

# Technical report

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# 1 Introduction

## 2 Theory and existing frameworks

### 2.1 $\mathcal{R}ml$

We have two representations of  $Rml$ , continuations and distributions. Both build on a monad, for ease of use.

The data structure used to represent  $Rml$  terms is as follows:

```
Inductive Rml :=
| Var : (N * Type) → Rml
| Const : ∀ (A : Type), A → Rml
| Let_stm : (N * Type) → Rml → Rml → Rml
| Fun_stm : Type → (N * Type) → Rml → Rml
| If_stm : Rml → Rml → Rml → Rml
| App_stm : Type → Rml → Rml → Rml
| Let_rec : Type → Type → N → N → Rml → Rml → Rml.
```

We use all cog types, as possible types of  $Rml$  expressions, since there are no real restrictions on the types. We encode variables, as a type and a natural number, so two variables are the same only if they have the same number and refer to the same type.

We have defined a relation `well_formed`, that checks that no variables are escaping the scope of an  $Rml$  program, that is there is always a binding for an expression of type `Var p`. We furthermore define a relation `rml_valid_type`, which checks that a given  $Rml$  expression can be typed under a given type. We have shown that if a  $Rml$  program is valid then it is well formed. We have then constructed a simplified form of  $Rml$  called  $sRml$  (for simple  $Rml$ ), to make it easier to reason about and evaluate expressions, with the following data structure:

```
Inductive sRml {A : Type} :=
| sVar : N → sRml
| sConst : A → sRml
| sFun : ∀ C (p : N * Type), A = (p.2 → C) → ·sRml C → sRml
| sIf : ·sRml bool → sRml → sRml → sRml
| sApp : ∀ T, ·sRml (T → A) → ·sRml T → sRml
| sFix : ∀ B (nf nx : N), ·sRml (B → A) → ·sRml B → sRml.
```

That is  $Rml$  where we remove expressions with variables, from `let_stm` statements (not `let_rec` statements). We then show that given a valid typing of an  $Rml$  expression, we can simplify that expression, and maintain the valid typing (under the same type). With this we can make an interpreter from an interpreter of  $sRml$ , which can be constructed as

(for continuations). We have a similar function for Rml, using the possibility distributions as interpretations. We see similar patterns arising, since both interpretations are monadic.

## 2.2 pwhile

## 2.3 Complete partial orders

A partially ordered set (poset) is a set with an associated binary ordering relation  $\leq$  which is both reflexive and transitive. The order is partial when the ordering relation is not defined on every pair of elements in the set.

There exist a number of different completeness properties that a poset can have. We will here have a look at  $\omega$ -complete partial orders, which we will use in order to interpret general recursion and randomised programs.

**Definition 1.**  *$\omega$ -complete partial order ( $\omega$ -cpo)*

An  $\omega$ -cpo is a partially ordered set that, additionally, has a distinct least element and where there exist least upper bounds on all monotonic sequences.

### 2.3.1 Recursive definitions as fixed point iterations

Before using  $\omega$ -cpo to interpret recursion, let us first have a look at some interesting things that our definition entails.

A monotonic sequence on an  $\omega$ -cpo  $X$  can be viewed as a monotonic function  $f : \mathbb{N} \xrightarrow{m} X$  where  $f(n)$  is the  $n$ th element of the sequence (or the least upper bound of the sequence, if  $n$  is larger than the length of the sequence).

There is a standard way of defining fixed point iterations on an  $\omega$ -cpo:

Consider an operator  $F : X \xrightarrow{m} X$  on some  $\omega$ -cpo  $X$ ; with this we define the monotonic sequence  $F_i \mapsto \underbrace{F(F(\dots F(0_X)\dots))}_{i \text{ times}}$  of repeated application of  $F$  to the least element of  $X$ .

By our choice of  $F$  and the definition of  $\omega$ -cpo, it is clear that there has to exist a least upper bound on  $F_i$ . This least upper bound is the fixed point of  $F$  and it will hold that  $\text{fix } F = F(\text{fix } F)$  if  $F$  is continuous.

For an  $\omega$ -cpo with underlying set  $B$  we can also define an  $\omega$ -cpo on functions from any set  $A$  whose co-domain is  $B$ .

To reiterate the definition, let us think of what we need for an  $\omega$ -cpo. We need an ordering relation, a least element, and a least upper bound operation. Those can be defined as follows:

$$\begin{aligned} f \leq_{A \rightarrow B} g &\Leftrightarrow \forall x : f(x) \leq_B g(x) && (\text{pointwise order}) \\ 0_{A \rightarrow B} &:= f(x) = 0_B && (\text{least element}) \\ \text{lub}_{A \rightarrow B} f_n &:= g(x) = \text{lub}_B(f_n(x)) && (\text{least upper bound operation}) \end{aligned}$$

The result of an interpretation of programs in the language of discourse will be in an  $\omega$ -cpo, so according to the above discussion functions will have an  $\omega$ -cpo structure as well. Together with the above definition of fixed points we can use this structure to interpret general recursive definitions.

We define a functional,  $F$ , taking as input a function and “adding a step to it”. Let us look at the example of the factorial function  $f(n) = n!$ . The recursive definition is well known:

$$fac(n) := \text{ if } n = 0 \text{ then } 1 \text{ else } n \cdot fac(n - 1)$$

For the interpretation of this definition, we want to define  $F : (\mathbb{N} \rightarrow \mathbb{N}) \rightarrow (\mathbb{N} \rightarrow \mathbb{N})$  in such a way that its fixed point is the same as the above recursive definition. We choose

$$F(F_i(n)) := \text{ if } n = 0 \text{ then } 1 \text{ else } n \cdot F_i(n - 1)$$

Where  $F_0(n)$  is  $0_{\mathbb{N} \rightarrow \mathbb{N}}$  (the function that takes a natural number and returns 0), by the above definition of the least element in the  $\omega$ -cpo defined on a function space. By repeated application of  $F$  the function will slowly approach the real factorial function, which is the fixed point of  $F$ . The beginning of the iteration will be

$$\begin{aligned} F_1(n) = F(F_0(n)) &= \begin{cases} 1 & \text{if } n \text{ is } 0 \\ 0 & \text{otherwise} \end{cases} \\ F_2(n) = F(F(F_0(n))) &= \begin{cases} 1 & \text{if } n \text{ is } 0 \\ 1 & \text{if } n \text{ is } 1 \\ 0 & \text{otherwise} \end{cases} \\ F_3(n) = F(F(F(F_0(n)))) &= \begin{cases} 1 & \text{if } n \text{ is } 0 \\ 1 & \text{if } n \text{ is } 1 \\ 2 & \text{if } n \text{ is } 2 \\ 0 & \text{otherwise} \end{cases} \end{aligned}$$

In this case it is easy to see that  $F_0 \leq F_1 \leq F_2 \leq F_3 \leq \dots$ . In the general case this follows from the fact that  $F$  has to be monotone and  $F_0$  is always the least element of the function space  $F$  operates on.

### 2.3.2 Interpreting random definitions

## 3 Our approach

### 3.1 Translating while to a functional language

In order to do the translations properly, let us first have a look at a translation from the simple, widely known **while** language to a simple functional language resembling **Rml**. The

thought behind this is that once this translation is in place, all we have to do to translate `pwhile` to  $\mathcal{Rml}$  is to add probability.

- (1)  $exp ::= x|n|\mathbf{true}|\mathbf{false}|f\ x$
- (2)  $stm ::= \mathbf{skip}|x := e|\mathbf{if}\ e\ \mathbf{then}\ s_1\ \mathbf{else}\ s_2|\mathbf{while}\ e\ \mathbf{do}\ s|s_1; s_2$

The syntax of our functional language is the same as  $\mathcal{Rml}$  modulo the pre-defined randomised functions.

The translation of expressions is completely straightforward: variables are mapped to variables, constants to constants, and function applications to function applications.

In order to translate statements we choose a set of SML-style matching rules; this choice is due to the translation of sequences being dependent on what the first statement is. We will in the following write the translation of a `while` statement  $s$  to an expression in our functional language as

Furthermore we need to handle the fact that while the imperative `while` has a return memory that one could extract the wished results from, a functional language has no such thing. We therefore need to choose the memory positions we are interested in and encapsulate those in a variable. We will, in the following, choose  $x_r$  to be the name of said variable.

- (3)  $\mathbf{skip}; s \mapsto \llbracket s \rrbracket$
- (4)  $\mathbf{skip} \mapsto x_r$
- (5)  $x := e; s \mapsto \mathbf{let}\ x := e\ \mathbf{in}\ \llbracket s \rrbracket$
- (6)  $x_r := e \mapsto e$
- (7)  $x := e \mapsto x_r$
- (8)
- (9)  $(\mathbf{if}\ e\ \mathbf{then}\ s_1\ \mathbf{else}\ s_2); s_3 \mapsto \mathbf{if}\ e\ \mathbf{then}\ \llbracket s_1; s_3 \rrbracket\ \mathbf{else}\ \llbracket s_2; s_3 \rrbracket$
- (10)  $\mathbf{if}\ e\ \mathbf{then}\ s_1\ \mathbf{else}\ s_2 \mapsto \mathbf{if}\ e\ \mathbf{then}\ \llbracket s_1 \rrbracket\ \mathbf{else}\ \llbracket s_2 \rrbracket$
- (11)  $(\mathbf{while}\ e\ \mathbf{do}\ s_1); s_2 \mapsto \mathbf{let}\ \mathbf{rec}\ f\ x := \mathbf{if}\ e\ \mathbf{then}\ \llbracket s_1; f\ x \rrbracket\ \mathbf{else}\ \llbracket s_2 \rrbracket\ \mathbf{in}\ f\ 0$
- (12)  $\mathbf{while}\ e\ \mathbf{do}\ s_1 \mapsto \mathbf{let}\ \mathbf{rec}\ f\ x := \mathbf{if}\ e\ \mathbf{then}\ \llbracket s_1; f\ x \rrbracket\ \mathbf{else}\ x_r\ \mathbf{in}\ f\ 0$

Note that in 10 and 11 we create recursive functions with a name and an argument, both of which are not present in the while construct we translate from. This means that we have to be careful about the translation: Both  $f$  and  $x$  have to be chosen fresh; and even fresher than that, they can not occur in the body of the while loop we are translating either, because that would break the recursive call.

Further notice that the recursive functions are always called with a dummy argument. This is because they act as procedures, but since our syntax requires an argument for recursive definitions, we give a dummy argument.

### 3.2 Translation from Rml to typed $\lambda$ -calculus

Rml	typed $\lambda$ -calculus
Var $(x, A)$	$x : A$
Const $A \ c$	$c : A$
Let $(x, A) \ e_1 \ e_2$	$(\lambda x : A. e_2) \ e_1$
Fun $(x, A) \ e$	$\lambda x : A. e$
App $e_1 \ e_2$	$e_1 \ e_2$
Let rec $(f, A \rightarrow B) (x, A) \ e_1 \ e_2$	$(\lambda f : A \rightarrow B. e_2) (Y (\lambda f : A \rightarrow B. \lambda x : A. e_1))$

The problem here is that we need to translate  $e_1$  and  $e_2$  to their simple forms, so we do an intermediate translation:

*Let*

#### 3.2.1 Example: Fib

Expression:

```
Let_rec (f,  $\mathbb{N} \rightarrow \mathbb{N}$ ) (x,  $\mathbb{N}$ )
  (if  $x \leq 0$ 
   then 0
   else  $f \ (x - 1) + f \ (x - 2)$ )
(f 3)
```

Typing:

```
Let_rec (f,  $\mathbb{N} \rightarrow \mathbb{N}$ ) (x,  $\mathbb{N}$ )
  ((if ( $x \leq 0 : \mathbb{B}$ )
    then ( $0 : \mathbb{N}$ )
    else ( $f : \mathbb{N} \rightarrow \mathbb{N}$ ) ( $x - 1 : \mathbb{N}$ ) + ( $f : \mathbb{N} \rightarrow \mathbb{N}$ ) ( $x - 2 : \mathbb{N}$ ) :  $\mathbb{N}$ )
  (( $f : \mathbb{N} \rightarrow \mathbb{N}$ ) ( $3 : \mathbb{N}$ ) :  $\mathbb{N}$ )
```

Semi-simple

```
Let_stm f
  sFix
    sFun (f,  $\mathbb{N} \rightarrow \mathbb{N}$ )
      sFun (x,  $\mathbb{N}$ )
        ((if ( $x \leq 0$ )
          then 0
          else  $f \ (x - 1) + f \ (x - 2)$ )))
(f 3)
```

Simple form:

```
sApp sFix
  sFun (f, ℕ → ℕ)
    sFun (x, ℕ)
      ((if (x ≤ 0)
        then 0
        else f (x - 1) + f (x - 2)))
3
```

### 3.3 All translations (forward)

Rml	@sRml A	typed $\lambda$ -calculus
Var (x, A)	sVar x	$x : A$
Const A c	sConst c	$c : A$
Let (x, T) e <sub>1</sub> e <sub>2</sub>	e' <sub>2</sub>	$(\lambda x : T, e_2 : A) (e_1 : T) : A$
Fun (x, T) e	sFun S (x, T) e'	$(\lambda x : T, e : S) : T \rightarrow S$
App T e <sub>1</sub> e <sub>2</sub>	sApp T e' <sub>1</sub> e' <sub>2</sub>	$(e_1 : T \rightarrow A) (e_2 : T) : A$
Let rec T S f x e <sub>1</sub> e <sub>2</sub>	sApp (T → S) (sFun A (f, T → S) e' <sub>2</sub> ) (sFun S (x, T) (sFix T f x e' <sub>1</sub> (sVar x)))	$(\lambda f : T \rightarrow S, e_2 : A)$ $(Y (\lambda f : T \rightarrow S, \lambda x : T, e_1 : S) : T \rightarrow S) : A$

## 4 Our contribution

## 5 Comparisons and future work

## 6 Conclusion

## 7 Appendix

Example - Error: Stack Overflow.

```

Fixpoint replace_all_variables_aux_type
  A (x : Rml) (env : seq (N * Type * Rml))
  (fl : seq (N * Type)) '{env_valid : valid_env env fl}
  '{x_valid : rml_valid_type A (map fst env) fl x} : sRml A

with replace_all_variables_aux_type_const
  A0 A a (env : seq (N * Type * Rml))
  (fl : seq (N * Type)) '{env_valid : valid_env env fl}
  '{x_valid : rml_valid_type A0 (map fst env) fl (Const A a)} : sRml A0
with replace_all_variables_aux_type_let
  A p x1 x2 (env : seq (N * Type * Rml))
  (fl : seq (N * Type)) '{env_valid : valid_env env fl}
  '{x_valid : rml_valid_type A (map fst env) fl (Let_stm p x1 x2)} : sRml A
with replace_all_variables_aux_type_fun
  A T p x (env : seq (N * Type * Rml))
  (fl : seq (N * Type)) '{env_valid : valid_env env fl}
  '{x_valid : rml_valid_type A (map fst env) fl (Fun_stm T p x)} : sRml A
with replace_all_variables_aux_type_if
  A x1 x2 x3 (env : seq (N * Type * Rml))
  (fl : seq (N * Type)) '{env_valid : valid_env env fl}
  '{x_valid : rml_valid_type A (map fst env) fl (If_stm x1 x2 x3)} : sRml A
with replace_all_variables_aux_type_app
  A T x1 x2 (env : seq (N * Type * Rml))
  (fl : seq (N * Type)) '{env_valid : valid_env env fl}
  '{x_valid : rml_valid_type A (map fst env) fl (App_stm T x1 x2)} : sRml A
with replace_all_variables_aux_type_let_rec A T T0 n n0 x1 x2 (env : seq (N * Type * Rml))
  (fl : seq (N * Type)) '{env_valid : valid_env env fl}
  '{x_valid : rml_valid_type A (map fst env) fl (Let_rec T T0 n n0 x1 x2)} : sRml A.

Proof.
  (** Structure **)
  {
    induction x ; intros ; refine (sVar (0,A)).
  }

  all: refine (sVar (0,A)).
Defined.

```