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Computational aspects of nodal multizone airflow systems

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Abstract

The multizone approach to steady-state airflow problems models a building as a network of discrete mass flow paths. A nodal formulation of the problem writes the governing equations in terms of the unknown pressures at the points where the flow paths connect. This paper proves conditions under which the nodal equations yield symmetric positive-definite matrices, guaranteeing a unique solution to the flow network. It also establishes relaxed conditions under which a nodal airflow system yields asymmetric matrices with positive eigenvalues, guaranteeing at least one solution.

Properly exploiting the system properties greatly reduces the cost of numerical solution. Thus, multizone airflow programs such as CONTAM and COMIS depend on symmetric positive-definite systems. However, the background literature neglects or simplifies the underlying assumptions, does not assert existence and uniqueness, and even contains factual errors. This paper corrects those errors, states the implicit assumptions made in the programs, and discusses implications for modelers and programmers.
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1. Introduction

Multizone airflow models idealize a building as a network of discrete *flow elements* such as doors, cracks, and ductwork. The flow elements connect at nodes, which represent either static *zones* such as rooms, or points where two elements meet, such as duct junctions. The governing equations represent: (1) pressure-flow relations in the flow elements, (2) mass conservation at the nodes, and (3) hydrostatic pressure variations in the zones.

A *nodal* formulation of the problem makes the node reference pressures the independent variables, calculating the hydrostatic effects and flows accordingly. The nodal formulation dominates among multizone airflow simulation programs such as CONTAM [1], COMIS [2], ESP [3], and MIX [4]. These programs solve the airflow network by adjusting the reference pressures in order to achieve mass balance. Typically they use variations on the well-known Newton–Raphson algorithm, adjusting the reference pressures simultaneously based on affine (linearized) models of the airflow network.

If linearizing the system always produces symmetric, positive-definite matrices, it allows considerable implementation efficiencies. Symmetry reduces, by almost half, the cost of storing and factoring the matrix. The positive-definite property guarantees its nonsingularity, and further simplifies the factorization. More important, an airflow system with these properties always has a unique solution, which may always be found using damped Newton–Raphson iteration.

Despite the centrality of symmetric positive-definite systems to programs such as COMIS and CONTAM, only a few sources treat the mathematical properties of nodal airflow systems in any detail [1,3,5–9]. However, they tend to neglect or oversimplify the conditions that guarantee these properties. Even those papers that provide greater detail and rigor [10–12] omit at least one of the underlying assumptions, and none shows that symmetric positive-definite matrices guarantee a unique solution to the nonlinear problem. Arguments for the existence and uniqueness of solutions to contaminant dispersal systems [13] do adapt to airflow systems, but only under the same conditions proved here.

This paper shows that the following conditions—all explicit or implicit assumptions of multizone airflow programs such as CONTAM and COMIS—ensure that a steady-state nodal airflow system produces symmetric positive-definite matrices when linearized:

(1) Every flow element model relates the steady-state mass flow through the element to the pressure drop across it.

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Nomenclature

Physical quantities

f^e_{i-j}	net mass flow through flow element e, positive from zone i to zone j
g	gravity acceleration
$\dot{m}_{i-j}$	net mass flow through all elements connecting zones i and j, positive from i to j
n	count of variable-pressure nodes in the network
N	count of all nodes in the network, $n < N$
p^e_i	pressure in zone i, at connection to flow element e
p_i	pressure in zone i, at zone reference level
Δp^e_{i-j}	pressure drop, across flow element e, from zone i to zone j
Δp^e_{L}	mechanical energy lost by dissipation in flow element e, on pressure basis
ρ^e	density of air in flow element e
ρ^e_i	density of air in zone i, at connection to flow element e
w^e	work per unit mass of air done by flow element e
z^e_i	absolute height of flow element e connection to zone i
z_i	absolute height of zone i reference level

Numerical quantities

J	Jacobian matrix of residual function derivatives
r	vector of residual errors in nonlinear system
x	vector of independent variables

Notation

x_i	element i of vector x
$A_{i,j}$	element in row i and column j of matrix A
$x_{[k]}$	vector x at iteration k
$\hat{r}_{[k]}$	affine model of r at iteration k
$f\{x\}$	function of x

(2) The parallel flow elements linking any two zones combine such that increasing the reference pressure in one zone makes the net outflow to the connected zone more positive, or at least leaves it unchanged. Call these *positive* and *nonnegative* responses, respectively.
(3) Every zone of unknown pressure connects to a zone of known pressure through a series of such net flow paths, all with positive responses. Adding nonnegative-response paths to such a system does not disturb its properties.
(4) Within a zone, the pressure varies hydrostatically; however, the change of pressure with height does not depend on the zone's reference pressure.
(5) After initialization, the zone density at each element connection does not change with the zone reference pressure.
(6) The zone temperatures and temperature distributions do not vary with the pressures or flows.
(7) The element pressure-flow curves and zone hydrostatic relations have continuous, bounded first derivatives with respect to the zone reference pressures.

This paper also shows that relaxing conditions 1, 4, 5 and 6 results in asymmetric but nonsingular matrices, and a flow network with at least one solution. None of the literature cited above contains this result, which pertains directly to coupled thermal-flow systems.

Section 2 characterizes the flow element and zone models used to assemble steady-state nodal airflow networks. Section 3 gives some numerical background, and Section 4 derives the important network properties. Finally, Section 5 discusses modeling implications of the nodal formulation. This paper does not treat other formulations, such as: (1) loop methods [14]; (2) symbolic and input-output-free formulations [15,16]; or (3) dynamic airflow models [17].

2. Airflow network background

This section describes the governing equations for flow elements and zones, then shows how a nodal formulation of the airflow network assembles the path and zone models into a complete system description.

2.1. Flow elements

Multizone models relate the flow through each element to the pressure and density of air in the zones it connects. Suppose element e connects zones i and j, and define f^e_{i-j} as the net mass flow through e, positive for net flow from zone i to zone j, and negative for flow in the opposite direction. A bidirectional element sums its two opposing flows to find f^e_{i-j}.

In a nodal formulation, the flow model finds f^e_{i-j} as a function of the pressures and densities at it terminals:

$$f^e_{i-j} = f^e_{i-j}\{p^e_i, p^e_j, \rho^e_i, \rho^e_j\}, \tag{2.1}$$

where p_i^e and ρ_i^e give the pressure and density of the air in zone i, at the point where element e connects to the zone.

Each flow element substitutes its own relations for Eq. (2.1). A typical model defines the flow in terms of the mechanical energy lost in the element due to viscous and turbulent dissipation. The "lost" energy, expressed on a pressure basis as Δp_{L}, re-appears as thermal energy in the downstream fluid. For example, the common *orifice* model sets $f_{i-j}^e = C_d A \sqrt{2\rho^e \Delta p_{\mathrm{L}}^e}$ for positive flows. For standard flow models, see [1,2,18].

A steady-state mechanical energy balance relates Δp_{L}^e to the connection pressures. Neglecting heat transfer and changes in kinetic energy,

$$p_i^e + \rho^e g z_i^e + w^e = p_j^e + \rho^e g z_j^e + \Delta p_{\mathrm{L}}^e, \tag{2.2}$$

where z_i^e gives the absolute height of the element's connection to zone i, measured from some global reference level; w^e gives the work per unit mass done on the fluid by an active element such as a fan; and ρ^e gives the density of air passing through the element. For net flow from zone j to zone i, $f_{i-j}^e < 0$ and $\Delta p_{\mathrm{L}}^e < 0$.

The flow element model, Eq. (2.1), incorporates both the mechanical energy balance and the relation between Δp_{L}^e and the flow. It also specifies the path density, ρ^e. Since the element transports air between the zones, typically ρ^e varies between ρ_i^e and ρ_j^e, with the exact value depending on the flow velocity.

2.2. Zones

In the nodal formulation, a zone model relates the pressure and density at the element connections to the zone's *reference pressure*. Pressure variations within a zone result from hydrostatic and wind effects only [1,2]. Unlike flow elements, zones have no associated energy losses; this includes heat transfer, since the space temperatures are (fixed) model parameters. Thus, the flow models must account for any dissipation associated with discharge from a flow element into a zone.

The volume of a zone does not affect the airflow system. Therefore this paper treats nodes, where flow elements connect directly, as volumeless zones.

Let p_i denote the reference pressure of zone i, measured at some absolute height z_i. Then the pressure and density at the connection to element e depend on p_i, on the heights z_i and z_i^e, and on the assumed temperature and wind profiles in the zone. Of these, the user specifies all but the zone reference pressure. Hence, in terms of variables controlled by the nodal solution algorithm, the element connection pressure and density vary with the reference pressure only:

$$p_i^e = p_i^e\{p_i\} \tag{2.3a}$$

and

$$\rho_i^e = \rho_i^e\{p_i\}. \tag{2.3b}$$

Zone models can accommodate temperature and density variations throughout the height of the space. However, a common zone idealization assumes a uniform zone density, ρ_i, given by the ideal gas law at the reference pressure. Then

$$p_i^e = p_i - \rho_i g(z_i^e - z_i) \tag{2.4a}$$

and

$$\rho_i^e = \rho_i = \frac{p_i}{R_{\mathrm{air}} T_i}, \tag{2.4b}$$

where the gas constant for dry air $R_{\mathrm{air}} = 287$ N m/kg K; and T_i gives the absolute temperature of the zone. For this steady-state model, any contribution due to wind would appear as a constant term on the right side of Eq. (2.4a).

2.3. Airflow networks

Consider an airflow system comprising N zones. Of these, the problem statement specifies the pressure of at least one zone—usually the ambient pressure of the building's surroundings. Thus the network has $n < N$ *variable-pressure* zones. The nodal formulation treats these n unknown reference pressures as the independent variables. For convenience, number the zones to make p_i a problem variable for $1 \leqslant i \leqslant n$, and a fixed parameter for $n+1 \leqslant i \leqslant N$.

Any two zones may share zero, one, or more flow paths. Define the net interzonal mass flow carried by these parallel paths as

$$\dot{m}_{i-j} = \sum_l f_{i-j}^l, \tag{2.5}$$

where index l extends over every element connecting zones i and j. If the zones share no flow elements, $\dot{m}_{i-j} = 0$. By definition, $\dot{m}_{i-i} = 0$.

At steady state, mass conservation requires $\sum_{k=1}^N \dot{m}_{i-k} = 0$. That is, the flows leaving zone i must sum to zero (here, as elsewhere, a positive flow entering a zone counts as a negative flow leaving the zone).

Mass conservation applies to all zones, $1 \leqslant i \leqslant N$. However, numerical solution demands only n equations in the n unknown pressures. Therefore the nodal formulation defines residual equations only for the variable-pressure zones:

$$r_i = \sum_{k=1}^N \dot{m}_{i-k}, \quad 1 \leqslant i \leqslant n. \tag{2.6}$$

The solution scheme adjusts the reference pressures to set the vector $r = 0$, i.e., so that the flows leaving each variable-pressure zone sum to zero.

3. Numerical background

Most multizone airflow programs use some variation on Newton–Raphson's method to solve Eq. (2.6) [18]. Newton–Raphson generates a series of matrices containing the derivatives of the residual equations. This section shows that: (1) when the nonlinear system always produces nonsingular matrices, it must have a solution; and (2) when those matrices are always symmetric positive-definite, the system has a unique solution. Guarantees of symmetry and invertibility also reduce the cost of finding a solution.

3.1. Newton–Raphson

Consider a system of nonlinear residual equations, $r\{x\}=0$. In a nodal airflow system, r describes mass conservation in the variable-pressure zones, while x represents the vector of pressures, p_1 through p_n. Here, x replaces p as the independent variable, both to match conventional linear algebra notation, and to stress that these results apply to any system of nonlinear equations.

Newton–Raphson's method seeks a solution by iteratively linearizing the residuals, and jumping to the solution of each linearized system [19]. The linearization step forms a first-order Taylor series of each residual about a current guess, $x_{[k]}$, and collects them into the affine model

$$\hat{r}_{[k]} = r_{[k]} + J_{[k]}(x - x_{[k]}), \tag{3.1}$$

where $r_{[k]} = r\{x_{[k]}\}$ gives the residuals evaluated at $x_{[k]}$. The Jacobian matrix, $J_{[k]}$, evaluates the derivatives of the residuals, again at $x_{[k]}$, such that the ith row and jth column has a component

$$J_{i,j} = \frac{\partial r_i}{\partial x_j}. \tag{3.2}$$

Note that Eq. (2.6) allows *assembly* of the Jacobian by directly summing the derivatives of the flow element relations [5].

Solving for $\hat{r}_{[k]}=0$ gives the Newton–Raphson solution, $x^{\mathrm{NR}}_{[k+1]} = x_{[k]} - J^{-1}_{[k]} r_{[k]}$. For efficiency, matrix factorization replaces inversion of the Jacobian. Still, factorization remains a costly operation, requiring order-n^3 floating-point operations [19]. Furthermore, the resulting iteration may not converge to a solution, and general techniques for stabilizing the method increase the computational burden [19].

Reducing the cost of numerical solution depends on exploiting properties of the Jacobian matrix. Two important properties, nonsingularity and symmetry, allow significant implementation efficiencies.

3.2. Nonsingular

If the residual equations yield a nonsingular Jacobian when linearized about any $x_{[k]}$, then they must have a solution. For proof, consider the "sum of squares" cost function, $r^{\mathrm{T}}r$. The cost function gradient, $2J^{\mathrm{T}}_{[k]}r_{[k]}$, gives a search direction in which a change in x reduces the cost function. A zero gradient at $x_{[k]}$ implies either: (1) $r_{[k]}=0$, so that $x_{[k]}$ solves the nonlinear system; or (2) $J^{\mathrm{T}}_{[k]}r_{[k]}=0$, making $J_{[k]}$ singular by definition. Therefore, if the residual equations always yield a nonsingular Jacobian, then any $x_{[k]}$ either solves the system, or gives a nonzero search direction that reduces the magnitude of the residual vector [19].

A nonsingular Jacobian also ensures that the local model of the residuals, $\hat{r}_{[k]}$, has a solution. Therefore, local to any $x_{[k]}$, the Newton–Raphson direction reduces every residual magnitude toward zero. Barring numeric effects, a series of line searches can always solve such a system, although more complicated methods may speed convergence [19]. Despite this fact, building airflow programs have only recently addressed known convergence problems using general line search techniques [20].

3.3. Symmetric

A square symmetric matrix, A, has $A_{i,j} = A_{j,i}$. Symmetry allows improved storage [11] and factorization routines [19], reducing the associated memory and floating-point operation counts by about half. For the sparsity patterns typical of building airflow systems, a skyline storage scheme has proved useful [8,9,11].

3.4. Positive-definite

By definition, a symmetric positive-definite matrix has $x^{\mathrm{T}}Ax > 0$ for any vector $x \neq 0$. It follows that A has all positive eigenvalues, and hence always has an inverse [21]. Furthermore, Gaussian elimination on A always finds positive pivots, even without row exchanges. This allows further simplification of the factorization routine.

If a nonlinear system always yields symmetric positive-definite Jacobians, then it has a unique solution. Consider an arbitrary point $x^* + s$, offset by nonzero vector s from a solution at x^*. To find $r\{x^*+s\}$, integrate the residuals along the line segment $x^* + ts$ between the two points. With $r=0$ at $t=0$, it follows that $r\{x^*+s\} = \int_0^1 (J\{t\}s)\,\mathrm{d}t$ [19]. Since $s^{\mathrm{T}}J\{t\}s > 0$ for all t, the inner product

$$s^{\mathrm{T}}r\{x^*+s\} = \int_0^1 (s^{\mathrm{T}}J\{t\}s)\,\mathrm{d}t > 0. \tag{3.3}$$

With $s \neq 0$, the result $s^{\mathrm{T}}r > 0$ implies $r\{x^*+s\} \neq 0$, making x^* the only solution to the residual equations.

4. System properties

Most multizone airflow programs assume symmetric positive-definite Jacobians, in order to reduce the computational burden of solution. This section establishes conditions under which a nodal formulation of an airflow network problem actually does yield symmetric positive-definite matrices. It also shows a new result, in which less restricted flow systems produce asymmetric, nonsingular matrices.

These system properties depend on finding nonpositive off-diagonals, and sufficiently positive diagonals, in the Jacobian matrix. Therefore this section begins by examining individual Jacobian elements, then shows how they lead to matrices with the desired properties. For notational convenience, this section drops the iteration subscript $[k]$; all results apply to any $J_{[k]}$ implicitly. It also assumes a Jacobian exists at any $x_{[k]}$. In other words, it assumes that the residual functions have continuous, bounded derivatives everywhere (condition 7 in the Introduction).

4.1. Jacobian elements

From Eqs. (2.6) and (3.2), $J_{i,j} = \sum_{k=1}^{N} (\partial \dot{m}_{i-k}/\partial p_j)$. First consider $i \neq j$. For these off-diagonal elements, $\dot{m}_{i-k}$ can vary with p_j only when $k = j$. That is, the off-diagonal elements represent links between variable-pressure zones. Thus

$$J_{i,j} = \frac{\partial \dot{m}_{i-j}}{\partial p_j} = -\frac{\partial \dot{m}_{j-i}}{\partial p_j} \quad \text{for } i \neq j. \tag{4.1}$$

The second result follows since $\dot{m}_{i-j} = -\dot{m}_{j-i}$ in a steady-state flow system. If no flow path connects zones i and j, then $\dot{m}_{i-j} = 0$ and $J_{i,j} = 0$.

On the diagonal, $J_{i,i} = \sum_{k=1}^{N} (\partial \dot{m}_{i-k}/\partial p_i)$. Breaking up the summation to distinguish between connections to zones of fixed and variable pressure, and substituting Eq. (4.1), gives

$$J_{i,i} = \sum_{k=n+1}^{N} \frac{\partial \dot{m}_{i-k}}{\partial p_i} - \sum_{i \neq k=1}^{n} J_{k,i}. \tag{4.2}$$

That is, the diagonal adds derivatives of any outflows to fixed-pressure zones, and subtracts those due to inflows from variable-pressure zones (given by the off-diagonals in the same column). Again, inflows count as negative outflows, and vice versa.

Both Eqs. (4.1) and (4.2) contain partial derivatives of the net interzonal mass flow leaving a zone, with respect to its reference pressure. From Eq. (2.5),

$$\frac{\partial \dot{m}_{i-j}}{\partial p_i} = \sum_l \frac{\partial f^l_{i-j}}{\partial p_i} = \sum_l \left(\frac{\partial f^l_{i-j}}{\partial p^l_i} \frac{\partial p^l_i}{\partial p_i} + \frac{\partial f^l_{i-j}}{\partial \rho^l_i} \frac{\partial \rho^l_i}{\partial p_i} \right), \tag{4.3}$$

where index l extends over every element connecting zones i and j. The second summation, which follows from Eqs. (2.1) and (2.3), shows that hydrostatic effects in the zone modify the contributions of the flow elements—a point ignored in the building airflow literature.

4.2. Sign of Jacobian elements

The partial derivatives of Eq. (4.3) determine the sign of the $\partial \dot{m}_{i-j}/\partial p_i$. First, note that most zone models will have $\partial p^e_i/\partial p_i \geqslant 0$ and $\partial \rho^e_i/\partial p_i \geqslant 0$, because the connection pressure and density should not decrease as the reference pressure increases. By similar reasoning, a typical flow element should have $\partial f^e_{i-j}/\partial \rho^e_i \geqslant 0$. Therefore $\dot{m}_{i-j}$ should tend to increase with p_i, provided the outflow has a positive derivative with respect to the connection pressure.

Most real flow components do have $\partial f^e_{i-j}/\partial p^e_i > 0$. That is, they give more positive outflows from zone i as the connection pressure p^e_i increases. The energy balance, Eq. (2.2), suggests why: as p^e_i increases, the conversion of mechanical to thermal energy in element e, given by Δp^e_{L}, becomes more positive. This implies more positive flow out of zone i (however, Section 5 gives some counterexamples). Thus parallel combinations of flow elements connecting two zones tend to have $\partial \dot{m}_{i-j}/\partial p_i > 0$.

Call the aggregate behavior of the parallel paths between zones i and j a *positive response* if increasing p_i always makes the net outflow to zone j more positive, $\partial \dot{m}_{i-j}/\partial p_i > 0$. Similarly, call the response *nonnegative* if increasing the reference pressure at least leaves the outflow unchanged, $\partial \dot{m}_{i-j}/\partial p_i \geqslant 0$.

For networks composed entirely of paths with positive responses (condition 2), Eqs. (4.1) and (4.2) imply

$$J_{i,j} \leqslant 0 \tag{4.4a}$$

and

$$J_{i,i} \geqslant - \sum_{i \neq k=1}^{n} J_{k,i} > 0. \tag{4.4b}$$

Strict equality in the second relation holds when variable-pressure zone i does not connect to a zone of constant pressure.

If a network satisfies Inequalities (4.4), then adding any number of flow paths with nonnegative responses does not affect those relations. Conditions 2 and 3 reflect this fact. In practical terms, this means that adding constant-flow elements, with their zero derivatives, to a flow system does not affect the existence and uniqueness of a solution (though it does change the solution itself).

Note that Inequality (4.4b) gives a more general result than the assertion usually found in discussions of airflow systems. That row-wise result, $J_{i,i} \geqslant \sum_{k \neq i} |J_{i,k}|$, assumes symmetry.

4.3. Nonsingularity

A steady-state nodal airflow system composed only of positive-response paths yields a nonsingular Jacobian provided every variable-pressure zone connects to one of fixed pressure via a series of such paths. Conversely, lacking any connection to a fixed-pressure zone, the system is singular. Adding any number of nonnegative-response paths to such a system does not alter these properties.

First consider a system with no connection to a fixed-pressure zone. By Eq. (4.2), the Jacobian rows sum to zero, making the matrix singular.

On the other hand, suppose the system has at least one positive-response connection to a fixed-pressure zone. To prove J has full rank, consider its eigenvalues, λ. Since J^{T} and J have the same eigenvalues, $J^{\mathrm{T}} v = \lambda v$ for some $v \neq 0$ [21]. The ith row of this equation give $\sum_{k=1}^{n} v_k J_{k,i} = \lambda v_i$. Let i select the component of v with largest absolute value. If the eigenvector has multiple elements with the same maximal value, then pick i such that zone i connects either to a fixed-pressure zone, or to a variable-pressure zone j with $|v_j| < |v_i|$. Such a zone must exist in the assumed system.

With this choice of i, $v_k/v_i \leqslant 1$ for all variable-pressure zones k. Then Inequalities (4.4) give

$$\lambda = \sum_{k=1}^{n} \frac{v_k}{v_i} J_{k,i} \geqslant \sum_{k=1}^{n} J_{k,i} \geqslant 0. \tag{4.5}$$

Equality cannot hold in both relations simultaneously. If zone i connects to a fixed-pressure zone, then $\lambda \geqslant \sum_{k=1}^{n} J_{k,i} > 0$. Otherwise, some $j \neq i$ has $(v_j/v_i)J_{j,i} > J_{j,i}$, and hence $\lambda > \sum_{k=1}^{n} J_{k,i} = 0$.

Thus, for every eigenvector of J^{T}, some choice of i shows $\lambda > 0$. Therefore the system matrix has positive eigenvalues, and cannot be singular. Adding nonnegative-response paths to the network does not affect these results, because at worst such paths can only contribute zero derivatives when assembling the Jacobian.

4.4. Comments on nonsingularity

As discussed in Section 3, if the system has a nonsingular Jacobian everywhere (i.e., for any choice of reference pressures), then it must have at least one solution. Note that the proof does not rely on symmetry. Therefore if a nodal system has continuous, bounded derivatives (condition 7) that satisfy Inequalities 4.4 (conditions 2 and 3), it has a solution. It need not meet any of conditions 1, 4, 5 or 6.

The conditions used here to prove the Jacobian is nonsingular also satisfy the "weak column sum criterion" needed to ensure that both the Jacobi and Gauss–Seidel methods can solve the linear system at each Newton–Raphson iteration [22]. However, numerical tests indicate the convergence may be slow [8,9].

4.5. Symmetry

Symmetry requires $J_{i,j} = J_{j,i}$ or, using Eq. (4.1), $\partial \dot{m}_{i-j}/\partial p_i = -\partial \dot{m}_{i-j}/\partial p_j$. From Eq. (4.3), this requires

$$\sum_{l} \left(\frac{\partial f_{i-j}^{l}}{\partial p_i^{l}} \frac{\partial p_i^{l}}{\partial p_i} + \frac{\partial f_{i-j}^{l}}{\partial \rho_i^{l}} \frac{\partial \rho_i^{l}}{\partial p_i} \right)$$
$$= -\sum_{l} \left(\frac{\partial f_{i-j}^{l}}{\partial p_j^{l}} \frac{\partial p_j^{l}}{\partial p_j} + \frac{\partial f_{i-j}^{l}}{\partial \rho_j^{l}} \frac{\partial \rho_j^{l}}{\partial p_j} \right) \tag{4.6}$$

for every combination of variable-pressure zones i and j in the system.

To force Eq. (4.6) to hold for any parallel combination of flow paths, it must hold for each path. Furthermore it must hold regardless of how the zone model changes p_i^e and ρ_i^e with changes to p_i. Therefore symmetry requires

$$\frac{\partial f_{i-j}^{e}}{\partial p_i^{e}} = -\frac{\partial f_{i-j}^{e}}{\partial p_j^{e}}, \tag{4.7}$$

so that the flow elements must calculate f_{i-j}^e as a function of the pressure drop $\Delta p_{i-j}^e = p_i^e - p_j^e$ across the element connections (condition 1). That is, symmetry requires flow models of the form

$$f_{i-j}^{e} = f_{i-j}^{e}\{\Delta p_{i-j}^{e}, \rho_i^{e}, \rho_j^{e}\}, \tag{4.8}$$

rather than of the form given by Eq. (2.1).

Even with flow a function of pressure drop, Eq. (4.6) still requires

$$\frac{\partial p_i^{e}}{\partial p_i} = 1 \tag{4.9a}$$

and

$$\frac{\partial \rho_i^{e}}{\partial p_i} = 0 \tag{4.9b}$$

for symmetry. According to Eq. (4.9a), any change to the zone reference pressure must produce an identical change at the flow element connection (condition 4). Eq. (4.9b) states that the zone density at each element connection cannot change with the reference pressure (condition 5). Note that while the latter relation prohibits updating the zone densities during the Newton–Raphson search, it does permit initializing them based on the initial reference pressures.

4.6. Positive-definite

Under the conditions described above, the airflow system matrix has positive eigenvalues. Therefore a symmetric Jacobian is also positive-definite [21]. Then by the discussion of Section 3, it may be factored without pivoting, and the nonlinear system has a unique solution.

5. Modeling implications

This section discusses the impact on models of the conditions required for a nodal airflow network to yield symmetric positive-definite system matrices. The most restrictive conditions affect the flow element models, on which multizone airflow programs such as COMIS and CONTAM chiefly rely in order to provide the user with rich modeling environments. Other requirements apply to the zone models.

5.1. Pressure drop

According to Eq. (4.8), a multizone airflow program must calculate the flow through each element as a function of the pressure drop across it, in order to produce symmetric systems. One important flow component, the duct T-junction, does not follow this pattern. Modeled as a three-port flow element, the duct junction would break symmetry. Of course, a multizone program can represent a T-junction as a zone, and try to account for mechanical energy losses in the adjoining duct elements. However, this approach misses the

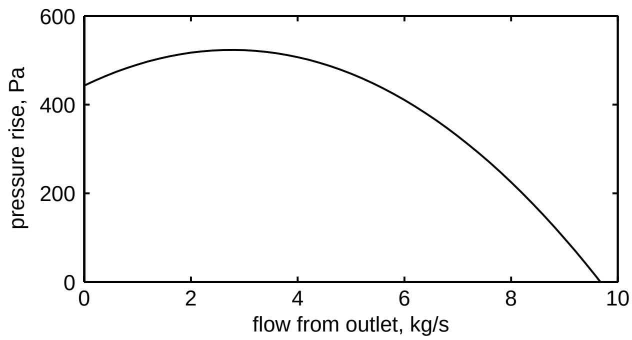

Fig. 1. As typically presented, a fan curve gives the pressure rise from inlet to outlet as a function of flow from inlet to outlet.

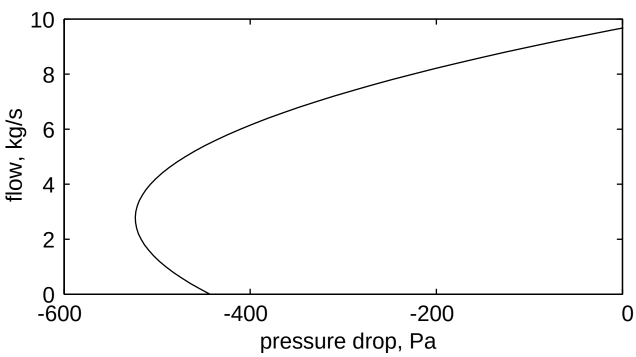

Fig. 2. Fan curve showing $f^e_{i-j}\{p^e_i - p^e_j\}$, with the fan inlet as zone i. Placing Δp^e_{i-j} in the neighborhood of the peak pressure rise gives multiple flows, and negative or infinite derivatives.

fact that the dissipated energy in each leg of a junction depends on the flows in its other legs [23].

Large flow elements such as doors and windows do not connect to a zone at a single height, as assumed in Eq. (2.2). Therefore they do not respond to a single pressure drop. However, because pressure and density variations over the height of the connection do not depend on the zone reference pressure (conditions 4 and 5), these models do not disturb the symmetric positive-definite properties.

5.2. *Positive derivatives*

Eqs. (4.3), (4.4a), and (4.9) require that increasing the pressure drop across a flow element should increase the flow in the direction of the pressure drop (or leave it unchanged for a nonnegative response).

As noted in the discussion above, most flow elements have $\partial f^e_{i-j}/\partial \Delta p^e_{i-j} > 0$. However, not every real flow element has monotone increasing flow over all pressure drops. Fig. 1 shows an important exception, a fan, as usually presented—with the pressure rise from inlet to outlet a function of the flow in the same direction. Fig. 2 shows the same data in the form $f^e_{i-j}\{p^e_i - p^e_j\}$, with the fan inlet at zone i. The fan, an active element with $w^e > 0$, gives negative pressure drops in the flow direction.

The curve in Fig. 2 has a negative slope at low flows. This creates a range of Δp^e_{i-j} with multiple associated flows—impossible to represent with a steady-state model of the form of Eq. (4.8). Furthermore the curve has an infinite derivative at the point of maximum pressure rise. Even though a properly-sized fan should not operate in this problematic region [23], a multizone solution algorithm might evaluate the fan model there, during its iterative search for the steady-state operating point. To avoid this, the fan model should enforce a positive derivative over the entire range of pressures, for example by linear extrapolation outside the region of expected operation [2].

While less common, passive flow elements also can give negative slopes in the pressure-flow curve. A self-regulating vent, for example, adjusts its flow area in order to deliver nearly constant flow under all conditions. Then, because larger pressure drops tend to increase infiltration, some self-regulating devices compensate further, by reducing the flow as the pressure drop grows larger [24]. This behavior presents no problem for the flow model—every Δp^e_{i-j} gives a unique flow—but the negative slope may produce an indefinite system, unless the vent operates in parallel with other flow elements of sufficiently positive slope (see Eq. (4.3)).

5.3. *Continuous derivatives*

The existence and uniqueness proofs of Section 3 assume a nonsingular Jacobian exists everywhere. This means the equations defining the flow models must have bounded, continuous derivatives with respect to the zone pressures. Of course, continuous derivatives imply continuous pressure-flow curves.

Piecewise flow relations, in particular, can violate either condition. For example, the Darcy–Weisbach duct model has a discontinuity at the transition between laminar and turbulent flow [23]. Less obviously, discontinuity may arise from a flow element's density relations. From Eqs. (2.1) and (2.2), if an element's connections have different absolute heights, then it must make ρ^e a continuous function of the terminal pressures in order to give a continuous pressure-flow curve.

Without special attention, even continuous pressure-flow curves can have discontinuous derivatives [11]. This does not, as sometimes stated, cause an asymmetric Jacobian, but rather makes the Jacobian undefined at certain pressures. The fact that this has not been identified as a source of problems in CONTAM and COMIS suggests that discontinuous derivatives do not, in themselves, cause numerical difficulty. In other words, while condition 7 requires continuous first derivatives, it seems reasonable to suppose that a continuous flow model whose one-sided derivatives satisfy condition 2 does not jeopardize the existence of a solution.

5.4. *Zone models*

Eqs. (4.9) restrict the form of zone models. Strictly speaking, a zone model should not change the densities at the flow element connections, as the solution algorithm changes the zone reference pressure (condition 5). Furthermore, the connection pressures should change in lockstep with the reference pressure (condition 4).

Some multizone programs enforce these relations, by not updating the zone densities during the Newton–Raphson search for the zone reference pressures. Probably this is an unnecessary precaution. At room temperature, the zone model given by Eqs. (2.4) has $\partial p_i^e / \partial p_i \approx 1.00001$. Computational experience with COMIS, which updates zone densities if requested by the user, suggests it is safe to ignore such small deviations from Eqs. (4.9)—that is, to approximate the Jacobian as symmetric.

The zone model of Eqs. (2.4) also shows that coupling the zone temperatures to the flows, in violation of condition 6, destroys symmetry. Consequently, coupled thermal systems allow multiple solutions to the airflow system [25].

6. Conclusions

This paper proves that under the conditions listed in the introduction, a nodal airflow network: (1) yields symmetric positive-definite systems; (2) has a unique solution; (3) produces system matrices with reduced storage and factorization costs; and (4) must be solvable, barring numeric effects, by descent-based line search methods. Several of these assumptions and system properties have not been described, or proved rigorously, in the building airflow literature.

The paper also proves that relaxing some of those conditions produces asymmetric, nonsingular system matrices, guaranteeing at least one solution, and likewise admitting line search methods.

In practice, some of those conditions can be relaxed still further without jeopardizing the numerical solution of the airflow network. The paper describes typical examples of how these restrictions affect zone and flow element models.

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