Trilateration

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1 The Trilateration Problem

A node N has determined the distances from itself to each of three (or more) other nodes A, B, C, and so on. The other nodes' geographic locations (coordinates) are known: $A = (a_x, a_y)$, $B = (b_x, b_y)$, $C = (c_x, c_y)$, and so on. The distances from node N to each of the other nodes are a_r, b_r, c_r, and so on. The **trilateration problem** is to find the coordinates of node $N = (n_x, n_y)$ from the given information. A complicating factor is that the known nodes' coordinates and distances typically include measurement errors.

Two methods of solving the trilateration problem are **nonlinear least squares** and **circle intersections with clustering**.

2 Nonlinear Least Squares

From the given information we can write a system of equations where each equation relates the coordinates of N, the coordinates of another node, and the distance to the other node:

$$\begin{aligned}(n_x - a_x)^2 + (n_y - a_y)^2 - a_r^2 &= 0 \\ (n_x - b_x)^2 + (n_y - b_y)^2 - b_r^2 &= 0 \\ (n_x - c_x)^2 + (n_y - c_y)^2 - c_r^2 &= 0 \\ \cdots\end{aligned} \tag{1}$$

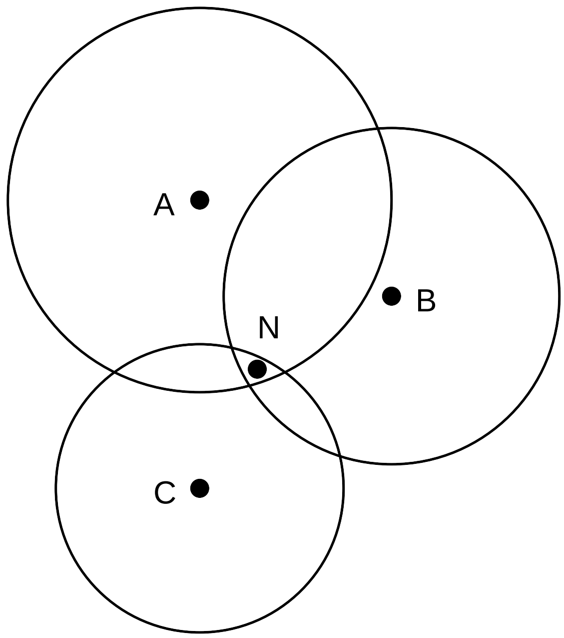
This is a system of three or more equations in the two unknowns (n_x, n_y). Since there are more equations than unknowns, the system is *overdetermined,* and in general there is not a unique solution. However, there is a *least squares* solution. Consider rewriting the equations as follows, where the zeroes on the right hand sides have been replaced by nonzero *residuals*:

$$\begin{aligned}(n_x - a_x)^2 + (n_y - a_y)^2 - a_r^2 &= a_\Delta^2 \\ (n_x - b_x)^2 + (n_y - b_y)^2 - b_r^2 &= b_\Delta^2 \\ (n_x - c_x)^2 + (n_y - c_y)^2 - c_r^2 &= c_\Delta^2 \\ \cdots\end{aligned} \tag{2}$$

The least squares solution is the unique solution (n_x, n_y) that minimizes the sum of the squares of the residuals $(a_\Delta^2 + b_\Delta^2 + c_\Delta^2 + \cdots)$.

The system of equations is also *nonlinear* in the solution parameters (n_x, n_y) because of the squared terms. Thus, a **nonlinear least squares** algorithm is needed to calculate the solution. The algorithm starts from an initial guess at the solution, then does a number of iterations. Each iteration applies a correction to the previous solution so as to reduce the sum of the squared residuals. The algorithm can also incorporate information about the uncertainty (variance) in the input quantities. For further information, see a numerical methods textbook, such as Press et al., *Numerical Recipes in C* (Cambridge University Press, 1992).

3 Circle Intersections With Clustering

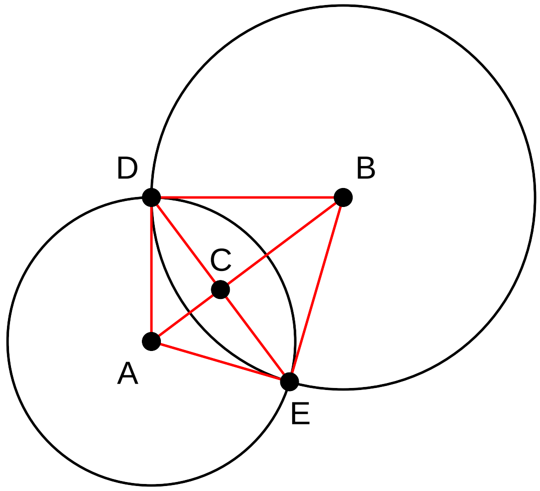


Draw a circle of radius a_r around point A, a circle of radius b_r around point B, a circle of radius c_r around point C, and so on. Since the circles' centers and radii are subject to measurement errors, the circles will overlap in a (hopefully small) region rather than intersecting at a single point. The unknown node location N is somewhere in this region.

Each pair of circles yields two intersection points (see Section 4 for formulas). With three circles, there are six intersection points. Three of these points are *clustered* closely together, while the rest are far apart. The node N is located in the middle of this cluster.

Here is one way to find the cluster. This works if the clustered points are much closer to each other than they are to the outlying points. Compute the distance between each pair of circle intersection points. Pick the two closest intersection points to be the initial cluster. Compute the **centroid** of the cluster. The centroid's X coordinate is the average of the X coordinates of the points in the cluster; the centroid's Y coordinate is the average of the Y coordinates of the points in the cluster. Next, find the circle intersection point that is closest to the cluster centroid. Add this intersection point to the cluster and recompute the cluster centroid. Continue in this way until k intersection points have been added to the cluster, where k is the number of circles. The final cluster centroid gives the location of node N.

4 Intersection of Two Circles



Let one circle's center be $A = (a_x, a_y)$ with radius a_r. Let another circle's center be $B = (b_x, b_y)$ with radius b_r. To minimize roundoff error in the calculations, the circles should be such that $a_r \leq b_r$. We also assume that $A \neq B$. We wish to find the circles' intersection points, $D = (d_x, d_y)$ and $E = (e_x, e_y)$.

First, find the distance between the circles' centers, Δ:

$$\Delta_x = b_x - a_x \tag{3}$$

$$\Delta_y = b_y - a_y \tag{4}$$

$$\Delta^2 = \Delta_x^2 + \Delta_y^2 \tag{5}$$

$$\Delta = \sqrt{\Delta^2} \tag{6}$$

If $\Delta > b_r + a_r$, then the circles do not intersect (they are too far apart). If $\Delta < b_r - a_r$, then the circles do not intersect (one is inside the other).

Next, find the point $C = (c_x, c_y)$. Let s be the distance AC, t be the distance BC, and u be the distance CD (or CE). Then:

$$s^2 + u^2 = a_r^2 \tag{7}$$

$$t^2 + u^2 = b_r^2 \tag{8}$$

Subtract the two equations:

$$s^2 - t^2 = a_r^2 - b_r^2 \tag{9}$$

Factor the left hand side:

$$(s - t)(s + t) = a_r^2 - b_r^2 \tag{10}$$

Since $s + t = \Delta$ and $t = \Delta - s$, this gives the following expression for s:

$$(s - (\Delta - s))\Delta = a_r^2 + b_r^2 \tag{11}$$

$$2s\Delta - \Delta^2 = a_r^2 - b_r^2 \tag{12}$$

$$s = \frac{\Delta^2 + a_r^2 - b_r^2}{2\Delta} \tag{13}$$

Then the coordinates of point C are:

$$c_x = a_x + \Delta_x s/\Delta \tag{14}$$

$$c_y = a_y + \Delta_y s/\Delta \tag{15}$$

Finally, find the points D and E. Substituting (13) into (7), we have:

$$u = \sqrt{a_r^2 - s^2} \tag{16}$$

Then the coordinates of point D are:

$$d_x = c_x - \Delta_y u/\Delta \tag{17}$$

$$d_y = c_y + \Delta_x u/\Delta \tag{18}$$

And the coordinates of point E are:

$$e_x = c_x + \Delta_y u/\Delta \tag{19}$$

$$e_y = c_y - \Delta_x u/\Delta \tag{20}$$