Mathematical Aspects of Shape Reconstruction from an Image Sequence

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Abstract

It is possible to decompose a three-dimensional objects to a collection of shadows. The geometric relation permits to decompose shadows of a three-dimensional object to shadows of planar objects. Using this geometric relations, we prove a class of non-convex objects is reconstructible from a series of shadows.

1 Introduction

In this paper, we deal with "Shape from shadows." [1, 2, 3]. The illumination problem [4] estimates the minimum and maximum numbers of view points for the reconstruction of a convex body from their views from an appropriate set of view points. The illumination problem is equivalent to shape reconstruction from silhouettes or shadows. However, it is in general difficult to answer the configuration of view points for a given object. There are many results for the reconstruction of a convex polygon from their shadows [5, 6]. Laurentini [7, 8] was concerned with geometric properties of silhouette-based shape reconstruction for polyhedrons, and clarified the relation among the visible hull and the convex hull of a polyhedron.

It is possible to decompose a three-dimensional objects to a collection of shadows. The geometric relation permits to decompose shadows of a three-dimensional object to shadows of planar objects. Using this geometric relations, we prove a class of non-convex objects is reconstructible from a series of shadows.

Let $x-y-z$ be an orthogonal coordinate system in $\mathbf{R}^3$. We call the system the world coordinate system. We denote a vector in the world coordinate $\boldsymbol{x}=(x,y,z)^\top$. Setting K to be a bounded closed convex set in $\mathbf{R}^3$, we denote the boundary of K as ∂K. If a plane touches K at a point on ∂K, this plane is called a support plane of K. We set that $h(\theta,\phi)$ is the Euclidean distance between the origin of the world coordinate system and the support plane of K. the normal vector of which is $\boldsymbol{n}$. $h(\theta,\phi)$ is a function on the unit sphere. K exists in a half space,

$$\mathbf{H}(\theta,\phi)=\{\boldsymbol{x}\,|\,\boldsymbol{x}^\top\boldsymbol{n}\le h(\theta,\phi)\}. \tag{1}$$

Therefore, we call this plane a support plane of K. The following proposition is a well-known result in convex geometry[9].

Proposition 1 *A finite closed convex object is fully reconstructed from the collection of support planes.* [1]

In the previous paper [10], we have proven the following negative theorem for the problem of shspe from shadows.

Theorem 1 *A serise of perspective projections whose camera center moves on a circle encircling a closed finite convex object cannot fully reconstruct an object.*

In this paper, we prove that a class of non-convex object can be fullly reconstructed from the collectionof perspective projections if the camera center moves on the whole sphere encircling an object.

2 Reconstruction of General Cylinder

Setting $(x,y)^\top$ to be an orthogonal coordinate system on a plane, a support line of a planar convex object is

[1]For the normal vector $\boldsymbol{n}=(\cos\phi\sin\theta,\cos\phi\sin\theta,\cos\phi)^\top$, setting $0\le\theta<\pi$, and $0\le\phi<2\pi$, we define a rotation matrix

$$\boldsymbol{R}=\begin{pmatrix}\cos\phi\sin\theta & \sin\phi\sin\theta & \cos\theta\\ \cos\phi\cos\theta & \sin\phi\cos\theta & -\sin\theta\\ -\sin\phi & \cos\phi & 0\end{pmatrix}.$$

and $\hat\nabla$ be the scaler gradient operator on the unit sphere $\mathbf{S}^2$; that is, for function h defined on the unit sphere, $\hat\nabla \boldsymbol{h}(\theta,\phi)=\left(\frac{\partial}{\partial\theta}h,\frac{1}{\sin\theta}\frac{\partial}{\partial\phi}h\right)^\top$, then $\boldsymbol{x}\in\partial K$ is obtained by $\boldsymbol{x}=\boldsymbol{x}R\hat\nabla\boldsymbol{h}$.

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expressed as

$$x \cos\theta + y \sin\theta = p(\theta), \qquad (2)$$

where $\boldsymbol{n} = (\cos\theta, \sin\theta)^\top$ and $p(\theta)$ are the unit normal of this line and the distance from the origin to this line, respectively, as shown in Figure 1 (a). The boundary of a closed convex object is reconstructed from the collection of support lines as

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} p(\theta) \\ \frac{dp(\theta)}{d\theta} \end{pmatrix}. \qquad (3)$$

Setting f to be the focal length of a pin-hole camera, if the length between the optical axis and the right edge of the boundary of the shadow measured from point $(R\cos\beta, R\sin\beta)^\top$ is $r(\alpha)$ as shown in Figure 1 (b), we obtain the relations $p(\pi-\alpha) = \frac{R}{f} r(\theta)\cos\alpha$ and $\boldsymbol{n} = (\cos(\pi-\alpha+\beta), \sin(\pi-\alpha+\beta))^\top$, where R is the radius of the circle on which the camera center moves. Therefore, we can reconstruct two-dimensional convex object from the collection of shadows. We call this geometry for the measurement of shadows the two-dimensional perspective projections.

This classical results of planar convex geometry [9] described above concludes that if an object is expressed as generalized cylinder of convex planar shape and if we know the axis of this object, we can reconstruct this object from a series of two-dimensional perspective projections. For the reconstruction of each slice, we are required to observe shadows from all points on a circle which encircles each slice. Figure 2 illustrates the reconstruction of generalized cylinder from a series of perspective projections.

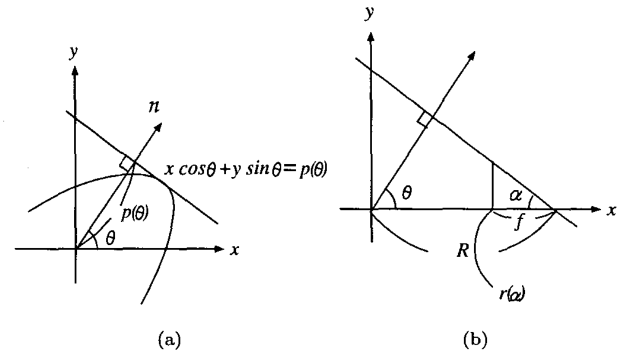

Figure 1: A support plane of a convex curve (a) and the parameters of two-dimensional perspective projection (b).

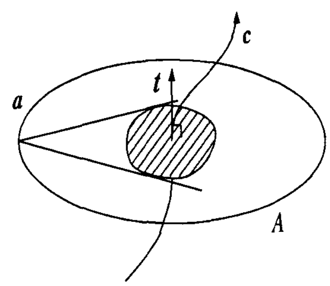

Figure 2: A slice and its support lines of a generalized cylinder whose slices are convex, where **c**, **t**, **a**, and **A** are the axis of a generalized cylinder, the tangent vector of the axis curve at a slice, the camera center, and a circle on which the camera center moves, respectively.

3 Reconstruction of Slice Convex Objects

In the following, we assume that on each slice there exists only one closed finite region. When a slice of an object is convex and we can observe this slice from all points on the circle which encircles this slice on the same plane, we can reconstruct this slice using two-dimensional method as shown in Figure 1. Since all slices of a convex object are convex, we can reconstruct a convex object from a collection of slices which are perpendicular to a axis of slice. For the reconstruction of an object from two-dimensional perspective projections, each point on the boundary is required to lie at least on a convex slice as shown in Figure 3. A banana is an example of such a shape.

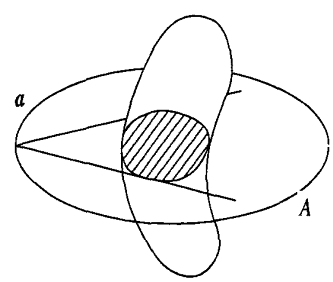

Figure 3: A slice of nonconvex object and its support lines, where $\boldsymbol{a}$ and $\boldsymbol{A}$ are the camera center, and a circle on which the camera center moves, respectively.

A view cone is a cone whose vertex is the camera center and which taches the boundary of an object.

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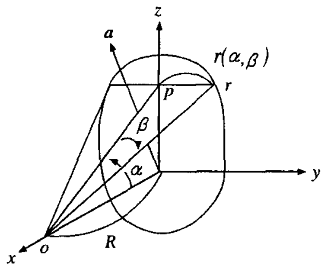

Figure 4: The parameters of a view cone.

For a view cone, when we set parameters α and β as shown in Figure 4, the axis of rotation of a camera, which is perpendicular to a slice is

$$a = R \begin{pmatrix} \sin\alpha \\ 0 \\ \cos\beta \end{pmatrix} \tag{4}$$

and the length between the camera axis and the right boundary of a shadow is $r(\alpha, \beta)\sin\beta$, where R is the rotation matrix which transforms the vector $\overline{pv}$, $\overline{pr}$ in Figure 4 and the rotation axis to λe_1, μe_2, and κe_3, respectively, for $\lambda > 0$, $\mu > 0$, and $\kappa > 0$ for $e_1 = (1, 0, 0)^\top$, $e_2 = (0, 1, 0)^\top$, and $e = (0, 0, 1)^\top$. Uisng this geometry transform, we have the following lemma, when we observe a series of shadows from vertices which lie on a sphere encircling this object.

Lemma 1 *From the collection of shadows which observed from vertices which lie on a sphere encircling this object, we can obtain the collection of two-dimensional perspective projections of a slice form a point which moves on a circle encircling this object.*

For any points on the boundary, if there exists at least one unique convex slice curve which contains this point, we call this object a slice convex object. A convex closed object is slice convex. A banana in Figure 6 is non-convex but slice convex. This geometric property and Lemma 1 derives the following theorem.

Theorem 2 *A slice convex object is uniquely reconstructible from the collection of shadows observed from vertices which lie on the whole sphere encircling this object.*

This theorem permits us for the reconstruction of a class of non-convex objects from shadows. Furthermore, In this expression, the axis for the reconstruction is not required to be a straight line.

For a slice convex object V with respect to axis λv_0 for $|v_0| = 1$ and $\lambda \neq 0$, setting $A[v]$ to be a reconstructed object with respect to the axis λv, for $\lambda \neq 0$, we have the following theorem

Theorem 3 *For an object V the relation*

$$V = \bigcap_{v \in S^2} A[v] \tag{5}$$

is satisfied if V is slice convex with respect to axis λv_0.

(Proof) For vectors $v_0 \neq v$

$$V = A[v_0] \supseteq \bigcap_{v \in S^2} A[v] \supseteq V. \tag{6}$$

Since V is slice convex with respect to axis λv_0, we have the relation

$$V = \bigcap_{v \in S^2} C[v]. \tag{7}$$

These two relations imply the theorem. (Q.E.D)

In the most of previous works for the reconstruction of object from shadows, a method is applied for objects which are rotationally symmetry. and, in these works, the symmetry axis with respect to which an object is slice convex are assumed to be pre-determined. However, this theorem implies that without predetecting the axes for the slice convex we can reconstruct this object if we can measure shadows of slices using perspective projections from vertices over the sphere. This is an important result derived from geometric analysis in this paper.

If an object is defined as the common region of a finite number of slice convex objects, that is, object V is expressed as

$$V = \bigcap_{\alpha=1}^{n} A[a_\alpha], \ |a_\alpha| = 1, \tag{8}$$

for $\lambda \neq 0$, where λa_α is the axis with respect to which slices of an object is convex. Since

$$V = \bigcap_{\alpha=1}^{n} A[a_\alpha] \supseteq \bigcap_{v \in S^2} A[\lambda v] \supseteq V. \tag{9}$$

we have the following theorem.

Theorem 4 *Object V is reconstructed as*

$$V = \bigcap_{v \in S^2} A[v]. \tag{10}$$

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4 Voting Method

Setting the characteristic function in the view cone to be

$$c(\boldsymbol{x};\boldsymbol{a},\boldsymbol{\omega}) = \begin{cases} 1, & \boldsymbol{x} \in C(\boldsymbol{a},\boldsymbol{\omega}) \\ 0, & \text{otherwise}, \end{cases} \tag{11}$$

if we vote $c(\boldsymbol{x};\boldsymbol{a},\boldsymbol{\omega})$ in to the space, we have a function

$$k(\boldsymbol{x}) = \sum_{\boldsymbol{a}\in\boldsymbol{A}} c(\boldsymbol{x};\boldsymbol{a},\boldsymbol{\omega}). \tag{12}$$

as the results of voting. For a positive integer τ, a set of points

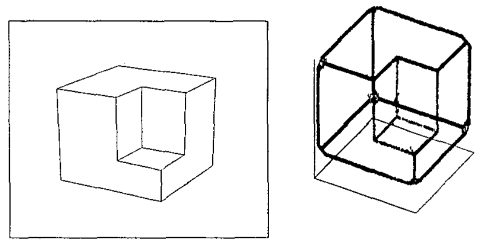
$$K_\tau = \{\boldsymbol{x}|, k(\boldsymbol{x}) \geq \tau\} \tag{13}$$

defines an object. The construction of shape by K_τ is called shape reconstruction by voting.

If an object is a collection of points in a space, projection $\tilde{K}(\boldsymbol{a})$ is a collection of points on a plane for each view point $\boldsymbol{a}$. Furthermore, if an object is a collection of line segments in a space, projection $\tilde{K}(\boldsymbol{a})$ is a collection of line segments on a plane for each view point $\boldsymbol{a}$. A point and a line segment on an imaging plane determine a half line and a fan whose origins are vector $\boldsymbol{a}$, respectively. We can reconstruct the position of a point and a line segment in a space as the common sets of a many half lines and many fans, respectively which are measured from verious directions. A point and a line segment are convex objects whose dimensions are one and two, respectively. A polyhedron is a collection of vertices (points) and edges (line segments) on a closed surface. These geometric properties conclude that the voting method permits us to reconstruct a class of nonconvex polyhedrons without holes from a series of images, if each vertex and each edge of a polyhedron are measured in several images [11]. In Figures 5 (a) and (b), we show an image of a nonconvex polyhedron, which are collection of points on a plane, and the reconstructed polyhedron by voting of half lines in a space, respectively. This polyhedron is not convex nor planar convex.

5 Conclusions

Our results have mathematically clarified that for full recovery of a class of non-convex objects. We also showed that the voting method permits us to reconstruct a class of nonconvex polyhedrons form a series of images using projections of vertices and edges which determine half lines and fans, respectively.



(a) (b)

Figure 5: An example of the reconstruction of a nonconvex polyhedron by voting of half lines in a space.

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