if n is larger than the length of the sequence).

There is a standard way of defining fixed point iterations on an ω -cpo: [1]

Consider an operator $F : X \xrightarrow{m} X$ on some ω -cpo X ; with this we define the monotonic sequence $F_i \mapsto \underbrace{F(F(\dots F(0_X)\dots))}_{i \text{ times}}$ of repeated application

of F to the least element of X . By our choice of F and the definition of ω -cpo, it is clear that there has to exist a least upper bound on F_i . This least upper bound is the fixed point of F and it will hold that $\text{fix } F = F(\text{fix } F)$ if F is continuous.

For an ω -cpo with underlying set B we can also define an ω -cpo on functions from any set A whose co-domain is B .

To reiterate the definition, let us think of what we need for an ω -cpo. We need an ordering relation, a least element, and a least upper bound operation. Those can be defined as follows:

$$\begin{aligned} f \leq_{A \rightarrow B} g &\Leftrightarrow \forall x : f(x) \leq_B g(x) && (\text{pointwise order}) \\ 0_{A \rightarrow B} &:= f(x) = 0_B && (\text{least element}) \\ \text{lub}_{A \rightarrow B} f_n &:= g(x) = \text{lub}_B(f_n(x)) && (\text{least upper bound operation}) \end{aligned}$$

The result of an interpretation of programs in the language of discourse will be in an ω -cpo, so according to the above discussion functions will have an ω -cpo structure as well. Together with the above definition of fixed points we can use this structure to interpret general recursive definitions.

We define a functional, F , taking as input a function and “adding a step to it”. Let us look at the example of the factorial function $f(n) = n!$ [3]. The recursive definition is well known:

$$fac(n) := \text{if } n = 0 \text{ then } 1 \text{ else } n \cdot fac(n - 1)$$

For the interpretation of this definition, we want to define $F : (\mathbb{N} \rightarrow \mathbb{N}) \rightarrow (\mathbb{N} \rightarrow \mathbb{N})$ in such a way that its fixed point is the same as the above recursive definition. We choose

$$F(F_i(n)) := \text{if } n = 0 \text{ then } 1 \text{ else } n \cdot F_i(n - 1)$$

Where $F_0(n)$ is $0_{\mathbb{N} \rightarrow \mathbb{N}}$ (the function that takes a natural number and returns 0), by the above definition of the least element in the ω -cpo defined on a function space. By repeated application of F the function will slowly approach the real factorial function, which is the fixed point of F . The beginning of the iteration will be

$$\begin{aligned} F_1(n) = F(F_0(n)) &= \begin{cases} 1 & \text{if } n \text{ is } 0 \\ 0 & \text{otherwise} \end{cases} \\ F_2(n) = F(F(F_0(n))) &= \begin{cases} 1 & \text{if } n \text{ is } 0 \\ 1 & \text{if } n \text{ is } 1 \\ 0 & \text{otherwise} \end{cases} \\ F_3(n) = F(F(F(F_0(n)))) &= \begin{cases} 1 & \text{if } n \text{ is } 0 \\ 1 & \text{if } n \text{ is } 1 \\ 2 & \text{if } n \text{ is } 2 \\ 0 & \text{otherwise} \end{cases} \end{aligned}$$

In this case it is easy to see that $F_0 \leq F_1 \leq F_2 \leq F_3 \leq \dots$. In the general case this follows from the fact that F has to be monotone and F_0 is always the least element of the function space F operates on.

2.2 Interpreting probabilistic definitions

In order to interpret probabilistic definitions, we need a way of expressing the distribution of possible results. For this we will use probability measures as described in [1].

2.2.1 The concept of measures

In layman's terms, we can describe a measure on a set A as exactly that: a way of measuring subsets of A . More precisely, a measure on A assigns a non-negative real number to every "suitable" subset of A , where "suitable" means fulfilling certain arbitrary conditions. We will henceforth write $\mu(X)$ to signify the value of $X \subseteq A$ under the measure μ .

In order for a function to be a measure, there are three properties it has to have: It must take only non-negative values, $\mu(\emptyset) = 0$, and it has to be countably additive. Being countably additive means that for every set of pairwise disjoint objects, the value of this set is equal to the sum of the values of each object:

$$\mu(X) = \mu\left(\bigcup_{x \in X} x\right) = \sum_{x \in X} \mu(x)$$

An example might be to choose A to be a set of 3-dimensional objects and a possible measure would be the total volume of objects in a subset.

We can understand a measure on A as integral over functions from A to \mathbb{R}^+ . From this perspective, the above example would consist of the function that given an object in A returns its volume. The integral over this function would be the volume of all objects in A . Now the challenge is to measure only a subset of A . We can do this by introducing the characteristic function of a subset X [4]:

Is this understood correctly?

$$\mathbb{I}_X(x) = \begin{cases} 1 & \text{if } x \in X \\ 0 & \text{otherwise} \end{cases}$$

By multiplying the characteristic function for X with the volume function on objects in A , we get a new function, f , such that $\int f \, d\mu = \mu(X)$. With this in place, we will allow ourselves to be sloppy in our notation and write $\mu(f)$ instead of $\int f \, d\mu$.

It is easy to see that the integral perspective still satisfies all the requirements that a function has to fulfil in order to be a measure, and the reader is invited to check this for herself.

Now why all this talk about measures? Wasn't it probabilistic programs we were talking about?

Well, yes. The cool thing is, that being able to measure function whose co-domain is the real numbers in the unit interval, $\tau \rightarrow [0, 1]$, gives us a way of representing probability distributions. A measure on type τ can be expressed with the type $(\tau \rightarrow [0, 1]) \rightarrow [0, 1]$. An interpretation of a probabilistic term whose type is τ can now be understood as a measure on type τ , or equivalently as a transformation of a probability distribution.

Something of type $\tau \rightarrow [0, 1]$ can be understood as the function that returns the probability of its input happening, or in other words, we can understand it as a probability density function. We can view our programs as a transformation of a distribution: If we input an initial distribution (this might also just be the characteristic function of a single value), we will as output receive the integral over the transformed distribution; with other words, we will receive the probability of the result of an actual pseudorandomised run of our program producing a result within the input distribution.

Or something like that... I am very unsure about what exactly we get when we input an initial distribution... help?

It seems we get the integral over the joint distribution (given that both distributions are independent, which we will have to assume).

Geistesblitz! The input is a function that expresses the probability of something being member of some set. The characteristic function of a set is non-probabilistic and hence the result of inputting this will give the exact probability of our result being part of that set. If we, for example, have a set where both `true` and `false` are members with probability $\frac{1}{2}$, the result of inputting this into, e.g., `unit true` would be the probability of the value (`true`) being a member of said set.

2.2.2 A monadic interpretation

We can represent measures with a monadic structure. This is sensible since once we go measure we never go back; once probability is involved in a computation, we will not get rid of it again.

Add something about how the measure type now is called M .

The monadic operators presented here have been formerly introduced by [1] and satisfy the usual monadic properties¹. They are also used by the Mathematical Components compliant Analysis Library² in their representation of distributions.

Footnote
too face-
tious?

```
unit :  $\tau \rightarrow M\tau$   
      = fun (x :  $\tau$ )  $\Rightarrow$  fun (f :  $\tau \rightarrow [0, 1]$ )  $\Rightarrow$  f x  
bind :  $M\tau \rightarrow (\tau \rightarrow M\sigma) \rightarrow M\sigma$   
      = fun ( $\mu$  :  $M\tau$ )  $\Rightarrow$  fun (M :  $\tau \rightarrow M\sigma$ )  $\Rightarrow$   
        fun (f :  $\sigma \rightarrow [0, 1]$ )  $\Rightarrow$   $\mu$  (fun (x :  $\tau$ )  $\Rightarrow$  M x f)
```

These definitions look big and scary, so let's break them down.

The `unit` function could be described as the “wrapper” that takes a value and wraps the monad around it. The outermost function binding takes the value that we want to produce the measure of and remembers it. The second function binding is to match the type of measures; this is where the probability distribution is received as input, and where we then give the output of said distribution function applied to our initial value.

Is this
the right
word for
the fun-
keyword)

A simple example could be `unit 5` applied to the uniform distributions of numbers between 1 and 5. The result would then be 0.2, since that is the probability of a (pseudo-)random sampling from our measure (we recall that in this case it is just the characteristic function $\mathbb{I}_{\{5\}}$) and a (pseudo-)random sampling from the initial distribution coinciding.

The `bind` function can be viewed as something transforming the value inside the monad. Let us take it from the inside and out. Clearly the last line of the definition is the result, the $M\sigma$ part; this is obvious from the types. μ and M are the usual arguments to a `bind` operation, μ being the initial value and M the transformation to be applied to it.

We can view the final construction like continuation passing style programming. We still have a value of type τ inside the monad before we bind

¹We refer the reader who is unfamiliar with monads (or requires proof of the authors' knowledge thereof) to an earlier work by the authors, which can be found at https://bitbucket.org/Ninjura/functionalprogramming/raw/47de0f3f259370f3214789bbc90511313be451f8/project/report_final.pdf

²<https://github.com/math-comp/analysis/tree/master/>

it into a function, and we want to retain that information in the result of the `bind`. But we can not access this information unless we supply the value of type $M\tau$ with a function of type $\tau \rightarrow [0, 1]$; this function is the one that μ is applied to in the final value. Since the result is of type $M\sigma$, it would make sense to use its input on something of type σ to get something in $[0, 1]$, which would satisfy the output type for the function we apply μ to.

The final function takes the input of type τ , that μ will give it, uses M on it to transform it to something of type $M\sigma$, and then supplies it with the last piece of the computation, f .

This way we retain all of the information that is contained in μ , transform it with M , and lastly transform it with f , once that is supplied.

Add an example.

2.3 The functional approach: $\mathcal{R}m1$

The first language for probabilistic programs implemented in Coq is called $\mathcal{R}m1$ ('Randomised monadic language') and is due to Philippe Audebaud and Christine Paulin-Mohring ([1]).

$\mathcal{R}m1$ could be described as a functional version of the `while` language with an interpretation that allows for probabilistic algorithms. The language itself does not contain probabilistic expressions, but rather makes use of a `random(n)` or `flip` function giving a uniform distribution of natural numbers between 0 and n , and `true` and `false`, respectively.

check this sentence

Since calling either function would be considered a valid $\mathcal{R}m1$ term, the result has to be a measure over natural numbers (in the case of `random(n)`) or booleans (in the case of `flip`).

The expressions that our language consists of (we recall that it is a functional language, so there are only expressions, no statements) are as follows:

Is this parenthetical needed?

$exp ::= x \mid c \mid \text{if } b \text{ then } e_1 \text{ else } e_2 \mid \text{let } x = e_1 \text{ in } e_2 \mid f \ e_1 \ \dots \ e_n$

Here x refers to a variable name previously bound in a let-binding, and c refers to a primitive constant. Since Paulin-Mohring and Audebaud don't give a clear definition of what a constant can be, it makes sense to assume, that it can be any Coq term.

The f in function application can be a "primitive or a user-defined function", where primitive would be `random(n)` or `flip`, and a user-defined

quote

function should be specified by `let $f\ x_1 \dots x_n = e$` , according to the language specification. Recursive functions can be defined with the keyword `let rec`.

The monadic interpretation $[e] : \mathbb{M}\tau$ of a term $e : \tau$ is given in figure 2.3.

| $\mathcal{R}ml$ term $e : \tau$ | Coq value $[e] : \mathbb{M}\tau$ |
|--|---|
| v | <code>unit v</code> v variable or constant |
| <code>let $x = a$ in b</code> | <code>bind [a] (fun x \Rightarrow [b])</code> |
| $f\ a_1 \dots a_n$ | <code>bind [a₁] (fun $x_1 \Rightarrow \dots$ bind [a_n] (fun $x_n \Rightarrow [f]\ x_1 \dots x_n) \dots)$</code> |
| <code>if b then a_1 else a_2</code> | <code>bind [b] (fun $x : \text{bool} \Rightarrow$ (if x then [a₁] else [a₂]))</code> |

What the hell is wrong with figure numbering?

Figure 2.1: Monadic interpretation of $\mathcal{R}ml$ terms as presented in [1].

2.4 EasyCrypt and `pwhile` (or ‘The imperative approach’)

EASyCRYPT is a framework that has been developed in order to help in the construction of machine-checkable proofs about cryptographic constructions and protocols [2]. A standard approach to this kind of proofs is based on so-called games; in EASyCRYPT cryptographic algorithms as well as games are modelled as *modules* consisting of procedures written in a simple imperative language called `pwhile`. The `p` in `pwhile` stands for “probabilistic”, so in total the name refers to a probabilistic extension of the well-known minimalistic `while` language.

We will in this section give an overview of the language as well as its interpretation in Coq, which is due to a development by Pierre-Yves Strub³. We will not concern ourselves with the module system of EASyCRYPT since the focus of the present development is on probabilistic languages and their interpretation rather than their use.

³<https://github.com/strub/xhl>

pwhile consists of the following expressions and commands:

$$\begin{aligned} \text{exp} &::= x \mid \text{const} \mid \text{prp } (p : \text{pred mem}) \mid e_1 \ e_2 \\ \text{cmd} &::= \text{abort} \mid \text{skip} \mid x := e \mid x \$ = e \\ &\mid \text{if } b \text{ then } c_1 \text{ else } c_2 \mid \text{while } b \text{ do } c \mid c_1; c_2 \end{aligned}$$

The embedding of **pwhile** in Coq is a so-called shallow embedding, which means that we use Coq terms and types as part of programs in **pwhile**. This is used in order to form expressions: Constants in **pwhile** are Coq constants, hence an expression in **pwhile** can have any Coq type. In the implementation, expressions are parameterised by their type.

Most of the above constructs are fairly standard and should be known to most readers. We give a full formal semantics in Figure .

Write the semantics for **pwhile**.

Insert reference to semantics-figure

In addition to the standard expressions and commands, **pwhile** has the expression **prp** p , which we will have a look at here. **prp** takes a single argument of type **pred mem**. This is a Coq type specified with the type constructor **pred** $= \forall \tau : \tau \rightarrow \mathbb{B}$ applied to the type of memories defined in the **xhl** development. The predicate over memories that is **prps** argument is mapped over the working memory at the time of evaluation.

For more clarity, we will look at an example. Let us take the situation where we only want to proceed with a computation, if a certain variable, x is defined in the memory; we want to access x , but want to avoid our program crashing if x has not been defined. We now write the function **x_defined** that, given a memory, returns true if x is defined in said memory and false otherwise. By branching on **prp** **x_defined**, we can now make sure that we only take the branch accessing x if it is present in the memory and do not end up with a program that may crash.

At this point a comment about “crashing” programs is in order. There are multiple ways in which a program can lead to an undefined result: encountering undefined behaviour, non-termination, and the **abort** command. The interpretation of all of these is the same: We recall that the result of interpreting a **pwhile** program in Coq is a probability distribution over memories; now the result of interpreting a “crash” is by simply returning the null-distribution over memories.

is this the right word choice? It doesn't seem that that's what a null distribution is. But the key-word use is **dnull**.

3 Our approach (Ready draft: 07.06.)

The goal of the present project was to gain an actionable understanding of $\mathcal{R}ml$ and `pwhile`, their respective interpretations, theoretic background, and similarities and differences between them. Our approach was to define a translation between `pwhile` and $\mathcal{R}ml$ as well as an interpretation of $\mathcal{R}ml$, and subsequently show that translating from `pwhile` to $\mathcal{R}ml$ and interpreting the resulting program would lead to the same interpretation as interpreting the `pwhile` program directly. This approach allowed us to get acquainted with both languages as well as the difficulties in interpretation that both of them share.

Triangle figure.

In this section we will concern ourselves with the translation between `pwhile` and $\mathcal{R}ml$, as well as their respective interpretations. We will first have a look at deterministic versions of both in order to define the main part of the translations (Sections 3.1, 3.2, 3.4, and 3.5), and afterwards we will have a look at how to add probabilistic constructs into all translations (Section 3.6).

The current section is fully theoretic, for a discussion of our implementation, we refer the reader to Section 4.

3.1 Translating while to a functional language

In order to do the translations properly, let us first have a look at a translation from the simple, widely known `while` language to a simple functional language resembling $\mathcal{R}ml$. The thought behind this is that once this translation is in place, all we have to do to translate `pwhile` to $\mathcal{R}ml$ is to add probability.

- (1) $exp ::= x | n | \text{true} | \text{false} | f\ x$
- (2) $stm ::= \text{skip} | x := e | \text{if } e \text{ then } s_1 \text{ else } s_2 | \text{while } e \text{ do } s | s_1; s_2$

The syntax of our functional language is the same as $\mathcal{R}ml$ modulo the pre-defined probabilistic functions.

The translation of expressions is completely straightforward: variables are mapped to variables, constants to constants, and function applications to function applications.

In order to translate statements we choose a set of SML-style matching rules; this choice is due to the translation of sequences being dependent on

what the first statement is. We will in the following write the translation of a **while** statement s to an expression in our functional language as $\llbracket s \rrbracket$.

The result of a computation in **while** is the state of a memory, while the result of a functional computation is a value. A simple way to make up for this difference is by choosing a variable name that is designated the return variable and encapsulates the information we are interested in after the computation. This is the result of a program translated from **while** to our functional language; in the following we choose x_r as the symbol for the chosen return variable.

$$(3) \quad \text{skip} ; s \mapsto \llbracket s \rrbracket$$

$$(4) \quad \text{skip} \mapsto x_r$$

$$(5) \quad x := e ; s \mapsto \text{let } x := e \text{ in } \llbracket s \rrbracket$$

$$(6) \quad x_r := e \mapsto e$$

$$(7) \quad x := e \mapsto x_r$$

$$(8) \quad (\text{if } e \text{ then } s_1 \text{ else } s_2) ; s_3 \mapsto \begin{array}{l} \text{if } e \\ \quad \text{then } \llbracket s_1 ; s_3 \rrbracket \\ \quad \text{else } \llbracket s_2 ; s_3 \rrbracket \end{array}$$

$$(9) \quad \text{if } e \text{ then } s_1 \text{ else } s_2 \mapsto \text{if } e \text{ then } \llbracket s_1 \rrbracket \text{ else } \llbracket s_2 \rrbracket$$

(10) $(\text{while } e \text{ do } s_1) ; s_2 \mapsto \text{let rec } f \ x := \text{if } e$
 $\quad \quad \quad \text{then } \llbracket s_1 ; f \ x \rrbracket$
 $\quad \quad \quad \text{else } \llbracket s_2 \rrbracket$
 $\quad \quad \quad \text{in } f \ 0$

(11) $\text{while } e \text{ do } s_1 \mapsto \text{let rec } f \ x := \text{if } e$
 $\quad \quad \quad \text{then } \llbracket s_1 ; f \ x \rrbracket$
 $\quad \quad \quad \text{else } x_r$
 $\quad \quad \quad \text{in } f \ 0$

Note that in 10 and 11 we create recursive functions with a name and an argument, both of which are not present in the while construct we translate from. This means that we have to be careful about the translation: Both f and x have to be chosen fresh; and even fresher than that, they can not occur in the body of the while loop we are translating either, because that would break the recursive call.

Further notice that the recursive functions are always called with a dummy argument. This is because they act as procedures, but since our syntax requires an argument for recursive definitions, we give a dummy argument.

3.2 Translation from Rml to typed λ -calculus

This section is preliminary and needs either huge changes or deletion before the report is finalised.

| Rml | @sRml A |
|-------------------------------------|------------------------------------|
| Var (x, A) | sVar x |
| Const $A \ c$ | sConst c |
| Let $(x, T) \ e_1 \ e_2$ | e'_2 |
| If $b \ e_1 \ e_2$ | sIf $b' \ e'_1 \ e'_2$ |
| Fun $(x, T) \ e$ | sFun $S \ (x, T) \ e'$ |
| App $T \ e_1 \ e_2$ | sApp $T \ e'_1 \ e'_2$ |
| Let rec $T \ S \ f \ x \ e_1 \ e_2$ | sFix $T \ S \ f \ x \ e'_1 \ e'_2$ |

| @sRml A | typed λ -calculus | distr |
|--------------------------|--|---|
| sVar x | $x : A$ | dunit (lookup |
| sConst c | $c : A$ | dunit c |
| sFun $S (x, T) e'$ | $(\lambda x : T, e : S) : T \rightarrow S$ | $\backslash x \rightarrow \text{dunit}$ |
| sApp $T e'_1 e'_2$ | $(e_1 : T \rightarrow A) (e_2 : T) : A$ | $e'_1 \gg e'_1 \rightarrow e'_2 \gg e'_2$ |
| sFix $T S f x e'_1 e'_2$ | $(\lambda f : T \rightarrow S, e_2) (Y (\lambda f : T \rightarrow S, (\lambda x : T, e_1 x))) : A$ | $(\backslash f \rightarrow e_2)(\backslash x \rightarrow \text{dlim}$ |

3.3 Example: Fact

Expression:

```
Let_rec (f,  $\mathbb{N} \rightarrow \mathbb{N}$ ) (x,  $\mathbb{N}$ )
  (If (x = 0) (1) (x * f (x - 1)))
  (f 3)
```

Simple form:

```
sFix f x
  (sIf (x = 0) 1 (x * f (x - 1)))
  (f 3)
```

Typed lambda expressions:

```
 $(\lambda f : \mathbb{N} \rightarrow \mathbb{N}, f\ 3) (Y(\lambda f : \mathbb{N} \rightarrow \mathbb{N}, \lambda x : \mathbb{N}, (x = 0) (1) (x * f (x - 1))))$ 
```

Distribution:

```
\dlet_(f  $\leftarrow$  dunit ( $\lambda x, \backslash \text{dlim}(\backslash \text{ubn} (\lambda f, (x = 0) (1) (x * f (x - 1)))))$ ) (f 3)
```

3.4 Interpreting λ -calculus in the space of ω -cpo

What do ω -cpo have to do with this?

3.5 Interpreting while directly

This should probably mainly refer back to the interpretation of `pwhile`.

3.6 Adding probabilistic definitions

4 Our contribution (Draft 31.05; Finished 03.06)

Need to fix the colour scheme for Coq-blocks.

According to Boileau, “what one understands well is expressed clearly”. Few forms of expressing an idea are as concise as code written for a computer.

Since the main goal of the present project was to gain an actionable understanding of a number of concepts within the field of language design, we chose to implement some of the theory present in the previous section in Coq.

Our development contains an implementation of a language based on $\mathcal{R}ml$, a number of definitions concerning the correctness of programs in said language, as well as an interpreter for such correct programs. Our goal was to implement $\mathcal{R}ml$ according to its specification, but due to a number of wrestling matches with Coq’s typechecker, we ended up deviating from what was presented in [1].

We will in the following call the language we actually implemented for $\mathcal{R}ml^*$.

4.1 The definition of $\mathcal{R}ml^*$

In order to write an interpreter for a language, we need to have a way to represent the language’s abstract syntax. To this end we define a datatype using the Coq keyword `Inductive`.

```
Inductive Rml :=
| Var : (ℕ * Type) → bool → Rml
| Const : ∀ {A : Type}, A → Rml
| Let_stm : (ℕ * Type) → Rml → Rml → Rml
| If_stm : Rml → Rml → Rml → Rml
| App_stm : Type → Rml → Rml → Rml
| Let_rec : Type → Type → ℕ → ℕ → Rml → Rml → Rml
| Random : Rml → Rml
| Flip : Rml.
```

Figure 4.2: The inductive definition of $\mathcal{R}ml^*$ abstract syntax.

The names of the constructors are straightforward and easily recognised from the discussion of $\mathcal{R}ml$ in Section 2.3. The arguments that each con-

structor takes are specific to our implementation so let us have a look at them.

Var takes a pair of a natural number and a type as well as a boolean. The number and the type represent the information to identify a variable with. We use natural numbers as identifiers because they are easy to work with, and the only requirement for an identifier is that there is an equality operation on its type. The second part of the pair is the type of the variable. The type information of a variable is needed for type-checking and interpretation of an expression, because it enables us to check that the value of said variable has the type we expect it to have.

The second argument for the construction of a variable is a boolean value. This is used to indicate if the variable refers to something bound by a previous let binding, otherwise it is something we need for type-checking the function body of a recursive definition. We will go into more detail about how this is used in Section 4.2.

Const constructs a constant and is straightforward. The only explicit argument it takes is a Coq value; the implicit argument **A** is the type of said value.

At this point it should be noted, that in Coq functions are values as well, so it is possible to have a constant whose type is a function type. In other words: When we want to access a Coq-toplevel function in our \mathcal{Rml}^* program, we can do this by encapsulating it in an \mathcal{Rml}^* constant.

Let_stm is rather straightforward. The pair of a natural number and a type is the identifier of the variable being defined. The first argument of type **Rml** is the value of the variable, and the other is the expression the value is being used in.

For example **Let_stm** (**x**,**T**) e_1 e_2 would, in \mathcal{Rml} syntax, be written as **Let** (**x**:**T**) := e_1 in e_2 .

If_stm is straightforward. All this constructor takes are the condition and the two branches, all in form of **Rml** values.

App_stm takes an argument more or one less than we would intuitively expect. It takes one type as argument instead of either inferring the types or asking for both the function type and the result type.

The trick here is that when type-checking, we know the type we expect the whole application expression to have, but we do not know what to expect from the function or its argument. We therefore explicitly supply the intermediate type.

An example would be `App_stm N (Const is_even) (Const 50)`, where we would typecheck the whole expression to have type `bool`, while we need the explicit `N` to check that the argument has the right type for the function.

This is because we can't construct on types in Coq, right...?

`Let_rec` is the most complicated of the linguistic constructs of \mathcal{Rml}^* . It takes two types, two identifiers (natural numbers) and two `Rml` expressions.

The two types are rather straightforwardly the domain and co-domain of the recursively defined function.

The identifiers are the name of the function and the name of its argument, respectively. Both can be referred to in the function body, which is the first argument of type `Rml`.

The last argument should be an `Rml` value of the same type as the newly defined function's argument. This is probably our largest deviation from the original specification of \mathcal{Rml} . We do not have any expressions of the form `Let_rec ...in ...`, instead we require to be recursive definitions to be of the form `Let_rec f x := ...in f(...)`. This way of writing and using recursive functions has the exact same expressiveness as the common form, but it is easier to interpret in Coq.

Note that this way of defining `Let_recs` removes the necessity of having a function environment. Each function is used right where it is defined and can not be called in the rest of the program, so effectively there is no binding of anything happening in our so-called `Let_rec`.

`Flip` and `Random` are the two probabilistic functions mentioned in Section 2.3. We added these to our abstract syntax, since this makes it easier for us to interpret them as `Rml` terms instead of just as distributions. Effectively the types are the same, but having them as part of our abstract syntax makes them more readily accessible in \mathcal{Rml}^* programs.

It is easy to see that all of the additional information we store in our abstract syntax tree is something that a parser would be able to infer and add. It would even be possible to transform the shape of a recursive let-binding at parse-time so it would fit our abstract syntax. This might introduce more recursive let-bindings, if the function is used in multiple places in the original code, but apart from this, the result would be equivalent.

4.2 Baby steps: The advent of $s\mathcal{R}ml$

In our first try to interpret $\mathcal{R}ml$ directly, we were not able to figure out how to use environments for the interpretation of let-bindings in a way that Coq would accept. So instead of going from $\mathcal{R}ml$ directly to its interpretation (as was suggested in [1]) we took a step in between. We defined a language of simplified $\mathcal{R}ml$, $s\mathcal{R}ml$. The main difference between $\mathcal{R}ml$ and $s\mathcal{R}ml$ is that $s\mathcal{R}ml$ does not have let-bindings; there is, however, still a construct that is the same as the previous `Let_rec` which we call `sFix`, but as mentioned earlier, this construction does not actually bind anything to an identifier, so there is no problem with environments here.

```
Inductive sRml {A : Type} :=
| sVar : N → sRml
| sConst : A → sRml
| sIf : @sRml bool → sRml → sRml → sRml
| sApp : ∀ T, @sRml (T → A) → @sRml T → sRml
| sFix : ∀ B (nf nx : N), @sRml A → @sRml B → sRml
| sRandom : (A = N) → @sRml N → sRml
| sFlip : (A = bool) → sRml.
```

Figure 4.3: The inductive definition of $s\mathcal{R}ml$ abstract syntax.

Having this definition in place, the next step is to translate $\mathcal{R}ml^*$ to $s\mathcal{R}ml$. This is done by the function `replace_all_variables_type`, which can be found in Appendix A.

We replace the variables by traversing our abstract syntax tree and building an environment that contains the identifier, type, and value of each variable that is bound. As we go, we construct the abstract syntax of an $s\mathcal{R}ml$ expression. Most of the cases are straightforward and an $\mathcal{R}ml^*$ construct is just replaced with the corresponding $s\mathcal{R}ml$ construct. The interesting cases are `Var` and `Let`.

Variables refer to let-bindings, so we want to get rid of them. But we still want to be able to have recursive function definitions, so we can not get rid of the variables that are the function name and argument in a recursive definition. This means we need a way to distinguish them. Here the extra boolean argument that the constructor `Var` takes comes into play:

- If it is `false`, we look up the variable in the environment and replace it with its value.

- If it is `true`, we know that we are currently looking at the body of a recursive definition and have no value with which to replace the variable, so we construct an `sVar` for it.

Let-bindings are what we want to get rid of in our simplification. We just saw how we get rid of variables referring back to bindings, so now for the actual binding part.

When we encounter a let-binding in our translation, we have to make sure that we can replace all variables in its body with their value by just looking into the environment. This means that we have to extend the environment with the newly bound variable and then replace all the variables in its body. And this is exactly what we do. We call the function to replace variables recursively, on the body of the let-binding with the newly extended environment.

4.3 Typing $\mathcal{R}m1^*$

But how can we be sure that the translation from $\mathcal{R}m1^*$ to $s\mathcal{R}m1$ actually works and does not “go wrong”? Having learned from Robin Milner [5] we know that in order to ensure this we must implement some typing rules and a way to check that a program is well-typed.

Before we commence the discussion of what it means for $\mathcal{R}m1^*$ or $s\mathcal{R}m1$ expressions to be well-typed, we should mention that we decided on using the axiom of decidable equality for types. We could alternatively have defined an even deeper embedding of $\mathcal{R}m1$ in Coq by defining a datastructure for $\mathcal{R}m1$ -types instead of using Coq-types. This, however, would not have changed our outcome, so we decided against the additional work.

too facetious?

The rules we implemented are straightforward and can be found in Appendix B.

The attentive reader will have noticed that $s\mathcal{R}m1$ is parameterised by a type representing the type of the constructed expression. This is in order to help us (and Coq) during the interpretation. The required types are consistent with the specified typing rules and are added during the translation from $\mathcal{R}m1^*$ to $s\mathcal{R}m1$. This means that in order to translate, the $\mathcal{R}m1^*$ expression we are trying to translate has to be well-typed; we assure this by our taking as argument a proof of the well-typedness of the expression to be translated.

The output of our translation is of type $\forall (A : \text{Type}), \text{verified_srml } A \text{ nil}$, whose definition can be seen in Figure 4.4. This means that the output of our translation effectively is a pair of the resulting $s\mathcal{R}m1$ expression and a

proof of its well-typedness. This, again, is to help us through the actual interpretation.

```

Inductive verified_srml (A : Type) (fl : seq ( $\mathbb{N}$  * Type)) : Type :=
  verified :  $\forall y : \text{srml}, \text{srml\_valid\_type } A \text{ fl } y \rightarrow \text{verified\_srml } A \text{ fl}$ 

```

Figure 4.4: The inductive definition of the type `verified_srml`.

When defining the translation from \mathcal{Rml}^* to $\text{s}\mathcal{Rml}$, we had to define a few helping lemmas along the way, in order to be able to prove everything correct. The most important one of these is weakening, which we had to prove for both the variable environment and the function environment for \mathcal{Rml}^* as well as for the function environment for $\text{s}\mathcal{Rml}$.

Weakening means that even though we extend our environment (with a let-binding), the interpretation of the expression is still well-typed. We had a tiny problem along the way here: What if we bind a new variable with the same name but of a different type? Then the expression inside the new let-binding might be ill-typed.

We fixed this by defining the lookup of a variable to not only refer to its name, but also its type (cf. Figure 4.2 and the subsequent discussion). This way we can be sure that the variable we retrieve is of the right type.

4.4 But what does it mean? Interpreting $\text{s}\mathcal{Rml}$

After all this talk about how we define well-typedness of \mathcal{Rml}^* and $\text{s}\mathcal{Rml}$, we want to finally make use of it.

Our interpretation of $\text{s}\mathcal{Rml}$ is into the type of $\forall (A : \text{Type}), \text{distr } \mathbb{R} (\text{Choice } A)$; or with other words: We interpret $\text{s}\mathcal{Rml}$ expressions of type τ as distributions over the type of τ . This should remind the careful reader of how we presented the interpretations of \mathcal{Rml} and `pwhile` in Sections 2.3 and 2.4.

The additional elements of \mathbb{R} and `Choice` are due to the implementation of distributions in the Mathematical Components compliant Analysis Library that we mentioned in Section 2.2.2.

Since most of the set-up for the interpretations has been presented throughout Section 2, we will here only present a formal version of the semantics whose implementation can be found in the Coq definitions `ssem_aux` and `ssem`.

5 Comparisons and future work (Draft 10.06.)

6 Conclusion (Finished 12.06.)

We did good.

References

- [1] P. Audebaud and C. Paulin-Mohring, “Proofs of randomized algorithms in coq,” *Science of Computer Programming*, no. 74, pp. 568–589, 2009.
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- [3] “Haskell/denotational semantics.” https://en.wikibooks.org/wiki/Haskell/Denotational_semantics#Recursive_Definitions_as_Fixed_Point_Iterations.
- [4] “Measure (mathematics).” [https://en.wikipedia.org/wiki/Measure_\(mathematics\)](https://en.wikipedia.org/wiki/Measure_(mathematics)).
- [5] R. Milner, “A theory of type polymorphism in programming,” *Journal of Computer and System Sciences*, no. 17, pp. 348–375, 1978.

Appendices

A Translating \mathcal{Rml}^* to $s\mathcal{Rml}$

This appendix presents the function translating \mathcal{Rml}^* abstract syntax to $s\mathcal{Rml}$ abstract syntax. The auxiliary function, which is the main workhorse, is defined recursively and written in the form of a proof. This is possible because Coq-proofs are program terms that can be run.

```
Fixpoint replace_all_variables_aux_type
  A (x : Rml) (env : seq (N * Type * Rml))
  (fl : seq (N * Type)) '{env_valid : valid_env env fl}
  '{x_valid : @rml_valid_type A (map fst env) fl x} {struct x}
  : verified_srml A fl.
Proof.
  (** Structure **)
  generalize dependent fl.
  generalize dependent env.
  generalize dependent A.
  induction x ; intros.
  (** Var *)
  {
    assert (List.In p (map fst env) ∨ List.In p fl)
    by (inversion x_valid ; subst ; auto).
    destruct p.
    assert (A = T) by (inversion x_valid ; subst ; reflexivity) ; subst.
    apply (@lookup (n,T) env fl env_valid H).
  }

  (** Const **)
  {
    assert (A0 = A) by (inversion x_valid ; subst ; reflexivity) ; subst.
    exists (sConst a).
    constructor.
  }

  (** Let-stm **)
  {
    assert (x1_valid : rml_valid_type p.2 (map fst env) fl x1)
    by (inversion x_valid ; subst ; assumption).

    pose (x1' := replace_all_variables_aux_type p.2 x1 env fl env_valid x1_valid).
    destruct x1' as [x1'].
    pose (x1'' := @sRml_to_rml p.2 x1').
```



```

assert (x1''_simple : @rml_is_simple fl x1'').
apply sRml_simple.
assumption.

assert (x1''_valid : @rml_valid_type p.2 (map fst env) fl x1'').
apply sRml_valid.
assumption.

pose (@rml_to_sRml_1 p.2 x1'' (map fst env) fl).

assert (x2_valid : rml_valid_type A (p :: [seq i.1 | i ← env]) fl x2)
by (inversion x_valid ; subst ; assumption).

assert (env_valid' : valid_env ((p,x1'') :: env) fl)
by (constructor ; assumption).

refine (replace_all_variables_aux_type A x2 ((p,x1'') :: env) fl env_valid' x2_valid).
}

(** If-stm **)
{
  assert (x1_valid : rml_valid_type bool (map fst env) fl x1)
  by (inversion x_valid ; subst ; assumption).
  assert (x2_valid : rml_valid_type A (map fst env) fl x2)
  by (inversion x_valid ; subst ; assumption).
  assert (x3_valid : rml_valid_type A (map fst env) fl x3)
  by (inversion x_valid ; subst ; assumption).

  pose (b' := replace_all_variables_aux_type bool x1 env fl env_valid x1_valid).
  pose (m1' := replace_all_variables_aux_type A x2 env fl env_valid x2_valid).
  pose (m2' := replace_all_variables_aux_type A x3 env fl env_valid x3_valid).

  destruct b' as [b'].
  destruct m1' as [m1'].
  destruct m2' as [m2'].

  pose (b'' := sRml_to_rml b').
  pose (m1'' := sRml_to_rml m1').
  pose (m2'' := sRml_to_rml m2').

  refine (rml_to_sRml_1 (If_stm b'' m1'' m2'') [seq i.1 | i ← env] fl).
  constructor ; eauto using sRml_simple.
  constructor ; eauto using sRml_valid.
}

```

```

(** App-stm **)
{
  assert (x1_valid : rml_valid_type (T → A) (map fst env) fl x1)
  by (inversion x_valid ; subst ; assumption).

  assert (x2_valid : rml_valid_type T (map fst env) fl x2)
  by (inversion x_valid ; subst ; assumption).

  pose (e1' := replace_all_variables_aux_type (T → A) x1 env fl env_valid x1_valid).
  pose (e2' := replace_all_variables_aux_type T x2 env fl env_valid x2_valid).

  destruct e1' as [e1'].
  destruct e2' as [e2'].

  pose (e1'' := sRml_to_rml e1').
  pose (e2'' := sRml_to_rml e2').

  refine (rml_to_sRml_1 (App_stm T e1'' e2'') [seq i.1 | i ← env] fl).
  constructor ; eauto 2 using sRml_simple.
  constructor ; eauto 2 using sRml_valid.
}

(** Let rec **)
{
  pose (fl_x1 := [:: (n0, T), (n, T → T0) & fl]).

  assert (x1_valid : rml_valid_type A [seq i.1 | i ← env] fl_x1 x1)
  by (inversion x_valid ; subst ; assumption).

  assert (x2_valid : rml_valid_type T [seq i.1 | i ← env] fl x2)
  by (inversion x_valid ; subst ; assumption).

  assert (env_valid_x1 : valid_env env fl_x1)
  by (repeat apply extend_fl_still_valid ; assumption).

  pose (x1' := replace_all_variables_aux_type A x1 env fl_x1 env_valid_x1 x1_valid).
  assert (env_valid_x2 : valid_env env fl) by (repeat apply extend_fl_still_valid ; assumption).

  pose (x2' := replace_all_variables_aux_type T x2 env fl env_valid_x2 x2_valid).

  destruct x1' as [x1'].
  destruct x2' as [x2'].

```

```

    assert (A = T0) by (inversion x_valid ; subst ; reflexivity) ; subst.

    exists (sFix T n n0 x1' x2').
    constructor ; assumption.
  }

  (** Random **)
  {
    assert (inner_x_valid : rml_valid_type ℕ(map fst env) fl x)
    by (inversion x_valid ; assumption).

    pose (x' := replace_all_variables_aux_type ℕx env fl env_valid inner_x_valid).

    assert (type_eq : A = ℕ) by (inversion x_valid ; reflexivity).

    destruct x' as [x' x'_valid].

    exists (sRandom type_eq x').
    constructor ; assumption.
  }

  (** Flip **)
  {
    assert (A = bool) by (inversion x_valid ; reflexivity).
    exists (sFlip H).
    constructor.
  }
Defined.

Definition replace_all_variables_type A (x : Rml)
  '{x_valid : rml_valid_type A nil nil x} :=
  @replace_all_variables_aux_type A x nil nil (env_nil nil) x_valid.

```

B \mathcal{Rml}^* typing rules

```

Inductive rml_valid_type : Type → seq (ℕ * Type) → seq (ℕ * Type) → Rml →
Prop :=
| valid_var : ∀vl fl p,
  List.In p vl →
  rml_valid_type p.2 vl fl (Var p false)

| valid_fun_var : ∀vl fl p,
  List.In p fl →
  rml_valid_type p.2 vl fl (Var p true)

| valid_const : ∀(A : Type) vl fl (c : A),
  rml_valid_type A vl fl (@Const A c)

| valid_let : ∀A vl fl p a b,
  @rml_valid_type p.2 vl fl a →
  @rml_valid_type A (p :: vl) fl b →
  rml_valid_type A vl fl (Let_stm p a b)

| valid_if : ∀A vl fl b m1 m2,
  rml_valid_type bool vl fl b →
  rml_valid_type A vl fl m1 →
  rml_valid_type A vl fl m2 →
  rml_valid_type A vl fl (If_stm b m1 m2)

| valid_app : ∀A vl fl (B : Type) e1 e2,
  rml_valid_type (B → A) vl fl e1 →
  rml_valid_type B vl fl e2 →
  rml_valid_type A vl fl (App_stm B e1 e2)

| valid_let_rec : ∀A vl fl B nf nx e1 e2,
  @rml_valid_type A vl ((nx,B) :: (nf,B → A) :: fl) e1 →
  @rml_valid_type B vl fl e2 →
  rml_valid_type A vl fl (Let_rec B A nf nx e1 e2)

| valid_random : ∀vl fl e,
  rml_valid_type ℕvl fl e →
  rml_valid_type ℕvl fl (Random e)

| valid_flip : ∀vl fl,
  rml_valid_type bool vl fl Flip.

```

C Example - Error: Stack Overflow.

```

Fixpoint replace_all_variables_aux_type
  A (x : Rml) (env : seq (N * Type * Rml))
  (fl : seq (N * Type)) '{env_valid : valid_env env fl}
  '{x_valid : rml_valid_type A (map fst env) fl x} : sRml A

with replace_all_variables_aux_type_const
  A0 A a (env : seq (N * Type * Rml))
  (fl : seq (N * Type)) '{env_valid : valid_env env fl}
  '{x_valid : rml_valid_type A0 (map fst env) fl (Const A a)} : sRml A0
with replace_all_variables_aux_type_let
  A p x1 x2 (env : seq (N * Type * Rml))
  (fl : seq (N * Type)) '{env_valid : valid_env env fl}
  '{x_valid : rml_valid_type A (map fst env) fl (Let_stm p x1 x2)} : sRml A
with replace_all_variables_aux_type_fun
  A T p x (env : seq (N * Type * Rml))
  (fl : seq (N * Type)) '{env_valid : valid_env env fl}
  '{x_valid : rml_valid_type A (map fst env) fl (Fun_stm T p x)} : sRml A
with replace_all_variables_aux_type_if
  A x1 x2 x3 (env : seq (N * Type * Rml))
  (fl : seq (N * Type)) '{env_valid : valid_env env fl}
  '{x_valid : rml_valid_type A (map fst env) fl (If_stm x1 x2 x3)} : sRml A
with replace_all_variables_aux_type_app
  A T x1 x2 (env : seq (N * Type * Rml))
  (fl : seq (N * Type)) '{env_valid : valid_env env fl}
  '{x_valid : rml_valid_type A (map fst env) fl (App_stm T x1 x2)} : sRml A
with replace_all_variables_aux_type_let_rec A T T0 n n0 x1 x2 (env : seq (N * Type * Rml))
  (fl : seq (N * Type)) '{env_valid : valid_env env fl}
  '{x_valid : rml_valid_type A (map fst env) fl (Let_rec T T0 n n0 x1 x2)} : sRml A.

Proof.
  (** Structure **)
  {
    induction x ; intros ; refine (sVar (0,A)).
  }

  all: refine (sVar (0,A)).
Defined.

```